Model Spaces in Riemannian and Sub-Riemannian Geometries

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Contents

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1 Introduction

Overview of the Problem

The development of Riemannian geometry has been highly influenced by certain spaces with maximal symmetry called *model spaces*. Their ubiquity presents itself throughout differential geometry from the classical Gaussian map for surfaces to comparison theorems based on volume, the Laplacian, or Jacobi fields [Pet16]. Following in the footsteps of Klein's Erlangen program, model spaces fit with the approach of investigating the symmetries of a geometric object to understand the object itself. In the Riemannian setting, work by Wilhelm Killing and Heinz Hopf among others resulted in the complete classification of the Riemannian model spaces in the celebrated Killing-Hopf theorem. The theorem states that the only model spaces in Riemannian geometry are the spheres, the hyperbolic spaces, and the Euclidean spaces with their standard structures. This result is not only influential but also remarkable due to the fact that all the model spaces were already known and well examined prior to the classification.

In recent years sub-Riemannian geometry has emerged as an active field of research with ties to optimal control theory, Hamiltonian mechanics, geometric measure theory, and harmonic analysis. Nevertheless, new results in this subject are often only derived for special classes of structures such as Carnot groups and contact structures. This is not due to a lack of ability, but rather that the absence of a canonical connection as present in Riemannian geometry has complicated issues. Thus it is of interest to enlarge the concept of model spaces to the sub-Riemannian setting to establish reference spaces.

We say that a Riemannian manifold (M, g) is a *(Riemannian) model space* if it is simply connected, complete, and has constant sectional curvature. A definition based on sectional curvature, although advantageous in the Riemannian setting, does not generalize to sub-Riemannian geometries in an obvious way. There is however an equivalent definition of being a model space by using the isometry group: An n-dimensional Riemannian manifold (M, g) is a model space if it is simply connected and

$$\dim(\mathrm{Isom}(M)) = \max(n) := \frac{n(n+1)}{2},$$

where $\mathrm{Isom}(M)$ denotes the Lie group of all isometries of M. This definition gives rigorous weight to the term "maximal symmetry" as the dimension of any n-dimensional Riemannian manifold's isometry group is always less than or equal to $\max(n)$. Utilizing this observation is the start of sub-Riemannian model spaces.

A *sub-Riemannian model space* is a bracket generating sub-Riemannian geometry $(Q, \mathcal{H}, g)$, where Q is a simply connected manifold satisfying the following *symmetry condition*: For any points $p, q \in Q$ and any linear isometry $\phi : \mathcal{H}_p \to \mathcal{H}_q$ there exists a smooth isometry $\Phi : Q \to Q$ such that $d\Phi|_{\mathcal{H}_p} = \phi$. When $\mathcal{H} = TQ$ one can conclude from Theorem 2.52 that this definition reduces to the definition of Riemannian model spaces. It has been showed in [Gro16, Proposition 3.2] that any sub-Riemannian model space possesses a canonical *partial connection*. This fact opens up the study of what is called the *horizontal holonomy* of a sub-Riemannian model space. The symmetric nature of the model spaces ensure that the horizontal holonomy is polarizing in the sense that it is either as large as possible or zero. These two extreme cases give information about potential Lie group structures and whether its bundle of orthonormal frames provides us with a new sub-Riemannian model space.

To determine which sub-Riemannian geometries are model spaces, the theory of Gromov-Hausdorff convergence will be employed to produce a powerful invariant: Roughly speaking, if (M, d, m_0) is a pointed metric space then the *tangent cone* of M at the point m_0 is given by

$$CT_{m_0}M = \lim_{\lambda \to \infty} (\lambda M, m_0),$$

where the limit is in the sense of local Gromov-Hausdorff convergence. This theory is developed in Gromov's paper and applied specifically to the sub-Riemannian setting in Bellaíche's paper, both of which can be found in [BR96]. Applying this procedure to a sub-Riemannian geometry $(Q, \mathcal{H}, g)$ where the distance is the *Carnot-Carathéodory distance* d_{CC} gives the resulting space a Carnot group structure. Actually, when $(Q, \mathcal{H}, g)$ is a sub-Riemannian model space then the tangent cone at any point is also a model space. This provides an invariant which is fundamental when sub-Riemannian model spaces are classified.

In [Gro16, Theorem 5.6], all sub-Riemannian model spaces of step two are classified. They turn out to be, up to a technicality, the isometry groups of the Riemannian model spaces with left-invariant structures. Although a fundamental result, it does not feature the same challenges as one faces for higher steps. The main goal of the thesis is to obtain some insight into the difficulty with higher steps and rank by classifying all sub-Riemannian model spaces of step and rank three. In Section 5.6 we have enough terminology to explain why this is the natural next step to consider for the classification problem of sub-Riemannian model spaces.

Motivated by tangent cones, the classification procedure will begin in Section 5.1 with the Carnot groups which are also sub-Riemannian model spaces. This involve understanding how spatial rotations and reflections can be represented as transformations of the free nilpotent Lie algebra of step and rank three. The result is captured in Theorem 5.1 and shows existence and uniqueness of three Carnot groups of step and rank three which are model spaces. They will be denoted by $\mathcal{C}_{3,3}$, $\mathcal{A}_{3,3}$, and $N[3,3]$ and their growth vectors are $(3,6,9)$, $(3,6,11)$, and $(3,6,14)$, respectively. Through the tangent cone construction, this will imply that any sub-Riemannian model space of step and rank three has dimension 9,11, or 14. The sub-Riemannian model spaces of step and rank three are thus divided into three classes based on their tangent cone and are classified in Section 5.3. Surprisingly, the number of parameters needed to describe each class varies. The sub-Riemannian model spaces with tangent cone $\mathcal{C}_{3,3}$ are described by two parameters. This contrasts the Riemannian model spaces which are specified solely by their sectional curvature after fixing a dimension.

In [Gro16, Example 4.1] the tangent cone of any sub-Riemannian model spaces with step two and rank n is showed to be isometric to the free nilpotent Lie Group $N[n,2]$ with n generators of step two. However, in Theorem 5.9 we will show that if $(Q, \mathcal{H}, g)$ is any sub-Riemannian model space with tangent cone $N[3,3]$ then

$$Q \simeq N[3,3].$$

Hence there are (up to isometry) no nontrivial sub-Riemannian model spaces of step and rank three which have as tangent cone the free nilpotent Lie group $N[3,3]$. We can, somewhat simplistically, summarize the main results of the thesis in the following theorem.

Theorem 1.1. *The sub-Riemannian model spaces with step and rank three have dimension* 9,11*, or* 14*.*

- *Those with dimension* 9 *are parametrized by a two-parameter family of non-isometric spaces.*
- *Those with dimension* 11 *are parametrized by a one-parameter family of non-isometric spaces.*
- *The free nilpotent Lie group* $N[3,3]$ *with three generators of step three is the unique sub-Riemannian model space of dimension* 14*.*

Prerequisites

The thesis is written assuming the reader is familiar with basic notions in differential topology including vector bundles and differentiable forms. We have dedicated Appendix A.1 to vector valued forms and constructions on them, as this topic might be unfamiliar to the reader. The theory of Lie groups will be employed throughout the thesis, see Appendix A.2 for a short introduction to the central ideas used. Some representation theory will be needed for Chapter 4 and Chapter 5. Basic definitions can be found in Appendix A.2, although a general familiarity with representation theory of either finite or continuous type will make certain arguments clearer. We do not assume any previous knowledge of Riemannian manifolds or differential geometry in general. The beginning of the thesis is partly dedicated to survey central notions in differential geometry as well as fixing terminology and notation. We have tried to avoid stating results in either excessive categorical language or lengthy coordinate descriptions to cause the least amount of distress for readers with different backgrounds.

Structure and Style of the Thesis

We have chosen to go through the classification of Riemannian model spaces in Chapter 2 in reasonable detail for three reasons: Firstly, it serves as a prelude and provides motivation for studying sub-Riemannian model spaces. Secondly, this allows us to gain familiarity with isometry groups and symmetric spaces which will be essential for the sub-Riemannian setting as well. Thirdly, how the theory of curvature, Killing vector fields, and symmetric spaces come together to classify the model spaces in Riemannian geometry is (in the author's opinion) one of the success stories of modern mathematics.

Although the main motivation for investigating principal bundles in Chapter 3 is as a technical tool needed in later arguments, we survey results and ideas making the exposition more complete and interesting. We go as far as presenting the classical Ambrose-Singer theorem on holonomy without loosing track of the main purpose of the chapter. In the beginning of Chapter 4 we develop theory and examples from sub-Riemannian geometry, providing gradually the final prerequisites needed for the classification such as Gromov-Hausdorff convergence and nilpotentization. The rest of the chapter examines the general properties of sub-Riemannian model spaces. In Chapter 5 we classify all sub-Riemannian model spaces of step and rank three. We have tried to divide the classification into manageable pieces for the reader as it has several technical steps. Finally, the thesis ends with a discussion on the results obtained and where to go next.

Original Results

In the first four chapters with the exception of Section 4.7, the only originality is in some examples and a choice of exposition. In Proposition 4.41 we classify the sub-Riemannian model spaces which are contact geometries and give an explicit description using a series of well known identifications. We use this together with results from [AB12] to give a criterion in Corollary 4.43 for when certain three dimensional sub-Riemannian geometries are model spaces based on their Reeb vector field. Except for this, the original part of the thesis is all of Chapter 5. The whole chapter deals with the classification of sub-Riemannian model spaces of step and rank three. Except for auxiliary lemmas and remarks, the major standouts in Chapter 5 are Theorem 5.1, Theorem 5.3, Theorem 5.8, and Theorem 5.9. These theorems provide the classification of sub-Riemannian model spaces of step and rank three, which is the scientific contribution of this thesis.

2 Riemannian Geometry and Their Model Spaces

This chapter has three main goals: To develop the language of Riemannian geometry, to study the relationship between symmetric and constant curvature spaces, and to prove the classification of model spaces in Riemannian geometry. The symmetric space approach will provide us with an equivalent definition of Riemannian model spaces based on their isometry groups suitable for generalization to the sub-Riemannian setting. It is assumed that the reader is familiar with basic differential topology, a comprehensive source is [Lee13]. We moreover use the Einstein summation convention for convenience, see [Lee97, Chapter 2] for a short explanation. Throughout the thesis a manifold will always refer to a second countable topological Hausdorff space with a maximal smooth structure.

2.1 Basic Notions in Riemannian Geometry

Everything in this section which is not referred to other sources can be found in [Lee97] or [O'N83]. We use the notation $\mathfrak{X}(M)$ for vector fields on M and $\mathfrak{X}^*(M)$ for the one-forms. The tensor bundles and the bundles of alternating tensors are denoted by

$$T^k_l M = \coprod_{p\in M} T^k_l(T_pM) \quad \text{and} \quad \Lambda^k M = \coprod_{p\in M} \Lambda^k(T_pM),$$

respectively. We follow the convention that $T^k_l(T_pM)$ denotes the space of multilinear maps from l copies of T^*_pM and k copies of T_pM to $\mathbb{R}$. The sections of $T^k_l M$ will be called *tensor fields of type* (k,l) while the sections of $\Lambda^k M$ will be called *(differential) k-forms*. A tensor field of type $(k,0)$ will simply be referred to as a k-tensor field to improve readability. Otherwise, if $E \to M$ is a (smooth) vector bundle we will denote the sections of E over M by $\Gamma(E)$.

Definition 2.1. A *Riemannian manifold* (M,g) is a manifold M equipped with a symmetric 2-tensor field g which is positive definite, that is, $g_p(X,X) > 0$ whenever $X \in T_pM$ is non-zero. We call the tensor field g a *Riemannian metric*. Sometimes the notation $\langle \cdot,\cdot\rangle$ will be used in place of $g_p(\cdot,\cdot)$ if p is understood and we want to emphasize that g induces an inner product in each tangent space.

2.1.1 Metric Structure and Isometries

The Riemannian metric allows us to measure the length of a curve as follows: Recall that a curve $\gamma : I = [a,b] \to \mathbb{R}^n$ is *absolutely continuous* if the derivative $\dot\gamma(t)$ exists for almost every $t \in I$ (with respect to the Lebesgue measure) and γ can be recovered from its derivative through the fundamental theorem of calculus, that is,

$$\gamma(t) = \gamma(c) + \int_c^t \dot\gamma(s)\, ds,$$

for every $c \in I$. Other equivalent definitions of absolutely continuous curves are used in the literature, see [Rud87, Theorem 7.18]. This definition is also valid for curves on a manifold due to the fact that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, then $F \circ \gamma$ is still absolutely continuous. We denote the absolutely continuous curves taking values in M by $AC(I,M)$, where we allow curves with different intervals as domains even though the notation might suggest otherwise.

Definition 2.2. Let (M,g) be a connected Riemannian manifold. For $p,q \in M$, define the *distance* $d(p,q)$ to be

$$d(p,q) = \inf_{\gamma\in AC(I,M)} \int_a^b \|\dot\gamma(t)\|\, dt, \quad \gamma(a) = p, \quad \gamma(b) = q,$$

where each integral is over the domain $[a,b]$ of the curve γ in question.

With this definition, (M, g, d) becomes a metric space whose topology induced by the metric d is the same as the original topology. We will refer to the metric d induced by the infimum over absolutely continuous curves as a distance function, so not to confuse it with the Riemannian metric g. Moreover, whenever we refer to the distance function it will be implicitly assumed that the manifold in question is connected.

Example 2.3. A vast class of Riemannian manifolds is provided by the following construction: Let G be a Lie group with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Choose an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ and translate it by left translations. More precisely, for $p \in G$ define

$$\langle v, w \rangle_p = \langle dL_{p^{-1}}(v), dL_{p^{-1}}(w) \rangle_{\mathfrak{g}}$$

for $v, w \in T_pG$, where $L_{p^{-1}}$ denotes left translation by p^{-1}. This gives a Riemannian metric on G which is invariant under left translations. The matrix Lie groups $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, and $O(n, \mathbb{R})$ can easily be given such structures by considering the inner product on their Lie algebras which is inherited from $gl(n, \mathbb{R}) \simeq \mathbb{R}^{2n}$. A *bi-invariant metric* on a Lie group is a Riemannian metric which is invariant under both left and right translations. Compact Lie groups always admit bi-invariant metrics, see [Hal15, Appendix D].

The following three examples are provisionally called the *canonical spaces* in Riemannian geometry. Their symmetric nature will determine them uniquely in a sense we shall describe during this chapter.

Example 2.4 (Euclidean Spaces)**.** The most straightforward example of a Riemannian manifold is $\mathbb{R}^n$ with the usual inner product in each tangent space. We have a canonical identification of the tangent spaces with $\mathbb{R}^n$ itself and the Riemannian metric $\bar{g}$ is given in standard coordinates by $\bar{g} = \delta_{ij} dx^i dx^j$.

Example 2.5 (Spheres)**.** Denote by S^n_R the n-sphere with radius R. The inclusion $S^n_R \hookrightarrow \mathbb{R}^{n+1}$ induces a Riemannian metric $g_R = i^* \bar{g}$ on S^n_R called the *round metric of radius* R. Because the inclusion is an immersion we obtain a positive definite metric.

Example 2.6 (Hyperbolic Spaces)**.** The *Lorentz metric* is a nondegenerate, although not positive definite, scalar product on $\mathbb{R}^{n+1}$. It is given in coordinates $(\xi^1, \dots, \xi^n, \tau)$ by

$$m = (d\xi^1)^2 + \cdots + (d\xi^n)^2 - (d\tau)^2.$$

When $n = 3$ the Lorentz metric is also called the *Minkowski-metric* and is fundamental in describing Einstein's special theory of relativity, see [O'N83, Chapter 6]. If we restrict the Lorentz metric to the upper sheet of the hyperboloid

$$\mathbb{H}^n_R = \left\{ (\xi^1, \dots, \xi^n, \tau) \in \mathbb{R}^{n+1} : \tau^2 - |\xi|^2 = R^2 \right\},$$

we obtain a positive definite metric denoted by h_R. We call $(\mathbb{H}^n_R, h_R)$ *hyperbolic space of radius* R and simply *hyperbolic space* when $R = 1$.

We will only be interested in the canonical spaces where $n \geq 2$. These spaces are all simply connected, see [Hat02, Proposition 1.14] for the less obvious case regarding the spheres. We also mention that our definition of simply connectedness includes connectedness, which is not always the case in the literature. The canonical spaces will be returned to several times to illustrate the concepts presented in this chapter. An underlying theme throughout this chapter is to grasp the apparent symmetric nature of these spaces in a precise way. To do this we need to define when two Riemannian manifolds should be considered identical. Two Riemannian manifolds (M, g) and (N, h) are said to be *isometric* if there exists a diffeomorphism $\phi : M \to N$ such that $\phi^* h = g$, where

$$(\phi^* h)_p(X, Y) = h_{\phi(p)}(d\phi(X), d\phi(Y)),$$

for every $p \in M$. Thus the differential of an isometry is a linear isometry at each point. It is called a *local isometry* if the map is only a local diffeomorphism. Riemannian geometry is mainly concerned with properties which is invariant under local or global isometries. All isometries from a Riemannian manifold (M, g) to itself form a group under composition called the *isometry group* of (M, g) which is denoted by $\mathrm{Isom}(M)$.

2.1.2 Connections and Parallel Transport

Definition 2.7. Let M be a manifold and $E \to M$ a vector bundle over M. A *connection* ∇ on E is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E), \quad (X, Y) \longmapsto \nabla_X Y$$

which is linear over $C^\infty(M)$ in the first component, linear over $\mathbb{R}$ in the second, and satisfies the Leibniz rule

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y,$$

where $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$, and $Y \in \Gamma(E)$. We call $\nabla_X Y$ the *covariant derivative* of Y in the direction of X. If $E = TM$ the connection is called *affine*.

If ∇ is a connection on a vector bundle $E \to M$, then $\nabla_X Y|_p$ only depends on the value of X at the point p and on the values of Y along a curve tangent to X_p. If $\{E_i\}$ is a local frame for an open subset $U \subset M$ we define the *Christoffel symbols* of the connection with respect to this frame to be $\Gamma_{ij}^k = \langle \nabla_{E_i} E_j, E_k \rangle$. Affine connections exist on any manifold and can be constructed in a single chart by the formula

$$\nabla_X Y = \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \partial_k,$$

and patched together using partitions of unity. Any affine connection ∇ on M admits a unique extension to a connection on all the tensor bundles. This extension will still be denoted by ∇ and is a derivation with respect to the tensor product. Given a tensor field F of type (k, l), the extension is given by

$$\begin{aligned}(\nabla_X F)(\omega^1, \ldots, \omega^l, Y_1, \ldots, Y_k) &= X\left(F(\omega^1, \ldots, \omega^l, Y_1, \ldots, Y_k)\right) \\ &- \sum_{i=1}^{l} F(\omega^1, \ldots, \nabla_X \omega^i, \ldots, \omega^l, Y_1, \ldots, Y_k) - \sum_{j=1}^{k} F(\omega^1, \ldots, \omega^l, Y_1, \ldots, \nabla_X Y_j, \ldots, Y_k),\end{aligned}$$

for $X, Y_1, \ldots, Y_k \in \mathfrak{X}(M)$ and $\omega^1, \ldots, \omega^l \in \mathfrak{X}^*(M)$.

We say that an affine connection ∇ on a Riemannian manifold is *compatible with the metric* if

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle,$$

for $X, Y, Z \in \mathfrak{X}(M)$. Moreover, we can associate the *torsion tensor*

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

to the affine connection ∇ for $X, Y \in \mathfrak{X}(M)$. If the torsion tensor is identically zero then the affine connection is said to be *symmetric*. Together, compatibility and symmetry is enough to determine a unique connection which is especially intertwined with the geometry of the Riemannian manifold.

Theorem 2.8. *On every Riemannian manifold (M, g) there exists a unique affine connection which is symmetric and compatible with the metric. This connection will be referred to as the Levi-Civita connection of (M, g).*

We now give a brief account of geodesics with a view towards Riemannian symmetric spaces in Section 2.5. If $\gamma : I \to M$ is a smooth curve on a manifold M we denote by $\mathcal{T}(\gamma)$ all vector fields along γ, that is, the smooth maps $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$. If $\widetilde{V}$ is a vector field on M which coincides with $V \in \mathcal{T}(\gamma)$ on $T_{\gamma(t)}M$, then $\widetilde{V}$ is said to be an *extension* of V. The following lemma enables us to differentiate vector fields along smooth curves.

Lemma 2.9. *Let M be a manifold with affine connection ∇ and let $\gamma : I \to M$ be a smooth curve in M. Then ∇ determines a unique operator*

$$D_t : \mathcal{T}(\gamma) \longrightarrow \mathcal{T}(\gamma)$$

which is linear over $\mathbb{R}$, satisfies the product rule

$$D_t(fV) = \dot{f}V + fD_tV,$$

and has the property that

$$D_tV(t) = \nabla_{\dot{\gamma}(t)}\widetilde{V}$$

whenever $\widetilde{V}$ is an extension of V. The operator D_t will be called the covariant derivative along γ.

Definition 2.10. Let γ be a smooth curve on a manifold M with an affine connection ∇. We call γ a *geodesic* if the vector field $D_t\dot{\gamma}$ is identically zero.

If we choose coordinates (x^i) on an open set U, then $\gamma : I \to U$ is a geodesic if and only if its component functions $\gamma(t) = \big(x^1(\gamma(t)), \ldots, x^n(\gamma(t))\big)$ satisfies the *geodesic equation*

$$\ddot{x}^k(\gamma(t)) + \dot{x}^i(\gamma(t))\dot{x}^j(\gamma(t))\Gamma^k_{ij}(\gamma(t)) = 0, \tag{2.1}$$

for every $k = 1, \ldots, n$. The following existence and uniqueness result ensures us that geodesics exist in great abundance. By specifying a point, initial time, and initial velocity we can always find geodesics satisfying these conditions.

Proposition 2.11. *Let M be a Riemannian manifold with an affine connection and suppose we are given $p \in M$, $V_0 \in T_pM$, and $t_0 \in \mathbb{R}$. Then there exists a geodesic $\gamma : I \to M$ with $t_0 \in I$ such that $\gamma(t_0) = p$ and $\dot{\gamma}(t_0) = V_0$. Moreover, any two such geodesics agree on their common domain.*

Hence any $V \in T_pM$ determines a unique maximal geodesic with initial velocity V passing through p at time $t = 0$. We denote this geodesic by γ_V. If $\gamma : I \to M$ is a smooth curve, then $V \in \mathcal{T}(\gamma)$ is said to be *parallel* along γ if $D_tV = 0$ for every $t \in I$. Given $t_0 \in I$ and $V_0 \in T_{\gamma(t_0)}M$ there exists a unique vector field $V \in \mathcal{T}(\gamma)$ which is parallel along γ such that $V(\gamma(t_0)) = V_0$. We call this vector field the *parallel translate* of V_0 along γ. If $t_1 \in I$ this gives a well defined linear isomorphism

$$P^{t_1}_{t_0} : T_{\gamma(t_0)}M \longrightarrow T_{\gamma(t_1)}M$$

sending $V_0 \in T_{\gamma(t_0)}M$ to $V(t_1)$, where V is the parallel translate of V_0 along γ. This map is called *parallel transport* and is a linear isometry if and only if the connection ∇ underlying the covariant derivative is compatible with the metric. From now on, we assume that the underlying connection is the Levi-Civita connection unless otherwise stated.

Let us view the distance between two points $p, q \in M$ as a functional acting on absolutely continuous curves starting at p and ending at q. Then the minimizing curves (those who realize the distance) are always geodesics, hence smooth. Moreover, geodesics are locally minimizing although not necessarily globally so. A Riemannian manifold (M, g) is said to be *geodesically complete* if the domain of every geodesic can be extended to the whole real line.

Theorem 2.12 (Hopf-Rinow)**.** *Geodesic completeness is equivalent to completeness as a metric space with respect to the distance function. In either case, any two points in the same connected component can be joined by a geodesic.*

One of the conditions is often significantly easier to check than the other. A simple application of the Hopf-Rinow Theorem is that compact manifolds are always geodesically complete, regardless of the choice of Riemannian metric.

2.1.3 The Exponential Map

Definition 2.13. Given a Riemannian manifold (M, g) we denote by $\mathcal{E} \subset TM$ the elements $V \in TM$ such that γ_V is defined on an interval containing $[0, 1]$. Then

$$\mathrm{Exp} : \mathcal{E} \longrightarrow M, \quad \mathrm{Exp}(V) = \gamma_V(1)$$

is called the *exponential map* of (M, g). For each $p \in M$, the exponential map restricted to $\mathcal{E}_p := \mathcal{E} \cap T_pM$ is denoted by Exp_p.

The exponential map provides a way to collectively examine geodesics and how they behave when we change the initial point or the initial velocity. We use a capital "E" in Exp to distinguish it from the Lie group exponential map $\exp : \mathfrak{g} \to G$ although they agree for bi-invariant metrics on Lie groups, see [O'N83, Proposition 11.9]. The domain of the exponential map $\mathcal{E} \subset TM$ is an open set and $\mathrm{Exp} : \mathcal{E} \to M$ is smooth. Moreover, the restriction Exp_p has a star-shaped domain with respect to the origin. For each $V \in TM$ the geodesic γ_V has the form

$$\gamma_V(t) = \mathrm{Exp}(tV),$$

whenever either side is defined. The exponential map satisfies the following naturality condition: If (N, h) is another Riemannian manifold and $\phi : M \to N$ is an isometry, then the following diagram commutes

$$\begin{array}{ccc} \mathcal{E}_p & \xrightarrow{d\phi_p} & \mathcal{E}_{\phi(p)} \\ \downarrow{\scriptstyle \mathrm{Exp}_p} & & \downarrow{\scriptstyle \mathrm{Exp}_{\phi(p)}} \\ M & \xrightarrow{\phi} & N \end{array}$$

for every $p \in M$. From this it follows, using the uniqueness of geodesics, that isometries map geodesics to geodesics. For every $p \in M$ the restricted exponential map is a local diffeomorphism at the origin, implying that the following definition is never vacuous.

Definition 2.14. A neighbourhood $\mathcal{U}$ of $p \in M$ is called a *normal neighbourhood* if

$$\mathrm{Exp}_p\Big|_V : V \to \mathcal{U}$$

is a diffeomorphism, where V is a star shaped neighbourhood of the origin in T_pM. If $V = B_\epsilon(0) \subset T_pM$, where $B_\epsilon(0)$ denotes the ball in T_pM centered at the origin with radius ϵ, then $\mathcal{U}$ is called a *geodesic ball.*

An important feature of the exponential map is that it furnishes us with normal coordinates which, although hard to compute explicitly, is useful for "straightening out geodesics": Let $\{E_i\}$ be an orthonormal basis for T_pM viewed as a map $E : \mathbb{R}^n \to T_pM$. Then if $\mathcal{U}$ is a normal neighbourhood of $p \in M$, the map

$$x = E^{-1} \circ {\mathrm{Exp}_p}^{-1} : \mathcal{U} \longrightarrow \mathbb{R}^n$$

is called *normal coordinates* centered at the point p. In such normal coordinates, the components of the Riemannian metric g at the point p are $g_{ij} = \delta_{ij}$. Furthermore, the geodesics starting at p with initial velocity $V = V^i\partial_i \in T_pM$ have normal coordinates

$$\gamma_V(t) = (tV^1, \ldots, tV^n).$$

The following result illustrates the rigidity of isometries and will be used several times throughout the thesis.

Proposition 2.15. *Let $\phi, \psi : (M, g) \to (N, h)$ be isometries between Riemannian manifolds with M connected and such that there exists a point $p \in M$ with $d\phi_p = d\psi_p$. Then $\phi \equiv \psi$.*

Proof. The proof is a connectedness argument together with using the interplay between isometries and geodesics. Let

$$\mathcal{P} = \{q \in M : \ d\phi_q = d\psi_q\}.$$

By assumption $\mathcal{P}$ is nonempty and, since isometries are continuous, closed. Given $p \in \mathcal{P}$, let $\mathcal{U}$ be a normal neighbourhood centered at p and $q \in \mathcal{U}$ be arbitrary. Then there exists a geodesic γ_V starting at p with initial velocity $V \in T_pM$ such that $\gamma_V(t_0) = q$ for $t_0 \in [0,1]$. Since isometries transform geodesics to geodesics, the smooth curves

$$\phi \circ \gamma_V(t) \quad \text{and} \quad \psi \circ \gamma_V(t)$$

are geodesics which starts at $\psi(p) = \phi(p)$ and have initial velocity $\phi_* V = \psi_* V$. Hence by uniqueness of geodesics they are equal for every $t \in [0,1]$. Evaluating at t_0 we obtain $\phi(q) = \psi(q)$. Since the two isometries are equal on the open subset $\mathcal{U}$, their differential is also equal there. Hence $\mathcal{P}$ is open and $\phi \equiv \psi$. □

2.2 Homogeneous Riemannian Manifolds and Curvature

In this section we will investigate a particular type of Riemannian manifolds, namely the homogeneous ones. They will be related to quotient manifolds formed by Lie groups and we will see how questions about their geometry can be transformed to questions in Lie group theory. Thereafter, we shall survey the different notions of curvature in Riemannian geometry leading us to a precise definition of a Riemannian model space. Basic terminology and results regarding Lie group actions on manifolds is reviewed in Appendix A.2.2.

Definition 2.16. A Riemannian manifold (M, g) is said to be *homogeneous* if there exists a Lie group acting transitively by isometries on (M, g).

Intuitively, homogeneous Riemannian manifolds “look the same at every point”. It is a rather strong property as it implies completeness by comparing geodesic balls at different points on the manifold, see [O’N83, Remark 9.37]. We will now provide an alternative view of homogeneous spaces by relating them to quotient manifolds arising from Lie groups.

Proposition 2.17. *Any homogeneous Riemannian manifold (M, g) is isometric to the quotient manifold of a Lie group G with a compact subgroup K. Moreover, the Lie group G can be taken to be the isometry group of M and K the isotropy group of any point in M with respect to the action of the isometry group on M.*

Proof. For any Riemannian manifold M there is a unique way to make the isometry group $\mathrm{Isom}(M)$ into a finite-dimensional Lie group such that the natural action of $\mathrm{Isom}(M)$ on M is smooth, see [Pal57, Chapter 4]. Fix $x_0 \in M$ and consider the set $\mathbb{M} = \mathrm{Isom}(M)/K_{x_0}$, where K_{x_0} denotes the isotropy group of x_0. We denote by $O(T_{x_0}M)$ the orthogonal transformations from $T_{x_0}M$ to itself with respect to the inner product g_{x_0}. Recall that isometries are determined by their differential at a single point by Proposition 2.15. The isotropy group K_{x_0} is compact as the embedding $K_{x_0} \to O(T_{x_0}M)$ given by $\phi \mapsto d\phi_{x_0}$ exhibits K_{x_0} as a closed subspace of $O(T_{x_0}M)$. Two different isotropy groups K_{x_0} and K_{x_1} corresponding to different points are conjugate by an inner automorphism. Hence we omit the subscript from the notation as it is of minor importance.

Since K is a closed subgroup of $\mathrm{Isom}(M)$ the quotient space $\mathbb{M}$ is a manifold by [War83, Theorem 3.58]. Moreover, the projection

$$\begin{aligned} \pi_{\mathbb{M}} : \mathrm{Isom}(M) &\longrightarrow \mathbb{M} \\ \phi &\longmapsto [\phi] \end{aligned}$$

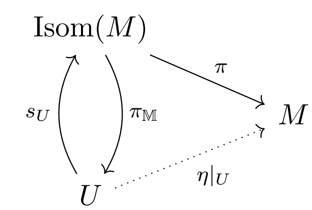
is smooth and admits local sections. The evaluation map $\pi : \mathrm{Isom}(M) \to M$ sending an isometry ϕ to $\phi(x_0)$ descends to a map

$$\begin{aligned} \eta : \mathbb{M} &\longrightarrow M \\ [\phi] &\longmapsto \phi(x_0) \end{aligned}$$

by the definition of K. Assume $\eta([\phi_1]) = \eta([\phi_2])$ for isometries $\phi_1, \phi_2 \in \mathrm{Isom}(M)$. Then $\phi_2^{-1} \circ \phi_1$ stabilizes x_0 and it follows that

$$[\phi_2] = [\phi_2 \circ (\phi_2^{-1} \circ \phi_1)] = [\phi_1],$$

showing injectivity of the map η. As M is homogeneous by assumption it is clear that η is also surjective. To show that η is smooth, let $[\phi]$ be an arbitrary element of $\mathbb{M}$. Choose an open set U containing $[\phi]$ such that there exists a map $s_U : U \to \mathrm{Isom}(M)$ with the property that $\pi_{\mathbb{M}} \circ s_U = Id_U$.



It follows that

$$\pi \circ s_U([\phi]) = \eta \circ \pi_{\mathbb{M}} \circ s_U([\phi]) = \eta([\phi]),$$

which exhibits η as the composition of smooth maps. By the Inverse Function Theorem it suffices to show that the differential $d\eta$ is everywhere invertible for η to be a diffeomorphism. This follows from the Equivariant Rank Theorem given in [Lee13, Theorem 7.25]. By pulling back the metric g on M to a Riemannian metric $g_{\mathbb{M}}$ on $\mathbb{M}$ gives that (M, g) is isometric to $(\mathbb{M}, g_{\mathbb{M}})$. ☐

Example 2.18. Let us consider the canonical spaces $\mathbb{R}^n$, S^n_R, and $\mathbb{H}^n_R$. Regarding Euclidean space the translations $\phi_r(s) = r + s$ are clearly isometries and act transitively since $\phi_{s-r}(r) = s$ for any $r, s \in \mathbb{R}^n$. The orthogonal transformations $O(n)$ also acts on $\mathbb{R}^n$ by isometries. It is a standard fact in elementary differential geometry that the isometry group $E(n)$ of $\mathbb{R}^n$ is the semi-direct product

$$E(n) = \mathbb{R}^n \rtimes O(n)$$

called the *Euclidean group*, where $\mathbb{R}^n$ represents the translations. The map $\phi : O(n) \to \mathrm{Aut}(\mathbb{R}^n)$ which induces the semi-direct product is simply evaluation.

We will now show that $O(n+1)$ act transitively by isometries on S^n_R. Let $p, q \in S^n_R$ and normalize them such that $\tilde{p} = p/R$ and $\tilde{q} = q/R$. Take two orthonormal bases $\{E_i\}_{i=1}^n$ and $\{F_j\}_{j=1}^n$ for $T_pS^n_R$ and $T_qS^n_R$, respectively. Viewing $\tilde{p}$ and $\tilde{q}$ as vectors in $\mathbb{R}^n$, notice that the collections $\{E_1, \ldots, E_n, \tilde{p}\}$ and $\{F_1, \ldots, F_n, \tilde{q}\}$ are both orthonormal in $\mathbb{R}^n$. We collect these vectors as columns of the matrices

$$\alpha = \begin{bmatrix} E_1 & \ldots & E_n & \tilde{p} \end{bmatrix}, \quad \beta = \begin{bmatrix} F_1 & \ldots & F_n & \tilde{q} \end{bmatrix}.$$

As the columns of both α and β are orthonormal they belong to $O(n+1)$ and it is straightforward to calculate that $\beta \circ \alpha^{-1}(p) = q$. Since the action of the orthogonal group on $\mathbb{R}^n$ preserves the Euclidean inner product, the restriction to S^n_R is an isometric action as well.

Let $O(n,1)$ be the linear transformations on $\mathbb{R}^{n+1}$ which preserves the Lorentz metric. Each element in $O(n,1)$ preserves the hyperboloid $\tau^2 - |\xi|^2 = R^2$ and we denote by $O_+(n,1)$ the ones which map the upper sheet to itself. A similar argument as for the spheres shows that $O_+(n,1)$ acts transitively by isometries on $\mathbb{H}^n_R$, for details see [Lee97, Proposition 3.6].

Notice that for all three canonical spaces we have something more than the homogeneous property. Each of them possesses an isometry group which acts transitively on orthonormal frames. Riemannian manifolds with this additional property will be called *frame-homogeneous*. This property will be essential for the description of spaces with maximal symmetry during the chapter. The infinite cylinder $S^1 \times \mathbb{R}$ with the product metric is homogeneous but not frame-homogeneous, see [O'N06, Chapter 8] for details.

Questions regarding the geometry of homogeneous spaces $M \simeq G/H := \mathrm{Isom}(M)/K_{x_0}$ can sometimes be answered by the following procedure: Notice that H acts on its own Lie algebra $\mathfrak{h}$ via the adjoint map, see Appendix A.2 for more details. The compactness of H implies that $\mathfrak{h}$ has an $\mathrm{Ad}(H)$-invariant complement $\mathfrak{m}$ in $\mathfrak{g} = \mathrm{Lie}(G)$. Since $\pi : G \to M$ is a submersion with $d\pi_e(\mathfrak{h}) = 0$ we get that

$$d\pi\Big|_{\mathfrak{m}} : \mathfrak{m} \longrightarrow T_{x_0}M$$

is an isomorphism. In effect, we have identified $\mathfrak{m}$ with the tangent space to M at x_0. If we require $d\pi|_{\mathfrak{m}}$ to be an isometry we get by [O'N83, Proposition 11.22] a one-to-one correspondence between $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$ and G-invariant metrics on M. Since M already came equipped with an invariant metric, this furnishes $\mathfrak{m}$ with an $\mathrm{Ad}(H)$-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{m}}$. If the inner product satisfies

$$\langle [X, Y]_{\mathfrak{m}}, Z \rangle = \langle X, [Y, Z]_{\mathfrak{m}} \rangle, \qquad \text{for } X, Y, Z \in \mathfrak{m}, \tag{2.2}$$

where $[X, Y]_{\mathfrak{m}}$ denotes the projection of $[X, Y]$ onto $\mathfrak{m}$, then M is said to be *naturally reductive*. If $H = \{e\}$ then condition (2.2) is equivalent to the bi-invariance of the metric on $M = G/\{e\} \simeq G$ by [O'N83, Proposition 11.9]. If M is naturally reductive then the geodesics on M starting at x_0 are given by projecting one-parameter subgroups arising from vectors in $\mathfrak{m}$, see [O'N83, Proposition 11.25]. The Riemannian symmetric spaces we define in Section 2.5 are all naturally reductive.

Example 2.19. We will now determine the geodesics in the canonical spaces as this is needed to compute their curvature and, moreover, illustrates the usefulness of frame-homogeneity. In Euclidean space the Christoffel symbols are all zero. Looking at the geodesic equation (2.1) reveals that the geodesics are precisely the straight lines with constant speed parametrizations. The computations are similar for spheres and hyperbolic spaces, so we will only provide a proof for the hyperbolic case. We are content with stating that the geodesics on S^n_R are the "great circles": the intersections of S^n_R with 2-planes through the origin with constant speed parametrizations. The reader should be aware that this can also be deduced through variational methods, see [vB04, Chapter 2] for the case of S^2.

For the hyperbolic space of radius R we now show that the geodesics are the "great hyperbolas": the intersections of $\mathbb{H}^n_R$ with 2-planes through the origin with constant speed parametrizations. Let us first consider the geodesic $\gamma : \mathbb{R} \to \mathbb{H}^n_R$ determined by

$$\gamma(0) = (0, \ldots, 0, R), \qquad \dot{\gamma}(0) = \frac{\partial}{\partial \xi^1}.$$

Assume that $\gamma(t_0)$ has a non-zero ξ^i coordinate for some $t_0 \in \mathbb{R}$ and $2 \leq i \leq n$. Then the map $\alpha \in O_+(n, 1)$ sending ξ^i to $-\xi^i$ and fixing the rest of the coordinates will map γ to itself, as it fixes both its initial point and initial velocity. This is clearly not possible since $\alpha(\gamma(t_0)) \neq \gamma(t_0)$. Hence γ must remain in the (ξ^1, τ) plane and is thus a constant speed parametrization of the hyperbola obtained by the intersection of this plane with $\mathbb{H}^n_R$. Let $\nu_v : \mathbb{R} \to \mathbb{H}^n_R$ be any other geodesic with $\dot{\nu}_v(0) = v \in T_p\mathbb{H}^n_R$ for $p \in \mathbb{H}^n_R$. By frame homogeneity there is an isometry $\beta \in O_+(n, 1)$ such that

$$\beta(0, \ldots, 0, R) = p, \quad d\beta\left(\frac{\partial}{\partial \xi^1}\right) = v.$$

Since isometries transform geodesics to geodesics we have that β maps γ to ν_v. Moreover, as $\beta \in O_+(n, 1)$ it also maps the (ξ^1, τ)-plane to another 2-dimensional plane Π through the origin. It follows that ν_v is the constant speed parametrization of $\mathbb{H}^n_R \cap \Pi$.

We will now present the various notions of curvature in Riemannian geometry, with the goal of describing sectional curvature as a higher dimensional generalization of Gaussian curvature for surfaces. The proof of all claims presented can be found in [Lee97, Chapters 7 and 8] except for the discussion on Gaussian curvature where details are given in [O'N06, Section 7.2].

Definition 2.20. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. The map

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for $X, Y, Z \in \mathfrak{X}(M)$ is called the *curvature endomorphism*.

The name and notation arises from fixing $X, Y \in \mathfrak{X}(M)$ so that $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is an endomorphism in the last input. It is straightforward to show that the curvature endomorphism on Euclidean space is identically zero. The curvature endomorphism measures how much the Riemannian manifold deviates from being *flat*, that is, locally isometric to Euclidean space. It satisfies the *Bianchi Identity*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0. \tag{2.3}$$

By using the Riemannian metric we can also consider it as a 4-tensor field by

$$Rm(X,Y,Z,W) = \langle R(X,Y)Z, W \rangle,$$

called the *curvature tensor*. It satisfies the symmetries

$$R_{ijkl} = -R_{jikl}, \quad R_{ijkl} = -R_{ijlk}, \quad R_{ijkl} = R_{klij}, \tag{2.4}$$

in any local frame $\{E_1, \ldots, E_n\}$, where R_{ijkl} is shorthand notation for $Rm(E_i, E_j, E_k, E_l)$. The curvature endomorphism and curvature tensor are both invariant under isometries. As they are both difficult to compute in general we can define the *Ricci curvature*, denoted by *Ric*, as the trace of the curvature endomorphism on the first and last indices. Hence the components of the Ricci curvature are given by

$$R_{ij} = R_{kij}{}^{k},$$

where $R_{kij}{}^{l}$ are the components of the curvature endomorphism. Other equivalent definitions of the Ricci curvature is also common in the literature by utilizing the symmetries (2.4). We may repeat this process, obtaining the *scalar curvature* by first raising the last entry of the Ricci curvature and then taking the trace again,

$$S := R_i{}^{i}.$$

Before describing sectional curvature we make a short excursion to look at Gaussian curvature as an intrinsic property of surfaces.

Let $M \subset \mathbb{R}^3$ be a smooth surface equipped with the induced metric g from $(\mathbb{R}^3, \bar{g})$. Recall from elementary differential geometry that the *principle curvatures* at $p \in M$, denoted by κ_1 and κ_2, are the minimum and maximum curvatures of M-geodesics passing through p. They depend on the specific embedding and are not intrinsic properties of the surface; consider for instance curling a sheet of paper slightly. Nevertheless, Gauss made the remarkable discovery that their product $K = \kappa_1 \kappa_2$ is an isometry invariant of (M, g). This result, known as Theorema Egregium, follows from the fact that the Gaussian curvature can be expressed via the curvature tensor as

$$K = \frac{Rm(X,Y,Y,X)}{|X|^2 |Y|^2 - \langle X, Y \rangle^2},$$

where X and Y constitute any basis for T_pM. Although Gaussian curvature is only applicable to smooth surfaces it has a higher dimensional analogue: sectional curvature.

Let us return to the setting where (M, g) is an arbitrary n-dimensional Riemannian manifold with $p \in M$. We will define sectional curvature as Gaussian curvature of certain 2-dimensional submanifolds of M containing p. Let Π be a 2-dimensional subspace of T_pM and choose a neighbourhood of zero $\mathcal{V} \subset T_pM$ where the restricted exponential map is a diffeomorphism. Then

$$S_\Pi := \mathrm{Exp}_p(\Pi \cap \mathcal{V})$$

is a 2-dimensional submanifold of M containing p called a *plane section*. Define the *sectional curvature* of M associated with Π at p to be the Gaussian curvature of S_Π at p with the induced metric from M. The notation $K(\Pi)$ or $K(X, Y)$ will be used when X, Y is any basis for Π. We regain the formula

$$K(X, Y) = \frac{Rm(X, Y, Y, X)}{|X|^2|Y|^2 - \langle X, Y\rangle^2},$$

which is nontrivial since the curvature tensor here is the one corresponding to M.

Two 4-tensor fields R_1 and R_2 which have the same symmetries as the curvature tensor (2.4) and satisfies

$$\frac{R_1(X, Y, Y, X)}{|X|^2|Y|^2 - \langle X, Y\rangle^2} = \frac{R_2(X, Y, Y, X)}{|X|^2|Y|^2 - \langle X, Y\rangle^2},$$

for any two linearly independent vectors $X, Y \in T_pM$ are equal. Hence sectional curvature recovers all the information in the curvature tensor. It also gives the following geometric interpretation of Ricci and scalar curvature: For any unit vector $E_1 \in T_pM$, extend it arbitrarily to an orthonormal basis $\{E_1, \ldots, E_n\}$. Then we have

$$Ric(E_1, E_1) = R_{k11}{}^k = \sum_{k=1}^n Rm(E_k, E_1, E_1, E_k) = \sum_{k=2}^n K(E_1, E_k).$$

Hence the Ricci curvature $Ric(E_1, E_1)$ is the sum of the sectional curvatures of the planes spanned by E_1 and the other elements of the orthonormal basis. Notice that the Ricci curvature is a symmetric 2-tensor field, hence completely determined by $Ric(V, V)$ where V runs through all unit vectors. Similarly, the scalar curvature can be written as

$$S = \sum_{j \neq k} K(E_j, E_k).$$

Henceforth, when we speak about curvature without specifying which type it will always be assumed to be sectional curvature.

Example 2.21. Let us compute the sectional curvatures of the canonical spaces $\mathbb{R}^n$, S^n_R, and $\mathbb{H}^n_R$. Since all the canonical spaces are frame-homogeneous, they all have constant sectional curvature since it is preserved by isometries. The sectional curvature of $\mathbb{R}^n$ is zero since its curvature tensor is zero. The argument for spheres and hyperbolic spaces is again similar, so we will only derive the sectional curvature of the sphere S^n_R and simply state that the sectional curvature of $\mathbb{H}^n_R$ is $-\frac{1}{R^2}$. Any plane section S_Π on the sphere S^n_R is isometric to S^2_R. The usual parametrization of the great circles on S^2_R show that $\kappa_1 = \kappa_2 = \pm\frac{1}{R}$, depending on whether an inward or an outward unit normal is chosen. Anyhow, the Gaussian curvature is $K = \frac{1}{R^2}$ which implies that the sectional curvature of S^n_R is $\frac{1}{R^2}$.

Definition 2.22. A *Riemannian model space* is a simply connected Riemannian manifold which is complete and has constant sectional curvature.

Notice that our work so far has shown that the canonical spaces $\mathbb{R}^n$, S^n_R, and $\mathbb{H}^n_R$ are Riemannian model spaces. Much of the remaining chapter will be dedicated to showing uniqueness of the Riemannian model spaces; any Riemannian model space will be isometric to one of the canonical spaces. Without the requirement of connectedness given implicitly through simply connectedness, disjoint unions of Riemannian model spaces would again be Riemannian model spaces. It is straightforward to check that the unit ball in $\mathbb{R}^n$ with the induced Riemannian metric from $\mathbb{R}^n$ satisfies all the conditions of a Riemannian model space except completeness. The abstract torus $S^1 \times S^1$ with the product metric is only lacking simply connectedness, as its fundamental group is two copies of the integers. Lastly, a straightforward computation shows that the product metric on $S^2 \times S^2$ does not have constant sectional curvature even through it is complete and $S^2 \times S^2$ is simply connected. Hence all of the conditions in the definition of a Riemannian model space are independent.

2.3 An Infinitesimal Model of Symmetry

In this section we shall describe an "infinitesimal model of symmetry" using Killing vector fields and see how this relates to the isometry group of a homogeneous Riemannian manifold. Through an identification with the Lie algebra of the isometry group, this will provide an upper bound on the dimension of the isometry group. We shall see that having an isometry group with maximal dimension is equivalent with a previously encountered property: frame-homogeneity. Firstly, we develop some theory on Jacobi fields we need both in this and the next section.

2.3.1 Jacobi Fields

The theory of Jacobi fields arises from trying to understand how geodesics spread apart. The proof of any statement in this subsection can be found in [Lee97, Chapter 10]. Given a geodesic segment $\gamma : [a, b] \to M$ and $\epsilon > 0$, we say that a map

$$\Gamma : (-\epsilon, \epsilon) \times [a, b] \longrightarrow M, \quad \Gamma(0, t) = \gamma(t)$$

is a *variation through geodesics* for γ whenever $\Gamma_s(t)$ is a geodesic segment for each $s \in (-\epsilon, \epsilon)$. We call the vector field $J(t) = \partial_s\Gamma(0, t)$ along γ the *variational field* of Γ as it provides an infinitesimal measure of how the geodesics spread apart. The following equation characterizes variational fields of variations through geodesics.

Proposition 2.23. *Let γ be a geodesic and J a vector field along γ. Then J is a variational field of a variation through geodesics for γ if and only if J satisfies the equation*

$$D_t^2 J + R(J, \dot{\gamma})\dot{\gamma} = 0.$$

The equation above is called the *Jacobi equation* and vector fields along curves which solves the Jacobi equation are called *Jacobi fields.* Let $\gamma : I \to M$ be a geodesic segment with $p = \gamma(a)$ for some $a \in I$ and $X, Y \in T_pM$. Then there exists a unique Jacobi field J along γ such that $J(a) = X$ and $D_tJ(a) = Y$. Hence the set of Jacobi fields $\mathrm{Jac}(\gamma)$ along γ is a $2n$-dimensional subspace of $\mathcal{T}(\gamma)$. The vector fields $\dot{\gamma}$ and $t\dot{\gamma}$ are always Jacobi fields along γ, and represent variations corresponding to simple reparametrizations of γ. In the following proposition we use the identification $T_x(T_pM) \simeq T_pM$ for $p \in M$ and $x \in T_pM$.

Proposition 2.24. *Let M be a Riemannian manifold with $p \in M$, $x \in T_pM$, and $v_x \in T_x(T_pM)$. Then*

$$dExp_p(v_x) = J(1),$$

where J is the unique Jacobi field along the geodesic γ_x such that $J(0) = 0$ and $D_tJ(0) = v_x$.

Remark. It is worthwhile to point out that although we will use Jacobi fields as a technicality for dealing with Killing vector fields and locally symmetric spaces, the theory of Jacobi fields could be given a much more prominent role in proving the classification of Riemannian model spaces, see [Lee97, Chapter 11] for such an approach. Our choice of going through symmetric and locally symmetric spaces for proving uniqueness of Riemannian model spaces is motivated by the sub-Riemannian case. This route will pave the way to define sub-Riemannian model spaces which are the main objects of interest in this thesis.

2.3.2 Killing Vector Fields

We now define Killing vector fields and work towards providing an upper bound for the isometry group of a homogeneous Riemannian manifold.

Definition 2.25. A vector field X on a Riemannian manifold M is said to be a *Killing vector field* if the stages ψ_t of its local flow are isometries.

We interpret Killing vector fields as an infinitesimal model for isometries, a viewpoint which will be given rigorous weight in Proposition 2.30. Recall that a vector field is said to be *complete* if its maximal integral curves are defined on the whole real line. Some care has to be taken in the definition of a Killing vector field when it is not complete. Then the domain of the local flow is a proper open subset of $\mathbb{R} \times M$ and the notion of isometry has to be adapted to this restriction, see [dC92, Exercise 3.5]. However, it follows from [O'N83, Proposition 9.30] that every Killing vector field on a complete Riemannian manifold is complete. As we will primarily be interested in homogeneous spaces, these technicalities will not play a role on our exposition. Many authors, e.g. [O'N83], define Killing vector fields by property (2) in the proposition below.

Proposition 2.26. *For a vector field X on a Riemannian manifold (M, g) the following are equivalent:*

(1) *X is a Killing vector field,*

(2) *$\mathcal{L}_X g \equiv 0$, where $\mathcal{L}$ denotes the Lie derivative,*

(3) *$X\langle Y, Z\rangle = \langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle$, for $Y, Z \in \mathfrak{X}(M)$,*

(4) *$\langle \nabla_Y X, Z\rangle = -\langle \nabla_Z X, Y\rangle$, for every $Y, Z \in \mathfrak{X}(M)$.*

The equivalence between property (2),(3), and (4) is straightforward by using that the Lie derivative is a derivation as explained in A.2. We refer the reader to [O'N83, Proposition 9.23] for the equivalence between property (1) and (2). Property (4) implies that parallel vector fields are Killing vector fields. Denote the Killing vector fields on a Riemannian manifold M by $i(M)$. The fact that

$$\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y],$$

for $X, Y \in \mathfrak{X}(M)$ shows that $i(M)$ is a subalgebra of $\mathfrak{X}(M)$ under the Lie bracket. The following Lemma gives the relationship between Killing vector fields and Jacobi fields.

Lemma 2.27. *Let M be a Riemannian manifold, γ a geodesic on M, and $X \in i(M)$. Then the restriction of X to γ is a Jacobi field.*

The proof of Lemma 2.27 can be found in [O'N83, Lemma 9.26]. Its main use is in the following theorem, which is an infinitesimal version of Proposition 2.15. It will later give us a quantitative bound on how many isometries a Riemannian manifold can have.

Proposition 2.28. *Let X and Y be Killing vector fields on a connected Riemannian manifold M and let $p \in M$. If $X_p = Y_p$ and $(\nabla X)_p = (\nabla Y)_p$, then $X \equiv Y$. Moreover,*

$$dim(i(M)) \leq \frac{n(n+1)}{2}.$$

Proof. It suffices to prove that $X \equiv 0$ whenever $X_p = 0$ and $(\nabla X)_p = 0$ by linearity. Let $\mathcal{A}$ be the set of points in M where both X and ∇X vanish. It is clearly closed and it is non-empty by assumption. The first statement will be proved if we can show that $\mathcal{A}$ is open. Let $p \in \mathcal{A}$ be arbitrary, $\mathcal{U}$ a normal neighbourhood centered at p, and τ a radial geodesic in $\mathcal{U}$. Then X restricted to τ, denoted by X_τ, is a Jacobi field by Lemma 2.27. Our assumptions imply that both $X_\tau(0)$ and $X'_\tau(0)$ are zero. Hence the uniqueness of Jacobi fields ensures that X_τ is identically zero. Both X and DX are zero on $\mathcal{U}$ since $\mathcal{U}$ is filled with radial geodesics.

For the second statement, define the linear map

$$\begin{aligned} \psi : i(M) &\longrightarrow T_pM \oplus \mathfrak{o}(T_pM) \\ X &\longmapsto (X_p, (\nabla X)_p), \end{aligned}$$

where $\mathfrak{o}(T_pM)$ denotes the skew-symmetric linear maps from T_pM to itself. It follows from what we just proved that ψ is injective. Therefore

$$\dim(i(M)) \leq \dim(T_pM) + \dim(\mathfrak{o}(T_pM)) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

□

Example 2.29. On $\mathbb{R}^3$ it is straightforward to check using property (3) in Proposition 2.26 that the following vector fields are Killing vector fields:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, & R_1 &= -y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}, \\ X_2 &= \frac{\partial}{\partial y}, & R_2 &= -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \\ X_3 &= \frac{\partial}{\partial z}, & R_3 &= -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}. \end{aligned}$$

It follows from Proposition 2.28 that these vector fields span $i(\mathbb{R}^3)$. Since R_1, R_2, and R_3 restrict to Killing vector fields on (S^2_R, g_R) they span $i(S^2_R)$. Let us consider the median geodesic $\gamma : \mathbb{R} \to S^2_R$ given by

$$\gamma(t) = (R\cos(t), R\sin(t), 0).$$

Restricting R_1, R_2, and R_3 to γ gives the Jacobi fields

$$\begin{aligned} \eta_1(t) &:= R_1\Big|_\gamma(t) = -R\sin(t)\frac{\partial}{\partial z}, \\ \eta_2(t) &:= R_2\Big|_\gamma(t) = R\cos(t)\frac{\partial}{\partial z}, \\ \eta_3(t) &:= R_3\Big|_\gamma(t) = -R\cos(t)\frac{\partial}{\partial y} + R\sin(t)\frac{\partial}{\partial x}. \end{aligned}$$

Notice that $\eta_3(t) = -\dot\gamma(t)$. Since the dimension of all Jacobi fields along γ is four we have

$$\mathrm{Jac}(\gamma) = \mathrm{span}\left\{\eta_1(t), \eta_2(t), \dot\gamma(t), t\dot\gamma(t)\right\}.$$

Recall that we can transform the geodesic γ into any other geodesic on the sphere with an orthogonal transformation. Thus we obtain all Jacobi fields on all geodesics on S^2_R by considering the image of $\mathrm{Jac}(\gamma)$ under orthogonal transformations.

Given a Riemannian manifold M, let us use the temporary notation $\mathfrak{g}$ for the Lie algebra of $\mathrm{Isom}(M)$. Let $X \in \mathfrak{g}$ have ψ_t in $\mathrm{Isom}(M)$ as its one-parameter subgroup. By the properties of the isometry group the map $\mathbb{R} \times M \to M$ sending (t, p) to $\psi_t(p)$ is smooth. We define X^+_p to be the initial velocity of the curve $t \mapsto \psi_t(p)$. Then X^+ is a smooth vector field on M such that the stages of its flow are given by ψ_t. Since one-parameter groups are defined on the whole line, X^+ is complete. Moreover, it is a Killing vector field since ψ_t is an isometry. The proof of the following proposition can be found in [O'N83, Proposition 9.33].

Proposition 2.30. *Let M be a complete Riemannian manifold and $\mathfrak{g}$ the Lie algebra of its isometry group. Then the assignment $X \mapsto X^+$ mapping $\mathfrak{g}$ into $i(M)$ described above is an anti-isomorphism, that is,*

$$[X^+, Y^+] = -[X, Y]^+, \quad X, Y \in \mathfrak{g}.$$

If M is not complete, it is an anti-isomorphism onto all complete Killing vector fields.

Thus we have a more rigorous interpretation of Killing vector fields as infinitesimal isometries and this justifies the notation $i(M)$. The following bound is immediate from Proposition 2.28 and Proposition 2.30.

Corollary 2.31. *Let M be any Riemannian manifold. Then*

$$dim(Isom(M)) \leq \frac{n(n+1)}{2}.$$

Homogeneous Riemannian manifolds have enough isometries to move between any two points via isometries. The following proposition quantifies this property by extending tangent vectors to Killing vector fields.

Proposition 2.32. *Let M be a homogeneous Riemannian manifold and $p \in M$. Every tangent vector $v \in T_pM$ extends to a Killing vector field on M.*

Proof. The map $\pi : \mathrm{Isom}(M) \to M$ given by $\psi \mapsto \psi(p)$ is a submersion, see [War83, Theorem 3.58]. Hence for $v \in T_pM$ there exists a vector $w \in T_e\mathrm{Isom}(M)$ such that $d\pi(w) = v$, where e is the identity of $\mathrm{Isom}(M)$. The Killing vector field W^+ which corresponds to w through the identification $i(M) \simeq T_e\mathrm{Isom}(M)$ given in Proposition 2.30 satisfies $W_p^+ = v$. □

Let M be a homogeneous Riemannian manifold. Then one can deduce, either from Proposition 2.32 or from the fact that $\pi : \mathrm{Isom}(M) \to M$ is a submersion, that the dimension of $\mathrm{Isom}(M)$ can not be smaller than the dimension of M. Putting together the previous results; any n-dimensional homogeneous Riemannian manifold M satisfies

$$n \leq \dim(\mathrm{Isom}(M)) \leq \frac{n(n+1)}{2}.$$

The following proposition implies that the manifold has constant curvature whenever the dimension of the isometry group is maximal.

Proposition 2.33. *For an n-dimensional homogeneous Riemannian manifold M, the following are equivalent:*

(i) $dim(Isom(M)) = \frac{n(n+1)}{2}$,

(ii) $dim(i(M)) = \frac{n(n+1)}{2}$,

(iii) *M is frame-homogeneous.*

Proof. The equivalence between the first two statements has already been established. Let K_p denote the isotropy group at $p \in M$. If we assume that (iii) holds, then the map

$$K_p \ni \phi \longmapsto d\phi_p \in O(T_pM)$$

is an isomorphism by Proposition 2.15. Thus

$$\dim(\mathrm{Isom}(M)) = \dim(M) + \dim(K_p) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

and (i) follows.

Proving the converse, namely that (i) implies (iii), is significantly more difficult and we will only provide a roadmap to this. One starts by proving that any Riemannian manifold (M, g) satisfying (i) has in fact constant sectional curvature, see [KN96, Theorem 6.3.3]. From there, it follows from [KN96, Appendix, Theorem 10.1] that any connected Riemannian manifold (M, g) with

$$\dim(\mathrm{Isom}(M)) = \frac{n(n+1)}{2}$$

necessarily has to be isomorphic to one of the canonical spaces $\mathbb{R}^n$, S^n_R, $\mathbb{H}^n_R$ or projective space $\mathbb{R}P^n_R \simeq S^n_R/\{\pm I\}$. Projective space is here given the Riemannian metric induced from the covering map $\pi : S^n_R \to \mathbb{R}P^n_R$, see [Lee97, Chapter 3]. By lifting the problem from $\mathbb{R}P^n_R$ to S^n_R, one can show by similar methods as in Example 2.18 that $\mathbb{R}P^n_R$ is frame homogeneous. This shows that (i) implies (iii) which finishes the proof. □

Remark. We have in this section focused on Riemannian manifolds with large isometry groups. It is worthwhile mentioning a theorem of Bochner [KN96, Theorem 5.3] at the other side of the spectrum. It states that on a Riemannian manifold with negative definite Ricci curvature, non-zero Killing vector fields can not obtain a relative maximum. If the manifold in addition is compact, this will force every Killing vector field to be zero. Thus Proposition 2.30 implies that the isometry group is finite and hence zero-dimensional.

2.4 Locally Symmetric Spaces and Isometries

Let ∇ be an arbitrary affine connection on a Riemannian manifold (M, g). We also use the notation ∇ for the *total covariant derivative* defined for an arbitrary tensor field F of type (k, l) by

$$(\nabla F)(\omega_1, \ldots, \omega_l, X_1, \ldots, X_{k+1}) = (\nabla_{X_{k+1}} F)(\omega_1, \ldots, \omega_l, X_1, \ldots, X_k),$$

for $\omega_1, \ldots, \omega_l \in \mathfrak{X}^*(M)$ and $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$. Any tensor field F such that $\nabla F \equiv 0$ is said to be *parallel*. The property that ∇ is compatible with the metric can be restated as $\nabla g \equiv 0$, that is, that the Riemannian metric g is parallel. The question we will answer in this section is the following:

How does the assumption that the curvature tensor is parallel affect the geometry?

Understanding this will give us insight into the local nature of symmetric spaces defined in the next section. As usual, we will only consider the Levi-Civita connection to get an association with the geometry of the manifold.

Definition 2.34. A Riemannian manifold is said to be *locally symmetric* if $\nabla R \equiv 0$, where ∇ is the Levi-Civita connection and R is the curvature tensor.

Proposition 2.35. *For a Riemannian manifold M the following are equivalent:*

- *The Riemannian manifold (M, g) is locally symmetric.*
- *If γ is a curve in M and $X, Y, Z \in \mathcal{T}(\gamma)$ are parallel along γ, then $R(X, Y)Z$ is parallel along γ.*
- *Sectional curvature is invariant under parallel translation.*

A proof of Proposition 2.35 can be found in [O'N83, Proposition 8.10]. Examples of locally symmetric spaces are the canonical spaces $\mathbb{R}^n$, S^n_R and $\mathbb{H}^n_R$, as well as $S^1 \times S^1$ with the product metric. A common theme in Riemannian geometry is that curvature plays a prominent role in the behaviour of nearby geodesics. We will work towards showing that this role is so dominant for locally symmetric spaces that it will imply the local version of the uniqueness of Riemannian model spaces in Corollary 2.39. Given two Riemannian manifolds M and N, we will first try to understand when a linear isometry $\psi : T_pM \to T_qN$ is the differential of an isometry defined on a normal neighbourhood of $p \in M$.

Definition 2.36. If $\psi : T_pM \to T_qN$ is a linear isometry and $\mathcal{U}$ a normal neighbourhood of $p \in M$ such that Exp_q is defined on $\psi(\operatorname{Exp}_p^{-1}(\mathcal{U}))$, then

$$P_\psi = \operatorname{Exp}_q \circ \psi \circ \operatorname{Exp}_p^{-1} : \mathcal{U} \longrightarrow N$$

is called the *polar map* of ψ on the neighbourhood $\mathcal{U}$.

Notice that if we choose $\mathcal{U}$ sufficiently small it follows that

$$\operatorname{Exp}_q\Big|_{\psi(\operatorname{Exp}_p^{-1}(\mathcal{U}))} : \psi(\operatorname{Exp}_p^{-1}(\mathcal{U})) \longrightarrow \mathcal{V}$$

is a diffeomorphism onto a normal neighbourhood $\mathcal{V}$ of q, hence $P_\psi : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism as well. Moreover, P_ψ is defined on any normal neighbourhood of $p \in M$ whenever N is complete since Exp_q is then defined on the whole of T_qN.

Lemma 2.37. *In the notation above, P_ψ maps radial geodesics to radial geodesics. Moreover, the differential of P_ψ at the point $p \in M$ is ψ.*

Proof. If $\gamma_v(t) = \operatorname{Exp}_p(tv)$ with $v \in T_pM$ it follows that

$$P_\psi(\gamma_v(t)) = \operatorname{Exp}_q(\psi(tv)) = \operatorname{Exp}_q(t\psi(v)),$$

for all t such that $\gamma_v(t)$ is in $\mathcal{U}$. This shows that P_ψ transforms radial geodesics to radial geodesics. For $v \in T_pM$ we have

$$dP_\psi(v) = dP_\psi(\dot\gamma_v(0)) = \frac{d}{dt}\Big|_{t=0} \operatorname{Exp}_q(t\psi(v)) = \psi(v).$$

□

We can conclude from Lemma 2.37 and Proposition 2.15 that if we seek an isometry defined on a normal neighbourhood $\mathcal{U}$ of p which extends ψ, then it has to be a polar map. Although we can always find polar maps which extend ψ, they are only diffeomorphisms (on suitably small normal neighborhoods) and not necessarily isometries. The following theorem, which is the main result in this section, gives a concrete condition under when a linear isometry can be extended to an isometry on a normal neighbourhood for locally symmetric spaces.

Theorem 2.38. *Let M and N be locally symmetric Riemannian manifolds with $\psi : T_pM \to T_qN$ a linear isometry that preserves curvature, that is,*

$$K(\Pi) = K(\psi\Pi)$$

where Π is a plane in T_pM and K is the sectional curvature. Then if $\mathcal{U}$ is a sufficiently small normal neighbourhood of p, there exists a unique isometry $P_\psi : \mathcal{U} \to \mathcal{V}$ where $\mathcal{V}$ is a normal neighbourhood of $q \in N$ such that $(dP_\psi)_p = \psi$.

Corollary 2.39. *If M and N are Riemannian manifolds of the same dimension and they both have the same constant curvature, then any two points $p \in M$ and $q \in N$ have isometric neighbourhoods. In particular, Riemannian model spaces are locally isometric if they have the same curvature.*

Corollary 2.39 follows from Theorem 2.38 since constant curvature spaces are locally symmetric by Proposition 2.35. We will extend Corollary 2.39 to a global result for Riemannian model spaces in Section 2.6. Let us turn to the proof of Theorem 2.38, which will utilize the theory we built up regarding Jacobi fields.

Proof. We have already established uniqueness and showed that if the isometry exists, then it has to be a polar map. Hence the result follows if we can show that every polar map P_ψ is a local isometry under the curvature assumption on ψ. We are going to compare the growth of Jacobi fields on the two manifolds. Let $\widetilde{\mathcal{U}}$ be the neighbourhood in T_pM corresponding to $\mathcal{U}$ via the exponential map. For $s \in \mathcal{U}$ and $v \in T_sM$ there is a unique $x \in \widetilde{\mathcal{U}}$ and $y_x \in T_x(T_pM)$ such that $d\mathrm{Exp}_p(y_x) = v$. It follows from Proposition 2.24 that

$$\langle v, v\rangle = \langle Y(1), Y(1)\rangle$$

where Y is the Jacobi field on γ_x such that $Y(0) = 0$ and $D_tY(0) = y$. Here $y_x \in T_x(T_pM)$ is identified with $y \in T_pM$ through the usual identification $T_x(T_pM) \simeq T_pM$.

By looking at the situation on N, we have that

$$dP_\psi(v) = d\mathrm{Exp}_p(\psi(y)_{\psi(x)})$$

since P_ψ is a polar map. Thus we can again appeal to Proposition 2.24 to obtain that

$$\langle dP_\psi(v), dP_\psi(v)\rangle = \langle \overline{Y}(1), \overline{Y}(1)\rangle,$$

where $\overline{Y}$ is the unique Jacobi field on $\gamma_{\psi(x)}$ such that $\overline{Y}(0) = 0$ and $D_t\overline{Y}(0) = \psi(y)$. Keep in mind that our goal is to show that

$$\langle Y(1), Y(1)\rangle = \langle \overline{Y}(1), \overline{Y}(1)\rangle.$$

Let $E_1, \dots, E_n$ and $\overline{E}_1, \dots, \overline{E}_n$ be parallel frames on γ_x and $\gamma_{\psi(x)}$, respectively, with $\psi(E_i(0)) = \overline{E}_i(0)$. Write

$$Y = \sum_{i=1}^n y^i E_i, \quad \overline{Y} = \sum_{i=1}^n \overline{y}^i \overline{E}_i, \quad \gamma_x' = \sum_{i=1}^n a^i E_i, \quad \gamma_{\psi(x)}' = \sum_{i=1}^n a^i \overline{E}_i.$$

Here a^i is constant since γ_x' and $\gamma_{\psi(x)}'$ are geodesics. Moreover, its the same constants for both γ_x' and $\gamma_{\psi(x)}'$ since $\psi(E_i(0)) = \overline{E}_i(0)$.

The result will now follow if we can show that $y^i = \overline{y}^i$ for every $i = 1, \dots, n$. Writing out the Jacobi equations for Y and $\overline{Y}$ gives

$$\frac{d^2\overline{y}^m}{dt^2} = \sum_{i,j,k} \overline{R}_{ijk}^m a^i \overline{y}^k a^k, \quad \frac{d^2 y^m}{dt^2} = \sum_{i,j,k} R_{ijk}^m a^i y^k a^k, \quad 1 \le m \le n.$$

Since $y^1, \dots, y^n$ and $\overline{y}^1, \dots, \overline{y}^n$ satisfy the same initial conditions we only have to show that $R_{ijk}^m = \overline{R}_{ijk}^m$. Now we use all our assumptions: The fact that ψ preserves curvature ensures that $\overline{R}_{ijk}^m(0) = R_{ijk}^m(0)$. It follows from Proposition 2.35 that

$$Rm(E_i, E_j, E_k, E_m) = \langle R(E_i, E_j)E_k, E_m\rangle$$

is constant as both $R(E_i, E_j)E_k$ and E_m are parallel. □

Finally, we will explain the term "locally symmetric" and pave the way for globally symmetric spaces in the next section. Given $p \in M$, let ξ_p be the polar map of the linear isometry $v \mapsto -v$ of T_pM on a normal neighbourhood $\mathcal{U}$ chosen so small that $\xi_p : \mathcal{U} \to \mathcal{U}$ is a diffeomorphism. It is easy to see by using the uniqueness of geodesics together with that polar maps transform radial geodesics to radial geodesics that ξ_p reverses the direction of geodesics passing through p. That is, if γ is a geodesic with $\gamma(0) = p$ then $\xi_p(\gamma(s)) = \gamma(-s)$. This property uniquely determines ξ_p and we call ξ_p the *local geodesic symmetry* of M at p. An equivalent definition of locally symmetric spaces can be given by requiring that the local geodesic symmetries ξ_p are isometries on suitable normal neighbourhoods, see [O'N83, Corollary 8.16].

2.5 Riemannian Symmetric Spaces

We will define Riemannian symmetric spaces motivated by how the local geodesic symmetries characterized locally symmetric spaces in the previous section. The geodesics on Riemannian symmetric spaces are rigid in the sense that they are either one-to-one or periodic. It will follow from Corollary 2.47 that the Riemannian model spaces are Riemannian symmetric spaces. By showing that Riemannian symmetric spaces are homogeneous we open up the Lie theory machinery developed in Section 2.2. In fact, we will see that Riemannian symmetric spaces are always naturally reductive. Finally, we provide a global version of Theorem 2.38 for symmetric spaces in Theorem 2.46 which will be our main tool for classifying the Riemannian model spaces in the next section.

Definition 2.40. Let M be a Riemannian manifold. A *global symmetry* at p is an isometry $\xi_p \in \mathrm{Isom}(M)$ such that

$$\xi_p(p) = p, \quad (d\xi_p)_p = -Id.$$

We call M a *(Riemannian) symmetric space* if every point possesses a global symmetry.

It is clear that such a symmetry has to be unique by Proposition 2.15. Moreover, by restricting ξ_p to a normal neighbourhood it is clear that symmetric spaces are locally symmetric. One way to see that there are locally symmetric spaces which are not globally symmetric is to realize that open subsets of locally symmetric spaces are again locally symmetric. This is not true for symmetric spaces due to the following proposition.

Proposition 2.41. *Symmetric spaces are homogeneous hence also complete. Thus any proper open submanifold of a symmetric space is locally symmetric, but not symmetric.*

Proof. We will prove that symmetric spaces are geodesically complete before proving that they are homogeneous. For this we will use the global symmetry maps to "reflect geodesics further". Let γ_p be a geodesic starting at $p \in M$ and let $\gamma_p(r) = q \in M$ for some $r > 0$. Then if we define $\gamma_q(t) = \gamma_p(t + r)$ we obtain a geodesic which starts at q, that is, $\gamma_q(0) = q$. Now reflecting twice with the global symmetry gives

$$\xi_q(\xi_p(\gamma_p(t))) = \xi_q(\gamma_p(-t)) = \xi_q(\gamma_q(-t - r)) = \gamma_q(t + r) = \gamma_p(t + 2r).$$

This shows that we can extend geodesics arbitrarily, hence M is a complete metric space by the Hopf-Rinow Theorem.

To show that symmetric spaces are homogeneous we let $x, y \in M$ be two arbitrary points. Let $\gamma : \mathbb{R} \to M$ be a geodesic connecting the two points, $\gamma(t_1) = x$ and $\gamma(t_2) = y$, which exists by completeness. If we make the substitution $\gamma_{\mathrm{mid}}(s) = \gamma\left(s + \frac{t_1+t_2}{2}\right)$ it follows that

$$\xi_{\gamma\left(\frac{t+s}{2}\right)}(x) = \xi_{\gamma\left(\frac{t+s}{2}\right)}\left(\gamma_{\mathrm{mid}}\left(\frac{t_1 - t_2}{2}\right)\right) = \gamma_{\mathrm{mid}}\left(\frac{t_2 - t_1}{2}\right) = y.$$

□

For a homogeneous Riemannian manifold M it is enough to possess a global symmetry ξ_p at one point $p \in M$ to be a symmetric space: Then for $q \in M$ and an isometry $\phi \in \mathrm{Isom}(M)$ taking p to q, the map $\xi_q = \phi^{-1} \circ \xi_p \circ \phi$ gives a global symmetry at q. Let G be a Lie group equipped with a bi-invariant Riemannian metric. Then [O'N83, Proposition 11.9] shows that the inversion map $g \mapsto g^{-1}$ is an isometry for every $g \in G$. This is clearly a global symmetry at the identity which implies that Lie groups with bi-invariant metrics are symmetric spaces.

Definition 2.42. Let M be a complete Riemannian manifold and let $\gamma : \mathbb{R} \to M$ be a geodesic. Then $\phi_c \in \mathrm{Isom}(M)$ is a *transvection* along γ of *shift* $c \in \mathbb{R}$ provided that

- ϕ_c translates γ, meaning that $\phi_c(\gamma(s)) = \gamma(s + c)$ for all $s \in \mathbb{R}$.

- If $v \in T_{\gamma(s)}M$, then $d\phi(v) \in T_{\gamma(s+c)}M$ is the parallel translate of v along γ.

If transvections of any shift exist for any geodesic, we simply say that M *possesses transvections*. In that case, if $\gamma : \mathbb{R} \to M$ is a geodesic with $\gamma(b) = \gamma(0)$ then

$$\gamma(s+b) = \phi_s(\gamma(b)) = \phi_s\gamma(0) = \gamma(s),$$

where ϕ_s is a transvection along γ of shift s. Thus for Riemannian manifolds which possess transvections, all geodesics have to be either one-to-one or simply periodic.

Proposition 2.43. *Riemannian symmetric spaces possess transvections. More concretely, the map $\xi_{\gamma(c/2)}\xi_{\gamma(0)}$ is the required transvection along γ of shift c.*

The translation property is straightforward to check for the proposed transvection while the parallel property can be found in detail in [O'N83, Proposition 9.30]. Before continuing to the main theorem on symmetric spaces, we examine the geometry of symmetric spaces through Lie theory similarly to how we did with homogeneous spaces. Recall that since a symmetric space M is homogeneous it can be described as $M \simeq G/H := \mathrm{Isom}(M)/K_{x_0}$, where K_{x_0} is the isotropy group at x_0. We will see that the geodesic symmetry $\xi := \xi_{x_0}$ provides additional information which ensures that symmetric spaces are always naturally reductive.

Lemma 2.44. *Let $M = G/H$ be a connected symmetric space with geodesic symmetry ξ at x_0. Then the map $\tau : G \to G$ given by $g \mapsto \xi g \xi$ is an involutive automorphism such that $Fix(\tau)_0 \subset H \subset Fix(\tau)$, where $Fix(\tau)_0$ denotes the connected component of $Fix(\tau)$ containing the identity.*

Proof. Since $\xi^{-1} = \xi$, τ is simply conjugation by ξ and hence an involutive automorphism. If $h \in H$ we have $d\tau(h) = d\xi_{x_0} dh_{x_0} d\xi_{x_0} = dh_{x_0}$. Thus the conectedness of M implies that $\tau(h) = h$ since the differentials agree at one point. This shows that $H \subset \mathrm{Fix}(\tau)$. Notice that $\mathrm{Fix}(\tau)_0$ is a connected Lie group and hence is generated by its one-parameter subgroups $\alpha(t)$. We are done if we can show that $\alpha(t)x_0 = x_0$, as this gives that $\alpha(t) \in H$. Since $\tau(\alpha(t)) = \alpha(t)$ it follows that

$$\xi(\alpha(t)x_0) = \alpha(t)\xi(x_0) = \alpha(t)x_0.$$

The fact that x_0 an isolated fixed point of ξ gives that $\alpha(t)x_0 = x_0$ for $|t|$ small. This ensures that $\alpha(t) \in H$ for all t. ☐

Proposition 2.45. *Using the notation from the previous lemma, with $\mathfrak{h} \subset \mathfrak{g}$ denoting the Lie algebras of H and G, we have*

- $\mathfrak{h} = \{X \in \mathfrak{g} : d\tau(X) = X\}$,
- $\mathfrak{g}$ *is the direct sum of* $\mathfrak{h}$ *and the Ad(H)-invariant subspace* $\mathfrak{m} = \{X \in \mathfrak{g} : d\tau(X) = -X\}$,
- $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$, *and* $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.

Proof. Since $\tau|_H$ is the identity by Lemma 2.44 it follows that $d\tau(X) = X$ whenever $X \in \mathfrak{h}$. Conversely, let $d\tau(X) = X$ and let α be the one-parameter subgroup corresponding to X. Then $\tau \circ \alpha = \alpha$ since they have the same initial velocity and are both one-parameter subgroups. Hence α lies in $\mathrm{Fix}(\tau)_0 \subset H$, so $X \in \mathfrak{h}$ and this establishes the first claim. For any $X \in \mathfrak{g}$ we write

$$X = X_{\mathfrak{h}} + X_{\mathfrak{m}} := \frac{X + d\tau(X)}{2} + \frac{X - d\tau(X)}{2}.$$

It is clear that $X_{\mathfrak{h}} \in \mathfrak{h}$ and $X_{\mathfrak{m}} \in \mathfrak{m}$. Hence $\mathfrak{h} \cap \mathfrak{m} = \{0\}$ implies that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. To see the $\mathrm{Ad}(H)$-invariance of $\mathfrak{m}$ we must show that $d\tau(\mathrm{Ad}_h(X)) = -\mathrm{Ad}_h(X)$ for $h \in H$. If we denote by C_h the map $g \mapsto hgh^{-1}$ for $h \in H$, then

$$\tau C_h(g) = \tau(hgh^{-1}) = h\tau(g)h^{-1}.$$

Hence C_h and τ commute and for $X \in \mathfrak{m}$,

$$d\tau(\mathrm{Ad}_h(X)) = d(\tau \circ C_h)(X) = d(C_h \circ \tau)(X) = \mathrm{Ad}_h(-X) = -\mathrm{Ad}_h(X).$$

The last assertion consists of straightforward computations. For $\mathfrak{m}$ the computation is

$$d\tau([X,Y]) = [d\tau(X), d\tau(Y)] = [-X,-Y] = [X,Y],$$

for $X, Y \in \mathfrak{m}$. □

Although $\mathrm{Ad}(H)$-invariant complements of $\mathfrak{h}$ always exist for homogeneous spaces, symmetric spaces have a canonical complement obtained from its geodesic symmetries as described above. The condition $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$ makes $M = G/H$ trivially into a naturally reductive space. Hence geodesics in M starting at x_0 is given by projections of one-parameter curves with initial velocity in $\mathfrak{m}$ as we mentioned when discussing naturally reductive spaces in general. Moreover, by [O'N83, Proposition 11.31] the curvature tensor of M at x_0 is given by

$$R(x,y)z = -d\pi\left([[X,Y],Z]\right),$$

where $X, Y, Z \in \mathfrak{m}$ correspond to $x, y, z \in T_{x_0}M$ under the isomorphism $d\pi|_{\mathfrak{m}} : \mathfrak{m} \to T_{x_0}M$. Hence curvature in symmetric spaces can be described purely by algebraic means through Lie theory. Let us now turn to the main theorem of this section. The term *Riemannian covering* for a map $\phi : M \to N$ between Riemannian manifolds is used for a covering map which is also a local isometry.

Theorem 2.46. *Let M and N be complete and locally symmetric Riemannian manifolds with M simply connected and N connected. If $\psi : T_pM \to T_qN$ is a linear isometry which preserves curvature, then there exists a unique Riemannian covering $\Psi : M \to N$ such that $d\Psi_p = \psi$.*

Remark. We will not prove Theorem 2.46, the proof can be found in [O'N83, Theorem 8.17]. The main idea of the proof is as follows: There exists by Theorem 2.38 an isometry P_ψ on some neighbourhood of p which extends ψ. Any point $p' \in M$ can be connected to p by a geodesic β, so we can parallel translate ψ along β to p' and obtain another isometry on a neighbourhood of p'. Finally, the simply connectedness of M will ensure that these isometries patch suitably together to form the desired Riemannian covering map.

Corollary 2.47. *A complete, simply connected, and locally symmetric Riemannian manifold is symmetric.*

Proof. For any point $p \in M$, the differential of the local geodesic symmetry $d\xi_p = -Id : T_pM \to T_pM$ is an isometry and preserves curvature. Applying Theorem 2.46 with $M = N$ gives a Riemannian covering map $\Phi : M \to M$ such that $(d\Phi)_p = -Id$. Since M is simply connected it follows from covering theory that the map is a diffeomorphism and hence an isometry, see [Hat02, Theorem 1.38]. This gives precisely a global symmetry at p. □

Most importantly for us, all Riemannian model spaces are symmetric spaces. In the next and final section of this chapter, this will provide us with a complete classification of the Riemannian model spaces.

2.6 Classification of Riemannian Model Spaces

We begin this section by surveying a few classical results which relate bounds on sectional curvature to the topology of the space, proofs can be found in [Lee97, Chapter 11]. These results motivate the classification of Riemannian model spaces, since they are the simplest spaces for which the assumptions in the theorems hold. Then the main result of this chapter, the classification of Riemannian model spaces, will be stated and proved using the theory we developed on symmetric spaces. Lastly, we will give several equivalent definitions of Riemannian model spaces based on "maximal symmetry", one of which generalizes to the sub-Riemannian setting in Chapter 4. The following theorem shows that manifolds for which a Riemannian metric with nonpositive sectional curvature exists are rather special.

Theorem 2.48 (Cartan-Hadamard)**.** *If (M, g) is a complete Riemannian manifold with sectional curvature nonpositive at every point, then the universal cover of M is diffeomorphic to $\mathbb{R}^n$. Hence if M in addition is simply connected, then M itself is diffeomorphic to $\mathbb{R}^n$.*

This theorem implies that it is not possible to find a Riemannian metric on real projective space $\mathbb{R}P^n$ with nonpositive sectional curvature at each point: Notice that any Riemannian metric on $\mathbb{R}P^n$ is complete due to compactness. Such a metric would imply a diffeomorphism between $\mathbb{R}^n$ and the universal cover of $\mathbb{R}P^n$, which is S^n. This is clearly not possible since S^n is compact while $\mathbb{R}^n$ is not. At the other end of the spectrum, we have the following topological consequences for Riemannian manifolds whose sectional curvature are bounded from below.

Theorem 2.49 (Bonnet's Theorem)**.** *Let (M, g) be a complete Riemannian manifold whose sectional curvature is bounded below by a positive constant $\frac{1}{R^2}$. Then the diameter of M (as a metric space) is less than or equal to πR. Moreover, M is compact and has finite fundamental group.*

It follows that the torus $S^1 \times S^1$ with *any* Riemannian metric can not have curvature bounded from below by a positive constant since its fundamental group is not finite. Bonnet's Theorem holds under the more general assumption that the Ricci curvature satisfies the estimate

$$Rc(V, V) \geq \frac{n-1}{R^2}|V|^2,$$

for every $p \in M$, $V \in T_pM$, and $n = \dim(M)$. We now come to the main theorem of the chapter: the classification of model spaces in Riemannian geometry.

Theorem 2.50 (Classification of Model Spaces)**.** *Let (M, g) be an n-dimensional Riemannian model space with $n \geq 2$ and sectional curvature K. Then $(M, g) \simeq \Sigma(n, K)$, where*

$$\Sigma(n, K) = \begin{cases} (S^n_R, g_R) & \text{if } K = \frac{1}{R^2} \\ (\mathbb{R}^n, \bar{g}) & \text{if } K = 0 \\ (\mathbb{H}^n_R, h_R) & \text{if } K = -\frac{1}{R^2} \end{cases}$$

Thus the Riemannian model spaces are (up to isometry) precisely the canonical spaces $\mathbb{R}^n$, S^n_R, and $\mathbb{H}^n_R$.

Proof. Clearly the three canonical spaces are not isometric since they have different sectional curvatures. Let $p \in M, q \in \Sigma(n, K)$, and pick a linear isometry $\psi : T_pM \to T_q\Sigma(n, K)$ which exists since the manifolds have the same dimension. Then ψ preserves curvature as both manifolds have the same constant sectional curvature. Since M and $\Sigma(n, K)$ are symmetric spaces and M is simply connected, we can apply Theorem 2.46 to find a Riemannian covering map $\Phi : M \to \Sigma(n, K)$ such that $(d\Phi)_p = \psi$. This is a diffeomorphism because $\Sigma(n, K)$ is simply connected and hence a global isometry. □

From now on we refer to the canonical spaces $\mathbb{R}^n$, S^n_R, and $\mathbb{H}^n_R$ as *the* model spaces in Riemannian geometry without any ambiguity. After fixing a dimension there is (up to topological and geometrical requirements) one parameter determining the model spaces in Riemannian geometry: sectional curvature.

We can conclude from Example 2.18 and Proposition 2.33 that $O(n+1) \subset \mathrm{Isom}(S^n_R)$ and $O_+(n, 1) \subset \mathrm{Isom}(\mathbb{H}^n_R)$ are co-dimension zero Lie subgroups. The reverse inclusion for $\mathrm{Isom}(S^n_R) \subset O(n+1)$ can be established by showing that any isometry of S^n_R is the restriction of an isometry of $\mathbb{R}^{n+1}$. Similarly, the case $\mathrm{Isom}(\mathbb{H}^n_R) \subset O_+(n, 1)$ can be deduced by showing that any isometry of $\mathbb{H}^n_R$ is the restriction of an isometry of $\mathbb{R}^{n+1}$ with the Minkowski metric given in Example 2.6. The results gathered about the Riemannian model spaces in this chapter can be summarized in the following table:

Model Spaces	$(\mathbb{R}^n, \bar{g})$	(S^n_R, g_R)	$(\mathbb{H}^n_R, h_R)$
Sectional Curvatures	0	$\frac{1}{R^2}$	$-\frac{1}{R^2}$
Geodesics	Straight Lines	Great Circles	Great Hyperbolas
Isometry Groups	$E(n)$	$O(n+1)$	$O_+(n, 1)$

Example 2.51. At the end of Section 2.2 we pointed out that the standard product metric on $S^2 \times S^2$ does not have constant sectional curvature. We will now apply the classification theorem to show that $M = S^n \times S^n$, $n \geq 2$ with *any* Riemannian metric g can not have constant sectional curvature. Notice that M is simply connected and any Riemannian metric on M is complete due to the compactness of M. Assume that (M, g) has constant sectional curvature. Then it is a Riemannian model space and must be isometric to one of the three Riemannian model spaces with dimension $2n$. Compactness excludes the possibilities $\mathbb{R}^{2n}$ and $\mathbb{H}^{2n}_R$, so only S^{2n}_R is an option. However, computing their homology groups with integer coefficients gives that

$$H_k(S^{2n}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2n \\ 0 & \text{otherwise} \end{cases}, \quad H_k(S^n \times S^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2n \\ \mathbb{Z} \times \mathbb{Z} & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

The computation for $H_k(S^n)$ can be done with a Mayer-Vietoris sequence while the computation for $H_k(S^n \times S^n)$ follows from the Künneth formula in homology, see [Hat02] for an explanation of these techniques. Moreover, the Cartan-Hadamard Theorem additionally impose that the curvature has to be strictly positive for at least one plane section.

Finally, we can combine results gathered throughout the chapter to give equivalent definitions of being a Riemannian model space. The last equivalent definition in Theorem 2.52 will be the basis for defining model spaces in sub-Riemannian geometry.

Theorem 2.52. *For a simply connected n-dimensional Riemannian manifold (M, g), the following are equivalent:*

(1) *(M, g) is a model space,*

(2) *dim(Isom(M))* $= \frac{n(n+1)}{2}$,

(3) *(M, g) is frame-homogeneous,*

(4) *any linear isometry $\psi : T_pM \to T_qM$ lifts to a unique isometry $\Psi : M \to M$ such that $d\Psi_p = \psi$, for every $p, q \in M$.*

Proof. We will begin by showing that (4) implies (3). For $p, q \in M$, let $E = (E_1, \ldots, E_n)$ and $F = (F_1, \ldots, F_n)$ be orthonormal bases at T_pM and T_qM, respectively. There exists a linear isometry $\psi : T_pM \to T_qM$ taking E to F. Then (4) gives an isometry $\Psi : M \to M$ whose differential at p takes E to F, showing that M is frame-homogeneous. This shows that (4) implies (3) and Proposition 2.33 gives the equivalence between (2) and (3). We have already pointed out that frame-homogeneity implies constant curvature and that homogeneous spaces are complete, showing that (3) implies (1). Finally, recall that Riemannian model spaces are symmetric spaces by Corollary 2.47. Thus Theorem 2.46 shows that the lift condition is satisfied due to the fact that any linear isometry preserves curvature on constant curvature spaces. This shows that (1) implies (4) and the result follows. □

Remark. The classification procedure we have presented in this chapter carries over with few alterations to the Semi-Riemannian setting. A *Semi-Riemannian manifold* $(M, \mathbf{g})$ is a manifold M together with a nondegenerate symmetric 2-tensor field $\mathbf{g}$. An example of a Semi-Riemannian manifold which is not Riemannian is $\mathbb{R}^4$ with the Minkowski-metric presented in Example 2.6. We refer the reader to [O'N83, Corollary 8.24] for the classification of model spaces in the Semi-Riemannian setting. The model spaces in Semi-Riemannian geometry are conceptually the nondegenerate generalizations of the Riemannian model spaces. Contrary to this, sub-Riemannian model spaces have a distinct flavor from their Riemannian counterparts as we shall see in Chapter 4 and Chapter 5.

3 Geometry of Bundles

In this chapter we will develop the theory of principal bundles and look closely at a particular example: the frame bundle. The main reason for developing this theory is to use it as a tool when studying sub-Riemannian geometry in Chapter 4 and Chapter 5. However, we will also survey some results not needed in later chapters to motivate concepts and make the exposition more complete. A principal connection will be an extra piece of information on a principal bundle which often encode the geometry of the original manifold. Curvature and holonomy of a principal connection will be developed, resulting in the Ambrose-Singer theorem providing the link between them. The frame bundle of a manifold will be closely related with Riemannian structures on the manifold, allowing us to define torsion and curvature of the frame bundle. Together with the structural equations and Bianchi identities, this provides a panoramic view of the original geometry through forms on the frame bundle. Any affine connection on a homogeneous manifold gives rise to principal connections on both the isometry group and the frame bundle. Understanding how these induced connections relate to each other gives a powerful tool for the classification of sub-Riemannian model spaces in the next chapters.

3.1 Principal and Frame Bundles

3.1.1 Principal Bundles

We will assume the reader is familiar with the definition of a fiber bundle, see [Lee13, Chapter 10] for a short introduction. Terminology regarding Lie group actions are refreshed in Appendix A.2.

Definition 3.1. A *principal bundle* is a smooth fiber bundle $\pi : P \to M$ together with a right Lie group action

$$\begin{aligned} P \times G &\longrightarrow P \\ (p, g) &\longmapsto p \cdot g = R_g(p), \end{aligned}$$

preserving the fibers and acting freely and transitively within each of them. We can identify each fiber with G and the local triviality condition reads as follows: There exists an atlas $\mathcal{A}$ for M such that the following diagram commutes,

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{(\pi,\phi)} & U \times G \\ \big\downarrow \pi & \swarrow Pr_U & \\ U & & \end{array}$$

for each $U \in \mathcal{A}$. The map $(\pi, \phi) : \pi^{-1}(U) \to U \times G$ is a diffeomorphism and $\phi : \pi^{-1}(U) \to G$ is G-equivariant. We use the expression *principal G-bundle* to emphasize the *structure group G* and the notation $G \to P \xrightarrow{\pi} M$ or simply $G \to P \to M$ to denote a principal bundle.

Standard terminology from fiber bundle theory such as *total space*, *base space*, and *fiber over* $x \in M$ will be used. Notice that $\dim(P) = \dim(G) + \dim(M)$ holds because $\phi : \pi^{-1}(U) \to U \times G$ is a diffeomorphism between open sets of P and $G \times M$. Equivalence between principal bundles is defined in the same way as for ordinary fiber bundles with the additional requirement that the group action is preserved under the map between the total spaces.

The most obvious example of a principal bundle is the product bundle

$$G \longrightarrow M \times G \xrightarrow{\pi} M,$$

where π is the projection onto the first factor and G acts on $M \times G$ by $(p, g_1) \cdot g_2 = (p, g_1 g_2)$. It is clear that the action is transitive and free within the fibers. Moreover, the global trivialization satisfies the G-invariance property. We call $G \to M \times G \xrightarrow{\pi} M$ the *trivial principal bundle* with base space

M and structure group G. Although simple, this example contains the only vector bundles which are simultaneously principal bundles. This is due to the fact that vector bundles always admit sections. A principal bundle on the other hand possesses globally defined sections if and only if it is equivalent to a trivial principal bundle: Clearly the bundle $G \to M \times G \xrightarrow{\pi} M$ has the global section $m \mapsto (m, e)$ for $m \in M$ and the identity $e \in G$. The result then holds for any bundle equivalent to a trivial principal bundle. Conversely, assume there exists a map $s : M \to P$ such that $\pi \circ s = id_M$. Then for any $p \in P$ there is a $g_p \in G$ such that $s(\pi(p)) \cdot g_p = p$. Define the map

$$\begin{aligned}(\pi, \phi) : P &\longrightarrow M \times G \\ p &\longmapsto (\pi(p), g_p).\end{aligned}$$

It is straightforward to check that (π, ϕ) is G-invariant and the induced map from M to M is the identity. These properties are sufficient for the bundles to be equivalent, see [Wal04, Theorem 3.1] for more details.

Remark. If $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is a local trivialization for $G \to P \xrightarrow{\pi} M$ then, whenever $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, there are two trivializations

$$U_{\alpha\beta} \times G \xleftarrow{(\pi,\phi_\alpha)} \pi^{-1}(U_{\alpha\beta}) \xrightarrow{(\pi,\phi_\beta)} U_{\alpha\beta} \times G.$$

Since $\phi_\alpha \circ \phi_\beta^{-1} : G \to G$ is G-invariant, it is straightforward to see that

$$(\pi, \phi_\alpha) \circ (\pi, \phi_\beta)^{-1}(x, h) = (x, g_{\alpha\beta}(x)h), \quad x \in U_{\alpha\beta}, \ \ h, g_{\alpha\beta}(x) \in G.$$

The functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ are called *transition functions* and satisfies the *cocycle condition*

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma},$$

whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. If we start with M, G, and the transition functions we can construct P such that $G \to P \xrightarrow{\pi} M$ becomes a principal bundle, see [KN96, Proposition 5.2].

Example 3.2. Let G be a Lie group with H a closed subgroup of G. We consider the homogeneous space G/H together with the projection $\pi : G \to G/H$. The canonical action of H on G given by $(g, h) \mapsto gh$ is smooth and acts freely and transitively within each fiber $\pi^{-1}([g])$ for $[g] \in G/H$ by definition. We pointed out in the proof of Proposition 2.17 that there exist smooth local sections for π. Hence for every $[g] \in G/H$ there is a neighbourhood U of $[g]$ and a map $s_U : U \to G$ such that $\pi \circ s_U = Id_U$. This gives a canonical map

$$\psi : U \times H \longrightarrow \pi^{-1}(U),$$

sending $([g], h)$ to $s_U([g]) \cdot h$. The inverse of ψ is a local trivializations given explicitly by

$$\begin{aligned}(\pi, \phi_U) := \psi^{-1} : \pi^{-1}(U) &\longrightarrow U \times H \\ g &\longmapsto ([g], s_U([g])^{-1} \cdot g).\end{aligned}$$

The G-invariance property follows from the definition, since

$$\phi_U(gh) = s_U([g])^{-1} \cdot gh = \left(s_U([g])^{-1} \cdot g\right) h = \phi_U(g)h,$$

for $h \in H$ and $g \in \pi^{-1}(U)$. Hence $H \to G \xrightarrow{\pi} G/H$ is a principal bundle.

3.1.2 Frame Bundles and Associated Bundles

Although vector bundles are not usually principal bundles, we can associate a principal bundle to every vector bundle. The construction that follows is most transparent from a geometric viewpoint when having the tangent bundle of M in mind. Let $\pi : E \to M$ be an arbitrary rank r vector bundle over M. We denote by $\mathcal{F}(E_x)$ all frames at $x \in M$, that is, all ordered bases for $E_x = \pi^{-1}(x)$. These spaces are collected together

$$\mathcal{F}(E) = \coprod_{x \in M} \mathcal{F}(E_x),$$

and equipped with the projection $\widetilde{\pi} : \mathcal{F}(E) \to M$ which sends a frame in $\mathcal{F}(E_x)$ to x. It will be useful to consider an ordered basis $X_1, \ldots, X_r$ for E_x as a mapping $u : \mathbb{R}^r \to E_x$ sending the standard basis $e_1, \ldots, e_r$ of $\mathbb{R}^r$ to $X_1, \ldots, X_r$, respectively. With this view, we have a canonical action of $GL(r, \mathbb{R})$ on each fiber given by precomposition,

$$\mathbb{R}^r \xrightarrow{A} \mathbb{R}^r \xrightarrow{u} E_x,$$

where $A \in GL(r, \mathbb{R})$ and $u \in \mathcal{F}(E_x)$. It follows from standard linear algebra that the action is smooth. Moreover, the action is transitive and free within each fiber. If $(\pi, \phi) : \pi^{-1}(U) \to U \times \mathbb{R}^r$ is a local trivialization for $\pi : E \to M$, then

$$\begin{aligned} (\widetilde{\pi}, \psi) : \widetilde{\pi}^{-1}(U) &\longrightarrow U \times GL(r, \mathbb{R}) \\ u &\longmapsto (\widetilde{\pi}(u), \phi|_{E_{\widetilde{\pi}(u)}} \circ u) \end{aligned}$$

is bijective and the transition functions are clearly smooth and take values in $GL(r, \mathbb{R})$. Hence the frame bundle is a principal $GL(r, \mathbb{R})$-bundle.

Analogous to the frame bundle construction, which associates a principal bundle to a vector bundle, one can associate a vector bundle to a principal bundle in the following way: Let $G \to P \xrightarrow{\pi} M$ be a principal bundle and $\rho : G \to GL(V)$ a finite dimensional representation of the structure group. Then $P \times_\rho V$ is defined to be the Cartesian product $P \times V$ under the equivalence relation $(p, v) \sim (pg, g^{-1}v)$. We have a projection $\beta : P \times_\rho V \to M$ given by $\beta([p, v]) = \pi(p)$ which is well-defined since the action of G on P preserves the fibers. Then $P \times_\rho V$ is a vector bundle called the *associated bundle* to $G \to P \xrightarrow{\pi} M$, see [Tu17, Proposition 31.1] for details. We use the notation $\mathrm{Ass}(P)$ for the associated bundle whenever the structure group and its representation is clear from the context. The vector space structure on the fiber is induced by the vector space structure on V. A Lie group G can always act on its Lie algebra $\mathfrak{g}$ via the adjoint representation, and the associated bundle $P \times_{Ad} \mathfrak{g}$ is called the *adjoint bundle*.

Remark. Beginning with a rank r vector bundle $\pi : E \to M$, we can form the frame bundle $\widetilde{\pi} : \mathcal{F}(E) \to M$ with structure group $GL(r, \mathbb{R})$. As $GL(r, \mathbb{R})$ already consists of linear isomorphisms from $\mathbb{R}^r$ to itself, we obtain the associated bundle

$$\mathrm{Ass}(\mathcal{F}(E)) = \mathcal{F}(E) \times_{Id_{GL(r,\mathbb{R})}} \mathbb{R}^r.$$

The map

$$\begin{aligned} \psi : \mathrm{Ass}(\mathcal{F}(E)) &\longrightarrow E \\ [p, v] &\longmapsto p(v) \end{aligned}$$

is a well-defined isomorphism of vector bundles over M. In particular,

$$\mathrm{Ass}(\mathcal{F}(TM)) \simeq TM.$$

3.1.3 The Fundamental Vector Field

If $G \to P \xrightarrow{\pi} M$ is a principal bundle, then $\mathcal{V} = \ker(d\pi) \subset TP$ is called the *vertical space* of the bundle. Elements of $\mathcal{V}_p$ are called *vertical vectors* at $p \in P$. We can use the Lie group action to represent the collection $\{\mathcal{V}_p, p \in P\}$ through the Lie algebra $\mathfrak{g}$ of G as follows: For every $p \in P$ the map $i_p : G \to P$ given by $i_p(g) = p \cdot g$ sends the identity of G to p, hence induces a map $di_p : \mathfrak{g} \to T_pP$. Letting $p \in P$ range over all different values gives the map

$$\xi : \mathfrak{g} \ni A \longmapsto \xi_A \in \mathfrak{X}(P),$$

called the *fundamental vector field* corresponding to $A \in \mathfrak{g}$. It can be written with the use of the Lie group exponential map $\exp : \mathfrak{g} \to G$ as

$$\xi_A(p) = \frac{d}{dt} p \cdot \exp(tA)\Big|_{t=0}.$$

For convenience, we employ the notation $e^A := \exp(A)$ for $A \in \mathfrak{g}$ and $\mathfrak{X}(\mathcal{V})$ for the vector fields on P taking values in $\mathcal{V}$. Since $\pi \circ i_p(g) = \pi(p)$ for every $g \in G$, the derivative of the composition is zero showing that $\xi_A \in \mathfrak{X}(\mathcal{V})$. As the following proposition shows, the fundamental vector field gives a way of identifying all the vertical spaces with the Lie algebra $\mathfrak{g}$.

Proposition 3.3. *The map $\xi(p) : \mathfrak{g} \to \mathcal{V}_p$ sending A to $\xi_A(p)$ is a vector space isomorphism for each $p \in P$. In particular, $\mathcal{V}$ is a trivial bundle.*

Proof. We have showed that $\xi_A \in \mathfrak{X}(\mathcal{V})$ and it is clear that $\xi(p)$ is linear since it is the derivative of the smooth map $i_p : G \to P$. To show injectivity we assume that

$$\xi_A(p) = \frac{d}{dt}\left(p \cdot e^{At}\right)\Big|_{t=0} = 0.$$

A simple computation shows that $\gamma(t) = p$ and $c_p(t) = p \cdot e^{At}$ are both integral curves of ξ_A through p. The uniqueness of integral curves gives that $p \cdot e^{At} = p$ for every $t \in (-\epsilon, \epsilon)$, for some $\epsilon > 0$. Since the group action is free, this forces e^{At} to be equal to the identity for the same values of t. Since the exponential map is a diffeomorphism in a neighbourhood of $0 \in \mathfrak{g}$ it follows that $A = 0$. Surjectivity follows once we know that the vector spaces $\mathfrak{g}$ and $\mathcal{V}_p$ have the same dimension. If $\mathcal{H}_p$ denotes an arbitrary complement to $\mathcal{V}_p$ in T_pP, then

$$\dim(\mathcal{V}_p) = \dim(T_pP) - \dim(\mathcal{H}_p) = \dim(P) - \dim(T_{\pi(p)}M) = \dim(P) - \dim(M) = \dim(\mathfrak{g}).$$

Hence $\xi(p) : \mathfrak{g} \to \mathcal{V}_p$ is a linear isomorphism. This implies that $\xi_{A_1}, \ldots, \xi_{A_r}$ is a global frame for $\mathcal{V}$ whenever $A_1, \ldots, A_r$ is a basis for $\mathfrak{g}$. ☐

The proof of Proposition 3.3 reveals that ξ_A is a non-vanishing vector field on P for any non-zero $A \in \mathfrak{g}$. This gives restrictions on which spaces can be total spaces in principal bundles. It is well known that there are no non-vanishing vector fields on S^2, see [Hat02, Theorem 2.28]. Thus S^2 can not be the total space of a principal bundle unless the structure group is zero-dimensional. Notice that the fundamental vector fields satisfy the following G-invariance property:

$$dR_g\xi_A(p) = \frac{d}{dt}\left(R_g\left(p \cdot e^{At}\right)\right)\Big|_{t=0} = \frac{d}{dt}\left(p \cdot gg^{-1}e^{At}g\right)\Big|_{t=0} = \xi_{Ad_{g^{-1}}(A)}(R_g(p)). \tag{3.1}$$

An algebraic reformulation of the situation so far is that we have for each $p \in P$ a short exact sequence of vector spaces

$$0 \longrightarrow \mathfrak{g} \xrightarrow{\xi(p)} T_pP \xrightarrow{d\pi_p} T_{\pi(p)}M \longrightarrow 0. \tag{3.2}$$

3.2 Connections and Curvature on Principal Bundles

We are now ready to define connections on principal bundles. By examining horizontal lifts and connection forms we will provide two equivalent definitions of a connection. After this we introduce the curvature of a connection through the framework of pseudotensorial forms. The structural equation and the Bianchi identity will be presented and provide a calculus for principal bundles.

For a principal bundle $G \to P \xrightarrow{\pi} M$, the vertical space $\mathcal{V} = \ker(d\pi) \subset TP$ is always given. However, a complement to $\mathcal{V}_p$ in T_pP for each $p \in P$ is not, hence must be considered as additional data. Since $\mathcal{V}$ is invariant under the group action, meaning that $dR_g\mathcal{V}_p = \mathcal{V}_{pg}$, we stipulate the same condition for its complement.

Definition 3.4. A *(principal) connection* on a principal bundle $G \to P \xrightarrow{\pi} M$ is a G-invariant distribution $\mathcal{H} \subset TP$ complementary to $\mathcal{V}$ at each point.

It will be clear from the context and notation whether we discuss principal connections or affine connections, abbreviating both to connections whenever convenient. At each point $p \in P$ we have

$$T_pP = \mathcal{H}_p \oplus \mathcal{V}_p, \quad dR_g\mathcal{H}_p = \mathcal{H}_{pg}.$$

Any vector $v \in T_pP$ can be decomposed into vertical and horizontal parts $v = v^{\mathcal{H}} + v^{\mathcal{V}}$. We call v a *horizontal* vector if $v^{\mathcal{V}} = 0$ and a *vertical vector* if $v^{\mathcal{H}} = 0$. On any principal bundle connections exist in abundance, see [KN96, Theorem 2.1].

One feature of a connection is that it allows us to lift tangent vectors on M to horizontal tangent vectors on P. Notice that for every $p \in P$ the map

$$d\pi\Big|_{\mathcal{H}_p} : \mathcal{H}_p \longrightarrow T_{\pi(p)}M$$

is injective, hence an isomorphism as the spaces have the same dimension. Thus for any vector $v \in T_xM$ and a choice $p \in \pi^{-1}(x)$, there exists a unique vector $w \in \mathcal{H}_p$ such that $d\pi(w) = v$. We call w the *horizontal lift* of v to $\mathcal{H}_p$ and employ the notation $w = h_pv$. Similarly, we can lift a vector field $X \in \mathfrak{X}(M)$ to a horizontal vector field $hX \in \mathfrak{X}(P)$ by

$$hX(p) = h_pX(\pi(p)).$$

If $X, Y \in \mathfrak{X}(M)$ then $h\,(X+Y) = hX + hY$ follows from linearity of the lift. Notice that the definition of horizontal lift of a vector field $X \in \mathfrak{X}(M)$ to $hX \in \mathfrak{X}(P)$ says precisely that hX is π-related to X, that is, $d\pi(hX) = X \circ \pi$. It follows from [War83, Proposition 1.55] that if hX is π-related to X and hY is π-related to Y, then $[hX, hY]$ is π-related to $[X, Y]$. Hence

$$d\pi\big([hX, hY]^{\mathcal{H}}\big) = d\pi\left([hX, hY]\right) = [X, Y] \circ \pi,$$

where the first equality follows from the definition of the vertical part. We can conclude that

$$h[X, Y] = [hX, hY]^{\mathcal{H}}$$

for all $X, Y \in \mathfrak{X}(M)$ since they are both horizontal and project to the same vector field on M.

The G-invariance of $\mathcal{H}$ implies the following G-invariance of the horizontal lifts,

$$dR_ghX(p) = dR_gh_pX(\pi(p)) = h_{pg}X(\pi(pg)) = hX(pg). \tag{3.3}$$

In fact, if we are given a system of lifts from $\mathfrak{X}(M)$ to $\mathfrak{X}(P)$ satisfying (3.3) we can define a distribution on P by

$$\mathcal{H}_p = \{hX(p) : X \in \mathfrak{X}(M),\ p \in P\}.$$

It is straightforward to check that these subspaces constitute a connection. Hence systems of G-invariant lifts are in one-to-one correspondence with connections on $G \to P \xrightarrow{\pi} M$.

In view of the exact sequence (3.2) this equivalence is, at least pointwise, that a map $h_p : T_{\pi(p)}M \to T_pP$ such that $d\pi \circ h_p = id_{T_{\pi(p)}M}$ is equivalent to a splitting

$$T_pP = \mathcal{H}_p \oplus \mathcal{V}_p \simeq T_{\pi(p)}M \oplus \mathfrak{g},$$

see [Lan02, Proposition 3.2]. The last equivalent definition of a split exact sequence is with the existence of a *contraction*, that is, by the existence of a map $\omega : T_pP \to \mathfrak{g}$ such that $\omega \circ \xi(p) = Id_{\mathfrak{g}}$. Let $\mathcal{H} \subset TP$ be a connection and for $p \in P$ define

$$\omega_p : T_pP \longrightarrow \mathfrak{g}, \quad \omega_p(X) = (\xi(p))^{-1}\,(v^{\mathcal{V}}),$$

where ξ is the fundamental vector field and $v \in T_pP$. We call ω the *connection one-form* corresponding to $\mathcal{H}$. Notice that $\ker(\omega) = \mathcal{H}$ and that ω_p provides a left inverse to $\xi(p)$.

Proposition 3.5. *The connection one-form ω satisfies the G-invariance property*

$$R_g^*\omega = Ad_{g^{-1}} \circ \omega$$

for every $g \in G$. Moreover, it is a smooth vector valued form.

Proof. We check that the G-invariance is true for horizontal and vertical vectors separately which suffices since both sides are linear. First consider a horizontal vector $X \in \mathcal{H}_p$. The vector dR_gX is still horizontal by the G-invariance of $\mathcal{H}$. Hence both sides are zero when evaluated at X. Now let $v \in T_pP$ be a vertical vector and write $v = \xi_A(p)$ for some $A \in \mathfrak{g}$. From the G-invariance of the fundamental vector field map (3.1), it follows that

$$R_g^*\omega(v) = R_g^*\omega(\xi_A(p)) = \omega\left(dR_g\xi_A(p)\right) = \omega\left(\xi_{Ad_{g^{-1}}(A)}(R_g(p))\right) = Ad_{g^{-1}}(A) = Ad_{g^{-1}} \circ \omega(v).$$

Smoothness is checked pointwise by patching together smooth local frames for $\mathcal{H}$ together with the fundamental vector fields corresponding to a basis for $\mathfrak{g}$, see [Tu17, Theorem 28.1] for details. □

Let ω be a smooth $\mathfrak{g}$-valued one-form which is a left inverse to $\xi(p)$ for each $p \in P$ and satisfies the G-invariance property in Proposition 3.5. Then $\mathcal{H} = \ker(\omega)$ defines a connection on the principal bundle $G \to P \xrightarrow{\pi} M$. The smoothness of $\mathcal{H}$ is the only thing which is not straightforward, see [Tu17, Theorem 28.5] for details. To summarize:

A connection can either be defined by a smoothly varying complement to $\mathcal{V} = ker(d\pi)$, by horizontal lifts, or by a connection one-form, all with suitable G-invariance properties.

We will now put the connection form in a broader perspective by introducing pseudotensorial forms, paving the way for defining curvature of a principal connection. Basic properties of vector valued forms are given in Appendix A.1.

Definition 3.6. Let $G \to P \xrightarrow{\pi} M$ be a principal bundle and $\rho : G \to GL(V)$ a representation of G on a finite-dimensional vector space V. A *pseudotensorial* r-form on P of type (ρ, V) is a r-form ϕ on P taking values in V such that

$$R_a^*\phi = \rho(a^{-1}) \cdot \phi,$$

for every $a \in G$. If a pseudotensorial r-form vanish whenever any of its inputs are vertical, we simply call it a *tensorial* r-form and we denote by $\Omega_\rho^r(P, V)$ all tensorial r-forms of type (ρ, V).

Example 3.7. Let $\pi_E : E = P \times_\rho V \to M$ be the associated bundle to $G \to P \xrightarrow{\pi} M$ with the representation $\rho : G \to GL(V)$. Associated to any $\phi \in \Omega_\rho^r(P, V)$ and $x \in M$ is a well-defined alternating multilinear map

$$\widetilde{\phi}_x : \overbrace{T_xM \times \cdots \times T_xM}^{r-\text{copies}} \longrightarrow \pi_E^{-1}(x)$$
$$\widetilde{\phi}_x(X_1, \ldots, X_r) = [u, \phi(X_1^*, \ldots, X_r^*)],$$

where $u \in \pi^{-1}(x)$ and $X_i^* \in T_uP$ is any vector such that $d\pi(X_i^*) = X_i$ for $i = 1, \ldots, r$. Conversely, given such a map $\widetilde{\phi}_x$ for each $x \in M$, we can recreate a tensorial form $\phi \in \Omega_\rho^r(P, V)$ by using that the map $v \mapsto [p, v]$ from V to E_x where $p \in \pi^{-1}(x)$ is an isomorphism, see [KN96, Example 5.2] for details. In particular, functions $f : P \to V$ such that $f(ua) = \rho(a^{-1})f(u)$ can be identified with sections of $\pi_E : E \to M$. This will be an insightful interpretation when we study the frame bundle in more detail in Section 3.5.

From now on, assume we are given a principal connection $\mathcal{H} \subset TP$. Notice that the connection one-form ω corresponding to $\mathcal{H}$ is a pseudotensorial 1-form of type $(Ad, \mathfrak{g})$. Let ϕ be an arbitrary pseudotensorial form and let $Pr_{\mathcal{H}}$ denote the projection of a tangent vector in TP to its horizontal part. Then $\phi \circ Pr_{\mathcal{H}}$ is clearly horizontal and is in fact in $\Omega^r_\rho(P, V)$ since $Pr_{\mathcal{H}}$ commutes with the group action due to the invariance of both $\mathcal{H}$ and $\mathcal{V}$. A way to form a pseudotensorial form of higher degree is to take the exterior differential $d\phi$ which again becomes a pseudotensorial form, see Appendix A.1 for details. Combining these operations gives the following definition.

Definition 3.8. If ϕ is a pseudotensorial r-form of type (ρ, V), then

$$D\phi = (d\phi) \circ Pr_{\mathcal{H}} \in \Omega^{r+1}_\rho(P, V)$$

is called the *exterior covariant derivative* of ϕ. If ω is the connection one-form we use the notation Ω in place of $D\omega$ and call it the *curvature form.*

Theorem 3.9 (Principal Structural Equation)**.** *Let ω be a connection form on the principal G-bundle $G \to P \to M$ and let Ω be its corresponding curvature form. Then*

$$\Omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)], \quad X, Y \in \mathfrak{X}(P).$$

The proof for the Principal Structural Equation can be found in [KN96, Proposition 5.5]. Whenever X and Y are both horizontal vector fields the Principal Structural Equation shows that

$$\Omega(X, Y) = d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) = -\omega([X, Y]). \tag{3.4}$$

This gives insight into what the curvature form measures; the curvature form is zero if and only if the horizontal distribution is *integrable*, that is, closed under Lie brackets. We say that a connection $\mathcal{H}$ on a principal bundle is *flat* if $\Omega \equiv 0$. In the next section we will use the theory of holonomy groups to justify this terminology.

We can use the curvature form Ω on the principal bundle $G \to P \to M$ to describe bracket relations for horizontal lifts of vector fields on M. If $X, Y \in \mathfrak{X}(M)$, then

$$[hX, hY] = h[X, Y] + \xi_{-\Omega(hX, hY)}. \tag{3.5}$$

To see this, decompose $[hX, hY]$ into horizontal and vertical parts

$$[hX, hY] = [hX, hY]^{\mathcal{H}} + [hX, hY]^{\mathcal{V}}.$$

We have previously showed that $[hX, hY]^{\mathcal{H}} = h[X, Y]$, so all that remains is to show that

$$[hX, hY]^{\mathcal{V}} = \xi_{-\Omega(hX, hY)}.$$

Since $[hX, hY]^{\mathcal{V}}$ is vertical it can for each $p \in P$ be written as $\xi_A(p)$ for some $A \in \mathfrak{g}$ depending on p. Then we use the connection one-form ω to obtain

$$A = \omega([hX, hY]^{\mathcal{V}}_p) = \omega([hX, hY]_p) = -\Omega(hX, hY)_p.$$

We now use the terminology and notation developed at the end of Appendix A.1 to write the Principal Structural Equation on the form

$$d\omega = -\frac{1}{2}[\omega, \omega] + \Omega.$$

Let $A_1, \dots, A_r$ be a basis for $\mathfrak{g}$ with structure constants c^i_{jk}. If we write $\omega = \sum_{i=1}^r \omega^i A_i$ and $\Omega = \sum_{i=1}^r \Omega^i A_i$, then using Proposition A.3 gives

$$d\omega^i = -\frac{1}{2} \sum_{j,k=1}^r c^i_{jk}(\omega^j \wedge \omega^k) + \Omega^i,$$

for $i = 1, \dots, r$.

Proposition 3.10 (Principal Bianchi Identity). *If Ω is the curvature form on a principal bundle with connection $\mathcal{H}$ then*

$$D\Omega \equiv 0.$$

Proof. If suffices to prove that $d\Omega(X,Y,Z) = 0$ whenever X, Y, and Z are horizontal vector fields. We apply the exterior derivative to the coordinate version of the Principal Structural Equation and get

$$0 = dd\omega^i = -\frac{1}{2}\sum_{j,k=1}^{r} c^i_{jk}\left(d\omega^j \wedge \omega^k - \omega^j \wedge d\omega^k\right) + d\Omega^i.$$

Evaluating the right hand side in X, Y, and Z gives the result since ω^s vanish on horizontal vectors for every $s = 1, \ldots, r$. □

3.3 Parallel Displacement and Holonomy Groups

In this section we will see that a connection on a principal bundle enables us to define horizontal lifting of curves. Analogous to the parallel transport defined for affine connections in Subsection 2.1.2, this will allow us to define parallel displacement in the total space along the curve in the base space. Using this we define holonomy groups, an invariant which will be used when we study sub-Riemannian model spaces in Chapter 4. The relationship between curvature and holonomy is given by the Ambrose-Singer Theorem which we shall present. Finally, the Ambrose-Singer Theorem will be used to give an interpretation of principal connections with zero curvature.

For the rest of the section, let $\mathcal{H}$ be a connection on a principal bundle $G \to P \xrightarrow{\pi} M$. If $\gamma : [0,1] \to M$ is a smooth curve in M, then a *horizontal lift* of γ is a smooth curve $\gamma^{\mathcal{H}} : [0,1] \to P$ such that $\pi(\gamma^{\mathcal{H}}) = \gamma$ and $\dot{\gamma}^{\mathcal{H}}(t)$ is a horizontal vector for every $t \in [0,1]$. We refer to $\gamma^{\mathcal{H}}$ simply as a *lift* of γ. Given an arbitrary point $u_0 \in \pi^{-1}(\gamma(0))$, there exists a unique lift $\gamma^{\mathcal{H}}$ of γ with $\gamma^{\mathcal{H}}(0) = u_0$, see [KN96, Proposition 3.1]. Then $\gamma^{\mathcal{H}}(1) \in \pi^{-1}(\gamma(1))$ and by varying $u_0 \in \pi^{-1}(\gamma(0))$ we obtain a map

$$\tau : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(1)),$$

which we call *parallel displacement* along γ. The parallel displacement commutes with the group action, that is, $R_a \circ \tau = \tau \circ R_a$. Unsurprisingly, the inverse of τ is given by the analogous operator for the curve $\gamma^{-1}(t) := \gamma(1-t)$. Moreover, concatenations of curves in M (whenever possible) result in compositions of their parallel displacements. Similarly to the construction of the fundamental group, we restrict our attention to loops in M and obtain the following definition.

Definition 3.11. Fix $x \in M$ and consider the set of all loops at x denoted by $C(x)$. Denote by $\Phi(x)$ the set of all isomorphisms $\tau : \pi^{-1}(x) \to \pi^{-1}(x)$ arising from parallel displacement along elements in $C(x)$. This is a group under composition called the *holonomy group* with reference point x relative to the connection $\mathcal{H}$. If $C_0(x)$ denotes the null-homotopic loops at x, then the corresponding group will be denoted by $\Phi_0(x)$ and is called the *restricted holonomy group* with reference point x relative to the connection $\mathcal{H}$.

We can realize the holonomy groups as subgroups of the structure group G as follows: If $u \in \pi^{-1}(x)$ then every loop $\gamma \in C(x)$ determines an element $a \in G$ such that $\tau(u) = ua$, where τ is the parallel displacement along γ. If $\mu \in C(x)$ determines $b \in G$, then the concatenation $\mu \cdot \gamma$ determines the element ba. Hence we obtain a subgroup of G which we denote by $\Phi(u)$ and refer to as the holonomy group relative to u. Similar notation and terminology will be used for the restricted holonomy group $\Phi_0(u)$, which is a subgroup of $\Phi(u)$. Notice that there are isomorphisms

$$\begin{array}{ccc} & C(x) & \\ \swarrow & & \searrow \\ \Phi(x) & \overset{\simeq}{\dashrightarrow} & \Phi(u) \\ \uparrow & & \uparrow \\ \Phi_0(x) & \overset{\simeq}{\dashrightarrow} & \Phi_0(u) \end{array}$$

We will mainly work with the holonomy group embedded as a subgroup of the structure group G. If two elements $u, v \in P$ can be joined by a horizontal curve, we use the notation $u \sim v$. The fact that parallel displacement commutes with the group action implies that $ua \sim va$ for every $a \in G$ whenever $u \sim v$.

Proposition 3.12. *Let u and v be elements in P with holonomy groups $\Phi(u)$ and $\Phi(v)$.*

(1) *If there exists some $a \in G$ such that $v = ua$, then $\Phi(u)$ is conjugate to $\Phi(v)$ in G.*

(2) *If $u \sim v$, then $\Phi(u) = \Phi(v)$.*

(3) *If M is connected then all the holonomy groups are conjugate, hence isomorphic.*

Moreover, all three statements hold for the restricted holonomy groups as well.

Proof. Assume that $v = ua$ for some $a \in G$ and let $b \in \Phi(u)$ so that $u \sim ub$. Then we also have $ua \sim (ub)a$ and we use this to conclude that

$$v = ua \sim (ub)a \sim va^{-1}ba.$$

This shows that $a^{-1}ba \in \Phi(v)$ and statement (1) follows. For statement (2), notice that $u \sim ua$ if and only if $v \sim va$ for $a \in G$. This is due to the transitivity of the equivalence relation $\sim$ and since $ua \sim va$ holds as well. Hence $a \in \Phi(u)$ if and only if $a \in \Phi(v)$, so they are equal. If M is connected, then for $u, v \in P$ we have $u \sim va$ for some $a \in G$ by lifting a curve in M from $\pi(u)$ to $\pi(v)$. Now statement (3) follows from statement (1). See [KN96, Proposition 4.1] for the case of the restricted holonomy groups. □

Theorem 3.13. *The holonomy group $\Phi(u)$ is a Lie group with $\Phi_0(u)$ as its connected component of the identity for every $u \in P$.*

We will not prove this result, it can be found in greater generality in [KN96, Theorem 4.2]. A remarkable feature of holonomy groups is that if we require less regularity, say by considering C^1 curves, we obtain the same holonomy groups anyhow by [KN96, Theorem 7.2]. For $u \in P$, let us denote by $P(u)$ the elements $v \in P$ which can be connected to u by a horizontal curve. We refer to $P(u)$ as the *holonomy bundle* through u. It is also common in the literature to refer to $P(u)$ as the *accessible set* of u.

Theorem 3.14 (Ambrose-Singer)**.** *Let $\mathcal{H}$ be a connection on a principal bundle $G \to P \xrightarrow{\pi} M$ with curvature form Ω. Denote the holonomy group and holonomy bundle with respect to $u \in P$ by $\Phi(u)$ and $P(u)$, respectively. Then the Lie algebra of $\Phi(u)$ is spanned by the elements on the form*

$$\Omega_v(X, Y), \quad v \in P(u), \quad X, Y \in \mathcal{H}_v.$$

The proof of the Ambrose-Singer Theorem can be found in [KN96, Theorem 8.1]. We say that a principal bundle $G_1 \to P_1 \xrightarrow{\pi_1} M$ is a *reduced bundle* of the principal bundle $G_2 \to P_2 \xrightarrow{\pi_2} M$ if there exists a principal bundle morphism $f : P_1 \to P_2$ which is an embedding and induces the identity on M. If $\mathcal{H}_1$ is a connection on $G_1 \to P_1 \xrightarrow{\pi_1} M$, then there exists a unique connection $\mathcal{H}_2$ on $G_2 \to P_2 \xrightarrow{\pi_2} M$ such that $df\,(\mathcal{H}_1) \subset \mathcal{H}_2$, see [KN96, Proposition 6.1]. We say that the connection $\mathcal{H}_2$ is *reducible* to the connection $\mathcal{H}_1$. The proof of the following result can be found in [KN96, Theorem 7.1].

Lemma 3.15. *Fix $u \in P$. Then $\Phi(u) \to P(u) \to M$ is a reduced bundle of $G \to P \to M$. Moreover, the connection $\mathcal{H}$ on $G \to P \to M$ is reducible to a connection on $P(u)$.*

Let us now consider the product bundle $G \to P = M \times G \to M$. The *canonical flat connection* on P is at each point $(x, a) \in P$ given by $T_xM \times \{0\} \simeq T_xM$. If θ is the canonical one-form of G, see Appendix A.2, then $\omega = Pr_G^*\theta$ is the connection one-form for the canonical flat connection. Using that the pullback commutes with the exterior differential and the Maurer-Cartan Equation (A.1), we have

$$d\omega = Pr_G^*(d\theta) = -\frac{1}{2}Pr_G^*([\theta, \theta]) = -\frac{1}{2}[\omega, \omega]. \tag{3.6}$$

The Principal Structural Equation presented in Theorem 3.9 implies that the canonical flat connection has zero curvature. A connection $\mathcal{H}$ on a principal bundle $G \to P \xrightarrow{\pi} M$ is *trivial* if for every $x \in M$ there exists a neighbourhood U of x and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times G$ such that $d\phi$ maps $\mathcal{H}$ onto the canonical flat connection on $U \times G$.

Proposition 3.16. *A connection $\mathcal{H}$ on $G \to P \to M$ is flat if and only if it is trivial.*

Proof. That any trivial connection is flat follows from Equation (3.6). Assume that $\mathcal{H}$ is flat and choose $x \in M$ together with a simply connected neighbourhood U of x. Then we have $\Phi_0(u) = \Phi(u)$ on $P|_U$ and the holonomy on the connection $\mathcal{H}|_U$ is trivial by the Ambrose-Singer Theorem. It follows from Lemma 3.15 that the connection $\mathcal{H}_U$ is reducible to a connection on $P(u) \simeq U \times \{e\}$. Since there is only one such connection, the result follows. □

If M is simply connected then the holonomy group of any connection on any principal bundle is connected and independent of the chosen point $x \in M$ up to conjugation. Moreover, by examining the proof of Proposition 3.16 we see that a flat connection $\mathcal{H}$ on $G \to P \to M$ induces a diffeomorphism of P into $G \times M$ whenever M is simply connected. Consider the principal bundle $O(2) \to O(3) \to S^2$ obtained from the homogeneous structure of S^2 equipped with an arbitrary connection $\mathcal{H}$. If $\mathcal{H}$ is flat we would have an isomorphism $O(3) \simeq S^2 \times O(2)$ which is not possible as $\pi_1(O(3)) \simeq \pi_1(\mathbb{R}P^3) = \mathbb{Z}/2\mathbb{Z}$ while $\pi_1(O(2)) \simeq \pi_1(S^1) = \mathbb{Z}$.

3.4 Induced Connections on Isometry Groups and Frame Bundles

We will now turn to describe how the affine connections introduced in Subsection 2.1.2 induce principal connections on frame bundles. For homogeneous Riemannian manifolds it is also possible to obtain a connection on the principal bundle representing the homogeneous structure. This gives two different ways of lifting the geometry of the base manifold to principal bundles, providing an efficient way to deal with problems arising in classifying sub-Riemannian model spaces in the next chapters. Finally, we show how principal connections on the isometry group induce connections on the orthonormal frame bundle for homogeneous Riemannian manifolds.

Consider a manifold M with an arbitrary connection ∇ on a vector bundle $\pi_E : E \to M$ of rank r. Let $\gamma : [0, 1] \to M$ be a curve with $\gamma(0) = x$ and choose $\phi \in \pi^{-1}(x)$, where π denotes the projection $\pi : \mathcal{F}(E) \to M$. Then there is a unique lift $\eta : [0, 1] \to \mathcal{F}(E)$ with $\eta(0) = \phi$ such that $\eta(t)$ is a parallel frame along $\gamma(t)$ with respect to the connection ∇. We call $\eta(t)$ the *parallel lift* of $\gamma(t)$ to ϕ and define

$$\mathcal{H}_\phi^\nabla = \{\dot{\eta}(0) \in T_\phi\mathcal{F}(E) : \eta(t) \text{ is the parallel lift of a curve } \gamma(t) \text{ in } M \text{ with } \eta(0) = \phi\}. \tag{3.7}$$

Proposition 3.17. *The distribution $\mathcal{H}^\nabla$ defines a connection on the principal bundle $GL(r, \mathbb{R}) \to \mathcal{F}(E) \xrightarrow{\pi} M$.*

Proof. The map

$$f_\phi : T_xM \longrightarrow T_\phi\mathcal{F}(E)$$
$$\dot\gamma(0) \longmapsto \dot\eta(0)$$

is linear [Tu17, Proposition 29.6], so $\mathcal{H}^\nabla_\phi$ is a vector subspace of $T_\phi\mathcal{F}(E)$ since $\mathcal{H}^\nabla_\phi = \operatorname{Im}(f_\phi)$. It is clear that $d\pi_\phi$ is a left inverse of f_ϕ, showing that f_ϕ is injective. Hence $\mathcal{H}^\nabla$ and $\mathcal{V} = \ker(d\pi)$ are transverse at every point. It follows from

$$\dim(\mathcal{V}_\phi) + \dim\left(\mathcal{H}^\nabla_\phi\right) = \dim(\mathfrak{gl}(r,\mathbb{R})) + \dim(M) = \dim(T_\phi\mathcal{F}(E))$$

that $\mathcal{H}^\nabla \oplus \mathcal{V} = T\mathcal{F}(E)$. We refer the reader to [Tu17, Proposition 29.8] which shows that $\mathcal{H}^\nabla$ is a smooth distribution which is invariant under $GL(r,\mathbb{R})$. ☐

In particular, any affine connection on M determines a principal connection on the frame bundle of the tangent bundle by (3.7). Let (M,g) be a Riemannian manifold with an affine connection ∇ which is compatible with the metric. The *orthonormal frame bundle* $\mathcal{F}^O(TM)$ is constructed in the same way as the frame bundle, with the additional requirement that its elements are orthonormal frames. We obtain a principal bundle

$$O(n) \longrightarrow \mathcal{F}^O(TM) \longrightarrow M,$$

where $O(n)$ denotes the n-dimensional orthogonal group. Recall from Subsection 2.1.2 that compatibility with the metric is equivalent to parallel translation being an isometry. Therefore (3.7) with the alteration $\eta : [0,1] \to \mathcal{F}^O(TM)$ is still valid. Hence any metric compatible connection ∇ induces a connection $\mathcal{H}^\nabla$ on the orthonormal frame bundle $\mathcal{F}^O(TM)$.

Let us now consider a homogeneous Riemannian manifold M with a fixed point $x_0 \in M$. Then Proposition 2.17 and Example 3.2 shows that $H \to G \to M$ is a principal bundle, where $G = \operatorname{Isom}(M)$ is the isometry group of M and $H = K_{x_0}$ is the isotropy group at $x_0 \in M$. Denoting the Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$, respectively, we have mentioned earlier that $\mathfrak{h}$ admits an $Ad(H)$-invariant complement $\mathfrak{m}$ in $\mathfrak{g}$. We consider the distribution $\mathcal{H}^\mathfrak{m}$ on G given by left translation of $\mathfrak{m}$.

Proposition 3.18. *The distribution $\mathcal{H}^\mathfrak{m}$ is a connection on the principal bundle $H \to G \to M$.*

Proof. Let $v \in \mathcal{H}^\mathfrak{m}_g$ be an arbitrary horizontal vector at $g \in G$. Then $dL_{g^{-1}}(v) \in \mathfrak{g}$ with zero $\mathfrak{h}$-part. Notice that dL_g restricts to the fundamental vector field on $\mathfrak{h}$, giving an isomorphism

$$dL_g\Big|_\mathfrak{h} : \mathfrak{h} \to \mathcal{V}_g.$$

Thus by applying dL_g to $dL_{g^{-1}}v$ shows that v has no $\mathcal{V}_g$-part. Hence $\mathcal{V}_g \cap \mathcal{H}^\mathfrak{m}_g = \{0\}$ and since $\dim(\mathcal{V}_g) = \dim(\mathfrak{h})$ and $\dim(\mathcal{H}^\mathfrak{m}_g) = \dim(\mathfrak{m})$, we can conclude that $\mathcal{V} \oplus \mathcal{H}^\mathfrak{m} = TG$. The $Ad(H)$-invariance of $\mathfrak{m}$ shows that

$$dR_g\mathcal{H}^\mathfrak{m}_s = dR_g \circ dL_s\mathfrak{m} = dL_{sg}ad_{g^{-1}}\mathfrak{m} \subset dL_{sg}\mathfrak{m} = \mathcal{H}^\mathfrak{m}_{sg},$$

for $s \in G$. ☐

It follows from [KN96, Theorem 11.1] that any left-invariant principal connection $\mathcal{H}^\mathfrak{m}$ on the principal bundle $H \to G \to M$ is in fact obtained in this way. If M is a symmetric space, then we will be particularly interested in the distribution $\mathcal{H}^\mathfrak{m}$ when $\mathfrak{m} \subset \mathfrak{g}$ is the canonical complement to $\mathfrak{h}$ described in Lemma 2.45. We will now provide a way to compute the curvature and holonomy of the left-invariant principal connections we have discussed.

Proposition 3.19. *Let $H \to G \to M$ be the principal bundle obtained from a connected Lie group G and a closed subgroup H described in Example 3.2. For any left-invariant connection $\mathcal{H}^{\mathfrak{m}}$ on $H \to G \to M$, the curvature form is given by*

$$\Omega(X, Y) = -[X, Y]_{\mathfrak{h}}, \quad X, Y \in \mathfrak{m}.$$

Moreover, the Lie algebra of the holonomy group $\Phi(e)$ at the identity is generated by all the elements of the form $[X, Y]_{\mathfrak{h}}$, for $X, Y \in \mathfrak{m}$.

Proof. Recall that Equation (3.4) states that

$$\Omega(X, Y) = -\omega\left([X, Y]\right),$$

where ω is the connection one-form. The connection one-form ω on this bundle is simply given by

$$\omega(v) = dL_{g^{-1}}(v^{\mathcal{V}}), \quad v \in T_g G,$$

so the curvature formula follows. The holonomy statement follows from the Ambrose Singer Theorem. □

Example 3.20. Let $V_k(\mathbb{R}^n)$ be the set of all k-tuples of orthonormal vectors in $\mathbb{R}^n$ with $0 < k < n$. Identify each such element with an $(n \times k)$ matrix where the columns are the orthonormal vectors. The space $V_k(\mathbb{R}^n)$ is compact in the subspace topology inherited from $\mathbb{R}^{nk}$. We have a group action

$$\begin{aligned} SO(n) \times V_k(\mathbb{R}^n) &\longrightarrow V_k(\mathbb{R}^n) \\ (A, v_1, \ldots, v_k) &\longmapsto Av_1, \ldots, Av_k, \end{aligned}$$

which is transitive. If $e_1, \ldots, e_n$ denotes the standard basis in $\mathbb{R}^n$, then the isotropy group of $(e_1, \ldots, e_k) \in V_k(\mathbb{R}^n)$ is

$$\left\{ A \in SO(n) : A = \begin{pmatrix} I_k & 0 \\ 0 & C \end{pmatrix}, \ C \in SO(n-k) \right\} \simeq SO(n-k).$$

Hence the homogeneous structure $V_k(\mathbb{R}^n) \simeq SO(n)/SO(n-k)$ induces a unique manifold structure on $V_k(\mathbb{R}^n)$ such that the action of $SO(n)$ on $V_k(\mathbb{R}^n)$ is smooth. We call $V_k(\mathbb{R}^n)$ the *Steifel manifold* of type (k, n).

Consider the Lie algebra $\mathfrak{o}(n)$ of $SO(n)$ consisting of skew-symmetric matrices and let

$$\langle X, Y \rangle_{\mathfrak{o}(n)} = \mathrm{tr}(XY^T) = -\mathrm{tr}(XY).$$

The inner product $\langle \cdot, \cdot \rangle_{\mathfrak{o}(n)}$ called the *trace form* of $\mathfrak{o}(n)$, see [O'N83, Lemma 11.6] for some properties. Consider the Lie algebra $\mathfrak{o}(n-k)$ of $SO(n-k)$ identified as a subset $\mathfrak{o}(n-k) \subset \mathfrak{o}(n)$ by

$$\mathfrak{o}(n-k) \ni Y \longmapsto \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} \in \mathfrak{o}(n).$$

Define $\mathfrak{m} = \mathfrak{o}(n-k)^{\perp}$ with respect to the trace form. If $Q \in \mathfrak{m}$ and $Y, Z \in \mathfrak{o}(n-k)$, then

$$\langle Ad_Y(Q), Z \rangle_{\mathfrak{o}(n)} = \langle YQY^{-1}, Z \rangle_{\mathfrak{o}(n)} = -\mathrm{tr}(YQY^{-1}Z) = -\mathrm{tr}(QZ) = 0,$$

showing that $\mathfrak{m}$ is $Ad(SO(n-k))$-invariant. Hence the left translation $\mathcal{H}_k^{\mathfrak{m}}$ of $\mathfrak{m}$ defines a principal connection on $SO(n-k) \to SO(n) \to V_k(\mathbb{R}^n)$.

Let us write an element $Q \in \mathfrak{m}$ as

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix},$$

with $Q_1 \in M_{k\times k}(\mathbb{R})$ and $Q_4 \in M_{n-k\times n-k}(\mathbb{R})$. The fact that $\mathfrak{m} \subset \mathfrak{o}(n)$ gives that $Q_1^T = -Q_1$, $Q_3^T = -Q_2$, and $Q_4^T = -Q_4$. Moreover, $\mathfrak{m}^\perp = \mathfrak{o}(n-k)$ implies that for $Y \in \mathfrak{o}(n-k)$,

$$\langle Q, Y\rangle_{\mathfrak{o}(n)} = -\mathrm{tr}\left[\begin{pmatrix} Q_1 & Q_2 \\ -Q_2^T & Q_4\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & Y\end{pmatrix}\right] = -\mathrm{tr}\,(Q_4 Y) = 0.$$

We will now show that the condition $\mathrm{tr}(Q_4Y) = 0$ for every $Y \in \mathfrak{o}(n-k)$ forces $Q_4 = 0$. Assume $Q_{4_{ij}} = a \neq 0$ with $i < j$ and let $Y = E_{ij} - E_{ji}$, where E_{ij} denotes the matrix with 1 at entry ij and zero otherwise. Then $\mathrm{tr}(Q_4Y) = -2a = 0$, showing that all the off-diagonal entries of Q_4 are zero. Since $Q_4 = -Q_4^T$, it follows that $Q_4 = 0$.

To find the curvature of the connection we use Proposition 3.19. For two elements $Q, R \in \mathfrak{m}$, we have

$$\begin{aligned}[Q,R] &= \begin{bmatrix} Q_1 & Q_2 \\ -Q_2^T & 0\end{bmatrix}\begin{bmatrix} R_1 & R_2 \\ -R_2^T & 0\end{bmatrix} - \begin{bmatrix} R_1 & R_2 \\ -R_2^T & 0\end{bmatrix}\begin{bmatrix} Q_1 & Q_2 \\ -Q_2^T & 0\end{bmatrix} \\ &= \begin{bmatrix} Q_1R_1 - Q_2R_2^T - R_1Q_1 + R_2Q_2^T & Q_1R_2 - R_1Q_2 \\ -Q_2^TR_1 + R_2^TQ_1 & -Q_2^TR_2 + R_2^TQ_2\end{bmatrix}.\end{aligned}$$

The $\mathfrak{o}(n-k)$-part is simply the last entry, so we get

$$\Omega(Q,R) = -[Q,R]_{\mathfrak{o}(n-k)} = Q_2^TR_2 - R_2^TQ_2.$$

Any element $S = \mathfrak{o}(n-k)$ can be written as $S = Q_2^TR_2 - R_2^TQ_2$ for $Q, R \in \mathfrak{m}$. By invoking Proposition 3.19 again it follows that the Lie algebra of the holonomy group $\Phi(e)$ is all of $\mathfrak{o}(n-k)$. Since $\Phi(e) \subset SO(n-k)$, we have that the holonomy group at the identity is the full structure group $SO(n-k)$. We will see in Section 4.1 that this computation implies that $\mathcal{H}_k^{\mathfrak{m}}$ is a bracket generating sub-Riemannian structure on $SO(n)$.

Given a homogeneous Riemannian manifold M, we have described two principal bundle structures related to it, namely

$$H \longrightarrow G \longrightarrow M \quad \text{and} \quad O(n) \longrightarrow \mathcal{F}^O(TM) \longrightarrow M,$$

where G denotes the isometry group of M and H the isotropy group at a point $x_0 \in M$. A choice of an $\mathrm{Ad}(H)$-invariant complement $\mathfrak{m} \subset \mathfrak{g}$ of $\mathfrak{h}$ gives a connection $\mathcal{H}^{\mathfrak{m}}$ on $H \to G \to M$. For every $a \in G$, the map $\Phi(a) : M \to M$ sending bH to abH is an isometry and hence induces a map

$$\begin{aligned} G \times \mathcal{F}^O(TM) &\longrightarrow \mathcal{F}^O(TM) \\ (a, \phi : \mathbb{R}^n \longrightarrow T_xM) &\longmapsto d\Phi(a)_x \circ \phi \end{aligned}$$

The map is a Lie group action and is abbreviated to $a \cdot \phi$ to avoid lengthy notation. It is clear by the definition that the action preserves the fibers $\mathcal{F}^O(TM)_x$. Assume that $a \cdot \phi = \phi$ for $\phi \in \mathcal{F}^O(TM)_x$. If $\psi \in \mathcal{F}^O(TM)_x$ then we can write $\psi = \phi \circ A$ for $A \in O(n)$. Thus

$$a \cdot \psi = a \cdot (\phi \circ A) = \phi \circ A = \psi,$$

simply by associativity of composition. Hence $d\Phi(a)_x = Id_{T_xM}$ and since $\Phi(a) : M \to M$ is an isometry, we have from Proposition 2.15 that $\Phi(a) = Id_M$. From this we conclude that $a = e$ and hence the action is free. The action is not transitive in general, as this is a reformulation of being frame-homogeneous. Now let us consider the distribution

$$\mathcal{H}^G = \left\{ \frac{d}{dt} a(t) \cdot \phi\Big|_{t=0} : \phi \in \mathcal{F}^O(TM),\ a(t) \text{ is a curve in } G \text{ tangent to } \mathcal{H}^{\mathfrak{m}} \right\}.$$

Proposition 3.21. *The distribution $\mathcal{H}^G$ induced from the connection $\mathcal{H}^{\mathfrak{m}}$ as described above is a principal connection on the orthonormal frame bundle. Moreover, this connection is invariant under the left action of G on $\mathcal{F}^O(TM)$.*

Proof. We will only show the invariance properties and leave the fact that $\mathcal{H}^G$ is a principal connection to the reader. The right-invariance with respect to $O(n)$ is clear since

$$dR_A \frac{d}{dt} a(t) \cdot \phi\Big|_{t=0} = \frac{d}{dt} a(t) \cdot (\phi \circ A)\Big|_{t=0},$$

and this is by definition in $\mathcal{H}^G_{\phi \circ A}$ since A is an isometry from $\mathbb{R}^n$ to itself. Hence $\mathcal{H}^G$ is a principal connection on $\mathcal{F}^O(TM)$. To show left-invariance with respect to G, notice that

$$dL_a \frac{d}{dt} a(t) \cdot \phi\Big|_{t=0} = \frac{d}{dt} aa(t) \cdot \phi\Big|_{t=0}.$$

As $\mathcal{H}^{\mathfrak{m}}$ is defined by left translation $a \cdot a(t)$ is still a horizontal curve, so $\mathcal{H}^G$ is preserved under the induced action of G on the left. □

Let $M = \Sigma(n, K)$ denote the n-dimensional model space with sectional curvature K. Fix $x \in M$ and $\phi_0 \in \mathcal{F}^O(TM)$ with $\pi(\phi_0) = x$ and consider the map

$$\begin{aligned} \Gamma : \mathrm{Isom}(M) &\longrightarrow \mathcal{F}^O(TM) \\ \Phi &\longmapsto d\Phi_x \circ \phi_0. \end{aligned}$$

Then since M is frame-homogeneous, the map Γ is a bijection. The inverse $\Gamma^{-1}(\phi)$ of a frame $\phi \in \mathcal{F}^O(TM)$ is the unique isometry $\Phi \in \mathrm{Isom}(M)$ such that $d\Phi_x$ takes ϕ_0 to ϕ. This identifies the fiber $\pi^{-1}(x)$ with the isotropy group K_x at the point $x \in M$. Since the fixing of $\phi_0 \in \pi^{-1}(x)$ also identifies $O(n)$ with $\pi^{-1}(x)$ as $O(n)$ acts freely and transitively within $\pi^{-1}(x)$, it follows that $K_x \simeq O(n)$ as Lie groups. As Γ is a diffeomorphism we can induce a Lie group structure on $\mathcal{F}^O(TM)$ such that ϕ_0 becomes the identity. Thus

$$\mathcal{F}^O(TS^n_R) \simeq O(n+1), \quad \mathcal{F}^O(T\mathbb{R}^n) \simeq E(n), \quad \mathcal{F}^O(T\mathbb{H}^n_R) \simeq O_+(n,1). \tag{3.8}$$

3.5 Geometry via Frame Bundles

In this section we focus on the frame bundle; the principal bundle which is most strongly related to the geometry of the base manifold. We will see how it carries more structure than an arbitrary principal bundle, ensuring among other things that the tangent bundle of the frame bundle is always trivializable. The pseudotensorial form approach will be united with curvature and torsion of affine connections on M. We will present a structural equation and a Bianchi identity intrinsic to the frame bundle. Together with the Principal Structural Equation and the Principal Bianchi Identity described in Section 3.2, they provide a calculus on the frame bundle for handling the geometry of the base manifold. The main purpose of this section is to unite the principal bundle approach with results in Chapter 2 and see how this produce a powerful machinery for answering nontrivial geometric questions.

Let M be an arbitrary manifold and consider the principal bundle $GL(n,\mathbb{R}) \to \mathcal{F}(TM) \xrightarrow{\pi} M$. The *solder form* θ is the $\mathbb{R}^n$-valued one-form on $\mathcal{F}(TM)$ defined by

$$\theta(X) = \phi^{-1}\left(d\pi(X)\right),$$

for $X \in T_\phi \mathcal{F}(TM)$. Notice that θ vanishes on $\mathcal{V} = \ker(d\pi)$ and satisfies

$$\theta(dR_a X) = (\phi \circ a)^{-1}(d\pi(dR_a X)) = a^{-1} \circ \phi^{-1}(d\pi(X)) = a^{-1}(\theta(X)),$$

for $a \in GL(n,\mathbb{R})$. This shows that θ is a tensorial one-form of type $(GL(n,\mathbb{R}), \mathbb{R}^n)$. Recall from Subsection 3.1.2 that the associated bundle of $\pi : \mathcal{F}(TM) \to M$ can be identified with the tangent bundle $\pi_{TM} : TM \to M$. Using the identification in Example 3.7, the solder form is identified with the identity map

$Id : T_xM \to T_xM$ for each $x \in M$, justifying that the solder form is called the *canonical form* in [KN96]. We use the expression *linear connection* on M for a principal connection on $\mathcal{F}(TM)$. Analogous to thinking of the fundamental vector field as connecting the vertical spaces of an arbitrary principal bundle, we have the following definition specifically for linear connections.

Definition 3.22. Let $\mathcal{H}$ be a linear connection on a manifold M. We associate to every $v \in \mathbb{R}^n$ a horizontal vector field $B_v \in \mathfrak{X}(\mathcal{H})$ whose value at $\phi \in \mathcal{F}(TM)$ is the unique horizontal vector $B_v(\phi)$ such that $d\pi(B_v(\phi)) = \phi(v)$. We call B_v the *standard horizontal vector field* corresponding to v.

Lemma 3.23. *The standard horizontal vector fields satisfy the following properties:*

- *The solder form provides a left inverse to all standard horizontal vector fields, that is, $\theta(B_v) = v$ for every $v \in \mathbb{R}^n$.*
- *Right invariance $dR_a(B_v) = B_{a^{-1}(v)}$ holds for $a \in GL(n, \mathbb{R})$ and $v \in \mathbb{R}^n$.*
- *The standard horizontal vector fields B_v are non-vanishing vector vector fields for $v \neq 0$.*

All three statements follows from the definition of standard horizontal vector fields and that

$$d\pi\Big|_{\mathcal{H}_\phi} : \mathcal{H}_\phi \longrightarrow T_{\pi(\phi)}M$$

is an isomorphism. The standard horizontal vector fields have the following impact on the frame bundle.

Proposition 3.24. *The tangent bundle of the frame bundle is trivializable.*

Proof. Let $v_1, \ldots, v_n$ be a basis for $\mathbb{R}^n$ and $A_1, \ldots, A_{n^2}$ a basis for $\mathfrak{gl}(n, \mathbb{R})$. We showed in Proposition 3.3 that the vector fields $\xi_{A_1}, \ldots, \xi_{A_{n^2}}$ trivialize $\mathcal{V} = \ker(d\pi)$. The first statement of Lemma 3.23 shows that $B_{v_1}, \ldots, B_{v_n}$ are linearly independent horizontal vector fields. As $\operatorname{rank}(\mathcal{H}) = \operatorname{rank}(TM) = n$, it follows that $\mathcal{H}$ is also trivializable. Hence the vector fields

$$B_{v_i}, \xi_{A_r}, \quad 1 \leq i \leq n,\, 1 \leq r \leq n^2$$

trivialize $T\mathcal{F}(TM)$. □

By considering the principal bundle arising from the definition of $\mathbb{R}P^2$, namely $\mathbb{Z}/2\mathbb{Z} \to S^2 \to \mathbb{R}P^2$, one sees that Proposition 3.24 is not valid for arbitrary principal bundles. Let us fix a linear connection $\mathcal{H}$ on M. Using our pseudotensorial approach, we define the *torsion form* of $\mathcal{H}$ to be

$$\Theta = D\theta,$$

where D denotes the exterior covariant differential. It is a tensorial 2-form on $\mathcal{F}(TM)$ of type $(GL(n, \mathbb{R}), \mathbb{R}^n)$. We employ the symbol $\circlearrowright$ to denote the cyclic sum of arguments, an example is

$$\circlearrowright \Omega(X, Y)\theta(Z) = \Omega(X, Y)\theta(Z) + \Omega(Y, Z)\theta(X) + \Omega(Z, X)\theta(Y).$$

Proposition 3.25 (Structural Equations and Bianchi Identities on the Frame Bundle)**.** *The connection form, solder form, curvature form, and torsion form satisfy:*

$$\textit{Frame Bundle Structural Equation: } \Theta(X, Y) = d\theta(X, Y) + \omega(X)\theta(Y) - \omega(Y)\theta(X);$$
$$\textit{Principal Structural Equation: } \Omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)].$$

Moreover, their exterior covariant differentials satisfy:

$$\textit{Frame Bundle Bianchi Identity: } 3D\Theta(X, Y, Z) = \circlearrowright \Omega(X, Y)\theta(Z);$$
$$\textit{Principal Bianchi Identity: } D\Omega = 0,$$

for every $X, Y, Z \in T_\phi(\mathcal{F}(TM))$.

The proof together with several applications can be found in [KN96, Section 3.2]. We can use the structural equations to show that the standard horizontal vector fields satisfy

$$[B_u, B_v] = -\xi_{\Omega(B_u,B_v)} - B_{\Theta(B_u,B_v)}, \quad u, v \in \mathbb{R}^n. \tag{3.9}$$

To illustrate, the computation for the horizontal part is given by

$$\begin{aligned}\theta([B_u, B_v]) &= -B_u(\theta(B_v)) + B_v(\theta(B_u)) - d\theta(B_u, B_v)\\ &= -d\theta(B_u, B_v)\\ &= -\Theta(B_u, B_v) + \omega(B_u)\theta(B_v) - \omega(B_v)\theta(B_u)\\ &= -\Theta(B_u, B_v).\end{aligned}$$

There are analogous relations involving the fundamental vector fields

$$[\xi_A, \xi_C] = \xi_{[A,C]}, \quad [\xi_A, B_u] = B_{A(u)}, \tag{3.10}$$

for $A, C \in \mathfrak{gl}(n, \mathbb{R})$ and $u \in \mathbb{R}^n$. Details can be found in [KN96, Proposition 1.4.1] and [KN96, Proposition 3.23], respectively.

Let $G \to P \xrightarrow{\pi} M$ be a principal bundle and let $\rho : G \to GL(V)$ be a finite dimensional representation. Then a connection $\mathcal{H}$ on $G \to P \xrightarrow{\pi} M$ always induces a connection on the associated bundle $E = P \times_\rho V$ as follows: Recall from Example 3.7 that there is a one-to-one correspondence, denoted by $\vee$, between sections of $\pi_E : E \to M$ and functions $f : P \to V$ such that $f(ua) = \rho(a^{-1})f(u)$ for $a \in G$ and $u \in P$. Define

$$\mathrm{Ass}(\mathcal{H})(X, Y) = (hXs)^\vee, \quad X, Y = s^\vee \in \mathfrak{X}(E),$$

where hX denotes the horizontal lift with respect to the connection $\mathcal{H}$. It is straightforward to check that hXs is still pseudotensorial and that $\mathrm{Ass}(\mathcal{H})$ satisfies the requirements for a connection on the vector bundle $\pi_E : E \to M$. Assume now that ∇ is an affine connection on M and let $\mathcal{H}^\nabla$ be the linear connection induced by ∇ as described in Proposition 3.17. Then $\mathcal{H}^\nabla$ induces a connection $\mathrm{Ass}(\mathcal{H})$ on the associated bundle $\pi_{TM} : TM \to M$. The two connections ∇ and $\mathrm{Ass}(\mathcal{H})$ coincide and ∇ can be computed as

$$(\nabla_X Y)_x = \phi\left(hX(\phi)\left[\theta(hY)\right]\right)$$

where $\pi(\phi) = x$ and $X, Y \in \mathfrak{X}(M)$, see [KN96, Section 3.5].

We will now describe how the torsion and curvature forms are related to torsion and curvature tensor fields on the base manifold. Let T^∇ and R^∇ denote the torsion and curvature tensor field with respect to ∇, and let Θ and Ω denote the torsion and curvature form with respect to $\mathcal{H}^\nabla$, respectively. The proof of the following theorem is given in [KN96, Theorem 5.1].

Proposition 3.26. *Using the notation above, we have that*

$$\Theta(v, w) = \phi^{-1}\left[T^\nabla(d\pi(v), d\pi(w))\right], \quad v, w \in T_\phi\mathcal{F}(TM);$$
$$\Omega(v, w)\phi^{-1}(Z) = \phi^{-1}\left[R^\nabla(d\pi(v), d\pi(w))Z\right], \quad v, w \in T_\phi\mathcal{F}(TM),\ Z \in T_{\pi(\phi)}M.$$

In particular, if (M, g) is a Riemannian manifold which is locally isometric to Euclidean space, then $\mathcal{H}^\nabla$ is trivial when ∇ is the Levi-Civita connection.

Let ∇ be an arbitrary affine connection on M and let $\mathcal{H}^\nabla$ be the induced linear connection. The geodesics in M with respect to ∇ are precisely the projections of the integral curves of the standard horizontal vector fields, see [KN96, Proposition 6.3]. Recall that a connection ∇ is said to be complete if every geodesic can be extended for all time. Completeness of ∇ is equivalent to the property that all horizontal vector fields with respect to $\mathcal{H}^\nabla$ are complete. If the torsion T^∇ of the affine connection is zero, then by Proposition 3.26 the Frame Bundle Bianchi Identity becomes

$$\circlearrowright \Omega(hX(\phi), hY(\phi))\theta(hZ(\phi)) = \circlearrowright \phi^{-1}R^\nabla(X, Y)Z = 0,$$

for $X, Y, Z \in T_xM$ and $\pi(\phi) = x$. Composing with ϕ, this gives exactly the classical Bianchi Identity given in Equation (2.3).

4 Sub-Riemannian Geometry and Their Model Spaces

We now embark on the main topic of the thesis: sub-Riemannian model spaces. Basic notions in sub-Riemannian geometry as well as nilpotentizations and Gromov-Hausdorff distance is developed before proceeding to define sub-Riemannian model spaces. Their description is based on maximal symmetry in the form of lifting linear isometries to global isometries. This approach is taken due to the lack of a satisfying notion of curvature in the sub-Riemannian setting. Several constructions, such as partial connections and horizontal holonomy, will be explored in this chapter and is based on the theory developed in Chapter 3. An invariant of a sub-Riemannian model space is its nilpotentization. This leads us to examine Carnot groups which are also model spaces as the first step in classifying model spaces in general. All model spaces of step two will be classified, revealing that they are not only Lie groups but are either (essentially) isometry groups of the Riemannian model spaces or free nilpotent Lie groups. Finally, we will in Section 4.7 provide a minor original result by classifying all sub-Riemannian model spaces with contact structures.

4.1 Sub-Riemannian Geometry

We will in the first section survey the main definitions and concepts in sub-Riemannian geometry. As our focus is towards defining sub-Riemannian model spaces, some important topics in sub-Riemannian geometry will be alluded to rather than fully explored.

Definition 4.1. A *sub-Riemannian geometry* is a triple $(Q, \mathcal{H}, g)$ where Q is a manifold, $\mathcal{H} \subset TQ$ is a subbundle, and g is a smooth fiber-metric defined on $\mathcal{H}$.

More explicitly, g defines an inner product between vectors in $\mathcal{H}_p$ for every $p \in Q$ and varies smoothly in the sense that if $X, Y \in \Gamma(\mathcal{H})$, then $g(X, Y)$ is a smooth function on Q. We refer to Q as a *sub-Riemannian manifold*, $\mathcal{H}$ as the *horizontal distribution*, and g as the *sub-Riemannian metric*. Elements in $\mathcal{H}$ will be called *horizontal vectors* while vector fields on Q taking values in $\mathcal{H}$ will be called *horizontal vector fields*. The reader should be aware that some authors use a different definition of a sub-Riemannian geometry, e.g. [Bel96] allowing $\mathcal{H}$ to be a distribution of non-constant rank. An absolutely continuous curve $\gamma : [a, b] \to Q$ will be called *horizontal* if $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$ for almost every $t \in [a, b]$. For any horizontal curve, we define its length analogous to the Riemannian setting by

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt.$$

Definition 4.2. Let $(Q, \mathcal{H}, g)$ be a sub-Riemannian geometry with Q connected. Then it is furnished with an extended distance function given by

$$d_{CC}(p, q) = \inf L(\gamma),$$

where the infimum is taken over all absolutely continuous horizontal curves connecting p and q.

This distance d_{CC} will be referred to as the *Carnot-Carathéodory distance* of $(Q, \mathcal{H}, g)$. We will be careful and not refer to d_{CC} as a metric to avoid confusing it with the fiber metric g. The word "extended" simply refers to that the distance between two points might not be finite. Given an arbitrary subbundle $\mathcal{H} \subset TQ$, we obtain a flag of subsheaves

$$\underline{\mathcal{H}} \subset \underline{\mathcal{H}}^2 \subset \cdots \subset \underline{\mathcal{H}}^j \subset \cdots \subset \mathfrak{X}(Q)$$

of the tangent sheaf $\mathfrak{X}(Q)$ defined inductively by

$$\underline{\mathcal{H}} = \Gamma(\mathcal{H}), \quad \underline{\mathcal{H}}^j = \underline{\mathcal{H}}^{j-1} + [\underline{\mathcal{H}}, \underline{\mathcal{H}}^{j-1}].$$

The notation $\mathcal{H}_q^j$ for $q \in Q$ will be used for the subset of T_qQ consisting of the elements $X(q)$ where $X \in \underline{\mathcal{H}}^j$.

Definition 4.3. For a sub-Riemannian geometry $(Q, \mathcal{H}, g)$, we say that $\mathcal{H}$ is *bracket generating* if for every $q \in Q$ there is a minimal number $r(q)$ such that $\mathcal{H}_q^{r(q)} = T_qQ$. By setting $n_i(q) = \mathrm{rank}(\mathcal{H}_q^i)$, the multi-index

$$\mathfrak{G}(q) = (n_1(q), \ldots, n_{r(q)}(q))$$

is called the *growth vector* at $q \in Q$. If the growth vector is constant, then the subbundle is said to be *equiregular* and r is called the *step* of the horizontal distribution.

For a bracket generating and equiregular horizontal distribution $\mathcal{H}$ we obtain a flag of subbundles

$$\mathcal{H} \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^r = TQ, \quad T_qQ\Big|_{\mathcal{H}^i} = \mathcal{H}_q^i.$$

Example 4.4 (Heisenberg Geometry)**.** The prototype example of a sub-Riemannian geometry is the Heisenberg geometry, which can be realized with $Q = \mathbb{R}^3$ and a basis for $\mathcal{H}$ given by

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}.$$

Requiring orthonormality of X and Y gives a metric g on $\mathcal{H}$ and we have $[X, Y] = \frac{\partial}{\partial z}$. This shows that the subbundle $\mathcal{H}$ is equiregular and bracket generating with growth vector $\mathfrak{G} = (2, 3)$.

Example 4.5 (Martinet Distribution)**.** As an example of a non-equiregular bracket generating subbundle, let us again consider $Q = \mathbb{R}^3$ and define $\mathcal{H}$ to be the annihilator of the one-form

$$\alpha = dz - \frac{1}{2}y^2 dx.$$

Then $\mathcal{H}$ is spanned by the vector fields $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial z}$. If $y \neq 0$ then the growth vector is $(2, 3)$, while for $y = 0$ it is $(2, 2, 3)$.

Example 4.6 (Symmetric Spaces of Non-Compact Type)**.** Let $M \cong G/H := \mathrm{Isom}(M)/K_{x_0}$ be a Riemannian symmetric space. By Proposition 2.45 we obtain a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{m}$ is $\mathrm{Ad}(H)$-invariant. We say that M is of *non-compact* type if the *Killing form* $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by $B(X, Y) = \mathrm{trace}(ad_X \circ ad_Y)$ is negative definite on $\mathfrak{h}$ and positive definite on $\mathfrak{m}$. By defining

$$\mathcal{H}_g^{\mathfrak{m}} = dL_g \mathfrak{m}$$

we obtain a sub-Riemannian structure $(G, \mathcal{H}^{\mathfrak{m}}, B)$ where

$$B(v, w) = B\Big|_{\mathfrak{m}} \left(dL_{g^{-1}}(v), dL_{g^{-1}}(v)\right),$$

for $v, w \in \mathcal{H}_g$.

A fundamental question is that of *horizontal connectivity*; can any two points in a sub-Riemannian geometry be connected by a horizontal curve? Recall that a distribution $\mathcal{H}$ is called *involutive* if $[X, Y]$ is a local section for $\mathcal{H}$ whenever X and Y are smooth local sections for $\mathcal{H}$. A classical theorem of Frobenius, see [Lee13, Theorem 19.12], states that if $(Q, \mathcal{H})$ is a manifold with an involutive distribution then for every point $p \in Q$ there exists an immersed submanifold $S \subset Q$ containing p such that $T_pS = \mathcal{H}_p$. We call S an *integral manifold* for $\mathcal{H}$ and involutive distributions are also called *integrable*. Thus horizontal connectivity for involutive distributions can only happen along integral manifolds for the distribution. Unless $\mathcal{H} = TQ$ there will be points $p, q \in Q$ which can not be connected by a horizontal curve whenever the distribution is involutive.

One might expect that the bracket generating condition will be sufficient for horizontal connectivity as this is the polar opposite of being involutive. The following theorem gives an affirmative answer to this and hence ensures when the Carnot-Carathéodory distance is a proper distance function. The theorem was proved by Wei-Liang Chow in 1939 and proved independently by Petr Konstanovich Rashevskii in 1938. We refer to [Cho40] for Chow's original paper and [Mon02, Chapter 2] for an exposition with modern notation.

Theorem 4.7 (Chow-Rashevskii Theorem). *Let $(Q, \mathcal{H}, g)$ be a sub-Riemannian geometry with Q connected and $\mathcal{H}$ bracket generating. Then any two points in Q can be connected by a piecewise smooth horizontal curve.*

Example 4.8. Let us revisit the principal bundle $SO(n-k) \to SO(n) \to V_k(\mathbb{R}^n)$ described in Example 3.20. The connection $\mathcal{H}_k^{\mathfrak{m}}$ given by left translation of $\mathfrak{m} = \mathfrak{o}(n-k)^\perp$ gives an equiregular sub-Riemannian structure $(SO(n), \mathcal{H}_k^{\mathfrak{m}}, g)$, where the metric g is simply left translation of the trace form. The fact that the holonomy group of this connection is all of $SO(n-k)$ gives that the subbundle is bracket generating. Hence the Chow-Rashevskii Theorem implies that any two points in $SO(n)$ can be joined by a horizontal curve. We will refer to these sub-Riemannian structures on $SO(n)$ as *Steifel structures*.

Let $(Q, \mathcal{H}, g)$ be a sub-Riemannian geometry with $v \in \mathcal{H}$ and $\alpha \in T^*Q$. Define the maps

$$\begin{aligned} \flat^g &: \mathcal{H} \longrightarrow \mathcal{H}^* & \sharp^g &: T^*Q \longrightarrow \mathcal{H} \\ \flat^g(v) &= g(v, \cdot) & \sharp^g(\alpha) &= \left(\flat^g\right)^{-1}(\alpha|_{\mathcal{H}}). \end{aligned}$$

We define the *cometric* corresponding to g to be the bilinear map

$$g^* : T^*Q \times T^*Q \longrightarrow \mathbb{R}, \quad \langle \alpha, \beta \rangle_{g^*} = \alpha\left(\sharp^g(\beta)\right),$$

for $\alpha, \beta \in T^*Q$. The cometric is zero on the annihilator of $\mathcal{H}$ and satisfies

$$\langle \alpha, \beta \rangle_{g^*} = \langle \sharp^g \alpha, \sharp^g \beta \rangle_g,$$

showing that is symmetric and semi-definite. While the sub-Riemannian metric is only defined on $\mathcal{H}$, the cometric is defined on the whole cotangent bundle.

A horizontal curve is said to be a *geodesic* if it is locally minimizing. Using the cometric, we can define the *sub-Riemannian Hamiltonian*

$$H : T^*Q \longrightarrow \mathbb{R}, \quad H(\alpha) = \frac{1}{2}\langle \alpha, \alpha \rangle_{g^*}.$$

Notice that the Hamiltonian determines the cometric by polarization. The Hamiltonian is related to the geodesics as follows: Recall that the cotangent bundle of any manifold always has a symplectic structure, that is, possesses a closed nondegenerate 2-form. If $(q^1, \ldots, q^n)$ are coordinates on $U \subset Q$, then $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ are coordinates on $\pi_{T^*Q}^{-1}(U) \subset T^*Q$ where $\alpha = p_i dq^i$ for $i = 1, \ldots, n$ and $\alpha \in \pi_{T^*Q}^{-1}(U)$. In these coordinates, the symplectic form is given by

$$\omega = dq^i \wedge dp_i.$$

Notice that we use lower indices for the last n coordinates so that the Einstein summation convention applies.

Using the symplectic form ω, we define the *Hamiltonian vector field* $\vec{H}$ by $\omega(\vec{H}, \cdot) = dH$, where H is the Hamiltonian function. Locally, the integral curves of the Hamiltonian vector field can be written as the solution to the ODE system

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i}, \end{aligned}$$

called the *Hamilton's equations* for the Hamiltonian function H. The projection down to Q of an integral curve to Hamilton's equations is a geodesics, see [Mon02, Theorem 1.14] for a proof. We call the geodesics obtained in this manner *normal geodesics*. Unlike Riemannian geometry where every geodesic can be obtained in this way, this is not the case in sub-Riemannian geometry. In fact, the Martinet distribution described in Example 4.5 possesses geodesics which are not obtained as projections of integral curves of the Hamiltonian vector field. This follows from the Minimality Theorem of Liu and Sussmann, see [Mon02, Theorem 3.3] for the statement and proof. Such geodesics are called *abnormal geodesic*. Unlike normal geodesics which are smooth as they are projections of smooth curves, it is not even known if abnormal geodesics are always smooth.

A final peculiar feature of sub-Riemannian geometries concerns their Hausdorff dimension. Recall that the r-dimensional *Hausdorff measure* of a metric space M is defined to be

$$\mathcal{H}^r(M) = \lim_{\delta \to 0^+} \inf \left\{ \sum_i \operatorname{diam}(E_i)^r : M \subset \bigcup_i E_i,\ \operatorname{diam}(E_i) < \delta \right\}.$$

There exists a number $s \in [0, +\infty]$ called the *Hausdorff dimension* of M such that $\mathcal{H}^r(M) = 0$ for every $r > s$ and $\mathcal{H}^r(M) = +\infty$ for every $r < s$, see [Fal14, Section 2.2]. Let $(Q, \mathcal{H}, g)$ be an equiregular sub-Riemannian geometry of step k. Define the *homogeneous dimension* of $(Q, \mathcal{H}, g)$ to be the number

$$D^{\mathcal{H}} = \sum_{i=1}^{k} i(n_i - n_{i-1}),$$

where $n_i = \operatorname{rank}(\mathcal{H}^i)$ are the components of the growth vector. The proof of the following theorem can be found in Mitchell's paper [Mit85].

Theorem 4.9 (Mitchell's Measure Theorem)**.** *For an equiregular and bracket generating sub-Riemannian geometry the Hausdorff dimension and homogeneous dimension coincide.*

4.2 Carnot Groups, Nilpotentization, and Gromov-Hausdorff Convergence

Before turning to the definition of sub-Riemannian model spaces, we describe a class of examples which will play an important role in the classification process: Carnot groups. They admit dilations and we think of them as sub-Riemannian analogues of Euclidean space. We introduce the nilpotentization of an equiregular, bracket generating sub-Riemannian geometry and explain how it associates a Carnot group structure to any such sub-Riemannian geometry. Through the development of Gromov-Hausdorff distance and convergence, we introduce tangent cones of sub-Riemannian geometries. See [Mon02, Section 8.5] for a derivation using normal coordinates showing that the tangent cone of a Riemannian manifold is Euclidean space with the same dimension as the manifold. Hence the model spaces in Riemannian geometry can be described by two parameters, namely sectional curvature and their tangent cone. Finally, we give Mitchell's theorem stating that the tangent cone construction coincides with the nilpotentization for equiregular, bracket generating sub-Riemannian geometries.

Definition 4.10. A Lie algebra $\mathfrak{g}$ is *nilpotent* if its *lower central series* defined inductively by

$$\mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}^{(i)}], \quad \mathfrak{g}^{(1)} = \mathfrak{g},$$

is eventually zero, that is, $\mathfrak{g}^{(k+1)} = \{0\}$ for some $k \in \mathbb{N}$. The smallest such k is called the *step* of $\mathfrak{g}$.

Notice that nilpotent Lie algebras always have nontrivial center. When presenting a Lie algebra with a basis $X_1, \ldots, X_n$, it suffices to describe $[X_i, X_j]$ for $i < j$. We use the convention that every bracket relation which is not mentioned and does not follow from other bracket relations through the Lie algebra axioms is assumed to be zero.

Definition 4.11. A *stratification* of a Lie algebra $\mathfrak{g}$ with *step* k is a decomposition

$$\mathfrak{g} = \bigoplus_{i=1}^{\infty} V_i$$

such that $[V_1, V_j] = V_{j+1}$ and $V_t = \{0\}$ for $t > k$. The vector space V_1 is called the *generating subspace* while V_i for $i \geq 1$ are in general called *layers*. We call a Lie algebra *stratifiable* if there exists a stratification.

Although stratified Lie algebras are clearly nilpotent, the converse is not true. A straightforward induction proof using the Jacobi identity shows that $[V_i, V_j] \subset V_{i+j}$ for any stratification.

Example 4.12. The $(2n+1)$-dimensional *Heisenberg algebra* is the step two stratified Lie algebra H^{2n+1} with basis $X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z$ and bracket relations

$$[X_i, Y_i] = Z, \quad i = 1, \ldots, n.$$

When $n = 1$ we call H^3 the *classical Heisenberg algebra.*

Example 4.13. The $(n+1)$-dimensional *filiform algebra* $\mathfrak{fil}(n+1)$ is spanned by $X, Y_1, \ldots, Y_n$ with bracket relations

$$[X, Y_i] = Y_{i+1}, \quad i = 1, \ldots, n-1.$$

It is stratified by $V_1 = \operatorname{span}\{X, Y_1\}$ and $V_i = \operatorname{span}\{Y_i\}$ for $i = 2, \ldots, n$. The case $n = 3$ is called the *Engel algebra.* The only intersection between the Heisenberg algebras and filiform algebras is the classical Heisenberg algebra H^3.

Example 4.14. Fix a set $\{X_1, \ldots, X_n\}$ of n elements. Consider the free vector space of all elements formally on the form

$$[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]]$$

where $i_j \in \{1, \ldots, n\}$ for every $j = 1, \ldots, k$. We identify two such elements if they can be transformed into one another via either bilinearity, skew-symmetry, or the Jacobi identity of the formal bracket $[\cdot, \cdot]$. The resulting space has a natural bracket operation turning it into a Lie algebra called the *free Lie algebra* of rank n and is denoted by $\mathfrak{f}^n$. Similarly as for the free vector space construction, the free Lie algebra only depends up to isomorphism on the cardinally of the generating set and not the choice of elements themselves. It has an obvious grading with the generating subspace spanned by $X_1, \ldots, X_n$. By identifying each element which consists of $k+1$ or more brackets with zero, we obtain the *free nilpotent Lie algebra* of rank n and step k, denoted by $\mathfrak{f}[n, k]$. We denote the l'th layer of the free Lie algebra by $\mathfrak{f}_l[n, k]$, or simply $\mathfrak{f}_l$ when the rank and step is clear from the context. Recall that the *Möbius function* μ is defined by

$$\mu(n) = \mu(p_1^{m_1} \cdots p_k^{m_k}) = \begin{cases} 1 & \text{if } m_s \leq 1 \text{ and } k \text{ is even,} \\ -1 & \text{if } m_s \leq 1 \text{ and } k \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

for $s = 1, \ldots, k$ and the prime decomposition of the positive integer $n = p_1^{m_1} \cdots p_k^{m_k}$. The dimension of the l'th layer is given by *Witt's formula*

$$\dim(\mathfrak{f}_l[n, k]) = \frac{1}{l} \sum_{d | l} \mu(d) n^{l/d}, \quad l \leq k. \tag{4.1}$$

We refer the reader to [MKO99] for some explicit calculations regarding the dimension of the free Lie algebras. We will later see that understanding representations on free Lie algebras have a central part in the classification of sub-Riemannian model spaces.

Definition 4.15. A *Carnot group* is a simply connected Lie group whose Lie algebra is stratifiable.

Although some authors assume that a Carnot group comes equipped with a sub-Riemannian structure, we will not assume this for simplicity. Let G be a Carnot group with stratified Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_k$. The maps $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ for $\lambda > 0$ sending $v \in V_j$ to $\lambda^j v$ are called *Lie algebra dilations.* As Carnot groups are simply connected, we obtain a one-parameter family of maps $\delta_\lambda^G : G \to G$ by Theorem A.5. These are called *Lie group dilations* or simply *dilations* whenever the context is clear. As the maps δ_λ are Lie algebra automorphisms, the induced group dilations δ_λ^G are Lie group automorphisms. It follows from [Hal15, Exercise 2.9] that the exponential map is a global diffeomorphism for Carnot groups. This allows us to express the dilations on the Carnot group as

$$\delta_\lambda^G(g) = \delta_\lambda^G \left[\exp\left(\sum_{i=1}^k v_i \right) \right] = \exp\left(\sum_{i=1}^k \lambda^i v_i \right),$$

for $v_i \in V_i$. It is simple to introduce a sub-Riemannian geometry on any Carnot group; introduce an inner product on the generating subspace of $\mathfrak{g}$ and use left translations to obtain a horizontal distribution with a sub-Riemannian metric. The terminology "Lie group dilations" is motivated by the fact that they are dilations with respect to the Carnot-Carathéodory distance,

$$d_{CC}(\delta_\lambda^G(g), \delta_\lambda^G(h)) = \lambda d_{CC}(g, h),$$

for $g, h \in G$.

Let $(Q, \mathcal{H}, g)$ be a sub-Riemannian geometry where $\mathcal{H}$ is equiregular of step k. Then

$$\mathfrak{nil}(q) = \mathcal{H}_q \oplus \left(\mathcal{H}_q^2/\mathcal{H}_q\right) \oplus \left(\mathcal{H}_q^3/\mathcal{H}_q^2\right) \oplus \cdots \oplus T_qQ/\mathcal{H}_q^{k-1},$$

is called the *nilpotentization* of $\mathcal{H}$ at $q \in Q$. The nilpotentization at q has the structure of a stratified Lie algebra where

$$\mathfrak{nil}(q) = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k, \quad \mathfrak{n}_j = \mathcal{H}_q^j/\mathcal{H}_q^{j-1},$$

with the convention that $\mathcal{H}_q^0 = \{0\}$ for every $q \in Q$. If v is any representative for $[v] \in \mathfrak{n}_i$ and w is any representative for $[w] \in \mathfrak{n}_j$, then their bracket is given by

$$[[v], [w]] := [X, Y](q) \mod \mathcal{H}_q^{i+j-1},$$

where $X \in \underline{\mathcal{H}^i}$ with $X(q) = v$ and $Y \in \underline{\mathcal{H}^j}$ with $Y(q) = w$. This bracket is well-defined, see [Mon02, Proposition 4.10] for details. Then g_q is an inner product on $\mathfrak{n}_1 = \mathcal{H}_q$ and we form the Carnot group $\text{Nil}(Q, q)$ corresponding to $\mathfrak{nil}(q)$ with subbundle and sub-Riemannian metric given by the translates of $\mathfrak{n}_1$ and g_q, respectively.

The rest of this section will focus on why the nilpotentization at a point $q \in Q$ plays an analogous role as the tangent space in Riemannian geometry. Namely, it gives the most accurate infinitesimal approximation to an equiregular bracket generating sub-Riemannian geometry in the setting of Gromov-Hausdorff convergence described below. Hence, one often thinks of Carnot groups as "linearized" sub-Riemannian geometries similarly to how one thinks of vector spaces as "linearized" Riemannian manifolds. Provisionally, we can define the *tangent cone* $CT_{m_0}M$ of a pointed metric space (M, d, m_0) to be the limit

$$CT_{m_0}M = \lim_{\lambda \to \infty} (\lambda M, m_0), \tag{4.2}$$

where λM denotes the metric space consisting of the same set as M but with a scaled distance function λd. We will introduce a distance function on the set of all metric spaces, making the limit in the definition rigorous. Heuristically, the formal computation for $t > 0$,

$$\delta_t CT_{m_0}M = \delta_t \lim_{\lambda \to \infty} (\lambda M, m_0) = \lim_{\lambda \to \infty} (t\lambda M, m_0) = \lim_{\lambda \to \infty} (\lambda M, m_0) = CT_{m_0}M$$

alludes to the fact that the tangent cone admits dilations, strengthening the intuitive relation to Carnot groups.

Definition 4.16. Let (X, d_X) be a metric space with distance function d_X and $A, B \subset X$. We define the *ϵ-neighbourhood* of A in X to be the set

$$N_\epsilon(A) = \{x \in X : d(x, a) < \epsilon \text{ for some } a \in A\}.$$

The *Hausdorff distance* between the sets A and B in X is then given by

$$d_H^X(A, B) = \inf\{\epsilon : A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A)\}.$$

As a distance between the metric spaces $(A, d_X|_A)$ and $(B, d_X|_B)$, this does not suffice as the distance is completely determined on how A and B are embedded into X, and not on their intrinsic properties. Consider two copies of the horizontal line $L_1 = L_2 = \{(x, 0) : x \in \mathbb{R}\}$ in the plane. The Hausdorff distance between them is zero. However, performing a rotation with angle $0 < \theta < \pi$ on L_2 produces a line L_2^θ such that

$$d_H^{\mathbb{R}^2}(L_1, L_2^\theta) = \infty,$$

no matter how small the rotation is.

Definition 4.17. The *Gromov-Hausdorff distance* between two metric spaces (A, d_A) and (B, d_B) is given by

$$d_{GH}(A, B) = \inf d_H^X(i(A), j(B)),$$

where the infimum is taken over all metric spaces (X, d_X) and all isometric embeddings $i : A \hookrightarrow X$ and $j : B \hookrightarrow X$.

Clearly, the Gromov-Hausdorff distance between isometric metric spaces is zero. The converse is not true: Consider a complete and separable metric space X without any isolated points and with a countable dense subset A. Then the Gromov-Hausdorff distance between A and X is zero, while they are not isometric since A is not complete by Baire's Theorem. A rather deep result proved by Misha Gromov states that if (A, d_A) and (B, d_B) are compact metric spaces then $d_{GH}(A, B) = 0$ implies that they are isometric, see [Mon02, Proposition 8.5]. Moreover, in the setting of compact metric spaces the finite metric spaces are Gromov-Hausdorff dense. This follows from covering a compact metric space with ϵ-balls and picking the centers of a finite sub-cover. Notice that $d_{GH}(\mathbb{R}, [-n, n]) = \infty$, which implies that $\lim_{n\to\infty}[-n, n] \neq \mathbb{R}$ with the Gromov-Hausdorff distance. To remedy this separation between bounded and unbounded spaces, we describe the desired convergence locally by using metric balls.

Definition 4.18. Let (A_i, a_i) be a sequence of pointed metric spaces and (B, b_0) another pointed metric space. Then (A_i, a_i) is said to *(local Gromov-Hausdorff) converge* to (B, b_0) if for every $\epsilon > 0$, the ϵ-balls centered at a_i in A_i converge to the ϵ-balls centered at b_0 in B in the sense of the Gromov-Hausdorff distance.

With this notion of convergence, the provisional definition of tangent cone given in (4.2) is now precise. Let $(Q, \mathcal{H}, g)$ be an equiregular and bracket generating sub-Riemannian geometry. We will be interested in the pointed metric space (Q, d_{CC}, q), where $q \in Q$ and d_{CC} is the Carnot-Carathéodory distance. The notation CT_qQ for its tangent cone will be used and we make sure that the sub-Riemannian structure is clear from the context.

Theorem 4.19 (Mitchell's Convergence Theorem). *The tangent cone of an equiregular and bracket generating sub-Riemannian geometry $(Q, \mathcal{H}, g)$ exists at any point $q \in Q$. Moreover, it is isometric to the nilpotentization at the same point*

$$CT_qQ \simeq Nil(Q, q).$$

Thus the tangent cone of an equiregular and bracket generating sub-Riemannian geometry at any point inherits a Carnot group structure. In [Bel96] the notion of tangent cone is extended to the non-regular setting. We will not pursue this as our main interest is in sub-Riemannian model spaces which are defined in the next section and are always equiregular.

4.3 Sub-Riemannian Model Spaces

In this section we define the main objects of the thesis: *sub-Riemannian model spaces*. The definition will be influenced by the maximal symmetry interpretation of Riemannian model spaces developed in Chapter 2. Arguments involving sub-Riemannian model spaces will often involve the interplay between their frame bundles and isometry groups we discussed in Chapter 3. The main difficulty with dealing with sub-Riemannian model spaces is that they are not defined by a numerical invariant in the way Riemannian model spaces are defined (up to well-behavedness of topological and metric properties) by sectional curvature. Classifying sub-Riemannian model spaces is particularly challenging as model spaces of different step or rank exhibit different features as will be seen later. The paper [Gro16] will be the main reference from this point on all the way to the end of Section 4.6.

Definition 4.20. A *sub-Riemannian model space* is a bracket generating sub-Riemannian geometry $(Q, \mathcal{H}, g)$, where Q is a simply connected manifold satisfying the following *symmetry condition*:

For any points $p, q \in Q$ and any linear isometry $\phi : \mathcal{H}_p \to \mathcal{H}_q$ there exists a smooth isometry $\Phi : Q \to Q$ such that $d\Phi|_{\mathcal{H}_p} = \phi$.

Before discussing anything else, the notion of isometry in the sub-Riemannian setting needs to be clarified. If $(Q^{(1)}, \mathcal{H}^{(1)}, g^{(1)})$ and $(Q^{(2)}, \mathcal{H}^{(2)}, g^{(2)})$ are two bracket generating sub-Riemannian geometries, a *(sub-Riemannian) isometry* is defined to be a homeomorphism $\Phi : Q^{(1)} \to Q^{(2)}$ which preserves the Carnot-Carathéodory distance, that is, a distance preserving map between the metric spaces $(Q^{(1)}, d_{CC}^{(1)})$ and $(Q^{(2)}, d_{CC}^{(2)})$. The following regularity result is assembled from [CLD16, Theorem 1.2] and [CLD16, Corollary 1.8].

Proposition 4.21. *Let $(Q, \mathcal{H}, g)$ be a bracket generating and equiregular sub-Riemannian geometry. Then any isometry $\Phi : Q \to Q$ is a smooth map satisfying $d\Phi(\mathcal{H}) \subset \mathcal{H}$ and restricts to a linear isometry*

$$d\Phi|_{\mathcal{H}_q} : \mathcal{H}_q \longrightarrow \mathcal{H}_{\Phi(q)},$$

for any $q \in Q$. Moreover, any isometry is uniquely determined by its restricted differential $d\Phi|_{\mathcal{H}_q}$ at a single point.

This proposition applies to sub-Riemannian model spaces as their definition implies that they are equiregular: The growth vector $\mathfrak{G}(q)$ at a point $q \in Q$ is determined by the Carnot-Carathéodory distance in a neighbourhood of q. Thus the fact that the model spaces are homogeneous, meaning that every two points can be connected by a sub-Riemannian isometry, force the growth vector to be constant. Hence the isometries appearing in the definition of sub-Riemannian model spaces have the properties described in Proposition 4.21. In particular, the smoothness requirement for the isometries in the definition of sub-Riemannian model spaces does not need to be checked. The last sentence in Proposition 4.21 is a sub-Riemannian analogue of Proposition 2.15.

Example 4.22. The Riemannian model spaces presented in Theorem 2.50 are sub-Riemannian model spaces by Theorem 2.52. In this case the horizontal distribution is the whole tangent bundle. It is useful to keep in mind that not all the Riemannian model spaces have a compatible Lie group structure, an example is S^2 as is not parallelizable.

Before describing more sub-Riemannian model spaces, we develop a few technical tools which will be essential for the next sections. Say we are given any diffeomorphism $\Phi : Q \to Q$ with $d\Phi(\mathcal{H}) \subset \mathcal{H}$, where $(Q, \mathcal{H}, g)$ is an equiregular sub-Riemannian geometry. For any vector field X, define an associated vector field $Ad(\Phi)X$ by the formula

$$Ad(\Phi)X(q) = d\Phi \circ X \circ \Phi^{-1}(q), \quad q \in Q.$$

This is simply the vector field which is Φ-related to X. As pointed out in Chapter 3, Φ-related vector fields commute with the Lie bracket,

$$[Ad(\Phi)X, Ad(\Phi)Y] = Ad(\Phi)[X, Y],$$

implying that $Ad(\Phi)$ maps the sections $\underline{\mathcal{H}}^j$ into itself for any $j \geq 1$. Since $\mathcal{H}$ is equiregular we have that

$$d\Phi(\mathcal{H}^j) \subset \mathcal{H}^j, \quad j \geq 1. \tag{4.3}$$

Lemma 4.23. *The horizontal distribution of any sub-Riemannian model space is orientable.*

Proof. Let $(Q, \mathcal{H}, g)$ be a sub-Riemannian model space with $\text{rank}(\mathcal{H}) = n$ and $q \in Q$. Fix $\omega \in \wedge^n \mathcal{H}_q$ with $|\omega|_g = 1$, where the norm is the one induced from g to all the tensor bundles. For every $p \in Q$, let Φ be an isometry of Q such that $\Phi(q) = p$. Then $|d\Phi(\omega)|_g = |\omega|_g \neq 0$ shows that this gives a non-vanishing n-form on the bundle $\mathcal{H}$ by varying Φ. □

For a sub-Riemannian model space $(Q, \mathcal{H}, g)$ we let G denote the isometry group $\text{Isom}(Q)$ of Q. It follows from [CLD16, Theorem 1.6] that G is a finite dimensional Lie group similarly as in the Riemannian case. For a fixed point $q \in Q$, let K_q denote the isotropy group corresponding to the point q. The isotropy groups corresponding to different points will all be conjugate and are compact by [GK03, Corollary 5.6]. We omit the explicit reference to the point q in the notation $K := K_q$ as this is of minor relevance in our arguments. Hence we can consider the principal bundle

$$K \to G \xrightarrow{\pi} Q \simeq G/K$$

described in Example 3.2, where $\pi(\Phi) = \Phi(q)$ for $\Phi \in G$. By compactness of K there is a K-invariant complement $\mathfrak{p}$ of $\mathfrak{k} \subset \mathfrak{g}$, where $\mathfrak{k}$ and $\mathfrak{g}$ are the Lie algebras of K and G, respectively. For any $v \in T_qQ$ let A_v denote the corresponding element in $\mathfrak{p}$, that is, the unique element such that $d\pi(A_v) = v$. Let $\theta(t) = \exp(tA_v)$ denote its one-parameter subgroup in the isometry group. Then for $\Phi \in K$, we have

$$d\pi(Ad(\Phi))A_v = d\Phi\left(\frac{d}{dt}\theta(t) \circ \Phi^{-1}(q)\Big|_{t=0}\right) = d\Phi(v). \tag{4.4}$$

That $\mathfrak{p}$ is K-invariant gives that $Ad(\Phi)A_v = A_{d\Phi(v)}$. This implies that

$$\mathfrak{p}^j = \left\{A \in \mathfrak{p} : d\pi(A) \in \mathcal{H}_q^j\right\} \tag{4.5}$$

is K-invariant by Equation (4.3).

Example 4.24. Recall that $\mathfrak{f}[n, k]$ denotes the free Lie algebra of rank n and step k introduced in Example 4.14. Let $X_1, \ldots, X_n$ be a basis for the generating subspace $\mathfrak{f}_1$ and denote by $N[n, k]$ the corresponding simply connected Lie group of $\mathfrak{f}[n, k]$ called the *free nilpotent Lie group* of rank n and step k. Fix an inner product on $\mathfrak{f}_1$ making $X_1, \ldots, X_n$ orthonormal and consider the left-translated structure $(\mathcal{H}, g)$ of $\mathfrak{f}_1$ with its inner product. This makes $(N[n, k], \mathcal{H}, g)$ into an equiregular and bracket generating sub-Riemannian geometry. We will show that $(N[n, k], \mathcal{H}, g)$ is in fact a sub-Riemannian model space. Notice that for any linear isomorphism $\phi : \mathfrak{f}_1 \to \mathfrak{f}_1$ there exists a corresponding Lie algebra automorphism $\overline{\phi} : \mathfrak{f}[n, k] \to \mathfrak{f}[n, k]$ preserving the grading and restricting to ϕ on the generating subspace: It is recursively defined on $\mathfrak{f}_j$ by

$$\overline{\phi}(A_j) = \overline{\phi}\left([A_{j-1}, X_i]\right) = [\overline{\phi}(A_{j-1}), \overline{\phi}(X_i)],$$

for $X_i \in \mathfrak{f}_1$, $A_j \in \mathfrak{f}_j$, and $A_{j-1} \in \mathfrak{f}_{j-1}$. This is well-defined since there are no relations in $\mathfrak{f}_j$ other than those imposed by the Lie algebra axioms when $j \leq k$. Since $N[n, k]$ is simply connected, we can associate to $\overline{\phi}$ a Lie group automorphism $\Phi : N[n, k] \to N[n, k]$ such that $d\Phi_1 = \overline{\phi}$ by Theorem A.5. By construction, if we initially chose ϕ to be a linear isometry then Φ will be an isometry. This suffices by left-invariance and hence the free nilpotent Lie groups are sub-Riemannian model spaces. See [LDO16, Theorem 1.1] for why any isometry $\Phi : C \to C$ with C a Carnot group and $\Phi(1) = 1$ is in fact a Lie group automorphism.

Sub-Riemannian model spaces exist in every dimension due to the Riemannian ones. Moreover, any combination of rank and step gives at least the free nilpotent Lie group as a model space. We can now state the main goal of the thesis clearly. It is to classify all sub-Riemannian model spaces of step and rank three. The next and final chapter is fully dedicated to this classification. All step two spaces of any rank are classified in [Gro16, Theorem 5.6], and this result will be presented in Subsection 4.6.3. In step two the classification is less involved that when the step is three, which is due to several results we will develop in the next three sections. The next three sections is also dedicated to survey results obtained in [Gro16] where the previously introduced concepts such as frame bundles, nilpotentizations, symmetric spaces, and holonomy are all present.

4.4 Canonical Partial Connections

In this section we will show, using the theory of principal bundles from Chapter 3, that every sub-Riemannian model space have a canonical partial connection. The partial connection coincides with the Levi-Civita connection when the horizontal distribution is the whole tangent space. This will allow us to define horizontal holonomy in Subsection 4.6.1. The existence of a canonical partial connection will also play a key part in the proof of our classification results in Section 5.4 and Section 5.5.

Definition 4.25. Let $(Q, \mathcal{H}, g)$ be a sub-Riemannian model space and let $\pi : E \to Q$ be any vector bundle. Then a *partial connection* $\nabla^{\mathcal{H}}$ on $\pi : E \to Q$ in the direction of $\mathcal{H}$ is a map

$$\begin{aligned} \nabla^{\mathcal{H}} : \Gamma(\mathcal{H}) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (X, Y) &\longmapsto \nabla^{\mathcal{H}}_X Y, \end{aligned}$$

which is $C^\infty(Q)$-linear in the first component, $\mathbb{R}$-linear in the second, and satisfies the Leibniz property

$$\nabla^{\mathcal{H}}_X fY = X(f)Y + f\nabla^{\mathcal{H}}_X Y.$$

The partial connections we will be interested in have $E = \mathcal{H}$ and can always be described as restrictions of affine connections to the vector bundle $\mathcal{H}$, see [CGJK18]. We say a partial connection $\nabla^{\mathcal{H}}$ on $\mathcal{H}$ in the direction of $\mathcal{H}$ is *compatible with the (sub-Riemannian) metric* if

$$X\langle Y, Z\rangle = \langle \nabla^{\mathcal{H}}_X Y, Z\rangle + \langle Y, \nabla^{\mathcal{H}}_X Z\rangle,$$

for $X, Y, Z \in \Gamma(\mathcal{H})$. If the partial connection $\nabla^{\mathcal{H}}$ satisfies

$$\nabla^{\mathcal{H}}_{d\Phi(X)} d\Phi(Y) = d\Phi\left(\nabla^{\mathcal{H}}_X Y\right),$$

for any $\Phi \in \mathrm{Isom}(Q)$ we refer to it as being *invariant under isometries*. The reader should compare the following theorem to the existence and uniqueness of the Levi-Civita connection on a Riemannian manifold in Theorem 2.8.

Theorem 4.26. *Let $(Q, \mathcal{H}, g)$ be a sub-Riemannian model space. There exists a unique partial connection $\nabla^{\mathcal{H}}$ on $\mathcal{H}$ in the direction of $\mathcal{H}$ which is compatible with the metric and invariant under isometries.*

Proof. (Existence) We will give a sketch of the proof for existence and refer the reader to [Gro16, Proposition 3.8] for more details. By considering horizontal orthonormal frames on $(Q, \mathcal{H}, g)$ we obtain the orthonormal frame bundle

$$O(n) \longrightarrow \mathcal{F}^O(\mathcal{H}) \xrightarrow{\pi} Q,$$

where $\mathrm{rank}(\mathcal{H}) = n$. Moreover, we have the isometry group

$$G = \mathrm{Isom}(Q) \xrightarrow{\pi_G} Q$$

together with the isotropy group $K = K_q$ for a chosen $q \in Q$. Let as usual $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively. The strategy is to reformulate the existence of a partial connection with the properties stated in the theorem so that it is equivalent to the existence of a subspace $\mathfrak{p}^1 \subset \mathfrak{g}$ which is transversal to $\mathfrak{k}$, invariant under the action of K, and satisfies

$$d\pi_G(\mathfrak{p}^1) = \mathcal{H}_q.$$

Assuming this was done, the compactness of K would imply the existence of an invariant complement $\mathfrak{p}$ to $\mathfrak{k}$. By defining $\mathfrak{p}^1$ as we did in Equation (4.5) we would have found the desired subspace $\mathfrak{p}^1 \subset \mathfrak{g}$.

To show the equivalence of the two statements, we will utilize the theory developed in Section 3.4. Since the partial connection $\nabla := \nabla^{\mathcal{H}}$ is compatible with the metric, it induces a principal connection $\mathcal{H}^\nabla$ on $\mathcal{F}^O(\mathcal{H})$ similarly as in Proposition 3.17. Analogously as in Section 3.4 we have that the action

$$\begin{aligned} G \times \mathcal{F}^O(\mathcal{H}) &\longrightarrow \mathcal{F}^O(\mathcal{H}) \\ (a, \phi : \mathbb{R}^n \longrightarrow \mathcal{H}_q) &\longmapsto da(\phi) \end{aligned}$$

is both free and transitive for $a \in G$ and $\phi \in \mathcal{F}^O(\mathcal{H})$. Choose a reference frame $\phi_0 \in \mathcal{F}^O(\mathcal{H})$ such that $\pi(\phi_0) = q$. Then we get an identification of G with $\mathcal{F}^O(\mathcal{H})$ through $a \mapsto da(\phi_0)$. The connection $\mathcal{H}^\nabla$ is identified with a distribution $\mathcal{H}^{\mathfrak{p}}$ on G which is left invariant under the action of G owing to the invariance under isometries

$$\nabla_{d\Phi(X)} d\Phi(Y) = d\Phi\left(\nabla_X Y\right).$$

Let us denote the restriction of $\mathcal{H}^{\mathfrak{p}}$ to the identity of G by $\mathfrak{p}$. Since $\mathcal{H}^\nabla$ is transverse to the vertical space $\mathcal{V} = \ker(d\pi)$ it follows that $\mathfrak{p}$ it transversal to the Lie algebra of K and $\mathfrak{p} \oplus \mathfrak{k} = \mathfrak{g}$. This gives $\mathfrak{p}^1$ by (4.4) and the required properties are satisfied. Moreover, it is clear how to reverse-engineer the identifications we have made so that the existence of the subspace $\mathfrak{p}^1 \subset \mathfrak{g}$ with the described properties implies the existence of a partial connection ∇ on $\mathcal{H}$ with the properties stated in the theorem. □

Before turning to the proof of uniqueness, we need some results regarding representation theory of orthogonal groups: Recall that $O(n)$ denotes the orthogonal group, which is a Lie group with Lie algebra consisting of the skew-symmetric matrices $\mathfrak{o}(n)$. It is sometimes convenient to identify $\wedge^2\mathbb{R}^n$ as a vector space with $\mathfrak{o}(n)$ through the map

$$x \wedge y \longmapsto yx^T - xy^T, \quad x, y \in \mathbb{R}^n. \tag{4.6}$$

The orthogonal group $O(n)$ acts on $\mathbb{R}^n$ by matrix multiplication and on $\mathfrak{o}(n)$ by conjugation, that is,

$$a \cdot v = av, \quad a \cdot A = aAa^{-1},$$

for $a \in O(n), v \in \mathbb{R}^n$, and $A \in \mathfrak{o}(n)$. It follows from [O'N83, Lemma 11.6] that the action on $\mathfrak{o}(n)$ is simply the adjoint map described more generally in Appendix A.2. The readers unfamiliar with representation theory can find the terminology used below explained in Appendix A.2.

Lemma 4.27. *For $n \geq 2$ the representations of $O(n)$ on $\mathbb{R}^n$ and $\mathfrak{o}(n)$ described above are irreducible. Moreover, $\mathbb{R}^n$ and $\mathfrak{o}(n)$ are not isomorphic as representations.*

Proof. Let $V \subset \mathbb{R}^n$ be a non-zero invariant subspace under the action of $O(n)$ and pick a vector $v \in V$ with $\|v\| = 1$. We know from Example 2.18 that $O(n)$ acts transitively on the sphere S^{n-1}. Hence there exist elements $a_1, \dots, a_n \in O(n)$ such that $a_i \cdot v = e_i$ for $i = 1, \dots, n$, showing that $V = \mathbb{R}^n$.

An invariant subspace of $\mathfrak{o}(n)$ yields an invariant subspace for the induced Lie algebra representation of $\mathfrak{o}(n)$ on itself. However, the computation given in [War83, Proposition 3.47] shows that the induced Lie algebra representation of the adjoint representation is the Lie bracket,

$$\mathfrak{o}(n) \ni X \longmapsto (A \longmapsto \mathrm{ad}_X(A) = [X, A]) \in \mathrm{End}(\mathfrak{o}(n)).$$

An invariant subspace of this action is simply an ideal in $\mathfrak{o}(n)$, that is, a subspace which is additionally closed under the Lie bracket. However, for $n \notin \{2,4\}$ the classification of real simple Lie algebras reveals that $\mathfrak{o}(n)$ is simple, that is, does not have any nontrivial proper ideals. Hence the adjoint action of $O(n)$ on $\mathfrak{o}(n)$ is irreducible whenever $n \notin \{2,4\}$. For $n = 2$ then

$$\dim(O(2)) = \dim(S^1) = 1,$$

showing that there is no nontrivial subspaces of $\mathfrak{o}(2)$. Finally, the irreducibility of the action of $O(4)$ on $\mathfrak{o}(4)$ can be found in [Don11, Chapter 2]. As the dimensions of $\mathbb{R}^n$ and $\mathfrak{o}(n)$ only coincide when $n = 3$, this is the only possible time they could be isomorphic as representations. However, a straightforward computation shows that any intertwining map between $\mathbb{R}^3$ and $\mathfrak{o}(3)$ will be zero by applying suitable rotations and reflections. □

Proof. (Uniqueness) We now turn to the uniqueness claim of Theorem 4.26. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two K-invariant subspaces of $\mathfrak{g}$ which are transversal to $\mathfrak{k}$ and such that $d\pi$ restricts to a linear isomorphism on both $\mathfrak{p}$ and $\mathfrak{q}$ onto $\mathcal{H}_q$. We use the notation $A_v \in \mathfrak{p}$ and $B_v \in \mathfrak{q}$ for the elements corresponding to $v \in \mathcal{H}_q$ under the isomorphism $d\pi$. Define the map

$$\eta : \mathfrak{p} \longrightarrow \mathfrak{k}, \quad \eta(A_v) = A_v - B_v.$$

Then the K-invariance of $\mathfrak{p}$ and $\mathfrak{q}$ shows that $Ad(a)\eta = \eta Ad(a)$ for $a \in K$. Hence η is an intertwining map of representations. Since we are dealing with a model space, we can identify K with $O(n)$ by maximal symmetry. Then the action of K on $\mathfrak{k}$ is simply the adjoint action of $O(n)$ on $\mathfrak{o}(n)$. Similarly, Equation (4.4) shows that the action of K on $\mathfrak{p}$ is isomorphic to the usual action of $O(n)$ on $\mathbb{R}^n$. Hence Lemma 4.27 shows that $\eta \equiv 0$ which proves uniqueness. □

We call the partial connection in Theorem 4.26 the *canonical partial connection* on $\mathcal{H}$ and the notation $\nabla^{\mathcal{H}}$ will from now on refer to this partial connection. It gives us a necessary condition for sub-Riemannian model spaces $(Q^{(1)}, \mathcal{H}^{(1)}, g^{(1)})$ and $(Q^{(2)}, \mathcal{H}^{(2)}, g^{(2)})$ to be isometric: Assume $\Phi : Q^{(1)} \to Q^{(2)}$ is an isometry. By choosing a point $q_1 \in Q^{(1)}$ we consider the isotropy groups

$$K_{q_1} \subset G^{(1)} = \mathrm{Isom}(Q^{(1)}), \quad K_{q_2} \subset G^{(2)} = \mathrm{Isom}(Q^{(2)}),$$

where $q_2 = \Phi(q_1)$. As the isotropy groups are all conjugate, we remove q_i from the notation and simply write $K^{(1)}$ and $K^{(2)}$ in place of K_{q_1} and K_{q_2}, respectively. The map Φ induces a group homomorphism

$$\overline{\Phi} : G^{(1)} \longrightarrow G^{(2)}$$

by conjugating with Φ, that is, $\overline{\Phi}(\varphi) = \Phi \circ \varphi \circ \Phi^{-1}$ for $\varphi \in G^{(1)}$. Then $\overline{\Phi}(K^{(1)}) = K^{(2)}$ and we use the notation $\mathfrak{g}^{(i)}$ and $\mathfrak{k}^{(i)}$ for the Lie algebras of $G^{(i)}$ and $K^{(i)}$ for $i = 1, 2$, respectively.

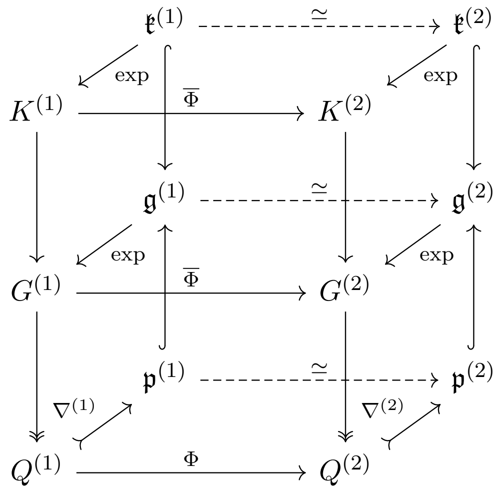
As described in the proof of Theorem 4.26, the connections $\nabla^{(i)} := \nabla^{\mathcal{H}^{(i)}}$ correspond after a choice of orthonormal frames to subspaces $\mathfrak{p}^{(i)} \subset \mathfrak{g}^{(i)}$, for $i = 1, 2$. We equip $\mathfrak{p}^{(i)}$ with an inner product making

$$d\pi\Big|_{\mathfrak{p}^{(i)}} : \mathfrak{p}^{(i)} \longrightarrow \mathcal{H}^{(i)}_{q_i}$$

a linear isometry for $i = 1, 2$. As $d\overline{\Phi}_1 : \mathfrak{g}^{(1)} \to \mathfrak{g}^{(2)}$ and

$$\nabla^{(2)}_{d\Phi(X)} d\Phi(Y) = d\Phi\left(\nabla^{(1)}_X Y\right),$$

we have that $d\overline{\Phi}_1$ maps $\mathfrak{p}^{(1)}$ isometrically onto $\mathfrak{p}^{(2)}$ and maps $\mathfrak{k}^{(1)}$ onto $\mathfrak{k}^{(2)}$. The following diagram illustrates the induced linear isomorphisms arising from the isometry Φ:



The arrows $\rightarrowtail$ in the diagram corresponding to $\nabla^{(i)}$ simply indicates an association in the diagram and nothing more. Summarizing, if the two model spaces are isometric then there is a Lie algebra isomorphism from $\mathfrak{g}^{(1)}$ to $\mathfrak{g}^{(2)}$ preserving the subspaces $\mathfrak{k}^{(i)}$ and $\mathfrak{p}^{(i)}$ as described in the diagram above.

4.5 Carnot Model Spaces

In this section we will give conditions for when a Carnot group is a model space; such spaces are called *Carnot model spaces*. This is not only an interesting class of examples in itself but, as we will see through the theory of tangent cones and nilpotentization, it gives a powerful invariant for any sub-Riemannian model space. In particular, only the growth vectors of Carnot model spaces can occur as growth vectors of general sub-Riemannian model spaces. We will begin by going through a few examples we have previously encountered and determining whether they correspond to Carnot model spaces.

Example 4.28. Let $\mathfrak{fil}(n+1)$ denote the $n+1$-dimensional filiform algebra introduced in Example 4.13. Fix an orthonormal basis X, Y_1 for the generating subspace V_1 and consider the isometry $\psi : V_1 \to V_1$ given by a 90 degree rotation, that is, $\psi(X) = Y_1$ and $\psi(Y_1) = -X$. If we assume ψ induces a Lie algebra automorphism

$$\overline{\psi} : \mathfrak{fil}(n+1) \longrightarrow \mathfrak{fil}(n+1),$$

this forces

$$\overline{\psi}(Y_2) = [\overline{\psi}(X), \overline{\psi}(Y_1)] = [Y_1, -X] = Y_2, \quad \overline{\psi}(Y_3) = [\overline{\psi}(X), \overline{\psi}(Y_2)] = [Y_1, Y_2] = 0,$$

which contradicts the injectivity of $\overline{\psi}$. Hence as long as $n \geq 3$, the Carnot group having $\mathfrak{fil}(n+1)$ as its Lie algebra is not a Carnot model space. For $n = 2$, the filiform algebra $\mathfrak{fil}(3)$ is the same as the classical Heisenberg algebra H^3 described in Example 4.12. The simply connected Lie group corresponding to H^3 is called the *(three-dimensional) Heisenberg group*, compare with Example 4.4. Since the Heisenberg group is the free nilpotent Lie group $N[2, 2]$, it is clear that $\mathfrak{fil}(3)$ corresponds to a Carnot model space.

Example 4.29. Let H^{2n+1} denote the $(2n+1)$-dimensional Heisenberg algebra introduced in Example 4.12. We choose an orthogonal basis $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ for the generating subspace V_1 and let ψ the isometry sending X_1 to $-X_1$ while fixing all the other basis elements. Then the Lie algebra extension

$$\overline{\psi} : H^{2n+1} \longrightarrow H^{2n+1}$$

satisfies

$$\overline{\psi}(Z) = \overline{\psi}\,[X_1, Y_1] = \left[\overline{\psi}(X_1), \overline{\psi}(Y_1)\right] = [-X_1, Y_1] = -Z.$$

On the other hand, this also becomes

$$\overline{\psi}(Z) = \overline{\psi}\,[X_i, Y_i] = \left[\overline{\psi}(X_i), \overline{\psi}(Y_i)\right] = [X_i, Y_i] = Z, \quad i = 2, \dots, n.$$

Hence the $(2n+1)$-dimensional Heisenberg algebra give rise to a Carnot model space only in the case where $n = 1$, that is, for the classical Heisenberg algebra H^3.

Notice that the intersection of the filiform algebras and the Heisenberg algebras is the only Lie algebra in either class which corresponds to a Carnot model space. This direct approach of checking that an orthogonal map on the generating subspace extends to a Lie algebra automorphism is efficient in low dimensions for excluding that a given Carnot group is a Carnot model space. However, it falls short of providing us with new Carnot model spaces. We now give another approach based on representation theory of the orthogonal group for determining whether a Carnot group is a Carnot model space.

Let $(C, \mathcal{H}, g)$ denote a Carnot group with a left-invariant rank n subbundle and metric given by left translating an inner product on the generating subspace V_1 of its stratified Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_k$. Fix an orthonormal basis $X_1, \dots, X_n$ for V_1 and consider the free nilpotent Lie algebra $\mathfrak{f}[n,k] = \mathfrak{f}_1 \oplus \cdots \oplus \mathfrak{f}_k$ of step k on the basis $X_1, \dots, X_n$. Let $(N[n,k], \mathcal{E}, h)$ denote the free nilpotent Lie group corresponding to $\mathfrak{f}[n,k]$. We have a surjective group homomorphism

$$\Phi : N[n,k] \longrightarrow C$$

by considering $\mathfrak{g}$ as the free nilpotent Lie algebra modulo some additional relations. The differential

$$d\Phi|_{\mathcal{E}_x} : \mathcal{E}_x \longrightarrow \mathcal{H}_{\Phi(x)}$$

is a linear isometry for every $x \in N[n,k]$. This is the canonical identification of $\mathfrak{f}_1$ with V_1 when x is the identify of $N[n,k]$. Notice that any Lie algebra automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ is determined by $\psi|_{V_1}$ as $\mathfrak{g}$ is stratified. Lifting $\psi|_{V_1}$ through $d\Phi_e$, the result determines a Lie algebra automorphism

$$\psi^F : \mathfrak{f}[n,k] \longrightarrow \mathfrak{f}[n,k]$$

by the procedure described in Example 4.24. Moreover, ψ^F clearly satisfies

$$d\Phi_e \circ \psi^F = \psi \circ d\Phi_e.$$

Conversely, any isomorphism ψ^F of $\mathfrak{f}[n,k]$ gives a linear isomorphism $\psi : V_1 \to V_1$ by the same procedure. However, this lifts to a Lie algebra automorphism of $\mathfrak{g}$ only if ψ^F preserves the ideal $\ker(d\Phi_e)$. Hence, if C is a model space, this property has to hold for every $\psi \in O(\mathfrak{f}_1)$. By summarizing this discussion we get the following proposition.

Proposition 4.30. *We can determine Carnot model spaces by examining ideals $\mathfrak{a}$ of $\mathfrak{f}[n,k]$ which are also sub-representations of the action of $O(\mathfrak{f}_1)$ on $\mathfrak{f}[n,k]$ given by lifting isometries as described above.*

We will apply this technique right away to classify all the Carnot model spaces with step two. This is straightforward due to the irreducibility results we showed in Lemma 4.27.

Proposition 4.31. *The only Carnot model spaces of step two are the free nilpotent Lie groups $N[n,2]$.*

Proof. The action of $O(\mathfrak{f}_1)$ on the first layer of $\mathfrak{f}[n,2]$ is isomorphic to the usual action of $O(n)$ on $\mathbb{R}^n$ given by matrix multiplication. On the second layer, it is isomorphic to the action of $O(n)$ on $\wedge^2\mathbb{R}^n$ given by

$$a \cdot (x \wedge y) = ax \wedge ay.$$

By using the identification (4.6), this is isomorphic to the adjoint action of $O(n)$ on its Lie algebra $\mathfrak{o}(n)$. It follows from Lemma 4.27 that both representations are irreducible. Hence $N[n,2]$ is the only Carnot model space with rank n and step two. □

Proposition 4.31 is a special feature of having step two. In the next chapter we will see that Carnot model spaces other than the free ones exist when the step is three. We will now describe how Carnot model spaces can be used as an invariant for general sub-Riemannian model spaces through the nilpotentization procedure developed in Section 4.2.

Let $(Q^{(1)}, \mathcal{H}^{(1)}, g^{(1)})$ and $(Q^{(2)}, \mathcal{H}^{(2)}, g^{(2)})$ be two sub-Riemannian model spaces and let $\mathrm{Nil}(Q^{(1)}, q_1)$ and $\mathrm{Nil}(Q^{(2)}, q_2)$ denote their Carnot groups arising from the nilpotentization at the point $q_i \in Q^{(i)}$ for $i = 1, 2$. Recall that any isometry $\Phi : Q^{(1)} \to Q^{(2)}$ with $\Phi(q_1) = q_2$ satisfies $d\Phi(\mathcal{H}^{(1)}) \subset \mathcal{H}^{(2)}$ by Proposition 4.21. This induces a map

$$\mathrm{Nil}_{q_1}(\Phi) : \mathrm{Nil}(Q^{(1)}, q_1) \longrightarrow \mathrm{Nil}(Q^{(2)}, q_2)$$

defined as the Lie group automorphism whose differential at the identity is given by

$$\mathfrak{nil}(q_1) \ni [v] \longmapsto [d\Phi(v)].$$

This is well-defined by (4.3) and determines a unique Lie group homomorphism by Theorem A.5. For model spaces the specification of a point for the nilpotentization procedure is immaterial, as the nilpotentizations at different points are isomorphic by homogeneity. In view of this, we shorten the notation for the nilpotentizations to $\mathrm{Nil}(Q^i)$ for $i = 1, 2$.

Proposition 4.32. *If $(Q, \mathcal{H}, g)$ is a sub-Riemannian model space then Nil(Q) is a Carnot model space with the same growth vector. In particular, the only growth vectors sub-Riemannian model spaces can have are those that occur on Carnot model spaces.*

Proof. For $\phi \in O(\mathcal{H}_q) = O(\mathfrak{nil}(q)_1)$, let $\Phi : Q \to Q$ be the isometry extending ϕ on Q. Then $\mathrm{Nil}_q(\Phi)$ is the desired isometry extending ϕ on $\mathrm{Nil}(Q)$, proving that $\mathrm{Nil}(Q)$ is a model space. □

This result implies together with Proposition 4.31 that co-dimension one model spaces with growth vector $(n, n+1)$ can exist only when

$$n + 1 = \dim(\mathfrak{f}[n, 2]) = \frac{n(n+1)}{2} \iff n = 2.$$

Example 4.33. Recall from Example 4.8 that we obtained a family $(SO(n), \mathcal{H}_k^{\mathfrak{m}}, g)$ of equiregular sub-Riemannian structures on $SO(n)$ with step two for $0 < k < n$ called Steifel structures. As $SO(n)$ is not simply connected, it can not be a model space with any sub-Riemannian structure. However, we consider the lifted structure to the universal covering group of $SO(n)$. The growth vectors are

$$\mathfrak{G} = \left(\frac{k(2n - k + 1)}{2}, \frac{n(n+1)}{2} \right).$$

It follows from Proposition 4.31 and Proposition 4.32 that the only growth vector compatible with a model space structure is when $k = 1$. In this case, the Steifel manifold $V_1(\mathbb{R}^n)$ is simply S^{n-1}. The sub-Riemannian structure on $SO(n)$ is the one induced by the Levi-Civita connection ∇ on S^{n-1} through the procedure described in Proposition 3.17. We will see in Theorem 4.38 that this structure is in fact a sub-Riemannian model space.

4.6 Model Spaces with Free Nilpotentization

In this section we classify all sub-Riemannian model spaces with step two. This will require us to develop horizontal holonomy and provide a connection with the frame bundles investigated in Chapter 3. We will see that horizontal holonomy can only take on extreme cases on sub-Riemannian model spaces; either the horizontal holonomy is trivial or it is of maximal dimension. More details and justifications of the statements we will present on horizontal holonomy can be found in [CGJK18] and [Gro16].

4.6.1 Horizontal Holonomy

Let $(Q, \mathcal{H}, g)$ be a sub-Riemannian model space of rank n with canonical partial connection $\nabla^{\mathcal{H}}$. Consider the orthonormal frame bundle $O(n) \to \mathcal{F}^O(\mathcal{H}) \xrightarrow{\pi} Q$ together with the principal connection $\mathcal{H}^{\nabla}$ induced by $\nabla^{\mathcal{H}}$ as described in the proof of Theorem 4.26. Recall from Section 3.2 that we might as well work with the associated connection one-form ω of $\mathcal{H}^{\nabla}$. Let $\Phi(u)$ denote the holonomy groups inside the structure group $O(n)$ where $u \in \pi^{-1}(p)$ with $p \in Q$ as described in Section 3.3. Consider the subgroup

$$\mathrm{Hol}^{\mathcal{H}}(\phi) = \{A \in O(n) : \phi \cdot A = \tau(\phi)\},$$

where τ is the parallel displacement corresponding to a horizontal curve $\gamma : [0,1] \to Q$ with $\gamma(0) = \gamma(1) = \pi(\phi)$. This will be called the *horizontal holonomy group* corresponding to the point $\phi \in \mathcal{F}^O(\mathcal{H})$.

As Q is simply connected, the horizontal holonomy groups are connected. Moreover, the bracket generating condition ensures that the groups $\mathrm{Hol}^{\mathcal{H}}(\phi)$ all coincide up to conjugation for different points $p \in Q$ and $\phi \in \pi^{-1}(p)$. Since we will only consider horizontal holonomy for model spaces, we tactically use the notation $\mathrm{Hol}(Q, \mathcal{H}) := \mathrm{Hol}^{\mathcal{H}}(\phi)$ whenever convenient. It follows from [Gro16, Theorem 3.5] that there always exists an affine connection ∇ on TQ such that

$$\nabla_{|\mathcal{H}} = \nabla^{\mathcal{H}},$$

and the horizontal holonomy of $\nabla^{\mathcal{H}}$ being equal the usual holonomy of ∇. If the curvature endomorphism corresponding to ∇ is zero, then it follows from Proposition 3.26 together with the Ambrose-Singer theorem that $\mathrm{Hol}(Q, \mathcal{H})$ is the trivial group. Horizontal holonomy for sub-Riemannian model spaces are polarized as the following proposition shows.

Proposition 4.34. *For a sub-Riemannian model space $(Q, \mathcal{H}, g)$ of rank n the horizontal holonomy group is either trivial or isomorphic to $SO(n)$. The horizontal holonomy group is trivial if and only if there exists for every $p \in Q$ a Lie group structure on Q such that p becomes the identity element, $(\mathcal{H}, g)$ is left-invariant, and every isometry fixing p is a Lie group automorphism. Moreover, if*

$$\mathit{Nil}(Q) \simeq N[n, 2k], \quad n, k \in \mathbb{N},$$

then the horizontal holonomy group is trivial.

This proposition will play an important part in the classification of step two model spaces in Theorem 4.38. We will not give the proof of Proposition 4.34 as this would require us to introduce *selectors*, which is a technical tool not needed in any of the subsequent sections. The proof can be found in [Gro16] and more details on selectors are given in [CGJK18].

4.6.2 Frame Bundles of Model Spaces

We will now work towards examining what happens when the horizontal holonomy group is all of $SO(n)$. For this, we will briefly introduce a generalization of the model space definition to geometries which are not necessarily bracket generating. For such geometries, the Chow-Rashevskii Theorem does not apply and we might not have a proper distance function given by taking infimum over horizontal curves. This requires us to redo some definitions which are based on isometries.

Definition 4.35. Let $(Q, \mathcal{H}, g)$ be a sub-Riemannian geometry where $\mathcal{H}$ is not necessarily bracket generating.

- We say that a diffeomorphism $\phi : Q \to Q$ with $d\phi(\mathcal{H}) \subset \mathcal{H}$ is a *loose isometry* if it satisfies

$$\langle d\phi(v), d\phi(w)\rangle_g = \langle v, w\rangle_g.$$

- If there exists a Lie group G acting smoothly on Q such that for every $\phi \in G$ the map $p \mapsto \phi \cdot p$ is a loose isometry, then we say that G acts *loosely isometric*.
- If G in addition acts transitively we refer to Q as a *loose homogeneous space*.
- We say that Q is a *loose model space* if Q is connected and there is a Lie group G which acts by loose isometries such that whenever $p, q \in Q$ and $\psi : \mathcal{H}_p \to \mathcal{H}_q$ is a linear isometry, then $\psi(v) = \phi \cdot v$ for some $\phi \in G$ and every $v \in \mathcal{H}_p$.

It follows from Equation (4.3) that the distribution $\mathcal{H}$ will still be equiregular for loose model spaces. We will refer to the usual model spaces as *true model spaces* for the rest of this Subsection to emphasize the distinction from loose model spaces. The observant reader will have noticed that we have not included simply connectedness in the definition of loose model space analogously to what we had for true model spaces. This is because we will now construct a true model space from any loose model space which is not dependent on requiring simply connectedness of the loose model space. We use the notation

$$\mathcal{O}_p = \{q \in Q : \text{there is a horizontal curve connecting } p \text{ and } q\},$$

and call this the *accessible set* corresponding to $p \in Q$.

Lemma 4.36. *Let $(Q, \mathcal{H}, g)$ be a loose model space with $p \in Q$. Then $(\mathcal{O}_p, \mathcal{H}|\mathcal{O}_p, g|\mathcal{O}_p)$ is a sub-Riemannian homogeneous space. The universal covering space $\widetilde{\mathcal{O}}_p$ of $\mathcal{O}_p$ together with the lifted sub-Riemannian structure is a true model space. Moreover, a different choice of point $q \in Q$ will result in isometric spaces $\mathcal{O}_q \simeq \mathcal{O}_p$.*

Proof. Notice that since $\mathcal{H}$ is equiregular, there is a well defined number r such that $\underline{\mathcal{H}}^r = \underline{\mathcal{H}}^{r+1}$. Hence by the theorem of Frobenius, see [Lee97, Theorem 19.12], it follows that $\mathcal{H}^r$ is an integrable distribution with the accessible sets $\{\mathcal{O}_p : p \in Q\}$ as connected integral manifolds. In particular, $\mathcal{H}|\mathcal{O}_p$ is a bracket generating distribution on $\mathcal{O}_p$. We claim that any loose isometry taking a point in $\mathcal{O}_p$ to another point in $\mathcal{O}_p$ is an isometry of $(\mathcal{O}_p, \mathcal{H}|\mathcal{O}_p, g|\mathcal{O}_p)$: If γ is a horizontal curve in Q with $\gamma(0) = p$, then $\phi \circ \gamma$ is a horizontal curve starting at $\phi(q)$ for any loose isometry ϕ. Hence $\phi(\mathcal{O}_p) = \mathcal{O}_p$ for any loose isometry ϕ with $\phi(p) \in \mathcal{O}_p$, showing the claim. Thus $(\mathcal{O}_p, \mathcal{H}|\mathcal{O}_p, g|\mathcal{O}_p)$ is a sub-Riemannian homogeneous space and the remaining statements are straightforward. □

Remark. Even if the loose model space $(Q, \mathcal{H}, g)$ is simply connected there is no guarantee that the accessible sets will be simply connected as well. Notice that Theorem 4.26 also holds for loose model spaces as the proof never depended on the bracket generating condition.

The previous construction will now be applied to lift the structure of a true sub-Riemannian model space $(Q, \mathcal{H}, g)$ to the orthonormal frame bundle $\pi : \mathcal{F}^O(\mathcal{H}) \to Q$. Let $\nabla^{\mathcal{H}}$ be the canonical partial connection on $\mathcal{H}$ whose existence is guaranteed by Theorem 4.26. Recall from the proof of Theorem 4.26 that the canonical partial connection induces a subbundle $\mathcal{H}^\nabla$ of $T\mathcal{F}^O(\mathcal{H})$. For $p \in Q$ and any frame $\phi \in \mathcal{F}^O(\mathcal{H})_p$, we have that

$$d\pi\Big|_{\mathcal{H}^\nabla_\phi} : \mathcal{H}^\nabla_\phi \longrightarrow \mathcal{H}_p$$

is a linear isomorphism. This allows us to lift the sub-Riemannian metric g to a metric $\widetilde{g}$ on the subbundle $\mathcal{H}^\nabla \subset T\mathcal{F}^O(\mathcal{H})$. Similarly as in Proposition 3.21, the subbundle $\mathcal{H}^\nabla$ is invariant under the left action of $\mathrm{Isom}(Q)$ and under the right action of $O(n)$, where n is the rank of $\mathcal{H}$. With a combination of these group actions, it follows that any linear isometry $\psi : \mathcal{H}^\nabla_{\phi_1} \to \mathcal{H}^\nabla_{\phi_2}$ can be lifted to a loose isometry $\Psi : \mathcal{F}^O(\mathcal{H}) \to \mathcal{F}^O(\mathcal{H})$ such that

$$d\Psi\Big|_{\mathcal{H}^\nabla_{\phi_1}} = \psi.$$

If $\mathcal{F}^O_0(\mathcal{H})$ denotes a connected component of $\mathcal{F}^O(\mathcal{H})$ we have showed that $(\mathcal{F}^O_0(\mathcal{H}), \mathcal{H}^\nabla, \widetilde{g})$ is a loose model space. Let us introduce the notation $\mathbb{F}(Q, \mathcal{H})$ for the universal covering space of $\mathcal{F}^O_0(\mathcal{H})$. By Lemma

4.36, the space $\mathbb{F}(Q,\mathcal{H})$ together with the lifted structure is only possibly lacking the bracket generating condition to be a true model space as we never restricted to the orbit of the group action. The choice of component for $\mathcal{F}^O_0(\mathcal{H})$ is immaterial, as there are loose isometries between the components of $\mathcal{F}^O(\mathcal{H})$. We now give the precise condition for when $\mathbb{F}(Q,\mathcal{H})$ becomes a true model space, relating this back to horizontal holonomy. For a proof of the following proposition together with more information on the rank of $\mathbb{F}(Q,\mathcal{H})$, see [Gro16, Proposition 5.4].

Proposition 4.37. *Let $(Q,\mathcal{H},g)$ be a true sub-Riemannian model space of rank n. Then $\mathbb{F}(Q,\mathcal{H})$ is a true model space as well if and only if*

$$Hol(Q,\mathcal{H}) \simeq SO(n).$$

4.6.3 Classification of Step Two Model Spaces

We will now provide the full classification of model spaces in step two, following [Gro16] closely. This involves a lot of previously encountered theory and stands as a guideline to how higher dimensional classification can be carried out. The reader should nevertheless be aware that the step two case does not reflect all the difficulties encountered in the higher step cases. This is mainly due to Proposition 4.31, making the step two case exceptional. We begin by providing an alternate description of the model spaces in Riemannian geometry. The reason for this will be apparent when the classification result is presented in Theorem 4.38.

For $\rho \in \mathbb{R}$ and $n \geq 2$, consider the Lie algebra $\mathfrak{g}(n,\rho)$ consisting of all matrices on the form

$$\begin{pmatrix} A & x \\ -\rho x^T & 0 \end{pmatrix}, \quad A \in \mathfrak{o}(n),\ x \in \mathbb{R}^n. \tag{4.7}$$

There is a decomposition $\mathfrak{g}(n,\rho) = \mathfrak{p} \oplus \mathfrak{k}$ where $\mathfrak{p}$ is the subspace given by $A = 0$ and $\mathfrak{k}$ is the subspace given by $x = 0$. We give $\mathfrak{p}$ the Euclidean metric $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$ in these coordinates and consider the simply connected Lie group $G(n,\rho)$ corresponding to $\mathfrak{g}(n,\rho)$. If B is the Killing form on $\mathfrak{g}(n,\rho)$, then the Euclidean metric on $\mathfrak{p}$ coincide (up to a constant) with the Killing form B restricted to $\mathfrak{p}$. Moreover, we have $\mathfrak{p} = \mathfrak{k}^{\perp}$ with respect to the Killing form B.

The Lie group $G(n,\rho)$ together with the translated structure $(\mathcal{H}^{\mathfrak{p}}, g_{\mathfrak{p}})$ is a sub-Riemannian geometry. As both $\mathfrak{p}$ and the Euclidean metric $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$ are invariant under the closed Lie subgroup $K \subset G(n,\rho)$ corresponding to $\mathfrak{k}$, this induces a well defined Riemannian metric on $G(n,\rho)/K$. This Riemannian manifold is isometric to the Riemannian model space $\Sigma(n,\rho)$ with sectional curvature ρ. Moreover, $G(n,\rho)$ is isomorphic as a Lie group to the frame bundle of the Riemannian model space $\Sigma(n,\rho)$, see [KN96, Lemma 5.1] for justification of these statements.

Remark. It might seem at first glance that we pulled the description of $\mathfrak{g}(n,\rho)$ out of thin air. In fact, the description is motivated by combining several results we discussed in Chapter 3. Recall that we showed at the end of Section 3.4 that the isometry groups of the Riemannian model spaces are isomorphic as Lie groups to their orthonormal frame bundles, that is,

$$\mathcal{F}^O(TS^n_R) \simeq O(n+1), \quad \mathcal{F}^O(T\mathbb{R}^n) \simeq E(n), \quad \mathcal{F}^O(T\mathbb{H}^n_R) \simeq O_+(n,1). \tag{4.8}$$

Moreover, recall that the torsion of the Levi-Civita connection ∇ is zero and the curvature of the Riemannian model spaces is constant. It thus follows from Proposition 3.26 that the curvature form is constant while the torsion form is zero on the orthonormal frame bundle of the Riemannian model spaces.

The Lie algebra structure of the orthonormal frame bundles is completely determined by the fundamental vector fields and the standard horizontal vector fields, see [KN96, Corollary 3.5.6]. It is clear that the bracket relations (3.9) and (3.10) on the orthonormal frame bundle of the Riemannian model spaces are a coordinate free version of the matrix form given in (4.7). The subbundle $\mathcal{H}^{\mathfrak{p}}$ defined above is precisely the principal connection induced by the Levi-Civita connection ∇ we described in Section 3.4.

Theorem 4.38. *Let $(Q, \mathcal{H}, g)$ be a sub-Riemannian model space of step two and rank n which is not a Carnot group. Then*

$$(Q, \mathcal{H}, g) \simeq (G(n,\rho), \mathcal{H}^{\mathfrak{p}}, g_{\mathfrak{p}}),$$

for some $\rho \in \mathbb{R} \setminus \{0\}$.

Proof. The reader should be aware that we will use a technical result regarding the representation theory of the orthogonal groups given in [Gro16, Lemma A.3] once during the proof. We will supply the reader with justifications for this in the remark given after the proof. If $(Q, \mathcal{H}, g)$ denotes a sub-Riemannian model space of step two and rank n, then it follows from Proposition 4.34 that there is a Lie group structure on Q such that $(\mathcal{H}, g)$ is left-invariant. Choose such a structure and let $\mathfrak{m}$ denote the Lie algebra of Q. As $\mathcal{H}$ is left-invariant we denote by $\mathfrak{m}^1$ the subspace of $\mathfrak{m}$ such that $\mathcal{H}_p = dL_p\mathfrak{m}^1$.

The following approach is influenced by the methods used when we dealt with Riemannian symmetric spaces in Section 2.5. Let $\sigma : Q \to Q$ be the unique isometry such that

$$d\sigma\Big|_{\mathfrak{m}^1} = -Id_{\mathfrak{m}^1}.$$

This gives a decomposition $\mathfrak{m} = \mathfrak{m}^- \oplus \mathfrak{m}^+$ into eigenspaces corresponding to ± 1. Clearly $\mathfrak{m}^1 \subset \mathfrak{m}^-$ and because we are in step two we can deduce from

$$\mathfrak{m}^2 := [\mathfrak{m}^1, \mathfrak{m}^1] \subset \mathfrak{m}^+, \quad \mathfrak{m}^1 + \mathfrak{m}^2 = \mathfrak{m},$$

that $\mathfrak{m}^1 = \mathfrak{m}^-$ and $\mathfrak{m}^2 = \mathfrak{m}^+$. Hence $\mathfrak{m}^1 \oplus \mathfrak{m}^2 = \mathfrak{m}$. We know that $\mathrm{Nil}(Q)$ is also a model space by Proposition 4.32. Moreover, Proposition 4.31 gives that $\mathrm{Nil}(Q) \simeq N[n,2]$. We can identify $\mathfrak{m}^-$ and $\mathfrak{m}^+$ with $\mathbb{R}^n$ and $\mathfrak{o}(n)$, respectively, where $\mathbb{R}^n$ has the standard Euclidean structure. Through this identification, we have that the bracket between the two elements $(x,0), (y,0) \in \mathbb{R}^n \oplus \mathfrak{o}(n) \simeq \mathfrak{m}$ is given by

$$\left[\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ x \wedge y \end{pmatrix}, \tag{4.9}$$

where the identification (4.6) between $\wedge^2\mathbb{R}^n$ and $\mathfrak{o}(n)$ is in effect.

For any linear isometry $\phi : \mathfrak{m}^- \to \mathfrak{m}^-$ there is an isometry $\Phi : Q \to Q$ fixing the identity and

$$d\Phi\Big|_{\mathfrak{m}^-} = \phi.$$

Hence Φ is a Lie group automorphism by Proposition 4.34 and acts on $\mathfrak{m}^-$ by

$$d\Phi(x, A) = \phi(x, A) = (\phi \cdot x, \phi A \phi^{-1}), \quad x \in \mathbb{R}^n, \ A \in \mathfrak{o}(n),$$

where we have identified ϕ with an orthogonal transformation on $\mathbb{R}^n$. It follows from the representation theory of the orthogonal group, see [Gro16, Lemma A.3], that there are constants ρ_1 and ρ_2 such that

$$\left[\begin{pmatrix} x \\ A \end{pmatrix}, \begin{pmatrix} y \\ B \end{pmatrix}\right] = \begin{pmatrix} \rho_1(Ay - Bx) \\ x \wedge y + \rho_2[A,B] \end{pmatrix}. \tag{4.10}$$

The bracket on the right indicates the usual commutator between the elements $A, B \in \mathfrak{o}(n)$. For this to satisfy the Jacobi identity it is straightforward to check that $\rho_1 = \rho_2$, and we use the notation $\rho = \rho_1 = \rho_2$. As Q is simply connected, it is completely determined as a Lie group by its Lie algebra. Assume first that $\rho \neq 0$. The Lie algebra $\mathfrak{m}$ is now clearly isomorphic to $\mathfrak{g}(n,\rho)$ for $\rho \neq 0$. Any linear isometry $\psi : \mathfrak{m} \to \mathfrak{g}(n,\rho)$ induces by simply connectedness a diffeomorphism

$$\Psi : Q \longrightarrow G(n,\rho).$$

As both sub-Riemannian geometries are given by left-translation it follows that $(Q, \mathcal{H}, g) \simeq (G(n,\rho), \mathcal{H}^{\mathfrak{p}}, g_{\mathfrak{p}})$ as sub-Riemannian geometries. When $\rho = 0$ we clearly have the Carnot model space $N[n,2]$. □

Remark. The need to cite a result regarding representations of the orthogonal group during the proof is not one necessity, but rather a choice of exposition. In fact, the result in [Gro16] we used will be part of Theorem 5.3, where we will provide a different proof than in [Gro16] for one of the statements. This result will not logically depend on anything done previously, so the reader can skip ahead and read the result now if so desired.

4.7 Contact Model Spaces

In this section we will classify the sub-Riemannian model spaces having a contact structure and give explicit descriptions. This will connect the classification result given in Theorem 4.38 to existing literature on sub-Riemannian geometry to provide context. Both Proposition 4.41 and Corollary 4.43 are original results.

Definition 4.39. Let Q be a manifold of dimension $2k+1$. We say that $(Q, \mathcal{H}, g)$ is a *contact sub-Riemannian geometry* if we can locally write $\mathcal{H} = \ker(\xi)$, where $\xi \in \mathfrak{X}^*(Q)$ satisfies the *non-integrability condition*

$$\xi \wedge \underbrace{d\xi \wedge \cdots \wedge d\xi}_{k-\text{copies}} \neq 0.$$

We call $\mathcal{H}$ a *contact distribution*. If $(Q, \mathcal{H}, g)$ is in addition a sub-Riemannian model space, we simply refer to it as a *contact model space*.

Contact distributions have been thoroughly studied in the sub-Riemannian setting, see [ABB17, Section 4.3] and [Mon02, Section 1.10] for some elementary properties. Every contact distribution is a *hyperplane distribution*, that is, it has codimension one. Hence the remark we made at the end of Section 4.5 forces any contact model space to have dimension three. As contact sub-Riemannian geometries have step two they can all be found in Theorem 4.38.

On the other hand, any three dimensional Lie group with a bracket generating left-invariant distribution is seen to be a contact sub-Riemannian geometry as follows: Let X_1 and X_2 constitute a global left-invariant frame for the distribution and let $\xi \in \mathfrak{X}^*(Q)$ be such that $\ker(\xi) = \langle X_1, X_2\rangle$. Then using formula (A.1) gives

$$\begin{aligned} d\xi(X_1, X_2) &= X_1\xi(X_2) - X_2\xi(X_1) - \xi\left([X_1, X_2]\right) \\ &= -\xi\left([X_1, X_2]\right) \neq 0, \end{aligned}$$

as $[X_1, X_2]$ is not contained in the distribution due to the bracket generating condition. The non-integrability condition is easily seen to be satisfied.

Definition 4.40. Let $(Q, \mathcal{H}, g)$ be a contact model space. We say that $(Q, \mathcal{H}, g)$ has *contact curvature* $\rho \in \mathbb{R} \setminus \{0\}$ if

$$(Q, \mathcal{H}, g) \simeq (G(2, \rho), \mathcal{H}^{\mathfrak{p}}, g_{\mathfrak{p}}).$$

If $Q \simeq N[2,2]$ we say that Q has contact curvature $\rho = 0$.

We will now give a more explicit description of the spaces $G(2, \rho)$ for $\rho \in \mathbb{R} \setminus \{0\}$. For $\rho > 0$ the space $G(2, \rho)$ is isomorphic to the universal covering space of the connected component of $\mathrm{Isom}(S^2_\rho)$. We determined in Chapter 2 that $\mathrm{Isom}(S^2_\rho) \simeq O(3)$, so it follows that

$$G(2, \rho) \simeq \mathrm{Spin}(3) := \widetilde{SO(3)}.$$

Let $SU(2)$ denote the special unitary group given by

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

It is a unique feature in three dimensions that we have a two-fold covering map $\psi : SU(2) \to SO(3)$ described in the following way: Identify $SU(2)$ with the unit quaternions $Sp(1)$ as Lie groups through the identification

$$SU(2) \ni \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} a+ib & -(c-id) \\ c+id & a-ib \end{pmatrix} \longmapsto a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in Sp(1).$$

If we identify the span of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ with $\mathbb{R}^3$ as a vector space, then $SU(2) \simeq Sp(1)$ acts on $\mathbb{R}^3$ by conjugation. It is clear that

$$qvq^{-1} = (-q)v(-q)^{-1}, \quad v \in \mathbb{R}^3, \quad q \in Sp(1),$$

and it can be checked that $v \mapsto qvq^{-1}$ is in fact a rotation. This gives the two-to-one map $\psi : SU(2) \to SO(3)$. See [Hal15, Proposition 1.19] for more details showing that it is in fact a smooth and surjective group homomorphism. By identifying the quaternions $\mathbb{H}$ with $\mathbb{R}^4$ as vector spaces we see that $Sp(1) \simeq S^3$ as topological spaces. This shows that $SU(2)$ is simply connected and covering theory implies that $SU(2) \simeq \text{Spin}(3)$ as topological spaces. If we let $\xi : \text{Spin}(3) \to SO(3)$ denote the covering map, then

$$\tau := d\psi^{-1} \circ d\xi : \mathfrak{g}(2, \rho) \longrightarrow \mathfrak{su}(2)$$

is a Lie algebra isomorphism by [War83, Proposition 3.26]. As both $SU(2)$ and $\text{Spin}(3)$ are simply connected, we have by Theorem A.5 that

$$G(2, \rho) \simeq \text{Spin}(3) \simeq SU(2)$$

as Lie groups. Through the isomorphism τ we obtain a left-invariant structure $\mathcal{H}^\rho$ on $SU(2)$. Finally, if we require that τ is a linear isometry we get an inner product on $\mathfrak{su}(2)$. By left-translating this inner product we acquire a sub-Riemannian metric g_ρ such that

$$(Q, \mathcal{H}, g) \simeq (G(2, \rho), \mathcal{H}^{\mathfrak{p}}, g_{\mathfrak{p}}) \simeq (SU(2), \mathcal{H}^\rho, g_\rho)$$

as sub-Riemannian geometries.

Let us write out the case $\rho = 1$ as an illustration: In this case,

$$\mathfrak{g}(2, 1) = \begin{pmatrix} 0 & a & x_1 \\ -a & 0 & x_2 \\ -x_1 & -x_2 & 0 \end{pmatrix}, \quad x_1, x_2, a \in \mathbb{R}.$$

Introduce the basis

$$E_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad E_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad E_3 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

for $\mathfrak{su}(2)$. The subspace $\mathfrak{m} \subset \mathfrak{su}(2)$ spanned by E_1 and E_2 corresponds under the isomorphism τ to the subspace $\mathfrak{p} \subset \mathfrak{g}(2, 1)$ described in Subsection 4.6.3. Then

$$\mathcal{H}^1_A = dL_A\mathfrak{m} = \text{span}\{AE_1, AE_2\} = \text{span}\left\{\frac{i}{2}\begin{pmatrix} -\overline{\beta} & \alpha \\ \overline{\alpha} & \beta \end{pmatrix}, \frac{i}{2}\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & -\overline{\alpha} \end{pmatrix}\right\} \subset T_A SU(2), \quad A = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}.$$

Similarly, the induced inner product on $\mathfrak{m}$ is the one making E_1 and E_2 orthonormal. Left translating this gives the sub-Riemannian metric

$$g_{1_A}\left(\frac{i}{2}\begin{pmatrix} -\eta\overline{\beta} + \gamma\alpha & \eta\alpha + \gamma\overline{\beta} \\ \eta\overline{\alpha} + \gamma\beta & \eta\beta - \gamma\overline{\alpha} \end{pmatrix}, \frac{i}{2}\begin{pmatrix} -\sigma\overline{\beta} + \tau\alpha & \sigma\alpha + \tau\overline{\beta} \\ \sigma\overline{\alpha} + \tau\beta & \sigma\beta - \tau\overline{\alpha} \end{pmatrix}\right) = \eta\sigma + \gamma\tau,$$

for $\eta, \gamma, \sigma, \tau \in \mathbb{R}$.

For $\rho < 0$, we have that $G(2,\rho)$ is isomorphic to the universal covering space of the connected component of $\mathrm{Isom}(\mathbb{H}^2_\rho)$. In Chapter 2 we determined that $\mathrm{Isom}(\mathbb{H}^2_\rho) \simeq O_+(2,1)$. We denote the connected component of $O_+(2,1)$ by $SO_+(2,1)$. It is well known that

$$SO_+(2,1) \simeq SL(2) = Sp(2),$$

where $SL(2)$ denotes the 2×2 real matrices with unit determinant and $Sp(2)$ denotes the 2×2 real matrices preserving the usual symplectic form

$$[(x_1,x_2),(y_1,y_2)] = x_1y_2 - x_2y_1, \quad x_i, y_i \in \mathbb{R}, \quad i = 1,2.$$

It is common to denote the universal covering group of $Sp(2)$ by $Mp(2)$ and refer to it as the *planar metaplectic group*. Thus by the same arguments as for the case when $\rho > 0$ we have

$$(Q,\mathcal{H},g) \simeq (G(2,\rho),\mathcal{H}^{\mathfrak{p}},g_{\mathfrak{p}}) \simeq (\widetilde{SL(2)},\mathcal{H}^\rho,g_\rho) \simeq (Mp(2),\mathcal{H}^\rho,g_\rho),$$

where $(\mathcal{H}^\rho, g_\rho)$ is obtained in the same way as when $\rho > 0$. Unfortunately, we cannot give a nice description of the planar metaplectic group $Mp(2)$ in terms of matrices as it one of the classical examples of a Lie group with no faithful finite-dimensional representation, see Appendix A.2 for the terminology. Moreover, while $SU(2)$ is compact the same does not hold for $SL(2)$ and its universal covering space since $SL(2) \simeq \mathbb{R}^2 \times S^1$ as topological spaces.

It should be noted that the sub-Riemannian structures we have presented on both $SU(2)$ and $Mp(2)$ have been studied previously in the literature, but under different guises. In [CCM09] the sub-Riemannian structure on $SU(2)$ was examined disguised by the isomorphism $SU(2) \simeq S^3$ given through the quaternions. In [GV11] the sub-Riemannian structure on the planar metaplectic group $Mp(2)$ was studied disguised by the isomorphism $SL(2) \simeq SU(1,1,\mathbb{C})$ to the generalized special unitary group. We refer to [GV11] for the definition of $SU(1,1,\mathbb{C})$ and further properties. Summarizing, we have proved the following result.

Proposition 4.41. *Let Q be a contact model space with non-zero contact curvature ρ. Then*

$$(Q,\mathcal{H},g) \simeq \begin{cases} (SU(2),\mathcal{H}^\rho,g_\rho) & \text{if } \rho > 0 \\ (Mp(2),\mathcal{H}^\rho,g_\rho) & \text{if } \rho < 0 \end{cases}.$$

The classification of all homogeneous sub-Riemannian geometries of dimension three has been done in [FG96]. In the classification, the authors also pointed out that six of the spaces in the classification are so called *sub-symmetric spaces*. This coincides with how we would have used the word, namely as a bracket generating sub-Riemannian geometry $(Q,\mathcal{H},g)$ such that for every $p \in Q$ there exists an isometry $\Phi : Q \to Q$ such that

$$\Phi(p) = p, \quad d\Phi\Big|_{\mathcal{H}_p} = -Id_{\mathcal{H}_p}.$$

Three of these spaces are of course the contact model spaces we have classified. The universal covering space of orientation preserving planar motions with a left-invariant structure is showed to be sub-symmetric in [FG96]. This gives an example of a simply connected sub-symmetric space with nontrivial distribution which is not a model space. Any left-invariant contact structure on a Lie group satisfies the assumption in the following definition.

Definition 4.42. Let Q be a manifold with a contact distribution $\mathcal{H}$ which is globally defined by the non-vanishing of a one-form ξ satisfying the non-integrability condition. The *Reeb vector field* of $(Q,\mathcal{H})$ is the unique vector field $X_0 \in \mathfrak{X}(Q)$ such that

$$\xi(X_0) = 1, \quad d\xi(X_0,\cdot) = 0.$$

For any three dimensional contact sub-Riemannian geometry one can define two functions κ and χ, as done for instance in [AB12], which reflect intrinsic geometrical properties of the geometry. The function χ is constructed from the Reeb vector field and is intimately related to it. The function κ was originally discovered in terms of the asymptotic expansion of the conjugate locus, see [Agr96] for more on this. It is clear from the description of χ and κ in [AB12] that they are constant functions whenever the manifold is a Lie group and the distribution is left-invariant. From the definition of χ given in [AB12], it follows that $\chi \geq 0$ and it vanishes if and only if the flow of the Reeb vector field X_0 consists of sub-Riemannian isometries. By extending our terminology from Section 2.3, we would say that the Reeb vector field is then a *sub-Killing field*.

If $(Q, \mathcal{H}, g)$ is in addition simply connected and complete as a metric space, then [AB12, Corollary 2] shows that the condition $\chi = 0$ implies that $(Q, \mathcal{H}, g)$ is isometric to precisely $N[2,2], SU(2)$, or $Mp(2)$ with the structures we have described depending on the values of κ. In this case, κ is equal to the contact curvature ρ and we have the following description.

Corollary 4.43. *Let Q be a three dimensional simply connected Lie group and let $(\mathcal{H}, g)$ be a bracket generating left-invariant sub-Riemannian structure on Q. Then $(Q, \mathcal{H}, g)$ is a contact model space if and only if the Reeb vector field is a sub-Killing field.*

Remark. The reason we can leave out the assumption of completeness in the hypothesis of Corollary 4.43 is that a left-invariant structure $(\mathcal{H}, g)$ on a Lie group Q will always be complete. To see this, choose a Riemannian metric g_R on Q which is left-invariant and satisfies

$$g_R\Big|_{\mathcal{H}} = g.$$

Since g_R is left-invariant the Riemannian manifold (Q, g_R) is homogeneous and hence complete. If we let d_R denote the associated distance function to g_R, then

$$d_R(p, q) \leq d_{CC}(p, q),$$

as the infimum is taken over a larger class of curves. Thus if $\{p_n\}$ is a Cauchy sequence for d_{CC} then it is also a Cauchy sequence for d_R, forcing $\{p_n\}$ to converge in the topology induced by d_R. As the topologies induced by d_R and d_{CC} are both equal to the usual topology on Q, this shows that d_{CC} is complete as well. For more general deductions about completeness, the reader is referred to [Str86, Theorem 7.4] and the published corrections to the paper given in [Str89].

5 Model Spaces of Step and Rank Three

We will in this chapter classify all sub-Riemannian model spaces of step and rank three. Several techniques discussed previously will be employed and theory from all the previous chapters will be used. However, the difficulty in several arguments will be rooted in the fact that the representation theory is less rigid than in step two. We will first classify the Carnot model spaces of step and rank three in Section 5.1. In Section 5.3 we will survey the main results of the thesis. The proofs of the main results are given in Section 5.4 and Section 5.5. Several of the technical details needed in those proofs are based on representation theory and will be developed in Section 5.2. The result of the classification is unexpected when comparing with the known results from step two given in Theorem 4.38. We will devote a section at the end of the chapter for a short discussion of this as well as commenting on which challenges lie ahead in the topic of sub-Riemannian model spaces. All results in this chapter are original results unless otherwise stated.

5.1 Carnot Model Spaces of Step and Rank Three

The Carnot model spaces of step and rank three will be used as an invariant for general sub-Riemannian model spaces through the nilpotentization procedure in the later sections. We will follow the approach in Proposition 4.30, although the determination of sub-representations which are also ideals is more complicated and interesting than in the step two case.

Recall that $\mathfrak{f}[3,3]$ denotes the free nilpotent Lie algebra of step and rank three. Then Witt's formula (4.1) implies that the third layer has dimension eight. Let us fix an orthonormal basis A_1, A_2, A_3 for the generating subspace $\mathfrak{f}_1 = \mathfrak{f}_1[3,3]$. Throughout this section we will use the notation

$$\mathfrak{f}_2 = \operatorname{span}\{A_{12}, A_{13}, A_{23}\},$$

$$\mathfrak{f}_3 = \operatorname{span}\{A_{112}, A_{113}, A_{221}, A_{231}, A_{223}, A_{312}, A_{331}, A_{332}\},$$

for the induced basis elements

$$A_{ij} = [A_i, A_j], \quad A_{ijk} = [A_i, [A_j, A_k]].$$

This gives the relations $A_{ij} = -A_{ji}$ and $A_{ijk} = -A_{ikj}$ from the skew-symmetry, while the identity

$$A_{123} = -A_{231} - A_{312}$$

is due to the Jacobi identity. Recall that the action of the orthogonal group $O(\mathfrak{f}_1)$ on $\mathfrak{f}[3,3]$ is given inductively by

$$a \cdot [A, B] = [a \cdot A, a \cdot B], \quad a \in O(\mathfrak{f}_1), \ \ A, B \in \mathfrak{f}[3,3],$$

where the action on the generating layer is the standard action of $O(\mathfrak{f}_1)$ on $\mathfrak{f}_1$. Since we have fixed an orthonormal basis A_1, A_2, A_3 for $\mathfrak{f}_1$ we can identify $\mathfrak{f}_1 \simeq \mathbb{R}^3$ and $O(\mathfrak{f}_1) \simeq O(3)$ as representations. Let $\mathrm{Sym}^T(n)$ denote the traceless symmetric $n \times n$ real matrices. The main result we will prove in this section is the following theorem.

Theorem 5.1. *There are three non-isomorphic Carnot model spaces with step and rank three. Besides the free nilpotent Lie group $N[3,3]$ they are denoted by $\mathcal{C}_{3,3}$ and $\mathcal{A}_{3,3}$. The growth vectors of $\mathcal{C}_{3,3}$, $\mathcal{A}_{3,3}$, and $N[3,3]$ are $(3,6,9)$, $(3,6,11)$, and $(3,6,14)$, respectively. The Lie algebras of $\mathcal{C}_{3,3}$ and $\mathcal{A}_{3,3}$ are as vector spaces given by*

$$\mathit{Lie}(\mathcal{C}_{3,3}) = \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{c}_{3,3} \simeq \mathfrak{f}[3,3]/\mathfrak{a}_{3,3}, \quad \mathit{Lie}(\mathcal{A}_{3,3}) = \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{a}_{3,3} \simeq \mathfrak{f}[3,3]/\mathfrak{c}_{3,3},$$

where

$$\mathfrak{c}_{3,3} = \mathit{span}\{A_{112} + A_{332}, A_{223} + A_{113}, A_{331} + A_{221}\},$$

$$\mathfrak{a}_{3,3} = span\{A_{213}, A_{312}, A_{112} - A_{332}, A_{223} - A_{113}, A_{331} - A_{221}\}.$$

Moreover, the representations of $O(\mathfrak{f}_1)$ on $\mathfrak{c}_{3,3}$ and $\mathfrak{a}_{3,3}$ are isomorphic to the representations of $O(3)$ on $\mathfrak{o}(3)$ and $Sym^T(3)$ given by

$$E \longmapsto \det(a)aEa^{-1}, \quad A \longmapsto \det(a)aAa^{-1}$$

for $E \in \mathfrak{o}(3)$, $A \in Sym^T(3)$, and $a \in O(3)$.

It follows from Proposition 4.32 that the sub-Riemannian model spaces of step and rank three have dimension $9, 11$, or 14. From Theorem 5.1 we obtain the identifications

$$\mathrm{Lie}(\mathcal{C}_{3,3}) \simeq \mathbb{R}^3 \oplus \mathfrak{o}(3) \oplus \mathfrak{o}(3), \quad \mathrm{Lie}(\mathcal{A}_{3,3}) \simeq \mathbb{R}^3 \oplus \mathfrak{o}(3) \oplus \mathrm{Sym}^T(3)$$

as $O(\mathfrak{f}_1) \simeq O(3)$-modules with the standard actions on the first two layers. We will give the explicit Lie brackets in the beginning of Section 5.3. The description as skew-symmetric and traceless symmetric matrices will be useful for computations and for showing irreducibility.

Proof. The proof will be based on the strategy given in Proposition 4.30 and divided into four parts: Firstly, we will introduce notation and provide the motivation for why we consider the two Lie algebras presented in the theorem. Secondly, we will check that these are in fact sub-representations of the action of $O(\mathfrak{f}_1)$ on $\mathfrak{f}[3,3]$. Thirdly, we will provide the isomorphisms to the concrete matrix realizations presented in the theorem. Lastly, we will check that these are irreducible, ensuring that there are no other Carnot model spaces than those proposed in the theorem.

Step 1: We will use the notation P_τ for the *permutation matrices* relative to the basis $\{A_1, A_2, A_3\}$. As an example, P_{12} interchanges A_1 and A_2 while leaving A_3 fixed, thus having the matrix representation

$$P_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, the notation R^i_θ for $i = 1, 2, 3$ and $0 \leq \theta < 2\pi$ denotes the *rotation matrices* given by

$$R^1_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad R^2_\theta = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}, \quad R^3_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying the QR-decomposition of an orthogonal matrix using Givens rotations, see [Bjo96, Section 2.3.2], shows that any element in $O(3)$ can be written as a composition of rotation matrices and the three *reflection matrices*

$$H_i = I - 2e_i e_i^T, \quad i = 1, 2, 3.$$

The first two layers of the Lie algebra $\mathfrak{c} = \mathfrak{c}_1 \oplus \mathfrak{c}_2 \oplus \mathfrak{c}_3$ of any Carnot model space C of step and rank three have to be equal to $\mathfrak{f}_1 \oplus \mathfrak{f}_2$: If not, then the induced action of $O(\mathfrak{f}_1)$ on $\mathfrak{c}_1 \oplus \mathfrak{c}_2$ is a sub-representation of $O(\mathfrak{f}_1)$ on $\mathfrak{f}_1 \oplus \mathfrak{f}_2$. This would together with Proposition 4.30 contradict Proposition 4.31.

We will now find candidates for sub-representations by looking at the action of the permutation matrices. By applying P_{123}, P_{12}, P_{13}, and P_{23} we obtain the following diagram:

$$\begin{array}{ccccccccc} A_{112} & \xrightarrow{P_{123}} & A_{223} & \xrightarrow{P_{123}} & A_{331} & \xrightarrow{P_{123}} & A_{112} & \xrightarrow{P_{123}} & \ldots \\ \updownarrow{\scriptstyle P_{13}} & & \updownarrow{\scriptstyle P_{12}} & & \updownarrow{\scriptstyle P_{23}} & & \updownarrow{\scriptstyle P_{13}} & & \\ A_{332} & \xrightarrow{P_{123}} & A_{113} & \xrightarrow{P_{123}} & A_{221} & \xrightarrow{P_{123}} & A_{332} & \xrightarrow{P_{123}} & \ldots \end{array} \tag{5.1}$$

It is straightforward to check that any invariant subspace containing one of the basis elements appearing in the diagram (5.1) can not be proper. Hence if we are looking for any proper invariant subspace having some of the basis elements as generators, it can only be A_{231} or A_{312}. The fact that $P_{123}A_{231} = A_{312}$ reduces this possibility to the space

$$\mathfrak{a}^1_{3,3} = \text{span}\{A_{231}, A_{312}\}.$$

However, this is not an invariant subspace since

$$\begin{aligned} R^3_\theta A_{231} &= -[R^3_\theta A_2, [R^3_\theta A_1, R^3_\theta A_3]] \\ &= [\sin(\theta)A_1 - \cos(\theta)A_2, \cos(\theta)A_{13} + \sin(\theta)A_{23}] \\ &= \frac{1}{2}\sin(2\theta)(A_{113} - A_{223}) - \sin^2(\theta)A_{312} + \cos(2\theta)A_{231}. \end{aligned}$$

By combining the columns in the diagram (5.1), we see that the space

$$\mathfrak{c}_{3,3} = \text{span}\{A_{112} + A_{332}, A_{223} + A_{113}, A_{331} + A_{221}\}$$

is invariant under all the permutation matrices. In fact, the same is true for

$$\mathfrak{a}^2_{3,3} = \text{span}\{A_{112} - A_{332}, A_{223} - A_{113}, A_{331} - A_{221}\}.$$

However, $\mathfrak{a}^2_{3,3}$ is not invariant under the full orthogonal group as shown by the calculation

$$R^2_\theta(A_{112} - A_{332}) = \cos(2\theta)(A_{112} - A_{332}) + \sin(2\theta)A_{231} + 2\sin(2\theta)A_{312}.$$

Thus $\mathfrak{a}^1_{3,3}$ and $\mathfrak{a}^2_{3,3}$ obstruct one another from being sub-representations. Hence we consider their direct sum

$$\mathfrak{a}_{3,3} = \mathfrak{a}^1_{3,3} \oplus \mathfrak{a}^2_{3,3} = \text{span}\{A_{231}, A_{312}, A_{112} - A_{332}, A_{223} - A_{113}, A_{331} - A_{221}\}.$$

Step 2: We now check that the spaces $\mathfrak{c}_{3,3}$ and $\mathfrak{a}_{3,3}$ indeed provide us with sub-representations. Notice that

$$P_{123}H_1 = H_2, \quad P_{123}H_2 = H_3,$$
$$P_{23}R^3_\theta P_{23} = R^2_\theta, \quad P_{13}R^3_\theta P_{13} = R^1_\theta.$$

This implies together with the fact that P_{123} and P_{12} generate the permutation matrices, that it suffices to show invariance under the four matrices R^1_θ, H_1, P_{12}, and P_{123}. It is straightforward to see that $\mathfrak{c}_{3,3}$ and $\mathfrak{a}_{3,3}$ are invariant under H_1, P_{123}, and P_{12} by looking at the indices of the basis elements in both spaces. Hence we only need to check invariance under the rotation matrix R^1_θ. As the computations are similar to those presented previously, we leave the details to the reader and simply state that the action of R^1_θ on $\mathfrak{c}_{3,3}$ becomes

$$\begin{aligned} A_{112} + A_{332} &\longmapsto \cos(\theta)(A_{112} + A_{332}) + \sin(\theta)(A_{223} + A_{113}), \\ A_{223} + A_{113} &\longmapsto -\sin(\theta)(A_{112} + A_{332}) + \cos(\theta)(A_{223} + A_{113}), \\ A_{331} + A_{221} &\longmapsto A_{331} + A_{221}. \end{aligned} \tag{5.2}$$

This shows that $\mathfrak{c}_{3,3}$ is invariant and hence a sub-representation. Similarly, the action of R^1_θ on $\mathfrak{a}_{3,3}$ becomes

$$\begin{aligned} A_{231} &\longmapsto \cos^2(\theta)A_{213} + \sin^2(\theta)A_{312} + \frac{1}{2}\sin(2\theta)(A_{331} - A_{221}), \\ A_{312} &\longmapsto \sin^2(\theta)A_{213} + \cos^2(\theta)A_{312} - \frac{1}{2}\sin(2\theta)(A_{331} - A_{221}), \\ A_{112} - A_{332} &\longmapsto \cos(\theta)(A_{112} - A_{332}) + \sin(\theta)(A_{223} - A_{113}), \\ A_{223} - A_{113} &\longmapsto \sin(\theta)(A_{112} - A_{223}) + \cos(\theta)(A_{223} - A_{113}), \\ A_{331} - A_{221} &\longmapsto \cos(2\theta)(A_{331} - A_{221}) - \sin(2\theta)A_{231} + \sin(2\theta)A_{312}, \end{aligned}$$

showing that $\mathfrak{a}_{3,3}$ is also a sub-representation. Thus we have a decomposition

$$\mathfrak{f}_3 = \mathfrak{c}_{3,3} \oplus \mathfrak{a}_{3,3} \tag{5.3}$$

as a representation of $O(\mathfrak{f}_1)$. As the orthogonal group is compact it has the complete reducibility property, see Appendix A.2. Hence if we show that the two sub-representations $\mathfrak{c}_{3,3}$ and $\mathfrak{a}_{3,3}$ are irreducible, then these are all the sub-representations which are also ideals, finishing the theorem.

Step 3: We will provide the details for the identification of $\mathfrak{c}_{3,3}$ with $\mathfrak{o}(3)$ and leave most of the grunt work to the reader in the case of the identification of $\mathfrak{a}_{3,3}$ with $\mathrm{Sym}^T(3)$. We use the basis

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

for $\mathfrak{o}(3)$, henceforth referred to as its *standard basis*. Assume we have an isomorphism of representations $\rho : \mathfrak{c}_{3,3} \to \mathfrak{o}(3)$ and let

$$\begin{aligned} \rho(A_{112} + A_{332}) &= a_1 E_{12} + a_2 E_{13} + a_3 E_{23}, \\ \rho(A_{223} + A_{113}) &= a_4 E_{12} + a_5 E_{13} + a_6 E_{23}, \\ \rho(A_{331} + A_{221}) &= a_7 E_{12} + a_8 E_{13} + a_9 E_{23}. \end{aligned}$$

Applying P_{12} and P_{13} gives the relations

$$a_1 = a_7 = -a_5, \quad a_2 = -a_4 = -a_9, \quad a_3 = a_6 = -a_8,$$

while applying H_1 and H_3 shows that $a_1 = a_3 = 0$. Thus any isomorphism of representations have to be on the form

$$\begin{aligned} \rho(A_{112} + A_{332}) &= E_{13}, \\ \rho(A_{223} + A_{113}) &= -E_{12}, \\ \rho(A_{331} + A_{221}) &= -E_{23}, \end{aligned}$$

after a normalization.

It is straightforward to check that this is invariant under H_1,P_{12} and P_{123}. The invariance under R^1_θ follows from the equations in (5.2) and the relations

$$\begin{aligned} R^1_\theta E_{12} R^1_{-\theta} &= \cos(\theta) E_{12} - \sin(\theta) E_{13}, \\ R^1_\theta E_{13} R^1_{-\theta} &= -\sin(\theta) E_{12} + \cos(\theta) E_{13}, \\ R^1_\theta E_{23} R^1_{-\theta} &= E_{23}. \end{aligned}$$

Hence $\mathfrak{c}_{3,3}$ is isomorphic to $\mathfrak{o}(3)$ with the action

$$E \longmapsto \det(a) a E a^{-1}, \quad a \in O(3), \;\; E \in \mathfrak{o}(3).$$

Similarly for the identification of $\mathfrak{a}_{3,3}$ with $\mathrm{Sym}^T(3)$, we use the *standard basis* for $\mathrm{Sym}^T(3)$

$$A_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_{D1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{D2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Assume an isomorphism of representations $\rho : \mathfrak{a}_{3,3} \to \mathrm{Sym}^T(3)$ is given in the presented bases by $\rho = (a_{ij})_{i,j=1}^5$. An application of the permutation matrices together with a rotation shows, similarly to when we considered $\mathfrak{c}_{3,3}$, that the transformation matrix has to be on the form

$$\rho = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \end{pmatrix},$$

after a normalization. It is now straightforward to check that this is in fact invariant under the whole orthogonal group by showing invariance under the permutation matrices P_{12}, P_{123}, the reflection H, and a rotation matrix, say R^3_θ. The most tedious computations is with the rotation matrix, and the relevant formulas are

$$\begin{aligned} R^3_\theta A_{12} R^3_{-\theta} &= (2\cos^2(\theta) - 1)A_{12} - \sin(2\theta)A_{11}, \\ R^3_\theta A_{13} R^3_{-\theta} &= \cos(\theta)A_{13} + \sin(\theta)A_{23}, \\ R^3_\theta A_{23} R^3_{-\theta} &= -\sin(\theta)A_{13} + \cos(\theta)A_{23}, \\ R^3_\theta A_{D1} R^3_{-\theta} &= \sin(2\theta)A_{12} + (2\cos^2(\theta) - 1)A_{11}, \\ R^3_\theta A_{D2} R^3_{-\theta} &= -\frac{1}{2}\sin(2\theta)A_{12} + \sin^2(\theta)A_{11} + A_{22}, \end{aligned}$$

for the readers convenience.

Step 4: We will now show that the sub-representations we have found are in fact irreducible. We will operate with their matrix forms described in the previous step as this will make matters more simple and concrete. In fact, the representation of $O(\mathfrak{f}_1)$ on $\mathfrak{c}_{3,3}$ is irreducible by Lemma 4.27 applied to the representation of $O(3)$ on $\mathfrak{o}(3)$ as the determinant factor in the representation plays no part in the existence of invariant subspaces.

We are left with checking that the action of $O(3)$ on $\mathrm{Sym}^T(3)$ is irreducible. This will be showed for the representation without the determinant factor to simplify notation and it will be based on an eigenvalue argument. It is straightforward to show that there are no one-dimensional invariant subspaces by performing a rotation. Let $V \subset \mathrm{Sym}^T(3)$ be an invariant subspace with $\dim(V) \geq 2$ and let $A \in V$ be non-zero. Since a symmetric matrix is diagonalizable, any non-zero symmetric matrix has at least one non-zero eigenvalue. By orthogonal decomposition, there exists an element $q \in O(3)$ such that

$$qAq^{-1} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \lambda_1 \neq 0,$$

where we have (possibly) performed a permutation to get λ_1 in the first entry. Hence V contains an element of the form

$$\widetilde{A} = \frac{qAq^{-1}}{\lambda_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix}, \quad \tau_i = \frac{\lambda_i}{\lambda_1}, \quad i = 2, 3.$$

We can not have that both $\tau_2 = \tau_3 = 0$, as this will imply that $\widetilde{A}$ has non-zero trace. On the other hand, we will now explain why we can find a matrix in V where precisely one of the eigenvalues is zero.

As $\dim(V) \geq 2$ there exists an element $B \in V$ which is linearly independent from A. Applying the same procedure for B, there exists a $p \in O(3)$ such that

$$\widetilde{B} = \frac{pBp^{-1}}{\lambda_1'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_2' & 0 \\ 0 & 0 & \tau_3' \end{pmatrix}, \quad \lambda_1' \neq 0, \quad \tau_i' = \frac{\lambda_i'}{\lambda_1'}, \quad i = 2, 3.$$

Hence V contains the element

$$C = \widetilde{A} - \widetilde{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tau_2 - \tau_2' & 0 \\ 0 & 0 & \tau_3 - \tau_3' \end{pmatrix}.$$

We need to take the following precaution: In case $C = 0$, acting with P_{23} on $\widetilde{B}$ before forming the difference gives

$$C' = \widetilde{A} - P_{23}\widetilde{B}P_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tau_2 - \tau_3' & 0 \\ 0 & 0 & \tau_3 - \tau_2' \end{pmatrix}$$

instead. If both C and C' are zero, then $\tau_2 = \tau_3 = \tau_2' = \tau_3' = -\frac{1}{2}$ by the zero trace condition. If this is the case, we form

$$C'' = \widetilde{A} + 2P_{12}\widetilde{B}P_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & -\frac{3}{2} \end{pmatrix}.$$

In any case, this shows that there exists a matrix in V with precisely two non-zero eigenvalues. We call the eigenvalues η_1 and η_2. The trace condition implies that $\eta_1 = -\eta_2$, so applying a permutation together with a scaling gives that $A_{D1} \in V$. However, as $O(3)$ has the complete reducibility property, this also has to be true for an invariant complement of V as long as $\dim(V^c) \geq 2$ for any complement V^c. This impossibility forces V to be of codimension one. Then any invariant complement of V has dimension one which is a contradiction to the fact that there are no invariant one-dimensional subspaces. Thus the representation of $O(3)$ on $\mathrm{Sym}^T(3)$ is irreducible. □

5.2 Invariant Maps

Before embarking on classifying all sub-Riemannian model spaces of step and rank three we will need some preliminary results. Looking at the proof of Theorem 4.38, we concluded at two places in the proof that the bracket of the Lie algebra had a particular form. The first time was in Equation (4.9) and was based on knowledge of the nilpotentization. In Section 5.1 we obtained the analogous knowledge for model spaces with step and rank three. This will allow us to determine the "brackets going forward" in a similar way as in the proof of Theorem 4.38.

The second time in the proof we made such a conclusion was in Equation (4.10), which was based on an argument found in [Gro16, Lemma A.3]. In this section we will build up the necessary results to make a similar deduction for the sub-Riemannian model spaces of step and rank three. This is really a statement about representation theory of the orthogonal group, and the reader unfamiliar with representation theory terminology is suggested to take a look at the end of Appendix A.2 before proceeding. The results we will need is collected in Proposition 5.2 and Theorem 5.3 below. The cases (M2) and (M5) in Theorem 5.3 have already been proved in [Gro16, Lemma A.3]. Nevertheless, we include them as they are needed later and provide a new proof for Equation (M5) based on Schur's Lemma and the Theorem of Highest Weight, see Appendix A.2 for those results.

We will be dealing with representations of $O(3)$ on the spaces $\mathbb{R}^3$, $\mathfrak{o}(3)$, and $\mathrm{Sym}^T(3)$. The action on $\mathbb{R}^3$ is the canonical one while the action on $\mathrm{Sym}^T(3)$ given by

$$A \longmapsto \det(a)aAa^{-1}, \quad a \in O(3), \ A \in \mathrm{Sym}^T(3).$$

We have encountered two representations of $O(3)$ on $\mathfrak{o}(3)$, namely with or without the factor $\det(a)$ in front of the conjugation. It is a special feature of three dimensions that the representation

$$E \longmapsto \det(a)aEa^{-1}, \quad a \in O(3), \ E \in \mathfrak{o}(3)$$

of $O(3)$ on $\mathfrak{o}(3)$ is isomorphic to the usual representation of $O(3)$ on $\mathbb{R}^3$ through the map

$$\mathfrak{o}(3) \ni \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \longmapsto (x, y, z) \in \mathbb{R}^3. \tag{5.4}$$

It is straightforward to check that the commutator between elements in $\mathfrak{o}(3)$ correspond to the cross product in $\mathbb{R}^3$ under the identification (5.4). Whenever we discuss the representation of $O(3)$ on $\mathfrak{o}(3)$ it will implicitly be with the usual adjoint action without the determinant term. This can of course also be identified as a vector space with $\mathbb{R}^3$ through (5.4), although the corresponding action of $O(3)$ on $\mathbb{R}^3$ will then be given by

$$v \longmapsto \det(a)av, \quad a \in O(3), \;\; v \in \mathbb{R}^3. \tag{5.5}$$

We will use the notation $\overline{\mathbb{R}}^3$ for $\mathbb{R}^3$ with the representation given in (5.5). Finally, the notation $\mathrm{Sym}(x, y)$ denotes the *traceless symmetrization map* of the vectors $x, y \in \mathbb{R}^3$

$$\mathrm{Sym}(x, y) = \frac{xy^T + yx^T}{2} - \frac{1}{3}\langle x, y\rangle I,$$

where $\langle \cdot, \cdot\rangle$ denotes the Euclidean inner product. We will need to determine the possible invariant bilinear maps between the different representations discussed above. This is equivalent to understanding invariant linear maps from their tensor product, and we have the following preliminary result.

Proposition 5.2. *There are no non-zero $O(3)$-invariant linear maps between the following representations:*

$$\begin{aligned}
\mathbb{R}^3 \otimes \mathbb{R}^3 &\longrightarrow \mathbb{R}^3, & \mathbb{R}^3 \otimes \mathbb{R}^3 &\longrightarrow Sym^T(3),\\
\mathbb{R}^3 \otimes \mathfrak{o}(3) &\longrightarrow \mathfrak{o}(3), & \mathbb{R}^3 \otimes Sym^T(3) &\longrightarrow \mathbb{R}^3,\\
\mathbb{R}^3 \otimes Sym^T(3) &\longrightarrow Sym^T(3), & \mathfrak{o}(3) \otimes \mathfrak{o}(3) &\longrightarrow \mathbb{R}^3,\\
\mathfrak{o}(3) \otimes \mathfrak{o}(3) &\longrightarrow Sym^T(3), & \mathfrak{o}(3) \otimes Sym^T(3) &\longrightarrow \mathfrak{o}(3),\\
Sym^T(3) \otimes Sym^T(3) &\longrightarrow \mathbb{R}^3, & Sym^T(3) \otimes Sym^T(3) &\longrightarrow Sym^T(3).
\end{aligned}$$

Proof. This follows from the fact that none of the maps are invariant under $-I \in O(3)$. To illustrate, assume that $\phi : \mathbb{R}^3 \otimes \mathrm{Sym}^T(3) \to \mathbb{R}^3$ is an invariant map under the action of $O(3)$. Then

$$-I \cdot \phi(v, A) = -\phi(v, A) \neq \phi(v, A) = \phi(-v, -A) = \phi\left(-I \cdot v, \det(-I)(-I)A(-I)^{-1}\right),$$

for $v \in \mathbb{R}^3$, $A \in \mathrm{Sym}^T(3)$, and $\phi(v, A) \neq 0$. Thus ϕ is invariant if and only if it is identically zero. □

It will require more work to show when there actually are invariant maps and moreover that they are unique. The following theorem is the main result of this section.

Theorem 5.3. *There is a unique (up to scaling) non-zero $O(3)$-invariant map between the following representations:*

$$\begin{aligned}
\mathbb{R}^3 \otimes \mathbb{R}^3 &\longrightarrow \mathfrak{o}(3), & (v, w) &\longmapsto v \times w, & \text{(M1)}\\
\mathbb{R}^3 \otimes \mathfrak{o}(3) &\longrightarrow \mathbb{R}^3, & (v, C) &\longmapsto Cv, & \text{(M2)}\\
\mathbb{R}^3 \otimes \mathfrak{o}(3) &\longrightarrow Sym^T(3), & (v, w) &\longmapsto Sym(v, w), & \text{(M3)}\\
\mathbb{R}^3 \otimes Sym^T(3) &\longrightarrow \mathfrak{o}(3), & (v, A) &\longmapsto Av, & \text{(M4)}\\
\mathfrak{o}(3) \otimes \mathfrak{o}(3) &\longrightarrow \mathfrak{o}(3), & (C, D) &\longmapsto [C, D], & \text{(M5)}\\
\mathfrak{o}(3) \otimes Sym^T(3) &\longrightarrow \mathbb{R}^3, & (v, A) &\longmapsto Av, & \text{(M6)}\\
\mathfrak{o}(3) \otimes Sym^T(3) &\longrightarrow Sym^T(3), & (C, A) &\longmapsto [C, A], & \text{(M7)}\\
Sym^T(3) \otimes Sym^T(3) &\longrightarrow \mathfrak{o}(3), & (A, B) &\longmapsto [A, B], & \text{(M8)}
\end{aligned}$$

after applying the identification (5.4) in (M1), (M3), (M4), and (M6).

The reader will notice in Section 5.4 and Section 5.5 that Proposition 5.2 and Theorem 5.3 will play prominent roles. We will divide the proof of Theorem 5.3 into three lemmas, based on which approaches are necessary.

Lemma 5.4. *There is a unique (up to scaling) non-zero $O(3)$-invariant linear map in the cases (M1) and (M3).*

Proof. Assume we have a non-zero $O(3)$-invariant linear map

$$\phi : \mathbb{R}^3 \otimes \mathbb{R}^3 \longrightarrow \mathfrak{o}(3)$$

and consider its kernel $N = \ker(\phi)$. Since the representation of $O(3)$ on $o(3)$ is irreducible we have that ϕ is surjective because the image of ϕ is an invariant subspace. Thus $\dim(N) = 6$. Decompose $\mathbb{R}^3 \otimes \mathbb{R}^3$ into symmetric and alternating part

$$\mathbb{R}^3 \otimes \mathbb{R}^3 \simeq M_{3\times 3}(\mathbb{R}) = \mathrm{Sym}(3) \oplus \mathfrak{o}(3),$$

where the identification is given by the outer product and the action of $O(3)$ on $M_{3\times3}(\mathbb{R})$ is conjugation. The action of $O(3)$ on $\mathfrak{o}(3)$ is irreducible by Lemma 4.27 while the action of $O(3)$ on the symmetric matricies $\mathrm{Sym}(3)$ decomposes into

$$\mathrm{Sym}(3) = \mathrm{Sym}^T(3) \oplus \{cI\}.$$

We showed in the proof of Theorem 5.1 that the representation of $O(3)$ on $\mathrm{Sym}^T(3)$ is irreducible. As N is invariant it follows by dimensional reasons that $N = \mathrm{Sym}(3)$. This leaves only the projection as an option, which is easily seen to be invariant. The associated map $S : \mathbb{R}^3 \otimes \mathbb{R}^3 \to \mathfrak{o}(3)$ is identified through (5.4) with the cross product $x \otimes y \mapsto x \times y$.

Let us now assume we have a non-zero $O(3)$-invariant linear map

$$\psi : \mathbb{R}^3 \otimes \mathfrak{o}(3) \longrightarrow \mathrm{Sym}^T(3).$$

By applying the previously discussed identifications, this can be identified with a map

$$\psi : \mathbb{R}^3 \otimes \mathfrak{o}(3) \simeq \mathbb{R}^3 \otimes \overline{\mathbb{R}}^3 \simeq M_{3\times3}(\mathbb{R}) \longrightarrow \mathrm{Sym}^T(3)$$

where the action of $O(3)$ on $M_{3\times3}(\mathbb{R})$ is now given by

$$B \longmapsto \det(a)aBa^{-1}, \quad B \in M_{3\times3}(\mathbb{R}),\ \ a \in O(3).$$

The existence and uniqueness of ψ is equivalent to the existence of a map

$$\overline{\psi} : M_{3\times3}(\mathbb{R}) = \mathrm{Sym}^T(3) \oplus \{cI\} \oplus \mathfrak{o}(3) \longrightarrow \mathrm{Sym}^T(3)$$

where the action of $O(3)$ on both $M_{3\times3}(\mathbb{R})$ and $\mathrm{Sym}^T(3)$ is without the determinant factor. For dimensional reasons together with irreducibility, the only invariant map up to scaling is the projection. Hence the original map

$$\psi : \mathbb{R}^3 \otimes \mathfrak{o}(3) \simeq \mathbb{R}^3 \otimes \overline{\mathbb{R}}^3 \longrightarrow \mathrm{Sym}^T(3)$$

is given by the traceless symmetrization map

$$x \otimes y \longmapsto \mathrm{Sym}(x, y) = \frac{x^T y + y^T x}{2} - \frac{1}{3}\langle x, y\rangle I.$$

□

Lemma 5.5. *There is a unique (up to scaling) non-zero $O(3)$-invariant linear map in the cases (M4), (M6), (M7), and (M8).*

Proof. Notice first of all that any map of type (M4) induces a map of type (M6) by using the identification (5.4) since we have the determinant factor once on each side. Thus if we can show that matrix multiplication is the unique $O(3)$-invariant map up to scaling in the case of (M4), the same holds for the case (M6). For the remaining three cases (M4), (M7) and (M8), the proposed maps in the theorem are clearly seen to be $O(3)$-invariant. To illustrate for (M7), the map $E \otimes A \mapsto [E, A] \in \mathrm{Sym}^T(3)$ for $E \in \mathfrak{o}(3)$ and $A \in \mathrm{Sym}^T(3)$ is $O(3)$-invariant as

$$\begin{aligned} a[E,A] &= \det(a)\left(aEAa^{-1} - aAEa^{-1}\right) \\ &= \det(a)\left(aEa^{-1}aAa^{-1} - aAa^{-1}aEa^{-1}\right) \\ &= [aE, aA]. \end{aligned}$$

As the computations are similar and quite lengthy we will only supply the details for the case (M8), as this is the most cumbersome of them all. We will work with the standard bases of $\mathfrak{o}(3)$ and $\mathrm{Sym}^T(3)$ given in the proof of Theorem 5.1. The strategy is to choose particular elements in $O(3)$ to show that invariant maps up to scaling are the ones presented in Theorem 5.3. Assume there is a non-zero bilinear map

$$S : \mathfrak{o}(3) \times \mathrm{Sym}^T(3) \longrightarrow \mathrm{Sym}^T(3).$$

We write S in the bases $\{E_{12}, E_{13}, E_{23}\}$ and $\{A_{12}, A_{13}, A_{23}, A_{D1}, A_{D2}\}$ as

$$S = \begin{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} & \begin{bmatrix} a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{bmatrix} & \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{15} \end{bmatrix} & \begin{bmatrix} a_{16} \\ a_{17} \\ a_{18} \\ a_{19} \\ a_{20} \end{bmatrix} & \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \\ a_{24} \\ a_{25} \end{bmatrix} \\ \begin{bmatrix} a_{26} \\ a_{27} \\ a_{28} \\ a_{29} \\ a_{30} \end{bmatrix} & \begin{bmatrix} a_{31} \\ a_{32} \\ a_{33} \\ a_{34} \\ a_{35} \end{bmatrix} & \begin{bmatrix} a_{36} \\ a_{37} \\ a_{38} \\ a_{39} \\ a_{40} \end{bmatrix} & \begin{bmatrix} a_{41} \\ a_{42} \\ a_{43} \\ a_{44} \\ a_{45} \end{bmatrix} & \begin{bmatrix} a_{46} \\ a_{47} \\ a_{48} \\ a_{49} \\ a_{50} \end{bmatrix} \\ \begin{bmatrix} a_{51} \\ a_{52} \\ a_{53} \\ a_{54} \\ a_{55} \end{bmatrix} & \begin{bmatrix} a_{56} \\ a_{57} \\ a_{58} \\ a_{59} \\ a_{60} \end{bmatrix} & \begin{bmatrix} a_{61} \\ a_{62} \\ a_{63} \\ a_{64} \\ a_{65} \end{bmatrix} & \begin{bmatrix} a_{66} \\ a_{67} \\ a_{68} \\ a_{69} \\ a_{70} \end{bmatrix} & \begin{bmatrix} a_{71} \\ a_{72} \\ a_{73} \\ a_{74} \\ a_{75} \end{bmatrix} \end{pmatrix}, \tag{5.6}$$

using the ordering of the bases as given above. By using the permutation, reflection, and rotation matrices introduced in the proof of Theorem 5.1 we will reduce this to the coordinate version of the commutator. By applying H_1 we obtain

$$S(H_1E_{12}, H_1A_{12}) = S(-E_{12}, A_{12}) = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \\ -a_4 \\ -a_5 \end{bmatrix}.$$

On the other hand, as S is invariant under $O(3)$ this becomes

$$H_1S(E_{12}, A_{12}) = \begin{bmatrix} a_1 \\ a_2 \\ -a_3 \\ -a_4 \\ -a_5 \end{bmatrix},$$

showing that $a_1 = a_2 = 0$. Similarly, using P_{12} shows that $a_3 = a_5 = 0$. We now normalize such that $a_4 = 2$, which is motivated by the fact that $[E_{12}, A_{12}] = 2A_{D1}$. To determine more entries, apply P_{23} to $S(E_{12}, A_{12})$ and obtain $a_{31} = a_{32} = a_{33} = 0$ along with $a_{34} = a_{35} = 2$. In the same vein, we get by applying P_{12} to $S(E_{13}, A_{13})$ that $a_{61} = a_{62} = a_{63} = a_{64} = 0$ and $a_{65} = 2$. That takes care of all the "diagonal entries" and we now work with the other entries in the left 3×3 block of (5.6). Using H_i on $S(E_{12}, A_{13})$ for $i = 1, 2$ shows that $a_6 = a_7 = a_9 = a_{10} = 0$. By translating this around with the help of P_{12}, P_{13}, and P_{23} determines all sub-entries in the left 3×3 block of (5.6), and the non-zero ones of the off diagonal are $a_{12} = a_{36} = a_{56} = -a_8$ and $a_{28} = a_{52} = a_8$.

Considering now $S(E_{12}, A_{D1})$, we have by utilizing P_{12} and H_1 that $a_{17} = a_{18} = a_{19} = a_{20} = 0$. By shifting this with P_{23} we get that $a_{71} = a_{72} = a_{74} = a_{75} = 0$ while $a_{73} = a_{16}$. Applying the rotation matrix R_3^θ on $S(E_{12}, A_{D1})$ gives

$$\begin{bmatrix} a_{16}\cos(2\theta) \\ 0 \\ 0 \\ 2\sin(2\theta) \end{bmatrix} = \begin{bmatrix} a_{16}\cos(2\theta) \\ 0 \\ 0 \\ -a_{16}\sin(2\theta) \end{bmatrix},$$

showing that $a_{16} = -2$ by choosing $\theta = \frac{\pi}{4}$. Acting with H_i on $S(E_{13}, A_{D1})$ for $i = 1, 2$ shows that we obtain $a_{41} = a_{43} = a_{44} = a_{45} = 0$, while using P_{13} gives that $a_{46} = a_{48} = a_{49} = a_{50} = 0$ as well as $a_{42} = a_{47}$. If we apply P_{23} to $S(E_{12}, A_{D1})$ we can now determine that $a_{42} = -1$. Using H_i on $S(E_{23}, A_{D1})$ for $i = 1, 2$ gives $a_{66} = a_{67} = a_{69} = a_{70} = 0$. Translating with P_{13} shows that $a_{21} = a_{68}$ and $a_{22} = a_{23} = a_{24} = a_{25} = 0$. Now that everything that is supposed to be zero is taken care of, we need only relate the final constants. Acting with R_3^θ on $S(E_{23}, A_{D2})$ gives the relation

$$\begin{bmatrix} 0 \\ \sin^3(\theta) + \sin(\theta) - a_8 \sin(\theta)\cos^2(\theta) \\ -\sin^2(\theta)\cos(\theta) + a_{68}\sin^2(\theta)\cos(\theta) - 2\cos(\theta) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\sin(\theta) \\ -2\cos(\theta) \\ 0 \\ 0 \end{bmatrix}.$$

The second row shows that $a_8 = -1$ while the third row reveals that $a_{68} = -1$. Summarizing, the mapping S has the coordinate form

$$S = \begin{pmatrix}
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}
\end{pmatrix}.$$

This is easily checked to be the coordinate form of the commutator. □

Lemma 5.6. *There is a unique (up to scaling) non-zero $O(3)$-invariant linear map in the cases (M2) and (M5).*

Proof. The cases (M2) and (M5) are both proved in [Gro16, Lemma A.3] and we will omit the proof of (M2). We will provide a new proof for the existence of a unique non-zero $O(3)$-invariant linear map in the case (M5) based on the Theorem of Highest Weight for $\mathfrak{sl}(2,\mathbb{C})$, see Theorem A.9. The Lie bracket is an invariant map since

$$a[B,C]a^{-1} = a(BC - CB)a^{-1} = aBa^{-1}aCa^{-1} - aCa^{-1}aBa^{-1} = [aBa^{-1}, aCa^{-1}],$$

for $a \in O(3)$ and $B, C \in \mathfrak{o}(3)$. We will show that the Lie bracket is the unique non-zero map up to scaling which is invariant under $SO(3)$, from which the result then follows for $O(3)$.

For an odd dimensional irreducible $SO(3)$-module V, we claim that $V \simeq V^*$, where V^* is the dual representation. This is because the odd dimensional irreducible representations of $SO(3)$ are in one-to one correspondence with the odd-dimensional irreducible representations of $\mathfrak{o}(3)$, which again is in one-to-one correspondence with the irreducible representations of its complexification $\mathfrak{sl}(2,\mathbb{C})$, see [Hal15, Section 4.7] for details. By the Theorem of Highest Weight, any two irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ of the same dimension are isomorphic, hence the same holds for $SO(3)$ for irreducible representations of odd degree. Thus as representation spaces,

$$\mathfrak{o}(3) \otimes \mathfrak{o}(3) \simeq \mathfrak{o}(3)^* \otimes \mathfrak{o}(3) \simeq \mathrm{End}(\mathfrak{o}(3)).$$

The action on $\mathrm{End}(\mathfrak{o}(3))$ is given by acting both in the domain and codomain, that is,

$$(a \cdot \phi)(B) = a\phi(aBa^{-1})a^{-1}.$$

Since $\mathfrak{o}(3)$ is semisimple it has trivial center. This implies, together with the Jacobi identity that the map $\mathrm{ad} : \mathfrak{o}(3) \to \mathrm{End}(\mathfrak{o}(3))$ given by $\mathrm{ad}_B(A) = [B, A]$ for $A, B \in \mathfrak{o}(3)$ is an isomorphism of Lie algebras onto its image, which we denote by $\mathrm{ad}(\mathfrak{o}(3))$. This is also an isomorphism of $SO(3)$-modules as

$$(a \cdot \mathrm{ad}_B)(C) = a[B, a^{-1}Ca]a^{-1} = aBa^{-1}C - CaBa^{-1} = [aBa^{-1}, C] = \mathrm{ad}_{a \cdot B}(C).$$

Hence the uniqueness of the original problem (M5) is guaranteed if we can show uniqueness of $SO(3)$-invariant maps

$$S : \mathrm{End}(\mathfrak{o}(3)) \longrightarrow \mathrm{ad}(\mathfrak{o}(3)).$$

As $SO(3)$ is compact it has the complete reducibility property, so we have

$$\mathrm{End}(\mathfrak{o}(3)) = \mathrm{ad}(\mathfrak{o}(3)) \oplus C,$$

where C is an invariant complement. As $\mathrm{ad}(\mathfrak{o}(3)) \simeq \mathfrak{o}(3)$ is irreducible as a $SO(3)$-module, the only map which intertwines these spaces is the projection: The kernel of any map

$$S : \mathrm{ad}(\mathfrak{o}(3)) \oplus C \longrightarrow \mathrm{ad}(\mathfrak{o}(3))$$

is invariant, and if any part of it ends up inside $\mathrm{ad}(\mathfrak{o}(3))$ we get a contradiction to irreducibility. Hence the kernel is exactly C and S is the projection from $\mathrm{End}(\mathfrak{o}(3))$ to $\mathrm{ad}(\mathfrak{o}(3))$. □

5.3 Statements of the Main Results

We will now state the main results of the thesis: the complete classification of sub-Riemannian model spaces with step and rank three. Recall from Theorem 5.1 that $\mathcal{C}_{3,3}$, $\mathcal{A}_{3,3}$, and $N[3,3]$ denote all the Carnot model spaces of step and rank three. The classification of sub-Riemannian model spaces of step and rank three is divided into cases based on their nilpotentization.

We can identify

$$\mathrm{Lie}(\mathcal{C}_{3,3}) \simeq \mathbb{R}^3 \oplus \mathfrak{o}(3) \oplus \mathbb{R}^3, \quad \mathrm{Lie}(\mathcal{A}_{3,3}) \simeq \mathbb{R}^3 \oplus \mathfrak{o}(3) \oplus \mathrm{Sym}^T(3) \tag{5.7}$$

as $O(\mathfrak{f}_1) \simeq O(3)$-modules by Theorem 5.1 and identification (5.4). The results in Theorem 5.3 show that the Lie bracket of $\mathrm{Lie}(\mathcal{C}_{3,3})$ after the identification (5.7) is given by

$$\left[\begin{pmatrix} x \\ A \\ u \end{pmatrix}, \begin{pmatrix} y \\ B \\ v \end{pmatrix}\right] = \begin{pmatrix} 0 \\ x \times y \\ Ay - Bx \end{pmatrix}, \tag{5.8}$$

for $x, y, u, v \in \mathbb{R}^3$ and $A, B \in \mathfrak{o}(3)$. Similarly, the Lie bracket of $\mathrm{Lie}(\mathcal{A}_{3,3})$ after the identification (5.7) is given by

$$\left[\begin{pmatrix} x_1 \\ y_1 \\ A_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ A_2 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ x_1 \times x_2 \\ \mathrm{Sym}(x_1, y_2) - \mathrm{Sym}(y_1, x_2) \end{pmatrix}, \tag{5.9}$$

for $x_1, x_2 \in \mathbb{R}^3$, $y_1, y_2 \in \mathfrak{o}(3)$, and $A_1, A_2 \in \mathrm{Sym}^T(3)$. Finally, the Lie bracket of $\mathrm{Lie}(N[3,3]) = \mathfrak{f}[3,3]$ is given by

$$\left[\begin{pmatrix} x_1 \\ y_1 \\ A_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ A_2 \\ z_2 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ x_1 \times x_2 \\ \mathrm{Sym}(x_1, y_2) - \mathrm{Sym}(y_1, x_2) \\ x_1 \times y_2 + y_1 \times x_2 \end{pmatrix}, \tag{5.10}$$

for $x_1, x_2, z_1, z_2 \in \mathbb{R}^3$, $y_1, y_2 \in \mathfrak{o}(3)$, and $A_1, A_2 \in \mathrm{Sym}^T(3)$. By noting that the map $(M2)$ in Theorem 5.3 can be identified with the cross product, the reader should feel confident in that (5.10) is compatible with (5.8) and (5.9). The sub-Riemannian model spaces with nilpotentization $\mathcal{C}_{3,3}$ was classified in [Gro16, Theorem 6.1] and have the following description.

Theorem 5.7. *Let $(Q, \mathcal{H}, g)$ be a sub-Riemannian model space with nilpotentization $Nil(Q) \simeq \mathcal{C}_{3,3}$ and fix $p \in Q$. Then $Q \simeq \mathcal{C}_{3,3}(a_1, a_2)$ for $(a_1, a_2) \in \mathbb{R}^2$, where $\mathcal{C}_{3,3}(a_1, a_2)$ is a model space with the following description: The Lie algebra $\mathfrak{g}$ of $G = Isom(\mathcal{C}_{3,3}(a_1, a_2))$ has the identification*

$$\mathfrak{g} \simeq \mathbb{R}^3 \oplus \mathfrak{o}(3) \oplus \mathbb{R}^3 \oplus \mathfrak{o}(3),$$

where the last $\mathfrak{o}(3)$-term is identified with the Lie algebra of the isotropy group at p. The Lie bracket between elements in $\mathfrak{g}$ is given by

$$\left[\begin{pmatrix} x \\ A \\ u \\ C \end{pmatrix}, \begin{pmatrix} y \\ B \\ v \\ D \end{pmatrix}\right] = \begin{pmatrix} a_1(Av - Bu) + Cy - Dx \\ x \times y + a_1(x \times v + u \times y) + a_1[A, B] + (a_1^2 + a_2) u \times w + [A, D] - [B, C] \\ Ay - Bx + a_1(Av - Bu) + Cv - Du \\ a_2(x \times v + u \times y) + a_2[A, B] + a_1 a_2 u \times v + [C, D] \end{pmatrix}.$$

Moreover, $\mathcal{C}_{3,3}(a_1, a_2)$ for $(a_1, a_2) \in \mathbb{R}^2$ form a non-isometric family of sub-Riemannian model spaces implying that $(Q, \mathcal{H}, g)$ is uniquely determined by the numbers a_1 and a_2.

Thus the model spaces $(Q, \mathcal{H}, g)$ with $\mathrm{Nil}(Q) \simeq \mathcal{C}_{3,3}$ are given by a two-parameter family. The proof of Theorem 5.7 will be omitted and can be found in [Gro16, Theorem 6.1], due to the fact that $\mathcal{C}_{3,3}$ was already known to be a model space prior to the complete classification of Carnot model spaces of step and rank three we gave in Theorem 5.1. Moreover, every aspect of the proof will be present when we provide the proofs for the other cases, only that they are more involved due to the increase in dimension. For the classification of sub-Riemannian model spaces with nilpotentization $\mathcal{A}_{3,3}$, we will prove the following result.

Theorem 5.8. *Let $(Q, \mathcal{H}, g)$ be a sub-Riemannian model space with nilpotentization $Nil(Q) \simeq \mathcal{A}_{3,3}$ and fix $p \in Q$. Then $Q \simeq \mathcal{A}_{3,3}(\kappa)$ for $\kappa \in \mathbb{R}$, where $\mathcal{A}_{3,3}(\kappa)$ is a model space with the following description: The Lie algebra $\mathfrak{g}$ of $G = Isom(\mathcal{A}_{3,3}(\kappa))$ has the identification*

$$\mathfrak{g} \simeq \mathbb{R}^3 \oplus \mathfrak{o}(3) \oplus Sym^T(3) \oplus \mathfrak{o}(3),$$

where the last $\mathfrak{o}(3)$-term is identified with the Lie algebra of the isotropy group at p. The Lie bracket between elements in $\mathfrak{g}$ is given by

$$\left[\begin{pmatrix} x_1 \\ y_1 \\ A_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ A_2 \\ w_2 \end{pmatrix}\right] = \begin{pmatrix} \kappa(x_1 \times y_2 + y_1 \times x_2) + \frac{24}{49}\kappa^2(A_2 y_1 - A_1 y_2) + x_1 \times w_2 + w_1 \times x_2 \\ x_1 \times x_2 + \frac{2}{7}\kappa y_1 \times y_2 - \frac{6}{7}\kappa(A_2 x_1 - A_1 x_2) + y_1 \times w_2 + w_1 \times y_2 \\ Sym(x_1, y_2) - Sym(y_1, x_2) + \frac{3}{7}\kappa([y_1, A_2] - [y_2, A_1]) + [A_1, w_2] - [A_2, w_1] \\ \frac{15}{49}\kappa^2 y_1 \times y_2 - \frac{144}{343}\kappa^3[A_1, A_2] + \frac{18}{49}\kappa^2(A_2 x_1 - A_1 x_2) + w_1 \times w_2 \end{pmatrix}.$$

Moreover, $\mathcal{A}_{3,3}(\kappa)$ for $\kappa \in \mathbb{R}$ form a non-isometric family of sub-Riemannian model spaces implying that $(Q, \mathcal{H}, g)$ is uniquely determined by the constant κ.

Notice that we use the identification (5.4) to identify $\mathfrak{o}(3)$ with $\mathbb{R}^3$ so that the symbol $\times$ indicates the usual cross product in $\mathbb{R}^3$ after the identification has been made. An important feature of the model spaces $(Q, \mathcal{H}, g)$ with $\text{Nil}(Q) \simeq \mathcal{A}_{3,3}$ is that they are parametrized by one parameter, not two as in the case of those with nilpotentization $\mathcal{C}_{3,3}$. We will refer to the model spaces with nilpotentization $\mathcal{A}_{3,3}$ as *model spaces of class* $(3, 6, 11)$ for convenience as this is their growth vector. The proof of Theorem 5.8 will be given in Section 5.4.

Finally, we are left with classifying the model spaces with free nilpotentization, that is, the model spaces $(Q, \mathcal{H}, g)$ with $\text{Nil}(Q) \simeq N[3, 3]$. Recall that all step two model spaces have free nilpotentization. Together with the naive assumption that there should be more spaces in higher dimensions, this might lead one to expect at least a two-parameter family of spaces with $\text{Nil}(Q) \simeq N[3, 3]$. As the following theorem shows, this conjecture is far from true.

Theorem 5.9. *For any sub-Riemannian model space $(Q, \mathcal{H}, g)$ with nilpotentization $Nil(Q) \simeq N[3, 3]$ we have*

$$Q \simeq N[3, 3].$$

We will provisionally refer to the model spaces with nilpotentization $N[3, 3]$ as *model spaces of class* $(3, 6, 14)$ for convenience. The proof of Theorem 4.9 will be given in Section 5.5.

5.4 Model Spaces of Class (3, 6, 11)

In this section we will focus fully on providing the classification of sub-Riemannian model spaces of class $(3, 6, 11)$ given in Theorem 5.8. The proof will have four main parts: Firstly, we will set up a correspondence between the model space structure on $(Q, \mathcal{H}, g)$ and a decomposition of the Lie algebra of its isometry group $G := \text{Isom}(Q)$. In contrast to the proof of Theorem 4.38, we do not know that the space $(Q, \mathcal{H}, g)$ has a Lie group structure and need to utilize the Lie group structure of the isometry group G. The correspondence will be influenced by our study of Riemannian symmetric spaces and the existence of a canonical partial connection on sub-Riemannian model spaces. Secondly, we will use the results developed in Section 5.2 to determine the structure of the Lie algebra of G, leaving us with several classes having concrete expressions. However, not all of these (and actually very few) will turn out to be Lie algebras. Thirdly, we will determine using the Jacobi identity which of the possible structures are in fact Lie algebras. Lastly, we will show that our construction determines the sub-Riemannian model spaces of class $(3, 6, 11)$ uniquely.

Proof (Theorem 5.8). Let $(Q, \mathcal{H}, g)$ be a sub-Riemannian model space of class $(3, 6, 11)$ and let G denote its isometry group. Fix a point $p \in Q$ and let $K := K_p$ be the isotropy group corresponding to p. The notation $\mathfrak{g}$ and $\mathfrak{k}$ indicates as usual the Lie algebras of G and K, respectively, while $\pi : G \to Q$ denotes the projection sending Φ to $\pi(\Phi) = \Phi(p)$. Inspired by Riemannian symmetric spaces, we let $\sigma : Q \to Q$ be the unique isometry such that

$$d\sigma\Big|_{\mathcal{H}_p} = -Id_{\mathcal{H}_p}.$$

It follows from Proposition 4.21 that $\sigma^{-1} = \sigma$ since σ^2 restricts to the identity on $\mathcal{H}_p$. A straightforward modification of Lemma 2.44 and Proposition 2.45 gives an eigenvalue decomposition

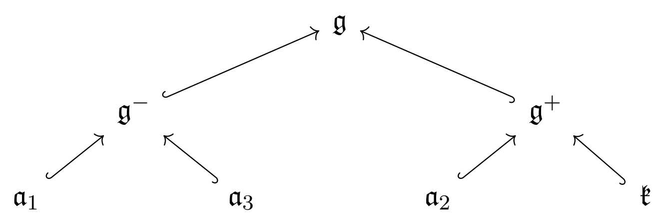
$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-, \quad \mathfrak{k} \subset \mathfrak{g}^+.$$

We will now use the canonical partial connection $\nabla^{\mathcal{H}}$ to give a decomposition of $\mathfrak{g}$. Let $\mathfrak{a}_1 \subset \mathfrak{g}^-$ be the subspace corresponding to the canonical partial connection $\nabla^{\mathcal{H}}$ as described in the proof of Theorem 4.26. As $\mathfrak{a}_1$ is invariant under the action of the compact group K we have that there exists an invariant complement $\mathfrak{a}_3 \subset \mathfrak{g}^-$ of $\mathfrak{a}_1$. We define

$$\mathfrak{a}_2 = [\mathfrak{a}_1, \mathfrak{a}_1] \subset \mathfrak{g}^+.$$

Notice that $\mathcal{H}_p^2 \subset d\pi_e(\mathfrak{a}_1 + \mathfrak{a}_2)$. However, the reverse inclusion also holds because $\mathrm{Nil}(Q) \simeq \mathcal{A}_{3,3}$. This implies that $\mathfrak{a}_2$ is transverse to $\mathfrak{k} = \ker(d\pi_e)$ inside $\mathfrak{g}^+$. Summarizing, we have decomposed

$$\mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3 \oplus \mathfrak{k}$$



Let us fix an orthonormal basis for $\mathcal{H}_p$ by choosing a linear isometry $\phi : \mathbb{R}^3 \to \mathcal{H}_p$. Then

$$\phi^{-1} \circ d\pi_e\Big|_{\mathfrak{a}_1} : \mathfrak{a}_1 \longrightarrow \mathbb{R}^3$$

identifies $\mathfrak{a}_1$ with $\mathbb{R}^3$, $\mathfrak{a}_2$ with $\mathfrak{o}(3)$, and $\mathfrak{a}_3$ with $\mathrm{Sym}^T(3)$ as vector spaces since $\mathrm{Nil}(Q) \simeq \mathcal{A}_{3,3}$. Recall that any isometry $\Phi \in K$ is uniquely determined by $d\Phi_p \in O(\mathcal{H}_p) \simeq O(3)$ by Proposition 4.21. We get an identification $K \simeq O(3)$ as Lie groups since $(Q, \mathcal{H}, g)$ is a model space. This identifies the Lie algebra $\mathfrak{k}$ of K with the Lie algebra $\mathfrak{o}(3)$ of $O(3)$. An element in $\mathfrak{g}$ will be denoted by (x, y, A, w) according to the above decomposition with $x \in \mathbb{R}^3$, $A \in \mathrm{Sym}^T(3)$, and $y, w \in \mathfrak{o}(3)$. Under these identifications, $K \simeq O(3)$ acts on $\mathfrak{g}$ by

$$a \cdot (x, y, A, w) = (a \cdot x, a \cdot y, a \cdot A, a \cdot w), \quad a \in O(3).$$

Here the action on $\mathbb{R}^3$, $\mathfrak{o}(3)$, and $\mathrm{Sym}^T(3)$ is given by matrix multiplication, the adjoint representation, and the adjoint representation with determinant term, respectively.

Thus we transformed the problem into classifying Lie algebras

$$\mathfrak{g} = \mathbb{R}^3 \oplus \mathfrak{o}(3) \oplus \mathrm{Sym}^T(3) \oplus \mathfrak{o}(3)$$

(up to isomorphism) with the following properties:

- The Lie brackets are invariant under $O(3)$, that is,

$$a \cdot [(x_1, y_1, A_1, w_1), (x_2, y_2, A_2, w_2)] = [a \cdot (x_1, y_1, A_1, w_1), a \cdot (x_2, y_2, A_2, w_2)], \tag{P1}$$

  for $a \in O(3)$.

- By construction, we have

$$[\mathfrak{a}_i, \mathfrak{k}] \subset \mathfrak{a}_i, \quad i = 1, 2, 3. \tag{P2}$$

- The choice $\mathfrak{a}_2 = [\mathfrak{a}_1, \mathfrak{a}_1]$ gives together with the Lie brackets in (5.9) that

$$[(x_1, 0, 0, 0), (x_2, 0, 0, 0)] = (0, x_1 \times x_2, 0, 0). \tag{P3}$$

- Since the Lie bracket of $\mathfrak{k} \simeq \mathfrak{o}(3)$ can be identified with the cross product through (5.4), it follows that

$$[(0,0,0,w_1),(0,0,0,w_2)] = (0,0,0,w_1 \times w_2). \tag{P4}$$

- Again due to $\mathrm{Nil}(Q) \simeq \mathcal{A}_{3,3}$ it follows from (5.9) that

$$Pr_{\mathfrak{a}_3}[(x,0,0,0),(0,y,0,0)] = (0,0,\mathrm{Sym}(x,y),0). \tag{P5}$$

We will now use the results obtained in Theorem 5.3 to put further restrictions on the Lie bracket based on property (P1). We often exclude zeroes from our brackets to avoid excessive notation, e.g. denoting the bracket $[(0,0,A,0),(0,0,0,w)]$ simply by $[A,w]$ for convenience. To avoid confusion, we will be consistent with denoting elements of $\mathfrak{a}_1$ by x or x_j for $j \in \{1,2\}$. Similar conventions will be used for $\mathfrak{a}_2, \mathfrak{a}_3$, and $\mathfrak{k}$ based on the letters y, A, and w, respectively. The bracket between two elements in the Lie algebra $\mathfrak{g}$ is given by

$$\left[\begin{pmatrix} x_1 \\ y_1 \\ A_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ A_2 \\ w_2 \end{pmatrix}\right] = \begin{array}{c} [x_1,x_2]+[y_1,y_2]+[A_1,A_2]+[w_1,w_2]+[x_1,y_2]+[y_1,A_2]+[A_1,w_2] \\ +[x_1,A_2]+[y_1,w_2]+[x_1,w_2]+\{\text{skew-symmetric terms}\}. \end{array}$$

We have already determined $[x_1,x_2]$ and $[w_1,w_2]$, which leaves us with the remaining eight terms written above. It follows from property (P2) together with Theorem 5.3 that

$$[x_1,w_2] = (x_1 \times w_2,0,0,0),$$
$$[y_1,w_2] = (0,y_1 \times w_2,0,0),$$
$$[A_1,w_2] = (0,0,[A_1,w_2],0).$$

The reason these terms do not contain arbitrary constants is because we can apply the Jacobi identity. To illustrate, assume $[y_1,w_2] = (0,\alpha y_1 \times w_2,0,0)$ for $\alpha \in \mathbb{R}$. Expanding the Jacobi identity

$$[w_1,[y,w_2]] + [y,[w_2,w_1]] + [w_2,[w_1,y]] = 0$$

gives the equation $\alpha^2 = \alpha$, forcing $\alpha = 1$. Let $c_1,\dots,c_9$ be arbitrary real constants. By considering the bracket $[y_1,y_2]$ together with Equation (M5) of Theorem 5.3 shows that

$$[y_1,y_2] = (0,c_1 y_1 \times y_2,0,c_2 y_1 \times y_2).$$

We have identified the commutator between elements of $\mathfrak{o}(3)$ with the cross product through the identification (5.4). Similarly, we get

$$[A_1,A_2] = (0,c_3[A_1,A_2],0,c_4[A_1,A_2])$$

from Equation (M8) of Theorem 5.3. Keeping property (P5) in mind we have

$$[x_1,y_2] = (c_5 x_1 \times y_2,0,\mathrm{Sym}(x_1,y_2),0).$$

Finally, the last two terms are

$$[y_1,A_2] = (c_6 A_2 y_1,0,c_7[y_1,A_2],0),$$
$$[x_1,A_2] = (0,c_8 A_2 x_1,0,c_9 A_2 x_1).$$

Here the identification (5.4) is in full use: The notation $A_2 y_1$ denotes matrix multiplication when thinking of y_1 as a vector in $\mathbb{R}^3$ while $[y_1,A_2]$ is the bracket when considering y_1 as a skew-symmetric matrix.

Moreover, pay attention to that we need to invoke Proposition 5.2 and Theorem 5.3 for each of these terms as well. Summarizing, the bracket between two arbitrary elements in $\mathfrak{g}$ is given by

$$\left[\begin{pmatrix} x_1 \\ y_1 \\ A_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ A_2 \\ w_2 \end{pmatrix}\right] = \begin{pmatrix} c_5(x_1 \times y_2 + y_1 \times x_2) + c_6(A_2 y_1 - A_1 y_2) + x_1 \times w_2 + w_1 \times x_2 \\ x_1 \times x_2 + c_1 y_1 \times y_2 + c_3[A_1, A_2] + c_8(A_2 x_1 - A_1 x_2) + y_1 \times w_2 + w_1 \times y_2 \\ \mathrm{Sym}(x_1, y_2) - \mathrm{Sym}(y_1, x_2) + c_7([y_1, A_2] - [y_2, A_1]) + [A_1, w_2] - [A_2, w_1] \\ c_2 y_1 \times y_2 + c_4[A_1, A_2] + c_9(A_2 x_1 - A_1 x_2) + w_1 \times w_2 \end{pmatrix} \tag{5.11}$$

We are now done with giving conditions on the structure of $\mathfrak{g}$ based on $(Q, \mathcal{H}, g)$ having nilpotentization $\mathcal{A}_{3,3}$ and the theory developed in Section 5.2. Although the bracket (5.11) is bilinear and skew-symmetric by construction, it does not necessarily satisfy the Jacobi identity for arbitrary constants $c_1, \ldots, c_9$. We will start by imposing constraints on these constants by forcing the Jacobi identity to hold for carefully selected basis elements. Recall that we denote the standard basis in $\mathbb{R}^3$ by e_1, e_2, e_3 and use the notation for the standard basis of $\mathrm{Sym}^T(3)$ given in the proof of Theorem 5.1. To achieve some simple relations we let

$$v_1 = \begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} e_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ e_1 \\ 0 \\ 0 \end{pmatrix}.$$

Then we obtain

$$[v_1, v_2] = \begin{pmatrix} 0 \\ e_3 \\ 0 \\ 0 \end{pmatrix}, \quad [v_2, v_3] = \begin{pmatrix} -c_5 e_3 \\ 0 \\ \frac{1}{2}A_{12} \\ 0 \end{pmatrix}, \quad [v_1, v_3] = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{3}A_{D1} + \frac{1}{3}A_{D2} \\ 0 \end{pmatrix}.$$

The Jacobi identity becomes

$$\left[\begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -c_5 e_3 \\ 0 \\ \frac{1}{2}A_{12} \\ 0 \end{pmatrix}\right] - \left[\begin{pmatrix} e_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{2}{3}A_{D1} + \frac{1}{3}A_{D2} \\ 0 \end{pmatrix}\right] + \left[\begin{pmatrix} 0 \\ e_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_3 \\ 0 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

By computing the final brackets we acquire the equations

$$\frac{5}{6}c_9 = c_2, \tag{E1}$$

$$c_5 + \frac{5}{6}c_8 = c_1. \tag{E2}$$

As the next four computations are of a similar nature, we will gradually provide fewer details. Let

$$v_1 = \begin{pmatrix} 0 \\ e_1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ e_2 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Writing out the Jacobi identity gives

$$\left[\begin{pmatrix} 0 \\ e_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -c_5 e_3 \\ 0 \\ -\frac{1}{2}A_{12} \\ 0 \end{pmatrix}\right] + \left[\begin{pmatrix} 0 \\ e_2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{2}{3}A_{D1} + \frac{1}{3}A_{D2} \\ 0 \end{pmatrix}\right] + \left[\begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ c_1 e_3 \\ 0 \\ c_2 e_3 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

From this we extract the equation $3c_7 = c_5 + c_1$. Using Equation (E2) we can rewrite this as

$$3c_7 = 2c_1 - \frac{5}{6}e_8. \tag{E3}$$

Next we start to involve $\mathrm{Sym}^T(3)$ and consider

$$v_1 = \begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ e_1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ A_{12} \\ 0 \end{pmatrix}.$$

The Jacobi identity then becomes

$$\left[\begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c_6 e_2 \\ 0 \\ c_7 A_{13} \\ 0 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 \\ e_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ c_8 e_2 \\ 0 \\ c_9 e_2 \end{pmatrix}\right] + \left[\begin{pmatrix} 0 \\ 0 \\ A_{12} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{2}{3}A_{D1} + \frac{1}{3}A_{D2} \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This time the equations we get are

$$c_3 = c_1 c_8 + c_9 - c_6 - c_7 c_8, \tag{E4}$$
$$c_4 = c_2 c_8 - c_7 c_9. \tag{E5}$$

By letting

$$v_1 = \begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} e_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ A_{13} \\ 0 \end{pmatrix},$$

we obtain the equations

$$c_6 = -c_5 c_8 - c_9, \tag{E6}$$
$$c_7 = -\frac{1}{2}c_8. \tag{E7}$$

Notice that we can use equations (E1) - (E7) to write all the coefficients in terms of c_1 and c_2. Finally, to obtain a dependence between c_1 and c_2 we consider

$$v_1 = \begin{pmatrix} 0 \\ e_1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ e_2 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ A_{13} \\ 0 \end{pmatrix}.$$

The Jacobi identity gives the equation

$$c_2 = c_7^2 - c_1 c_7 + \frac{1}{2}c_6. \tag{E8}$$

It is now straightforward to use equations (E1) - (E8) to write the coefficients $c_1, \ldots, c_9$ in terms of a single coefficient. We rename $\kappa := c_5$ and get after some straightforward manipulations the equations

$$c_1 = \frac{2}{7}\kappa, \quad c_2 = \frac{15}{49}\kappa^2, \quad c_3 = 0, \quad c_4 = -\frac{144}{343}\kappa^3,$$
$$c_5 = \kappa, \quad c_6 = \frac{24}{49}\kappa^2, \quad c_7 = \frac{3}{7}\kappa, \quad c_8 = -\frac{6}{7}\kappa, \quad c_9 = \frac{18}{49}\kappa^2.$$

Inserting this back into the Lie bracket (5.11) gives

$$\left[\begin{pmatrix} x_1 \\ y_1 \\ A_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ A_2 \\ w_2 \end{pmatrix}\right] = \begin{pmatrix} \kappa(x_1 \times y_2 + y_1 \times x_2) + \frac{24}{49}\kappa^2(A_2 y_1 - A_1 y_2) + x_1 \times w_2 + w_1 \times x_2 \\ x_1 \times x_2 + \frac{2}{7}\kappa y_1 \times y_2 - \frac{6}{7}\kappa(A_2 x_1 - A_1 x_2) + y_1 \times w_2 + w_1 \times y_2 \\ \mathrm{Sym}(x_1, y_2) - \mathrm{Sym}(y_1, x_2) + \frac{3}{7}\kappa([y_1, A_2] - [y_2, A_1]) + [A_1, w_2] - [A_2, w_1] \\ \frac{15}{49}\kappa^2 y_1 \times y_2 - \frac{144}{343}\kappa^3[A_1, A_2] + \frac{18}{49}\kappa^2(A_2 x_1 - A_1 x_2) + w_1 \times w_2 \end{pmatrix}.$$

To see that this in fact satisfies the Jacobi identity one simply has to observe that everything cancels when expanding the identity. However, we will leave this to the reader as it is very tedious and results in little conceptual understanding. Using a symbolic computing environment such as MAPLE to check this is highly recommended.

We will now show that the constant κ uniquely determines the sub-Riemannian model spaces of class $(3, 6, 11)$. Let us use the temporary notation $\mathfrak{g}(\kappa_i)$ for the Lie algebra of $\mathrm{Isom}(Q_i)$, where $(Q_i, \mathcal{H}^i, g_i)$ are sub-Riemannian model spaces of class $(3, 6, 11)$ and $i = 1, 2$. Assume there exists an isometry $\Phi : Q_1 \to Q_2$. The discussion at the end of Section 4.4 implies that we get an induced Lie algebra map

$$\varphi := d\overline{\Phi} : \mathfrak{g}(\kappa_1) = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3 \oplus \mathfrak{k} \longrightarrow \mathfrak{g}(\kappa_2) = \widetilde{\mathfrak{a}_1} \oplus \widetilde{\mathfrak{a}_2} \oplus \widetilde{\mathfrak{a}_3} \oplus \widetilde{\mathfrak{k}}$$

that maps $\mathfrak{a}_1$ isometrically onto $\widetilde{\mathfrak{a}_1}$ and $\varphi(\mathfrak{k}) = \widetilde{\mathfrak{k}}$. As $[\mathfrak{a}_1, \mathfrak{a}_1] = \mathfrak{a}_2$ and similarly for $\widetilde{\mathfrak{a}_2}$, we get that after identifying $\mathfrak{a}_1$ with $\widetilde{\mathfrak{a}_1}$ through φ that $\mathfrak{a}_2$ gets identified with $\widetilde{\mathfrak{a}_2}$ as well. However, we can not conclude that $\mathfrak{a}_3$ is mapped onto $\widetilde{\mathfrak{a}_3}$. Nevertheless, we know that φ will map $[\mathfrak{a}_1, \mathfrak{a}_2]$ into $[\widetilde{\mathfrak{a}_1}, \widetilde{\mathfrak{a}_2}]$. As $\mathfrak{a}_3 \subset [\mathfrak{a}_1, \mathfrak{a}_2]$ and similarly for $\widetilde{\mathfrak{a}_3}$, we can at least ensure that

$$\varphi\Big|_{\mathfrak{a}_3} : \mathfrak{a}_3 \longrightarrow [\widetilde{\mathfrak{a}_1}, \widetilde{\mathfrak{a}_2}] \subset \widetilde{\mathfrak{a}_1} \oplus \widetilde{\mathfrak{a}_3}.$$

If $A \in \mathfrak{a}_3$ we will use the notation $\varphi(A)_i$ according to the decomposition of $\varphi(\mathfrak{a}_3) \subset \widetilde{\mathfrak{a}_1} \oplus \widetilde{\mathfrak{a}_3}$, for $i = 1, 3$. For $x \in \mathfrak{a}_1$ and $y \in \mathfrak{a}_2$, we have

$$\varphi([x, y]) = \varphi \begin{pmatrix} \kappa_1 x \times y \\ 0 \\ \mathrm{Sym}(x, y) \\ 0 \end{pmatrix} = \begin{pmatrix} \kappa_1 \varphi(x) \times \varphi(y) + \varphi(\mathrm{Sym}(x, y))_1 \\ 0 \\ \varphi(\mathrm{Sym}(x, y))_3 \\ 0 \end{pmatrix}.$$

In the last equality, we used that φ is invariant and maps $\mathfrak{a}_1$ onto $\widetilde{\mathfrak{a}_1}$. On the other hand, we also have that

$$[\varphi(x), \varphi(y)] = \begin{pmatrix} \kappa_2 \varphi(x) \times \varphi(y) \\ 0 \\ \mathrm{Sym}(\varphi(x), \varphi(y)) \\ 0 \end{pmatrix},$$

where we used that φ sends $\mathfrak{a}_2$ onto $\widetilde{\mathfrak{a}_2}$. Rewriting the first row gives

$$(\kappa_2 - \kappa_1)\varphi(x) \times \varphi(y) = \varphi(\mathrm{Sym}(x, y))_1 . \tag{5.12}$$

However, notice that the left hand side of Equation (5.12) is skew-symmetric while the right hand side is symmetric. Thus both sides are identically zero and

$$\kappa_1 \varphi(x) \times \varphi(y) = \kappa_2 \varphi(x) \times \varphi(y).$$

It follows from picking $x \in \mathfrak{a}_1$ and $y \in \mathfrak{a}_2$ such that $\varphi(x \times y) = \varphi(x) \times \varphi(y) \neq 0$ that $\kappa_1 = \kappa_2$. Thus if $\kappa_1 \neq \kappa_2$ it follows that $Q_1 \not\cong Q_2$. Hence different values of $\kappa \in \mathbb{R}$ parametrize a non-isometric family $\mathcal{A}_{3,3}(\kappa)$ of class $(3, 6, 11)$ model spaces. Through the construction we have given it is clear that every model space of class $(3, 6, 11)$ have the form presented in Theorem 5.8. $\square$

5.5 Model Spaces of Class (3, 6, 14)

We will in this section provide the classification of all sub-Riemannian model spaces of class $(3,6,14)$. The result is given in Theorem 5.9 and states that the only sub-Riemannian model space of step and rank three with free nilpotentization is the free nilpotent Lie group $N[3,3]$. It will be apparent that the proof has a similar structure as in the $(3,6,11)$-classification. Hence we will provide fewer details at places where the arguments are repetitions of those given in the $(3,6,11)$-classification. Except for some changes and observations, the major difference is that the equations we obtain when trying to satisfy the Jacobi identity will be enough to show that there are no model spaces of class $(3,6,14)$ except for the free nilpotent Lie group $N[3,3]$.

Proof (Theorem 5.9). Let $(Q,\mathcal{H},g)$ denote a sub-Riemannian model space of class $(3,6,14)$. We will use the same notation as introduced in the beginning of Section 5.4. Similarly as before, we get an eigenvalue decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ of the Lie algebra of the isometry group. The canonical partial connection $\nabla^{\mathcal{H}}$ is again used to obtain a subspace $\mathfrak{a}_1 \subset \mathfrak{g}$ which is invariant under the action of the isotropy group K. We define $\mathfrak{a}_2 = [\mathfrak{a}_1,\mathfrak{a}_1]$ and note that $\mathfrak{a}_2$ is transverse to $\mathfrak{a}_1$ since $\mathfrak{a}_1 \subset \mathfrak{g}^-$ while $\mathfrak{a}_2 \subset \mathfrak{g}^+$. Moreover, the argument presented in the proof of Theorem 5.8 carries over to show that $\mathfrak{a}_2$ is transverse to $\mathfrak{k}$.

Define $\widetilde{\mathfrak{a}_3} = [\mathfrak{a}_1,\mathfrak{a}_2] \subset \mathfrak{g}^-$. Then $\widetilde{\mathfrak{a}_3}$ is eight-dimensional since

$$d\pi_e(\mathfrak{a}_1 + \mathfrak{a}_2 + \widetilde{\mathfrak{a}_3}) = T_pQ.$$

The subspace $\widetilde{\mathfrak{a}_3}$ is clearly transverse to both $\mathfrak{a}_2$ and $\mathfrak{k}$ due to the eigenvalue decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$. A nonempty intersection of $\mathfrak{a}_1$ and $\widetilde{\mathfrak{a}_3}$ would contradict the properties

$$\dim(\mathfrak{a}_1 + \mathfrak{a}_2 + \widetilde{\mathfrak{a}_3}) < 14, \quad \dim(T_pQ) = 14,$$

showing that $\widetilde{\mathfrak{a}_3}$ is transverse to $\mathfrak{a}_1$ as well. We emphasize to the reader that this argument only works because $\mathrm{Nil}(Q) \simeq N[3,3]$, and should be compared with the different strategy used in the proof of Theorem 5.8.

Similarly as in the proof of Theorem 5.8, we get an identification $K \simeq O(3)$ by fixing an orthonormal basis for $\mathcal{H}_p$. In the same vein, $\mathfrak{a}_i \simeq \mathfrak{f}_i$ for $i = 1,2$ and $\widetilde{\mathfrak{a}_3} \simeq \mathfrak{f}_3$ as representations, where $\mathfrak{f}_i$ denotes the i'th layer of the free nilpotent Lie algebra $\mathfrak{f}[3,3]$. We can as usual identify $\mathfrak{a}_1$ with $\mathbb{R}^3$ and $\mathfrak{a}_2$ with $\mathfrak{o}(3)$. Finally, we use the concrete description of the action on $\mathfrak{f}_3$ given in Theorem 5.1 to decompose $\widetilde{\mathfrak{a}_3} = \mathfrak{a}_3 \oplus \mathfrak{a}_4$ as representation spaces, where $\mathfrak{a}_3 \simeq \mathrm{Sym}^T(3)$ and $\mathfrak{a}_4 \simeq \mathfrak{o}(3)$. We use (5.4) to write $\mathfrak{a}_4 \simeq \mathbb{R}^3$, where the action of $O(3)$ on $\mathbb{R}^3$ is the standard action. Summarizing these identifications gives

$$\mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3 \oplus \mathfrak{a}_4 \oplus \mathfrak{k} \simeq \mathbb{R}^3 \oplus \mathfrak{o}(3) \oplus \mathrm{Sym}^T(3) \oplus \mathbb{R}^3 \oplus \mathfrak{o}(3) \tag{5.13}$$

as a $K \simeq O(3)$-module. We will denote an arbitrary element in $\mathfrak{g}$ by (x,y,A,z,w) according to the decomposition (5.13). The Lie bracket of $\mathfrak{g}$ satisfies obvious modifications of the properties (P1) - (P4) presented in the proof of Theorem 5.8 for the same reasons as before. Since we were able to choose $\widetilde{\mathfrak{a}_3} = [\mathfrak{a}_1,\mathfrak{a}_2]$ due to having free nilpotentization it follows from (5.10) that

$$[(x,0,0,0,0),(0,y,0,0,0)] = (0,0,\mathrm{Sym}(x,y),x \times y,0). \tag{5.14}$$

Hence property (P5) presented in in the proof of Theorem 5.8 holds without the projection to $\widetilde{\mathfrak{a}_3}$ for spaces of class $(3,6,14)$.

Similarly as in the proof of Theorem 5.8, we will exclude zeroes from the bracket notation and use reasonable notation to avoid confusion about where elements belong. Moreover, we will use the identification (5.4) without mention from now on whenever convenient. The bracket between two elements in the Lie algebra is given by

$$\left[\begin{pmatrix} x_1 \\ y_1 \\ A_1 \\ z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ A_2 \\ z_2 \\ w_2 \end{pmatrix}\right] = \begin{array}{l} [x_1, x_2] + [y_1, y_2] + [A_1, A_2] + [z_1, z_2] + [w_1, w_2] + [x_1, y_2] + [y_1, A_2] + [A_1, z_2] + [z_1, w_2] \\ \\ + [x_1, A_2] + [y_1, z_2] + [A_1, w_2] + [x_1, z_2] + [y_1, w_2] + [x_1, w_2] + \{\text{skew-symmetric terms}\}. \end{array}$$

In the following, the symbols $a_1, a_2, b_1, \ldots, b_6, c_1, c_2, d_1, d_2, f_1, \ldots, f_6$ are arbitrary real constants. The terms $[x_1, x_2]$ and $[w_1, w_2]$ are given through the modifications of (P3) and (P4), while the term $[x_1, y_2]$ is given by (5.14). With the use of Theorem 5.3 together with the eigenvalue decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ we have that

$$\begin{aligned} [y_1, y_2] &= (0, b_1 y_1 \times y_2, 0, 0, f_1 y_1 \times y_2), \\ [A_1, A_2] &= (0, b_2 [A_1, A_2], 0, 0, f_2 [A_1, A_2]), \\ [z_1, z_2] &= (0, b_3 z_1 \times z_2, 0, 0, f_3 z_1 \times z_2), \\ [A_1, z_2] &= (0, b_4 A_1 z_2, 0, 0, f_4 A_1 z_2), \\ [x_1, A_2] &= (0, b_5 A_2 x_1, 0, 0, f_5 A_2 x_1), \\ [x_1, z_2] &= (0, b_6 x_1 \times z_2, 0, 0, f_6 x_1 \times z_2). \end{aligned}$$

The fact that $[\mathfrak{a}_i, \mathfrak{k}] \subset \mathfrak{a}_i$ for $i = 1, 2, 3, 4$ together with Theorem 5.3 shows that

$$\begin{aligned} [x_1, w_2] &= (x_1 \times w_2, 0, 0, 0, 0), \\ [z_1, w_2] &= (0, 0, 0, z_1 \times w_2, 0), \\ [A_1, w_2] &= (0, 0, [A_1, w_2], 0, 0), \\ [y_1, w_2] &= (0, y_1 \times w_2, 0, 0, 0). \end{aligned}$$

We do not end up with arbitrary constants in these four cases. This is due to the Jacobi identity in the same way as in the proof of Theorem 5.8. There are two terms left which have not been determined, and another application of Theorem 5.3 shows that

$$\begin{aligned} [y_1, A_2] &= (a_1 A_2 y_1, 0, c_1 [y_1, A_2], d_1 A_2 y_1, 0), \\ [y_1, z_2] &= (a_2 y_1 \times z_2, 0, c_2 \mathrm{Sym}(y_1, z_2), d_2 y_1 \times z_2, 0). \end{aligned}$$

The Lie bracket between two arbitrary elements is thus given by

$$\begin{aligned} \left[\begin{pmatrix} x_1 \\ y_1 \\ A_1 \\ z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ A_2 \\ z_2 \\ w_2 \end{pmatrix}\right] &= \begin{pmatrix} a_1 A_2 y_1 + a_2 y_1 \times z_2 + x_1 \times w_2 \\ x_1 \times x_2 + b_1 y_1 \times y_2 + b_2 [A_1, A_2] + b_3 z_1 \times z_2 + b_4 A_1 z_2 + b_5 A_2 x_1 + b_6 x_1 \times z_2 + y_1 \times w_2 \\ \mathrm{Sym}(x_1, y_2) + c_1 [y_1, A_2] + c_2 \mathrm{Sym}(y_1, z_2) + [A_1, w_2] \\ x_1 \times y_2 + d_1 A_2 y_1 + d_2 y_1 \times z_2 + z_1 \times w_2 \\ f_1 y_1 \times y_2 + f_2 [A_1, A_2] + f_3 z_1 \times z_2 + f_4 A_1 z_2 + f_5 A_2 x_1 + f_6 x_1 \times z_2 + w_1 \times w_2 \end{pmatrix} \\ &\quad - \begin{pmatrix} a_1 A_1 y_2 + a_2 y_2 \times z_1 + x_2 \times w_1 \\ b_4 A_2 z_1 + b_5 A_1 x_2 + b_6 x_2 \times z_1 + y_2 \times w_1 \\ \mathrm{Sym}(y_1, x_2) + c_1 [y_2, A_1] + c_2 \mathrm{Sym}(z_1, y_2) + [A_2, w_1] \\ x_2 \times y_1 + d_1 A_1 y_2 + d_2 y_2 \times z_1 + z_2 \times w_1 \\ f_4 A_2 z_1 + f_5 A_1 x_2 + f_6 x_2 \times z_1 \end{pmatrix}. \end{aligned}$$

We will now derive no less than eighteen equations by using the Jacobi identity. As their derivations are straightforward and already illustrated in the proof of Theorem 5.8, we will only provide an outline and not the explicit calculations. From the obtained equations we will show that all the constants present in the Lie bracket are in fact zero. If we start by choosing

$$v_1 = \begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} e_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

then the Jacobi identity becomes

$$\left[\begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}A_{12} \\ -e_3 \\ 0 \end{pmatrix}\right] - \left[\begin{pmatrix} e_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{2}{3}A_{D1} + \frac{1}{3}A_{D2} \\ 0 \\ 0 \end{pmatrix}\right] + \left[\begin{pmatrix} 0 \\ e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Computing the final brackets gives the equations

$$b_1 = b_6 + \frac{5}{6}b_5, \qquad \text{(Eq1)}$$

$$f_1 = f_6 + \frac{5}{6}f_5, \qquad \text{(Eq2)}$$

by looking at the second and fifth row. A similar computation by using the vectors

$$v_1 = \begin{pmatrix} 0 \\ e_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ e_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

gives by looking at the third row the equation

$$b_1 = 3c_1 + c_2. \qquad \text{(Eq3)}$$

We now start to involve the symmetric matrices and choose

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ A_{12} \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ A_{23} \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_1 \\ 0 \end{pmatrix}.$$

The Jacobi identity becomes

$$\left[\begin{pmatrix} 0 \\ 0 \\ A_{23} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b_4e_2 \\ 0 \\ 0 \\ f_4e_2 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ e_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b_2e_2 \\ 0 \\ 0 \\ f_2e_2 \end{pmatrix}\right],$$

and the equations we obtain are

$$f_2 = -d_1b_4 - d_2b_2, \qquad \text{(Eq4)}$$

$$f_4 = \frac{1}{2}c_2b_2 - c_1b_4. \qquad \text{(Eq5)}$$

We will provide fewer details for the rest of the equations as the computations are similar. By choosing

$$v_1 = \begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ A_{12} \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_1 \\ 0 \end{pmatrix},$$

we obtain the equations

$$f_4 = -b_5 a_2, \tag{Eq6}$$
$$b_4 = c_2 b_5, \tag{Eq7}$$
$$f_5 = -d_2 b_5 - b_4. \tag{Eq8}$$

If we take

$$v_1 = \begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} e_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ A_{13} \\ 0 \\ 0 \end{pmatrix},$$

then the equations become

$$f_5 = -a_1, \tag{Eq9}$$
$$b_5 = -2c_1, \tag{Eq10}$$
$$b_5 = -d_1. \tag{Eq11}$$

Reversing this, we choose two of the elements in $\mathrm{Sym}^T(3)$,

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ A_{12} \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ A_{13} \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The equation we obtain by looking at the third row is

$$b_2 = -6(f_5 + c_1 b_5). \tag{Eq12}$$

Finally, we will derive the last six equations by looking at the interplay between the first and fourth layer. Choosing

$$v_1 = \begin{pmatrix} e_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} e_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_2 \\ 0 \end{pmatrix},$$

give the equations

$$f_6 = a_2, \tag{Eq13}$$
$$b_6 = -c_2, \tag{Eq14}$$
$$b_6 = d_2. \tag{Eq15}$$

Reversing this, we choose

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_2 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} e_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and obtain

$$f_3 = a_2 b_6, \tag{Eq16}$$

$$b_3 = -c_2 b_6, \tag{Eq17}$$

$$f_6 = b_3 - d_2 b_6. \tag{Eq18}$$

A careful look at the formulas (Eq1)-(Eq18) reveals that all the constants have to be zero once we have showed that $c_1 = c_2 = 0$. To show this we proceed as follows: The equations (Eq1) and (Eq3) combine to give

$$3c_1 + c_2 = b_6 + \frac{5}{6} b_5.$$

Together with the formulas (Eq10) and (Eq14) this shows that

$$c_1 = -\frac{3}{7} c_2.$$

We will now show that $c_2 = 0$, from which the rest of the constants will follow.

Writing out (Eq18) by applying (Eq17), (Eq15), and (Eq14) give

$$f_6 = b_3 - d_2 b_6 = (-c_2 b_6) - b_6^2 = b_6^2 - b_6^2 = 0.$$

Together with (Eq13) this shows that $a_2 = 0$. Looking at (Eq6) shows now that $f_4 = 0$. However, we have another expression for f_4, namely

$$f_4 = \frac{1}{2} c_2 b_2 - c_1 b_4.$$

We will use this expression to show that

$$f_4 = \frac{72}{49} c_2^3,$$

forcing both c_1 and c_2 to be equal to zero. Firstly, we have from equation (Eq8) that

$$f_5 = -d_2 b_5 - b_4 = -b_6 b_5 - c_2 b_5 = -b_6 b_5 + b_6 b_5 = 0,$$

by using formulas (Eq15), (Eq7), and (Eq14). With this, expressing f_4 with the help of equation (Eq5) shows that

$$\begin{aligned} f_4 = \frac{1}{2} c_2 b_2 - c_1 b_4 &= \frac{1}{2} c_2(-6 c_1 b_5) - c_1 (c_2 b_5) \\ &= -4 c_1 c_2 b_5 \\ &= 8 c_1^2 c_2 = \frac{72}{49} c_2^3. \end{aligned}$$

Here we used formulas (Eq12), (Eq7), and (Eq10). Thus all the constants in the Lie bracket are zero and it is on the form

$$\left[\begin{pmatrix} x_1 \\ y_1 \\ A_1 \\ z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ A_2 \\ z_2 \\ w_2 \end{pmatrix} \right] = \begin{pmatrix} x_1 \times w_2 + w_1 \times x_2 \\ x_1 \times x_2 + y_1 \times w_2 + w_1 \times y_2 \\ \mathrm{Sym}(x_1, y_2) - \mathrm{Sym}(y_1, x_2) + [A_1, w_2] - [A_2, w_1] \\ x_1 \times y_2 + y_1 \times x_2 + z_1 \times w_2 + w_1 \times z_2 \\ w_1 \times w_2 \end{pmatrix}.$$

We observe that

$$[\mathfrak{a}_i, \mathfrak{a}_j] \cap \mathfrak{k} = \{0\}, \quad i, j \in \{1, 2, 3, 4\}.$$

Hence Proposition 3.19 and the remarks made in Subsection 4.6.1 imply that the horizontal holonomy $\mathrm{Hol}(Q, \mathcal{H})$ is trivial. Thus by Proposition 4.34 there exists a Lie group structure on Q such that $\mathcal{H}$ is left-invariant and the chosen point p is the identity. Then observation (4.5) together with the fact that the growth vector is maximal show that $\mathfrak{a}_1$, $\mathfrak{a}_2$, and $\widetilde{\mathfrak{a}_3}$ are isomorphic to $\mathcal{H}_p$, $[\mathcal{H}_p, \mathcal{H}_p]$, and $[\mathcal{H}_p, [\mathcal{H}_p, \mathcal{H}_p]]$, respectively. Hence the Lie algebra $\mathrm{Lie}(Q)$ of Q is isomorphic as a Lie algebra to

$$\widetilde{\mathfrak{g}} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3 \oplus \mathfrak{a}_4$$

with the Lie bracket

$$\left[\begin{pmatrix} x_1 \\ y_1 \\ A_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ A_2 \\ z_2 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ x_1 \times x_2 \\ \mathrm{Sym}(x_1, y_2) - \mathrm{Sym}(y_1, x_2) \\ x_1 \times y_2 + y_1 \times x_2 \end{pmatrix}.$$

Reversing the identification $\widetilde{\mathfrak{a}_3} = \mathfrak{a}_3 \oplus \mathfrak{a}_4$ together with the identifications done in Theorem 5.3 shows that

$$\mathrm{Lie}(Q) \simeq \widetilde{\mathfrak{g}} \simeq \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \widetilde{\mathfrak{a}_3} \simeq \mathfrak{f}[3, 3].$$

Since simply connected Lie groups are completely determined by their Lie algebras, it follows that

$$Q \simeq N[3, 3]$$

as Lie groups. Since $\mathcal{H}$ is simply left translation of the subspace $\mathfrak{a}_1 \subset \widetilde{\mathfrak{g}}$ the result follows. □

5.6 Final Remarks and Further Research

We have in the course of the thesis classified all sub-Riemannian model spaces of step and rank three. This resulted in the spaces $\mathcal{C}(a_1, a_2)$, $\mathcal{A}(\kappa)$, and $N[3, 3]$ described Section 5.3. A careful reading of their derivations in [Gro16, Section 6.1], Section 5.4, and Section 5.5 reveal both the influence and importance of the theory developed on symmetric spaces and principal bundles in Chapters 2 and Chapter 3, respectively.

In hindsight, there are a few remarks we would like to make about extending the classification to spaces with different step or rank. Firstly, the choice of classifying precisely the spaces with step and rank three is that it is the next truly nontrivial case after the step two case was settled in [Gro16]. One might initially consider to classify model spaces with rank two and step three instead. However, when starting with the Carnot model spaces in that class one quickly realizes that the representation of $O(2) \simeq S^1$ on $\mathbb{R}^2$ is radically different from the action of $O(n)$ on $\mathbb{R}^n$ for $n \geq 3$ in rigidity. One can check by using the simple technique showcased in Example 4.29 and Example 4.28 that the only Carnot model space with rank two and step three is $N[2, 3]$. Hence model spaces of rank two and step three have only one choice of nilpotentization similar to the model spaces with step two. This is exclusively a feature of model spaces with low step or low rank as a nontrivial Carnot model space of rank n and step k for $n \geq 2$, $k \geq 3$ with $(n, k) \neq (2, 3)$ was constructed in [Gro16, Example 4.3]. For the case $r = n = 3$, this coincides with the model space $\mathcal{C}_{3,3}$ we found in Theorem 5.1.

Secondly, some difficulties of extending the methods and techniques we have used to higher step or rank should be mentioned. The immediate difficulty which arises is that the dimensional increase of $\mathfrak{f}[n, r]$ makes classifying the Carnot model spaces through the method described in Proposition 4.30 unfeasible. For this to still be a valid approach when dealing with higher step or rank, one need to get a more coherent understanding of the representation theory of the orthogonal group on the free Lie algebras. The increase in dimension will also require different tools for proving analogous statements of those found in Theorem 5.3. As the reader will recall, the uniqueness of the maps (M4), (M6), (M7), and (M8) in Lemma 5.5 was proved by working in coordinates. This approach is only possible due to the small dimensions of the spaces, and would need to be altered to fit a more general scheme. The identification (5.4) is used to great convenience throughout most of Chapter 5. This clearly does not generalize to higher dimensions due to the fact that the dimensions of $\mathbb{R}^n$ and $\mathfrak{o}(n)$ coincide if and only if $n = 3$. The final difficulty is the lack of a clear conjecture for what happens when the step or rank is higher. This looks even harder to get a grasp on after Theorem 5.9, as this strikes a clear difference with the previously known case of model spaces with step two. Except for extending the classification to model spaces with higher step or rank, possible future research on sub-Riemannian model spaces is to:

- **Prove comparison results for sub-Riemannian model spaces**: There are several results in Riemannian geometry commonly known as comparison theorems, most of the well known ones can be found in [Pet16]. What the comparison theorems have in common is that they all compare geometric quantities (Laplacian, curvature, volume etc.) on a suitably well-behaved class of Riemannian manifolds with the corresponding quantities on the Riemannian model spaces. An example is the *Volume Comparison Theorem* which applies to n-dimensional Riemannian manifolds (M, g) which are complete and whose Ricci curvature satisfies

  $$Ric(M) \geq (n-1)K, \quad K \in \mathbb{R}.$$

  Then

  $$Vol(B(p,r)) \leq Vol_K(B(r)),$$

  for any $p \in M$, where $Vol_K(B(r))$ is the volume of a ball $B(r)$ of radius r in the Riemannian model space with section curvature K. Can one find such comparison theorems for sub-Riemannian model spaces? Which geometric quantities should be attempted? One can define an analogous operator to the Laplacian in sub-Riemannian geometry, called the *sub-Laplacian* [Mon02, Chapter 10.5], so this might be a possibility.

- **Find global invariants for sub-Riemannian model spaces**: What can be said about global topological invariants and global geometrical invariants of sub-Riemannian model spaces? A topological space X is said to be *aspherical* if all the higher homotopy groups vanish, that is, $\pi_n(X) = 0$ for $n \geq 2$. In Riemannian geometry the model spaces which are not compact are aspherical. By looking at $O_+(3,1)$ described in Subsection 4.6.3 it is clear that this does not hold for sub-Riemannian model spaces since well known identifications imply that

  $$\pi_3(O_+(3,1)) = \pi_3(SO(3)) = \pi_3(\mathbb{R}P^3) = \pi_3(S^3) = \mathbb{Z}.$$

  However, in step two the non-zero homotopy groups of the sub-Riemannian model spaces are those of $SO(n)$. These repeat due to Bott periodicity and are are well known to be either the integers or zero. One can hence ask

  Conjecture: Given a sub-Riemannian model space which is not Riemannian, do any of the homotopy groups have torsion?

  A geometric invariant of a metric space is its large scale geometry, see [NY12] for a monograph on this topic. Two metric spaces (X, d_X) and (Y, d_Y) are said to be *quasi-isometric* if there exists a map $f : X \to Y$ such that $f(X)$ is a net in Y and there exists constants $L, C > 0$ such that

  $$\frac{1}{L} d_X(x,y) - C \leq d_Y(f(x), f(y)) \leq L d_X(x,y) + C,$$

for all $x, y \in X$. Large scale geometry is mainly concerned with properties which are preserved under quasi-isometries. In the quasi-isometric category, we have that

$$\mathbb{R}^n \simeq \mathbb{Z}^n, \quad S^n \simeq \{pt\},$$

and $\mathbb{H}^n \not\simeq \mathbb{R}^n \not\simeq S^n$ for $n \geq 2$. Hence the Riemannian model spaces are well understood in the setting of large scale geometry. A natural question is to ask which quasi-isometry classes can occur as sub-Riemannian model spaces through the Carnot-Carathéodory distance. More refined (and realistically solvable) questions can be made about asymptotic dimension, amenability, or growth properties. The interested reader should consult [NY12, Chapter 2 and 3] for the definition of these large scale invariants.

- **Develop a satisfactory notion of curvature for sub-Riemannian model spaces**: As we have seen in Chapter 2, a powerful tool in Riemannian geometry is the notion of curvature. Can we develop a notion of *canonical curvature* for sub-Riemannian model spaces which is isometry-invariant, reduces to sectional curvature for the Riemannian model spaces, and treats the Carnot model spaces as "flat spaces"?

A Appendix: Various Prerequisites

A.1 Exterior Calculus

We will summarize a few definitions and formulas related to the exterior derivative and Lie derivative, both in the scalar valued and vector valued cases. No proofs will be provided, and the reader may consult [Lee13, Chapter 14] for the scalar valued forms and [Tu17, Chapter 21] for all the results regarding vector valued forms. Throughout this section we let M denote a manifold, $\Omega^k(M)$ the space of differentiable k-forms on M, and $\Omega^*(M)$ the algebra of all differentiable forms together with the wedge product.

Proposition A.1. *There exists a unique $\mathbb{R}$-linear map $d : \Omega^*(M) \to \Omega^*(M)$ of degree one called the exterior derivative such that*

- *for any function $f \in C^\infty(M)$ we have that df is the total differential;*
- *if $\omega \in \Omega^k(M)$ and $\mu \in \Omega^l(M)$, then*

$$d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^k \omega \wedge d\mu;$$

- $d^2 = 0$.

In terms of a local coordinate system, if $\omega \in \Omega^k(M)$ is written as

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1 \dots i_k} du^{i_1} \wedge \cdots \wedge du^{i_k},$$

then the exterior derivative has the form

$$d\omega = \sum_{i_1 < \cdots < i_k} df_{i_1 \dots i_k} \wedge du^{i_1} \wedge \cdots \wedge du^{i_k}.$$

Moreover, if $\omega \in \Omega^k(M)$ then

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^i X_i(\omega(X_1, \dots \widehat{X}_i, \dots, X_{k+1})) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}). \end{aligned}$$

We will mostly be interested in the case where ω is a one-form, in which case the formula simplifies to

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]), \tag{A.1}$$

for $X, Y \in \mathfrak{X}(M)$.

Remark. Our explicit formulas for the exterior derivative are the same as the ones found in [Lee97] and [Tu17], but are different from those in [KN96]. The differences arise from how the wedge product is defined, compare for instance the definitions in [Lee97] and [KN96].

Recall that $\mathcal{T}^k_l(M)$ denotes the *tensor fields* of type (k, l). To collectively speak about tensor fields, we employ the notation

$$\mathcal{T}^*(M) = \bigoplus_{l,k \in \mathbb{N}} \mathcal{T}^k_l(M).$$

A mapping from $\mathcal{T}^*(M)$ to itself is said to be *type-preserving* if it preserves the layers $\mathcal{T}^k_l(M)$ for any whole numbers k and l.

Proposition A.2. *For each $X \in \mathfrak{X}(M)$ there exists a unique type-preserving linear derivation*

$$\mathcal{L}_X : \mathcal{T}^*(M) \to \mathcal{T}^*(M)$$

with respect to the tensor product, called the Lie derivative in the direction of X. It is given by

$$\mathcal{L}_X(f) = Xf, \quad \mathcal{L}_X(Y) = [X, Y],$$

and satisfies

$$\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y],$$

for $f \in C^\infty(M)$ and $X, Y \in \mathfrak{X}(M)$. Moreover, it commutes with the exterior derivative.

For each $X \in \mathfrak{X}(M)$ there is a map

$$i_X : \Omega^*(M) \to \Omega^*(M)$$

relating the exterior derivative and the Lie derivative called *interior multiplication*. It can be described as the unique skew-derivation of degree -1 given on one-forms by $i_X(\omega) = \omega(X)$, for $\omega \in \mathfrak{X}^*(M)$. It is related to the exterior derivative and Lie derivative through *Cartan's formula*

$$\mathcal{L}_X = i_X \circ d + d \circ i_X.$$

Let us now turn to describe vector valued forms and the operations on them. If T and V are finite-dimensional vector spaces, then a V-valued k-covector on T is an alternating multilinear map

$$\phi : \overbrace{T \times \cdots \times T}^{k-\text{copies}} \longrightarrow V.$$

This can also be considered as a linear map from $\bigwedge^k T$ to V by the universal property of the exterior power. A V-valued k-form on M assigns to each $x \in M$ a V-valued k-covector on T_xM. The notation $E \otimes V$ when $\pi : E \to M$ is a vector bundle and V is a vector space refers to the tensor product of the bundles E and the trivial bundle $M \times V \to M$. Using the isomorphism

$$\mathrm{Hom}_{\mathbb{R}}\left(\bigwedge^k T_xM, V\right) \simeq \bigwedge^k T_x^*M \otimes V,$$

it makes sense to speak of *smooth* V-valued k-forms as elements in

$$\Omega^k(M, V) := \Gamma\left[\left(\bigwedge^k T^*M\right) \otimes V\right].$$

If $v_1, \ldots, v_n$ constitute a basis for V, then any element $\alpha \in \Omega^k(M, V)$ can be written as $\alpha = \sum_{i=1}^n \alpha^i v_i$, with $\alpha^i \in \Omega^k(M)$. We define the exterior derivative of any $\alpha \in \Omega^k(M, V)$ by

$$d\alpha = \sum_{i=1}^n (d\alpha^i) v_i.$$

This is independent of the choice of basis. Moreover, if N is a manifold and $f : N \to M$ is smooth then $f^* d\alpha = df^*\alpha$ for $\alpha \in \Omega^k(M, V)$, when the pullback is defined on vector valued forms by

$$f^*\alpha = \sum_{i=1}^n \left(f^*\alpha^i\right) v_i.$$

Assume now that $\alpha \in \Omega^k(M,\mathfrak{g})$ and $\beta \in \Omega^l(M,\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra with Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$. Recall that a (k,l)-*shuffle* τ is an element in S_{k+l} such that

$$\tau(1) < \cdots < \tau(k), \quad \tau(k+1) < \cdots < \tau(k+l).$$

Then the Lie bracket of α and β, denoted by $[\alpha,\beta]$ is defined to be the element in $\Omega^{k+l}(M,\mathfrak{g})$ given by

$$[\alpha,\beta]_p(u_1,\ldots,u_{k+l}) = \sum_{(k,l)-\text{shuffles } \tau} \operatorname{sgn}(\tau) \left[\alpha_p(u_{\tau(1)},\ldots,u_{\tau(k)}), \beta_p(u_{\tau(k+1)},\ldots,u_{\tau(k+l)})\right]_{\mathfrak{g}},$$

whenever $u_1,\ldots,u_{k+l} \in T_pM$. In particular, if α and β are $\mathfrak{g}$-valued one-forms then

$$[\alpha,\beta](X,Y) = [\alpha(X),\beta(Y)] - [\alpha(Y),\beta(X)],$$

for $X,Y \in \mathfrak{X}(M)$. When $\alpha = \beta$ this reduces to the simple expression

$$[\alpha(X),\alpha(Y)] = \frac{1}{2}[\alpha,\alpha](X,Y).$$

Proposition A.3. *Let $A_1,\ldots,A_n$ be a basis for a Lie algebra $\mathfrak{g}$ and let $\alpha \in \Omega^k(M,\mathfrak{g})$ and $\beta \in \Omega^l(M,\mathfrak{g})$ be $\mathfrak{g}$-valued forms written in the basis as $\alpha = \sum_{i=1}^n \alpha^i A_i$ and $\beta = \sum_{j=1}^n \beta^j A_j$. Then*

$$[\alpha,\beta] = \sum_{i,j=1}^n (\alpha^i \wedge \beta^j)[A_i,A_j]_{\mathfrak{g}} \in \Omega^{k+l}(M,\mathfrak{g}).$$

A.2 Lie Theory

A.2.1 Lie Groups and Their Lie Algebras

This subsection will serve as a reminder of Lie theory, specifically about the basic relations between Lie groups and Lie algebras. The reader may consult [War83, Chapter 3] or [Hel01, Chapter 2] for proofs and further results.

Definition A.4. A *Lie group* is a differentiable manifold G which in addition carries a group structure such that the multiplication map is smooth.

Remark. Some authors require, moreover, that the inversion map sending an element $g \mapsto g^{-1}$ to be smooth. This is redundant, which can be proved using the inverse function theorem.

For $g \in G$ the map given by $g \mapsto hg$ will be denoted by L_h and referred to as *left translation* by $h \in G$. Similarly, R_h denotes *right translation* by $h \in G$. All the vector fields $X \in \mathfrak{X}(G)$ such that $dL_hX(g) = X(hg)$ for all $g,h \in G$ constitute a Lie algebra under the Lie bracket which is denoted by $\mathfrak{g}$. The map

$$\begin{aligned} \mathfrak{g} &\longrightarrow T_eG \\ X &\longmapsto X(e) \end{aligned}$$

is an isomorphism of vector spaces. Hence T_eG inherits a Lie algebra structure and we identify $\mathfrak{g} \simeq T_eG$ as Lie algebras whenever convenient.

Similarly as for vector fields, a differentiable form $\omega \in \Omega(G)$ is called *left-invariant* if $L_g^*\omega = \omega$ for every $g \in G$. The vector space of all left-invariant one-forms will be denoted by $\mathfrak{g}^*$, which is consistent since it isomorphic to the dual space of $\mathfrak{g}$. It is clear that $\omega(X)$ is a constant function on G whenever $X \in \mathfrak{g}$ and $\omega \in \mathfrak{g}^*$. If $\omega \in \mathfrak{g}^*$, then applying formula (A.1) gives the *Maurer-Cartan Equation*

$$d\omega(X,Y) = -\omega([X,Y]), \quad X,Y \in \mathfrak{g}. \tag{A.2}$$

In Section 3.3 we show that the Maurer-Cartan Equation implies that the canonical flat connection on the trivial bundle has zero curvature. The *canonical one-form* on a Lie group G is the left-invariant $\mathfrak{g}$-valued one-form θ which is uniquely determined by $\theta(A) = A$ for $A \in \mathfrak{g}$.

A smooth group homomorphism $\phi : G \to H$ between Lie groups is called a *Lie group homomorphism* and it induces a Lie algebra homomorphism $d\phi_e : \mathfrak{g} \to \mathfrak{h}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively. The following theorem implies that if simply connected Lie groups have isomorphic Lie algebras, then they are isomorphic.

Theorem A.5. *Assume that G is simply connected. Then any Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{h}$ induces a unique Lie group homomorphism $\phi : G \to H$ such that $d\phi_e = \psi$.*

A Lie group homomorphism from $\mathbb{R}$ to a Lie group G is called a *one-parameter subgroup* of G. If $X \in \mathfrak{g}$, then the map

$$\lambda \frac{d}{dt} \longmapsto \lambda X$$

is a Lie algebra homomorphism from the Lie algebra of $\mathbb{R}$ to $\mathfrak{g}$. By Theorem A.5 there exists a one-parameter subgroup $\exp^X : \mathbb{R} \to G$ whose tangent vector at 0 is $X(e)$. The map

$$\begin{aligned} \exp : \mathfrak{g} &\longrightarrow G \\ X &\longmapsto \exp^X(1) \end{aligned}$$

is called the *exponential map* for the Lie group G. It is a smooth map and the differential at the origin is the identity map after the obvious identifications. For a Lie group homomorphism $\phi : G \to H$ the exponential map satisfies

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle \exp}\uparrow & & {\scriptstyle \exp}\uparrow \\ \mathfrak{g} & \xrightarrow{d\phi} & \mathfrak{h} \end{array} .$$

A.2.2 Lie Group Actions and Representation Theory

This subsection will recall terminology and definitions about Lie group actions. We will in particular consider representations of Lie groups and induced representations of their Lie algebras. The reader may consult [Hal15] for more on representation theory of Lie groups.

A *right Lie group action* of a Lie group G on a manifold M is a smooth map $\mu : G \times M \to M$ such that $\mu(e, p) = p$ and $\mu(g, \mu(h, p)) = \mu(hg, p)$. We usually employ the notation $p \cdot g$ for the group action to avoid proliferation of parentheses. Notice that a right action can be converted to a left action by sending $(g, p) \mapsto \mu(g^{-1}, p)$. For any $p \in M$ the *orbit* of p is the set

$$\mathcal{O}(p) = \{q \in M : p \cdot g = q \text{ for some } g \in G\}$$

and the *isotropy group* of p consists of every $g \in G$ such that $g \cdot p = p$. The action is said to be *transitive* if for any $p, q \in M$ there exists an element $g \in G$ such that $p \cdot g = q$, that is, the orbit of any element is all of M. The action is said to be *free* if for any $p \in M$, the property $p \cdot g = p$ forces g to be equal to the identity. Any Lie group G acts on its Lie algebra $\mathfrak{g}$ through the *adjoint map*

$$\begin{aligned} \mathrm{Ad} : G &\longrightarrow \mathrm{Aut}(\mathfrak{g}) \\ h &\longmapsto d_e C_h, \end{aligned}$$

where C_h denotes conjugation by h. One might consider the derivative of the adjoint map, denoted by $\mathrm{ad} : \mathfrak{g} \mapsto \mathrm{End}(\mathfrak{g})$. This is in fact given by the bracket, that is, $\mathrm{ad}(X)$ is the map sending Y to $[X, Y]$.

Definition A.6. Let G be a Lie group and V a vector space. A smooth homomorphism $\rho : G \to \mathrm{GL}(V)$ is called a *(Lie group) representation* of G on V. We call V a *G-module* and say that G *acts* on V. If ρ is a monomorphism, then the representation is said to be *faithful*.

Since a representation is a special case of a Lie group action, we again write $g \cdot v$ instead of $\rho(g)(v)$ when no confusion will arise. The observant reader will notice that although we typically work with right group actions, representations will be left actions due to familiarity with applying linear transformations on the left. A subspace of V which is invariant under the action of G will be referred to as an *invariant subspace* of the representation. If no nontrivial nonempty invariant subspaces of V exist, then the representation is said to be *irreducible*.

Let V_1 and V_2 be G-modules, where G is any Lie group. Then G also acts on their tensor product $V_1 \otimes V_2$ given by

$$g \cdot (v_1 \otimes v_2) = g \cdot v_1 \otimes g \cdot v_2, \quad v_1 \in V_1,\, v_2 \in V_2.$$

Not every element of the tensor product $V_1 \otimes V_2$ is on such a form, but a spanning set is and we can thus extend the action by linearity to all of $V_1 \otimes V_2$. This is independent of the choice of spanning set and is simply referred to as the *tensor product representation* of the G modules V_1 and V_2.

Definition A.7. Let G be a Lie group with two representations $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(W)$ of G. We call a linear map $\psi : V \to W$ an *intertwining map* or *G-equivariant* if it commutes with the action of G, that is, the following diagram commutes for every $g \in G$

$$\begin{array}{ccc} V & \xrightarrow{\;\psi\;} & W \\ \downarrow{\scriptstyle \rho(g)} & & \downarrow{\scriptstyle \rho'(g)} \\ V & \xrightarrow{\;\psi\;} & W \end{array} .$$

If ψ is a bijection, it is called an *equivalence of representations*.

Theorem A.8 (Schur's Lemma)**.** *Let V and W be irreducible representations of a Lie group G and let $\psi : V \to W$ be a non-zero intertwining map. Then ψ is an equivalence of representations.*

Let $\mathfrak{g}$ be a Lie algebra and V a vector space. Then a *(Lie algebra) representation* of $\mathfrak{g}$ on V is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V),$$

where the endomorphism ring $\mathrm{End}(V)$ is considered with the usual commutator as Lie bracket. We will in the proof of Lemma 5.6 need a specific result about irreducible representations of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ consisting of traceless 2×2 matrices with complex coefficients.

Theorem A.9 (Theorem of Highest Weight)**.** *For any positive integer m there is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ on the complex vector space V_m consisting of homogeneous polynomials of degree m in two complex variables. Moreover, any irreducible complex representation is isomorphic to one of these representations.*

We say that a Lie group has the *complete reducibility property* if every finite-dimensional Lie group representation can be written as a direct sum of finitely many irreducible representations. For such Lie groups, every invariant subspace of a given finite-dimensional representation have an invariant complement. There are two important classes of Lie groups which have this property; the compact and semisimple ones. A Lie group is said to be *semisimple* if its Lie algebra is semisimple, that is, have no non-zero abelian ideals. In both cases, the complete reducibility property relies on the existence of a bi-invariant Haar measure, see [Hal15, Appendix D] for details.

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