Well-posedness Issues for Nonlinear Partial Differential Equations Appearing in the Modeling of Long Water Waves

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Thesis for the Degree of Philosophiae Doctor (PhD)
University of Bergen, Norway
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Preface

This thesis is submitted for the degree of Doctor of Philosophy in Applied Mathematics at the Department of Mathematics, University of Bergen. The research work described in this thesis was conducted between October 2014 and September 2018 under the supervision of Professor Henrik Kalisch in the Department of Mathematics, University of Bergen, Norway.

The thesis is centred on nonlinear Partial Differential Equations that appear in the modelling of shallow water waves. In particular, the Riemann problem for such models is investigated and the results, to the best of my knowledge, are original. In a case where the result appear in a previous work, acknowledgements and references are made accordingly. The thesis has not been submitted for other degree or diploma in any other university.

The content of this thesis is in two parts. Part I consists of general background information about water wave theory and summary of the main research results. Part II consists of the following papers written during the work of the thesis:

Paper A: H. Kalisch, D. Mitrovic and V. Teyekpiti, *Delta shock waves in shallow water flow*, Physics Letters A **381**, (2017), 1138-1144.

Paper B: H. Kalisch and V. Teyekpiti, *Hydraulic jumps on shear flows with constant vorticity*, Eur. J. Mech. B Fluids, **72**, (2018), 594–600.

Paper C: H. Kalisch and V. Teyekpiti, *A shallow-water system with vanishing buoyancy*, Appl. Anal., (2017). (submitted).

Paper D: H. Kalisch, D. Mitrovic and V. Teyekpiti, *Existence and Uniqueness of Singular Solutions for a Conservation Law Arising in Magnetohydrodynamics*, Nonlinearity, (2018). (Accepted).

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Abstract

Modelling of wave motion in a fluid is usually based on classical systems which are obtained by the hypotheses that the flow is irrotational and the bottom is even. In such a context, the influence of vorticity is entirely disregarded in the formulation of the governing equations. Although this consideration is justified in many circumstances, there are also a fair number of observed cases in near-shore hydrodynamics and open channel flow where this approach is unsuitable. In this thesis, the influence of constant background vorticity on the properties of shock waves in a shallow water system is considered and the governing equations are derived. An analysis of the shock-wave solutions of the system detailed in the body of this paper shows that stationary jumps can be described in terms of two non-dimensional parameters, one being the Froude number and the other incorporating the background vorticity. It is shown that these two parameters completely determine the strength of the jump. Moreover, in many practical situations, the assumption of a flat bottom is too restrictive. If this theory is to describe the physics of an underlying problem adequately, then it is important to introduced uneven bed in the formulation of the governing equations. This is done in this thesis where it is shown that the combination of discontinuous free-surface solutions and bottom step transitions naturally lead to singular solutions featuring Dirac delta distributions. These singular solutions feature a Rankine-Hugoniot deficit and the method of complex-valued weak asymptotic is used to provide a firm link between the Rankine-Hugoniot deficit and the singular parts of the weak solutions.

Furthermore, it is shown that a shallow water system for interfacial waves in the case of a neutrally buoyant two-layer fluid setup ceases to be strictly hyperbolic and the standard theory of hyperbolic conservation laws cannot be used to solve the Riemann problem. Nevertheless, it is shown that the Riemann problem can still be solved uniquely using singular shocks which contain Dirac delta distributions travelling with the shock. The solution is characterized in terms of the complex-valued weak asymptotic method and it is established that the two solution concepts coincide.

The thesis also made a significant contribution to the Brio system which is a two-by-two system of conservation laws arising as a simplified model in ideal magnetohydrodynamics (MHD). It was found in previous works that the standard theory of hyperbolic conservation laws does not apply to this system since the characteristic fields are not genuinely nonlinear on the set $v = 0$.

In the present contribution, the focus is on such an example, a hyperbolic conservation law appearing in ideal magnetohydrodynamics. For this conservation law, solutions cannot be found using the classical techniques of conservation laws. Consequently, certain Riemann problems have no weak solutions in the traditional Lax admissible sense. It was argued by some authors that in order to solve the system, singular solutions containing Dirac masses along the shock waves might have to be used. Although solutions of this type were exhibited, uniqueness was not obtained. In this thesis, a nonlinear change of variables which makes it possible to solve the Riemann problem in the framework of the standard theory of conservation laws is introduced. In addition, a criterion which leads to an admissibility condition for singular solutions of the original system is developed and it is shown that such admissible solutions are unique in the framework developed in this thesis.

Contents

Part I

General Background

Chapter 1

Introduction

The theory of water waves is a fascinating subject which embodies the general ideas about the generation and propagation of water waves. The theory is one of the basic features of all physical phenomena which occur in nature and offers great variety of applications to mathematically and physically related problems. Waves serve as the means by which information is transmitted between two distinct points in space and time without necessarily moving the medium across the two points. Such waves exist in different media but we shall limit the discussion to the case where the considered medium is water. Water waves are generated due to the presence of restoring forces and are usually classified into three different categories namely interface gravity waves, internal gravity waves and compression and expansion waves. On the one hand, these waves can be cherished in a non-restrictive way by people with less technical knowledge of the subject. On the other hand, the theory of water waves can be formulated quantitatively with rigorous mathematical complexities that can be mostly appreciated by specialists. In the later case, an earnest attempt at developing feasible mathematical concepts and techniques to investigate the subject from theoretical point of view lead to the conception of linear wave theory.

There has been a valuable development of the linear wave theory since Joseph-Louis Lagrange [44, 45] derived the governing equations for small amplitude waves. To authenticate the validity of his derivation, he obtained a limiting solution for plane waves in shallow water and found during the process that the propagation speed of such waves is $\sqrt{gh}$, where h is the fluid depth and g is the gravitational acceleration. If the wavelength is small compared to the fluid depth, then the wave speed is independent of the wavelength. Although Lagrange's linear wave theory remain valid for modelling water waves, it has been observed that as waves propagate into shallow water their steepness increase and their profiles become nonlinear. Consequently, further assumptions are needed to derive equations that account for the nonlinearity.

The conceptually more difficult nonlinear wave theory dates back to the work of Stokes [71] and Riemann [59]. The theory has seen a considerable development and serves as a mathematical tool for modelling shallow water waves. The purpose of this thesis is to investigate well-posedness for Nonlinear Partial Differential Equations appearing in the study of shallow water waves. The theory of evolution equations is an important field of research within the nonlinear sciences where traditional disciplines of mathematics, physics, chemistry and biology merge, interact and change ideas. Even in mathematics it covers several areas such as fluid mechanics, optics, numerical analysis, dynamical systems and partial differential equations. Evolution equations, particularly Partial Differential Equations, are present in nature and technology. They arise mainly in physical problems, typically whenever a wave motion is observed. Electromagnetic waves, seismic waves, shock waves and many other types of waves can be modelled by hyperbolic equations. These PDEs occur in mathematical models of conservation laws and are found in problems such as transport process and wave propagation. Therefore, a better and deeper mathematical understanding of conservation processes is essential. Such knowledge will provide accurate strategies for influencing and controlling the dynamics of the conservation process.

In the context of surface waves, the shallow-water system describes the flow of an inviscid fluid in a long channel of small uniform width. The system is used as a standard model in hydraulics and is also fundamental in the study of bores and storm surges in rivers and channels [20, 76]. If the bottom of the channel is flat as is normally done in a laboratory situation, then the system is usually written in the form

$$\begin{aligned} \partial_t h + \partial_x (uh) &= 0, \\ \partial_t (uh) + \partial_x \left(u^2 h + g \frac{h^2}{2} \right) &= 0, \end{aligned} \tag{1.1}$$

where h represents the total fluid depth, u denotes the average horizontal velocity and g is the gravitational constant. The first equation in (1.1) represents mass conservation which from physical principles states that the rate of change of the total mass in a control volume is equal to the net mass flux through the boundary of the control volume. The second equation in (1.1) denotes momentum conservation which states that the rate of change momentum is equal to the net momentum inflow plus the net forces acting on the boundary.

The above system is derived by the assumption that the channel bed is flat. However, in many practical situations, the assumption of a flat bottom is excessively limiting. The goal is to introduce an uneven bottom bed in the above system and show that an admissible weak solution that conserves mass and momentum and dissipates mechanical energy must give rise to a Rankine-Hugoniot deficit for the conservation equation for the total head. This result is presented fully in Paper

A which is attached in Part II of this thesis. The system of interest is of the form

$$\begin{aligned} \partial_t h + \partial_x (uh) &= 0, \\ \partial_t (uh) + \partial_x \left(u^2 h + g\frac{h^2}{2} \right) &= -g\partial_x b, \end{aligned} \tag{1.2}$$

where $b(x)$ describes the bottom topography. A courageous attempt is made to describe how such a singular solution can be understood as a δ-shock wave in the sense of the complex-valued weak asymptotic method [31, 32]. Complex-valued approximations which become real-valued in the distributional limit is used to establish a resolute connection between the singular parts of the weak solutions and the Rankine-Hugoniot deficit. The interaction of a travelling hydraulic jump with a bottom step is studied and Lax admissible solutions are found.

Modelling of surface wave motion in shallow water is normally based on classical systems which are derived in the framework of irrotational flow. In this context, the influence of vorticity is in principle neglected in the formulation of the governing equations. In spite of the fact that this consideration is justified in many circumstances, there are a reasonable number of observed cases in open channel flow and near-shore hydrodynamics where this treatment is unsuitable. As a matter of fact, there is sufficient evidence that vorticity may have a strong impact on the propagation of wave in a variety of circumstances. For instance, it was recently shown that vorticity has a major influence on the modulational stability of quasi-periodic wavetrains [6, 75] as well as the streamline pattern and pressure profiles in shallow water waves [1, 5, 57, 63, 64, 73]. In addition, the significant influence of vorticity in the modelling of surface waves has been demonstrated in recent studies of wave-current interaction [74], the interaction of vortex patches and point vortices with the free surface [9, 66], the influence of non-constant vorticity on small amplitude waves [37] and the generation of vorticity in long-wave models [4].

In this work, the interaction of surface waves with an existing shear current is studied. Such currents are created by the action of viscous stress at the channel bed, wind stress at the free surface and tidal forcing. Once they are formed, these shear currents may be taken as background conditions when studying individual surface waves. Therefore, it is necessary to make restrictive assumptions which yield manageable mathematical formulations for modelling the influence of constant background vorticity on the properties of shock waves. A shallow water system incorporating constant background shear has been found independently by a number of authors [2, 24, 41] and written as

$$\begin{aligned} \partial_t H + \partial_x \left(\tfrac{\Gamma}{2} H^2 + uH \right) &= 0, \\ \partial_t \left(\tfrac{\Gamma}{2} H^2 + uH \right) + \partial_x \left(\tfrac{\Gamma^2}{3} H^3 + \Gamma u H^2 + u^2 H + \tfrac{1}{2} g H^2 \right) &= 0. \end{aligned} \tag{1.3}$$

An analysis of shock wave solutions of the system (1.3), shows that stationary hydraulic jumps can be described in terms of two non-dimensional parameters, namely, the Froude number which is suitably defined in the presence of the shear current and a non-dimensional background vorticity. Moreover, it is shown in Paper B [35] that stronger background vorticity has the effect of moderating the strength of the hydraulic jump. The analysis performed on (1.3) is partly motivated by the work in [58] where hydraulic jumps were studied in a meaningful manner. The authors of [58] were the first to be able to simulate oscillation of the hydraulic jump toe in a physically reasonable way and obtained a very close match with experimental data. In the current work, we study hydraulic jumps using the simplifying assumption of constant background shear. Although (1.3) is not able to explain the creation of vorticity in a hydraulic jump which featured prominently in [58], it is able to quantify the dependence of the strength of the hydraulic jump on the vorticity and also give fairly simple closed-form solutions.

As mentioned earlier, waves can propagate internally at the interface of a two-fluid system and in this thesis, such waves are studied using a triangular system of conservation laws of the form

$$\begin{aligned} \partial_t u + \partial_x \left(\tfrac{u^2}{2} + g \tfrac{\rho_1 - \rho_2}{\rho_1} \eta \right) &= 0, \\ \partial_t \eta + \partial_x (u + \eta u) &= 0. \end{aligned} \tag{1.4}$$

The interest in this system is physically motivated by considering large pools of heavy liquid located at the bottom of a deep ocean. It has been reported in [25] that such pools of heavy liquids may occur naturally and have also been put forward as a potential long-term site for captured CO_2 storage. Climate change and global warming are understood to result partially from increasing concentration of CO_2 in the atmosphere and techniques that aim at stabilizing atmospheric CO_2 level are the focus of recent scientific research. One promising procedure is to capture the CO_2 in fossil-burning processes and sequester it in storage sites such as depleted petroleum and natural gas reservoirs, unminable coal beds, saline aquifers and the ocean. As a result of ecological and climate considerations, the ocean is the preferable storage site for undissolved CO_2 and it is reported in [21, 68] that at predominant oceanic temperatures, CO_2 condenses to liquid at about 400m depth. Owing to the relatively higher compressibility of liquid CO_2 than seawater, liquid CO_2 become denser than seawater at a depth of about 3000m and it is thus, theoretically possible to store CO_2 in the ocean at depths exceeding 3000m. It is also generally known that the interface of CO_2 and sea-water is characterized by rapid nucleation of CO_2 and H_2O into an ice-like compound. This hydrate layer serves as a protective membrane which may prevent the liquid CO_2 from escaping even at depths smaller than 3000m (see [25]). As established in [22, 77], modelling the hydrate layer is accomplished by using surface tension.

The highly compressible nature of liquid CO_2 imply that at certain depth its density will coincide with that of the surrounding seawater. Furthermore, changes in the surrounding seawater temperature may render an initially stable configuration unstable by making the CO_2 either stable, unstable or neutrally stable. In the current contribution, the neutrally stable case where the two densities ρ_2 and ρ_1 are equal is studied and the considerations are restricted to a shallow-water-like system of equations of the form

$$\begin{aligned} \partial_t u + \partial_x \left(\frac{u^2}{2} \right) &= 0, \\ \partial_t \eta + \partial_x (u + \eta u) &= 0. \end{aligned} \tag{1.5}$$

In terms of hyperbolic conservation laws, the characteristic speeds of this system coincide in phase space and hence, the classical theory of strictly hyperbolic conservation laws cannot be used to completely resolve the Riemann problem. Nevertheless, it is shown in this work (Paper C) that it is possible to construct a unique solution to the Riemann initial value problem associated with (1.5). In other words, the Riemann problem can be solved uniquely using singular shocks containing Dirac delta distributions travelling with the shock. Moreover, it is also shown that the solution contains a singular δ-shock whose strength is an exact measure of the corresponding Rankine-Hugoniot deficit. Although non-strictly hyperbolic systems of conservation laws have no definitive theory which guarantee existence of weak solutions, the interest in studying such systems is partially motivated by their significant applications in oil reservoir simulation, gas dynamics and magnetohydrodynamics [38, 61, 67].

Magnetohydrodynamics (MHD) is a field that deals with the study of the magnetic properties of electrically charged fluids. In other words, it is the study of how electric currents in a propagating conductive fluid interact with the magnetic field created by the moving fluid itself. Such magnetofluids include but not limited to plasma, salt water and liquid metals. The MHD equations are generally coupled in such a way that make the system highly complex requiring the solution to be constructed simultaneously. The complexity of the full system imply that one has to rely on numerical approximation of solutions in order to understand the dynamics of the system. To overcome this difficulty, restrictive assumptions were made in [3] in other to obtain simplified equations which are easier to solve but preserve some of the important features observed in the full MHD systems. The resulting system is a two-by-two system of conservation laws with a relatively simple structure. It was put forth in [19] that in order to resolve the system adequately, singular solutions containing Dirac masses along the shock waves might have to be used. Such singular solutions were presented in [31, 60] but the authors fail to obtain uniqueness. The goal in this thesis is to solve the Riemann problem for the simplified MHD, commonly called the Brio system, in the framework of the standard theory of hyperbolic conservation laws by using an approach which guarantees existence and uniqueness of admissible solutions. The full result is presented neatly in Paper D (attached in Part II of this thesis).

The thesis is organized as follows: Chapter 1 introduces the wave problem and describes the various models that govern wave motion in shallow water. In Chapter 2, the governing equations are derived. Chapter 3 focuses on hyperbolic conservation laws where admissibility concept for solutions is introduced and existence and uniqueness of a solution to the Riemann problem is established. In Chapter 4, the notion of weak asymptotic is presented and δ-shock solution concept is explained.

1.1 Formulation of the wave problem

The theory of water waves is a fascinating subject which incorporates the fundamental ideas about the generation and propagation of water waves and other mathematically and physically related problems. Waves are generated due to the existence of restoring forces and are commonly classified into three different categories. These include interface gravity waves where the restoring forces are gravity and surface tension, internal gravity waves where the restoring force is gravity and compression and expansion waves where the restoring force is due to the compressibility of the fluid. These waves occur in different media but our discussion shall be restricted in general to the case where the medium is water. Considering the motion of waves in the $x-z$ plane and suppose that the x-axis is in the horizontal direction and that the waves propagate in this direction only. In addition, assume that the z-axis is vertically upwards from the free surface located at $z=0$. Assume further that the fluid has a uniform undisturbed depth H and the deflection of the free surface from its rest state is $z=\eta(x,t)$ so that the actual surface is located at $z=h=\eta+H$. If the fluid is inviscid, incompressible and homogeneous with constant density ρ, then following [43], the continuity equation is

$$\nabla \cdot \mathbf{u} = 0, \tag{1.6}$$

where $\mathbf{u}=(u,w)$ is the velocity vector. In the classical case, it is assumed that the flow is irrotational so that waves propagating at zero initial vorticity remain so for all times [76]. The velocity potential, φ, in this case is defined by

$$u = \frac{\partial \varphi}{\partial x}, \quad w = \frac{\partial \varphi}{\partial z}. \tag{1.7}$$

Substituting this equation into the continuity equation (1.6) gives the Laplace equation

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2}. \tag{1.8}$$

To solve this equation one has to specify appropriate boundary conditions at both the free surface and the bottom. The bottom boundary condition is the zero normal velocity which is expressed as

$$w = \frac{\partial \varphi}{\partial z} = 0 \quad \text{at} \quad z = -H. \tag{1.9}$$

As stated in [76], two boundary conditions are required at the free surface, namely, a kinematic and a dynamic boundary conditions. The reason for this is the fact that the free surface elevation, $\eta(x,t)$, and the velocity potential, φ, are not yet known. The kinematic boundary condition of the free surface states that fluid particles at the surface remain there at any given time so that the expression

$$\frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \varphi}{\partial z} \quad \text{at} \quad z = \eta. \tag{1.10}$$

holds. The dynamic condition on the other hand states that the pressure just below the surface is equal to the atmospheric pressure. If we take the atmospheric pressure to be zero, then the dynamic boundary condition can be written as

$$P = 0 \quad \text{at} \quad z = \eta. \tag{1.11}$$

Since the flow is assumed to be irrotational, the pressure can be calculated from the Bernoulli equation [43]

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(u^2 + w^2) + \frac{p}{\rho} + gz = F(t), \tag{1.12}$$

where $F(t)$ is independent of location and can be absorbed into the term $\frac{\partial \varphi}{\partial t}$ by basically redefining the velocity potential φ so that (1.12) becomes

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(u^2 + w^2) + g\eta = 0 \quad \text{at} \quad z = \eta. \tag{1.13}$$

The equation derived above is nonlinear and finding solution may be demanding. In addition, the elevation of the free surface, $z = \eta(x,t)$, from its rest state as well as the domain in which the velocity potential, $\varphi(x,z,t)$, is to be determined are not yet known. Moreover, if it is assumed that $\varphi(x,z,t)$ is analytic and uniformly bounded throughout the fluid medium, then the solution would possibly not exist for all times $t > 0$. The reason for this is the fact that any mathematical formulation of the problem which would match observed wave occurrence would certainly require an assumption that singularities exist in an unknown location in space and time [70]. Consequently, it is necessary to make simplifying assumptions that preserve the physical characteristics of the nonlinear equation but lead to a manageable formulation which is mathematically interesting.

1.2 The linearised formulation

Simplifying assumptions are imposed on the nonlinear wave equation which is formulated in the previous section to reduce it to relatively less demanding linear equation. If the amplitude of the surface wave is assumed small such that the velocity potential, the free surface elevation and their derivatives are all small quantities, then the term $\partial_x\varphi\partial_x\eta$ in (1.10) will be significantly small reducing that equation to

$$\partial_t\eta = \partial_z\varphi \quad \text{at} \quad z = \eta. \tag{1.14}$$

Using Taylor series it is possible to evaluate $\partial_z\varphi$ around $z = 0$ to get

$$\partial_z\varphi|_{z=\eta} = \partial_z\varphi|_{z=0} + \eta\partial_{zz}\varphi|_{z=0} + \eta^2\partial_{zz}\varphi|_{z=0} + \cdots .$$

Approximating the above equation to first order term yields

$$\partial_z\varphi|_{z=\eta} \approx \partial_z\varphi|_{z=0}$$

and inserting it into equation (1.14) gives

$$\partial_t\eta = \partial_z\varphi \quad \text{at} \quad z = 0. \tag{1.15}$$

Since we have assumed that the wave propagates with small amplitude, the nonlinear term $u^2 + w^2$ in (1.13) can be disregarded and the equation eventually reduces to the linear form of the unsteady Bernoulli equation

$$\frac{\partial\varphi}{\partial t} + \frac{p}{\rho} + g\eta = 0 \quad \text{at} \quad z = \eta. \tag{1.16}$$

Applying the pressure term in (1.11) reduces this equation to

$$\frac{\partial\varphi}{\partial t} = -g\eta \quad \text{at} \quad z = 0. \tag{1.17}$$

Notice that the terms in (1.17) were evaluated at $z = 0$ and not at $z = \eta$ because the wave amplitude is assumed small. Although an approximation error may occur in the above calculations, it is significantly small that the linearised formulation is considered a valid approximation in shallow water theory.

1.3 Elementary solution of the wave problem

The goal of this section is to establish the solution of (1.8)

$$\partial_{xx}\varphi + \partial_{zz}\varphi = 0$$

subject to the boundary conditions

$$\frac{\partial\varphi}{\partial z} = 0 \qquad \text{at} \quad z = -H,$$
$$\frac{\partial\varphi}{\partial z} = \frac{\partial\eta}{\partial t} \qquad \text{at} \quad z = 0,$$
$$\frac{\partial\varphi}{\partial t} = -g\eta \quad \text{at} \quad z = 0.$$

Notice that these boundary conditions have already been obtained in the preceding section. In particular, they correspond to equations (1.9), (1.15) and (1.17) respectively. For frequency ω and wavenumber k, assume an expression for the surface

$$\eta(x,t) = a\cos(kx - \omega t), \tag{1.18}$$

where a is the wave amplitude. Now express the wavenumber as $k = \frac{2\pi}{\lambda}$, where λ is the wavelength. The dependence of η on $(kx - \omega t)$ and the boundary conditions (1.15) and (1.17) require the velocity potential, φ, to be written as a sine function of $(kx - \omega t)$. Eventually, the analysis is based on the assumption that the Laplace equation (1.8) has a separable solution

$$\varphi(z,x,t) = f(z)\sin(kx - \omega t), \tag{1.19}$$

where $f(z)$ and $\omega = \omega(k)$ are to be determined. If (1.19) is substituted into the Laplace equation (1.8), then it becomes

$$\partial_{zz} f - k^2 f = 0, \tag{1.20}$$

which admits a general solution of the form

$$f(z) = Ae^{kz} + Be^{-kz}, \tag{1.21}$$

where A and B are constants to be determined. Using this equation, the velocity potential can be written as

$$\varphi(z,x,t) = (Ae^{kz} + Be^{-kz})\sin(kx - \omega t). \tag{1.22}$$

If (1.22) is differentiated with respect to t, then by using the bottom boundary condition (1.9) it is immediate that

$$B = Ae^{-2kH}. \tag{1.23}$$

Substituting (1.18) and (1.22) into the boundary condition in equation (1.15) yield

$$k(A - B) = \omega a. \tag{1.24}$$

It is now an easy exercise to determine the constants A and B by solving equations (1.23) and (1.24) to get

$$A = \frac{a\omega}{k(1-e^{-2kH})}, \qquad B = \frac{a\omega e^{-2kH}}{k(1-e^{-2kH})}. \tag{1.25}$$

Substituting these into (1.23) gives the desired solution

$$\varphi = \frac{a\omega}{k}\frac{\cosh(k(z+H))}{\sin(kH)}\sin(kx-\omega t). \tag{1.26}$$

As stated above, the surface elevation must also be found and this is done by applying the dynamic boundary condition (1.17). Substitution of (1.18) and (1.26) into (1.17) give

$$\frac{-a\omega^2}{k}\frac{\cosh(kH)}{\sinh(kH)}\cosh(kx-\omega t) = -ag\cosh(kx-\omega t).$$

Algebraic simplification of this equation yields the relation

$$\omega = \sqrt{gk\tan(kH)}. \tag{1.27}$$

This equation expresses the connection between the frequency and the wave number and is commonly referred to as the *dispersion relation*. The phase speed of the surface wave can be obtained from (1.27) as

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tan(kH)} = \sqrt{\frac{g\lambda}{2\pi}\tan\left(\frac{2\pi H}{\lambda}\right)}. \tag{1.28}$$

This equation shows that the wave speed does not dependent on the wavelength $\lambda = \frac{2\pi}{k}$. When the water depth is greater compared with the wavelength, thus $kH \gg 1$, then $\tan(kH) \approx 1$ and the dispersion relation (1.27) simplifies to $\omega = \sqrt{gk}$. Waves with this dispersive nature are called *deep water waves*. However, when the ratio of the water depth to the wavelength is small, thus $\frac{H}{\lambda} \ll 1$, the waves are called *shallow water waves*. In this case, $\tan(kH) \approx kH$ and the dispersion relation simplifies to $\omega = gk^2H$ [43].

1.4 Shallow water theory

In Section 1.2, the assumption leading to the linearisation of the wave equation is that the amplitude of the surface wave is small. In this section, a different kind of approximation is obtained by the hypothesis that the depth of the water is significantly small when compared to the wavelength. This assumption leads to an interesting equation which in its lowest order becomes useful in studying wave propagation in compressible gases [70]. Assume a setup where the y-axis is pointing vertically upwards and the x-axis is in the flow direction. Suppose that h represents the undisturbed water depth such that the bottom is located at $y = -h$.

Nonlinear theory

If $u(x,y,t)$ and $v(x,y,t)$ denote the velocity components, then the continuity equation can be written as

$$\partial_x u + \partial_y v = 0. \tag{1.29}$$

By following Stoker [70], the bottom boundary condition is given by

$$u\partial_x h + v = 0 \quad \text{at} \quad y = -h. \tag{1.30}$$

In addition to the bottom boundary condition, free surface conditions must be specified. These are the kinematic boundary condition

$$\partial_t \eta + u\partial_x \eta - v = 0 \quad \text{at} \quad y = \eta, \tag{1.31}$$

and the dynamic boundary condition

$$P = 0 \quad \text{at} \quad y = \eta. \tag{1.32}$$

Integrating the continuity equation (1.29) with respect to y gives

$$\int_{-h}^{\eta} (\partial_x u) dy + v|_{-h}^{\eta} = 0. \tag{1.33}$$

From the bottom boundary condition (1.30) and the kinematic condition (1.31), we have

$$v|_{-h}^{\eta} = \partial_t \eta + u|_{\eta} \partial_x \eta + u|_{-h} \partial_x h.$$

Substitution of this equation into (1.33) yields

$$\int_{-h}^{\eta} (\partial_x u) dy + \partial_t \eta + u|_{\eta} \partial_x \eta + u|_{-h} \partial_x h = 0. \tag{1.34}$$

If we introduce the relation

$$\frac{\partial}{\partial_x} \int_{-h(x)}^{\eta(x)} u dy = u\partial_x \eta|_{y=\eta} + u\partial_x h|_{y=-h} + \int_{-h}^{\eta} (\partial_x u) dy,$$

then equation (1.34) simplifies to

$$\frac{\partial}{\partial_x}\int_{-h}^{\eta} u\,dy = -\partial_t\eta. \tag{1.35}$$

Given the water density ρ and acceleration due to gravity g, the hydrostatic pressure is written as

$$P = g\rho(\eta - y), \tag{1.36}$$

which agrees with the dynamic boundary condition at the surface. Differentiating this equation with respect to x yields

$$\partial_x P = g\rho\partial_x\eta. \tag{1.37}$$

This basically means that $\partial_x P$ is independent of the y-axis and that the acceleration component of the x-axis is also independent of y. Eventually, the velocity component of the x-axis is independent of y for all times t, thus, $u = u(x,t)$. By applying (1.37), the equation of motion in the x-direction can be written in the Eulerian form

$$\partial_t u + u\partial_x u = -g\partial_x\eta. \tag{1.38}$$

Notice that $\partial_y u = 0$ since u is independent of y. As a result of this observation, equation (1.35) can be written as

$$\partial_x(u(\eta + h)) = -\partial_t\eta. \tag{1.39}$$

Equations (1.38) and (1.39) are the nonlinear Partial Differential Equations for $u(x,t)$ and $\eta(x,t)$ that govern shallow water waves.

Linear theory

The goal of this section is to linearise the nonlinear shallow water equations derived in the previous section. Assume that the hypotheses made in Section (1.2) hold and that $u(x,t)$ and $\eta(x,t)$ and all their derivatives are small. Then both $u\partial_x u$ and $\partial_x(u\eta)$ will be small compared to the linear terms and equations (1.38) and (1.39) become

$$\begin{aligned} \partial_t u &= -g\partial_x\eta, \\ \partial_x(uh) &= -\partial_t\eta, \end{aligned}$$

respectively. If we differentiate the first equation with respect to t and the second with respect to x, then η can be eliminated to get

$$\partial_{tt}u - c^2\partial_{xx}u = 0, \tag{1.40}$$

where $c = \sqrt{gh}$ is the speed of propagation and h is the constant water depth.

Chapter 2

Derivation of shallow water models

The previous chapter presented a theoretical description of the classical shallow water model and the various formulations that can be obtained by making further assumptions. The classical system is obtained from the incompressible Euler equations [15] which consist of the mass conservation and momentum conservation. In addition to these conservation equations is the the conservation of energy which turns out to be useful when analysing the Riemann problem for such systems. The goal in this chapter is to derive model equations for shallow water flow over a variable bottom topography and also formulate a model system that governs the flow in the presence of a constant background vorticity. Furthermore, a derivation of a shallow-water-like system that models wave propagation at the interface of a two fluids system having distinct densities is presented.

2.1 Shallow water flow over an uneven bottom bed

Consider a shallow water flow in the $x-z$ plane as depicted in Figure 2.1. Assume that the x-axis is in the horizontal direction and that the frictionless flow is in this direction only. The z-axis is along the vertical cross-section of the fluid. The fluid is assumed to be inviscid, incompressible and homogeneous with constant density ρ. The flow is over a bottom bed described by the function $\beta(x)$. The bottom of the control volume of width w delimited by $[x_1, x_2]$ at any given time is assumed to be free from the processes of erosion and sedimentation. The deflection of the free surface from its rest state is denoted by $\eta(x,t)$ so that the actual surface is located at

$$z(x,t) = h_0(x,t) + \eta(x,t) + \beta(x).$$

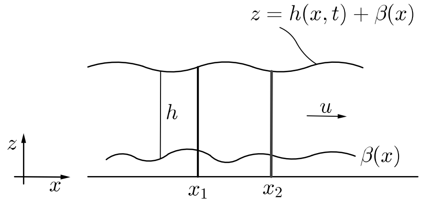


Figure 2.1: Shallow water flow over a bottom step

Pressure

This section presents the pressure associated with the balance laws in the control volume. The pressure is constructed from the primary flow variables and is essential for the computation of energy and momentum balances. The importance of the pressure force on the control volume and the work done by the pressure force is stated in the discussion leading to the derivation of (2.3) and (2.4). The presence of the bottom step means that the bottom pressure is not the same on either side of the jump. The net upward force due to pressure difference on the bottom bed and the free surface must be balanced by the weight of the fluid in the control volume. This gives the expression

$$\frac{\partial P}{\partial z} = -\rho g.$$

Vertically integrating the above equation along the fluid column and normalizing the pressure at the surface yields

$$P = (\beta(x) + h - z)\rho g, \tag{2.1}$$

where $h(x,t) = h_0 + \eta(x,t)$.

Mass Balance

Consider a control volume bounded by the free surface, the bottom and the fixed lateral sides x_1 and x_2 where $x_1 < x_2$. Then the total mass of the fluid contained in the control volume at time t is given by

$$\mathcal{M} = \int_{x_1}^{x_2} \int_{\beta(x)}^{h+\beta(x)} \rho w \, dz \, dx,$$

where w is the average width of the control volume. Mass conservation in this volume is based on the physical principle that the mass of the fluid particles in the volume is constant. This requires that the time rate of change of the total mass in the control volume is equal to the net mass flux through the boundaries of the control volume. The mass flux is only through the lateral sides x_1 and x_2 and the conservation of mass in terms of the flow variables is given by

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2}\int_{\beta(x)}^{h+\beta(x)} \rho w \, dz \, dx = -\left[\int_{\beta(x)}^{h+\beta(x)} \rho w u \, dz\right]_{x_1}^{x_2}.$$

This can be simplified further to get

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho w h(x,t) \, dx = -\left[\rho w u(x,t) h(x,t)\right]_{x_1}^{x_2}.$$

Algebraic manipulation and rearrangement of terms yield

$$\int_{x_1}^{x_2}\left[\frac{\partial}{\partial t}\big(\rho w h(x,t)\big) + \frac{\partial}{\partial x}\big(\rho w u(x,t) h(x,t)\big)\right] dx = 0.$$

Dividing through by $(x_2 - x_1)$ and taking limit as $\Delta x \to 0$, we obtain the local mass balance equation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0. \tag{2.2}$$

Notice that the constant density ρ and the uniform channel width cancel out in the limit.

Momentum Balance

This section is devoted to the formulation of the momentum balance law that models the flow in the control volume described in the previous section. If the control volume has a uniform width w and contains a constant density fluid, then the total horizontal momentum from the bottom to the free surface and delimited by $[x_1, x_2]$ such that $x_1 < x_2$ is expressed mathematically as

$$I = \frac{\partial}{\partial t}\int_{x_1}^{x_2}\int_{\beta(x)}^{h+\beta(x)} \rho w u \, dz \, dx.$$

The time rate of change of momentum in the control volume is balanced by the net influx of momentum through the boundaries plus the net work done on the boundary by the volume. This fundamental hypothesis leads to the expression

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2}\int_{\beta(x)}^{h+\beta(x)} \rho w u \, dz \, dx + \left[\int_{\beta(x)}^{h+\beta(x)} \rho w u^2 \, dz\right]_{x_1}^{x_2} + \left[\int_{\beta(x)}^{h+\beta(x)} P w \, dz\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} P|_{z=\beta(x)} \partial_x \beta(x) \, dx = 0.$$

The first term accounts for momentum accumulation in the control volume and the second term represents momentum outflow. Using equation (2.1) and the fact that there is no shear stress in a frictionless horizontal flow suggest by logical necessity that

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2}\int_{\beta(x)}^{h+\beta(x)} \rho w\, dz\, dx + \left[\int_{\beta(x)}^{h+\beta(x)} \rho w u^2\, dz\right]_{x_1}^{x_2} + \left[\int_{\beta(x)}^{h+\beta(x)} \rho g w\big(\beta(x)+h-z\big)\, dz\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \rho g w(\beta(x)+h-z)|_{z=\beta(x)}\partial_x\beta(x)\, dx = 0.$$

Using the assumptions that ρ and w are constants yield

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} uh\, dx + \left[u^2 h\right]_{x_1}^{x_2} + \left[\frac{g}{2}h^2\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} gh(x,t)\partial_x\beta(x)\, dx = 0.$$

This can be and simplified further to give

$$\int_{x_1}^{x_2}\left[\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2\right) + gh\partial_x\beta(x)\right] dx = 0.$$

Dividing through by $(x_2 - x_1)$ and taking the limit as $\Delta x \to 0$ give the momentum conservation equation

$$\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2\right) = -gh\partial_x\beta(x). \tag{2.3}$$

Energy Balance

This section focuses on the derivation of the total mechanical energy inside the control volume described above. The total mechanical energy in the control volume is the sum of the kinetic energy and the potential which is express as

$$\mathcal{E} = \frac{\partial}{\partial t}\int_{x_1}^{x_2}\int_{\beta(x)}^{h+\beta(x)}\left(\frac{1}{2}\rho w u^2 + \rho w g z\right) dz\, dx.$$

Physical principles require that the time rate of change of total mechanical energy in the control volume be balanced by the net energy flux plus the work done by pressure forces acting on the control volume. Conservation of total energy is given by

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2}\int_{\beta(x)}^{h+\beta(x)}\left(\frac{1}{2}\rho w u^2+\rho w g z\right)\,dz\,dx + \left[\int_{\beta(x)}^{h+\beta(x)}\left(\frac{1}{2}\rho w u^2+\rho w g z\right)u\,dz\right]_{x_1}^{x_2} + \left[\int_{\beta(x)}^{h+\beta(x)}\rho w g\big(\beta(x)+h-z\big)u\,dz\right]_{x_1}^{x_2}=0.$$

This can be simplified further by integrating with respect to z to get

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2}\rho w\left(\frac{1}{2}u^2+\frac{1}{2}gh^2+gh\beta(x)\right)\,dx + \left[\rho w\left(\frac{1}{2}u^3h+guh^2+ghu\beta(x)\right)\right]_{x_1}^{x_2}=0.$$

Since the density and the channel width are constants, the equation can be written as

$$\int_{x_1}^{x_2}\left[\frac{\partial}{\partial t}\left(\frac{1}{2}u^2+\frac{1}{2}gh^2+gh\beta(x)\right)+\frac{\partial}{\partial x}\left(\frac{1}{2}u^3h+guh^2+ghu\beta(x)\right)\right]\,dx=0.$$

Dividing through by (x_2-x_1) and taking the limit as $\Delta x\to 0$, the integrand must vanish identically to give energy conservation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2h+\frac{1}{2}gh^2+gh\beta(x)\right)+\frac{\partial}{\partial x}\left(\frac{1}{2}u^3h+ugh(h+\beta(x))\right)=0. \tag{2.4}$$

2.2 Shallow water dynamics on shear flow

Modelling of wave motion in a fluid is normally based on classical systems which are obtained by assuming that the flow is irrotational. Such assumption produces model equations that are restrictive and unsuitable in certain open channel flows. To overcome this limitation, the influence of constant background vorticity on the properties of shallow water flow is studied and mathematical equations that efficiently model such a flow are formulated. For the purpose of preventing an undue mathematical complexity, it is assumed that the shallow water waves to be described are perturbations of an existing background flow with a shear profile that is linear. This procedure has been used in [7, 14, 26, 27, 36, 57] and it has been indicated to approximate naturally occurring shear flows fairly well [28]. In particular, the dispersion relation associated with a linear background

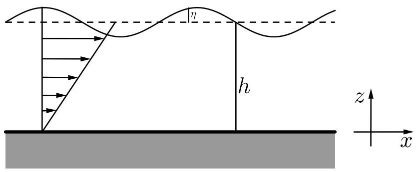


Figure 2.2: Schematic representation of a linear shear flow over an even bottom.

shear efficiently approximates experimentally measured dispersion relation [29]. In addition, it was shown in [73] that linear shear flows can be used as a first approximation to more general shear profiles in the shallow water wave regime since the wavelength of such waves is on a different scale than the variation of the shear profile. The governing shallow water equations that describe the motion of incompressible and inviscid constant shear fluid are derived in this section. Consider a flat-bottom channel of unit width and even bottom containing a fluid with undisturbed depth h. Assume that the elevation of the free surface from its equilibrium state is given by $\eta(x,t)$ such that the total depth at a point x and time t is defined by $H(x,t) = h + \eta(x,t)$. A schematic representation of the problem setup is shown in Figure 2.2. If the average horizontal velocity is denoted by $u(x,t)$, then the total velocity component is written as

$$v(x,t,z) \equiv U(z) + u(x,t) = \Gamma z + u, \tag{2.5}$$

where $U(z)$ is the linear shear current and Γ is constant. The mass of the incompressible, inviscid fluid of unit depth is

$$\mathcal{M} = \int_{x_1}^{x_2} \int_0^{H(x,t)} \rho \, dz \, dx.$$

If the free surface and the flat bottom are impermeable so that no transfer of mass occur there, then the physical hypothesis of mass conservation requires that the rate of change of mass per unit time is proportional to the mass flux through the lateral boundaries. The mathematical idealization of this concept is expressed by the integral equation

$$\frac{d}{dt} \int_{x_1}^{x_2} \int_0^{H(x,t)} \rho \, dz \, dx = \int_0^{H(x,t)} \rho v(x_1,t) \, dz - \int_0^{H(x,t)} \rho v(x_2,t) \, dz.$$

If the flow variables as well as the domain are smooth, then the above equation can be written as

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho H(x,t) \, dx + \left[\int_0^{H(x,t)} \rho(\Gamma z + u(x,t)) \, dz \right]_{x_1}^{x_2} = 0.$$

Assume that the density ρ is constant and divide through by $(x_2 - x_1)\rho$. Then by letting $x_2 - x_1 \to 0$ will cause the integrand to converge pointwisely to zero on the arbitrary interval $[x_1, x_2]$ to give an equation representing mass conservation in the form

$$\partial_t H + \partial_x \left(\frac{\Gamma}{2} H^2 + uH \right) = 0. \tag{2.6}$$

In a similar way, an expression representing momentum conservation in the control volume can be derived. If we suppose that pressure force is the only force acting on the control volume, then the conservation of momentum is based on the physical principle that the fluid is in a hydrostatic balance such that the pressure $p = p(x,z,t)$ is introduced. Applying this assumption in an arbitrary fluid column $[x_1, x_2] \times [z, z + \Delta z]$ gives

$$\big(p(\bar{x}, z + \Delta z, t) - p(\bar{x}, z, t)\big)(x_2 - x_1) = -(x_2 - x_1)\rho g \Delta z,$$

where $\bar{x} \in [x_1, x_2]$. If the flow variables are smooth, then dividing through by $(x_2 - x_1)\Delta z$ and taking the limit as $\Delta z \to 0$ gives

$$\frac{dp}{dz} = -\rho g.$$

Integrate and normalize the pressure to be zero at the surface to get

$$p(x,z,t) = \rho g \big(H(x,t) - z\big). \tag{2.7}$$

Considering the control volume described above, the total momentum is given by

$$I = \int_{x_1}^{x_2} \int_0^{H(x,t)} \rho v(x,t)\, dz\, dx.$$

Momentum conservation is obtained from Newton's second law which requires that the rate of change of total momentum is equal to the net momentum flux through the lateral boundaries plus the pressure forces acting on the boundaries. This is expressed mathematically as

$$\begin{aligned} \frac{d}{dt} \int_{x_1}^{x_2} \int_0^{H(x,t)} \rho v(x,t)\, dz\, dx = \int_0^H \rho v^2(x_1,t)\, dz - \int_0^H \rho v^2(x_2,t)\, dz \\ + \int_0^H p(x_1,z,t)\, dz - \int_0^H p(x_2,z,t)\, dz. \end{aligned}$$

Substituting the total velocity in (2.5) and the pressure term in (2.7) and simplifying yield

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho \left[\left(\frac{\Gamma}{2} H^2 + uH \right)_t + \left(\frac{\Gamma^2}{3} H^3 + \Gamma u H^2 + u^2 H + \frac{g}{2} H^2 \right)_x \right] dx = 0$$

If we suppose that the density is constant, then dividing through by $(x_2 - x_1)\rho$ and taking the limit as $(x_2 - x_1) \to 0$ will make the integrand to vanish at every point (x,t). This is simply a result of the fact that the interval $[x_1, x_2]$ is arbitrarily chosen. The result is momentum conservation equation expressed as

$$\left(\frac{\Gamma}{2}H^2 + uH\right)_t + \left(\frac{\Gamma^2}{3}H^3 + \Gamma uH^2 + u^2H + \frac{1}{2}gH^2\right)_x = 0. \tag{2.8}$$

Total mechanical energy in the control volume is expressed as

$$\frac{d}{dt}\int_{x_1}^{x_2}\int_0^{h+\eta} \rho\left(\tfrac{1}{2}\tilde{v}^2 + gz\right) dz dx = \\ \left[\int_0^{h+\eta} \rho\left(\tfrac{1}{2}\tilde{v}^2 + gz\right)\tilde{v} dz + \int_0^{h+\eta} \rho\left(h + \eta - z\right)\tilde{v} dz\right]_{x_2}^{x_1}.$$

Substituting (2.5) and integrating in the z-direction gives

$$\int_{x_1}^{x_2} \left(\tfrac{\Gamma}{2}uH^2 + \tfrac{\Gamma^2}{6}H^3 + \tfrac{1}{2}u^2H + \tfrac{1}{2}gH^2\right)_t + \\ \left(\tfrac{3\Gamma}{4}u^2H^2 + \tfrac{\Gamma}{2}gH^3 + \tfrac{\Gamma^2}{2}uH^3 + \tfrac{\Gamma^3}{8}H^4 + \tfrac{1}{2}u^3H + guH^2\right)_x dx = 0.$$

Dividing through by $(x_1 - x_2)$ and taking the limit as $(x_1 - x_2) \longrightarrow 0$ yield the energy conservation equation

$$\left(\tfrac{\Gamma}{2}uH^2 + \tfrac{\Gamma^2}{6}H^3 + \tfrac{1}{2}u^2H + \tfrac{1}{2}gH^2\right)_t + \\ \left(\tfrac{3\Gamma}{4}u^2H^2 + \tfrac{\Gamma}{2}gH^3 + \tfrac{\Gamma^2}{2}uH^3 + \tfrac{\Gamma^3}{8}H^4 + \tfrac{1}{2}u^3H + guH^2\right)_x = 0. \tag{2.9}$$

Equations (2.6), (2.8) and (2.9) form the basis of Paper B [35] where it is explained how to construct a steady state solution for the linear shear flow by analysing a stationary hydraulic jump. It is also shown in the same paper that the flow-depth ratio of stationary shocks can be written as a function of two non-dimensional parameters: the Froude number, suitably defined in the presence of the shear flow, and a non-dimensional vorticity.

2.3 A shallow water model with vanishing buoyancy

The focus of this section is on waves at the at the interface of a two-fluid setup in which a finite uniform layer of fluid with density ρ_1 and approximate equilibrium depth h_1 is located below an upper fluid layer of density ρ_2 and is infinitesimally deep. Furthermore, assume that $\rho_1 > \rho_2$ so that the setup is initially stable. A

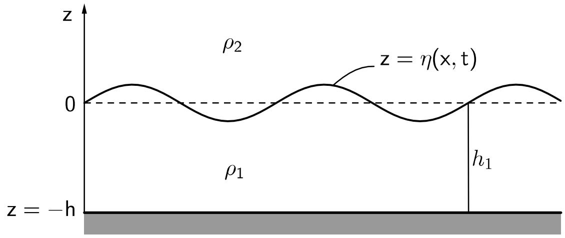


Figure 2.3: Long waves at the interface in a two-layer system of fluids

schematic representation of the setup is shown in Figure 2.3. The bottom is located at $z = -h_1$ and the interface, denoted by $z = \eta(x,t)$, is under the influence of interfacial tension whose proportionality constant τ is proportional to the curvature of the surface. It is assumed that viscosity is zero and the flow is irrotational so that it is governed by the Euler equations which can be written in terms of two velocity potentials; φ in the finite lower fluid layer and ψ in the infinite upper layer. Derivations similar to the one in this section can found in [8, 30, 76]. The equations that describe the flow are written in the form

$$\tilde{\varphi}_{\tilde{x}\tilde{x}} + \tilde{\varphi}_{\tilde{z}\tilde{z}} = 0, \quad \text{in } -h_1 < \tilde{z} < \tilde{\eta}, \tag{2.10}$$

$$\tilde{\psi}_{\tilde{x}\tilde{x}} + \tilde{\psi}_{\tilde{z}\tilde{z}} = 0, \quad \text{in } \quad \tilde{\eta} < \tilde{z} < \infty, \tag{2.11}$$

$$\tilde{\varphi}_{\tilde{z}} = 0, \quad \text{at} \quad \tilde{z} = -h_1, \tag{2.12}$$

$$\tilde{\psi}_{\tilde{z}} = 0, \quad \text{if} \quad \tilde{z} \to \infty. \tag{2.13}$$

In addition, the following boundary conditions must be satisfied at the interface

$$\tilde{\eta}_{\tilde{t}} + \tilde{\varphi}_{\tilde{x}}\tilde{\eta}_{\tilde{x}} - \tilde{\varphi}_{\tilde{z}} = 0, \quad \text{at} \quad \tilde{z} = \tilde{\eta}, \tag{2.14}$$

$$\tilde{\eta}_{\tilde{t}} + \tilde{\psi}_{\tilde{x}}\tilde{\eta}_{\tilde{x}} - \tilde{\psi}_{\tilde{z}} = 0, \quad \text{at} \quad \tilde{z} = \tilde{\eta}, \tag{2.15}$$

$$p_1 - p_2 + \tau\tilde{\eta}_{\tilde{x}\tilde{x}} = 0, \quad \text{at} \quad \tilde{z} = \tilde{\eta}. \tag{2.16}$$

Denote the typical wavelength by λ and let a represent the wave amplitude so that the following dimensionless parameters are conveniently introduced

$$\alpha = \frac{a}{h_1}, \qquad \beta = \frac{h_1}{\lambda}, \qquad \mu = \frac{\tau}{g\lambda^2(\rho_1 - \rho_2)}. \tag{2.17}$$

It is assumed that these parameters are small and of the same order. To make clear the difference in the z-scales in the two fluid layers, a proper nondimensional formulation of the original equations is necessary. Two distinct normalisations are used for the purpose of reflecting the problem geometry and bringing to light the relative significance of terms by the size of appearing coefficients. Thus,

$$\tilde{z} = \lambda Z, \quad \text{in } -h_1 < \tilde{z} < \tilde{\eta}, \tag{2.18}$$

$$\tilde{z} = h_1 z, \quad \text{in} \quad \tilde{\eta} < \tilde{z} < \infty. \tag{2.19}$$

In addition, the following dimensionless variables are introduced

$$\tilde{x} = \lambda x, \quad \tilde{t} = \frac{\lambda t}{c_0}, \quad \tilde{\eta} = a\eta, \quad \tilde{\varphi} = \frac{ga\lambda}{c_0}\varphi, \quad \tilde{\psi} = \frac{ga\lambda}{c_0}\psi, \tag{2.20}$$

where $c_0 = \sqrt{gh}$. In these new variables, the original equations become

$$\beta^2\varphi_{xx} + \varphi_{zz} = 0, \qquad \text{in} \quad -1 < z < \alpha\eta, \tag{2.21}$$
$$\psi_{xx} + \psi_{ZZ} = 0, \qquad \text{in} \quad \alpha\beta\eta < \mathbb{Z} < \infty, \tag{2.22}$$
$$\varphi_z = 0, \qquad \text{at} \quad z = -1, \tag{2.23}$$
$$\psi_Z = 0, \qquad \text{if} \quad Z \to \infty. \tag{2.24}$$

At the interface, the kinematic boundary conditions (2.14) and (2.15) can be combined as

$$\tilde{\varphi}_{\tilde{z}} = \tilde{\psi}_{\tilde{z}} + \tilde{\eta}_{\tilde{x}}\big(\tilde{\varphi}_{\tilde{x}} - \tilde{\psi}_{\tilde{x}}\big).$$

In dimensionless variables, this becomes

$$\varphi_z = \beta\psi_Z + \alpha\beta^2\eta_x\big(\varphi_x - \psi_x\big). \tag{2.25}$$

To solve for the velocity potential in the lower fluid, it is written as a formal series about $z = -1$

$$\varphi = \sum_{k=0}^{\infty} \beta^k (z+1)^k f_k(x,t), \tag{2.26}$$

which when substituted into (2.21) and the bottom boundary condition (2.23) is applied yield

$$\varphi = \sum_{n=0}^{\infty} (-1)^n \beta^{2n} \frac{(z+1)^{2n}}{(2n)!} \frac{\partial^{2n}}{\partial x^{2n}} f(x,t). \tag{2.27}$$

From this equation, it is not hard to check that

$$\varphi_z = -\beta^2(z+1)f_{xx} + O(\beta^4), \tag{2.28}$$

so that when substituted into (2.25), it gives

$$\psi_Z = -\beta f_{xx} + O(\alpha\beta, \beta^2) \quad \text{at} \quad Z = \alpha\beta\eta. \tag{2.29}$$

From (2.22) and (2.24), it is found that

$$\psi_{txx} + \psi_{tZZ} = 0, \quad \text{in} \quad Z > 0,$$
$$\psi_{tZ} = -\beta f_{txx} + O(\alpha\beta, \beta^2), \quad \text{at} \quad Z = 0,$$

By applying the Hilbert transform $\mathcal{H}$ and the Poisson kernel $\mathcal{P}$, the above boundary value problem can solved and the solution is of the form

$$\psi_t = -\beta\mathcal{H}\left(\partial_x^{-1}\mathcal{P}(z) f_{xxt}\right).$$

This implies that at the interface $Z = \alpha\beta\eta$, it is true that

$$\psi_t = -\beta\mathcal{H} f_{xt} + O(\alpha\beta, \beta^2). \tag{2.30}$$

By applying the Bernoulli's law

$$p = -\rho\left(\varphi_t + \frac{1}{2}|\bigtriangledown\varphi|^2 + g\eta\right)$$

in each fluid layer, the boundary condition (2.16) can be written in normalised variables as

$$g\rho_1\left(\eta + \varphi_t + \frac{1}{2}\alpha\varphi_x^2 + \frac{\alpha}{2\beta^2}\varphi_z^2\right) - g\rho_2\left(\eta + \psi_t + \frac{1}{2}\alpha\psi_x^2 + \frac{1}{2}\alpha\psi_Z^2\right) = \frac{\tau}{\lambda^2}\eta_{xx}.$$

Substituting (2.26)-(2.30) into the equation and keeping terms up to order α and β yield

$$g\rho_1\left(\eta + f_t + \frac{1}{2}\alpha f_x^2\right) - g\rho_2\left(\eta - \beta\mathcal{H} f_{xt}\right) = \frac{\tau}{\lambda^2}\eta_{xx} + O(\alpha\beta, \beta^2).$$

Differentiating once with respect to x and setting $u = f_x$ give

$$u_t + g\left(\frac{\rho_1-\rho_2}{\rho_1}\right)\eta_x + \alpha u u_x + \frac{\rho_2}{\rho_1}\beta\mathcal{H}u_t = \mu\left(\frac{\rho_1-\rho_2}{\rho_1}\right)\eta_{xxx} + O(\alpha\beta, \beta^2). \tag{2.31}$$

The kinematic boundary condition (2.14) in the normalised variables reads

$$\eta_t + \alpha\varphi_x\eta_x = \frac{1}{\beta^2}\varphi_z, \quad \text{at} \quad z = \alpha\eta.$$

Substituting (2.27) and (2.28) into the above equation and simplifying give

$$\eta_t + \alpha f_x\eta_x + (\alpha\eta + 1) f_{xx} + O(\alpha\beta, \beta^2) = 0.$$

Setting $u = f_x$ once more, the above equation can be written as

$$\eta_t + \left((\alpha\eta + 1)u\right)_x + O(\alpha\beta, \beta^2) = 0. \tag{2.32}$$

If all terms containing β and higher derivative of η are dropped in (2.31) and (2.32) and α is set to unity, then the system8 (1.4) appears.

Chapter 3

Hyperbolic conservation laws

The theory of conservation laws is concerned with solutions to time-dependent hyperbolic systems of Partial Differential Equations (PDEs) which have the general form

$$\frac{\partial}{\partial t}\mathbf{u}(x,t)+\frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}(x,t))=\mathbf{0}, \tag{3.1}$$

in one dimensional space where $\mathbf{u}(x,t)$ is an n-dimensional vector representing conserved quantities which in this study are mass, momentum and energy. $\mathbf{f}(\mathbf{u}(x,t))$ which is an $n\times n$ matrix, is called the flux function for the conservation laws and is usually a nonlinear function of $\mathbf{u}$. This leads to nonlinear PDEs whose solutions are in general not exact. Nonetheless, these types of conservation laws are increasingly popular among scientists and engineers because many practical problems in these fields are formulated in terms of conservation laws that consequently result in nonlinear PDEs belonging to this class. If (3.1) represents a scalar conservation law, then integration over the closed interval $[x_1,x_2]$ gives

$$\begin{aligned}\frac{\partial}{\partial t}\int_{x_1}^{x_2}u(x,t)\,dx &= \int_{x_1}^{x_2}u_t(x,t)\,dx,\\ &= \int_{x_1}^{x_2} f\big(u(x,t)\big)_x\,dx,\\ &= f\big(u(x_1,t)\big)-f\big(u(x_1,t)\big).\end{aligned}$$

The first term on the right hand side represents influx through the boundary at x_1 whiles the second term denotes outflux through the boundary at x_2. The expression implies that the quantity represented by u is conserved if the inflow is balanced by the outflow. The equation in (3.1) can be posed as a Cauchy problem. Thus, we can specify the initial data

$$u(x,0)=u_0(x),\qquad x\in\mathbb{R},\quad t\in\mathbb{R}_+. \tag{3.2}$$

A special case of the Cauchy problem is an initial data with a single discontinuity. Such an initial value problem is called the Riemann problem and it is the fundamental focus of this chapter.

The concept of hyperbolicity

Consider the $n \times n$ system of conservation laws (3.1) in the form

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u}, x, t)\mathbf{u}_x = \mathbf{0}, \tag{3.3}$$

where $\mathbf{A}$ is an $n \times n$ Jacobian matrix

$$\mathbf{A}(\mathbf{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_n} \end{bmatrix}. \tag{3.4}$$

Let $\lambda_i,\ i = 1, 2, \cdots n$ denote the eigenvalues of $\mathbf{A}$. Then the system (3.3) is said to be hyperbolic if all the λ_i are real and $\mathbf{A}$ is diagonalisable. Thus

$$\mathbf{A} = \mathbf{R}\Lambda\mathbf{R}^{-1},$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is the diagonal matrix of eigenvalues and $\mathbf{R} = [r_1|r_2|\cdots|r_n]$ is the matrix of right eigenvectors. The system (3.3) is strictly hyperbolic if the real eigenvalues are distinct, thus, $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. The eigenvalues are called characteristics and satisfy the characteristic equation

$$\frac{dx}{dt} = \lambda_i, \quad i = 1, 2, \cdots, n.$$

In the case of a scalar equation of the form

$$u_t + \lambda u_x = 0,$$

with smooth initial data (3.2), the characteristics in the $x-t$-plane satisfy the Ordinary Differential Equation (ODE)

$$\frac{dx}{dt} = \lambda, \quad x(0) = \xi.$$

The solution is given by

$$u(x, t) = u_0(x - \lambda t),$$

and propagates to the right if $\lambda > 0$ and to the left if $\lambda < 0$.

3.1 Generalized weak solution

It is important to note that as time evolves, the solution propagates unchanged along the characteristic curves, thus,

$$\begin{aligned}\frac{d}{dt}u\big(x(t),t\big) &= \frac{d}{dt}u\big(x(t),t\big) + \frac{dx}{dt}\frac{d}{dx}u\big(x(t),t\big)\\ &= u_t + \lambda u_x\\ &= 0.\end{aligned}$$

This imply that the solution profile remains unchanged over time. In other words, the solution $u(x,t)$ is constant along the characteristic curves in the $x-t$-plane and depends only on the initial data. The solution can be determined uniquely for small enough time t by following characteristics which are straight lines in the $x-t$-plane. However, if

$$x_1 < x_2 \quad \Rightarrow \quad f\big(u_0(x_1)\big) > f\big(u_0(x_1)\big), \tag{3.5}$$

then the characteristics may cross and uniqueness may no longer be established [48, 49]. For arbitrary scalar equation

$$u_t + f(u)_x = 0, \tag{3.6}$$

with smooth initial data (3.2) where $f(u)$ is convex, consider characteristics originating at two initial points ξ and $\xi + dx$. Since the characteristics are straight lines, they are given by

$$x = \xi + f'\big(u_0(\xi)\big)t, \tag{3.7}$$

$$x = \xi + dx + f'\big(u_0(\xi + dx)\big)t. \tag{3.8}$$

Taylor expansion of equation (3.8) gives

$$x = \xi + dx + f'\big(u_0(\xi)\big)t + f''\big(u_0(\xi)\big)u_0'(\xi)dxt + O\big((dx)^2\big)t. \tag{3.9}$$

Subtracting equation (3.7) from equation (3.9) gives

$$dx = -f''\big(u_0(\xi)\big)u_0'(\xi)dxt,$$

which implies that

$$t = -\frac{1}{f''\big(u_0(\xi)\big)u_0'(\xi)}.$$

If we let t^* represent the minimum time where the characteristics first cross, then it can be written as

$$t^* = -\frac{1}{\min\limits_{-\infty<\xi<\infty}\Big(f''\big(u_0(\xi)\big)u_0'(\xi)\Big)}. \tag{3.10}$$

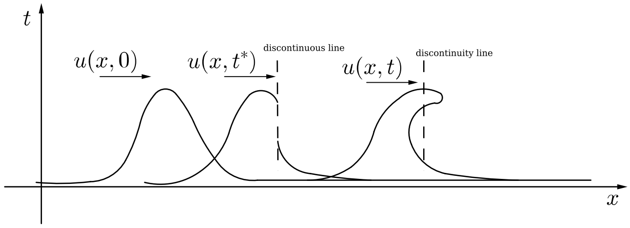


Figure 3.1: Changing solution profile leading to shock formation. The initial wave profile propagates until t^* when the characteristics first cross and shock forms. At time $t > t^*$, classical solutions may not exist.

At t^*, the wave breaks leading to the formation of shocks. A schematic representation of the solution profile as time evolves is shown in Figure 3.1. Beyond this time t^*, no classical solutions of the PDE exist but it is possible to find generalized weak solutions. If $\varphi(x,t) \in C_0^1(\mathbb{R} \times \mathbb{R}_+)$ is a test function, then integrating equation (3.6) over space and time yield

$$\int_0^\infty \int_{-\infty}^\infty \left[\varphi u_t + \varphi f(u)_x\right] \, dx\, dt = 0.$$

Applying integration by parts to the above gives

$$\int_0^\infty \int_{-\infty}^\infty \left[\varphi_t u + \varphi_x f(u)\right] \, dx\, dt = -\int_{-\infty}^\infty \varphi(x,0)u(x,0)\, dx. \tag{3.11}$$

We say that $u(x,t)$ is a weak solution of the conservation law (3.6) if equation (3.11) holds for all test functions $\varphi \in C_0^1$. If the weak solution that is constructed is not unique, further conditions are imposed on the solution in order to select the one that is physically meaningful. Such conditions are called entropy conditions.

However, if the initial data is not smooth, thus, if there is a singularity in $u_0(\xi)$, then the solution will have a singularity of the same order along characteristics emanating from ξ but will remain smooth along characteristics through smooth portions of the data. In particular, it is generally known that for linear hyperbolic equations, discontinuities propagate along characteristics. Nevertheless, not all discontinuities are allowed as part of a weak solution. An acceptable discontinuity has to satisfy a jump condition. Consider a domain $\Omega = \Omega_- \cup \Omega_+$ where we define

$$\begin{aligned}
\Omega_- &= \{(x,t) : 0 < t < \infty, \ -\infty < x < \xi(t)\}, \\
\Omega_+ &= \{(x,t) : 0 < t < \infty, \ \xi(t) < x < \infty\}, \\
u(x,t) &= g(x), \\
\varphi(x,t) &= 0.
\end{aligned}$$

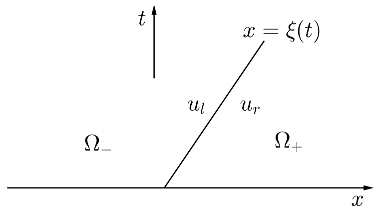


Figure 3.2: The domain Ω

$x = \xi(t)$ is a curve along which the solution is discontinuous. A sketch of the domain is shown in Figure 3.2. Applying (3.11) on this domain gives

$$\begin{aligned}&\iint_{\Omega} \left[\varphi_t u + \varphi_x f(u)\right] \, dx\, dt = \\ &\quad \iint_{\Omega_-} \left[\varphi_t u + \varphi_x f(u)\right] \, dx\, dt + \iint_{\Omega_+} \left[\varphi_t u + \varphi_x f(u)\right] \, dx\, dt = 0.\end{aligned} \tag{3.12}$$

If $u, v \in C^1(\bar{U})$, then by using the integration by parts formula [16]

$$\int_U u_{x_i} v \, dx = -\int_U u v_{x_i} \, dx + \int_{\partial U} u v \nu^i \, dS, \quad i = 1, 2, \cdots, n,$$

it is not hard to check that

$$\begin{aligned}&\iint_{\Omega_-} \left[\varphi_t u + \varphi_x f(u)\right] \, dx\, dt = \\ &\qquad -\iint_{\Omega_-} \left[\varphi u_t + \varphi f(u)_x\right] \, dx\, dt + \int_{x=\xi(t)} \left[\varphi u_l \nu^2 + \varphi f(u_l) \nu^1\right] \, dS,\end{aligned}$$

where ν^i, $i = 1, 2, \cdots, n$ are the outward unit vectors. The equation above can be simplified further to get

$$\begin{aligned}&\iint_{\Omega_-} \left[\varphi_t u + \varphi_x f(u)\right] \, dx\, dt = \\ &\qquad -\iint_{\Omega_-} \varphi\left[(u_l)_t + f\big((u_l)\big)_x\right] \, dx\, dt + \int_{x=\xi(t)} \varphi\left[u_l \nu^2 + f(u_l) \nu^1\right] \, dS.\end{aligned}$$

Notice here that u_l is a weak solution of (3.6), thus

$$(u_l)_t + f\big((u_l)\big)_x = 0,$$

so that

$$\iint_{\Omega_-} \left[\varphi_t u + \varphi_x f(u)\right] \, dx\, dt = \int_{x=\xi(t)} \varphi\left[u_l \nu^2 + f(u_l) \nu^1\right] \, dS. \tag{3.13}$$

Similarly, applying integration by parts in the domain Ω_+ gives

$$\iint_{\Omega_+} \left[\varphi_t u + \varphi_x f(u)\right] dx\, dt = -\int_{x=\xi(t)} \varphi\left[u_r \nu^2 + f(u_r)\nu^1\right] dS. \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12), one obtains

$$\int_{x=\xi(t)} \varphi\left[u_l \nu^2 + f(u_l)\nu^1 - u_r \nu^2 - f(u_r)\nu^1\right] dS = 0.$$

Since this holds for all test functions φ, the integrand must vanish along $x = \xi(t)$, thus

$$u_l \nu^2 + f(u_l)\nu^1 - u_r \nu^2 - f(u_r)\nu^1 = 0.$$

Algebraic manipulation of the above equation yields

$$-\frac{\nu_2}{\nu_1} = \frac{f(u_l) - f(u_r)}{u_l - u_r}. \tag{3.15}$$

The curve has a slope which is equal to the negative reciprocal of the normal vectors, thus

$$\frac{dt}{dx} = \frac{1}{\xi'(t)} = -\frac{\nu_1}{\nu_2} \quad \Rightarrow \quad \xi'(t) = -\frac{\nu_2}{\nu_1}.$$

Consequently, we find that

$$\xi'(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r} \equiv \frac{[f]}{[u]}. \tag{3.16}$$

The expression in (3.16) is called the Rankine-Hugoniot jump condition and holds for vectors **f(u)** and **u** as well. However, in the case of vectors, only jumps for which [**f**] and [**u**] are linearly dependent are allowed as part of the solution.

3.2 The Riemann problem

The Riemann problem is fundamental in the study of hyperbolic conservation laws and solutions to such a problem consist of elementary waves such as shock waves and rarefaction curves. In some cases, the solutions may contain contact discontinuities. In what follows, we shall study the properties of these elementary waves and describe the equations that define them. The Riemann problem is a special case of the Cauchy problem where the initial data contains a single discontinuity. In particular, equation (3.6) together with the data

$$u(x,0) = \begin{cases} u_l, & \text{if} \quad x < 0, \\ u_r, & \text{if} \quad x > 0, \end{cases} \tag{3.17}$$

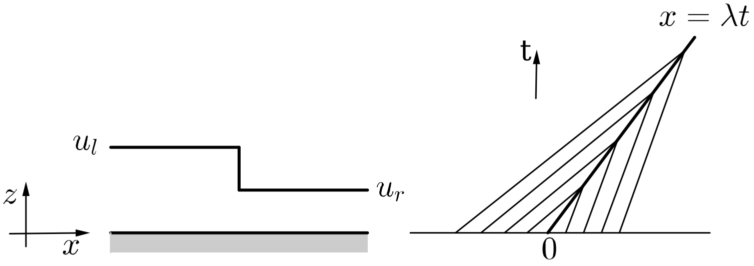


Figure 3.3: A shock wave propagating to the right

whose solution consists of elementary waves is called the Riemann problem. As an example to demonstrate the solution profile of a shock wave, we shall consider the first equation in (1.5) which is repeated below

$$u_t + uu_x = 0,$$

and it is called the Burgers' equation. The solution structure of a shock wave profile is shown in Figure 3.3 and Figure 3.4. To establish the solution, we shall prove that

Lemma 3.2.1 *The function $u(x,t)$ defined by*

$$u(x,t) = \begin{cases} u_l, & x < \lambda t, \\ u_r, & x > \lambda t, \end{cases}$$

where $\lambda = (u_l + u_r)/2$ represents weak solution of the inviscid Burgers' equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0,$$

if

$$\int_0^\infty \int_{-\infty}^\infty (\varphi_t u + \varphi_x f(u))\, dx\, dt = -\int_{-\infty}^\infty \varphi(x,0)u(x,0)\, dx \tag{3.18}$$

is satisfied for all functions $\varphi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)$.

Proof For $t = 0$, we have the initial data

$$u(x,0) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases}$$

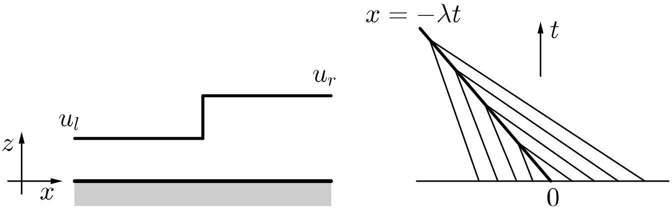


Figure 3.4: A shock wave propagating to the left

The right-hand side of (3.18) could be expressed as

$$-\int_{-\infty}^{\infty} \varphi(x,0)u(x,0)\, dx = -u_l \int_{-\infty}^{0} \varphi(x,0)dx - u_r \int_{0}^{\infty} \varphi(x,0)\, dx.$$

The left-hand side of (3.18) could also be expressed as

$$\int_0^\infty \int_{-\infty}^{\infty} \left(\varphi_t u + \varphi_x \frac{u^2}{2}\right) dx\, dt = \int_0^\infty \int_{-\infty}^{\lambda t} \left(\varphi_t u_l + \varphi_x \frac{u_l^2}{2}\right) dx\, dt + \int_0^\infty \int_{\lambda t}^{\infty} \left(\varphi_t u_r + \varphi_x \frac{u_r^2}{2}\right) dx\, dt,$$

which when expanded gives

$$\int_0^\infty \int_{-\infty}^{\infty} \left(\varphi_t u + \varphi_x \frac{u^2}{2}\right) dx\, dt = u_l \int_0^\infty \int_{-\infty}^{\lambda t} \varphi_t\, dx\, dt + \frac{u_l^2}{2} \int_0^\infty \int_{-\infty}^{\lambda t} \varphi_x\, dx\, dt + u_r \int_0^\infty \int_{\lambda t}^{\infty} \varphi_t\, dx\, dt + \frac{u_r^2}{2} \int_0^\infty \int_{\lambda t}^{\infty} \varphi_x\, dx\, dt.$$

Assume that the shock speed $\lambda > 0$ (noting that the same procedure works for the case $\lambda < 0$). Using the appropriate limits for the first and the third integrands, we obtain

$$\int_0^\infty \int_{-\infty}^{\infty} \left(\varphi_t u + \varphi_x \frac{u^2}{2}\right) dx\, dt = u_l \left(\int_{-\infty}^{0} \int_0^\infty \varphi_t + \int_0^\infty \int_{\frac{x}{\lambda}}^{\infty} \varphi_t\right) dt\, dx + \frac{u_l^2}{2} \int_0^\infty \int_{-\infty}^{\lambda t} \varphi_x\, dx\, dt + u_r \int_0^\infty \int_0^{\frac{x}{\lambda}} \varphi_t\, dt\, dx + \frac{u_r^2}{2} \int_0^\infty \int_{\lambda t}^{\infty} \varphi_x\, dx\, dt.$$

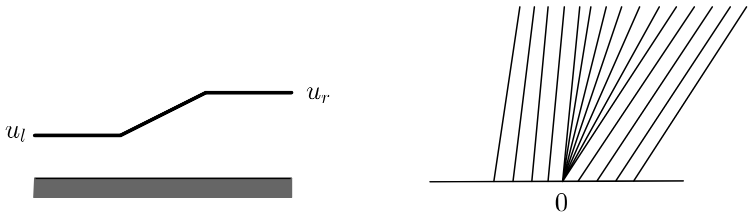


Figure 3.5: Left panel: Rarefaction solution of Burgers equation. Right panel: corresponding characteristics.

This equation can be simplified to yield

$$\int_0^\infty \int_{-\infty}^\infty \left(\varphi_t u + \varphi_x \frac{u^2}{2}\right) dx\,dt =$$
$$-u_l \int_{-\infty}^0 \varphi(x,0)dx - u_l \int_0^\infty \varphi\left(x, \frac{x}{\lambda}\right) dx + u_r \int_0^\infty \left(\varphi\left(x, \frac{x}{\lambda}\right) - \varphi(x,0)\right) dx$$
$$+ \frac{u_l^2}{2} \int_0^\infty \varphi(\lambda t, t)\,dt - \frac{u_r^2}{2} \int_0^\infty \varphi(\lambda t, t)\,dt.$$

By letting $y = \lambda t$ in the above equation, we obtain

$$\int_0^\infty \int_{-\infty}^\infty \left(\varphi_t u + \varphi_x \frac{u^2}{2}\right) dx\,dt = -u_l \int_{-\infty}^0 \varphi(x,0)\,dx - u_r \int_0^\infty \varphi(x,0)\,dx$$
$$-(u_l - u_r) \int_0^\infty \varphi\left(x, \frac{x}{\lambda}\right) dx + \frac{(u_l^2 - u_r^2)}{2\lambda} \int_0^\infty \varphi\left(y, \frac{y}{\lambda}\right) dy.$$

Using the expression for the shock speed, λ, defined in the Lemma reduces the equation to

$$\int_0^\infty \int_{-\infty}^\infty \left(\varphi_t u + \varphi_x \frac{u^2}{2}\right) dx\,dt = -u_l \int_{-\infty}^0 \varphi(x,0)dx - u_r \int_0^\infty \varphi(x,0)\,dx,$$

which is exactly the same as the right hand side of (3.18).

□

The next group of elementary wave solutions that are of interest to our discussion are rarefaction waves which take the form [20, 48]

$$u(x,t) = \begin{cases} u_l, & \frac{x}{t} \leq \lambda(u_l), \\ u\left(\frac{x}{t}\right), & \lambda(u_l) \leq \frac{x}{t} \leq \lambda(u_r), \\ u_r, & \frac{x}{t} \geq \lambda(u_r), \end{cases}$$

where $\lambda(u_l) \leq \lambda(u_r)$. A schematic representation of a rarefaction wave profile is shown in Figure 3.5.

Furthermore, we focus on the nonlinear properties of the system of equations given in (1.5) which we recall here

$$u_t + (u^2/2)_x = 0, \tag{3.19}$$

$$\eta_t + \big((\eta+1)u\big)_x = 0. \tag{3.20}$$

In particular, we study the shock curves of these equations and their properties by noting first that the system (3.19), (3.20) can be written in the general form

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0,$$

where

$$\mathbf{U} = \begin{pmatrix} u \\ \eta \end{pmatrix} \quad \text{and} \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} g(u,\eta) \\ h(u,\eta) \end{pmatrix} = \begin{pmatrix} u^2/2 \\ (\eta+1)u \end{pmatrix}.$$

The flux Jacobian of $\mathbf{F}(\mathbf{U})$ is given by

$$J = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{pmatrix} u & 0 \\ \eta+1 & u \end{pmatrix}, \tag{3.21}$$

which has the repeated eigenvalue

$$\lambda_{1,2} = u,$$

with the corresponding right eigenvector

$$r_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It is well-known that the classical theory of hyperbolic laws in one dimensional space usually demands that the system be strictly hyperbolic with either genuinely nonlinear or linearly degenerate characteristic fields. In this case, existence of entropy weak solutions can be found when the initial data have small total variation [17, 47, 51, 65]. Weak solutions of this type are usually discontinuous and consist of elementary Lax-admissible waves. Nevertheless, many nonlinear hyperbolic

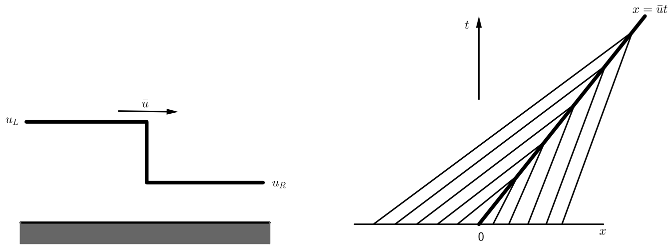


Figure 3.6: Left panel: Shock solution of Burgers equation. Right panel: corresponding characteristics.

systems used by physicists and engineers to model physical phenomena do not satisfy the above hypotheses entirely. For example, modelling a two-phase flow of a mixture of steam and water in a cooling process of conventional nuclear reactors by water under pressure is described by a system of equations which do not satisfy the basic assumptions of the theory of nonlinear hyperbolic systems [69]. In contrast to strictly hyperbolic conservation laws, non-strictly hyperbolic systems do not have a definitive theory which guarantees existence of weak solutions. An interesting feature of (3.19), (3.20) is that the characteristic speeds coincide in phase space. Consequently, the classical theory of strictly hyperbolic conservation laws does not apply. However, it is shown in this work that it is possible to construct a unique solution to the Riemann initial value problem associated with (3.19), (3.20).

For an arbitrary constant left state (u_L, η_L) and a right state (u_R, η_R), the Rankine-Hugoniot jump conditions for (3.19), (3.20) are respectively

$$-c[u] + [u^2/2] = 0, \tag{3.22}$$

$$-c[\eta] + [(\eta+1)u] = 0, \tag{3.23}$$

where $[u] = u_R - u_L$ and $[\eta] = \eta_R - \eta_L$. The shock speed in (3.22) is well known and is written as

$$c = (u_L + u_R)/2 \equiv \bar{u}, \tag{3.24}$$

and satisfies the Lax entropy condition

$$\lambda_i(u_R) \leq c \leq \lambda_i(u_L), \quad i = 1, 2.$$

For $[u] \neq 0$, a piecewise continuous function $u(x,t)$ with a single discontinuity travelling with speed $\bar{u}$ and having u_L and u_R on opposite side of the discontinuity is a weak solution of (3.19). Thus, we obtain the following lemma:

Lemma 3.2.2 *The function* $u(x,t)$ *defined by*

$$u(x,t) \;=\begin{cases} u_L, & \text{if } x < \bar{u}t, \\ u_R, & \text{if } x > \bar{u}t, \end{cases} \tag{3.25}$$

where $\bar{u}$ *is given in* (3.24)*, represents a weak solution of* (3.19) *if*

$$\int_0^\infty \int_{-\infty}^\infty \left(\phi_t u + \phi_x \frac{u^2}{2}\right) dxdt + \int_{-\infty}^\infty \phi(x,0)u(x,0)dx = 0,$$

holds for all functions $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)$

Remark *Lemma* (3.2.2) *represents the case where* $u_L > u_R$ *and is illustrated in Figure 3.6. Similarly, if* $u_L < u_R$ *then the solution of* (3.19) *is given by a rarefaction wave*

$$u(x,t) = \begin{cases} u_L, & \text{if } \quad x \leq u_L t, \\ x/t, & \text{if } \; u_L t < x < u_R t, \\ u_R, & \text{if } \quad x \geq u_R t, \end{cases} \tag{3.26}$$

which is illustrated in Figure 3.7.

Note also that weak solutions of (3.19),(3.20) satisfy (3.23), so that with the help of (3.24) we can obtain the condition

$$\begin{aligned} -c(\eta_R - \eta_L) + (\eta_R + 1)u_R - (\eta_L + 1)u_L &= 0 \\ \eta_R(u_R - c) - \eta_L(u_L - c) + u_R - u_L &= 0 \\ \eta_R(u_R - u_L) + \eta_L(u_L - u_R) + 2u_R - 2u_L &= 0 \end{aligned}$$

which can be simplified further to obtain

$$\eta_R = -(\eta_L + 2). \tag{3.27}$$

However, this last relation does not hold for singular solutions. For constant states u_L, u_R, η_L and η_R, let the initial data for (3.19), (3.20) be given by

$$u(\xi,0) = \begin{cases} u_L, & \text{if } \; \xi < 0, \\ u_R, & \text{if } \; \xi > 0, \end{cases} \qquad \eta(\xi,0) = \begin{cases} \eta_L, & \text{if } \; \xi < 0, \\ \eta_R, & \text{if } \; \xi > 0, \end{cases} \tag{3.28}$$

respectively. The main goal here is to solve the Riemann problem for (3.19), (3.20) subject to the initial data (3.28).

Theorem 3.2.3 *Assume that the constant states* u_L, u_R, η_L *and* η_R *are given such the* (3.28) *represents Riemann initial data for the system* (3.19), (3.20).

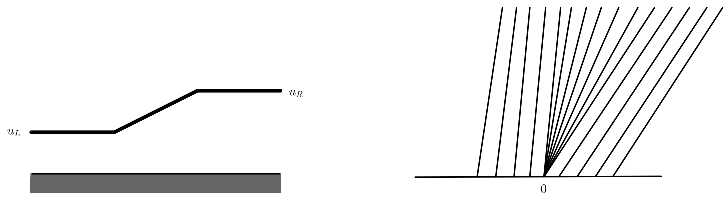


Figure 3.7: Left panel: Rarefaction solution of Burgers equation. Right panel: corresponding characteristics.

(a) If $u_L > u_R$, then u has a single shock given in (3.25)*, whereas η has a single jump*

$$\eta(x,t) = \begin{cases} \eta_L, & \text{if} \quad x < \bar{u}t, \\ \eta_R, & \text{if} \quad x > \bar{u}t, \end{cases}$$

together with a propagating Dirac mass whose strength is given by

$$[w] = (t/2)\big((u_L - u_R)(\eta_L + \eta_R + 2)\big). \tag{3.29}$$

(b) If $u_L < u_R$, then the weak solution of u is a rarefaction given by (3.26) *whereas η has two jump discontinuities given by*

$$\eta(x,t) = \begin{cases} \eta_L, & \text{if} \quad x < u_L t, \\ -1, & \text{if} \;\; u_L y \le x \le u_R t, \\ \eta_R, & \text{if} \;\; u_R t < x. \end{cases} \tag{3.30}$$

Proof To prove this theorem we define a function $w(x,t)$ by

$$w(x,t) = \int_{-\kappa}^{x} \eta(s,t)ds, \tag{3.31}$$

for an arbitrary positive constant κ. From this equation the following are obtained

$$\frac{\partial w}{\partial x} = \eta, \quad \frac{\partial^2 w}{\partial x^2} = \eta_x, \quad \frac{\partial^3}{\partial x^3} = \eta_{xx}, \quad \frac{\partial^2 w}{\partial t \partial x}.$$

By using these derivatives, the system (3.19), (3.20) transforms into

$$u_t + uu_x = 0, \tag{3.32}$$
$$w_t + uw_x = -u. \tag{3.33}$$

The first equation (3.32) in the new system is the well known Burgers' equation which together with the Riemann initial data in u given in (3.28) admits shock and rarefaction solutions which are given in (3.25) and (3.26). Moreover, it is found that

$$w(\xi,0) = \int_{-\kappa}^{\xi} \eta(s,0)\, ds,$$

so that

$$\begin{aligned} \xi < 0 \quad &\Rightarrow \quad w(\xi,0) = \int_{-\kappa}^{\xi} \eta_L\, ds = \eta_L(\xi+\kappa), \\ \xi > 0 \quad &\Rightarrow \quad w(\xi,0) = \int_{-\kappa}^{0} w(s,0)\, ds + \int_{0}^{\xi} w(s,0)\, ds = \eta_L\kappa + \eta_R\xi. \end{aligned}$$

Notice that the second equation (3.33) is a nonhomogeneous transport equation in w and has the Riemann initial data

$$w(\xi,0) \equiv w_0(\xi) = \begin{cases} \eta_L\kappa+\eta_L\xi, & \text{if} \quad \xi \le 0, \\ \eta_L\kappa+\eta_R\xi, & \text{if} \quad \xi \ge 0. \end{cases} \tag{3.34}$$

It is quite clear that this initial data is linear and piecewise continuous. However, the nonhomogeneous part, as well as the variable coefficient $u(x,t)$ in equation (3.33), contains a discontinuity which may lead to the development of discontinuities in $w(x,t)$. The propagation of w is along curves $x(t)$ which satisfy the characteristic equation

$$\frac{dx}{dt} = u(x,t), \qquad x(0) = \xi. \tag{3.35}$$

Nevertheless, $w(x,t)$ is not constant along these curves but satisfies the differential equation

$$\frac{d}{dt} w(x,t) = -u(x,t), \tag{3.36}$$

and is found by solving the ODEs (3.35) and (3.36) [48]. In the first case where $u_L > u_R$, the solution of (3.32) is given by a shock wave (3.25) propagating at a speed given in (3.24). The technique used here is to substitute this travelling shock wave into (3.33) and solve the resulting equation by using the method of characteristics. If $\xi \le 0$, then (3.33) simplifies to

$$w_t + u_L w_x = -u_L,$$

so that

$$\frac{dx}{dt} = u_L \quad \Rightarrow \quad x = u_L t + C. \tag{3.37}$$

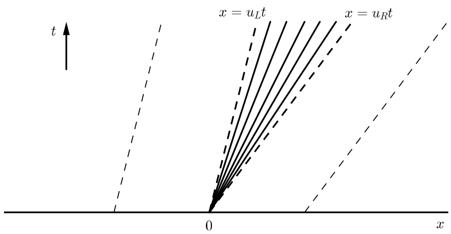


Figure 3.8: Schematic representation of the characteristics for w (dashed) when $u_L < u_R$. The rarefaction fan in u (solid lines) is bounded by $x = u_L t$ and $x = u_R t$.

At $(\xi, 0)$, it is easy to see that $\xi = C$. In a similar way, if $\xi \geq 0$, then (3.33) takes the form

$$w_t + u_R w_x = -u_R,$$

such that

$$\frac{dx}{dt} = u_R \quad \Rightarrow \quad x = u_R t + C.$$

Just as done above, it is easy to find that $\xi = C$ at $(\xi, 0)$. The characteristics in this case are

$$x(t) = \begin{cases} u_L t + \xi, & \text{if} \quad \xi < (\bar{u} - u_L)t, \\ u_R t + \xi, & \text{if} \quad \xi > (\bar{u} - u_R)t, \end{cases}$$

and they propagate into the shock. If $\xi \leq 0$, then $u(x,t) = u_L$ and (3.33) becomes

$$w_t + u_L w_x = -u_L,$$

which has the solution

$$\begin{aligned} w(x,t) &= w_0(x - u_L t) - \int_0^t u_L \, ds \\ &= \eta_L (x - u_L t + \kappa) - u_L t. \end{aligned}$$

Similarly, if $\xi \geq 0$, then $u(x,t) = u_R$ and (3.33) becomes

$$w_t + u_R w_x = -u_R,$$

which has the solution

$$w(x,t) = \eta_L \kappa + \eta_R (x - u_R t) - u_R t,$$

such that $w(x,t)$ is given by

$$w(x,t) = \begin{cases} \eta_L\kappa + \eta_L(x - u_L t) - u_L t, & \text{if} \quad x < \bar{u}t, \\ \eta_R\kappa + \eta_R(x - u_R t) - u_R t, & \text{if} \quad x > \bar{u}t. \end{cases}$$

The solution $\eta(x,t)$ in (a) is obtained by evoking the expression in (3.31). Since the initial assumption is that $u_L > u_R$, the characteristics emanating from $\xi < 0$ will propagate values of $w(x,t)$ which are different from those propagated by the characteristics originating from $\xi > 0$. Consequently, if the jump in $u(x,t)$ is large, then it will lead to a jump discontinuity in $w(x,t)$ across the shock. The characteristics together with the initial Riemann data (3.34) give

$$\begin{aligned} w_L &= \eta_L\kappa + \big(\eta_L(\bar{u} - u_L) - u_L\big)t, \\ w_R &= \eta_L\kappa + \big(\eta_R(\bar{u} - u_R) - u_R\big)t. \end{aligned}$$

where w_L and w_R represent respectively the left and right limits of $w(x,t)$ at the shock $x = \bar{u}t$. From these expressions, we obtain the jump in $w(x,t)$ whose strength is given by

$$\begin{aligned} [w] = w_R - w_L &= \big(\eta_R(\bar{u} - u_R) - u_R\big)t - \big(\eta_L(\bar{u} - u_L) - u_L\big)t \\ &= \left(\eta_R\left(\frac{u_L - u_R}{2}\right) - u_R\right)t - \left(\eta_L\left(\frac{u_R - u_L}{2}\right) - u_L\right)t, \end{aligned}$$

which simplifies to

$$[w] = \frac{t}{2}\big(u_L - u_R\big)\big(\eta_L + \eta_R + 2\big). \tag{3.38}$$

In the second case where $u_L < u_R$, the characteristics emanating from $\xi < 0$ propagate at a speed of the rarefaction tail whereas those originating at $\xi > 0$ propagate parallel to the rarefaction head of the wave as shown in Figure 3.9. The characteristic equations in this case are given by

$$x(t) = \begin{cases} u_R t + \xi, & \text{if} \quad \xi > 0, \\ \gamma t, & \text{if} \quad \xi = 0, \\ u_L t + \xi, & \text{if} \quad \xi < 0, \end{cases}$$

where γ is an arbitrary constant only subject to the constraint $u_L < \gamma < u_R$. It is not hard to find the solution $w(x,t)$ in each of the three regions in Figure 3.9 as the process is identical to the previous calculations. A graphical representation of the characteristics is shown in Figure 3.8. By applying (3.34), the solution of (3.33) is

$$w(x,t) = \begin{cases} \eta_L\kappa + \eta_L(x - u_L t) - u_L t, & \text{if} \quad x < u_L t, \\ \eta_L\kappa - x, & \text{if} \quad u_L t \le x \le u_R t, \\ \eta_l\kappa + \eta_R(x - u_R t) - u_R t, & \text{if} \quad u_R t < x. \end{cases}$$

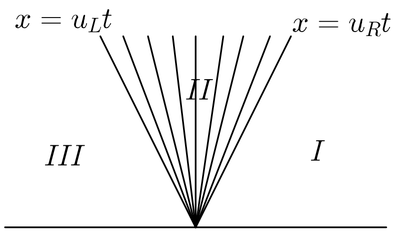


Figure 3.9: The rarefaction tail described by $x = u_L t$ and its head is given by $x = u_R t$.

A partial derivative with respect to x gives the solution $\eta(x,t)$ of (3.20) defined in (3.30).

□

3.3 Stationary hydraulic jump

A shallow water system incorporating constant background shear has been given in Chapter 1, equation (1.3) and we recall it here

$$\begin{aligned} \partial_t H + \partial_x \left(\tfrac{\Gamma}{2} H^2 + uH \right) &= 0, \\ \partial_t \left(\tfrac{\Gamma}{2} H^2 + uH \right) + \partial_x \left(\tfrac{\Gamma^2}{3} H^3 + \Gamma u H^2 + u^2 H + \tfrac{1}{2} g H^2 \right) &= 0. \end{aligned} \tag{3.39}$$

The first equation in this system represents mass conservation and the second represents momentum conservation. If the flow variables are smooth so that the solutions are also smooth, then an equivalent system has the form

$$\partial_t H + \partial_x \left(\tfrac{\Gamma}{2} H^2 + uH \right) = 0, \tag{3.40}$$

$$\partial_t u + \partial_x \left(\tfrac{1}{2} u^2 + gH \right) = 0. \tag{3.41}$$

The purpose of this section is to analyse the above equations and explain how to construct a steady state solution. The analysis of the hydraulic jump is done using the Froude number which is defined in such a way that it takes into account the

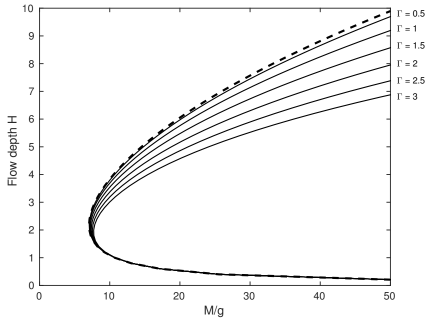


Figure 3.10: The momentum function M/g is plotted against the flow depth for various strengths of vorticity. The dashed line shows the irrotational case $\Gamma = 0$. The other curves show the respective cases of the strengths of the vorticity Γ.

average flow velocity (see equation (2.5)) over the entire fluid depth. In dimensionless variables, the depth averaging integral is defined as

$$\mathrm{Fr} = \frac{\frac{1}{H}\int_0^H U + u\,dz}{\sqrt{gH}},$$

which simplifies readily to

$$\mathrm{Fr} = \frac{u + \frac{\Gamma}{2}H}{\sqrt{gH}}.$$

The analysis is based on the assumption that the hydraulic jump is stationary and that the velocity and water depth increase across the jump. If Λ represents the volume flow rate per unit width, then the conservation of mass necessitates that

$$\Lambda = u_L H_L + \frac{\Gamma}{2}H_L^2 = u_R H_R + \frac{\Gamma}{2}H_R^2, \qquad (3.42)$$

be satisfied. Subscript L indicates upstream variables and subscript R represents downstream variables. Using the concept of momentum conservation explained in the previous chapter, an expression for the momentum conservation across the discontinuity is obtained as

$$\frac{\Gamma^2}{3}H_L^3 + \Gamma u_L H_L^2 + u_L^2 H_L + \frac{1}{2}gH_L^2 = \frac{\Gamma^2}{3}H_R^3 + \Gamma u_R H_R^2 + u_R^2 H_R + \frac{1}{2}gH_R^2. \qquad (3.43)$$

Specifically, if we define $M = \frac{\Gamma^2}{3}H^3 + \Gamma u H^2 + u^2 H + \frac{1}{2}gH^2$, then the quantity M/g is the analogue of the momentum function used in hydraulic engineering.

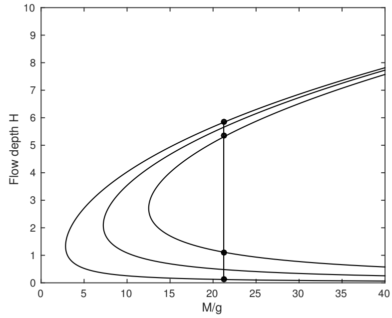


Figure 3.11: The momentum function M/g is plotted against the flow depth H for constant vorticity $\Gamma = 1.5$ and different values of the flow rate per unit width Λ.

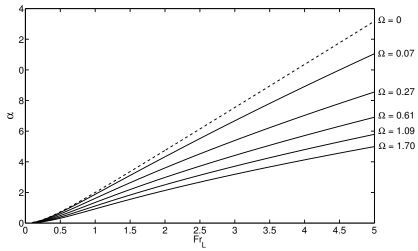


Figure 3.12: The ratio of right to left Froude numbers α plotted against the left Froude number Fr_L for various strengths of vorticity Ω. The dashed curve depicts the irrotational case.

Since M needs to be preserved through a stationary jump, for a given volume flow rate per unit width Λ, it is possible to find the conjugate flow depths by plotting the curve

$$M = \frac{1}{12}\Gamma^2 H^3 + \frac{\Lambda^2}{H} + \frac{1}{2} g H^2.$$

The plot is shown in Figure 3.10 for $\Lambda = 10$ and a variety of background vorticities ranging from $\Gamma = 0$ to $\Gamma = 3$. On the other hand, Figure 3.11 shows the graphs for a fixed $\Gamma = 1.5$ but for a variety of values of Λ.

If the expression for Λ is substituted into relation (3.43), it yields

$$\Lambda^2 \left(\frac{1}{H_R} - \frac{1}{H_L} \right) = \frac{\Gamma^2}{12} \left(H_L^3 - H_R^3 \right) + \frac{1}{2} g \left(H_L^2 - H_R^2 \right). \tag{3.44}$$

Substituting the expression for the Froude number stated above and doing further algebraic simplification, the cubic function

$$\frac{\Gamma^2}{6g} H_L \alpha^3 + \left(1 + \frac{\Gamma^2}{6g} H_L\right) \alpha^2 + \left(1 + \frac{\Gamma^2}{6g} H_L\right) \alpha - 2\mathrm{Fr}_L^2 = 0, \tag{3.45}$$

appears where $\alpha = H_R/H_L$ is the ratio of depths. Notice that the strength of the vorticity depends on the non-dimensional parameter $\Omega = H_L \Gamma^2/6g$ so that we can write the equation in the final form

$$\Omega \alpha^3 + (1 + \Omega)\alpha^2 + (1 + \Omega)\alpha - 2\mathrm{Fr}_L^2 = 0. \tag{3.46}$$

The cubic equation can be solved for any value of Ω. Figure 3.12 shows a plot of α as a function of Fr_L for a number of values of the vorticity Γ. It is apparent that larger values of Ω have the effect of moderating the strength of the hydraulic jump. This result is published in Paper B [35].

Chapter 4

The weak asymptotic method

The method of weak asymptotic is used on conservation laws to establish singular solutions which can contain Dirac-δ distributions and the goal in this section is to apply the method to find a unique δ-type solution to the Brio system

$$\begin{aligned} \partial_t u + \partial_x\left(\tfrac{u^2+v^2}{2}\right) &= 0, \\ \partial_t v + \partial_x\big(v(u-1)\big) &= 0. \end{aligned} \tag{4.1}$$

This system is obtained from the highly complex and mathematically challenging ideal MHD system which requires numerical approximation of solutions in order to understand its dynamics. An interesting feature of this system is that the quantities u and v denote the velocity components of the fluid whose dynamics is determined by MHD forces. This feature implies that the system represents the conservation of the velocities. However, it is worth noting that velocity conservation in this form holds only in idealized situations in which the solutions are smooth. This means that in the physical world, this assumption is restrictive and the limitation is manifested in the fact that the system cannot be adequately solved even for the simplest piece-wise constant Riemann initial data

$$u|_{t=0} = \begin{cases} U_L, & x \leq 0 \\ U_R, & x > 0 \end{cases}, \qquad v|_{t=0} = \begin{cases} V_L, & x \leq 0 \\ V_R, & x > 0 \end{cases}. \tag{4.2}$$

Furthermore, the characteristic fields of this system are neither genuinely nonlinear nor linearly degenerate in certain regions in the (u,v)-plane as pointed out in [19]. This implies that the standard theory of hyperbolic conservation laws which can be found in [10, 20, 48] does not entirely apply and the classical Riemann solution admissible in the sense of Lax [46] or Liu [50] cannot be found. In order to resolve the problem of non-existence of solutions to the Riemann problem for certain conservation laws, the concept of singular solutions admitting δ-distributions along shock curves was proposed in [42]. This solution technique was advanced further in [40] where singular solution for the Riemann problem for strictly hyperbolic system of conservation laws was found for states which are sufficiently close together. Some authors have defined theories of distribution products in order to incorporate the δ-distributions into the concept of weak solutions [10, 23, 60]. For the purpose of finding admissibility conditions for such singular solutions, some authors have relied on the weak asymptotic method [12, 13, 53, 54]. To deal with the nonlinearity which features in the system (4.1), the weak asymptotic method was extended to include complex-valued approximations [31]. The authors of [31] convincingly showed that it was possible to construct singular solutions of (4.1) even in cases which could not be resolved earlier. However, the authors did not succeed in proving uniqueness. Existence of singular solutions to (4.1) was also proved in [60] using the theory of distribution products but uniqueness could not be obtained. In the current work, a nonlinear change of variables which makes it possible to resolve the Riemann problem in the framework of the standard theory of conservation laws is introduced. In addition, a criterion which leads to an admissibility condition for singular solutions of the original system is developed and it can be shown that admissible solutions are unique in the framework developed in this work.

It is worth noting that there are a number of systems which could be resolved only by introducing the δ-solution concept. These systems are usually nonlinear with respect to both of the variables such as in the Chaplygin gas system [52] or the chromatography system [72]. In all these systems, it was possible to control the nonlinear operation over an approximation of the δ-distribution. However, this is not the case with (4.1) since the term u^2+v^2 will necessarily tend to infinity for any real approximation of the δ-function. This problem can be dealt with by introducing complex-valued approximations of the δ-distribution.

4.1 Complex-valued approximations

Let us now define what we mean by a complex-valued weak asymptotic solution and emphasize some techniques to restrict the solution concept with the goal of establishing uniqueness. Firstly, define $\mathcal{D}(\mathbb{R})$ to be the standard space of test functions and denote by $\mathcal{D}'(\mathbb{R})$ the dual space of distributions (see e.g. [56]). In order to define complex-valued weak asymptotic solutions of (4.1), we recall the definition of a vanishing family of distributions.

Definition 4.1.1 *Let $f_\varepsilon(x) \in \mathcal{D}'(\mathbb{R})$ represent a family of distributions depending on $\varepsilon \in (0,1)$. We say that $f_\varepsilon = o_{\mathcal{D}'}(1)$ if for any test function $\varphi(x) \in \mathcal{D}(\mathbb{R})$, the estimate*

$$\langle f_\varepsilon, \varphi \rangle = o(1), \text{ as } \varepsilon \to 0, \tag{4.3}$$

holds.

Thus a family of distributions disappears in the sense of definition (4.1.1) if for a given test function φ, the pairing $\langle f_\varepsilon, \varphi \rangle$ tends to zero with ε. For families of distributions depending on t, we say that $f_\varepsilon = o_{\mathcal{D}'}(1) \subset \mathcal{D}'(\mathbb{R})$ if (4.3) holds uniformly in t. In other words, f_ε vanishes if

$$\langle f_\varepsilon(\cdot,t), \varphi \rangle \leq C_T g(\varepsilon) \text{ for } t \in [0,T],$$

for a function g depending on $\varphi(x,t)$ and tending to zero with $\varepsilon \to 0$, and where the constant C_T should only depend on T [31, 34]. Using the weak asymptotic approach, a somewhat general theory can be developed by considering the system

$$\begin{aligned} \partial_t u + \partial_x f(u,v) &= 0, \\ \partial_t v + \partial_x g(u,v) &= 0. \end{aligned} \tag{4.4}$$

Next we define solutions of (4.4) in the weak asymptotic sense.

Definition 4.1.2 *The collection of smooth complex-valued distributions* (u_ε) *and* (v_ε) *represent a weak asymptotic solution to* (4.4) *if there exist real-valued distributions* $u, v \in C(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}))$*, such that for every fixed* $t \in \mathbb{R}_+$

$$u_\varepsilon \rightharpoonup u, \;\; v_\varepsilon \rightharpoonup v \;\text{ as }\; \varepsilon \to 0,$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R})$*, and*

$$\begin{aligned} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon, v_\varepsilon) &= o_{\mathcal{D}'}(1), \\ \partial_t v_\varepsilon + \partial_x g(u_\varepsilon, v_\varepsilon) &= o_{\mathcal{D}'}(1). \end{aligned} \tag{4.5}$$

In addition, if initial data are given, we require

$$u_\varepsilon(x,0) \rightharpoonup u(x,0) \;\text{ and }\; v_\varepsilon(x,0) \rightharpoonup v(x,0), \tag{4.6}$$

where the weak convergence denotes convergence in the framework of distributions as ε converges to 0.

4.2 Generalised weak solution

Suppose $\Gamma = \{\gamma_i \mid i \in I\}$ is a graph in the closed upper half plane, containing Lipschitz continuous arcs γ_i, $i \in I$, where I is a finite index set. Let I_0 be the subset of I containing all indices of arcs that connect to the x-axis. Furthermore, let $\frac{\partial \varphi(x,t)}{\partial \mathrm{l}}$ represent the tangential derivative of a function φ on the graph γ_i and let $\int_{\gamma_i}$ denote the line integral over the arc γ_i with respect to arclength. Then the following definition gives the concept of δ-shock solution to system (4.4).

Definition 4.2.1 *The pair of distributions*

$$u = U + \alpha(x,t)\delta(\Gamma) \ \text{ and } \ v = V + \beta(x,t)\delta(\Gamma) \tag{4.7}$$

where $\alpha(x,t)\delta(\Gamma) = \sum_{i\in I}\alpha_i(x,t)\delta(\gamma_i)$ *and* $\beta(x,t)\delta(\Gamma) = \sum_{i\in I}\beta_i(x,t)\delta(\gamma_i)$ *are called a generalized* δ*-shock wave solution of system* (4.4) *with the initial data* $U_0(x)$ *and* $V_0(x)$ *if the integral identities*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} (U\partial_t\varphi + f(U,V)\partial_x\varphi)\, dx\, dt + \sum_{i\in I}\int_{\gamma_i}\alpha_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}} + \int_{\mathbb{R}} U_0(x)\varphi(x,0)\, dx = 0, \tag{4.8}$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} (V\partial_t\varphi + g(U,V)\partial_x\varphi)\, dx\, dt + \sum_{i\in I}\int_{\gamma_i}\beta_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}} + \int_{\mathbb{R}} V_0(x)\varphi(x,0)\, dx = 0, \tag{4.9}$$

hold for all test functions $\varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}_+)$.

This definition may be interpreted as an extension of the classical notion of weak solutions. The definition agrees with the concept of measure solutions proposed in [11, 23] in the sense that the two singular parts of the solution coincide whereas the regular parts are dissimilar on a set of Lebesgue measure zero. Nonetheless, Definition 4.2.1 can be applied unreservedly to any hyperbolic system of conservation laws while the notion of solution defined in [11] only applies in a special situation when the δ-distribution is connected to an unknown which appears linearly in the flux f or g or when nonlinear operations on δ can somehow be controlled in another way.

Although Definition 4.2.1 is quite general and allows a combination of initial jumps and δ-distributions, its effectiveness is demonstrated by considering the Riemann problem with a single jump. Consider the Riemann problem for (4.4) with initial data $u(x,0) = U_0(x)$ and $v(x,0) = V_0(x)$, where

$$U_0(x) = \begin{cases} u_1, & x<0, \\ u_2, & x>0, \end{cases} \qquad V_0(x) = \begin{cases} v_1, & x<0, \\ v_2, & x>0. \end{cases} \tag{4.10}$$

Then, the following theorem holds:

Theorem 4.2.2 **a)** *If* $u_1 \neq u_2$ *then the pair of distributions*

$$u(x,t) = U_0(x-ct), \tag{4.11}$$

$$v(x,t) = V_0(x-ct) + \beta(t)\delta(x-ct), \tag{4.12}$$

where

$$c = \frac{[f(U,V)]}{[U]} = \frac{f(u_2,v_2)-f(u_1,v_1)}{u_2-u_1}, \ \textit{and} \ \beta(t) = (c[V]-[g(U,V)])t, \quad (4.13)$$

represents the δ*-shock wave solution of* (4.4) *with initial data* $U_0(x)$ *and* $V_0(x)$ *in the sense of Definition 4.2.1 with* $\alpha(t) = 0$.

b) *If* $v_1 \neq v_2$ *then the pair of distributions*

$$u(x,t) = U_0(x-ct) + \alpha(t)\delta(x-ct), \quad (4.14)$$
$$v(x,t) = V_0(x-ct), \quad (4.15)$$

where

$$c = \frac{[g(U,V)]}{[V]} = \frac{g(u_2,v_2)-g(u_1,v_1)}{v_2-v_1}, \ \alpha(t) = (c[U]-[f(U,V)])t \quad (4.16)$$

represents the δ*-shock solution of* (4.4) *with initial data* $U_0(x)$ *and* $V_0(x)$ *in the sense of Definition 4.2.1 with* $\beta(t) = 0$.

Proof Only the first part of the theorem is proved as the second part can be proved in a similar way. It is immediate to see that u and v given in (4.11) and (4.12) satisfy equation (4.8) since the propagation speed c is given exactly by the Rankine-Hugoniot condition derived from that system. By substituting u and v into (4.9), and performing standard transformations we obtain

$$\int_{\mathbb{R}_+} (c[V]-[g(U,V)])\,\varphi(ct,t)\,dt - \int_{\mathbb{R}_+} \beta'(t)\varphi(ct,t)\,dt = 0.$$

It is now an easy algebraic exercise to check that from here and the fact that $\alpha(0) = 0$, the conclusion follows immediately.

□

As the solution framework of Definition 4.2.1 is very weak, one might expect non-uniqueness issues to arise. This is indeed the case and the proof of the following proposition is an easy exercise and hence omitted.

Proposition 4.2.3 *System* (4.4) *with the zero initial data:* $u|_{t=0} = v|_{t=0} = 0$ *admits* δ*-shock solutions of the form:*

$$u(x,t) = 0, \ \ v(x,t) = \beta\delta(x-c_1t) - \beta\delta(x-c_2t),$$

for arbitrary constants β, c_1 *and* c_2.

4.3 Energy-velocity conservation

In this section, the necessity of δ-type solutions for (4.1) shall be explained by following considerations from [39] where it was debated (in a quite different setting) that the wrong variables are conserved. In other words, the presence of a δ-distribution in a weak solution actually signifies the inadequacy of the corresponding conservation law in the case of weak solutions. A similar consideration was recently put forward in the case of singular solutions in the shallow-water system [34].

To introduce the new conservation law, we define an energy function

$$q(u,v) = \frac{u^2+v^2}{2}, \tag{4.17}$$

which is a mathematical entropy for the system (4.1). Then using the transformation

$$(u,v) \to \big(u, \tfrac{u^2+v^2}{2}\big),$$

equation (4.1) can readily be transformed into the new system

$$\begin{aligned} \partial_t u + \partial_x q &= 0, \\ \partial_t q + \partial_x\big((2u-1)q + \tfrac{u^2}{2} - \tfrac{2u^3}{3}\big) &= 0. \end{aligned} \tag{4.18}$$

The system (4.1) and the transformed new system (4.18) are equivalent for differentiable solutions. Nonetheless, the introduction of the nonlinear transformation changes the character of the system. It shall be shown that while (4.1) is not always genuinely nonlinear, the new system (4.18) is always strictly hyperbolic and genuinely nonlinear making it possible to apply the standard theory of hyperbolic conservation laws without reservations.

In what follows, the mathematical properties of (4.18) are analysed and elementary wave solutions are found. The flux function of the new system is given by

$$F = \begin{pmatrix} q \\ (2u-1)q + \frac{u^2}{2} - \frac{2u^3}{3} \end{pmatrix}$$

with flux Jacobian

$$DF = \begin{pmatrix} 0 & 1 \\ 2q+u-2u^2 & 2u-1 \end{pmatrix}.$$

The characteristic velocities are given by

$$\lambda_{-,+} = \frac{2u-1 \mp \sqrt{8q-4u^2+1}}{2}. \tag{4.19}$$

A direct consequence of (4.17) gives the relation $2q \geq u^2 \geq 0$ which means that the quantity under the square root is non-negative. That is, $8q - 4u^2 + 1 > 0$ and the eigenvalues are real and distinct which make the system strictly hyperbolic. The right eigenvectors in this case are

$$\begin{aligned} r_- &= \begin{pmatrix} 1 \\ u - \frac{1}{2} - \sqrt{2q - u^2 + \frac{1}{4}} \end{pmatrix}, \\ r_+ &= \begin{pmatrix} 1 \\ u - \frac{1}{2} + \sqrt{2q - u^2 + \frac{1}{4}} \end{pmatrix}. \end{aligned} \tag{4.20}$$

It is not hard to verify that these eigenvectors are linearly independent and span the (u,q)-plane. The corresponding characteristic fields

$$\nabla\lambda_- \cdot r_- = 2 + \frac{1}{\sqrt{8q - 4u^2 + 1}}, \tag{4.21}$$

$$\nabla\lambda_+ \cdot r_+ = 2 - \frac{1}{\sqrt{8q - 4u^2 + 1}}, \tag{4.22}$$

are genuinely nonlinear and admit both shock and rarefaction waves. If a shock profile connects a constant left state $(u,q) = (u_L, q_L)$ to a constant right state $(u,q) = (u_R, q_R)$, then the Rankine-Hugoniot jump conditions for (4.18) are given by

$$c(u_L - u_R) = \ (q_L - q_R), \tag{4.23}$$

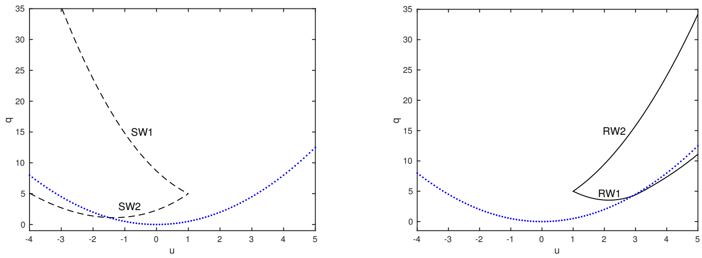
$$c(q_L - q_R) = \left((2u_L - 1)q_L + \tfrac{u_L^2}{2} - \tfrac{2u_L^3}{3} - (2u_R - 1)q_R - \tfrac{u_R^2}{2} + \tfrac{2u_R^3}{3}\right), \tag{4.24}$$

where c is the shock speed. The speed in (4.23), (4.24) must satisfy the Lax admissibility condition which is stated here as

$$\lambda_\mp(u_L, q_L) \geq c \geq \lambda_\mp(u_R, q_R). \tag{4.25}$$

In order to determine the set of all states which can be connected to a fixed left state (u_L, q_l), the shock speed, c, is eliminated from the above equations so that the shock curves

$$(q_R)_{1,2} = \frac{2q_L - (u_L - u_R)(2u_R - 1)}{2} \pm \frac{\sqrt{[-2q_L + (u_L - u_R)(2u_R - 1)]^2 + 4\left[(u_L - u_R)\left((2u_L - 1)q_L + \frac{u_L^2}{2} - \frac{u_R^2}{2} - \frac{2u_L^3}{3} + \frac{2u_R^3}{3}\right) - q_L^2\right]}}{2}.$$



(a) (b)

Figure 4.1: (a) Shock waves of the first and the second families at the left state $(u_L, q_L) = (1,5)$. (SW1) is indicated by the upper curve, while (SW2) is the lower curve. The blue dotted curve shows the critical curve $q = u^2/2$. (b) Rarefaction waves of the first and the second families at the left state $(u_L, q_L) = (1,5)$. (RW1) is indicated by the lower curve while (RW2) is the upper curve.

are obtained. This can be simplified by performing basic algebraic manipulations to obtain

$$(q_R)_{1,2} = q_L - \frac{1}{2}(u_L - u_R)(2u_R - 1) \\ \pm \mid u_L - u_R \mid \sqrt{2q_L + \tfrac{1}{4} + \tfrac{1}{2}(u_L - u_R) - \tfrac{1}{3}\left(2u_L^2 + 2u_L u_R - u_R^2\right)} \tag{4.26}$$

From this expression, equation (4.25) and by taking (u_R, q_R) in a small neighbourhood of (u_L, q_L), it is reasonable to conclude that the shock wave of the second family (SW2), the shock wave of the first family (SW1), the rarefaction wave of the second family (RW2) and the rarefaction wave of the first family (RW1) are given as follow:

$$\text{(SW1)} \qquad q_R = q_L - \frac{1}{2}\left(u_L - u_R\right)\left(2u_R - 1\right) \\ + \mid u_L - u_R \mid \left(2q_L + \tfrac{1}{2}(u_L - u_R) - \tfrac{1}{3}\left(2u_L^2 + 2u_L u_R - u_R^2\right) + \tfrac{1}{4}\right)^{\frac{1}{2}}, \tag{4.27}$$

if $u_R < u_L$. To prove that this equation is indeed the shock wave of the first family, we obtain from (4.23) and (4.25) that

$$\lambda_-(u_L, q_L) \geq c = \frac{2u_R - 1 - \sqrt{8q_L + 1 + \frac{4u_R^2}{3} - \frac{8u_L u_R}{3} - \frac{8u_L^2}{3} - 2u_R + 2u_L}}{2}.$$

Using the expression for λ_- given above, it follows immediately from the above equation that

$$2(u_L-u_R) \geq \sqrt{8q_L+1-4u_L^2} - \sqrt{8q_L+1+\frac{4u_R^2}{3}-\frac{8u_Lu_R}{3}-\frac{8u_L^2}{3}-2u_R+2u_L}.$$

This can be further simplified to get

$$2 \geq \frac{-\frac{4}{3}(u_L-u_R)-2}{\sqrt{8q_L+1-4u_L^2}+\sqrt{8q_L+1+\frac{4u_R^2}{3}-\frac{8u_Lu_R}{3}-\frac{8u_L^2}{3}-2u_R+2u_L}},$$

which is obviously correct. The second part of the Lax admissibility condition,

$$\lambda_-(u_R,q_R) \leq c,$$

can be verified in a similar manner. Furthermore, it is not hard to verify the additional inequality $\lambda_+(u_R,q_R) \geq c$, which ensures that three characteristic curves enter the shock trajectory and one characteristic curve leaves the shock.

(SW2) $$q_R = q_L - \frac{1}{2}\big(u_L-u_R\big)\big(2u_R-1\big) - \mid u_L-u_R \mid \Big(2q_L+\tfrac{1}{2}(u_L-u_R)-\tfrac{1}{3}\big(2u_L^2+2u_Lu_R-u_R^2\big)+\tfrac{1}{4}\Big)^{\frac{1}{2}}, \tag{4.28}$$

if $u_R < u_L$. The proof is omitted since it is identical to the (SW1) above. Next, we have s.

The rarefaction curve fo the first family (RW1), is obtained using the method from [10, Theorem 7.6.5] and this wave can be written as

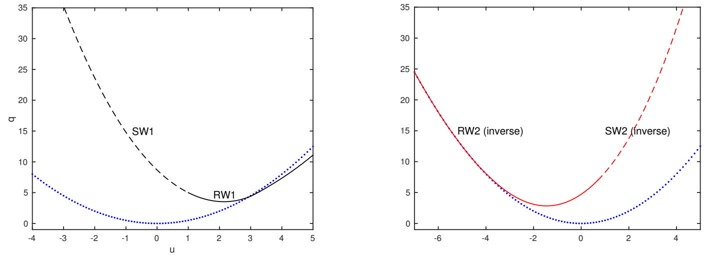
$$\frac{dq}{du} = \frac{2u-1-\sqrt{8q-4u^2+1}}{2} = \lambda_-(u,q), \quad q(u_L) = q_L, \tag{4.29}$$

for $u_R > u_L$. It is quite obvious that if $u_R < u_L$, then one cannot have (RW1) since in this domain, states are connected by (SW1) as shown above. To prove that (4.29) indeed provides RW1, one has to show that

$$\lambda_-(u_L,q_L) < \lambda_-(u_R,q_R) \text{ if } u_R > u_L. \tag{4.30}$$

By introducing the change of variables $\tilde{q} = 8q-4u^2+1$, it is possible to rewrite (4.29) in the form

$$\frac{d\tilde{q}}{du} = -4(1+\sqrt{\tilde{q}}) < 0.$$



(a) (b)

Figure 4.2: Shock and rarefaction wave curves of the first and the second families: (a) shows SW1 (dashed) and RW1 (solid) at the left state $(u_L, q_L) = (1, 5)$. (b) shows inverse SW2 (dashed, red) and inverse RW2 (solid, red) at the right state $(u_R, q_R) = (0.7, 7)$.

From here, it is clear that $\tilde{q}$ is decreasing with respect to u such that if $u_L < u_R$, then the relation

$$8q_L - 4u_L^2 + 1 = \tilde{q}_L > \tilde{q}_R = 8q_R - 4u_R^2 + 1,$$

holds. This relation together with the inequality $u_L < u_R$ immediately implies (4.30).

(RW2) Using [10, Theorem 7.6.5] one more time, we have

$$\frac{dq}{du} = \frac{2u - 1 + \sqrt{8q - 4u^2 + 1}}{2} = \lambda_+(u, q), \quad q(u_L) = q_L, \tag{4.31}$$

for $u_R > u_L$. It can be shown that (4.31) gives the rarefaction wave (RW2) in the same way explained above for (RW1). The wave fan issuing from the left state (u_L, q_L) is shown in Figure 4.1(a), Figure 4.1(b) and Figure 4.2(a) while the inverse wave fan issuing from the right state (u_R, q_R) is given in Figure 4.2(b).

The next goal is to prove existence of solution for arbitrary Riemann initial data without necessarily assuming a small enough initial jump. The essential hypothesis that must be fulfilled is that both left and right states are above the critical curve $q_{crit} = u^2/2$:

$$q_L \geq u_L^2/2, \quad q_R \geq u_R^2/2. \tag{4.32}$$

This assumptions is of course natural if the change of variables $q = \frac{u^2+v^2}{2}$ is considered. However, this condition makes it a bit complicated to prove the task since it must also be shown that the Lax admissible solution to a Riemann problem remains in the region $q \geq u^2/2$. To this end, the following lemma will be useful.

Lemma 4.3.1 *The function* $q_{crit}(u) = \frac{u^2}{2}$ *satisfies* (4.31).

The proof is obvious and therefore omitted. The above lemma is important since, according to the uniqueness of solutions to the Cauchy problem for ordinary differential equations, it shows that if the left and right states (u_L, q_L) and (u_R, q_R) are above the curve $q_{crit}(u) = \frac{u^2}{2}$, then the simple waves (SW1, SW2, RW1, RW2) connecting such states will also remain above it and this implies that we can use the solution to (4.18) to obtain the solution of (4.1) since the square root which gives the function $v = \sqrt{2q - u^2}$ will be well defined. Concerning the Riemann problem, we have the following theorem.

Theorem 4.3.2 *Given a left state* (u_L, q_L) *and a right state* (u_R, q_R)*, so that both are above the critical curve* $q_{crit}(u) = \frac{u^2}{2}$ *i.e. we have* $q_L \geq u_L^2/2$ *and* $q_R \geq u_R^2/2$*, the states* (u_L, q_L) *and* (u_R, q_R) *can be connected Lax admissible shocks and rarefaction waves via a middle state belonging to the domain* $q > u^2/2$*.*

The proof of this theorem and the one on the admissible δ-type solution are presented neatly in Paper D [33] which is attached in Part II of this thesis.

Future work

As mentioned earlier, a shallow water system incorporating constant background shear has been found independently by a number of authors [2, 24, 41] and written as

$$\begin{aligned} \partial_t H + \partial_x \left(\tfrac{\Gamma}{2} H^2 + uH \right) &= 0, \\ \partial_t \left(\tfrac{\Gamma}{2} H^2 + uH \right) + \partial_x \left(\tfrac{\Gamma^2}{3} H^3 + \Gamma u H^2 + u^2 H + \tfrac{1}{2} g H^2 \right) &= 0. \end{aligned} \tag{F1}$$

The derivation was based on a flow whose total velocity component is

$$v(x,t,z) \equiv U(z) + u(x,t) = \Gamma z + u, \tag{F2}$$

where $U(z)$ is the linear shear current and Γ is constant. An analysis of shock wave solutions of the system (F1), shows that stationary hydraulic jumps can be described in terms of two non-dimensional parameters, namely, the Froude number which is suitably defined in the presence of the shear current and a non-dimensional background vorticity. Moreover, it is shown in Paper B [35] that stronger background vorticity has the effect of moderating the strength of the hydraulic jump. It is important to notice that the velocity profile leading to the derivation of (F1) is linear. However, an observation of the velocity distribution, at least as far as the time-averaged velocity and pressure fields are concerned, described in [55, 62] indicates that at certain time and location, the velocity profile is parabolic. To solidify this observation, experimental results on velocity distribution given by Hager [18] were analysed. The figure below shows the time-averaged velocity distribution of a 50cm wide channel:

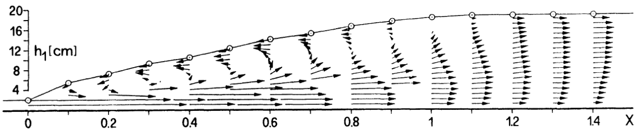


Fig.2.9 Velocity Distribution in Classical Hydraulic Jump for F_1 = 6.85, h_1 = 2.05 cm.

Hager's data include five runs with $4.3 < \mathrm{Fr} < 8.9$ as shown below:

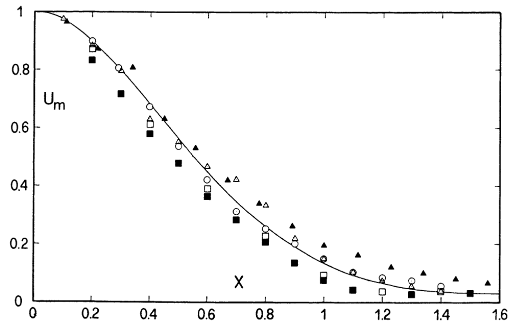


Fig.2.11 Classical Hydraulic Jump, Maximum Forward Velocity U_m as a Function of X. Notation Fig.2.10, (—) Eq.(2.18). F_1 = (○)4.3, (▲)4.95, (△)5.5, (■)6.85, (□)8.9.

It is noted from the above figures that out of the hydraulic jump the distribution of the velocity profile of the flow is parabolic. Based on this observation, we seek to reformulate the mathematical model by defining the velocity profile as

$$v(x,t) = u(x,t) + U(z) = u - \Gamma z^2 - \Gamma_0 z \tag{F3}$$

This velocity profile gives the following conservation laws:
Mass conservation:

$$H_t + \left(uH - \frac{\Gamma}{3}H^3 - \frac{\Gamma_0}{2}H^2 \right)_x = 0 \tag{F4}$$

Momentum conservation:

$$\begin{aligned}&\left(uH - \frac{\Gamma}{3}H^3 - \frac{\Gamma_0}{2}H^2 \right)_t + \\ &\left(u^2H - \frac{2\Gamma}{3}uH^3 - \Gamma_0 uH^2 + \frac{\Gamma^2}{5}H^5 + \frac{\Gamma\Gamma_0}{2}H^4 + \frac{\Gamma_0^2}{3}H^3 + \frac{1}{2}gH^2 \right)_x = 0\end{aligned} \tag{F5}$$

It would be interesting to analyse (F4), (F5) in a similar as in done for (F1) in Paper B and compare the two results. Unlike (F1) where the stationary hydraulic jumps can be described in terms of two non-dimensional parameters, namely, the Froude number and a non-dimensional background vorticity, the initial analysis performed on (F4), (F5) shows that the strength of the hydraulic jump is described by four non-dimensional parameters: the Froude number and the following three parameters

$$\Omega = \frac{8\Gamma^2}{45g}H_L^3, \qquad \Omega_1 = \frac{\Gamma_0\Gamma}{3g}H_L^2, \qquad \Omega_0 = \frac{\Gamma_0^2}{6g}H_L. \tag{F6}$$

Even the Riemann problem can be solved for these systems in a similar as done in Paper A and Paper D. Some preliminary work has been done on these systems but no conclusive results have been established.

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Part II

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Delta shock waves in shallow water flow *

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Delta shock waves in shallow water flow

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ABSTRACT

The shallow-water equations for two-dimensional hydrostatic flow over a bottom bathymetry $b(x)$ are

$$h_t + (uh)_x = 0,$$

$$u_t + \left(gh + u^2/2 + gb\right)_x = 0.$$

It is shown that the combination of discontinuous free-surface solutions and bottom step transitions naturally lead to singular solutions featuring Dirac delta distributions. These singular solutions feature a Rankine–Hugoniot deficit, and can readily be understood as generalized weak solutions in the variational context, such as defined in [13,22]. Complex-valued approximations which become real-valued in the distributional limit are shown to extend the range of possible singular solutions. The method of complex-valued weak asymptotics [22,23] is used to provide a firm link between the Rankine–Hugoniot deficit and the singular parts of the weak solutions. The interaction of a surface bore (traveling hydraulic jump) with a bottom step is studied, and admissible solutions are found.

1. Introduction

The standard theory of hyperbolic conservation laws in one spatial dimension can be applied to systems which are strictly hyperbolic and genuinely nonlinear. If initial data are given which have small enough total variation, then it can be shown that there is a solution which exists for all times [16,18,33]. This solution will in general be discontinuous, featuring a number of jumps. However, if one of the above hypotheses is not satisfied, the initial-value problem cannot in general be resolved (see e.g. [4–6,8,11,27,19,29,34]) and further restrictions on the data need to be introduced, such as for example in [34]. In fact, in some cases, even the Riemann problem cannot be solved.

Starting with the work reported on in [26], existence of solutions was shown to be possible if the space of solutions was extended to include Radon measures. In particular, such non-standard solutions were shown to contain Dirac δ-distributions attached to the location of certain discontinuities. As was shown in [25], the incorporation of such δ-shocks is equivalent to relaxing one or more of the required Rankine–Hugoniot conditions for classical shocks, and it may be shown that the strength of the Dirac δ-distribution associated to a certain shock is a precise measure of the deficit in the Rankine–Hugoniot conditions which are required to obtain a solution.

In the present work, we consider the shallow-water system, and show how δ-shocks arise naturally if this theory is to describe the physics of the underlying problem adequately. Indeed, unlike the situation from [23] where the δ-distribution was adjoined to the surface excursion, here we shall see that δ-naturally appears as part of the velocity as a measure of the Rankine–Hugoniot deficit. An alternative approach for physical explanation of the appearance of delta functions and Rankine–Hugoniot deficits in this context was given in [14], where a localized jet is considered. Singular solutions may also occur in shallow-water systems for two-layer flow [7,21] and in mixing closures for two-layer systems [20].

In the context of surface waves, the shallow-water system describes the flow of an inviscid fluid in a long channel of small uniform width, is used as a standard model in hydraulics, and is fundamental in the study of bores and storm surges in rivers and channels [18,35]. If the bottom is flat (such as in a laboratory situation), the system is usually written in the form

$$\partial_t h + \partial_x (uh) = 0, \text{ (mass conservation)}, \tag{1.1}$$

$$\partial_t (uh) + \partial_x \left(u^2 h + g\frac{h^2}{2}\right) = 0, \text{ (momentum balance)}, \tag{1.2}$$

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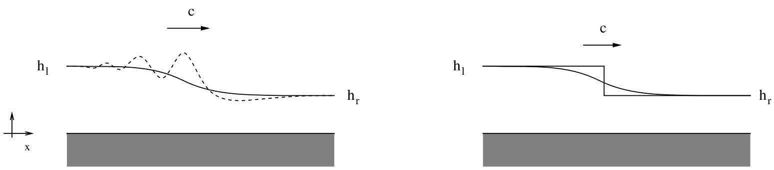

Fig. 1. Left panel: Surface profile of a traveling hydraulic jump (undular bore). Right panel: shallow-water approximation.

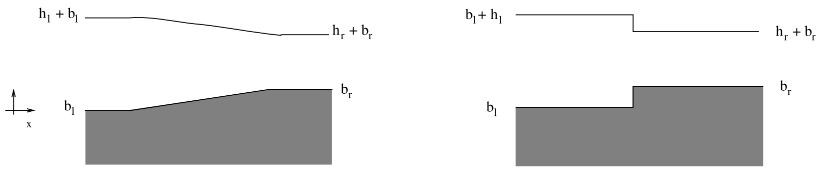

Fig. 2. Left panel: Surface profile over a bottom transition. Right panel: shallow-water approximation.

where h denotes the total flow depth, u represents an average horizontal velocity, and g is the gravitational constant. For smooth solutions, an equivalent system is

$$\partial_t h + \partial_x (uh) = 0, \tag{1.3}$$

$$\partial_t u + \partial_x \Big(\tfrac{u^2}{2} + gh \Big) = 0, \tag{1.4}$$

and it is immediately clear that mass and momentum conservation in discontinuous solutions lead to a Rankine–Hugoniot deficit in (1.4). One might conclude that it would therefore be best to avoid the system (1.3)–(1.4) in favor of the system (1.1)–(1.2). The theory for this system is well developed, and both the initial-value problem and the Riemann problem can be solved [18]. It is well known that the conservation of energy is formulated as

$$\partial_t \left(h\tfrac{u^2}{2} + g\tfrac{h^2}{2} \right) + \partial_x \left(guh^2 + h\tfrac{u^3}{2} \right) = 0 \tag{1.5}$$

and this then serves as a mathematical entropy [2,3,35].

On the other hand, in many practical situations, the assumption of a flat bottom is too restrictive. If an uneven bed is introduced, the equations take the form

$$\partial_t h + \partial_x (uh) = 0 \quad \text{(mass conservation)} \tag{1.6}$$

$$\partial_t u + \partial_x \left(gh + \tfrac{u^2}{2} \right) = -gb_x \tag{1.7}$$

$$\partial_t (uh) + \partial_x \left(u^2 h + g\tfrac{h^2}{2} \right) = -ghb_x \quad \text{(momentum balance)} \tag{1.8}$$

$$\partial_t \left(h\tfrac{u^2}{2} + g\tfrac{h^2}{2} + bh \right) + \partial_x \left(guh(h+b) + h\tfrac{u^3}{2} \right) = 0 \quad \text{(energy balance)} \tag{1.9}$$

In this system, the function $b(x)$ measures the rise of the bed above a certain reference level at $z = 0$. The function $h(x,t)$ measures the flow depth of the fluid, so that $b(x) + h(x,t)$ measures the position of the free surface relative to the reference point $z = 0$ (see Fig. 1 and Fig. 2).

Again, for discontinuous solutions, mass and momentum conservation are to be satisfied, so that (1.7) and the energy equation (1.9) will feature a Rankine–Hugoniot deficit. In the case of a shock over a bottom step, momentum is not conserved because of the lateral pressure force appearing in (1.8), and in this case energy conservation needs to be specified. Therefore, in this case a Rankine–Hugoniot deficit will be introduced in (1.8).

In this paper we will address the relatively simple situation of a flow of a shock wave over a bottom step. The shock wave is governed by the Rankine–Hugoniot conditions originating from mass and momentum conservation, i.e. by (1.6) and (1.8). On the other hand, as explained above, a discontinuity over a bottom step is governed by the Rankine–Hugoniot conditions originating from mass and energy conservation, i.e. by (1.6) and (1.9). Thus it is plain that it is not possible to resolve the underlying physical problem with the use of only two governing equations. If the goal is to maintain the classical modeling approach of describing a situation with a certain fixed set of equations so that the number of equations and unknowns is the same, it is necessary to allow for Rankine–Hugoniot deficits and the corresponding incorporation of singular delta shocks.

Thus in order to salvage the classical modeling approach, we propose the following procedure. Use the system (1.6)–(1.7) as the system to be solved, and use the corresponding Rankine–Hugoniot conditions for momentum or energy conservation in the appropriate places. Since these can be made explicit via delta-shock waves, the system is self-sufficient. For further study, the system (1.6)–(1.7) can be cast in conservative form by writing

$$\left.\begin{aligned} \partial_t h + \partial_x (uh) &= 0, \\ \partial_t u + \partial_x \left(gh + \tfrac{u^2}{2} + gb \right) &= 0. \end{aligned}\right\} \tag{1.10}$$

The plan of the present paper is as follows. In Section 2, surface discontinuities over a flat bottom are studied, and it is shown that if these discontinuous solutions satisfy mass and momentum conservation, and the required energy loss, then the total head $\frac{1}{2g}u^2 + h$ cannot be conserved. Thus a Rankine–Hugoniot deficit is needed in the second equation in (1.10). The solution is verified both in the weak asymptotic context, and in the weak variational context. In Section 3, bottom step transitions are studied. In Section 4, the interaction of a discontinuous moving surface profile with a bottom step is investigated.

2. Surface discontinuities

In this section, we briefly review the theory surrounding discontinuous solutions of the shallow-water system, and we show that an admissible weak solution conserving mass and momentum, and dissipating mechanical energy must give rise to a Rankine–Hugoniot deficit for the conservation equation for the total head. Then, it is described how such a singular solution can be understood as a delta shock wave in the framework of the weak asymptotic method, and in the generalized variational framework.

2.1. Traveling hydraulic jump

A traveling hydraulic jump traveling over an even bottom must respect the conservation of mass (1.1) and momentum (1.2). In the shallow-water theory, it is useful to consider the jump as having a discontinuity at the bore front. This modeling approach leads to the Rankine–Hugoniot conditions

$$\begin{aligned} c(h_r - h_l) &= u_r h_r - u_l h_l, \\ c(u_r h_r - u_l h_l) &= \left(u_r^2 h_r + \frac{1}{2} g h_r^2\right) - \left(u_l^2 h_l + \frac{1}{2} g h_l^2\right). \end{aligned} \tag{2.1}$$

Here the subscripts l and r indicate the left and right states of the shock, respectively. From these relations, the velocity can be expressed as

$$c = \frac{u_r h_r - u_l h_l}{h_r - h_l} = \frac{\left(u_r^2 h_r + \frac{1}{2} g h_r^2\right) - \left(u_l^2 h_l + \frac{1}{2} g h_l^2\right)}{u_r h_r - u_l h_l}. \tag{2.2}$$

Algebraic manipulation of equation (2.2) gives the expression

$$u_r - u_l = \pm(h_r - h_l)\sqrt{\frac{g}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right)}. \tag{2.3}$$

In particular, we see from this relation that $u_r \neq u_l$ if and only if $h_r \neq h_l$. Inserting this relation into (2.2) gives

$$c = u_l \pm h_r \sqrt{\frac{g}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right)} = u_r \pm h_l \sqrt{\frac{g}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right)}. \tag{2.4}$$

It is clear that solving the square root gives rise to two possible solutions. In order to pick the one which is physically reasonable, use is made of the conservation of mechanical energy (1.5).

Mass conservation through the jump discontinuity is derived from the first equation in (2.1) and equation (2.4) as

$$m = h_r(u_r - c) = h_l(u_l - c) = \mp h_r h_l \sqrt{\frac{g}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right)}. \tag{2.5}$$

Momentum conservation through the discontinuity is derived similarly from the second equation in (2.1) and equation (2.4) in the form

$$h_r(u_r - c)u_r + \frac{1}{2} g h_r^2 = h_l(u_l - c)u_l + \frac{1}{2} g h_l^2.$$

Using the expression in (2.5) simplifies this equation to

$$m u_r + \frac{1}{2} g h_r^2 = m u_l + \frac{1}{2} g h_l^2. \tag{2.6}$$

The mechanical energy dissipates in the jump but remains balanced in areas where the solution is smooth. Using the Rankine–Hugoniot condition for equation (1.9) which has the form

$$\begin{aligned} &c\left[\left(\frac{1}{2} u_r^2 h_r + \frac{1}{2} g h_r^2\right) - \left(\frac{1}{2} u_l^2 h_l + \frac{1}{2} g h_l^2\right)\right] \\ &\quad = \left(\frac{1}{2} u_r^3 h_r + g u_r h_r^2\right) - \left(\frac{1}{2} u_l^3 h_l + g u_l h_l^2\right), \end{aligned}$$

an expression for the mechanical energy loss per unit time is

$$\begin{aligned} \frac{1}{\rho Y}\Delta E = &\left(\frac{1}{2} u_r^2 + \frac{1}{2} g h_r\right) h_r(u_r - c) - \left(\frac{1}{2} u_l^2 + \frac{1}{2} g h_l\right) h_l(u_l - c) \\ &+ \frac{g}{2}(u_r h_r^2 - u_l h_l^2). \end{aligned}$$

This can be simplified further by using equations (2.5) and (2.6) to obtain[1]

$$\frac{1}{\rho Y}\Delta E = -\frac{m g (h_r - h_l)^3}{4 h_r h_l}. \tag{2.7}$$

This shows that the mechanical energy, which should be chosen as the entropy condition for picking the valid solution, decreases as the solution passes through the discontinuity. As a consequence of (2.7), it is noted that

$$\Delta E < 0 \quad \text{if} \quad m(h_r - h_l) > 0. \tag{2.8}$$

Since energy gain is impossible, it is clear that these inequalities together with the expression in equation (2.5) lead to the relations

$$\begin{aligned} h_r > h_l &\Rightarrow u_r > c \text{ and } u_l > c, \\ h_r < h_l &\Rightarrow u_r < c \text{ and } u_l < c. \end{aligned} \tag{2.9}$$

What is not yet established is the relation between the left state, u_l, and the right state, u_r, variables. From equations (2.3) and (2.5) it is found that

$$u_r - u_l = -\frac{m(h_r - h_l)}{h_r h_l}, \tag{2.10}$$

and the inequalities in equation (2.8) require that this expression obeys the condition

$$u_r < u_l. \tag{2.11}$$

This condition plays an important role in analyzing the Rankine–Hugoniot jump condition for the equation (1.4). Indeed, as it turns out, energy per unit mass cannot be constant through the shock, and we have the following theorem.

Theorem 2.1. *An admissible shock-wave solution satisfying the Rankine–Hugoniot conditions arising from mass and momentum conservation, and featuring the required energy loss must feature a Rankine–Hugoniot deficit in equation (1.4).*

Equation (1.4) can be interpreted as a balance equation involving horizontal velocity and total hydraulic head $\mathcal{H} = \frac{u^2}{2g} + h$, and it is convenient to state the result in the following form:

$$g\Delta\mathcal{H} = (u_r - c)u_r - (u_l - c)u_l + \frac{1}{2}(u_l^2 - u_r^2) + g(h_r - h_l).$$

Using the expression for m in (2.5) gives[2]

$$g\Delta\mathcal{H} = m\left(\frac{u_r}{h_r} - \frac{u_l}{h_l}\right) + \frac{1}{2}\left(u_l^2 - u_r^2\right) + g(h_r - h_l). \tag{2.12}$$

Considering different cases for h_r and h_l and using the above inequalities, it is not hard to check that $\Delta\mathcal{H}$ cannot be zero. From the above equation we obtain the expression

$$g\Delta\mathcal{H} = \frac{1}{2}\left((u_r - c)^2 - (u_l - c)^2 + 2g(h_r - h_l)\right). \tag{2.13}$$

To simplify this expression further we obtain from (2.5) the following relations

$$\begin{aligned} (u_r - c)^2 &= \frac{g h_l^2}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right), \\ (u_l - c)^2 &= \frac{g h_r^2}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right). \end{aligned}$$

[1] Note that the quantity on the left has been divided by the density ρ and the width of the channel Y in order to get the units of energy per unit time. It will be convenient in the following to assume that both ρ and Y are unity.

[2] It appears most convenient here to present this quantity as head loss $\Delta\mathcal{H}$ per unit time. The quantity has been multiplied by g in order to get the right units.

Insert these expressions into (2.13) to obtain

$$g\Delta\mathcal{H} = \frac{1}{2}\left(\frac{g}{2}(h_l^2 - h_r^2)\left(\frac{1}{h_r} + \frac{1}{h_l}\right) + 2g(h_r - h_l)\right),$$

which is simplified by algebraic manipulations to

$$g\Delta\mathcal{H} = \frac{g(h_l - h_r)^3}{4h_l h_r}. \tag{2.14}$$

From this expression it is obvious that $g\Delta\mathcal{H}$ is nonzero so long as $h_l \neq h_r$.

2.2. Weak asymptotics

One available tool for the description of singular shock waves is the method of weak asymptotics [9,10,12,15,28,30,32]. This method was recently extended to the case where complex-valued approximations are allowed which significantly extended its range of applicability [22–24].

Define $\mathcal{D}(\mathbb{R})$ to be the standard space of test functions, and let $\mathcal{D}'(\mathbb{R})$ be the dual space of distributions (see e.g. [31]). In order to define complex-valued weak asymptotic solutions of (1.10), we first recall the definition of a vanishing family of distributions.

Definition 2.1. Let $f_\varepsilon(x) \in \mathcal{D}'(\mathbb{R})$ be a family of distributions depending on $\varepsilon \in (0,1)$. We say that $f_\varepsilon = o_{\mathcal{D}'}(1)$ if for any test function $\phi(x) \in \mathcal{D}(\mathbb{R})$, we have the estimate

$$\langle f_\varepsilon, \phi\rangle = o(1), \quad \text{as } \varepsilon \to 0. \tag{2.15}$$

Thus a family of distributions vanishes in the sense defined above if for a given test function ϕ, the pairing $\langle f_\varepsilon, \phi\rangle$ converges to zero with ε. For families of distributions depending on t, we say $f_\varepsilon = o_{\mathcal{D}'}(1) \subset \mathcal{D}'(\mathbb{R})$ if (2.15) holds uniformly in t. In other words, f_ε vanishes if

$$\langle f_\varepsilon(\cdot,t), \varphi\rangle \leq C_T g(\varepsilon) \text{ for } t \in [0,T],$$

for a function g depending on $\varphi(x,t)$ and tending to zero with $\varepsilon \to 0$, and where the constant C_T should only depend on T. Next we define solutions of (1.10) in the weak asymptotic sense.

Definition 2.2. The collection of smooth complex-valued distributions (u_ε) and (h_ε) represent a weak asymptotic solution to (1.10) if there exist real-valued distributions $u, v \in C(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}))$, such that for every fixed $t \in \mathbb{R}_+$

$$u_\varepsilon \rightharpoonup u, \;\; h_\varepsilon \rightharpoonup h \;\text{ as }\; \varepsilon \to 0,$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R})$, and

$$\left.\begin{aligned} \partial_t h_\varepsilon + \partial_x(u_\varepsilon h_\varepsilon) &= o_{\mathcal{D}'}(1), \\ \partial_t u_\varepsilon + \partial_x\left(gh_\varepsilon + \tfrac{u_\varepsilon^2}{2} + gb\right) &= o_{\mathcal{D}'}(1). \end{aligned}\right\} \tag{2.16}$$

In addition, if initial data are given, we require

$$u_\varepsilon(x,0) \rightharpoonup u(x,0) \;\text{ and }\; h_\varepsilon(x,0) \rightharpoonup h(x,0), \tag{2.17}$$

where the weak convergence designates convergence in the sense of distributions as ε tends to 0.

In order to state an existence theorem in the context of the above definitions, we define the functions

$$H_0(x) = \begin{cases} h_l, & \text{if } x < 0, \\ h_r, & \text{if } x > 0, \end{cases} \tag{2.18}$$

and

$$U_0(x) = \begin{cases} u_l, & \text{if } x < 0, \\ u_r, & \text{if } x > 0. \end{cases} \tag{2.19}$$

Theorem 2.2. *If the constants h_l, h_r, u_l and u_r are chosen such that the functions $H_0(x-ct)$ and $U_0(x-ct)$ (with c given by (2.2)) represent an admissible (energy-dissipating) shock wave which conserves both mass and momentum, then there are weak asymptotic solutions h_ε and u_ε of the system (1.3)–(1.4), such that the families (h_ε) and (u_ε) have distributional limits*

$$h(x,t) = H_0(x-ct), \tag{2.20}$$

$$u(x,t) = U_0(x-ct) + \alpha(t)\delta(x-ct), \tag{2.21}$$

where

$$\alpha'(t) = g\Delta\mathcal{H}.$$

In order to prove this theorem, let $\rho \in C_c^\infty(\mathbb{R})$ be non-negative, smooth, compactly supported even function such that

$$\text{supp}\,\rho \subset (-1,1), \quad \int_{\mathbb{R}} \rho(z)dz = 1, \quad \rho \geq 0.$$

Let $C_{\rho,2} = \int_{\mathbb{R}} \rho^2(z)dz$, and take

$$\delta_\varepsilon(x,t) = \frac{1}{2\varepsilon}\rho\left(\frac{x-ct-4\varepsilon}{\varepsilon}\right) + \frac{1}{2\varepsilon}\rho\left(\frac{x-ct+4\varepsilon}{\varepsilon}\right),$$

$$R_\varepsilon(x,t) = \frac{i}{2\varepsilon}\rho\left(\frac{x-ct-2\varepsilon}{\varepsilon}\right) - \frac{i}{2\varepsilon}\rho\left(\frac{x-ct+2\varepsilon}{\varepsilon}\right),$$

$$S_\varepsilon(x,t) = \frac{1}{\sqrt{\varepsilon}}\frac{1}{\sqrt{C_{\rho,2}}}\rho\left(\frac{x-ct}{\varepsilon}\right).$$

Now let the functions U_ε and H_ε be defined by

$$U_\varepsilon(x,t) = \begin{cases} u_l, & x < ct - 30\varepsilon, \\ 0, & ct - 20\varepsilon \leq x \leq ct + 20\varepsilon, \\ u_r, & x \geq ct + 30\varepsilon, \end{cases}$$

$$H_\varepsilon(x,t) = \begin{cases} h_l, & x < ct - 30\varepsilon, \\ 0, & ct - 20\varepsilon \leq x \leq ct + 20\varepsilon, \\ h_r, & x \geq ct + 30\varepsilon. \end{cases}$$

Notice in particular that we have

$$R_\varepsilon \rightharpoonup 0, \;\text{ and }\; S_\varepsilon \rightharpoonup 0.$$

Moreover, we also have the identities

$$U_\varepsilon\delta_\varepsilon = 0, \quad U_\varepsilon R_\varepsilon = 0, \quad U_\varepsilon S_\varepsilon = 0, \quad \delta_\varepsilon R_\varepsilon = 0, \quad \delta_\varepsilon S_\varepsilon = 0, \text{ and } R_\varepsilon S_\varepsilon = 0.$$

Furthermore, it is not hard to check that

$$H_\varepsilon\delta_\varepsilon = 0, \quad H_\varepsilon R_\varepsilon = 0, \;\text{ and }\; H_\varepsilon S_\varepsilon = 0.$$

In addition, the following limit is obtained:

$$S_\varepsilon^2 \rightharpoonup \delta. \tag{2.22}$$

Now make the ansatz

$$h_\varepsilon(x,t) = H_\varepsilon(x-ct),$$

$$u_\varepsilon(x,t) = U_\varepsilon(x-ct) + \alpha(t)(\delta_\varepsilon(x-ct) + R_\varepsilon(x-ct)) + \sqrt{2c\alpha(t)}S_\varepsilon(x-ct),$$

and substitute it into equations (1.10). Notice first of all that

$$u_\varepsilon^2(x,t) = U_\varepsilon^2 + \alpha^2(t)(R_\varepsilon^2 + \delta_\varepsilon^2) + c\alpha(t)S^2.$$

Focusing on the expression $R_\varepsilon^2 + \delta_\varepsilon^2$, we take $\varphi \in C_c^\infty(\mathbb{R})$ and consider the integral

$$\int_{\mathbb{R}} (R_\varepsilon^2 + \delta_\varepsilon^2)\varphi \, dx.$$

Noting the relation

$$\int_{\mathbb{R}} \tfrac{1}{\varepsilon^2}\Big(\rho^2((x-ct+\alpha\varepsilon)/\varepsilon) + \rho^2((x-ct-\beta\varepsilon)/\varepsilon)\Big)\varphi(x)\,dz$$
$$= \int_{\mathbb{R}} \tfrac{1}{\varepsilon}\rho^2(z)\left(\varphi(ct+\varepsilon(z-\alpha)) + \varphi(ct+\varepsilon(z+\beta))\right)dz$$
$$= \int_{\mathbb{R}} \tfrac{1}{\varepsilon}\rho^2(z)\big(2\varphi(ct) + \varepsilon\varphi'(ct)(\beta-\alpha)\big)\,dz + \mathcal{O}(\varepsilon),$$
for $\alpha, \beta \in \mathbb{R}$,

which follows by making the changes of variables $(x-ct+\alpha\varepsilon)/\varepsilon = z$ and $(x-ct-\beta\varepsilon)/\varepsilon = z$, and observing that $\int z\rho^2(z)dz = 0$ since ρ is an even function, the above integral can be rewritten as

$$\frac{1}{4}\int_{\mathbb{R}} \tfrac{1}{\varepsilon^2}\Big(-\rho^2((x-ct+2\varepsilon)/\varepsilon) - \rho^2((x-ct-2\varepsilon)/\varepsilon)$$
$$+ \rho^2((x-ct+4\varepsilon)/\varepsilon) + \rho((x-ct-4\varepsilon)/\varepsilon)\Big)\varphi\,dx = \mathcal{O}(\varepsilon).$$

Finally, collecting terms, we have

$$\partial_t U_\varepsilon + \frac{1}{2}\partial_x U_\varepsilon^2 + \partial_x H_\varepsilon + \alpha'(t)\delta_\varepsilon - c\alpha(t)\delta' + c\alpha\partial_x S_\varepsilon^2 = o_{\mathcal{D}'}(1). \quad (2.23)$$

Note that the last two terms on the left cancel by (2.22). Therefore, taking into account Definition 2.2, we see that the Rankine–Hugoniot deficit is

$$\alpha'(t) = (u_r - u_l)c + \frac{1}{2}(u_l^2 - u_r^2) + g(h_r - h_l) = g\Delta\mathcal{H}.$$

From Theorem 2.1, we see that $\alpha'(t)$ must be nonzero. The first equation in (2.16) is verified in a similar fashion, but this is even simpler thanks to the choice of the constant c which was found from the Rankine–Hugoniot condition corresponding to the first equation.

2.3. Generalized weak solutions

We will show that the weak asymptotic solutions constructed above represent solutions to the shallow-water system also in the framework introduced in [13]. Following [13], we let $\Gamma = \{\gamma_i \mid i \in I\}$ be a graph in the closed upper half plane, consisting of Lipschitz curves γ_i, $i \in I$, with I a finite index set. I_0 is the subset of I containing the indices of all curves which touch the x-axis, and $\Gamma_0 = \{x_k^0 \mid k \in I_0\}$ is the set of initial points of the curves γ_k with $k \in I_0$. We denote the singular part of the solution by $\alpha(x,t)\delta(\Gamma) = \sum_{i\in I}\alpha_i(x,t)\delta(\gamma_i)$. The expression $\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}$ designates the tangential derivative of a function φ on the arc γ_i, and $\int_{\gamma_i}$ denotes the line integral over the set γ_i.

Definition 2.3. A graph Γ and a couple of distributions (h, u) where U is given by

$$u(x,t) = U(x,t) + \sum_{i\in I}\alpha_i(x,t)\delta(\gamma_i),$$

with $h, U \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$, $\alpha_i \in C^1(\Gamma)$, $i \in I$, is called a generalized δ-shock wave solution of system (1.10) with initial data $h_0(x)$ and $U_0(x) + \sum_{I_0}\alpha_k(x_0^k, 0)\delta\big(x - x_k^0\big)$ if the integral identities

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} (h\partial_t\varphi + (Uh)\partial_x\varphi)\,dxdt + \int_{\mathbb{R}} h_0(x)\varphi(x,0)\,dx = 0, \quad (2.24)$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \Big(U\partial_t\varphi + \Big(\tfrac{U^2}{2} + g(h+b)\Big)\partial_x\varphi\Big)\,dxdt \quad (2.25)$$
$$+ \sum_{i\in I}\int_{\gamma_i}\alpha_i(x,t)\tfrac{\partial\varphi(x,t)}{\partial \mathbf{l}} + \int_{\mathbb{R}} U^0(x)\varphi(x,0)\,dx$$
$$+ \sum_{k\in I_0}\alpha_k(x_k^0, 0)\varphi(x_k^0, 0) = 0,$$

hold for all test functions $\varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}_+)$.

It is straightforward to check the solutions defined by (2.20) and (2.21) satisfy this weak definition. Indeed, the requirement (2.24) is exactly of the same form as the usual definition of a weak solution. Requirement (2.25) contains the more interesting singular part. As above, consider the case of a flat bed at $b = 0$. Using (2.20) and (2.21), standard computations lead to the identity

$$\int_{\mathbb{R}_+}\Big(c[U] - [U^2/2 + gh]\Big)\varphi(ct,t)\,dt - \int_{\mathbb{R}_+}\alpha'(t)\varphi(ct,t)\,dt = 0,$$

where $[U] = u_r - u_l$ and similarly $[U^2/2 + gh] = \big(u_r^2/2 + gh_r\big) - \big(u_l^2/2 + gh_l\big)$. Since $\alpha(0) = 0$, the conclusion follows from the form of $\alpha(t)$ defined in (2.14). Thus we have the following theorem:

Theorem 2.3. *If the constants h_l, h_r, u_l and u_r are chosen such that the functions $H_0(x - ct)$ and $U_0(x - ct)$ (with c given by (2.2)) represent an admissible (energy-dissipating) shock wave which conserves both mass and momentum, and $\alpha(t)$ is given by (2.14), then the functions defined in (2.20) and (2.21) represent a solution of the Riemann problem corresponding to the system (1.10) in the sense of Definition 2.3.*

3. Bottom step transitions

Consider a smooth bottom topography function defined by

$$b(x) = \begin{cases} b_l, & \text{if } x < 0, \\ b_r, & \text{if } x > 0. \end{cases} \quad (3.1)$$

For this bottom step, in the shallow-water approximation, the mass and energy of a flow have to be conserved. Since the shock-wave solution over a bottom step is stationary, the Rankine–Hugoniot conditions are written as

$$u_l h_l = u_r h_r \quad (3.2)$$

$$g u_r h_r (h_r + b_r) + h_r \frac{u_r^3}{2} = g u_l h_l (h_l + b_l) + h_l \frac{u_l^3}{2}. \quad (3.3)$$

As it turns out, the second condition can be replaced by the simpler condition

$$g(h_r + b_r) + \frac{u_r^2}{2} = g(h_l + b_l) + \frac{u_l^2}{2}, \quad (3.4)$$

and the conditions (3.2) and (3.4) are the standard relations in hydraulic theory [17]. Since the hydraulic fall over a step does not require a Rankine–Hugoniot deficit in the second equation of (1.10), it is clear that this is a weak solution in the classical sense, and satisfies Definition 2.3 without the singular part.

4. Flow of a bore over a bottom step

To be concrete, we study a bore (traveling hydraulic jump) approaching a bottom step from the left. In order to describe the interaction of the traveling jump with the bottom step, we need to solve the Riemann problem over a bottom step. In [1], it was found that there are 26 different solutions, but the authors did not investigate the admissibility of these solutions. Here, we find *admissible* solutions involving two shocks, one propagating to the left and the other propagating to the right of the step.

Definition 4.1. The shock defined by

$$u(x,t)=\begin{cases}u_l, & \text{if } x<ct,\\ u_r, & \text{if } x>ct,\end{cases} \tag{4.1}$$

connecting a left state (h_l,u_l) and a right state (h_r,u_r) is i-admissible if the shock speed c satisfies the Lax entropy conditions

$$\lambda_i(h_r,u_r)\le c\le \lambda_i(h_l,u_l), \tag{4.2}$$

for $i=1,2$.

Consider a bottom step function where $b_l=0$ and $b_r=1$ and the initial data

$$h\,|_{t=0}=\begin{cases}4, & \text{if } x<-1,\\ 1, & \text{if } x>-1,\end{cases}\qquad u\,|_{t=0}=\begin{cases}5.14, & \text{if } x<-1,\\ -2.29, & \text{if } x>-1.\end{cases} \tag{4.3}$$

For the given initial data the shock

$$h(t,x)=\begin{cases}4, & \text{if } x<c_1t-1,\\ 1, & \text{if } x>c_1t-1,\end{cases}$$

$$u(t,x)=\begin{cases}5.14, & \text{if } x<c_1t-1,\\ -2.29, & \text{if } x>c_1t-1,\end{cases} \tag{4.4}$$

where $c_1=7.61$ is a 2-shock and reaches the step at $t=1/c_1$. At that moment we need to solve the system (1.10) at the step. It is important to note that out of the step the process is still governed by (1.1) and (1.2). The shock (4.4) will be split into a 1-shock corresponding to (1.1) and (1.2), a 2-shock corresponding to (1.1) and (1.2) and a stationary shock (SS) corresponding to (1.10). The goal is to obtain one system which describes the entire flow phenomenon. This can be done through the δ-shock concept where δ will actually describe deficiency of the model. To achieve this we consider the system (1.10) with initial data given in (4.3). The next task is to find two constant states (h_2,u_2) and (h_3,u_3) which are located to the left and right of the bottom step respectively. These states are obtained by simultaneously solving the equations

$$\left.\begin{aligned}u_2-u_l&=-(h_2-h_l)\sqrt{\frac{g}{2}\left(\frac{1}{h_2}+\frac{1}{h_l}\right)},\\ u_r-u_3&=(h_r-h_3)\sqrt{\frac{g}{2}\left(\frac{1}{h_r}+\frac{1}{h_3}\right)},\\ g(h_2+b_l)+\frac{u_2^2}{2}&=g(h_3+b_r)+\frac{u_3^2}{2},\\ u_2h_2&=u_3h_3.\end{aligned}\right\} \tag{4.5}$$

The first two equations in (4.5) are obtained from the Rankine–Hugoniot conditions that describe the relations between the states on both sides of the left and right going shocks and the last two equations describe the bottom condition. To the left of the bottom step, the constant state (h_2,u_2) is connected to (h_l,u_l) by a left going 1-shock whereas to the right of the step the constant state (h_3,u_3) is connected to (h_r,u_r) by a right going 2-shock and a stationary shock is located at the step $x=0$ as shown in Fig. 3. The state-wave diagram for this case is

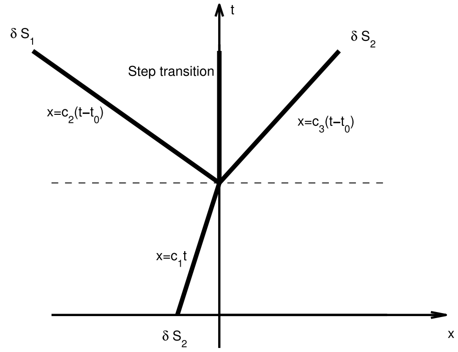

Fig. 3. Flow of a bore over a bottom step. The incoming shock δS_2 meets the bottom step at time $t_0=1/c_1$. For $t>t_0$, δS_1 is a left going delta shock, the bottom transition is at $x=0$, and δS_2 is a right going delta shock.

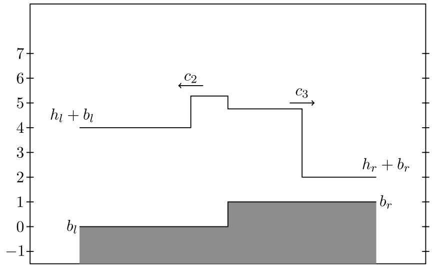

Fig. 4. A 1-shock moving left, a stationary step transition located at $x=0$ and a 2-shock moving right.

$$(h_l,u_l)\xrightarrow{\delta S_1}(h_2,u_2)\xrightarrow{\text{SS}}(h_3,u_3)\xrightarrow{\delta S_2}(h_r,u_r). \tag{4.6}$$

The flow pattern corresponding to the above diagram is shown in Fig. 4. The undisturbed water surface is located at $\eta(x,t)=h_j+b(x)$ for $j\in\{l,r,2,3\}$. The left going shock is travelling at the speed $c_2=-2.62$ and the right going shock has the approximate speed $c_3=7.07$. The physically relevant solution has the form

$$\begin{aligned}h(t,x)={}&\chi_{[0,t_0]}(t)\begin{cases}4, & \text{if } x<c_1t-1,\\ 1, & \text{if } x>c_1t-1,\end{cases}\\ &+\chi_{(t_0,+\infty)}(t)\begin{cases}4, & \text{if } x<c_2(t-t_0),\\ 5.28, & \text{if } c_2(t-t_0)<x<0,\end{cases}\\ &+\chi_{(t_0,+\infty)}(t)\begin{cases}3.76, & \text{if } 0<x<c_3(t-t_0),\\ 1, & \text{if } c_3(t-t_0)<x,\end{cases}\end{aligned} \tag{4.7}$$

and

$$u(t,x)=\chi_{[0,t_0]}(t)\alpha_1\delta(x-c_1t-1)$$

$$+ \chi_{[0,t_0]}(t) \begin{cases} 5.14, & \text{if } x < c_1 t - 1, \\ -2.29, & \text{if } x > c_1 t - 1, \end{cases}$$
$$+ \chi_{(t_0,+\infty)}(t) \begin{cases} 5.14, & \text{if } x < c_2(t - t_0), \\ 3.25, & \text{if } c_2(t - t_0) < x < 0, \end{cases}$$
$$+ \chi_{(t_0,+\infty)}(t) \begin{cases} 4.58, & \text{if } 0 < x < c_3(t - t_0), \\ -2.29, & \text{if } c_3(t - t_0) < x, \end{cases}$$
$$+ \chi_{(t_0,+\infty)}(t)\alpha_2\delta(x - c_2(t - t_0))$$
$$+ \chi_{(t_0,+\infty)}(t)\alpha_3\delta(x - c_3(t - t_0)), \tag{4.8}$$

where the Rankine–Hugoniot deficits are given by

$$\alpha_1 = \Big(c_1[u] - [gh + u^2/2 + gb]\Big)\, t,$$
$$\alpha_2 = \Big(c_2[u] - [gh + u^2/2]\Big)\,(t - t_0) + \Big(c_2[u] - [gh + u^2/2 + gb]\Big)\, t_0,$$
$$\alpha_3 = \Big(c_3[u] - [gh + u^2/2 + 1]\Big)\,(t - t_0).$$

That is

$$\alpha_1 = \Big(c_1(u_l - u_r) - (gh_l + u_l^2/2) + (gh_r + u_r^2/2 + 1)\Big)\, t,$$
$$\alpha_2 = \Big(c_2(u_l - u_2) - (gh_l + u_l^2/2) + (gh_2 + u_2^2/2)\Big)\,(t - t_0) + \Big(c_2(u_l - u_r) - (gh_l + u_l^2/2) + (gh_r + u_r^2/2 + 1)\Big)\, t_0,$$
$$\alpha_3 = \Big(c_3(u_3 - u_r) - (gh_3 + u_3^2/2 + 1) + (gh_r + u_r^2/2 + 1)]\Big) \times (t - t_0).$$

It is now straightforward to show that these functions define a solution in the sense of Definition 2.3. Moreover, evaluating the eigenvalues of the derivative matrix for the states (h_l, u_l), (h_2, u_2), (h_3, u_3) and (h_r, u_r) reveals that all three delta shocks δS_1, δS_2 and δS_3, are admissible, and so is the bottom step transition.

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Paper B

Hydraulic jumps on shear flows with constant vorticity *

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Hydraulic jumps on shear flows with constant vorticity

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A B S T R A C T

In this work, the influence of constant background vorticity on the properties of shock waves in a shallow water system are considered. It is shown that the flow-depth ratio of stationary shocks can be written as a function of two non-dimensional parameters: the Froude number, suitably defined in the presence of the shear flow, and a non-dimensional vorticity. Some properties of these hydraulic jumps are explored, and it is shown that stronger background vorticity has the effect of moderating the strength of the hydraulic jump.

1. Introduction

Modeling of surface wave motion in a fluid is normally based on classical systems which are obtained in the framework of irrotational flow. In such a context, the influence of vorticity is completely disregarded in the formulation of the governing equations. Although this consideration is justified in many circumstances, there are also a fair number of observed cases in near-shore hydrodynamics and open channel flow where this approach is unsuitable.

Indeed, there is ample evidence that vorticity may have a large impact on wave motion in a variety of situations. For example, it was recently shown that vorticity has significant influence on the modulational stability of quasi-periodic wavetrains [1,2] as well as the streamline pattern and pressure profiles in long waves [3–8]. The importance of vorticity in the modeling of free surface waves has also been exhibited in recent studies of wave–current interaction [9], the interaction of point vortices and vortex patches with the free surface [10,11], the influence of non-constant vorticity on small amplitude waves [12] and the creation of vorticity in long-wave models [13].

In the current work, we are concerned with the interaction of surface waves with an existing shear current. Such currents are created by the action of wind stress at the free surface and viscous stress at the bed, as well as tidal forcing. Once established, these shear currents may be taken as background conditions when studying individual surface waves. For example, the time scales needed to create such currents through wind forcing are typically much larger then the typical interaction time of a single wave with such a current. Moreover, on small time scales, surface waves are relatively unaffected by viscosity, so that an inviscid theory may be used.

In order to avoid undue mathematical complexity, it is assumed that the long waves to be described are perturbations of an existing background flow with a linear shear profile. This approach has been used by a number of authors (see for example [6,14–19]), and it has been indicated to approximate naturally occurring shear flows fairly well [20]. In particular, it was shown in [21] that the dispersion relation associated with a linear background shear gives rather good agreement with experimentally measured dispersion relation. It was also argued in [5] that linear shear flows can be used as a first approximation to more general shear profiles in the long-wave regime since the wavelength of the waves is then on a different scale than the variation of the shear profile.

Very recently, a new shallow-water system incorporating constant background shear has been found independently in [22–24]. Supposing a background shear flow of the form $U(z) = \Gamma z$, the system is written in terms of the total flow depth H and horizontal velocity perturbation u as

$$H_t + \left(\tfrac{\Gamma}{2}H^2 + uH\right)_x = 0, \tag{1.1}$$

$$\left(\tfrac{\Gamma}{2}H^2 + uH\right)_t + \left(\tfrac{\Gamma^2}{3}H^3 + \Gamma uH^2 + u^2H + \tfrac{1}{2}gH^2\right)_x = 0. \tag{1.2}$$

The first of these equations describes mass conservation, and the second arises from momentum conservation.

In the present contribution, this new system is used to understand the influence of vorticity on the properties of hydraulic jumps. An analysis of shock-wave solutions of the system (1.1), (1.2) detailed in the body of this paper shows that stationary jumps can be described in terms of two non-dimensional numbers, one being the Froude number, and the other incorporating the background vorticity Γ. To be more precise, if H_L is the upstream flow depth and u_L is the fluid velocity upstream of a stationary

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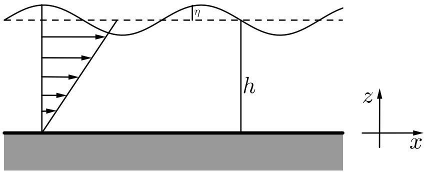

Fig. 1. Schematic representation of a linear shear flow over an even bottom.

jump, we define the Froude number by

$$Fr = \frac{u_L + \Gamma/2}{\sqrt{gH_L}},$$

and the non-dimensional vorticity by

$$\Omega = \frac{\Gamma^2 H}{6g},$$

where g is the gravitational constant, and the factor 6 is chosen for the sake of reaching a tidy expression in the final relation. These two parameters completely determine the strength of the jump, which is described mathematically as the ratio $\alpha = H_R/H_L$ through the cubic equation

$$\Omega\alpha^3 + (1+\Omega)\alpha^2 + (1+\Omega)\alpha - 2\mathrm{Fr}_L^2 = 0.$$

The derivation of this equation will be given in Section 4.

Our work is motivated in part by [25], where hydraulic jumps were studied in the presence of variable vorticity. This paper made an important contribution in the understanding of hydraulic jumps as transient phenomena, and the authors of [25] were the first to be able to simulate oscillation of the jump toe in a physically reasonable way, obtaining a very close match with experimental data. While hydraulic jumps are classically compartmentalized using the Froude number, one of the findings of [25] was that a hydraulic jump does not simply depend on a single parameter, the upstream Froude number, but also on the vorticity in the flow. In the current work, we study hydraulic jumps using the simplifying assumption of constant background shear. While our model is not able to explain the creation of vorticity in a hydraulic jump which featured prominently in [25], we are able to quantify the dependence of the strength of the hydraulic jump on the vorticity, and give fairly simple closed-form solutions which we hope will prove useful in practical studies investigating the influence of background shear on the properties of hydraulic jumps.

A schematic representation of the problem setup is shown in Fig. 1 where a channel of unit width and even bottom containing a fluid with undisturbed depth h is considered. The elevation of the free surface from its rest state is given by $\eta(x,t)$ so that the total depth at a point x and time t is given by $H(x,t) = h + \eta(x,t)$. The average horizontal velocity is denoted by $u(x,t)$ and the total velocity component is

$$v(x,t,z) \equiv U(z) + u(x,t) = \Gamma z + u, \tag{1.3}$$

where $U(z)$ is the linear shear current and Γ is constant. As shown in [22–24], if the waves at the free surface are long enough compared to the fluid depth h, the appropriate equations describing the shear flow are given by (1.1) and (1.2). If the flow variables are smooth so that the solutions are also smooth, then an equivalent system has the form

$$\partial_t H + \partial_x\left(\tfrac{\Gamma}{2}H^2 + uH\right) = 0, \tag{1.4}$$

$$\partial_t u + \partial_x\left(\tfrac{1}{2}u^2 + gH\right) = 0. \tag{1.5}$$

It is well known that in the case of discontinuous solutions, mass and momentum conservation could introduce a Rankine–Hugoniot deficit in (1.5) which may raise an argument against the use of (1.4)–(1.5) in favor of the system (1.1)–(1.2). However, (1.4)–(1.5) is mathematically interesting and the theory contributing to the progress of this system is fully elaborated and the Riemann problem and the initial value problem for this system can be resolved adequately [26–28]. An analysis of these systems in the case of zero vorticity is neatly presented in [29] where it is shown that the combination of discontinuous free-surface solutions and bottom step transitions naturally lead to singular solutions featuring a Rankine–Hugoniot deficit. While the Eqs. (1.1) and (1.2) were found in the works [22–24], these authors did not consider the total mechanical energy equation which must be considered in order to provide a physical selection criterion on the admissibility of shock waves. In the current work, we derive the energy equations and use it to discard non-admissible shocks. As will be shown in Section 2, energy conservation for the shear flow is formulated as

$$\left(\tfrac{\Gamma}{2}uH^2 + \tfrac{\Gamma^2}{6}H^3 + \tfrac{1}{2}u^2H + \tfrac{1}{2}gH^2\right)_t + \left(\tfrac{3\Gamma}{4}u^2H^2 + \tfrac{\Gamma}{2}gH^3 + \tfrac{\Gamma^2}{2}uH^3 + \tfrac{\Gamma^3}{8}H^4 + \tfrac{1}{2}u^3H + guH^2\right)_x = 0. \tag{1.6}$$

The plan of the paper is as follows: In Section 2, we formulate the free surface problem for a shear flow over a flat bottom. The derivation is based on the Euler equations for an incompressible and inviscid fluid. In Section 3, the equations are analyzed and mathematical properties of the flow variables are presented. Explicit expressions representing mass conservation and momentum conservation are obtained and it is shown that a shallow water flow over a flat bed in the presence of a linear shear current features energy loss. In Section 4, we explain how to construct a steady state solution for the linear shear flow by defining the Froude number in terms of a depth-averaged integral.

2. Linear shear flow over flat bottom topography

The governing shallow water equations that describe the motion of incompressible and inviscid fluid are derived in this section. We consider a flat-bottom channel with uniform width and set the x-coordinate in the flow direction whilst the z-coordinate is vertically upwards.

Consider a control volume of unit width enclosed by the flat bottom, the free surface and the interval $[x_1, x_2]$ such that $x_1 < x_2$ on the lateral sides. The mass of the incompressible, inviscid fluid of uniform depth contained in the control volume is

$$\mathcal{M} = \int_{x_1}^{x_2}\int_0^{H(x,t)} \rho \, dz \, dx.$$

If the free surface and the flat bottom are impermeable so that no transfer of mass occur there, then the physical hypothesis of mass conservation requires that the rate of change of mass per unit time is proportional to the mass flux through the lateral boundaries. The mathematical idealization of this concept is expressed by the integral equation

$$\frac{d}{dt}\int_{x_1}^{x_2}\int_0^{H(x,t)} \rho \, dz \, dx = \int_0^{H(x,t)} \rho v(x_1,t)\, dz - \int_0^{H(x,t)} \rho v(x_2,t)\, dz.$$

If the flow variables as well as the domain are smooth, then the above equation can be written as

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho H(x,t)\, dx + \left[\int_0^{H(x,t)} \rho(\Gamma z + u(x,t))\, dz\right]_{x_1}^{x_2} = 0.$$

If we have constant density ρ, we can divide through by $(x_2 - x_1)\rho$. Then by letting $x_2 - x_1 \to 0$, we see that the integrand must be zero, so that we obtain the equation representing mass conservation in the form

$$\partial_t H + \partial_x \left(\frac{\Gamma}{2} H^2 + uH \right) = 0.$$

In a similar manner, an expression representing momentum conservation in the control volume can be derived. If we suppose that pressure force is the only force acting on the control volume, then conservation of momentum is based on the physical principle that the fluid is in a hydrostatic balance so that the pressure $p = p(x, z, t)$ is introduced. Applying this assumption in a fluid column $[x_1, x_2] \times [z, z + \Delta z]$ gives

$$\big(p(\bar{x}, z + \Delta z, t) - p(\bar{x}, z, t)\big)(x_2 - x_1) = -(x_2 - x_1)\rho g \Delta z,$$

for $\bar{x} \in [x_1, x_2]$. If the flow variables are smooth, then dividing through by $(x_2 - x_1)\Delta z$ and taking the limit as $\Delta z \to 0$ gives

$$\frac{dp}{dz} = -\rho g.$$

Integrating and normalizing the pressure to be zero at the surface yields the relation

$$p(x, z, t) = \rho g \big(H(x, t) - z\big) \tag{2.1}$$

for the hydrostatic pressure. Considering the control volume described above, the total horizontal momentum in the control volume is

$$\mathcal{I} = \int_{x_1}^{x_2} \int_0^{H(x,t)} \rho v(x, t)\, dz\, dx.$$

Momentum conservation is obtained from Newton's second law which requires that the rate of change of total momentum is equal to the net momentum flux through the lateral boundaries plus the pressure forces acting on the boundaries. This is expressed mathematically as

$$\begin{aligned}\frac{d}{dt} &\int_{x_1}^{x_2} \int_0^{H(x,t)} \rho v(x, t)\, dz\, dx \\ &= \int_0^H \rho v^2(x_1, t)\, dz - \int_0^H \rho v^2(x_2, t)\, dz \\ &\quad + \int_0^H p(x_1, z, t)\, dz - \int_0^H p(x_2, z, t)\, dz.\end{aligned}$$

Substituting the total velocity in (1.3) and the pressure term in (2.1) and simplifying yield

$$\begin{aligned}\frac{d}{dt} \int_{x_1}^{x_2} \rho \Bigg[&\left(\frac{\Gamma}{2} H^2 + uH \right)_t \\ &+ \left(\frac{\Gamma^2}{3} H^3 + \Gamma u H^2 + u^2 H + \frac{g}{2} H^2 \right)_x \Bigg] dx = 0\end{aligned}$$

Dividing through by $(x_2 - x_1)\rho$ and taking the limit as $(x_2 - x_1) \to 0$ shows that the momentum conservation equation can be expressed as

$$\left(\frac{\Gamma}{2} H^2 + uH \right)_t + \left(\frac{\Gamma^2}{3} H^3 + \Gamma u H^2 + u^2 H + \frac{1}{2} g H^2 \right)_x = 0.$$

As can be readily seen, for smooth solutions an equivalent formulation is obtained by further algebraic manipulation to yield the variant form (1.4) and (1.5). Finally, we turn to the conservation of energy. Total mechanical energy in the control volume is expressed as

$$\begin{aligned}\frac{d}{dt} &\int_{x_1}^{x_2} \int_0^{h+\eta} \rho \left(\tfrac{1}{2} \tilde{v}^2 + gz \right) dz dx \\ &= \left[\int_0^{h+\eta} \rho \left(\tfrac{1}{2} \tilde{v}^2 + gz \right) \tilde{v} dz + \int_0^{h+\eta} \rho \big(h + \eta - z \big) \tilde{v} dz \right]_{x_2}^{x_1}\end{aligned}$$

Substituting (1.3) and integrating in the z-direction gives

$$\begin{aligned}\int_{x_1}^{x_2} &\left(\tfrac{\Gamma}{2} u H^2 + \tfrac{\Gamma^2}{6} H^3 + \tfrac{1}{2} u^2 H + \tfrac{1}{2} g H^2 \right)_t + \\ &\left(\tfrac{3\Gamma}{4} u^2 H^2 + \tfrac{\Gamma}{2} g H^3 + \tfrac{\Gamma^2}{2} u H^3 + \tfrac{\Gamma^3}{8} H^4 + \tfrac{1}{2} u^3 H + g u H^2 \right)_x dx = 0.\end{aligned}$$

Dividing through by $(x_1 - x_2)$ and taking the limit as $(x_1 - x_2) \longrightarrow 0$ yield the energy conservation equation

$$\begin{aligned}&\left(\tfrac{\Gamma}{2} u H^2 + \tfrac{\Gamma^2}{6} H^3 + \tfrac{1}{2} u^2 H + \tfrac{1}{2} g H^2 \right)_t + \\ &\left(\tfrac{3\Gamma}{4} u^2 H^2 + \tfrac{\Gamma}{2} g H^3 + \tfrac{\Gamma^2}{2} u H^3 + \tfrac{\Gamma^3}{8} H^4 + \tfrac{1}{2} u^3 H + g u H^2 \right)_x = 0,\end{aligned}$$

which is the same as (1.6).

3. Mathematical description of the flow properties

We provide a discussion on the flow properties associated with waves propagation in shallow water in the presence of linear shear. The characteristics in this case are found by first putting the equation in characteristic form. In conservative variables $U = (H, u)^T$, (1.4) and (1.5) are expressed in matrix notation as

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0, \tag{3.1}$$

where $\mathbf{F}(\mathbf{U})_x = \mathbf{F}'(\mathbf{U})\mathbf{U}_x$. The flux Jacobian $\mathbf{F}'(\mathbf{U})$ is

$$\mathbf{F}'(\mathbf{U}) = \begin{pmatrix} \Gamma H + u & H \\ 1 & u \end{pmatrix}. \tag{3.2}$$

The eigenvalues of the Jacobian matrix are

$$\begin{aligned}\lambda_- &= u + \frac{1}{2}\Gamma H - \sqrt{H + \left(\tfrac{1}{2}\Gamma H\right)^2} \quad \text{and} \\ \lambda_+ &= u + \frac{1}{2}\Gamma H + \sqrt{H + \left(\tfrac{1}{2}\Gamma H\right)^2},\end{aligned} \tag{3.3}$$

with $\lambda_- < \lambda_+$ for $H \neq 0$ so that the system is strictly hyperbolic. The corresponding right eigenvectors

$$\begin{aligned}r_- &= \begin{pmatrix} 1 \\ -\frac{1}{2}\Gamma - \frac{1}{H}\sqrt{H + \left(\frac{1}{2}\Gamma H\right)^2} \end{pmatrix} \quad \text{and} \\ r_+ &= \begin{pmatrix} 1 \\ -\frac{1}{2}\Gamma + \frac{1}{H}\sqrt{H + \left(\frac{1}{2}\Gamma H\right)^2} \end{pmatrix},\end{aligned} \tag{3.4}$$

are linearly independent and therefore, span the eigenspace in the (H, u)-plane. Both characteristic fields are genuinely nonlinear with

$$\begin{aligned}\nabla\lambda_-(H, u) \cdot r_-(H, u) &= -\frac{6gH + 2\Gamma^2 H^2}{\sqrt{4gH + \Gamma^2 H^2}} < 0 \\ \nabla\lambda_+(H, u) \cdot r_+(H, u) &= +\frac{6gH + 2\Gamma^2 H^2}{\sqrt{4gH + \Gamma^2 H^2}} > 0.\end{aligned} \tag{3.5}$$

It is well-known that a traveling hydraulic jump over a flat bottom obeys mass conservation and momentum conservation. To describe the relation between the state variables on each side of the jump, we assume that the discontinuity is located at the bore front. If we consider two constant states $(H, u) = (H_L, u_L)$ and $(H, u) = (H_R, u_R)$, then we obtain from (1.1) and (1.2) the Rankine–Hugoniot conditions

$$-s[H] + \left[\frac{\Gamma}{2}H^2 + uH\right] = 0, \tag{3.6}$$

$$-s\left[\frac{\Gamma}{2}H^2 + uH\right] + \left[\frac{\Gamma^2}{3}H^3 + \Gamma uH^2 + u^2H + \frac{1}{2}gH^2\right] = 0. \tag{3.7}$$

Define $[H] = H_R - H_L$ and $[u] = u_R - u_L$, where the subscripts L and R denote the left and right states of the hydraulic jump respectively. Then we can explicitly write

$$s(H_R - H_L) = \left(\frac{\Gamma}{2}H_R^2 + u_RH_R - \frac{\Gamma}{2}H_L^2 + u_LH_L\right),$$

$$s\left(\frac{\Gamma}{2}H_R^2 + u_RH_R - \frac{\Gamma}{2}H_L^2 + u_LH_L\right) = \left(\frac{\Gamma^2}{3}H_R^3 + \Gamma u_RH_R^2 + u_R^2H_R + \frac{1}{2}gH_R^2 - \frac{\Gamma^2}{3}H_L^3 + \Gamma u_LH_L^2 + u_L^2H_L + \frac{1}{2}gH_L^2\right).$$

The shock speed, s, satisfies the Lax entropy condition [26,30]

$$\lambda_-(H_R, v_R) < s < \lambda_-(H_L, v_L), \quad s < \lambda_+(H_R, v_R), \tag{3.8}$$

for a 1-shock and

$$\lambda_+(H_R, v_R) < s < \lambda_+(H_L, v_L), \quad s > \lambda_-(H_L, v_L), \tag{3.9}$$

for a 2-shock and is expressed as

$$s = \frac{\frac{\Gamma}{2}H_R^2 + u_RH_R - \frac{\Gamma}{2}H_L^2 + u_LH_L}{H_R - H_L} = \frac{\frac{\Gamma^2}{3}H_R^3 + \Gamma u_RH_R^2 + u_R^2H_R + \frac{1}{2}gH_R^2 - \frac{\Gamma^2}{3}H_L^3 + \Gamma u_LH_L^2 + u_L^2H_L + \frac{1}{2}gH_L^2}{\frac{\Gamma}{2}H_R^2 + u_RH_R - \frac{\Gamma}{2}H_L^2 + u_LH_L}. \tag{3.10}$$

Considering the expressions on both sides of the last equality, we obtain

$$(u_R - u_L)^2 = \frac{\Gamma^2}{12H_LH_R}(H_R - H_L)^4 + \Gamma(H_R - H_L)(u_L - u_R) + \frac{g}{2}(H_R + H_L)(H_R - H_L)^2$$

Simplifying this expression further gives the relation

$$u_R - u_L = \frac{\Gamma}{2}(H_R - H_L) \times \left(-1 \pm \sqrt{\frac{1}{3} + \frac{1}{H_L}\left(\frac{H_R}{3} + \frac{2g}{\Gamma^2}\right) + \frac{1}{H_R}\left(\frac{H_L}{3} + \frac{2g}{\Gamma^2}\right)}\right). \tag{3.11}$$

It is obvious to see that the relation between the states u_R and u_L is either positive or negative and is dependent on the fluid depths H_R and H_L on each side of the shock. It shall be shown how the state variables H, u and s are related on each side of the hydraulic jump but firstly, we write the shock speed in terms of the above expression. Substituting this relation into Eq. (3.10) gives

$$\begin{aligned} s &= u_R + \frac{\Gamma}{2}H_R \pm \frac{\Gamma}{2}H_L\sqrt{\frac{1}{3} + \frac{1}{H_L}\left(\frac{H_R}{3} + \frac{2g}{\Gamma^2}\right) + \frac{1}{H_R}\left(\frac{H_L}{3} + \frac{2g}{\Gamma^2}\right)} \\ &= u_L + \frac{\Gamma}{2}H_L \pm \frac{\Gamma}{2}H_R\sqrt{\frac{1}{3} + \frac{1}{H_L}\left(\frac{H_R}{3} + \frac{2g}{\Gamma^2}\right) + \frac{1}{H_R}\left(\frac{H_L}{3} + \frac{2g}{\Gamma^2}\right)}. \end{aligned} \tag{3.12}$$

Schematics of shocks of the first and the second families and the characteristic curves are shown in Figs. 2 and 3. As mentioned earlier, a discontinuity propagating over a flat-bottom at a speed s, given in (3.12), must respect mass conservation. In what follows, a mathematical expression representing mass conservation through the discontinuity is obtained in terms of the shock speed. From Eq. (3.6) we get

$$\mu \equiv \frac{\Gamma}{2}H_R^2 + (u_L - s)H_L = \frac{\Gamma}{2}H_L^2 + (u_R - s)H_R. \tag{3.13}$$

Inserting the shock speed in (3.12) gives

$$\mu = \mp\frac{\Gamma}{2}H_LH_R\sqrt{\frac{1}{3} + \frac{1}{H_L}\left(\frac{H_R}{3} + \frac{2g}{\Gamma^2}\right) + \frac{1}{H_R}\left(\frac{H_L}{3} + \frac{2g}{\Gamma^2}\right)}. \tag{3.14}$$

In a similar manner, conservation of momentum through the hydraulic jump is derived from (3.7) as

$$\left(\frac{\Gamma}{2}H_R^2 + u_RH_R\right)(u_R - c) + \frac{\Gamma}{2}u_RH_R^2 + \frac{\Gamma}{3}H_R^3 + \frac{1}{2}gH_R^2 = \left(\frac{\Gamma}{2}H_L^2 + u_LH_L\right)(u_L - c) + \frac{\Gamma}{2}u_LH_L^2 + \frac{\Gamma}{3}H_L^3 + \frac{1}{2}gH_L^2.$$

Inserting (3.13) into the above expression leads to

$$\mu\left(\frac{\Gamma}{2}H_R + u_R\right) + \frac{\Gamma}{12}H_R^3 + \frac{1}{2}gH_R^2 = \mu\left(\frac{\Gamma}{2}H_L + u_L\right) + \frac{\Gamma}{12}H_L^3 + \frac{1}{2}gH_L^2. \tag{3.15}$$

The Rankine–Hugoniot condition for Eq. (1.6) is

$$-s\left[\frac{\Gamma^2}{6}H^3 + \frac{\Gamma}{2}uH^2 + \frac{1}{2}u^2H + \frac{1}{2}gH^2\right] + \left[\frac{\Gamma^3}{8}H^4 + \frac{\Gamma^2}{2}uH^3 + \frac{3\Gamma}{4}u^2H^2 + \frac{\Gamma}{2}gH^3 + \frac{1}{2}u^3H + guH^2\right] = 0. \tag{3.16}$$

The mechanical energy associated with the above Rankine–Hugoniot condition dissipates in the discontinuity. Out of the discontinuity where the solution is smooth, the mechanical energy is conserved. The hydraulic jump can therefore, be interpreted as heat dump which absorbs the excess energy of the fluid. We derive in what follows a mathematical expression that represents the energy loss. In order words, we show that an admissible shock wave solution that satisfies the Rankine–Hugoniot condition in (3.16) dissipates mechanical energy. From (3.16) we have

$$\Delta E = E_R - E_L, \tag{3.17}$$

where

$$E_R = \left(\frac{\Gamma^3}{6}H_R^4 + \frac{\Gamma^2}{2}u_RH_R^3 - s\frac{\Gamma^2}{6}H_R^3 + \frac{3\Gamma}{4}u_R^2H_R^2 + \frac{\Gamma}{2}gH_R^3 - s\frac{\Gamma}{2}u_RH_R^2 + \frac{1}{2}u_R^3H_R + gu_RH_R^2 - s\frac{1}{2}u_R^2H_R - s\frac{1}{2}gH_R^2\right), \tag{3.18}$$

and

$$E_L = \left(\frac{\Gamma^3}{6}H_L^4 + \frac{\Gamma^2}{2}u_LH_L^3 - s\frac{\Gamma^2}{6}H_L^3 + \frac{3\Gamma}{4}u_L^2H_L^2 + \frac{\Gamma}{2}gH_L^3 - s\frac{\Gamma}{2}u_LH_L^2 + \frac{1}{2}u_L^3H_L + gu_LH_L^2 - s\frac{1}{2}u_L^2H_L - s\frac{1}{2}gH_L^2\right). \tag{3.19}$$

E_R and E_L represent the energy on the right and the left of the hydraulic jump. It is obvious from (3.12) that the discontinuity can propagate either to the right or to the left. For a right going shock for instance, E_R represents the mechanical energy after the discontinuity while E_L symbolizes the energy before the discontinuity. Through the jump, we have

$$\begin{aligned} \Delta E = &\left(\tfrac{1}{2}u_R^2 + \tfrac{1}{2}gH_R\right)\left((u_R - s)H_R + \tfrac{\Gamma}{2}H_R^2\right) \\ &- \left(\tfrac{1}{2}u_L^2 + \tfrac{1}{2}gH_L\right)\left((u_L - s)H_L + \tfrac{\Gamma}{2}H_L^2\right) \\ &+ \tfrac{\Gamma^3}{8}\left(H_R^4 - H_L^4\right) + \tfrac{\Gamma^2}{2}\left(u_RH_R^3 - u_LH_L^3\right) \\ &- \tfrac{s\Gamma^2}{6}\left(H_R^3 - H_L^3\right) + \tfrac{\Gamma}{2}\left(u_R^2H_R^2 - u_L^2H_L^2\right) \\ &- \tfrac{s\Gamma}{2}\left(u_RH_R^2 - u_LH_L^2\right) + \tfrac{g\Gamma}{4}\left(H_R^3 - H_L^3\right) + \tfrac{g}{2}\left(u_RH_R^2 - u_LH_L^2\right). \end{aligned}$$

Applying the expressions for mass conservation and momentum conservation through the discontinuity (see Eqs. (3.13) and (3.15))

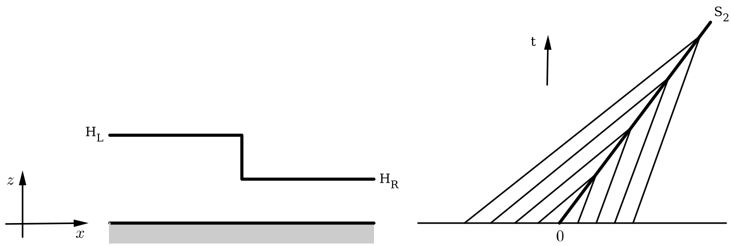

Fig. 2. Schematic representation of a linear shear flow over an even bottom. S_2 is a shock of the second family.

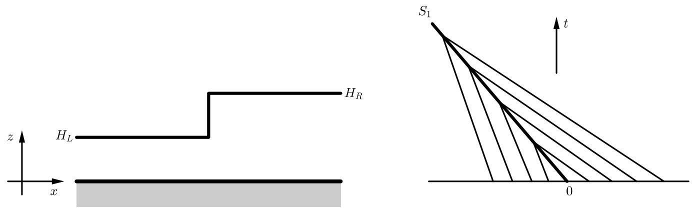

Fig. 3. Schematic representation of a linear shear flow over an even bottom. S_1 is a shock of the first family.

give

$$\Delta E = \frac{\mu}{2}\left(\left((u_R - s) + \tfrac{\Gamma}{2}H_R\right)^2 - \left((u_L - s) + \tfrac{\Gamma}{2}H_L\right)^2 + \tfrac{\Gamma^2}{4}(H_R^2 - H_L^2) + 2g(H_R - H_L)\right).$$

By focusing on the first two terms on the right hand side, it is noted from (3.13) that

$$\left((u_R - s) + \tfrac{\Gamma}{2}H_R\right)^2 = \left(\tfrac{\Gamma}{2}H_L\right)^2\left(\tfrac{1}{3} + \tfrac{1}{H_L}\left(\tfrac{H_R}{3} + \tfrac{2g}{\Gamma^2}\right) + \tfrac{1}{H_R}\left(\tfrac{H_L}{3} + \tfrac{2g}{\Gamma^2}\right)\right),$$

$$\left((u_L - s) + \tfrac{\Gamma}{2}H_L\right)^2 = \left(\tfrac{\Gamma}{2}H_R\right)^2\left(\tfrac{1}{3} + \tfrac{1}{H_L}\left(\tfrac{H_R}{3} + \tfrac{2g}{\Gamma^2}\right) + \tfrac{1}{H_R}\left(\tfrac{H_L}{3} + \tfrac{2g}{\Gamma^2}\right)\right).$$

Substituting these relations into the preceding expression for the energy loss, we get

$$\Delta E = -\mu(H_R - H_L)^3\left(\frac{\Gamma^2(H_L + H_R) + 6g}{24H_LH_R}\right). \tag{3.20}$$

Notice firstly, that the fractional term on the right hand side is strictly positive. It is noted also that $H_L \neq H_R$ if and only if $v_L \neq v_R$. Consequently, the mechanical energy loss through the discontinuity requires that

$$\mu(H_R - H_L)^3 > 0. \tag{3.21}$$

This inequality together with mass conservation through the discontinuity (see Eq. (3.13)) and the total velocity component (1.3) give the following conditions

$$\begin{aligned} &H_R > H_L \iff v_R > s \text{ and } v_L > s,\\ &H_R < H_L \iff v_R < s \text{ and } v_L < s. \end{aligned} \tag{3.22}$$

The first condition in (3.22) simply describes a hydraulic jump in which the fluid depth on the right is larger than that on the left. In this case, the propagating shock speed s is greater than the fluid velocities on both sides of the jump. In other words, we say that the shear flow traveling at speed v_L hit the shock from the left and become mitigated as they emerge from the shock with traveling speed v_R. The same explanation holds for the second condition. From the mass conservation through the hydraulic jump given in (3.13) and Eqs. (3.11) and (3.12), it is found that the relation

$$v_R - v_L = -\frac{\mu(H_R - H_L)}{H_RH_L}, \tag{3.23}$$

holds. Applying the strictly positive inequality in (3.21) gives the relation

$$v_R < v_L. \tag{3.24}$$

In Fig. 3, a propagating 1-shock moving to the left is shown where the shock speed is larger than the characteristic speeds on left side of the shock but lower than those on the right. The characteristics emanating from $x > 0$ and $x < 0$ propagate into the shock. The conditions (3.22) and (3.24) play an important role in analyzing the Rankine–Hugoniot jump condition for the shear flow. In fact it can be shown that the shock given in (3.12) satisfies these conditions. For $H_R > H_L$, we have a 1-shock, $s = S_1$, such that

$$v_R > u_R + \frac{\Gamma}{2}H_R > u_R + \frac{\Gamma}{2}H_R - \sqrt{\frac{1}{3} + \frac{1}{H_L}\left(\frac{H_R}{3} + \frac{2g}{\Gamma^2}\right) + \frac{1}{H_R}\left(\frac{H_L}{3} + \frac{2g}{\Gamma^2}\right)} = S_1.$$

The inequality $v_L > s$ is proved in like manner and justifies the first condition in (3.22) Similarly, $H_R < H_L$ gives a 2-shock, $s = S_2$, satisfying

$$\begin{aligned} v_R &< u_R + \frac{\Gamma}{2}H_R + \frac{\Gamma}{2}H_L\\ &< u_R + \frac{\Gamma}{2}H_R\\ &\quad + \frac{\Gamma}{2}H_L\sqrt{\frac{1}{3} + \frac{1}{H_L}\left(\frac{H_R}{3} + \frac{2g}{\Gamma^2}\right) + \frac{1}{H_R}\left(\frac{H_L}{3} + \frac{2g}{\Gamma^2}\right)} = S_2. \end{aligned}$$

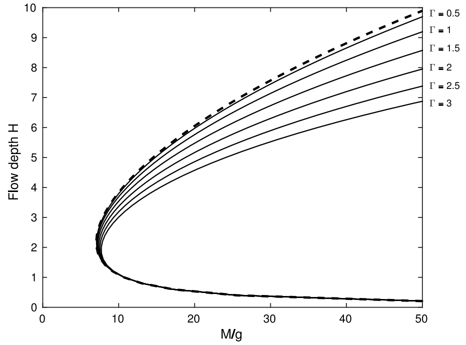

Fig. 4. The momentum function M/g is plotted against the flow depth for various strengths of vorticity. The dashed line shows the irrotational case $\Gamma = 0$. The other curves show the respective cases of the strengths of the vorticity Γ.

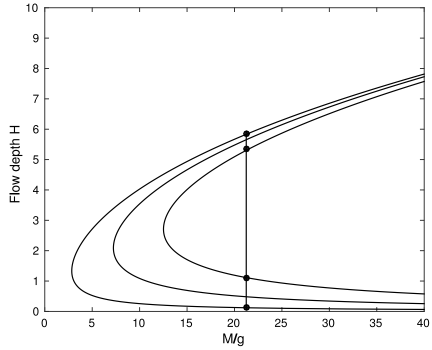

Fig. 5. The momentum function M/g is plotted against the flow depth H for constant vorticity $\Gamma = 1.5$ and different values of the flow rate per unit width Λ.

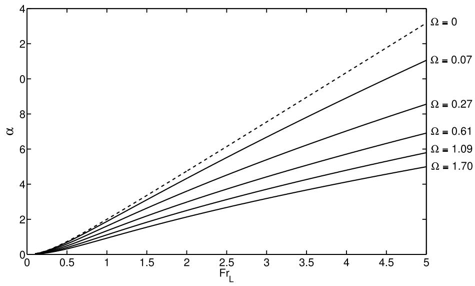

Fig. 6. The ratio of right to left Froude numbers α plotted against the left Froude number Fr_L for various strengths of vorticity Ω. The dashed curve depicts the irrotational case.

4. Steady state solution

We analyze the hydraulic jump by using the Froude number which is defined in such a way that it takes into account the average flow velocity over the entire fluid depth. In dimensionless variables, the depth averaging integral gives

$$\mathrm{Fr} = \frac{\frac{1}{H}\int_0^H U + u\,dz}{\sqrt{gH}}.$$

By using (1.3), the Froude number simplifies to

$$\mathrm{Fr} = \frac{u + \frac{\Gamma}{2}H}{\sqrt{gH}}.$$

The analysis is based on the hypothesis that the hydraulic jump is stationary and that the velocity and water depth increase across the jump. If we let Λ represent the volume flow rate per unit width, then the conservation of mass necessitates that

$$\Lambda = u_L H_L + \frac{\Gamma}{2}H_L^2 = u_R H_R + \frac{\Gamma}{2}H_R^2, \tag{4.1}$$

be satisfied. Using the concept of momentum conservation explained above, an expression for the momentum conservation across the discontinuity is obtained in the form

$$\frac{\Gamma^2}{3}H_L^3 + \Gamma u_L H_L^2 + u_L^2 H_L + \frac{1}{2}gH_L^2 = \frac{\Gamma^2}{3}H_R^3 + \Gamma u_R H_R^2 + u_R^2 H_R + \frac{1}{2}gH_R^2. \tag{4.2}$$

In particular, if we define $M = \frac{\Gamma^2}{3}H^3 + \Gamma u H^2 + u^2 H + \frac{1}{2}gH^2$ the quantity M/g is the analogue of the momentum function used in hydraulic engineering. Since M needs to be preserved through a stationary jump, for a given volume flow rate per unit width Λ, one can find the conjugate flow depths by plotting the curve

$$M = \frac{1}{12}\Gamma^2 H^3 + \frac{\Lambda^2}{H} + \frac{1}{2}gH^2.$$

Such a plot is shown in Fig. 4 for $\Lambda = 10$ and a variety of background vorticities ranging from $\Gamma = 0$ to $\Gamma = 3$. On the other hand, Fig. 5 shows the graphs for a fixed $\Gamma = 1.5$ but for a variety of values of Λ.

If we substitute the expression for Λ into relation (4.2), then we get

$$\Lambda^2\left(\frac{1}{H_R} - \frac{1}{H_L}\right) = \frac{\Gamma^2}{12}\left(H_L^3 - H_R^3\right) + \frac{1}{2}g\left(H_L^2 - H_R^2\right). \tag{4.3}$$

Substituting the expression for the Froude number stated above and carrying out further algebraic simplification gives the cubic function

$$\frac{\Gamma^2}{6g}H_L\alpha^3 + \left(1 + \frac{\Gamma^2}{6g}H_L\right)\alpha^2 + \left(1 + \frac{\Gamma^2}{6g}H_L\right)\alpha - 2\mathrm{Fr}_L^2 = 0, \tag{4.4}$$

where $\alpha = H_R/H_L$ is the ratio of depths. Note that the strength of vorticity depends on the non-dimensional parameter $\Omega = H_L\Gamma^2/6g$, so that we can write the equation in the final form

$$\Omega\alpha^3 + (1+\Omega)\alpha^2 + (1+\Omega)\alpha - 2\mathrm{Fr}_L^2 = 0. \tag{4.5}$$

The cubic equation can be solved for any value of Ω. Fig. 6 shows a plot of α as a function of Fr_L for a number of values of the vorticity Γ. It is apparent that larger values of Ω have the effect of moderating the strength of the hydraulic jump.

Acknowledgment

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Paper C

A shallow-water system with vanishing buoyancy *

* *Submitted to Applicable analysis, 2017*

A shallow-water system with vanishing buoyancy

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May 24, 2018

Abstract

In this work, a shallow-water system for interfacial waves in the case of a neutrally buoyant two-layer fluid system is considered. Such a situation arises in the case of large underwater lakes of compressible liquids such as CO_2 in the deep ocean which may happen naturally or may be man-made. Depending on temperature and depth, such deposits may be either stable, unstable or neutrally stable, and in the current contribution, the neutrally stable case is considered.

The motion of the long waves at the interface can be described by a shallow-water system which becomes triangular in the neutrally stable case. In this case, the system ceases to be strictly hyperbolic, and the standard theory of hyperbolic conservation laws may not be used to solve the initial-value or even the Riemann problem.

It is shown that the Riemann problem can still be solved uniquely using singular shocks containing Dirac delta distributions traveling with the shock. We characterize the solutions in integrated form, so that no measure-theoretic extension of the solution concept is needed. Uniqueness follows immediately from the construction of the solution. We characterize solutions in terms of the complex vanishing viscosity method, and show that the two solution concepts coincide.

Keywords: Rankine-Hugoniot deficit; singular solutions; weak asymptotics; travelling waves; Riemann problem

1 Introduction

In this paper we study a triangular system of conservation laws of the form

$$u_t + uu_x = 0, \tag{1.1}$$

$$\eta_t + h_1 u_x + (\eta u)_x = 0. \tag{1.2}$$

This system is derived as a model for internal waves at the interface of a two-fluid system where a finite uniform layer fluid of density ρ_1 and approximate depth h_1 is located below an upper layer of density ρ_2 and very large depth as shown in Figure 1. The interest in this system is physically motivated by considering large pools of heavy liquid located at the bottom of a deep ocean. Such pools of heavy liquids may occur naturally [15], and have

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also been put forward as a possible long-term storage site for CO_2 captured from fossil fuel combustors.

As global warming and climate change are now understood to be in part due to larger and larger concentrations of CO_2 in the atmosphere, stabilizing the level of CO_2 in the atmosphere has been the focus of a large body of research. One potential method of reducing the rise of atmospheric CO_2 levels is to capture it in fossil-burning processes and sequester it elsewhere. Potential storage sites include depleted petroleum and natural gas reservoirs, saline aquifers, unminable coal beds and the world's oceans. While the oceans are the largest potential reservoir for dissolved CO_2, it appears to be preferable from ecological and climate considerations to store CO_2 in undissolved form.

It is well established (see [34, 12]) that at predominant oceanic temperatures, CO_2 condenses to the liquid phase at depths of about 400m. Due to the relatively higher compressibility of liquid CO_2 than seawater, liquid CO_2 is denser than seawater at about 3000m depth. Storage of liquid CO_2 in the deep ocean is thus at least theoretically possible at depths exceeding 3000m. However, it is also well known that the interface of CO_2 and seawater is characterized by the rapid nucleation of H_2O and CO_2 into an icelike compound. This hydrate layer acts as a membrane which may prevent the CO_2 from escaping even at depths smaller than 3000m (see [15]). The hydrate layer is sometimes modeled physically by introducing interfacial tension (cf. [13, 39]). However, as experiments show, this approach may not be optimal since the hydrate layer is highly non-uniform and behaves more like a brittle solid, breaking up if a certain threshold stress is exceeded due to wave motion at the interface.

The changes in the CO_2 density imply that at a certain depth it will coincide with the density of the ambient seawater. Moreover, unexpected large changes in the temperature of the ambient seawater may render a previously stable configuration unstable by making the CO_2 buoyant or neutrally buoyant. In the present work we focus on the borderline case of vanishing buoyancy which leads to a two-fluid system with fluids of equal density. For a large underwater pool of CO_2, long waves will be the dominant wave phenomenon, and the effects of the hydrate layer and interfacial tension will be mostly felt on a smaller scale than that of a long wave, so we restrict our considerations to a shallow-water-like system of equations of the form

$$\begin{aligned} \eta_t + h_1 u_x + (\eta u)_x &= 0, \\ u_t + g\tfrac{\rho_1 - \rho_2}{\rho_1}\eta_x + u u_x &= 0. \end{aligned}$$

Such a system can be obtained by by following the analysis of Craig *et al.* [4], Section 5.4, or the derivation of a long-wave system used in [16]). In the neutrally buoyant case, the densities ρ_2 and ρ_1 will be equal, and the system reduces to (1.1), (1.2).

As is well-known, the classical theory of hyperbolic laws in one space dimension usually requires that the system be strictly hyperbolic with either genuinely nonlinear or linearly degenerate characteristic fields. In this case, existence of entropy weak solutions can be obtained when the initial data have small total variation [10, 24, 26, 33]. Such weak solutions are usually discontinuous and consist of elementary Lax-admissible waves. However, many nonlinear hyperbolic systems used by physicists and engineers to model physical phenomena do not satisfy the above hypotheses entirely. For instance, modelling a two-phase flow of a mixture of steam and water in a cooling process of conventional nuclear reactors by water under pressure is described by a system of equations which do not satisfy the basic assumptions of the theory of nonlinear hyperbolic systems [35]. In contrast to strictly

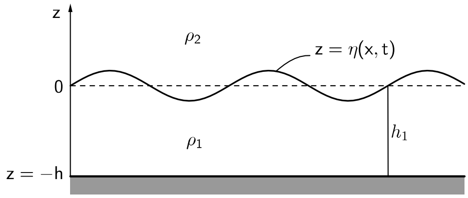

Figure 1: Long waves at the interface in a two-layer system of fluids

hyperbolic conservation laws, nonstrictly hyperbolic systems have no definitive theory which governs existence of weak solutions and the interest in studying such systems is partially motivated by their significant applications in gas dynamics, magnetohydrodynamics, high pressure cavitating liquid jets and oil reservoir simulation [31, 32, 21].

An interesting feature of (1.1), (1.2) is that the characteristic speeds coincide in phase space. Consequently, the classical theory of strictly hyperbolic conservation laws does not apply. Nevertheless, in this work we show that it is possible to construct a unique solution to the Riemann initial value problem associated with (1.1), (1.2). We also show that the solution contains a singular δ-shock whose strength is an exact measure of the corresponding Rankine-Hugoniot deficit. In particular, the exact construction of the solution coincides with the solution provided by the weak asymptotic method [17, 18] which in a sense validates the weak asymptotic method. An extension of the weak asymptotic method to complex-valued approximation has been shown to be essential for the correct physical interpretation of delta shock waves [20].

In context of δ-shock solutions, our results are similar in spirit to the work of Hayes and LeFloch [11]. In their paper, they established a Riemann solution to the system

$$\begin{aligned} u_t + (u^2/2)_x &= 0, \\ v_t + \big((u-1)v\big)_x &= 0, \end{aligned}$$

by introducing an integrated variable $V(x,t)$ defined by

$$V(x,t) = \int_{-C}^{x} v(\tau,t)d\tau \quad \Longrightarrow \quad \frac{\partial V}{\partial x} = v,$$

where $C > 0$ is an arbitrary constant. This variable transforms the system into an equivalent homogeneous system but the initial data for V are now piecewise linear and continuous. Our case requires rigorous computations and care to arrive at the desired result due to the structure of our system. In fact, our construction is more difficult because of an inhomogeneous term appearing in the integrated equation.

Similar systems of equations have also been studied in [23], [37], [5] and [3], where various regularizations were explored. In particular, in [37] parabolic regularizations were used, and uniqueness was obtained, while the weak asymptotic method was used in [5]. The general initial-value problem was considered in [14] and [9]. As mentioned above, the system considered here differs from the equations in all the above works since the integrated equations are inhomogeneous. Recently, there has also been some interest in interactions of delta shock waves, for example in [2, 3, 36, 38], and in some cases uniqueness can also be proved [27, 28].

The plan of the paper is as follows: In Section 2, we solve the Riemann problem for (2.1), (2.2) and show that a unique solution can be constructed in all cases. The construction corresponds exactly to the weak asymptotic approach in a formalized way. In Section 3, we present a δ-shock solution as a combination of a Dirac-δ distribution and a shock wave and verify the solution in the context of the weak asymptotic method and in the weak variational formulation. Finally, in Section 4, we explain how a regularized system can be solved exactly in terms of a travelling wave profile.

2 The Riemann Problem

This section focuses on the nonlinear properties of the system of equations

$$u_t + (u^2/2)_x = 0, \tag{2.1}$$
$$\eta_t + \big((\eta+1)u\big)_x = 0, \tag{2.2}$$

which is obtained from (1.1), (1.2) by an appropriate rescaling. Specifically, we study the shock curves of these equations and their properties by noting first that the system (2.1), (2.2) is of the general form

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0,$$

where

$$\mathbf{U} = \begin{pmatrix} u \\ \eta \end{pmatrix} \quad \text{and} \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} g(u,\eta) \\ h(u,\eta) \end{pmatrix} = \begin{pmatrix} u^2/2 \\ (\eta+1)u \end{pmatrix}.$$

The flux Jacobian of $\mathbf{F}(\mathbf{U})$ is given by

$$J = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{pmatrix} u & 0 \\ \eta+1 & u \end{pmatrix}, \tag{2.3}$$

which has the repeated eigenvalue

$$\lambda_{1,2} = u,$$

with the corresponding right eigenvector

$$r_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For an arbitrary constant left state (u_L, η_L) and a right state (u_R, η_R), the Rankine-Hugoniot conditions for (2.1), (2.2) are respectively

$$-c[u] + [u^2/2] = 0, \tag{2.4}$$
$$-c[\eta] + [(\eta+1)u] = 0, \tag{2.5}$$

where $[u] = u_R - u_L$ and $[\eta] = \eta_R - \eta_L$. The shock speed in (2.4) is well known and has the form

$$c = (u_L + u_R)/2 \equiv \bar{u}, \tag{2.6}$$

and satisfies the Lax entropy condition

$$\lambda_i(u_R) \le c \le \lambda_i(u_L), \quad i = 1, 2.$$

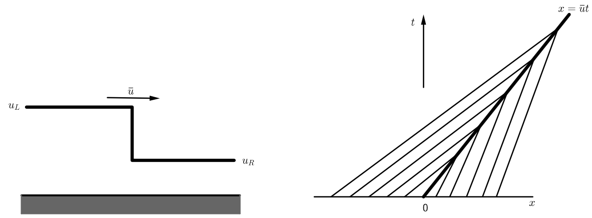

Figure 2: Left panel: Shock solution of Burgers equation. Right panel: corresponding characteristics.

For $[u] \neq 0$, a piecewise continuous function $u(x,t)$ with a single discontinuity travelling with speed $\bar{u}$ and having u_L and u_R on opposite side of the discontinuity is a weak solution of (2.1). Thus, we obtain the following lemma:

Lemma 2.1. *The function $u(x,t)$ defined by*

$$u(x,t) \;\; = \begin{cases} u_L, & \text{if } \; x < \bar{u}t, \\ u_R, & \text{if } \; x > \bar{u}t, \end{cases} \tag{2.7}$$

where $\bar{u}$ is given in (2.6) represents weak solution of (2.1) if

$$\int_0^\infty \int_{-\infty}^\infty \left(\phi_t u + \phi_x \frac{u^2}{2} \right) dxdt + \int_{-\infty}^\infty \phi(x,0)u(x,0)dx = 0,$$

holds for all functions $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)$

Remark 2.1. *Lemma (2.1) represents the case where $u_L > u_R$ and is illustrated in Figure 2. Similarly, if $u_L < u_R$ then the solution of (2.1) is given by a rarefaction wave*

$$u(x,t) = \begin{cases} u_L, & \text{if} \quad x \leq u_L t, \\ x/t, & \text{if} \;\; u_L t < x < u_R t, \\ u_R, & \text{if} \quad x \geq u_R t, \end{cases} \tag{2.8}$$

which is illustrated in Figure 3.

Note also that weak solutions of (2.1),(2.2) satisfy (2.5), so that with the help of (2.6) we can obtain the condition

$$\eta_R = -(\eta_L + 2). \tag{2.9}$$

However, this last relation does not hold for singular solutions. For constant states u_L, u_R, η_L and η_R, let the initial data for (2.1), (2.2) be given by

$$u(\xi,0) = \begin{cases} u_L, & \text{if } \; \xi < 0, \\ u_R, & \text{if } \; \xi > 0, \end{cases} \qquad \eta(\xi,0) = \begin{cases} \eta_L, & \text{if } \; \xi < 0, \\ \eta_R, & \text{if } \; \xi > 0, \end{cases} \tag{2.10}$$

respectively. The main objective is to solve the Riemann problem for (2.1), (2.2) subject to the initial data (2.10).

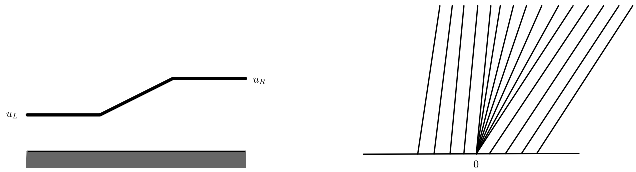

Figure 3: Left panel: Rarefaction solution of Burgers equation. Right panel: corresponding characteristics.

Theorem 2.2. *Let the constant states u_L, u_R, η_L and η_R be given such the* (2.10) *represents Riemann initial data for the system* (2.1), (2.2).

(a) If $u_L > u_R$, then u has a single shock given in (2.7), *whereas η has a single jump*

$$\eta(x,t) = \begin{cases} \eta_L, & \text{if} \quad x < \bar{u}t, \\ \eta_R, & \text{if} \quad x > \bar{u}t, \end{cases}$$

together with a propagating Dirac mass whose strength is given by

$$[w] = (t/2)\big((u_L - u_R)(\eta_L + \eta_R + 2)\big). \tag{2.11}$$

(b) If $u_L < u_R$, then the weak solution of u is a rarefaction given by (2.8) *whereas η has two jump discontinuities given by*

$$\eta(x,t) = \begin{cases} \eta_L, & \text{if} \quad x < u_L t, \\ -1, & \text{if} \quad u_L y \le x \le u_R t, \\ \eta_R, & \text{if} \quad u_R t < x. \end{cases} \tag{2.12}$$

Proof. To prove this theorem we define a function $w(x,t)$ by

$$w(x,t) = \int_{-\kappa}^{x} \eta(s,t)ds, \tag{2.13}$$

for an arbitrary positive constant κ. It is not hard to check that $w(x,t)$ is related to $\eta(x,t)$ by $w_x = \eta$. With this relation the system (2.1), (2.2) transforms into

$$u_t + uu_x = 0, \tag{2.14}$$

$$w_t + uw_x = -u. \tag{2.15}$$

The first equation (2.14) in this system is the well known Burgers' equation which together with the Riemann initial data in u given in (2.10) admits shock and rarefaction solutions which are specified in (2.7) and (2.8). The second equation (2.15) is a nonhomogeneous transport equation in w and has the Riemann initial data

$$w(\xi,0) \equiv w_0(\xi) = \begin{cases} \eta_L\kappa + \eta_L\xi, & \text{if} \quad \xi \le 0, \\ \eta_L\kappa + \eta_R\xi, & \text{if} \quad \xi \ge 0. \end{cases} \tag{2.16}$$

It is obvious that this initial data is linear and piecewise continuous. However, the nonhomogeneous part, as well as the variable coefficient $u(x,t)$ in (2.15), contains a discontinuity which may lead to the development of discontinuities in $w(x,t)$. The progressive evolution of w is along curves $x(t)$ which satisfy the characteristic equation

$$\frac{dx}{dt} = u(x,t), \qquad x(0) = \xi. \tag{2.17}$$

However, $w(x,t)$ is not constant along these curves but satisfies

$$\frac{d}{dt}w(x,t) = -u(x,t), \tag{2.18}$$

and is found by solving the ODEs (2.17) and (2.18) [25]. In the first case where $u_L > u_R$ the solution of (2.14) is given by a shock wave (2.7) travelling at a speed given in (2.6). The technique we used is to substitute this travelling shock wave into (2.15) and solve the resulting equation by the method of characteristics. The characteristics in this case are

$$x(t) = \begin{cases} u_L t + \xi, & \text{if} \quad \xi < (\bar{u} - u_L)t, \\ u_R t + \xi, & \text{if} \quad \xi > (\bar{u} - u_R)t, \end{cases}$$

and they propagate into the shock. Consequently, the solution for $w(x,t)$ is given by

$$w(x,t) = \begin{cases} \eta_L \kappa + \eta_L(x - u_L t) - u_L t, & \text{if} \quad x \le \bar{u}t, \\ \eta_R \kappa + \eta_R(x - u_R t) - u_R t, & \text{if} \quad x > \bar{u}t. \end{cases}$$

The solution $\eta(x,t)$ in (a) is obtained by evoking the expression in (2.13). Since the initial assumption is that $u_L > u_R$, the characteristics emanating from $\xi < 0$ will propagate values of $w(x,t)$ which are different from those propagated by the characteristics originating from $\xi > 0$. Consequently, if the jump in $u(x,t)$ is large, then it will lead to a jump discontinuity in $w(x,t)$ across the shock. The characteristics together with the initial Riemann data (2.16) give

$$\begin{aligned} w_L &= \eta_L \kappa + \big(\eta_l(\bar{u} - u_L) - u_L\big)t, \\ w_R &= \eta_L \kappa + \big(\eta_R(\bar{u} - u_R) - u_R\big)t. \end{aligned}$$

where w_L and w_R represent respectively the left and right limits of $w(x,t)$ at the shock $x = \bar{u}t$. From these expressions, we obtained the jump in $w(x,t)$ whose strength is given by

$$[w] = w_R - w_L = \frac{t}{2}\big(u_L - u_R\big)\big(\eta_L + \eta_R + 2\big). \tag{2.19}$$

In the second case where $u_L < u_R$, the characteristics originating at $\xi < 0$ propagate at a speed of the rarefaction tail whereas those emanating from $\xi > 0$ propagate parallel to the rarefaction head of the wave. The characteristic equations in this case are given by

$$x(t) = \begin{cases} u_R t + \xi, & \text{if} \quad \xi > 0, \\ \gamma t, & \text{if} \quad \xi = 0, \\ u_L t + \xi, & \text{if} \quad \xi < 0, \end{cases}$$

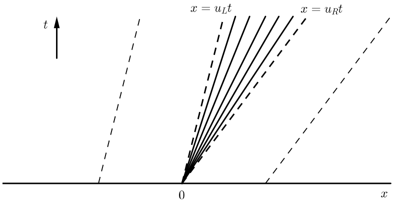

Figure 4: Schematic representation of the characteristics for w (dashed) when $u_L < u_R$. The rarefaction fan in u (solid lines) is bounded by $x = u_L t$ and $x = u_R t$.

where γ is an arbitrary constant only subject to the constraint $u_L < \gamma < u_R$. A graphical representation of the characteristics is shown in Figure 4. By applying (2.16), the solution of (2.15) is

$$w(x,t) = \begin{cases} \eta_L \kappa + \eta_L (x - u_L t) - u_L t, & \text{if} \quad x < u_L t, \\ \eta_L \kappa - x, & \text{if} \quad u_L t \leq x \leq u_R t, \\ \eta_l \kappa + \eta_R (x - u_R t) - u_R t, & \text{if} \quad u_R t < x. \end{cases}$$

A partial derivative with respect to x gives the solution $\eta(x,t)$ of (2.2) defined in (2.12). $\square$

3 Weak asymptotic solution

In this section, we present a delta-shock solution to the system (2.1), (2.2) in the context of weak asymptotics defined in [1, 6, 8]. What is interesting about this method is that it allows the approximate solutions to be complex-valued thereby expanding the range of possible singular solutions as demonstrated in [17, 18, 19]. In [18] it was shown that the method can be used to construct solutions which accommodate combinations of Dirac-δ distributions and shock waves. The goal of this section is to present a δ-shock wave solution in the framework of the weak asymptotics method.

In order to construct a δ-shock solution to (2.1), (2.2), we first review the notion of weak asymptotic and define vanishing family of distributions. Let $\mathcal{D}$ denote the space of smooth functions with compact support and $\mathcal{D}'$ represent the space of distributions as defined in [29].

Definition 3.1. *Let $f_\varepsilon(x,t) \in \mathcal{D}'(\mathbb{R})$ denote a collection of distributions which depend on $\varepsilon \in (0,1)$. If the estimate*

$$\langle f_\varepsilon(x,t), \varphi(x) \rangle = o(1), \quad \textit{as} \quad \varepsilon \to 0, \tag{3.1}$$

holds uniformly for any test function $\varphi(x) \in \mathcal{D}(\mathbb{R})$, then we have $f_\varepsilon = o_{\mathcal{D}'}(1)$.

Definition 3.1 is equivalent to saying that a collection of distributions approaches zero in the context defined above if the pairing $\langle f_\varepsilon(x,t), \varphi(x) \rangle$ converges to zero for a given smooth, compactly supported function φ. For the collection of distributions $f_\varepsilon(x,t)$, we say that $f_\varepsilon = o_{\mathcal{D}'}(1) \subset \mathcal{D}'(\mathbb{R})$ if (3.1) holds uniformly in $t \in \mathbb{R}_+$. Essentially, we require that

$$\langle f_\varepsilon(\cdot,t), \varphi(\cdot)\rangle \le C_T g_\varepsilon(\cdot) \quad \text{for} \quad t \in [0,T],$$

where C_T is a constant which depends on T and $g_\varepsilon(\cdot)$ depends on the $\varphi(x,t)$ and vanishes as $\varepsilon \to 0$.

Definition 3.2. *The family of smooth, complex-valued (real-valued) distributions* (u_ε) *and* (η_ε) *is a weak asymptotic solution to the system* (2.1), (2.2) *if* u, $\eta \in C\big(\mathbb{R}_+; \mathcal{D}'(\mathbb{R})\big)$ *are real-valued distributions such that*

$$u_\varepsilon \rightharpoonup u, \qquad \eta_\varepsilon \rightharpoonup \eta \qquad \text{as} \quad \varepsilon \to 0,$$

holds for any fixed $t \in (0,\infty)$ *in the sense of distributions in* $\mathcal{D}'(\mathbb{R})$ *and*

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{1}{2}\frac{\partial u_\varepsilon^2}{\partial x} = o_{\mathcal{D}'}(1), \tag{3.2}$$

$$\frac{\partial \eta_\varepsilon}{\partial t} + \frac{\partial\big((\eta_\varepsilon + 1)u_\varepsilon\big)}{\partial x} = o_{\mathcal{D}'}(1). \tag{3.3}$$

In the case of complex-valued distributions, it is obvious in this definition that the imaginary part of the solution vanishes in the limit as $\varepsilon \to 0$. Moreover, it is required that convergence to the initial data be satisfied so that

$$u_\varepsilon(\xi,0) \to u(\xi,0) \qquad \text{and} \qquad \eta_\varepsilon(\xi,0) \to \eta(\xi,0),$$

in the sense of distributions as $\varepsilon \to 0$ where $u(\xi,0)$ and $\eta(\xi.0)$ are defined in equation (2.10).

The weak asymptotic method admits a solution comprising of a combination of jump discontinuities and delta distributions in the context of the above definitions and we have the following theorem:

Theorem 3.1. *Let the constant states* u_L, u_R, η_L *and* η_R *be given such that* (2.10) *represents Riemann initial data for the system* (2.1), (2.2) *and* c *is the admissible shock speed given in* (2.6). *Then there exist weak asymptotic solutions* u_ϵ *and* η_ϵ *such that the families* (u_ϵ) *and* (η_ϵ) *have distributional limits given by*

$$u(x,t) = u_L + (u_R - u_L)H(x - ct), \tag{3.4}$$

$$\eta(x,t) = \eta_L + (\eta_R - \eta_L)H(x - ct) + \alpha(t)\delta(x - ct), \tag{3.5}$$

where H *is the Heaviside function,* δ *is the Dirac delta distribution, and*

$$\alpha(t) = [w].$$

Proof. In order to construct the required approximate solution that satisfies (3.4) and (3.5), we define an approximate delta distribution

$$\delta_\varepsilon(x,t) = \frac{1}{2\varepsilon}\rho\left(\frac{x - ct - 3\varepsilon}{\varepsilon}\right) + \frac{1}{2\varepsilon}\rho\left(\frac{x - ct + 3\varepsilon}{\varepsilon}\right).$$

In addition, we define a regularized smooth function

$$H_\varepsilon(x,t) = \begin{cases} 1, & \text{if} \quad x \le ct - 10\varepsilon, \\ \frac{1}{2}, & \text{if} \quad ct - 5\varepsilon < x < ct + 5\varepsilon, \\ 0, & \text{if} \quad x \ge ct + 10\varepsilon, \end{cases}$$

which continuous smoothly in $(-10\varepsilon, -5\varepsilon)$ and $(5\varepsilon, 10\varepsilon)$. Notice in particular that

$$\delta_\varepsilon(x - ct) \rightharpoonup \delta(x - ct) \quad \text{as} \quad \varepsilon \to 0. \tag{3.6}$$

The interaction between the Heaviside function and the delta function gives the weak limit

$$H_\varepsilon(x - ct)\delta_\varepsilon(x - ct) \rightharpoonup \frac{1}{2}\delta(x - ct). \tag{3.7}$$

Furthermore, the following asymptotic expansion holds

$$H_\varepsilon(x,t)\frac{\partial H_\varepsilon(x,t)}{\partial x} = \frac{1}{2}\delta_\varepsilon(x,t) + o_{\mathcal{D}'}(1). \tag{3.8}$$

We start with the singular ansatz:

$$u_\varepsilon(x,t) = u_L + (u_R - u_L)H_\varepsilon(x - ct), \tag{3.9}$$

$$\eta_\varepsilon(x,t) = \eta_L + (\eta_R - \eta_L)H_\varepsilon(x - ct) + \alpha(t)\delta_\varepsilon(x - ct). \tag{3.10}$$

Now, it remains to insert this ansatz into (2.1) and (2.2). Focussing first on (2.2), we get

$$\begin{aligned} &(\eta_R - \eta_L)\partial_t H_\varepsilon + \alpha'(t)\delta_\varepsilon - c\alpha(t)\delta' + (u_R - u_L)\partial_x H_\varepsilon + u_L(\eta_R - \eta_L)\partial_x H_\varepsilon + u_L\alpha(t)\delta' \\ &\quad + \eta_L(u_R - u_L)\partial_x H_\varepsilon + (u_R - u_L)(\eta_R - \eta_L)\partial_x H_\varepsilon^2 + \frac{\alpha(t)}{2}(u_R - u_L)\delta' = o_{\mathcal{D}'}(1). \end{aligned}$$

All terms containing δ' cancel out by equation (3.7). By using (3.6)–(3.8), it follows from Definition 3.2 that

$$\alpha'(t) = \frac{1}{2}\big(u_L - u_R\big)\big(\eta_L + \eta_R + 2\big),$$

which can be interpreted as the measure of the rate of change of the strength of the discontinuity given in (2.11). Indeed, we have $\alpha'(t) = \partial_t[w]$, so that it is clear that the weak asymptotic method yields the same solution as the direct construction used in Section 2. Equation (2.1) is verified similarly and the choice of the shock speed c which is obtained from the Rankine-Hugoniot condition associated with (2.1) gives the desired result readily.

4 Generalized weak solutions

In what follows, we generalize the weak asymptotic solution constructed above by following the solution concept introduced in [7]. This concept is an extension of the traditional framework of weak solutions, such as used for example in Lemma 2.1, which essentially allows the inclusion of singular δ-shocks.

Suppose that $\Gamma = \{\gamma_i \mid i \in I\}$ is a graph in the upper half plane containing Lipschitz continuous arcs γ_i for $i \in I$, where I denotes a finite index set. Let I_0 represent a subset of I containing all arcs that originate at points on the x-axis and assume $\Gamma_0 = \{x_k^0 \mid k \in I_0\}$ is the set of initial points of the arcs γ_k, where $k \in I_0$. We also denote by $\int_{\gamma_i}$ the line integral over the arc γ_i whiles $\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}$ denotes the tangential derivative of a function $\varphi(x,t)$ on the graph γ_i. Define

$$U_0(x) = \begin{cases} u_L, & \text{if} \quad x < 0, \\ u_R, & \text{if} \quad x > 0, \end{cases} \tag{4.1}$$

and

$$G_0(x) = \begin{cases} \eta_L, & \text{if} \quad x < 0, \\ \eta_R, & \text{if} \quad x > 0, \end{cases} \tag{4.2}$$

and define δ-shock initial data by

$$u_0(x) = U_0(x), \tag{4.3}$$

$$\eta_0(x) = G_0(x) + \sum_{k \in I_0} \alpha_k(x_k^0, 0)\delta(x - x_k^0) \tag{4.4}$$

where x_k^0 and $\alpha_k(x_k^0, 0)$ are real constants. We denote the singular part of a solution by

$$\alpha(x,t)\delta(\Gamma) = \sum_{i \in I} \alpha_i(x,t)\delta(\gamma_i).$$

Now we can make the following definition:

Definition 4.1. *The graph Γ together with the pair of distributions $(u(x,t), \eta(x,t))$ defined such that*

$$u(x,t) = U(x,t),$$
$$\eta(x,t) = G(x,t) + \alpha(x,t)\delta(\Gamma),$$

for piecewise continuous functions $U(x,t)$ and $G(x,t)$ is called a generalized δ-shock wave solution of (2.1), (2.2) *with the initial data $(u_0(x), \eta_0(x))$ if the integral identities*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(U\partial_t\varphi + (U^2/2)\partial_x\varphi\right) dxdt + \int_{\mathbb{R}} U_0(x)\varphi(x,0)\ dx = 0, \tag{4.5}$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(G\partial_t\varphi + ((G+1)u)\,\partial_x\varphi\right)\ dxdt + \sum_{i \in I}\int_{\gamma_i} \alpha_i(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}} + \int_{\mathbb{R}} G_0(x)\varphi(x,0)\ dx + \sum_{k \in I_0} \alpha_k(x_k^0, 0)\varphi(x_k^0, 0) = 0, \tag{4.6}$$

hold for all test functions $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$.

The integral identities (4.5) and (4.6) in Definition 4.1 can be interpreted as natural a generalization of the classical weak solution integral identity given in Lemma 2.1. It is not hard to check that the solution (3.4) and (3.5) defined in Theorem 3.1 satisfy Definition 4.1. Consequently, we have the theorem:

Theorem 4.1. *Let the constant states u_L, u_R, η_L and η_R be given such that* (2.10) *represents Riemann initial data for the system* (2.1), (2.2)*. If c is the admissible shock speed given in* (2.6)*, then the functions u and η defined in* (3.4) *and* (3.5) *represent a weak solution to the Riemann problem associated with the system* (2.1), (2.2) *in the framework of Definition 4.1.*

5 Wavefront profile

The goal in this section is to establish a wave front solution for (2.1), (2.2) by introducing viscous terms into the equations describing the flow. That is, we study a non-travelling wave profile of the system

$$u_t + (u^2/2)_x = \nu_1 u_{xx}, \tag{5.1}$$

$$\eta_t + \big((\eta+1)u\big)_x = \nu_2 \eta_{xx}, \tag{5.2}$$

where ν_1 and ν_2 are small viscosity coefficients. We shall show that there exists a non-travelling wave solution that is bounded at $-\infty$ by the left state (u_L, η_L) and at $+\infty$ by the right state (u_R, η_R).

Theorem 5.1. *For a given left state (u_L, η_L) and right state (u_R, η_R), there exists non-travelling wave solutions to* (5.1), (5.2) *for certain values of the viscous ratio $\frac{\nu_1}{\nu_2} = \frac{n}{2}$, for $n \in \mathbb{Z}_+$.*

Proof. We seek a wavefront type solution of the form

$$u(x,t) = \mathcal{U}(x-ct) = \mathcal{U}(s),$$
$$\eta(x,t) = \mathcal{H}(x-ct) = \mathcal{H}(s),$$

where the travelling wave speed c, is given in (2.6). Substituting this ansatz into the system (5.1), (5.2) gives

$$-c\mathcal{U}' + \left(\frac{\mathcal{U}^2}{2}\right)' = \nu_1 \mathcal{U}'' \tag{5.3}$$

$$-c\mathcal{H}' + \big((\mathcal{H}+1)U\big)' = \nu_2 \mathcal{H}'', \tag{5.4}$$

which satisfy the asymptotic conditions

$$\mathcal{U}(-\infty) = u_L, \qquad \mathcal{U}(+\infty) = u_R,$$
$$\mathcal{H}(-\infty) = \eta_L, \qquad \mathcal{H}(+\infty) = \eta_R.$$

Solving (5.3) gives the familiar wave solution

$$\mathcal{U}(s) = c - \frac{u_L - u_R}{2} \tanh\left((u_L - u_R)\frac{s}{4\nu_1}\right).$$

Denoting the shock strength by $\gamma = (u_L - u_R)/2$ simplifies the solution to

$$\mathcal{U}(s) = c - \gamma \tanh\left(\frac{\gamma}{2\nu_1} s\right). \tag{5.5}$$

Next, we integrate (5.4) once with respect to s to obtain a first order ODE

$$\nu_2 \frac{d\mathcal{H}}{ds} = -c\big(\mathcal{H} - \eta_L\big) + \big(\mathcal{H}+1\big)\mathcal{U} - \big(\eta_L + 1\big)u_L.$$

Substituting $\mathcal{U}(s)$ given in (5.5) and simplifying the structure of the equation by the change of variable $s = \nu_2 \xi$ give

$$\frac{d\mathcal{H}}{d\xi} + \gamma \tanh(\beta\xi)\mathcal{H} = -\big(\gamma(\eta_L + 1) + \gamma \tanh(\beta\xi)\big),$$

where $\beta = \gamma\nu_2/2\nu_1$. We solve this equation by using the integrating factor

$$I(\xi) = \exp\left(\int^{\xi} \gamma \tanh(\beta\tau) d\tau\right) = \big(\cosh(\beta\xi)\big)^{\frac{\gamma}{\beta}}, \tag{5.6}$$

to obtain the solution

$$\mathcal{H}(\xi) = -\gamma\big(\cosh(\beta\xi)\big)^{-\frac{\gamma}{\beta}} \left((\eta_L + 1)\int^{\xi} \big(\cosh(\beta\tau)\big)^{\frac{\gamma}{\beta}} d\tau + \int^{\xi} \big(\cosh(\beta\tau)\big)^{\frac{\gamma}{\beta}} \tanh(\beta\tau) d\tau\right).$$

We can obtain unreserved expression for $H(\xi)$ if the exponent $\frac{\gamma}{\beta}$ is an integer n. Thus, an explicit solution can be found if the viscosity relation

$$\frac{\nu_2}{\nu_1} = \frac{2}{n}, \quad \text{for} \quad n \in \mathbb{Z},$$

holds. This leads to the wave profile

$$\mathcal{H}(\xi) = -\gamma(\eta_L + 1)\left(\sum_{k=0}^{n} \binom{n}{k} e^{(2k-n)\beta\xi}\right)^{-1} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{e^{(2k-n)\beta\xi}}{(2k-n)\beta} + 2^{-n}\mu\right) - 1, \tag{5.7}$$

where μ is a constant of integration. It is obvious that the solution converges only if $\gamma \neq 2k\beta$ so we consider two different cases for the solution.

(a) If $\gamma < 2k\beta$, then the estimate

$$e^{(2k\beta-\gamma)\xi} \to 0 \quad \text{as} \quad \xi \to -\infty,$$

holds in (5.7) and the ideal choice of the constant that guarantees bounded solution is $\mu = 0$. This choice simplifies the solution to

$$\mathcal{H}(\xi) = -\gamma(\eta_L + 1)\left(\sum_{k=0}^{n} \binom{n}{k} e^{(2k-n)\beta\xi}\right)^{-1} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{e^{(2k-n)\beta\xi}}{(2k-n)\beta}\right) - 1.$$

We next find the limits of $\mathcal{H}(\xi)$ by noting first that as $\xi \to -\infty$, the term $e^{\gamma\xi}$ dominates and corresponds to the term $k = 0$. Consequently, we have

$$\lim_{\xi \to -\infty} \mathcal{H}(\xi) = -\gamma(\eta_L + 1)\frac{1}{-\gamma} - 1 = \eta_L. \tag{5.8}$$

In like manner as $\xi \to +\infty$, the term $e^{2k\beta\xi} = e^{\gamma\xi}$, for $n = 2k$, dominates and corresponds to the term $k = n$. This gives the estimate

$$\lim_{\xi \to +\infty} \mathcal{H}(\xi) = -\gamma(\eta_L + 1)\frac{1}{\gamma} - 1 = \eta_R, \tag{5.9}$$

where the relation in equation (2.9) is used.

(b) If $\gamma > 2k\beta$, then we have

$$e^{(2k\beta-\gamma)\xi} \to 0 \quad \text{as} \quad \xi \to +\infty.$$

In this case, the appropriate choice of the integration constant which gives a bounded solution is $\mu = 0$, just as in the above case and the left and right limits in (5.8) and (5.9) apply.

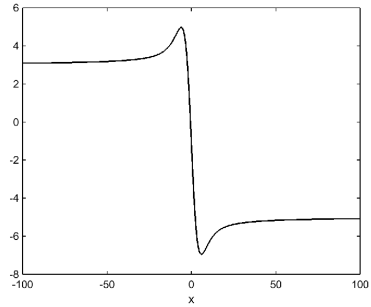

Figure 5: A graphical representation of the solution $\mathcal{H}(\xi)$ for $u_L = 2$, $u_R = 1$, $\eta_L = 3$ and $\eta_R = -5$.

□

Note that the asymptotic boundary conditions of these viscous solutions satisfy the relation (2.9) as though they are approximating a weak solution of the system. However, it appears that as ν_1 and ν_2 approach zero, the solutions do not tend to weak solutions of the system (2.1), (2.2). Indeed, as shown in Figure 5, the viscous system always features a non-monotonic profile. Moreover, as shown in Sections 2 and 3, the system does not admit weak solutions, but necessitates the inclusion of the delta singularity in the shock wave such as in (3.4), (3.5). As the strength of the delta distribution grows linearly in time (cf. (2.19)), this solution cannot be described in terms of a traveling-wave profile.

6 Conclusion

In this paper, a hyperbolic system arising in the study of long waves in two-fluid systems has been studied. This system (1.1), (1.2) is relevant for example in the neutrally buoyant case of a large pool of liquid located at the bottom of a deep ocean. The system is given in terms of an average velocity u in the lower layer, and in terms of the vertical displacement of the interface from the rest position, denoted by η.

As the flux Jacobian (2.3) of the system has repeated eigenvalues, the system is not strictly hyperbolic, and the standard theory of hyperbolic conservation laws cannot be used to find admissible weak solutions. However, thanks to the special structure of the system, it is possible to reformulate the second equation in terms of the primitive w of the unkown η.

Using the fact that the system is triangular, solving the first equation (Burgers' equation) explicitly, and using the transformed equation for the the second unknown, an exact weak solution to the Riemann problem associated to the original system (1.1), (1.2) has been found in Section 2. Since this solution is constructed explicitly by solving a transport equation, the solution is automatically unique. Depending on the disposition of the Riemann data, the unknown w may be given by a shock wave. Since w is the primitive of the original unknown η, the jump in w may be interpreted as a Dirac delta distribution traveling with the shock in η. Thus we have proved that in certain cases, the unique solutions to a hyperbolic but not strictly hyperbolic system is given by a singular solution featuring a Dirac delta distribution traveling with the discontinuity, or in other words, a so-called Delta shock.

In Section 3, we have defined the solution of the original system (1.1), (1.2) in terms of weak asymptotic solutions, such as defined for example in [8]. Using this theory also leads to a Delta shock with a strength $\alpha(t)$, where the derivative $\alpha'(t)$ represents the Rankine-Hugoniot deficit as defined by various authors [23, 22, 17]. The solution also displays the reassuring feature that $\alpha(t)$ matches the amplitude of the Dirac delta distribution obtained in Section 2 exactly.

In Section 4, a more general type of weak solution is defined. This definition is given along the lines of the definition fond in [8, 17], and the solution obtained in Section 3 also satisfies the equation in terms of this more general definition. Finally, Section 5 investigates the problem in the framework of a viscous regularization. Using the resulting nonlinear parabolic system, steady traveling wave profiles may be found in exact form using extensive computations. These solutions feature smoothed spikes, such as may be expected from the Delta shocks found in the previous sections. However, as the viscous parameters tend to zero, the spikes do not grow in size which shows that the solutions found in the previous sections cannot be framed in terms of traveling wave solutions. Indeed these solutions feature growing Delta singularities, and therefore cannot be approximated by smooth steady profiles.

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Paper D

Existence and uniqueness of singular solutions for a conservation law arising in magnetohydrodynamics *

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Existence and Uniqueness of Singular Solutions for a Conservation Law Arising in Magnetohydrodynamics

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Abstract

The Brio system is a two-by-two system of conservation laws arising as a simplified model in ideal magnetohydrodynamics (MHD). The system has the form

$$\partial_t u + \partial_x\big(\tfrac{u^2+v^2}{2}\big) = 0,$$
$$\partial_t v + \partial_x\big(v(u-1)\big) = 0.$$

It was found in previous works that the standard theory of hyperbolic conservation laws does not apply to this system since the characteristic fields are not genuinely nonlinear on the set $v = 0$. As a consequence, certain Riemann problems have no weak solutions in the traditional Lax admissible sense.

It was argued in [8] that in order to solve the system, singular solutions containing Dirac masses along the shock waves might have to be used. Solutions of this type were exhibited in [11, 23], but uniqueness was not obtained.

In the current work, we introduce a nonlinear change of variables which makes it possible to solve the Riemann problem in the framework of the standard theory of conservation laws. In addition, we develop a criterion which leads to an admissibility condition for singular solutions of the original system, and it can be shown that admissible solutions are unique in the framework developed here.

1 Introduction

Conservation laws have been used as a mathematical tool in a variety of situations in order to provide a simplified description of complex physical phenomena which nevertheless keeps the essential features of the processes to be described, and the general theory of hyperbolic conservation laws aims to provide a unified set of techniques needed to understand the mathematical properties of such equations. However, in some cases, the general theory fails to provide a firm mathematical description for a particular case because some of the assumptions needed in the theory are not in place.

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In the present contribution we focus on such an example, a hyperbolic conservation law appearing in ideal magnetohydrodynamics. For this conservation law, solutions cannot be found using the classical techniques of conservation laws, and a new approach is needed.

Magnetohydrodynamics (MHD) is the study of how electric currents in a moving conductive fluid interact with the magnetic field created by the moving fluid itself. The MHD equations are a combination of the Navier-Stokes equations of fluid mechanics and Maxwell's equations of electromagnetism, and the equations are generally coupled in such a way that they must be solved simultaneously. The ideal MHD equations are based on a combination of the Euler equations of fluid mechanics (i.e. for an inviscid and incompressible fluid) and a simplified form of Maxwell's equations. The resulting system is highly complex and one needs to rely on numerical approximation of solutions in order to understand the dynamics of the system.

As even the numerical study of the full system is very challenging, it can be convenient to introduce some simplifying assumptions – valid in some limiting cases – in order to get a better idea of the qualitative properties of the system, and in order to provide some test cases against which numerical codes for the full MHD system can be tested.

The emergence of coherent structures in turbulent plasmas has been long observed both in numerical simulations and experiments. Moreover, the tendency of the magnetic field to organize into low-dimensional structures such as two-dimensional magnetic pancakes and one-dimensional magnetic ropes is well known. As a consequence, in certain cases it makes sense to use simplified one or two dimensional model equations. Such simplified equations will be easier to solve, but nevertheless preserve some of the important features observed in MHD systems. In [1], a simplified model system for ideal MHD was built using such phenomenological considerations. The system is written as

$$\begin{aligned} \partial_t u + \partial_x\big(\tfrac{u^2+v^2}{2}\big) &= 0, \\ \partial_t v + \partial_x\big(v(u-1)\big) &= 0. \end{aligned} \tag{1}$$

The quantities u and v are the velocity components of the fluid whose dynamics is determined by MHD forces, and the system represents the conservation of the velocities. Velocity conservation in this form holds only in idealized situations in the case of smooth solutions, and the limitation of this assumption manifests itself in the non-solvability of the system even for the simplest piece-wise constant initial data, i.e. for certain dispositions of the Riemann initial data

$$u|_{t=0} = \begin{cases} U_L, & x \le 0 \\ U_R, & x > 0 \end{cases}, \qquad v|_{t=0} = \begin{cases} V_L, & x \le 0 \\ V_R, & x > 0 \end{cases}. \tag{2}$$

From a mathematical point of view, the characteristic fields of this system are neither genuinely nonlinear nor linearly degenerate in certain regions in the (u, v)-plane (see [8]). In this case the standard theory of hyperbolic conservation laws which can be found in e.g. [3] does not apply and one cannot find a classical Riemann solution admissible in the sense of Lax [17] or Liu [18].

In order to deal with the problem of non-existence of solutions to the Riemann problem for certain conservation laws, the concept of singular solutions incorporating δ-distributions along shock trajectories was introduced in [16]. The idea was pursued further in [8, 15], and by now, the literature on the subject is rather extensive. Some authors have defined theories of distribution products in order to incorporate the δ-distributions into the notion of weak solutions [4, 10, 23]. In other works, the need to multiply δ-distributions has been avoided either by working with integrated equations [9, 13], or by making an appropriate definition of singular solutions [6]. In order to find admissibility conditions for such singular solutions, some authors have used the weak asymptotic method [5, 6, 21, 22] or simply look for the limit of the vanishing viscosity approximation [15, 24, 25]. With the aim of dealing with the nonlinearity featured by the system (1), the weak asymptotic method was also extended to include complex-valued approximations [11]. The authors of [11] were able to provide singular solutions of (1) even in cases which could not be resolved earlier. However, even if [11] provides some admissibility conditions, the authors of [11] did not succeed to prove uniqueness. Existence of singular solutions to (1) was also proved in [23] using the theory of distribution products, but uniqueness could not be obtained.

Therefore, it was natural to ask whether the Brio system should be solved in the framework of δ-distributions as conjectured in [8] where the system was first considered from the viewpoint of the theory of hyperbolic conservation laws. The authors of [8] compared (1) with the triangular system

$$\begin{aligned} \partial_t u + \partial_x\big(\tfrac{u^2}{2}\big) &= 0, \\ \partial_t v + \partial_x\big(v(u-1)\big) &= 0. \end{aligned} \tag{3}$$

which differs from (1) in the quadratic term v^2. However, the system (3) is linear with respect to v and it naturally admits δ-type solutions (obtained e.g. via the vanishing viscosity approximation). To this end, let us remark that most of the systems admitting δ-shock wave solutions are linear with respect to one of the unknown functions [4, 6, 8, 10, 15]. There are also a number of systems which can be solved only by introducing the δ-solution concept and which are non-linear with respect to both of the variables such as the chromatography system [25] or the Chaplygin gas system [20]. However, in all such systems, it was possible to control the nonlinear operation over an approximation of the δ-distribution. This is not the case with (1) since the term u^2+v^2 will necessarily tend to infinity for any real approximation of the δ-function. This problem can be dealt with by introducing complex-valued approximations of the δ-distribution. Using this approach, a somewhat general theory can be developed as follows. Consider the system

$$\begin{aligned} \partial_t u + \partial_x f(u,v) &= 0, \\ \partial_t v + \partial_x g(u,v) &= 0, \end{aligned} \tag{4}$$

Suppose $\Gamma = \{\gamma_i \mid i \in I\}$ is a graph in the closed upper half plane, containing Lipschitz continuous arcs γ_i, $i \in I$, where I is a finite index set. Let I_0 be the subset of I containing all indices of arcs that connect to the x-axis. Let $\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}$ denote the tangential derivative of a function φ on the graph γ_i, and let $\int_{\gamma_i}$ denote the line integral over the arc γ_i with respect to arclength.

The following definition gives the notion of δ-shock solution to system (4).

Definition 1.1. *The pair of distributions*

$$u = U + \alpha(x,t)\delta(\Gamma) \quad \textit{and} \quad v = V + \beta(x,t)\delta(\Gamma) \tag{5}$$

where $\alpha(x,t)\delta(\Gamma) = \sum_{i\in I}\alpha_i(x,t)\delta(\gamma_i)$ *and* $\beta(x,t)\delta(\Gamma) = \sum_{i\in I}\beta_i(x,t)\delta(\gamma_i)$ *are called a generalized* δ*-shock wave solution of system* (4) *with the initial data* $U_0(x)$ *and* $V_0(x)$ *if the integral identities*

$$\int_{I\!R_+}\int_{I\!R}\left(U\partial_t\varphi + f(U,V)\partial_x\varphi\right)\,dxdt + \sum_{i\in I}\int_{\gamma_i}\alpha_i(x,t)\tfrac{\partial\varphi(x,t)}{\partial \mathbf{l}} + \int_{I\!R}U_0(x)\varphi(x,0)\,dx = 0, \tag{6}$$

$$\int_{I\!R_+}\int_{I\!R}\left(V\partial_t\varphi + g(U,V)\partial_x\varphi\right)\,dxdt + \sum_{i\in I}\int_{\gamma_i}\beta_i(x,t)\tfrac{\partial\varphi(x,t)}{\partial \mathbf{l}} + \int_{I\!R}V_0(x)\varphi(x,0)\,dx = 0, \tag{7}$$

hold for all test functions $\varphi \in \mathcal{D}(I\!R \times I\!R_+)$.

This definition may be interpreted as an extension of the classical notion of weak solutions. The definition is consistent with the concept of measure solutions as put forward in [4, 10] in the sense that the two singular parts of the solution coincide, while the regular parts differ on a set of Lebesgue measure zero. However, Definition 1.1 can be applied to any hyperbolic system of equations while the solution concept from [4] only works in the special situation when the δ-distribution is attached to an unknown which appears linearly in the flux f or g, or when nonlinear operations on δ can somehow be controlled in another way.

Definition 1.1 is quite general, allowing a combination of initial steps and delta distributions; but its effectiveness is already demonstrated by considering the Riemann problem with a single jump. Indeed, for this configuration it can be shown that a δ-shock wave solution exists for any 2×2 system of conservation laws.

Consider the Riemann problem for (4) with initial data $u(x,0) = U_0(x)$ and $v(x,0) = V_0(x)$, where

$$U_0(x) = \begin{cases} u_1, & x<0, \\ u_2, & x>0, \end{cases} \quad V_0(x) = \begin{cases} v_1, & x<0, \\ v_2, & x>0. \end{cases} \tag{8}$$

Then, the following theorem holds:

Theorem 1.2. a) *If* $u_1 \neq u_2$ *then the pair of distributions*

$$u(x,t) = U_0(x-ct), \tag{9}$$

$$v(x,t) = V_0(x-ct) + \beta(t)\delta(x-ct), \tag{10}$$

where

$$c = \frac{[f(U,V)]}{[U]} = \frac{f(u_2,v_2)-f(u_1,v_1)}{u_2-u_1}, \quad \text{and} \quad \beta(t) = (c[V]-[g(U,V)])t, \tag{11}$$

represents the δ*-shock wave solution of* (4) *with initial data* $U_0(x)$ *and* $V_0(x)$ *in the sense of Definition 1.1 with* $\alpha(t) = 0$.

b) *If* $v_1 \neq v_2$ *then the pair of distributions*

$$
\begin{aligned}
u(x,t) &= U_0(x-ct) + \alpha(t)\delta(x-ct), \quad (12)\\
v(x,t) &= V_0(x-ct), \quad (13)
\end{aligned}
$$

where

$$
c = \frac{[g(U,V)]}{[V]} = \frac{g(u_2,v_2) - g(u_1,v_1)}{v_2 - v_1}, \quad \alpha(t) = (c[U] - [f(U,V)])t \quad (14)
$$

represents the δ*-shock solution of* (4) *with initial data* $U_0(x)$ *and* $V_0(x)$ *in the sense of Definition 1.1 with* $\beta(t) = 0$.

Proof. We will prove only the first part of the theorem as the second part can be proved analogously. We immediately see that u and v given by (9) and (10) satisfy (6) since c is given exactly by the Rankine-Hugoniot condition derived from that system. By substituting u and v into (7), we get after standard transformations:

$$
\int_{\mathbb{R}_+} (c[V] - [g(U,V)])\, \varphi(ct,t)\ dt - \int_{\mathbb{R}_+} \beta'(t)\varphi(ct,t)\ dt = 0.
$$

From here and since $\alpha(0) = 0$, the conclusion follows immediately. □

As the solution framework of Definition 1.1 is very weak, one might expect non-uniqueness issues to arise. This is indeed the case, and the proof of the following proposition is an easy exercise.

Proposition 1.3. *System* (4) *with the zero initial data:* $u|_{t=0} = v|_{t=0} = 0$ *admits* δ*-shock solutions of the form:*

$$
u(x,t) = 0, \quad v(x,t) = \beta\delta(x - c_1 t) - \beta\delta(x - c_2 t),
$$

for arbitrary constants β, c_1 *and* c_2.

As already alluded to, a different formal approach for solving (1) was used by [23]. However, just as in [11] the definition of singular solutions used in [23] is so weak that uniqueness cannot be obtained. Another problem left open in [11, 23] is the physical meaning of the δ-distribution appearing as the part of the solution. Considering systems such as the Chaplygin gas system or (3), the use of the δ-distribution in the solution can be justified by invoking extreme concentration effects if we assume that v represents density. However, in the case of the Brio system, u and v are velocities and unbounded velocities cannot be explained in any reasonable physical way.

In the present contribution, we shall try to explain necessity of δ-type solutions for (1) following considerations from [14] where it was argued (in a quite different setting) that the wrong variables are conserved. In other words, the presence of a δ-distribution in a weak solution actually signifies the inadequacy of the corresponding conservation law in the case of weak solutions. Similar consideration were recently put forward in the case of singular solutions in the shallow-water system [12].

Starting from this point, we are able to formulate uniqueness requirement for the Riemann problem for (1). First, we shall rewrite the system using the energy $q = (u^2 + v^2)/2$ as one of the conserved quantities (which is actually an entropy function corresponding to

(1)). Thus, we obtain a strictly hyperbolic and genuinely nonlinear system which admits a Lax admissible solution for any Riemann problem. Such a solution is unique and it will give a unique δ-type solution to the original system. The δ-distribution will necessarily appear due to the nonlinear transformation that we apply.

The paper is organized as follows: In Section 2, we shall rewrite (1) in the new variables q and u, and exhibit the admissible shock and rarefaction waves. In Section 3, we shall introduce the admissibility concept for solutions of the original system (1), and prove existence and uniqueness of a solution to the Riemann problem in the framework of that definition.

2 Energy-velocity conservation

As mentioned above, conservation of velocity is not necessarily a physically well defined balance law, and it might be preferable to specify conservation of energy for example. Actually, in some cases, conservation of velocity does give an appropriate balance law, such as for example in the case of shallow-water flows [7]. In the present situation, it appears natural to replace at least one of the equations of velocity conservation. As will be seen momentarily, such a system will be strictly hyperbolic with genuinely nonlinear characteristic fields, so that the system will be more amenable to standard method of hyperbolic conservation laws. To introduce the new conservation law, we define an energy function

$$q(u,v) = \frac{u^2+v^2}{2}, \tag{15}$$

and note that this function is a mathematical entropy for the system (1). Then we use the transformation

$$(u,v) \to \big(u, \tfrac{u^2+v^2}{2}\big),$$

to transform (1) into the system

$$\begin{aligned} \partial_t u + \partial_x q &= 0, \\ \partial_t q + \partial_x\big((2u-1)q + \tfrac{u^2}{2} - \tfrac{2u^3}{3}\big) &= 0. \end{aligned} \tag{16}$$

System (1) and the transformed system (16) are equivalent for differentiable solutions. However, as will be evident momentarily, the nonlinear transformation changes the character of the system, and while (1) is not always genuinely nonlinear, the new system (16) is always strictly hyperbolic and genuinely nonlinear.

In the following, we analyze (16), and find the elementary waves for the solution of (16). The flux function of the new system is given by

$$F = \begin{pmatrix} q \\ (2u-1)q + \frac{u^2}{2} - \frac{2u^3}{3} \end{pmatrix}$$

with flux Jacobian

$$DF = \begin{pmatrix} 0 & 1 \\ 2q + u - 2u^2 & 2u-1 \end{pmatrix}.$$

The characteristic velocities are given by

$$\lambda_{-,+} = \frac{2u - 1 \mp \sqrt{8q - 4u^2 + 1}}{2}. \tag{17}$$

A direct consequence of (15) gives the relation $2q \geq u^2 \geq 0$ which implies that the quantity under the square root is non-negative. Thus, $8q-4u^2+1 > 0$ and the eigenvalues are real and distinct so that the system is strictly hyperbolic. The right eigenvectors in this case are given by

$$\begin{aligned} r_- &= \begin{pmatrix} 1 \\ u - \frac{1}{2} - \sqrt{2q - u^2 + \frac{1}{4}} \end{pmatrix}, \\ r_+ &= \begin{pmatrix} 1 \\ u - \frac{1}{2} + \sqrt{2q - u^2 + \frac{1}{4}} \end{pmatrix}. \end{aligned} \tag{18}$$

It can be verified easily that these eigenvectors are linearly independent and span the (u, q)-plane. The associated characteristic fields

$$\nabla\lambda_- \cdot r_- = 2 + \frac{1}{\sqrt{8q - 4u^2 + 1}}\ , \tag{19}$$

$$\nabla\lambda_+ \cdot r_+ = 2 - \frac{1}{\sqrt{8q - 4u^2 + 1}}\ , \tag{20}$$

are genuinely nonlinear and admit both shock and rarefaction waves. For a shock profile connecting a constant left state $(u, q) = (u_L, q_L)$ to a constant right state $(u, q) = (u_R, q_R)$, the Rankine-Hugoniot jump conditions for (16) are

$$c(u_L - u_R) = \ (q_L - q_R), \tag{21}$$

$$c(q_L - q_R) = \big((2u_L - 1)q_L + \tfrac{u_L^2}{2} - \tfrac{2u_L^3}{3} - (2u_R - 1)q_R - \tfrac{u_R^2}{2} + \tfrac{2u_R^3}{3}\big), \tag{22}$$

where c is the shock speed. We want the speed in (21), (22) to satisfy the Lax admissibility condition

$$\lambda_\mp(u_L, q_L) \geq c \geq \lambda_\mp(u_R, q_R). \tag{23}$$

To determine the set of all states that can be connected to a fixed left state (u_L, q_l), we eliminate the shock speed, c, from the above equations to obtain the shock curves

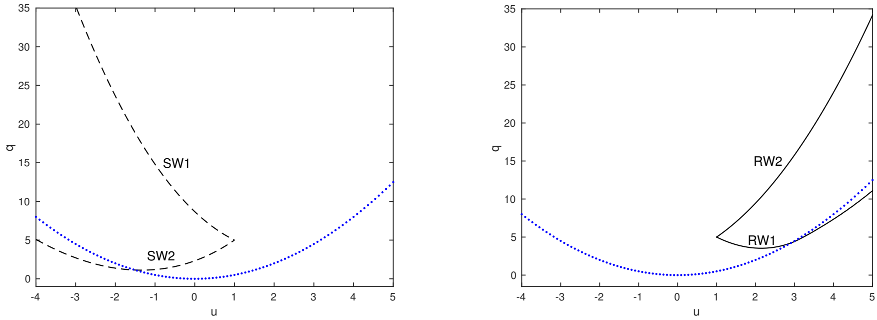
$$(q_R)_{1,2} = \frac{2q_L - (u_L - u_R)(2u_R - 1)}{2} \quad \pm$$

$$\frac{\sqrt{[-2q_L + (u_L - u_R)(2u_R - 1)]^2 + 4\left[(u_L - u_R)\left((2u_L - 1)q_L + \frac{u_L^2}{2} - \frac{u_R^2}{2} - \frac{2u_L^3}{3} + \frac{2u_R^3}{3}\right) - q_L^2\right]}}{2}.$$

After basic algebraic manipulations, we obtain

$$\begin{aligned} (q_R)_{1,2} = q_L - \frac{1}{2}(u_L - u_R)(2u_R - 1) \\ \pm \mid u_L - u_R \mid \sqrt{2q_L + \tfrac{1}{4} + \tfrac{1}{2}(u_L - u_R) - \tfrac{1}{3}\big(2u_L^2 + 2u_L u_R - u_R^2\big)} \end{aligned} \tag{24}$$

From here and (23), by considering (u_R, q_R) in a small neighborhood of (u_L, q_L), we conclude that the shock wave of the first family (SW1), the shock wave of the second family (SW2), the rarefaction wave of the first family (RW1) and the rarefaction wave of the second family (RW2) are given as follows:



(a) (b)

Figure 1: (a) Shock waves of the first and the second families at the left state $(u_L, q_L) = (1, 5)$. (SW1) is indicated by the upper curve, while (SW2) is the lower curve. The blue dotted curve shows the critical curve $q = u^2/2$. (b) Rarefaction waves of the first and the second families at the left state $(u_L, q_L) = (1, 5)$. (RW1) is indicated by the lower curve while (RW2) is the upper curve.

$$\text{(SW1)} \qquad q_R = q_L - \frac{1}{2}\big(u_L - u_R\big)\big(2u_R - 1\big) + \mid u_L - u_R \mid \Big(2q_L + \tfrac{1}{2}(u_L - u_R) - \tfrac{1}{3}\big(2u_L^2 + 2u_L u_R - u_R^2\big) + \tfrac{1}{4}\Big)^{\frac{1}{2}}, \qquad (25)$$

for $u_R < u_L$. To verify that this indeed is the shock wave of the first family, we obtain from (21) and (23) that

$$\lambda_-(u_L, q_L) \geq c = \frac{2u_R - 1 - \sqrt{8q_L + 1 + \frac{4u_R^2}{3} - \frac{8u_L u_R}{3} - \frac{8u_L^2}{3} - 2u_R + 2u_L}}{2}.$$

Taking into account the form of λ_-, we conclude from the above equation that

$$2(u_L - u_R) \geq \sqrt{8q_L + 1 - 4u_L^2} - \sqrt{8q_L + 1 + \frac{4u_R^2}{3} - \frac{8u_L u_R}{3} - \frac{8u_L^2}{3} - 2u_R + 2u_L}.$$

Further simplification leads to

$$2 \geq \frac{-\frac{4}{3}(u_L - u_R) - 2}{\sqrt{8q_L + 1 - 4u_L^2} + \sqrt{8q_L + 1 + \frac{4u_R^2}{3} - \frac{8u_L u_R}{3} - \frac{8u_L^2}{3} - 2u_R + 2u_L}},$$

which is obviously correct. In a similar way, the second part of the Lax condition,

$$\lambda_-(u_R, q_R) \leq c,$$

can be verified. Moreover, it is trivial to verify the additional inequality $\lambda_+(u_R, q_R) \geq c$, so that we have three characteristic curves entering the shock trajectory, and one characteristic curve leaving the shock.

(SW2)
$$q_R = q_L - \frac{1}{2}\big(u_L - u_R\big)\big(2u_R - 1\big) - \mid u_L - u_R \mid \Big(2q_L + \tfrac{1}{2}(u_L - u_R) - \tfrac{1}{3}\big(2u_L^2 + 2u_L u_R - u_R^2\big) + \tfrac{1}{4}\Big)^{\frac{1}{2}}, \quad (26)$$

for $u_R < u_L$. We will skip the proof since it is the same as in the case of (SW1). Next, we have the rarefaction curves.

(RW1), Using the method from [3, Theorem 7.6.5], this wave can be written as

$$\frac{dq}{du} = \frac{2u - 1 - \sqrt{8q - 4u^2 + 1}}{2} = \lambda_-(u, q), \qquad q(u_L) = q_L, \quad (27)$$

for $u_R > u_L$. Clearly, for $u_R < u_L$ we cannot have (RW1) since in that domain, states are connected by (SW1) (see (SW1) above). In order to prove that (27) indeed provides RW1, we need to show that

$$\lambda_-(u_L, q_L) < \lambda_-(u_R, q_R) \ \text{ if } \ u_R > u_L. \quad (28)$$

Introducing the change of variables $\tilde{q} = 8q - 4u^2 + 1$ in (27), we can rewrite it in the form

$$\frac{d\tilde{q}}{du} = -4(1 + \sqrt{\tilde{q}}) < 0.$$

From here, we see that $\tilde{q}$ is decreasing with respect to u and thus, for $u_L < u_R$, we must have

$$8q_L - 4u_L^2 + 1 = \tilde{q}_L > \tilde{q}_R = 8q_R - 4u_R^2 + 1.$$

This, together with $u_L < u_R$ immediately implies (28).

(RW2) Using again [3, Theorem 7.6.5], we have

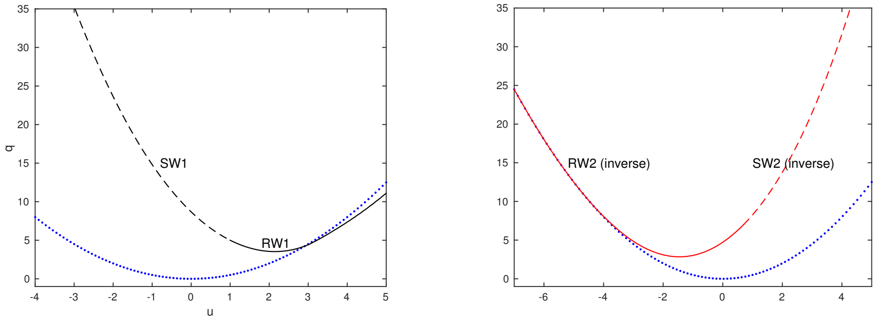
$$\frac{dq}{du} = \frac{2u - 1 + \sqrt{8q - 4u^2 + 1}}{2} = \lambda_+(u, q), \quad q(u_L) = q_L, \quad (29)$$

for $u_R > u_L$. It can be shown that (29) gives the rarefaction wave (RW2) in the same way explained above for (RW1). The wave fan issuing from the left state (u_L, q_L) and the inverse wave fan issuing from the right state (u_R, q_R) are given in Figure 2(a) and Figure 2(b), respectively.

We next aim to prove existence of solution for arbitrary Riemann initial data without necessarily assuming a small enough initial jump. The only essential hypothesis is that both left and right states are above the critical curve $q_{crit} = u^2/2$:

$$q_L \geq u_L^2/2, \qquad q_R \geq u_R^2/2. \quad (30)$$

This assumptions is of course natural given the change of variables $q = \frac{u^2+v^2}{2}$. Nevertheless, this condition complicates our task since it also needs to be shown that the Lax admissible solution to a Riemann problem remains in the area $q \geq u^2/2$. To this end, the following lemma will be useful.



(a) (b)

Figure 2: Shock and rarefaction wave curves of the first and the second families: (a) shows SW1 (dashed) and RW1 (solid) at the left state $(u_L, q_L) = (1, 5)$. (b) shows inverse SW2 (dashed, red) and inverse RW2 (solid, red) at the right state $(u_R, q_R) = (0.7, 7)$.

Lemma 2.1. *The function* $q_{crit}(u) = \frac{u^2}{2}$ *satisfies* (29)*.*

Proof. The proof is obvious and we omit it. □

The above lemma is important since, according to the uniqueness of solutions to the Cauchy problem for ordinary differential equations, it shows that if the left and right states (u_L, q_L) and (u_R, q_R) are above the curve $q_{crit}(u) = \frac{u^2}{2}$, then the simple waves (SW1, SW2, RW1, RW2) connecting the states will remain above it which means that we can use the solution to (16) to obtain a solutions of (1) since the square root giving the function $v = \sqrt{2q - u^2}$ will be well defined. Concerning the Riemann problem, we have the following theorem.

Theorem 2.2. *Given a left state* (u_L, q_L) *and a right state* (u_R, q_R)*, so that both are above the critical curve* $q_{crit}(u) = \frac{u^2}{2}$ *i.e. we have* $q_L \geq u_L^2/2$ *and* $q_R \geq u_R^2/2$*, the states* (u_L, q_L) *and* (u_R, q_R) *can be connected Lax admissible shocks and rarefaction waves via a middle state belonging to the domain* $q > u^2/2$*.*

Proof. In order to find a connection between (u_L, q_L) and (u_R, q_R), we first draw the waves of the first family (SW1 and RW1) through (u_L, q_L) and waves of the second family (SW2 and RW2) through (u_R, q_R). The point of intersection will be the middle state through which we connect (u_L, q_L) and (u_R, q_R) (see Figure 4 for different dispositions of (u_L, q_L) and (u_R, q_R)). In this case, the intersection point will be unique which can be seen by considering the four possible dispositions of the states (u_L, q_L) and (u_R, q_R) shown in Figure 4:

- For right states in region I: RW1 followed by RW2;
- For right states in region II: SW1 followed by RW2;
- For right states in region III: RW1 followed by SW2;

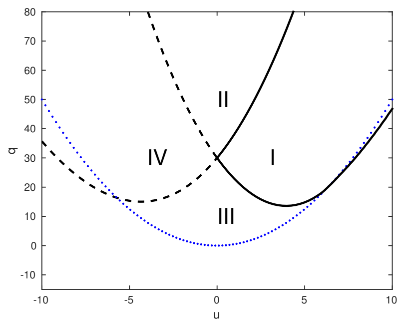


Figure 3: Admissible connections between a given left state (u_L, q_L) and a right state can be classified into four regions in the phase plane.

- For right states in region IV: SW1 followed by SW2;

Properties of the curves of the first and second families are provided in a)-d) above. The growth properties give also existence as we shall show in detail in the sequel of the proof.

Firstly, we remark that SW1 and RW1 emanating from (u_L, q_L) cover the entire $q \geq u^2/2$ domain (see Figure 2(a)). In other words, we have for the curve q_R defining the SW1 by (25):

$$\lim_{u_R \to -\infty} q(u_R) = \infty,$$

implying that the SW1 will take all q-values for $q_R > q_L$. More precisely, for every $q_R > q_L$ there exists $u_R < u_L$ such that $q_R(u_R) = q_R$ where q_R is given by (25).

As for the RW1, it holds for q given by (27) that

$$\frac{dq}{du} - u \leq -1 \implies \frac{dq}{du} \leq u - 1,$$

which means that the RW1 curve emanating from any (u_L, q_L) for which $q_L > u_L^2/2$ will intersect the curve $q_{crit} = \frac{u^2}{2}$ (since $\frac{dq_{crit}}{du} = u > u - 1 \geq \frac{dq}{du}$) at some $u_R > u_L$ as shown in Figure 1, b).

Now, we turn to the waves of the second family. Let us fix the right state (u_R, q_R). We need to compute the inverse waves (i.e. for the given right state, we need to compute curves consisting of appropriate left states (see Figure 2(b)). The inverse rarefaction curve of the second family is given by the equation (29), but we need to take values for $u_R < u_L$ (opposite to the ones given in (29)). As for the inverse SW2, we compute from (21) and (22) the value q_L:

$$q_L = q_R - \frac{1}{2}\big(u_L - u_R\big)\big(2u_L - 1\big) + \frac{(u_L - u_R)}{2}\sqrt{8q_R + 1 + \tfrac{4u_L^2}{3} - \tfrac{8u_L u_R}{3} - \tfrac{8u_R^2}{3} - 2u_L + 2u_R}, \tag{31}$$

for $u_R < u_L$. Clearly, the RW2 cannot intersect the critical line $q_{crit} = \frac{u^2}{2}$ since q_{crit} satisfy (29) (see Lemma 2.1) and the intersection would contradict uniqueness of solution

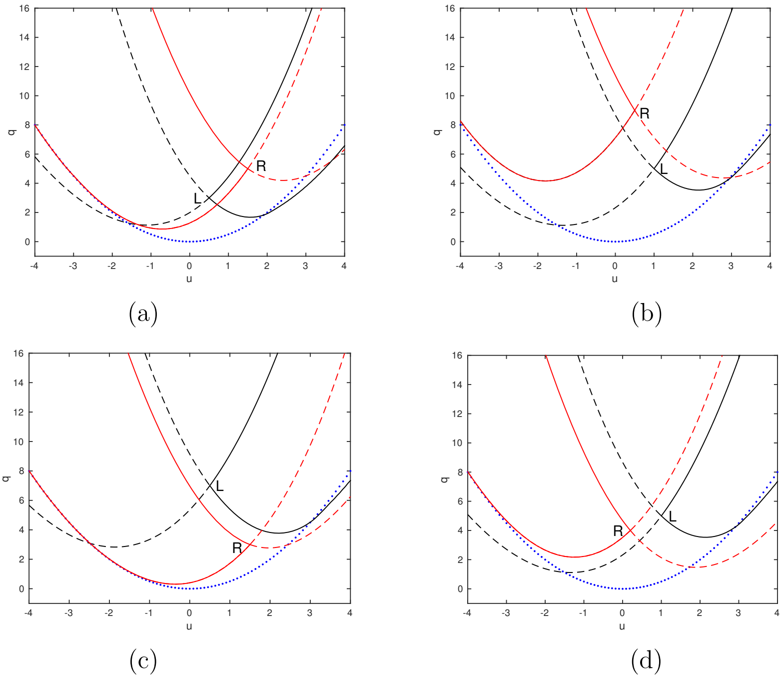


Figure 4: Shock and rarefaction wave curves of the first and the second families. At the left state $L = (u_L, q_L)$, the curves SW1 (dashed), SW2 (dashed), RW1 (solid), and RW2 (solid) are drawn in black. The inverse curves at the right state $R = (u_R, q_R)$ are indicated in red: SW1 (dashed), SW2 (dashed), RW1 (solid) and RW2 (solid). Panel (a) shows the situation for region I, Panel (b) shows the situation for region II, Panel (c) shows the situation for region III and Panel (d) shows the situation for region IV.

to the Cauchy problem for (29). However, a solution to (29) with the initial conditions $q(u_R) = q_R > u_R^2/2$ will converge toward the line $q_{crit} = u^2/2$ since for q given by (29) we have

$$\frac{dq}{du} - u \geq 0 \quad \text{and} \quad \frac{dq}{du}\Big|_{(u,u^2/2)} - u = 0,$$

implying that q will decrease toward $q_{crit} = u^2/2$ and that they will merge as $u_L \to -\infty$ (see Figure 2(b)). As for the inverse SW2 given by (31), we see that

$$\lim_{u_L \to \infty} q(u_L) = \infty,$$

which eventually imply that the 1-wave family emanating from (u_L, q_L) must intersect with the inverse 2-wave family emanating from (u_R, q_R) somewhere in the domain $q > u^2/2$ (see Figure 4 for several dispositions of the left and right states).

Finally, we remark that according to the previous analysis, it follows that the intersection between curves of the first and the second family is unique. □

3 Admissibility conditions for δ-shock wave solution to the original Brio System

Our starting point is that the original Brio system (1) is based on conservation of quantities which are not necessarily physically conserved, and that the transformed system (16) is a closer representation of the physical phenomenon to be described. The second principle is that in the present context, a δ-distribution is a manifestation of a deficiency in the model and therefore it should necessarily be present as a part of non-regular solutions to (1). Moreover, the regular part of a solution to (1) should be an admissible solution to (16). Guided by these requirements, we are able to introduce admissibility conditions for a δ-type solution to (1).

Let us first recall the characteristic speeds for (1). Following [8], we see immediately that

$$\lambda_1(u,v) = u - 1/2 - \sqrt{v^2 + 1/4}, \quad \lambda_2(u,v) = u - 1/2 + \sqrt{v^2 + 1/4}. \tag{32}$$

The shock speed for (1) for the shock determined by the left state (U_L, V_L) and the right state (U_R, V_R) is given by

$$s = \frac{U_L + U_R}{2} - \frac{1}{2} \pm \sqrt{\tfrac{V_L + V_R}{2} + \tfrac{1}{4}}. \tag{33}$$

Now we can formulate admissibility conditions for δ-type solution to (1) in the sense of Definition 1.1. We shall require that the real part of δ-type solution to (1) satisfy the energy-velocity conservation system (16) and that the number of δ-distributions appearing as part of the solution to (1) is minimal.

Definition 3.1. *We say that the pair of distributions* $u = U + \alpha(x,t)\delta(\Gamma)$ *and* $v = V + \beta(x,t)\delta(\Gamma)$ *satisfying Definition 1.1 with* $f(u,v) = \frac{u^2+v^2}{2}$ *and* $g(u,v) = v(u-1)$ *is an admissible* δ*-type solution to* (1), (2) *if*

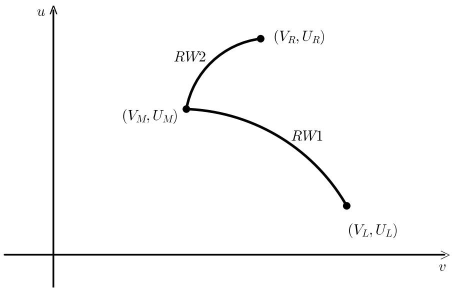


Figure 5: Admissible connection between rarefaction wave curves of the first and second families in the case $V_L, V_R > 0$.

- *The regular parts of the distributions u and v are such that the functions U and $q = (U^2 + V^2)/2$ represent Lax-admissible solutions to* (16) *with the initial data*

$$u|_{t=0} = U_0, \quad q|_{t=0} = q_0 = (U_0^2 + V_0^2)/2. \tag{34}$$

- *For every $t \geq 0$, the support of the δ-distributions appearing in u and v is of minimal cardinality.*

To be more precise, the second requirement in the last definition means that the admissible solution will have "less" δ-distributions as summands in the δ-type solution than any other δ-type solution to (1), (2). We have the following theorem:

Theorem 3.2. *There exists a unique admissible δ-type solution to* (1)*,* (2)*.*

Proof. We divide the proof into two cases:

In the first case, we consider initial data such that both left and right states of the function V_0 have the same sign. In the second case, we consider the initial data where left and right states of the function V_0 have the opposite sign.

In the first case, we first solve (16) with the initial data U_0 and $q_0 = (U_0^2 + V_0^2)/2$. According to Theorem 2.2, there exists a unique Lax admissible solution to the problem denoted by (U, q). Using this solution, we define $V = \sqrt{2q - U^2}$ if the sign of V_0 is positive and $V = -\sqrt{2q - U^2}$ if the sign of V_0 is negative.

To compute α and β in (5), we compute the Rankine-Hugoniot deficit if it exists at all. According to Theorem 2.2 there are four possibilities.

- Region I: The states (U_L, q_L) and (U_R, q_R) are connected by a combination of RW1 and RW2 via the state (U_M, q_M). In this situation, we do not have any Rankine-Hugoniot deficit since the solution (u, q) to (16) is continuous. Thus, we simply write $(u, v) = (u, \sqrt{2q - u^2})$ and this is the solution to (1), (2). The solution is plotted in Figure 5.

  As for the uniqueness, we know that the function u is unique since it is the Lax admissible solution to (16) with the initial data (34). The function v is determined by the unique functions u and q via

$$v = \pm\sqrt{2q - u^2}.$$

Thus, v could change sign so that we connect V_L by V_{M1} and then skip to $-V_{M1}$ on $v = -\sqrt{2q - u^2}$ and then connect it by $-V_{M2}$. From here we connect to V_{M2} located on the original curve $v = \sqrt{2q - u^2}$ and then connect V_{M2} to V_M. Finally, we connect V_M with V_R. The procedure is illustrated in Figure 6. However, since we imposed the requirement that the solutions have a minimal number of δ-distributions and we cannot connect the states (U_{M1}, V_{M1}) and $(U_{M1}, -V_{M1})$ using the δ-shock since such a choice would yield a solutions with a higher number of singular parts than the previously described solution.

Thus the shock connecting the states (U_{M1}, V_{M1}) and $(U_{M1}, -V_{M1})$ cannot be singular, (i.e. there can be no Rankine-Hugoniot deficit). Moreover, these states do not constitute a jump in the first equation of (1), and therefore the speed c of the shock must satisfy the Rankine-Hugoniot condition $-c[v] + [v(u-1)] = 0$ which is equivalent to (22) and results in the shock speed

$$c = U_{M1} - 1.$$

On the other hand, the characteristic speeds of (U_{M1}, V_{M1}) and $(U_{M1}, -V_{M1})$ are $\lambda_1(U_{M1}, V_{M1}) = \lambda_1(U_{M1}, -V_{M1}) \neq c$, and since these are equal, the shock connection between (U_{M1}, V_{M1}) and $(U_{M1}, -V_{M1})$ is impossible with the Rankine-Hugoniot condition satisfied.

Similarly, the same requirement makes it impossible to connect (U_{M2}, V_{M2}) and $(U_{M2}, -V_{M2})$ by a δ-shock. In this case, the shock speed would have to satisfy the Rankine-Hugoniot condition

$$c = U_{M2} - 1.$$

Furthermore, we have equality of speeds $\lambda_2(U_{M2}, V_{M2}) = \lambda_2(U_{M2}, -V_{M2})$, but we have the contrasting inequality $\lambda_2(U_{M2}, V_{M2}) = \lambda_2(U_{M2}, -V_{M2}) \neq c$ implying that a shock connection between (U_{M2}, V_{M2}) and $(U_{M2}, -V_{M2})$ is not possible if the Rankine-Hugoniot condition is satisfied. The same procedure leads to the conclusion that a δ-shock connection between (U_M, V_M) and $(U_M, -V_M)$ is impossible with the Rankine-Hugoniot condition satisfied.

Hence, the only possible connection of (U_L, V_L) and (U_R, V_R) is by the combination RW1 and RW2 via the state (U_M, V_M). Consequently, we remark that RW1 and RW2 corresponding to (16) are transformed via $(u, q) \mapsto (u, \sqrt{2q - u^2})$ into RW1 and RW2 corresponding to (1) (since q is the entropy function for (1), and RW1 and RW2 are smooth solutions to (16)).

- Region II: The states (U_L, q_L) and (U_R, q_R) are connected by the combination SW1 and RW2 via the state (U_M, q_M).

  Unlike the previous case, we have a shock wave in (16), and we will necessarily have a Rankine-Hugoniot deficit in the original system (1). We thus define

$$(u, v) = (u, \sqrt{2q - u^2}) + (0, \beta(t)\delta(x - ct)), \tag{35}$$

  where c is the speed of the SW1 connecting the states (U_L, q_L) and (U_M, q_M) in (16). According to (11), the speed c and the corresponding Rankine-Hugoniot deficit $\beta(t)$:

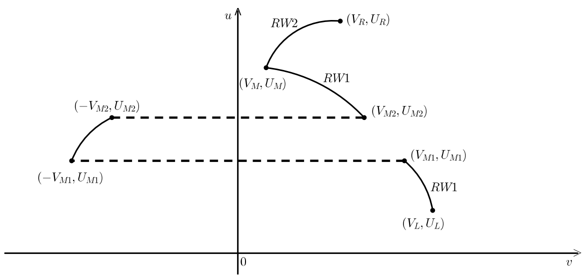


Figure 6: Nonadmissible connection between rarefaction wave curves of the first and the second families

are given in the form

$$c = \frac{\frac{U_L^2+V_L^2}{2} - \frac{U_M^2+V_M^2}{2}}{U_L - U_M}, \quad \beta(t) = (c(V_L - V_M) - (V_L(U_L-1) - V_M(U_M-1))t. \quad (36)$$

Concerning the other possible solutions, as in the previous item, we can only split the curve connecting (U_L, V_L) and (U_M, V_M) into several new curves e.g. by connecting the states (U_L, V_L) and (U_{M1}, V_{M1}), then the (opposite with respect to v) states (U_{M1}, V_{M1}) and $(U_{M1}, -V_{M1})$, then $(U_{M1}, -V_{M1})$ and $(U_{M2}, -V_{M2})$, then $(U_{M2}, -V_{M2})$ and (U_{M2}, V_{M2}) etc. until we reach (U_M, V_M). The states (U_{M1}, V_{M1}) and $(U_{M1}, -V_{M1})$ can be connected only by the shock satisfying the Rankine-Hugoniot conditions corresponding to (1) (due to the minimality condition on δ-shocks, we cannot have a Rankine-Hugoniot deficit).

Since we cannot have a Rankine-Hugoniot deficit, as in the previous item, we must connect the various states with shock waves satisfying the Rankine-Hugoniot conditions (corresponding to (1)), and at the same time being equal to the speed c (the speed of the SW1 connecting the states (U_L, q_L) and (U_M, q_M) in (16)). Indeed, according to the admissibility conditions, the states $(U_L, q_L) = (U_L, \frac{U_L^2+V_L^2}{2})$ and $(U_{M1}, q_{M1}) = (U_{M1}, \frac{U_{M1}^2+V_{M1}^2}{2})$, the states (U_{M1}, q_{M1}) and (U_{M2}, q_{M2}), etc. until the states (U_{Mk}, q_{Mk}) and (U_M, q_M), must be connected by admissible shock waves lying on the same shock curve (since (u, q) must satisfy the Riemann problem for (16) with the left state (U_L, q_L) and the right state (U_R, q_R) and this is done by at most two curves – in our case those are SW1 and RW2). Since all the states lie on the same curve they actually form only one shock which is determined by the end states (U_L, q_L) and (U_M, q_M). Therefore, the shocks connecting the states (U_L, V_L) and (U_{M_1}, V_{M_1}), then $(U_{M1}, -V_{M1})$ and $(U_{M2}, -V_{M2})$ etc. must have the speed c which is obviously never fulfilled i.e. the only solution in this case is (35).

- Region III:

  The states (U_L, q_L) and (U_R, q_R) are connected by the combination RW1 and SW2 via the state (U_M, q_M).

The analysis for the existence and uniqueness proceeds along the same lines as the first two cases. The admissible (and thus unique) δ-type solution in this case has the form:

$$(u, v) = \big(u, \sqrt{2q - u^2}\big) + \big(0, \beta(t)\delta(x - ct)\big), \tag{37}$$

where c in this case represents the speed of the SW2 connecting the states (U_R, q_R) and (U_M, q_M) in (16). The speed c and the corresponding Rankine-Hugoniot deficit $\beta(t)$ are given in (11) and explicitly expressed as in (36).

As in the case of the Regions I and II, notice that it is possible to generate infinitely many non-admissible (in the sense of Definition 3.1) solutions (in the sense of Definition 1.1) by partitioning the rarefaction wave of the first family that connects the states (U_L, V_L) and (U_M, V_M) or the shock wave of the second family connecting the states (U_M, V_M) and (U_R, V_R) as done in the considerations for Region II and Region I, respectively .

Consequently, the only solution admissible in this sense is (37).

- Region IV: The states (U_L, q_L) and (U_R, q_R) are connected by the combination SW1 and SW2 via the state (U_M, q_M).

  The presence of shocks in this case will necessarily introduce a Rankine-Hugoniot deficit in (1). The solution is constructed by solving (16) for the solution (u, q) and then go back to (1) to obtain the admissible δ-type solution

$$(u, v) = \big(u, \sqrt{2q - u^2}\big) + \big(0, \beta_1(t)\delta(x - c_1 t)\big) + \big(0, \beta_2(t)\delta(x - c_2 t)\big), \tag{38}$$

  where c_1 and c_2 given by the expressions

$$c_1 = \frac{\frac{U_L^2 + V_L^2}{2} - \frac{U_M^2 + V_M^2}{2}}{U_L - U_M} \quad \text{and} \quad c_2 = \frac{\frac{U_M^2 + V_M^2}{2} - \frac{U_R^2 + V_R^2}{2}}{U_M - U_R}, \tag{39}$$

  are the speeds of the shocks SW1 and SW2 respectively. The Rankine-Hugoniot deficits $\beta_1(t)$ and $\beta_2(t)$ are expressed as in (36) for the appropriate states. The analysis for uniqueness of (38) is similar to the above cases except that all the elementary waves involved in this case are shocks.

Next we will treat the case when V_L and V_R do not have the same sign. Let us focus on the particular case where $V_L > 0$ and $V_R < 0$. The case where $V_L < 0$ and $V_R > 0$ is then handled analogously.

It was shown in [8] that in this case, the Riemann problem (1), (2) does not admit a Lax admissible solution, even for initial data with small variation.

In order to get an admissible δ-type solution, as before, we solve (16) with (U_0, q_0) as the initial data. The obtained solution connects (U_L, q_L) with (U_R, q_R) by Lax admissible waves through a middle state (U_M, q_M). Next, we go back to the original system (1) by connecting (U_L, V_L) with $(U_M, \sqrt{2q_M - U_M^2})$ by an elementary wave containing the corresponding Rankine-Hugoniot deficit corrected by the δ-shock wave. Then, we connect $(U_M, \sqrt{2q_M - U_M^2})$ with $(U_M, -\sqrt{2q_M - U_M^2})$ by the shock wave whose speed will be $U_M - 1$ as explained above. Finally, we connect $(U_M, -\sqrt{2q_M - U_M^2})$ with (U_R, V_R) by

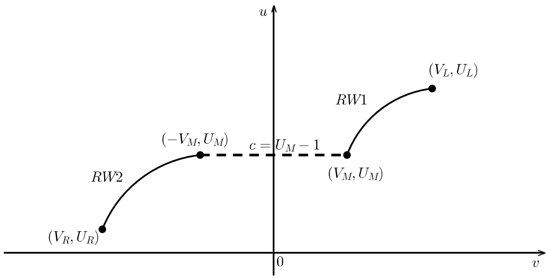


Figure 7: Admissible connection between rarefaction wave curves of the first and second families in the case when the left state has $V_L > 0$ and the right state has $V_R < 0$. In this case, a shock connecting the states $(U_M, -V_M)$ and (U_M, V_M) has to be fitted between the rarefaction curves. It is shown in the part of the proof pertaining to region I that this shock has the required speed.

an elementary wave containing the corresponding Rankine-Hugoniot deficit corrected by the δ-shock wave.

Let us first show it is possible to apply the described procedure. We again need to split considerations into four possibilities depending on how the states (U_L, q_L) and (U_R, q_R) are connected.

- Region I: The states (U_L, q_L) and (U_R, q_R) are connected by RW1 and RW2 via the middle state (U_M, q_M).

  It is clear that we can connect (U_L, V_L) with $(U_M, \sqrt{2q_M - U_M^2})$ using RW1 (it is the same for both equations since RW1 and RW2 are smooth solutions to (16)). Also, we can connect $(U_M, -\sqrt{2q_M - U_M^2})$ with (U_R, V_R) using RW2. We need to prove that the shock wave connecting $(U_M, \sqrt{2q_M - U_M^2})$ and $(U_M, -\sqrt{2q_M - U_M^2})$ has a speed which is between $\lambda_1(U_M, \sqrt{2q_M - U_M^2})$ and $\lambda_2(U_M, -\sqrt{2q_M - U_M^2})$. Note that since there is no jump in u, and there is no jump in v^2, the first equation of (1) is satisfied in the classical sense, and the shock speed is determined by the Rankine-Hugoniot condition associated to the second equation in (1), which yields $c = U_M - 1$.

  Thus we need to check whether we have

  $$U_M - \frac{1}{2} - \sqrt{V_M^2 + \tfrac{1}{4}} \leq U_M - 1 \leq U_M - \frac{1}{2} + \sqrt{V_M^2 + \tfrac{1}{4}}$$

  which is obviously correct. This configuration is depicted in Figure 7. We remark that it is possible to connect the states $(U_M, \sqrt{2q_M - U_M^2})$ and $(U_M, -\sqrt{2q_M - U_M^2})$ using a δ-shock but it would violate the principle of a minimal number of δ-shocks.

- Region IV:

  The states (U_L, q_L) and (U_R, q_R) are connected by SW1 and SW2 via the middle state (U_M, q_M).

  As in the previous item, we connect (U_L, V_L) with $(U_M, \sqrt{2q_M - U_M^2})$ this time using the SW1 from (16) which will induce the Rankine-Hugoniot deficit in (1).

Then, we skip from $(U_M, \sqrt{2q_M - U_M^2})$ to $(U_M, -\sqrt{2q_M - U_M^2})$ using the standard shock wave (the one satisfying the Rankine-Hugoniot conditions), and finally we go from $(U_M, -\sqrt{2q_M - U_M^2})$ to (U_R, V_R) using the SW2 from (16) and corrected with an appropriate δ-shock. More precisely, the admissible δ-type solution will have the form:

$$\begin{aligned} u(x,t) &= U_L + (U_M - U_L)(H(x - c_1t) - H(x - ct)) \\ &\quad + (U_M - U_L)(H(x - ct) - H(x - c_2t)) + (U_R - U_L)H(x - c_2t) \\ v(x,t) &= V_L + (V_M - V_L)(H(x - c_1t) - H(x - ct)) \\ &\quad + (V_M - V_L)(H(x - ct) - H(x - c_2t)) + (V_R - V_L)H(x - c_2t) \\ &\quad + \beta_1(t)\delta(x - c_1t) + \beta_2(t)\delta(x - c_2t), \end{aligned} \tag{40}$$

where c_1 is the speed of the SW1 connecting (U_L, q_L) with (U_M, q_M) in (16), c_2 is the speed of the SW2 connecting (U_M, q_M) with (U_R, q_R) in (16), while c is the speed of the shock connecting $(U_M, -\sqrt{2q_M - U_M^2})$ with $(U_M, \sqrt{2q_M - U_M^2})$ and it is given by the Rankine-Hugoniot conditions from (1). The deficits β_1 and β_2 are given by Theorem 1.2 (see (36) for the analogical situation).

However, we still need to prove that (40) is well defined, i.e. that $c_1 \le c \le c_2$. Using (22), the relation to be shown is

$$\frac{2U_M - 1 - \sqrt{8q_L + 1 + \frac{4U_M^2}{3} - \frac{8U_LU_M}{3} - \frac{8U_L^2}{3} - 2U_M + 2U_L}}{2} \\ \le U_M - 1 \le \frac{2U_M - 1 + \sqrt{8q_R + 1 + \frac{4U_M^2}{3} - \frac{8U_MU_R}{3} - \frac{8U_R^2}{3} - 2U_M + 2U_R}}{2}$$

which is also clearly true.

- Region III: The states (U_L, q_L) and (U_R, q_R) are connected by RW1 and SW2 via the middle state (U_M, q_M).

  This case, as well as the following one, is handled by combining the previous two cases.

- Region II:

  The states (U_L, q_L) and (U_R, q_R) are connected by SW1 and RW2 via the middle state (U_M, q_M).

Uniqueness is obtained by arguing as in the first part of the proof.

□

4 Conclusion

The standard theory of hyperbolic conservation laws is concerned with solutions which are at worst locally integrable. More precisely, if the general system (4) is genuinely nonlinear and strictly hyperbolic, and if the total variation of the initial data is small enough, then the corresponding Riemann problem has a unique solution consisting of rarefaction waves and compressive shock waves (Lax-admissible waves) [2]. On the other hand, if these conditions are not fulfilled, then the Riemann problem may not admit a Lax admissible weak solution or even any weak solution (see e.g. [15, 16, 24, 26]).

In the current contribution, we show that transforming a system using an appropriate entropy which represents a meaningful physical quantity that is actually conserved (in this case the energy) may help in the study of the original system. We focus on the system (1) which arises in the study of magnetohydrodynamics. The system (1) was first studied in [8], where it was conjectured that in order to find solutions, one would have to resort to singular shocks. In [11, 23] solutions were found, but the definition of singular solutions used in these works was extremely weak, and thus uniqueness could not be obtained.

In the present work, we are able to show the existence of a unique solution to the Riemann problem associated with the system (1) through use of Rankine-Hugoniot deficits and the related concept of singular shocks given by Dirac δ-distributions copropagating with shocks. In contrast to a number of previous studies such as [19, 27], where the δ-distribution represented concentration effects, in the current context it provides a measure of the discrepancy between using formally equivalent systems with different unknown variables.

It should be pointed out that the flux function in (1) is nonlinear with respect to both unknowns, whereas so far virtually all known uniqueness results for singular solutions have been obtained in the context of flux functions which are linear with respect to one of the unknowns. For example the uniqueness results obtained in [10, 21] concern pressureless gas dynamics systems which have a flux function linear with respect to one of the unknowns.

Whether a general recipe for existence and uniqueness of singular solutions can be given in a way that works for a general class of systems is unclear at this point. However, one may surmise that the physical background against which the system is derived may play a role in the question of whether or not this can be done, and how one should proceed.

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Addendum

Paper A

There is a minor misprint in **Theorem 2.2** (p.1141) which also appears in the proof on page 1142. The relation for the Rankine-Hugoniot deficit $\alpha'(t) = g\Delta\mathcal{H}$ is missing a minus sign. The correct expression should be $\alpha'(t) = -g\Delta\mathcal{H}$. In addition, equation (2.23) is missing g in the third term. The correct equation should read

$$\partial_t U_\varepsilon + \frac{1}{2}\partial_x U_\varepsilon^2 + g\partial_x H_\varepsilon + \alpha'(t)\delta_\varepsilon - c\alpha(t)\delta' + c\alpha\partial_x S_\varepsilon^2 = o_{\mathcal{D}'}(1).$$

Furthermore, the variable v in **Definition 2.2** is a misprint and should be h so that $u, v \in C(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}))$ becomes $u, h \in C(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}))$. These misprints are minor typos and do not affect the result of the paper.

Paper B

The gravitational constant g is missing in equations (3.2), (3.3) and (3.4). The correct equations should be as follow:

$$\mathbf{F}'(\mathbf{U}) = \begin{pmatrix} \Gamma H + u & H \\ g & u \end{pmatrix}, \tag{3.2}$$

$$\lambda_- = u + \frac{1}{2}\Gamma H - \sqrt{gH + \left(\tfrac{1}{2}\Gamma H\right)^2} \text{ and } \lambda_+ = u + \frac{1}{2}\Gamma H + \sqrt{gH + \left(\tfrac{1}{2}\Gamma H\right)^2}. \tag{3.3}$$

$$r_- = \begin{pmatrix} 1 \\ -\frac{1}{2}\Gamma - \frac{1}{H}\sqrt{gH + \left(\frac{1}{2}\Gamma H\right)^2} \end{pmatrix} \quad \text{and} \quad r_+ = \begin{pmatrix} 1 \\ -\frac{1}{2}\Gamma + \frac{1}{H}\sqrt{gH + \left(\frac{1}{2}\Gamma H\right)^2} \end{pmatrix}. \tag{3.4}$$

Furthermore, there is a minor misprint in (3.5). The correct equation is

$$\begin{aligned} \nabla\lambda_-(H,u)\cdot r_-(H,u) &= -\frac{3gH + \Gamma^2 H^2}{\sqrt{4gH + \Gamma^2 H^2}} < 0 \\ \nabla\lambda_+(H,u)\cdot r_+(H,u) &= +\frac{3gH + \Gamma^2 H^2}{\sqrt{4gH + \Gamma^2 H^2}} > 0. \end{aligned} \tag{3.5}$$

These minor misprints do not affect the results established in the paper.

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