STATISTICAL REPORT

A note on the Wick product[1]

by

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SUMMARY

The present paper is a discussion of some special properties of the Wick product in the White Noise setting. The most important result states that the space of Hida distributions $(\mathcal{S})^*$ is an integral domain under the binary relation defined by the Wick product. This has the consequence that the values of the Wick product can not be determined locally. Furthermore, we give several examples to demonstrate the irregular behavior of the Wick product in many connections, motivated by questions in [GHLØUZ]. Finally, we prove a result on L_1-bounds on Wick powers of white noise. This is to shed some light on the L_1-convergence of Itô-Wiener expansions.

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1 Introduction

The fundamental probability space that we will use is the White Noise probability space, $(\mathcal{S}'(\mathbf{R}^d), \mathcal{B}(\mathcal{S}'(\mathbf{R}^d)), \mu)$, where $\mathcal{S}'(\mathbf{R}^d)$ is the space of tempered distributions, $\mathcal{B}(\mathcal{S}'(\mathbf{R}^d))$ is the σ-algebra on $\mathcal{S}'(\mathbf{R}^d)$ generated by the finite dimensional sylinders and μ is the Gaussian measure determined by its characteristic functional

$$\int_{\mathcal{S}'} e^{i\langle\omega,\xi\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\xi\|^2}, \qquad \forall \;\; \xi \in \mathcal{S}(\mathbf{R}^d) \quad .$$

Under this measure, $\langle\omega,\xi\rangle$ is a Gaussian random variable with expectation zero and variance $\| \xi \|^2$ for fixed $\xi \in \mathcal{S}$ (where $\| \cdot \|$ is the $L^2(\mathbf{R}^d)-$norm). The Wiener-Itô expansion now represents every element in $L^2(\mu)$ as an infinite sum over the complete orthogonal set $\{H_\alpha(\omega), \quad \alpha \quad \text{multiindex with } |\alpha| = \sum \alpha_i < \infty\}$:

$$X(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega).$$

Here $H_\alpha(\omega) = \prod_{i=1}^\infty h_{\alpha_i}(\langle\omega,\phi_i\rangle)$, where $\{\phi_1, \phi_2, \ldots\}$ is any CONS of $L^2(\mathbf{R}^d)$ and $h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2})$ are the Hermite polynomials. The Wick product of X and $Y(\omega) = \sum_\beta d_\beta H_\beta(\omega)$ is then defined as

$$X(\omega) \diamond Y(\omega) = \sum_\gamma \left(\sum_{\alpha+\beta=\gamma} c_\alpha d_\beta \right) H_\gamma(\omega).$$

Since $\| H_\gamma \|^2_{L^2(\mu)} = \gamma!$, for the Wick product to be defined as an $L^2(\mu)-$random variable is is necessary and sufficient that

$$\sum_\gamma \left(\sum_{\alpha+\beta=\gamma} c_\alpha d_\beta \right)^2 \gamma! < \infty.$$

However, the same definition can be used for the Wick product even when $X, Y \in (\mathcal{S})^*$. The sequence representation must then be regarded as a formal series representation with certain growth conditions on the coefficients. See [Z] for this representation of Hida distributions. One important tool of White Noise-analysis is the S-transform, defined for $X \in L^2(\mu)$ as

$$SX(\xi) = \int_{\mathcal{S}'} X(\omega + \xi) d\mu(\omega)$$

for $\xi \in \mathcal{S}(\mathbf{R}^d)$. The S-transform can also be extended to $X \in (\mathcal{S})^*$. For our purpose the most prominent feature of the S-transform is its effect on Wick products:

$$S(X \diamond Y)(\xi) = SX(\xi) \cdot SY(\xi),$$

i.e. the Wick product is translated into an ordinary product of functionals on $\mathcal{S}(\mathbf{R}^d)$ by the S-transform.
For further details on the White Noise probability space and the Wick product, consult [HKPS] and [GHLØUZ] respectively.

2 Main Properties

2.1 $(\mathcal{S})^*$ as an integral domain under $\diamond$

The first thing to observe is that the space $(\mathcal{S})^*$ is closed under the Wick product. This follows from the characterization of $(\mathcal{S})^*$ given in [Z]. It also follows from the characterization in [PS], which we will briefly restate here:

Definition 2.1 *Let F be a complex-valued functional on $\mathcal{S}(\mathbf{R}^d)$. We call F a U-functional if and only if the following two conditions are satisfied:*
C1. For all ξ, $\eta \in \mathcal{S}(\mathbf{R}^d)$, the mapping $\lambda \mapsto F(\eta + \lambda\xi)$, $\lambda \in \mathbf{R}$, has an entire extension.
C2. There exists a $p \in \mathbf{N}_0$ and $C > 0$ so that for all $\xi \in \mathcal{S}(\mathbf{R}^d)$ with $|\xi|_{2,p} \leq 1$ and all sufficiently large $r > 0$,

$$\sup_{z \in \mathbf{C}, |z| \leq r} |F(z\xi)| \leq e^{Cr^2}.$$

Theorem 2.2 *If $\Phi \in (\mathcal{S})^*$ then $S\Phi(\cdot)$ is a U-functional. Conversely, if F is a U-functional, then there is a unique Φ in $(\mathcal{S})^*$ so that $S\Phi = F$.*

See [PS] for a proof.
It can then be checked that the space of U-functionals is closed under usual pointwise multiplication.
That the Wick product is commutative, distributive and associative on $(\mathcal{S})^*$ is immediately clear from the definition. The crucial property is:

Theorem 2.3 If

$$X \diamond Y = 0 \text{ as an element of } (\mathcal{S})^*,$$

then

$$X = 0 \text{ or } Y = 0 \text{ as elements of } (\mathcal{S})^*.$$

Proof.
For any $Z \in (\mathcal{S})^*$ it holds that

$$S(Z)(\xi) = 0, \ \ \forall\, \xi \in \mathcal{S}(\mathbf{R}^d) \quad \Leftrightarrow \quad Z = 0 \text{ in } (\mathcal{S})^* \tag{2.1}$$

Thus

$$\begin{aligned} S(X \diamond Y)(\xi) &= 0, \ \forall \xi \\ \text{and consequently} \quad S(X)(\xi)S(Y)(\xi) &= 0, \ \forall \xi. \end{aligned}$$

Let $A = \{\xi \in \mathcal{S}(\mathbf{R}^d) : S(X)(\xi) = 0\}$. We look at the two cases
1) A is dense in $\mathcal{S}(\mathbf{R}^d)$.
Then, since $S(X)(\cdot)$ is continuous on $\mathcal{S}(\mathbf{R}^d)$, $\mathrm{S}(X)(\xi) = 0$, $\forall \xi$ and the theorem follows.

2) There exists an open set $E \in \mathcal{S}(\mathbf{R}^d)$ such that $S(Y)(\cdot)$ is zero on E.
Choose $\gamma_0 \in E$ and let $\gamma \in \mathcal{S}(\mathbf{R}^d)$ be arbitrary.
Then

$$f(t) \stackrel{\text{def}}{=} S(Y)(\gamma_0 + t(\gamma - \gamma_0))$$

is analytic in t, and there exists $\epsilon > 0$ such that $f(t) = 0$ for $|\, t\,| < \epsilon$. Hence $f(t) = 0, \quad \forall t$, in particular for $t = 1$, i.e.

$$S(Y)(\gamma) = 0, \quad \forall \gamma.$$

□

Theorem 2.3 has a direct bearing on the questions of local properties.

2.2 When is ⋄ local?

Definition 2.4 *Let $\mathcal{M}$ be a subset of (L^2). We will say that $\diamond$ is* local *on $\mathcal{M}$ if for all X_1,X_2,Y $\in\mathcal{M}$ and $A \in \mathcal{B}(\mathcal{S}'(\mathbf{R}^d))$ with $X_1 = X_2$ a.s. on A,*

$$X_1 \diamond Y = X_2 \diamond Y \quad a.s. \ on \ A.$$

The definition of $\diamond$ suggests that $\diamond$ local on $\mathcal{M}$ gives a strong restriction as to what elements $\mathcal{M}$ can contain. This is made precise by the following simple result:

Proposition 2.1 *If $\mathcal{M}$ contains 0 and X, Y (both nonzero) such that $X \cdot Y = 0$ a.s., then $\diamond$ is not local on $\mathcal{M}$.*

Proof. Assume that $\diamond$ is local on $\mathcal{M}$. Let $A = \{\omega : X(\omega) = 0\}$ and $B = \{\omega : Y(\omega) = 0\}$. Then

$$X \diamond Y = \begin{cases} 0 \diamond Y & \text{on} \quad A \\ X \diamond 0 & \text{on} \quad B \end{cases}$$

hence $X \diamond Y = 0$ on $A \cup B$, i.e $X \diamond Y = 0$ a.s. . But according to theorem 2.3 this implies that either $X = 0$ a.s. or $Y = 0$ a.s., which contradicts our assumption. □

Corollary 2.1 *Let $\mathcal{M}$ be a linear subspace of (L^2), let A, B be sets in $\mathcal{B}(\mathcal{S}'(\mathbf{R}^d))$ with $\mu(A)$, $\mu(B) > 0$ and $\mu(A \cup B) = 1$. If $\mathcal{M}$ contains X_1, X_2, Y_1, Y_2 such that $X_1 = X_2$ a.s. on A and $Y_1 = Y_2$ on B, then $\diamond$ is not local on $\mathcal{M}$.*

Proof. We have $X_1 - X_2 \in \mathcal{M}$, $Y_1 - Y_2 \in \mathcal{M}$ and $(X_1 - X_2) \cdot (Y_1 - Y_2) = 0$ a.s. . Since $0 \in\mathcal{M}$, we can use proposition 2.2. □

3 Counterexamples

In [GHLØUZ] several questions, important for further development of the ideas in the paper, were left unanswered. We will here give counterexamples to three different claims. The counterexamples show that the Wick product is not too well behaved compared to the ordinary product. But on the other hand, the importance of the Wick product stems exactly from those situations where *the ordinary product* is not well behaved but the Wick product *is.* This is the case in many stochastic integrals and stochastic differential equations. See e.g. [ØZ], [HLØUZ] and [G]. The numbering of the problems refers to the numbers used in [GHLØUZ].

3.1 The Wick product, problem 1

Apparently it is hard to obtain estimates of $X \diamond Y$ in terms of, for instance, $\|X\|_{L^2}$ and $\|Y\|_{L^2}$. This counterexample shows that it is virtually impossible in general.
Fix $\phi_1 \in \mathcal{S}(\mathbf{R}^d)$ with $\|\phi_1\|_{L^2} = 1$, and let $\Theta_1 =< \omega, \phi_1 >$, so that $\Theta_1 \sim N(0,1)$.
Put

$$X = 1_{\{\Theta_1 \geq 0\}} = \begin{cases} 1 & \text{if } \Theta_1 \geq 0 \\ 0 & \text{if } \Theta_1 < 0 \end{cases} .$$

Let $a_\alpha = E(XH_\alpha)/\alpha!$, which gives $X = \sum_\alpha a_\alpha H_\alpha$.

$$E(XH_\alpha) = \int 1_{\{\Theta_1 \geq 0\}} H_\alpha \, d\mu = 0$$

when $\alpha \neq (n,0,0,...)$ for any n, so consider

$$\begin{aligned} E(Xh_n(\Theta_1)) &= \int_0^\infty h_n(\theta)\, d\lambda(\theta) = \int_0^\infty (\theta h_{n-1} - (n-1)h_{n-2})\, d\lambda(\theta) \\ &= \int_0^\infty h_{n-1}(\theta)\theta \frac{1}{\sqrt{2\pi}} e^{-\theta^2/2}\, d\theta - (n-1)\int_0^\infty h_{n-2}(\theta)\, d\lambda(\theta) \\ &= -\frac{1}{\sqrt{2\pi}} e^{-\theta^2/2} h_{n-1}(\theta) \mid_0^\infty + \int_0^\infty h'_{n-1}(\theta)\, d\lambda(\theta) - (n-1)\int_0^\infty h_{n-2}(\theta)\, d\lambda(\theta) \\ &= \frac{1}{\sqrt{2\pi}} h_{n-1}(0) \qquad ,n \geq 2 \end{aligned}$$

Thus

$$a_n (= a_{(n,0,0,...)}) = \begin{cases} 1/2 & \text{when } n = 0 \\ \frac{1}{\sqrt{2\pi}}(-1)^{(n-1)/2}(n-2)!!/n! & \text{when } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

where $n!! = n(n-2)(n-4)\cdots$, and (-1)!! is taken to be 1. Now, the Wick square becomes:

$$X^{\diamond 2} \overset{\text{def}}{=} X \diamond X = \frac{1}{4} + X + \sum_{l=2,4,6,...} \Big(\sum_{\substack{i+j=l \\ i,j=1,3,5,...}} a_i a_j \Big) h_l(\Theta_1)$$

Since $\frac{1}{4} + X \in L^2$, $X^{\diamond 2} \in L^2$ if and only if $\sum_{l=2,4,6,...}(\sum_{\substack{i+j=l \\ i,j=1,3,5,...}} a_i a_j) h_l(\Theta_1) \in L^2$. But

$$\begin{aligned} \|X^{\diamond 2} - \frac{1}{4} - X\|_{L^2}^2 &= \sum_{l=2,4,6,...} \Big(\sum_{\substack{i+j=l \\ i,j=1,3,5,...}} a_i a_j \Big)^2 l! \\ &= \sum_{l=2,4,6,...} \left(\sum_{\substack{i+j=l \\ i,j=1,3,5,...}} \frac{\frac{1}{\sqrt{2\pi}}(-1)^{(i-1)/2}(i-2)!! \frac{1}{\sqrt{2\pi}}(-1)^{(j-1)/2}(j-2)!!}{i!\, j!} \sqrt{l!} \right)^2 \\ &= \frac{1}{(2\pi)^2} \sum_{l=2,4,6,...} \left(\sum_{\substack{i+j=l \\ i,j=1,3,5,...}} \frac{(i-2)!!(j-2)!!}{i!j!} \sqrt{l!} \right)^2 \\ &= \frac{1}{(2\pi)^2} \sum_{l=2,4,6,...} \left(\sum_{\substack{i+j=l \\ i,j=1,3,5,...}} \frac{\sqrt{l!}}{i(i-1)!!j(j-1)!!} \right)^2 \end{aligned}$$

$$\geq \frac{1}{(2\pi)^2} \sum_{l=2,4,6,\dots} \left(\frac{\sqrt{l!}}{((l/2)(l/2-1)!!)^2} \right)^2 \quad \text{by choosing } i = j = l/2.$$

By Stirling's formula

$$\frac{l!}{((l/2)(l/2-1)!!)^4} \sim \frac{\sqrt{2\pi} l^{l+1/2} e^{-l}}{\{(l/2)K_l(l/2-1)^{(l/2-1)/2+1/2} e^{-(l/2-1)/2}\}^4} = \frac{\sqrt{2\pi} l^{l+1/2} e^{-l}}{l^4/2^4 K_l^4 (l/2-1)^l e^{-(l-2)}}$$

$$\sim \frac{\sqrt{2\pi}\, 2^4}{K_l^4} 2^l l^{-7/2} \to \infty \quad \text{when } l \to \infty$$

where

$$K_l = \begin{cases} \sqrt{\pi} & \text{when} \quad l/2-1 \text{ is even} \\ \sqrt{2} & \text{when} \quad l/2-1 \text{ is odd} \end{cases}$$

This shows that $X^{\diamond 2}$ is not in L^2 , even though X is bounded (hence in L^∞).

3.2 The Wick product, problems 4 and 7

The relation between independence and the Wick product is partially answered by the fact that *strong* independence implies $X \diamond Y = X \cdot Y$ (see [GHLØUZ]). Independence itself is not enough to ensure $X \diamond Y = X \cdot Y$, though the simplest examples do not reveal this. The counterexample below is based on finding independent variables for which it is possible to estimate the second order moment for the Wick product.
Let X be as in 3.1 and write $X = \sum_{n=0}^{\infty} a_n h_n(\Theta_1)$ for the expansion. The r.v. $\Theta_1^{\diamond 2} = \Theta_1^2 - 1$ is independent of X. Now,

$$\| X \cdot \Theta_1^{\diamond 2} \|_{L^2(\mu)}^2 = \| X \|_{L^2(\mu)}^2 \cdot \| \Theta_1^{\diamond 2} \|_{L^2(\mu)}^2 = \frac{1}{2} \cdot 2! = 1$$

whereas

$$\begin{aligned} \| X \diamond \Theta_1^{\diamond 2} \|_{L^2(\mu)}^2 &= \| \textstyle\sum_{n=0}^{\infty} a_n h_{n+2}(\Theta_1) \|_{L^2(\mu)}^2 = \textstyle\sum_{n=0}^{\infty} a_n^2 (n+2)! \\ &\geq 2 \cdot \textstyle\sum_{n=0}^{\infty} a_n^2 n! + \sum_{n=1}^{\infty} a_n^2 n! > 2 \cdot \| X \|_{L^2(\mu)}^2 = 1 \end{aligned}$$

Thus, not even the second moments of $X \cdot \Theta_1^{\diamond 2}$ and $X \diamond \Theta_1^{\diamond 2}$ are equal. In fact, a direct application of Stirling's formula reveals that $a_n^2 n! \sim Cn^{-3/2}$ and hence that $a_n^2 (n+2)! \sim Cn^{1/2}$, so $X \diamond \Theta_1^{\diamond 2}$ is not in $L^2(\mu)$!
This argument can easily be adapted to show that the (vector) distribution of (X, Y) does not determine the distribution of $X \diamond Y$:
With $\phi_2 \in \mathcal{S}(\mathbf{R}^d)$, $\| \phi_2 \| = 1$ and $\langle \phi_1, \phi_2 \rangle = 0$, put $\Theta_2 = \langle \omega, \phi_2 \rangle$. Then $(X, \Theta_2^{\diamond 2})$ has the same joint distribution as $(X, \Theta_1^{\diamond 2})$ because of independence. As before,

$$\| X \diamond \Theta_2^{\diamond 2} \|_{L^2(\mu)}^2 = \| X \|_{L^2(\mu)}^2 \cdot \| \Theta_2^{\diamond 2} \|_{L^2(\mu)}^2 = 1$$

but

$$X \diamond \Theta_1^{\diamond 2} \notin L^2(\mu)$$

so they must indeed have different distributions.

4 An estimate for Wick powers of White Noise

We would like to find an estimate for

$$\|W_\phi^{\diamond n}\|_{L^1(\mu)} = E \mid h_n(W_\phi(\omega)) \mid .$$

This estimate can be useful in e.g. establishing the L^1–convergence of chaos expansions. There already exists an estimate for the L^p–norms in general, [CK], but in the L^1–case the exact asymptotic order of $\|W_\phi^{\diamond n}\|_{L^1(\mu)}$ in n can be found.
In [S], p324 there are the following approximation formulas for the Hermite polynomials:

$$\begin{aligned} H_{2n}(x) &= e^{x^2/2}\sqrt{2^{2n}(2n)!}\ \pi^{-1/4}n^{-1/4} \\ &\times \left[(-1)^n(1-\frac{\epsilon_1}{8n})\cos x\sqrt{4n+1} - \frac{\pi^{1/2}n^{1/4}}{\sqrt{4n+1}}k(2n,x)\right] \\ H_{2n+1}(x) &= e^{x^2/2}\sqrt{2^{2n+1}(2n+1)!}\ \pi^{-1/4}n^{-1/4} \\ &\times \left[(-1)^{n+1}(1-\frac{\epsilon_2}{8n})\sin x\sqrt{4n+3} - \frac{\pi^{1/4}n^{1/4}}{\sqrt{4n+3}}k(2n+1,x)\right] \end{aligned}$$

where $H_n(x) = e^{x^2}\frac{d^n\exp(-x^2)}{dx^n}$, $0<\epsilon_1<1$, $0<\epsilon_2<2$ and $\mid k(n,x)\mid<\mid x\mid^{5/2}$ for every n.
Since h_n and H_n are related through $h_n(x) = 2^{-n/2}(-1)^nH_n(x/\sqrt{2})$, the above estimates imply

$$\int \mid h_{2n}(x)\mid d\lambda(x) = \frac{\sqrt{(2n)!}}{\sqrt{2}}\pi^{-3/4}n^{-1/4}$$

$$\times \int \left|(-1)^n(1-\frac{\epsilon_1}{8n})\cos\left(x\frac{\sqrt{4n+1}}{\sqrt{2}}\right) - \frac{\pi^{1/2}n^{1/4}}{\sqrt{4n+1}}k(2n,\frac{x}{\sqrt{2}})\right| e^{-x^2/4}dx \tag{4.3}$$

$$\int \mid h_{2n+1}(x)\mid d\lambda(x) = \frac{\sqrt{(2n+1)!}}{\sqrt{2}}\pi^{-3/4}n^{-1/4}$$

$$\times \int \left|(-1)^{n+1}(1-\frac{\epsilon_2}{8n})\sin\left(x\frac{\sqrt{4n+3}}{\sqrt{2}}\right) - \frac{\pi^{1/4}n^{1/4}}{\sqrt{4n+3}}k(2n+1,\frac{x}{\sqrt{2}})\right| e^{-x^2/4}dx \tag{4.4}$$

Let us first estimate the value of

$$\begin{aligned} \int \mid\cos(xN)\mid e^{-x^2/4}dx &= \frac{1}{N}\int\mid\cos x\mid e^{-x^2/4N^2}dx \\ = \frac{1}{N}\sum_{k=-\infty}^{\infty}\int_{-\pi/2}^{\pi/2}\cos x e^{-(x+k\pi)^2/4N^2}dx &= \frac{1}{N}\int_{-\pi/2}^{\pi/2}\cos x\left(\sum_{k=-\infty}^{\infty}e^{-(x+k\pi)^2/4N^2}\right)dx \end{aligned}$$

Now, let $f(y) = e^{-(x+y\pi)^2/4N^2}$.
By comparison (when $x\in[-\pi/2,\pi/2]$)

$$\begin{aligned} \int f(y)dy + f(0) \ \geq\ \Sigma_k f(k) &\geq \int f(y)dy - \int_{-1}^{1} f(y)dy + f(0) \\ \frac{2N}{\sqrt{\pi}} + 1 \ \geq\ \Sigma_k f(k) &\geq \frac{2N}{\sqrt{\pi}} - 2 \end{aligned}$$

which gives us the estimate

$$\frac{4}{\sqrt{2}} - \frac{4}{N} \leq \int |\cos(xN)| e^{-x^2/4} dx \leq \frac{4}{\sqrt{2}} + \frac{4}{N} \tag{4.5}$$

similarly

$$\frac{4}{\sqrt{2}} - \frac{4}{N} \leq \int |\sin(xN)| e^{-x^2/4} dx \leq \frac{4}{\sqrt{2}} + \frac{4}{N}. \tag{4.6}$$

Then we note that

$$\int |k(2n, \frac{x}{\sqrt{2}})| e^{-x^2/4} dx \leq \int | \frac{x}{\sqrt{2}} |^{5/2} e^{-x^2/4} dx \leq 8\sqrt{\pi} \tag{4.7}$$

and

$$\int |k(2n+1, \frac{x}{\sqrt{2}})| e^{-x^2/4} dx \leq 8\sqrt{\pi} \tag{4.8}$$

Combining 4.3, 4.5, and 4.7 we obtain

$$\begin{aligned} \int |h_{2n}(x)| \, d\lambda(x) &\leq 2^{-1/4}\pi^{-3/4}\sqrt{(2n)!}\,(2n)^{-1/4} \\ &\times \left\{ \frac{4}{\sqrt{\pi}} + \frac{4\sqrt{2}}{\sqrt{4n+1}} + \frac{\pi^{1/2}n^{1/4}}{\sqrt{4n+1}} 8\sqrt{\pi} \right\} \\ &\leq 2^{7/4}\pi^{-5/4}\sqrt{(2n)!}\,(2n)^{-1/4}\left\{1 + 9(2n)^{-1/4}\right\} \end{aligned}$$

and

$$\begin{aligned} \int |h_{2n}(x)| \, d\lambda(x) &\geq 2^{7/4}\pi^{-5/4}\sqrt{(2n)!}\,(2n)^{-1/4} \\ &\times \left\{ (1 - \frac{1}{8n})(\frac{4}{\sqrt{\pi}} - \frac{4\sqrt{2}}{\sqrt{4n+1}}) - \frac{\pi^{1/2}n^{1/4}}{\sqrt{4n+1}} 8\sqrt{\pi} \right\} \\ &\geq 2^{7/4}\pi^{-5/4}\sqrt{(2n)!}\,(2n)^{-1/4}\left\{1 - 9(2n)^{-1/4}\right\} \end{aligned}$$

by some trivial calculations. Quite similarly we combine 4.4, 4.6, and 4.8 and get completely equivalent results. Combining all results in a single formula, we finally get

$$\begin{aligned} \|\phi\|_{L^2}^n 2^{7/4}\pi^{-5/4}\sqrt{(n)!}(n)^{-1/4}\left\{1 - 9(n)^{-1/4}\right\} &\leq \|W_\phi^{\diamond n}\|_{L^1(\mu)} \\ &\leq \|\phi\|_{L^2}^n 2^{7/4}\pi^{-5/4}\sqrt{(n)!}(n)^{-1/4}\left\{1 + 9(n)^{-1/4}\right\} \end{aligned}$$

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