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A Hybrid Domain Decomposition Method for Advection-Diffusion Problems

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Abstract

Two nonoverlapping domain decomposition algorithms are proposed for convection dominated convection-diffusion problems. In each subdomain, Dirichlet boundary condition is used on the inflow boundary and an artificial boundary condition is used on the outflow boundary. If the flow is simple, each subdomain problem only needs to be solved once. If there are closed streamlines, an iterative algorithm is needed and the convergence is proved. Analysis and numerical tests reveal that the methods are advantageous when the diffusion parameter ϵ is small. In such cases, the error introduced by the domain decomposition methods is neglectable in comparison with the error in the singular layers, and it allows easy and efficient grid refinement in the singular layers.

Key words. advection dominated problem, domain decomposition.

1. Introduction

Domain decomposition methods have been intensively studied for partial differential equations. When they are used for elliptic diffusion problems, they give two benefits. First, domain decomposition methods are iterative methods. In the iteration procedure, they produce good preconditioners. So, in order to reach a certain accuracy, the iteration number can be greatly reduced. Second, domain decomposition methods reduce a large problem into many smaller size problems on the subdomains, and the computation of the subdomain problems can be done by parallel processors. Convection-diffusion problems with a dominating convection term is still an elliptic equation. However, the dominating convection feature has a hyperbolic nature. When domain decomposition are used for convection dominated problems, the flow directions must be carefully considered. Some of the good properties of domain decomposition methods for diffusion problems are not anymore enjoyed by the convection dominated problems.

In literature, results of domain decomposition methods for convection dominated problems are not as rich as for diffusion dominated problems. In Wang and Yan [9], a nonoverlapping domain decomposition method combined with a mixed finite element method was proposed. In Tai, Johansen, Dahle and Espedal [7], a characteristic method is used for the convection term and a domain decomposition method is used for the diffusion problem, and so it enjoys the good properties of both of the methods. In Rannacher and Zhou [5, 12], streamline-diffusion finite element methods (SDFEM) were used with an overlapping domain decomposition. In Rognes and Tai [6], a general space decomposition method is proposed for convection dominated problems. In Kapurkin and Lube [3], a modified Schwarz iteration methods was discussed. Compared with the literature results, the proposed methods of this work are easy to implement and easy to do local refinement.

This work is inspired by [4] and [12]. Two nonoverlapping domain decomposition methods are proposed. When the flow is very simple, the noniterative domain decomposition can be used. The subdomains in the upwind side shall be computed first and the subdomains in downwind direction are computed one after another. For each subdomain, Dirichlet condition is used on the inflow boundary and an artificial boundary condition is used on the outflow boundary. When the flow is complicated, then it is not possible to use this marching process in the flow direction for the subdomains. Instead, an iterative method is proposed in §3. Both methods are suitable for problem (2.1) when the diffusion parameter ϵ is relatively small. In this case, the error introduced by the domain decomposition is small, and one can easily use finer meshes in the subdomains that intersect with singular layers. When the proposed methods are used for time dependent problems, the convergence properties are even better, see §4 for details.

For works related to diffusion dominated convection diffusion problems, we refer to Cai and Widlund [2], Wang [8], Cai and Xu [11], Xu [10], ect.

2. The Noniterative Domain Decomposition Method

Consider the advection diffusion problem:

$$\begin{cases} -div(\epsilon \bigtriangledown u) + div(\vec{\beta} u) + \alpha u = f, & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{2.1}$$

where α, f are bounded functions, $\vec{\beta}$ is a vector-valued function. For simplicity, it is assumed that ϵ is a small constant, all results can be extended to the case that ϵ is a symmetric and positive definite matrix-valued function with small entries ϵ_{ij}.

The standard Galerkin method for (2.1) is to seek $u \in S_0^h$ such that

$$(\epsilon \bigtriangledown u^h, \bigtriangledown v) + (div(\vec{\beta} u^h) + \alpha u^h, v) = (f, v), \quad \forall v \in S_0^h, \tag{2.2}$$

where $S_0^h \in H_0^1(\Omega)$ is the finite element space on Ω. It can be proved that when $\alpha + \frac{1}{2} div\vec{\beta} \geq \gamma > 0$,

$$\|u - u^h\|_0 \leq C \|u - u^I\|_1, \tag{2.3}$$

where $u^I \in S_0^h$ is the interpolation of u, and C is a constant. Here and also later, the constant C always denotes a positive constant that is independent of h, ϵ and u, but could differ from context to context.

For convection dominated problem (2.1), the finite element scheme (2.2) is stable only if the mesh size h is small enough, i.e. $h << \epsilon$. For practically usable h, stabilised finite element methods, like streamline-diffusion, upwinding, or characteristic type of methods should be used. However, to avoid to confine our analysis for a specific stabilised method, we shall present our analysis for the standard finite element method and in practical computations, the solution in each subdomain shall be computed by a stabilised finite element method.

To describe the domain decomposition algorithms, we first divide the domain Ω into some subdomains Ω_i satisfying

$$\bar{\Omega} = \bigcup_i \bar{\Omega}_i, \qquad \Omega_i \bigcap \Omega_j = \emptyset, \ \ i \neq j.$$

Let $S^h(\Omega_i) \subset H^1(\Omega_i)$ be the finite element space on Ω_i, we define

$$V_i = \{v \in S^h(\Omega_i);\ v = 0\ on\ \ \partial\Omega_i \bigcap \partial\Omega\},$$

$$\hat{S}_0^h = \sum_i V_i = \{v \in S^h(\Omega_i),\ \forall i,\ \ v = 0\ on\ \partial\Omega\}.$$

Notice that functions from $\hat{S}_0^h$ can have jumps along the interfaces. Bilinear form $A_i(\cdot,\cdot)$ is defined as:

$$A_i(w,v) = (\epsilon \bigtriangledown w, \bigtriangledown v)_{\Omega_i} + (div(\vec{\beta} w) + \alpha w, v)_{\Omega_i} - \int_{\partial\Omega_i^-} w_+ v_+ \vec{n}\vec{\beta} ds, \tag{2.4}$$

where $\vec{n}$ is the unit outer normal vector on $\partial\Omega_i$ and

$$w_{\pm}(x) = \lim_{s\to 0^{\pm}} w(x + s\vec{\beta}), \qquad (w,v)_{\Omega_i} = \int_{\Omega_i} wv dx,$$

$$\partial\Omega_i^- = \{x \in \partial\Omega_i,\ \ \vec{\beta}(x)\cdot\vec{n}(x) \le 0\}.$$

Our hybrid domain decomposition finite element solution is to find $\hat{u}^h = \sum \hat{u}_i^h$ such that $\hat{u}_i^h = 0$ in $\Omega \setminus \Omega_i$, and in Ω_i, $\hat{u}_i^h \in V_i$ satisfies

$$A_i(\hat{u}_i^h, v) = (f,v)_{\Omega_i} - \int_{\partial\Omega_i^-} (\hat{u}^h)_- v_+ \vec{n}\vec{\beta} ds, \quad \forall v \in V_i, \tag{2.5}$$

where $(\hat{u}^h)_-$ is the boundary value of the solution of the adjacent subdomains in the upwind direction.

In order to solve the subdomain problem (2.5) to get $\hat{u}_i^h$, the inflow boundary condition $\hat{u}^h|_{\partial\Omega_i^-}$ must be known. Therefore, we need to assume that the flow is simple so that the domain Ω can be divided into subdomains and when the subdomain problems are solved one after another in the flow direction, the inflow boundary condition is always known from the neighbouring subdomains. If the flow does not have closed streamlines, this kind of division is always possible. By suitably organising the subdomains, the computation of the subdomains in the crosswind direction can be done by parallel processors. This respect is similar as Zhou [12].

In the domain decomposition scheme (2.5), an artificial boundary condition on the outflow boundary is introduced, and so an error will be produced by the artificial boundary condition. In the next theorem we shall prove that when ϵ is small, the effect from the artificial boundary condition is small. This is also confirmed in our numerical experiments.

Theorem 2.1 Suppose u is the solution of (2.1), $\hat{u}^h$ is the hybrid domain decomposition finite element solution of (2.5), $\alpha + \frac{1}{2} div\vec{\beta} \ge \gamma > 0$ in Ω, and $|\vec{n}\vec{\beta}| \ge \gamma_1 > 0$ on inner boundaries $\partial\Omega_i^- \backslash \partial\Omega$, $\forall i$, then

$$\|u - \hat{u}^h\|_0 \le C(\|u - u^I\|_1 + \epsilon \sum_i \|\frac{\partial u}{\partial n}\|_{0,\partial\Omega_i^- \backslash \partial\Omega}), \tag{2.6}$$

where $u^I \in S_0^h$ is the interpolation of the solution u.

Proof. Define for any $w,\ v \in \hat{S}_0^h$,

$$\begin{aligned} A(w,v) &:= \sum_i A_i(w,v) + \sum_i \int_{\partial\Omega_i^-} w_- v_+ \vec{n}\vec{\beta} ds \\ &= \sum_i (\epsilon \bigtriangledown w, \bigtriangledown v)_{\Omega_i} + \sum_i (div(\vec{\beta} w) + \alpha w, v)_{\Omega_i} - \sum_i \int_{\partial\Omega_i^-} [w] v_+ \vec{n}\vec{\beta} ds, \end{aligned}$$

where

$$[w] = w_+ - w_-.$$

By (2.5),

$$A(\hat{u}^h, v) = (f, v), \qquad \forall v \in \hat{S}_0^h.$$

Integrating by part, it can be proved that for all $w \in \hat{S}_0^h$,

$$(\nabla(\vec{\beta} w), \nabla w)_{\Omega_i} = -(w, \vec{\beta} \cdot \nabla w)_{\Omega_i} + \int_{\partial\Omega_i^-} w_+^2 \vec{n}\vec{\beta} ds + \int_{\partial\Omega_i^+} w_-^2 \vec{n}\vec{\beta} ds.$$

Note that

$$div(\vec{\beta} w) = \vec{\beta} \cdot \nabla w + w div \vec{\beta}.$$

Hence

$$(div(\vec{\beta} w), w)_{\Omega_i} = \frac{1}{2}((div \vec{\beta}) w, w)_{\Omega_i} + \frac{1}{2}\int_{\partial\Omega_i^-} w_+^2 \vec{n}\vec{\beta} ds + \frac{1}{2}\int_{\partial\Omega_i^+} w_-^2 \vec{n}\vec{\beta} ds. \tag{2.7}$$

Then, by using (2.7), and noticing that $w = 0$ on $\partial\Omega_i \bigcap \partial\Omega$, for all $w \in \hat{S}_0^h$, we get

$$\begin{aligned}
A(w, w) &= \sum_i (\epsilon \nabla w, \nabla w)_{\Omega_i} + \sum_i (div(\vec{\beta} w) + \alpha w, w)_{\Omega_i} - \sum_i \int_{\partial\Omega_i^-} [w] w_+ \vec{n}\vec{\beta} ds \\
&= \sum_i (\epsilon \nabla w, \nabla w)_{\Omega_i} + \frac{1}{2}\sum_i (div(\vec{\beta}) w, w)_{\Omega_i} + \frac{1}{2}\sum_i \int_{\partial\Omega_i^-} w_+^2 \vec{n}\vec{\beta} ds \\
&\quad + \sum_i \frac{1}{2}\int_{\partial\Omega_i^+} w_-^2 \vec{n}\vec{\beta} ds + \sum_i (\alpha w, w)_{\Omega_i} - \sum_i \int_{\partial\Omega_i^-} [w] w_+ \vec{n}\vec{\beta} ds \\
&= \sum_i (\epsilon \nabla w, \nabla w)_{\Omega_i} + ((\alpha + \frac{1}{2} div \vec{\beta}) w, w) - \frac{1}{2}\sum_i \int_{\partial\Omega_i^-} w_+^2 \vec{n}\vec{\beta} ds \qquad (2.8) \\
&\quad - \frac{1}{2}\sum_i \int_{\partial\Omega_i^-} w_-^2 \vec{n}\vec{\beta} ds + \sum_i \int_{\partial\Omega_i^-} w_+ w_- \vec{n}\vec{\beta} ds \\
&= \sum_i (\epsilon \nabla w, \nabla w)_{\Omega_i} + ((\alpha + \frac{1}{2} div \vec{\beta}) w, w) \\
&\quad + \frac{1}{2}\sum_i \int_{\partial\Omega_i^-} [w]^2 |\vec{n}\vec{\beta}| ds \geq 0.
\end{aligned}$$

Let $\|w\|_A^2 = A(w, w)$, $u^I \in S_0^h$ be the interpolation of the solution u and $e^h = \hat{u}^h - u^I$.

Using the fact that $[u^I]|_{\partial\Omega_i^-} = 0$, there comes

$$\begin{aligned}
\|e_h\|_A^2 &= A(\hat{u}^h - u^I, e^h) = (f, e^h) - A(u^I, e^h) \\
&= (-div(\epsilon \bigtriangledown u) + div(\vec{\beta} u) + \alpha u, e^h) - A(u^I, e^h) \\
&= -\sum_i \int_{\partial\Omega_i} \epsilon \frac{\partial u}{\partial n} e^h + \sum_i (\epsilon \bigtriangledown (u - u^I), \bigtriangledown e^h)_{\Omega_i} + \sum_i (div(\vec{\beta}(u - u^I)), e^h)_{\Omega_i} \\
&\quad + \sum_i (\alpha(u - u^I), e^h)_{\Omega_i} + \sum_i \int_{\partial\Omega_i^-} [u^I] e_+^h \vec{n}\vec{\beta} ds \\
&= -\sum_i \int_{\partial\Omega_i^- \backslash \partial\Omega} \epsilon \frac{\partial u}{\partial n} [e^h] ds + \sum_i (\epsilon \bigtriangledown (u - u^I), \bigtriangledown e^h)_{\Omega_i} + \sum_i (div(\vec{\beta}(u - u^I)), e^h)_{\Omega_i} \\
&\quad + \sum_i (\alpha(u - u^I), e_h)_{\Omega_i} \\
&\leq C\left[\epsilon \sum_i \|\frac{\partial u}{\partial n}\|_{\partial\Omega_i^- \backslash \partial\Omega} \|\, [e_h]\, \|_{\partial\Omega_i^- \backslash \partial\Omega} + \|u - u^I\|_1 (\epsilon^{\frac{1}{2}} \|e^h\|_1 + \|e^h\|_0)\right].
\end{aligned} \tag{2.9}$$

By (2.8), when $|\vec{n}\vec{\beta}| \geq \gamma_1 > 0$ on $\bigcup_i \partial\Omega_i^- \backslash \partial\Omega$,

$$\sum_i \|\, [e_h]\, \|_{0,\partial\Omega_i^- \backslash \partial\Omega} \leq \gamma_1^{-\frac{1}{2}} \sum_i \|\, [e_h] |\vec{n}\vec{\beta}|^{\frac{1}{2}} \|_{0,\partial\Omega_i^-} \leq 2\gamma_1^{-\frac{1}{2}} \|e_h\|_A,$$

and when $\alpha + \frac{1}{2} div\vec{\beta} \geq \gamma > 0$,

$$\epsilon^{\frac{1}{2}} \|e_h\|_1 + \|e_h\|_0 \leq \gamma^{-\frac{1}{2}} \|e_h\|_A.$$

Hence, combing them with (2.9), one deduces

$$\|e_h\|_A \leq C\left[\epsilon (\sum_i \|\frac{\partial u}{\partial n}\|_{\partial\Omega_i^- \backslash \partial\Omega}) + \|u - u^I\|_1\right],$$

which proves theorem 2.1.

Remark 2.1 Compare (2.6) with (2.3), one sees that the error resulted from the artificial boundary condition is only

$$O(\epsilon)(\sum_i \|\frac{\partial u}{\partial n}\|_{0,\partial\Omega_i \backslash \partial\Omega}).$$

For convection dominated problems, boundary layer and transient layers can appear inside the domain Ω. However, the width of the singular layers are at most of the order $O(\epsilon)$. The singularity of the solution is stronger in the crosswind direction. At a point x inside the singular layers, let n_c be the unit vector in the crosswind direction, then it is known that

$$|\frac{\partial u}{\partial n_c}| \leq O(\epsilon^{-1}).$$

For a subdomain Ω_i, let H_i be its diameter. If the boundary $\partial\Omega_i^-$ is parallel to the streamlines, i.e. $\vec{n}\vec{\beta} = 0$, then

$$\|\frac{\partial u}{\partial n}\|_{0,\partial\Omega_i^-} = \|\frac{\partial u}{\partial n_c}\|_{0,\partial\Omega_i^-} = O(\epsilon^{-1}H_i^{\frac{1}{2}}).$$

In this case, the error produced by the artificial boundary condition is

$$\epsilon\|\frac{\partial u}{\partial n}\|_{0,\partial\Omega_i^-} = O(H_i^{\frac{1}{2}}),$$

which is very large.

The above calculation explains one of the practical implications of the condition $\vec{n}\vec{\beta} \geq \gamma_1 > 0$. In getting the subdomains, we shall avoid the situation that the subdomain boundaries are parallel to the streamlines in the singular layers. Outside the singular layers, there is no problem. Due to the reason that the boundary layers are always narrow, i.e. of width $O(\epsilon)$, we can construct the subdomains in such a way that the part of $\partial\Omega_i^-$ contained in the singular layers is only of the size $O(\epsilon)$. For example, if the streamlines and $\partial\Omega_i^-$ in the singular layers are all straight lines, and $|\vec{n}\vec{\beta}| \geq \gamma_1 > 0$, then there exists a positive constant θ such that the angle between $\partial\Omega_i^-$ and the streamlines is larger than θ. So, the size of the part of $\partial\Omega_i^-$ contained in the singular layers is $O(\frac{\epsilon}{sin\theta}) = O(\epsilon)$. As a consequence,

$$\int_{\partial\Omega_i^-}(\frac{\partial u}{\partial n})^2 ds = \int_l(\frac{\partial u}{\partial n})^2 ds + \int_{\partial\Omega_i^-\setminus l}(\frac{\partial u}{\partial n})^2 ds \leq C(\frac{1}{\epsilon})^2\frac{\epsilon}{sin\theta} + C \leq C\epsilon^{-1}.$$

Above, l denotes the part of $\partial\Omega_i^-$ that is contained in the singular layers. Therefore, in the worst case, the summation of the error from all the subdomains is

$$\epsilon\sum_i\|\frac{\partial u}{\partial n}\|_{\partial\Omega_i^-\setminus\partial\Omega} \leq C\epsilon\sum_i(\epsilon^{-\frac{1}{2}}) = O(\epsilon^{\frac{1}{2}}).$$

Remark 2.2 The streamline diffusion finite element method is stable and shall be used to compute the subdomain solutions preferably. Then, on every subdomain Ω_i, problem (2.5) shall be revised to find $\tilde{u}^h|_{\Omega_i} = \tilde{u}_i^h \in V_i$ such that

$$\tilde{A}_i(\tilde{u}_i^h, v) = (f, v + h\ div(\vec{\beta}v))_{\Omega_i} - \int_{\partial\Omega_i^-}(1 + h\alpha)\tilde{u}_-^h v_+\vec{n}\vec{\beta}ds \quad \forall v \in V_i,$$

with

$$\tilde{A}_i(\tilde{u}_i^h, v) = (\epsilon\nabla\tilde{u}_i^h, \nabla v)_{\Omega_i} + (div(\vec{\beta}\tilde{u}_i^h) + \alpha\tilde{u}_i^h, v + h\ div(\vec{\beta}v))_{\Omega_i} - \int_{\partial\Omega_i^-}(1 + \alpha h)(\tilde{u}_i^h)_+ v_+\vec{n}\vec{\beta}ds.$$

Using similar analyses as used in theorem 2.1, it can be proved that when $\alpha + \frac{1}{2}div\vec{\beta} \geq \gamma > 0$, and h is small enough (such that $\alpha + \frac{1}{2}div\vec{\beta} + \frac{h}{2}(\alpha div\vec{\beta} - \vec{\beta}\cdot\nabla\alpha) \geq \tilde{\gamma} > 0$ and $1 + \alpha h \geq \hat{\gamma} > 0$), then

$$\|\tilde{u}^h - u\|_0 \leq C\left[\epsilon\sum_i\|\frac{\partial u}{\partial n}\|_{0,\Omega_i^-\setminus\partial\Omega} + \epsilon h^{\frac{1}{2}}\sum_i\|u\|_{2,\Omega_i} + (\epsilon^{\frac{1}{2}} + h^{\frac{1}{2}})\|u - u^I\|_1 + h^{-\frac{1}{2}}\|u - u^I\|_0\right],$$

where the error

$$\epsilon h^{\frac{1}{2}}\sum_i\|u\|_{2,\Omega_i} + (\epsilon^{\frac{1}{2}} + h^{\frac{1}{2}})\|u - u^I\|_1 + h^{-\frac{1}{2}}\|u - u^I\|_0$$

is the same as the standard error comes from the SDFEM scheme. So that, the error comes from the artificial boundary condition is still $O(\epsilon)(\sum_i \|\frac{\partial u}{\partial n}\|_{\partial\Omega_i^- \backslash \partial\Omega})$.

3. The Iterative Domain Decomposition Method

When the flow is complicated and there are closed streamlines, it could be difficult to construct the subdomains in such a way that the subdomain solutions can be computed in the flow direction and inflow boundary condition is always available when we come to compute the solution of a new subdomain. In this case, we only need to construct the subdomains to guarantee $\vec{n}\vec{\beta} \geq \gamma_1 > 0$ on $\partial\Omega_i^-$, $\forall i$. Now, the subdomain solutions are all coupled to each other. An iterative scheme is needed. During the iteration, the inflow boundary condition is taken from the previous iterative step, and the algorithm can be written as:

Step 1. Choose initial value $\hat{u}_h^0$;

Step 2. For $n \geq 1$, in every subdomain Ω_i, find $\hat{u}_h^{n+1}|_{\Omega_i} = \hat{u}_i^{n+1} \in V_i$ such that

$$A_i(\hat{u}_i^{n+1}, v) = (f, v) - \int_{\partial\Omega_i^-} (\hat{u}_h^n)_- v_+ \vec{n}\vec{\beta} ds, \quad \forall v \in V_i \; ; \tag{3.1}$$

Step 3. Go to the next iteration.

For the above scheme, we can prove that it is convergent, and the spectral radius for the iteration operator can be estimated as in theorem 3.2.

Theorem 3.1 Let $\hat{u}^h$ be the solution of (2.5), $\hat{u}_h^n$ be the solution of (3.1) and $\alpha + \frac{1}{2} div\vec{\beta} \geq 0$, then the iterative scheme (3.1) is convergent, i.e.

$$\|\hat{u}^h - \hat{u}_h^n\|_0 \to 0 \qquad as\ n \to \infty.$$

Proof. Let $\hat{e}^n = \hat{u}_h^n - \hat{u}^h$. By (3.1) and (2.5), for any $v \in \hat{S}_0^h(\Omega)$,

$$\sum_i (\epsilon \bigtriangledown \hat{e}^n, \bigtriangledown v)_{\Omega_i} + \sum_i (div(\vec{\beta}\hat{e}^n) + \alpha\hat{e}^n, v)_{\Omega_i}$$

$$-\sum_i \int_{\partial\Omega_i^-} \hat{e}_+^n v_+ \vec{n}\vec{\beta} ds + \sum_i \int_{\partial\Omega_i^-} \hat{e}_-^{n-1} v_+ \vec{n}\vec{\beta} ds = 0.$$

Take $v = \hat{e}^n \in \hat{S}_0^h(\Omega)$, then

$$\sum_i (\epsilon \bigtriangledown \hat{e}^n, \bigtriangledown \hat{e}^n)_{\Omega_i} + \sum_i (div(\vec{\beta}\hat{e}^n) + \alpha\hat{e}^n, \hat{e}^n)_{\Omega_i}$$
$$-\sum_i \int_{\partial\Omega_i^-} (\hat{e}_+^n)^2 \vec{n}\vec{\beta} ds + \sum_i \int_{\partial\Omega_i^-} \hat{e}_-^{n-1} \hat{e}_+^n \vec{n}\vec{\beta} ds = 0. \tag{3.2}$$

Let $E^n = E(\hat{e}^n)$, where

$$E(\theta) = \sum_i \int_{\partial\Omega_i^-} (\theta_-)^2 |\vec{n}\vec{\beta}| ds. \tag{3.3}$$

Using the fact that $|\vec{n}\vec{\beta}| = -\vec{n}\vec{\beta}$ on $\partial\Omega_i^-$, equality (3.2) and relation (2.7), it can be shown that

$$\begin{aligned}
E^n &= E^{n-1} - \sum_i \int_{\partial\Omega_i^-} (\hat{e}_-^n)^2 \vec{n}\vec{\beta} ds + \sum_i \int_{\partial\Omega_i^-} (\hat{e}_-^{n-1})^2 \vec{n}\vec{\beta} ds - 2\sum_i (\epsilon \bigtriangledown \hat{e}^n, \nabla \hat{e}^n)_{\Omega_i} \\
&\quad -2\sum_i (div(\vec{\beta}\hat{e}^n) + \alpha\hat{e}^n, \hat{e}^n)_{\Omega_i} + 2\sum_i \int_{\partial\Omega_i^-} (\hat{e}_+^n)^2 \vec{n}\vec{\beta} ds - 2\sum_i \int_{\partial\Omega_i^-} \hat{e}_+^n \hat{e}_-^{n-1} \vec{n}\vec{\beta} ds \\
&= E^{n-1} + \sum_i \int_{\partial\Omega_i^+} (\hat{e}_-^n)^2 \vec{n}\vec{\beta} ds + \sum_i \int_{\partial\Omega_i^-} (\hat{e}_-^{n-1})^2 \vec{n}\vec{\beta} ds - 2\sum_i (\epsilon \bigtriangledown \hat{e}^n, \nabla \hat{e}^n)_{\Omega_i} \\
&\quad -\sum_i ((div\vec{\beta} + 2\alpha)\hat{e}^n, \hat{e}^n)_{\Omega_i} - \sum_i \int_{\partial\Omega_i^-} (\hat{e}_+^n)^2 \vec{n}\vec{\beta} ds - \sum_i \int_{\partial\Omega_i^+} (\hat{e}_-^n)^2 \vec{n}\vec{\beta} ds \\
&\quad +2\sum_i \int_{\partial\Omega_i^-} (\hat{e}_+^n)^2 \vec{n}\vec{\beta} ds - 2\sum_i \int_{\partial\Omega_i^-} \hat{e}_+^n \hat{e}_-^{n-1} \vec{n}\vec{\beta} ds \\
&= E^{n-1} - 2\sum_i (\epsilon \bigtriangledown \hat{e}^n, \nabla \hat{e}^n)_{\Omega_i} - \sum_i ((div\vec{\beta} + 2\alpha)\hat{e}^n, \hat{e}^n)_{\Omega_i} \\
&\quad +\sum_i \int_{\partial\Omega_i^-} (\hat{e}_-^{n-1})^2 \vec{n}\vec{\beta} ds - 2\sum_i \int_{\partial\Omega_i^-} \hat{e}_+^n \hat{e}_-^{n-1} \vec{n}\vec{\beta} ds + \sum_i \int_{\partial\Omega_i^-} (\hat{e}_+^n)^2 \vec{n}\vec{\beta} ds \\
&= E^{n-1} - 2\sum_i (\epsilon \bigtriangledown \hat{e}^n \bigtriangledown \hat{e}^n)_{\Omega_i} - \sum_i ((div\vec{\beta} + 2\alpha)\hat{e}^n, \hat{e}^n)_{\Omega_i} \\
&\quad +\sum_i \int_{\partial\Omega_i^-} (\hat{e}_+^n - \hat{e}_-^{n-1})^2 \vec{n}\vec{\beta} ds \\
&= E^{n-1} - F^n,
\end{aligned} \tag{3.4}$$

where

$$F^n = 2\sum_i (\epsilon \bigtriangledown \hat{e}^n, \hat{e}^n)_{\Omega_i} + \sum_i ((div\vec{\beta} + 2\alpha)\hat{e}^n, \hat{e}^n)_{\Omega_i} + \sum_i \int_{\partial\Omega_i^-} (\hat{e}_+^n - \hat{e}_-^{n-1})^2 |\vec{n}\vec{\beta}| ds \geq 0. \tag{3.5}$$

By (3.4) and (3.5), $\{E^n\}$ is a decreasing sequence of nonnegative numbers and

$$\sum_n F^n < \infty,$$

thus

$$F^n \to 0, \quad as \ \ n \to \infty.$$

By (3.5), if $\alpha + \frac{1}{2} div\vec{\beta} \geq \gamma > 0$,

$$\|\hat{e}^n\|_0 \leq \frac{1}{\gamma} F^n \to 0 \quad as \ \ n \to \infty,$$

else, if only $\alpha + \frac{1}{2} div\vec{\beta} \geq 0$ is satisfied, it still can be proved that

$$\|\hat{e}^n\|_0 \leq C|\hat{e}^n|_1 \leq \frac{C}{\epsilon} F^n \to 0 \quad as \ \ n \to \infty,$$

which proves theorem 3.1.

Let T_0 be the affine mapping from $\hat{S}_0^h$ to itself such that, for any $\theta \in \hat{S}_0^h$, $\psi = T_0(\theta)$ is the solution of the following equation:

$$\begin{aligned} &\sum_i (\epsilon \bigtriangledown \psi, \nabla v)_{\Omega_i} + \sum_i (div(\vec{\beta}\psi) + \alpha\psi, v)_{\Omega_i} - \sum_i \int_{\partial\Omega_i^-} \psi_+ v_+ \vec{n}\vec{\beta} ds \\ = & -\sum_i \int_{\partial\Omega_i^-} \theta_- v_+ \vec{n}\vec{\beta} ds, \qquad \forall v \in \hat{S}_0^h. \end{aligned} \tag{3.6}$$

Then, the spectral radius of this iteration operator T_0 will be discussed in the next theorem.

Theorem 3.2 Let $\rho(T_0)$ is the spectral radius of T_0, which is defined in (3.6), suppose $\alpha + \frac{1}{2} div\vec{\beta} \geq \gamma \geq 0$, then

$$\rho(T_0) \leq \left(\frac{1}{1 + C\epsilon + C\gamma h} \right)^{\frac{1}{2}}.$$

Proof. Let λ be an eigenvalue of T_0, and θ be the corresponding eigenvector, so that

$$T_0(\theta) = \lambda\theta.$$

It follows from (3.3) that

$$E(T_0(\theta)) = |\lambda|^2 E(\theta). \tag{3.7}$$

Also, by (3.4) and (3.5),

$$E(T_0(\theta)) = E(\theta) - F(T_0(\theta), \theta), \tag{3.8}$$

where

$$\begin{aligned} F(T_0(\theta), \theta) &= 2\sum_i (\epsilon \bigtriangledown T_0(\theta), \nabla T_0(\theta))_{\Omega_i} + \sum_i ((div\vec{\beta} + 2\alpha) T_0(\theta), T_0(\theta))_{\Omega_i} \\ & \quad + \sum_i \int_{\partial\Omega_i^-} (T_0(\theta)_+ - \theta_-)^2 |\vec{n}\vec{\beta}| ds \\ &\geq \epsilon \| \bigtriangledown T_0(\theta) \|_0^2 + \gamma \| T_0(\theta) \|_0^2 = |\lambda|^2 (\epsilon \| \bigtriangledown \theta \|_0^2 + \gamma \|\theta\|_0^2) \\ &\geq C|\lambda|^2 (\epsilon + h\gamma) (\sum_i \int_{\partial\Omega_i^-} (\theta_-)^2 |\vec{n}\vec{\beta}| ds) \\ &= C(\epsilon + h\gamma) |\lambda|^2 E(\theta). \end{aligned} \tag{3.9}$$

Hence, by (3.7), (3.8), (3.9),

$$|\lambda|^2 E(\theta) \leq E(\theta)(1 - C(\epsilon + h\gamma)|\lambda|^2),$$

and so,

$$|\lambda|^2 \leq 1 - C(\epsilon + h\gamma)|\lambda|^2.$$

That is

$$|\lambda| \leq \left(\frac{1}{1 + C\epsilon + Ch\gamma} \right)^{\frac{1}{2}},$$

which proves theorem 3.2.

Remark 3.1 When ϵ is not small, different kinds of boundary condition on the outflow boundary should be used to improve the accuracy. For example, Lagrange multiplier can be used on the inner boundaries, see [9] for the details.

4. Application to Time Dependent Problems

Consider the time dependent convection-diffusion problem:

$$\begin{cases} u_t - div(\epsilon \bigtriangledown u) + div(\vec{\beta} u) = f, & in \quad \Omega \times [0, T], \\ u(x,t) = 0, & on \quad \partial\Omega \times [0,T], \\ u(x,0) = u_0(x) & in \quad \Omega. \end{cases} \tag{4.1}$$

Let us use the backward difference scheme:

$$\begin{cases} \dfrac{\bar{u}^{k+1} - \bar{u}^k}{\triangle t} - div(\epsilon \bigtriangledown \bar{u}^{k+1}) + div(\vec{\beta}\bar{u}^{k+1}) = f, & in \quad \Omega, \\ \bar{u}^{k+1}(x,t) = 0, & on \quad \partial\Omega, \\ \bar{u}^0(x) = u_0(x), & in \quad \Omega, \end{cases} \tag{4.2}$$

In every time step, we just need to solve the problem

$$\begin{cases} -div(\epsilon \bigtriangledown \bar{u}^{k+1}) + div(\vec{\beta}\bar{u}^{k+1}) + \dfrac{\bar{u}^{k+1}}{\triangle t} = f + \dfrac{\bar{u}^k}{\triangle t}, & in \quad \Omega, \\ \bar{u}^{k+1} = 0, & on \quad \partial\Omega, \end{cases} \tag{4.3}$$

which is same as (2.1) with $\alpha = \frac{1}{\triangle t}$. So, the noniterative and iterative domain decomposition scheme (2.5) or (3.1) can be used to solve (4.3).

Similar as in theorem 2.1, it can be proved that

$$\|\bar{u}^k - \hat{u}_h^k\|_0 \le C(\epsilon\sqrt{\triangle t}\sum_i (\|\frac{\partial u}{\partial n}\|_{0,\partial\Omega_i^- \setminus \partial\Omega}) + \triangle t\ \|u - u^I\|_1 + \|u - u^I\|_0), \ \forall k, \tag{4.4}$$

where $\hat{u}_h^k$ is the domain decomposition solution of (4.3). When $\triangle t$ is small (for example, $\triangle t \sim h$ or $\triangle t \sim h^2$), the convergence is better than (2.6).

If the iterative domain decomposition is used to solve (4.3), because $\alpha = \frac{1}{\triangle t}$ is large, $\alpha + \frac{1}{2} div\vec{\beta} \ge 0$, so the iteration is convergent, and the spectral radius of the iteration operator is:

$$\rho(T_0) \le \left(\frac{1}{1 + C\epsilon + Ch(\triangle t)^{-1}}\right)^{\frac{1}{2}}. \tag{4.5}$$

Hence, when $\triangle t = O(h)$,

$$\rho(T_0) \le C < 1,$$

i.e. the error reduction of the iteration is uniform. Especially, when $\triangle t = O(h^2)$,

$$\rho(T_0) \le C\sqrt{h},$$

therefore only a few iteration steps are required at every time level.

5. Numerical Experiments

As a test example, we calculate the model problem

$$\begin{cases} -\epsilon \triangle u + \nabla u + 2u = f, & in \quad \Omega, \\ u = 0, & on \quad \partial\Omega, \end{cases} \tag{5.1}$$

with $\Omega = [0,1] \times [0,1]$, and

$$f = C_1(e^{a(1-x)} + e^{a(1-y)}) + C_2(e^{b(1-x)} + e^{b(1-y)}) + 2,$$

$$C_1 = \frac{1-e^b}{e^b - e^a}, \quad C_2 = \frac{e^a - 1}{e^b - e^a}, \quad a = \frac{-1+\sqrt{1+4\epsilon}}{2\epsilon}, \quad b = \frac{-1-\sqrt{1+4\epsilon}}{2\epsilon},$$

which has an analytical solution

$$u = (C_1 e^{a(1-x)} + C_2 e^{b(1-x)} + 1)(C_1 e^{a(1-y)} + C_2 e^{b(1-y)} + 1).$$

In computations, the domain Ω is divided into 5×5 subdomains, and the piecewise linear function on uniform triangular meshes is used. In each subdomain, a first order upwind approximation is used for the convection term and the inflow boundary condition is realised exactly which is taken from the subdomains in the upwind direction. Let i=1,2,3,4,5, and j=1,2,3,4,5 be the numbers associated with the subdomains in the x- and y-directions. We solve the subdomain problems by first sweeping over i=1,2,3,4,5 and then sweeping over j=1,2,3,4,5. By solving the subdomain problems in this order, the inflow boundary condition is always available when we come to compute a subdomain solution.

In table 1, some numerical results for different ϵ and different mesh sizes h are shown, where $\|e_g\|_0$ and $\|e_d\|_0$ represent the error of the global finite element solution and the error of the domain decomposition solution for problem (5.1) in L^2-norm, respectively.

Figure 1 shows the computed solutions and their errors for $\epsilon = 0.01$ and $h = 0.025$, where u, uhg and ud represent the exact solution, the global finite element solution and the domain decomposition solution of (5.1), respectively.

From table 1 and figure 1, one observes that when ϵ is small, the error of the domain decomposition solution is of the same order as the global finite element solution (see table 1 for ϵ =0.01, 0.001, 0.00001). From figure 1, one finds that the large errors both for the global FEM solution and the domain decomposition solution are concentrated in the neighbourhood of the outflow boundary. Due to the relative large mesh size used near the outflow boundary, the boundary layer is not properly resolved. Here comes the advantage of the proposed domain decomposition methods. Once we know that which subdomain contains the singular layers, we can use finer mesh in this subdomain. By doing so, the error introduced by the artificial boundary condition does not increase, but the singular layers can be efficient resolved by using the known boundary conditions from the neighbouring subdomains and a sufficient fine mesh in this subdomain. Different examples have been tested by the proposed algorithms. The numerical results always show that when ϵ is small, the domain decomposition solution and the global finite element solution have similar errors and the large errors are in the singular layers. To do grid refinement for the global problem is not easy, but it is very easy to use fine meshes for the subdomains that contains the singular layers.

	$\epsilon = 0.1$		$\epsilon = 0.01$		$\epsilon = 0.001$		$\epsilon = 0.00001$	
	$\|e_g\|_0$	$\|e_d\|_0$	$\|e_g\|_0$	$\|e_d\|_0$	$\|e_g\|_0$	$\|e_d\|_0$	$\|e_g\|_0$	$\|e_d\|_0$
h=0.1	0.0155	0.0354	0.0126	0.0200	0.0066	0.0095	0.0061	0.0084
h=0.05	0.0085	0.0323	0.0131	0.0186	0.0038	0.0051	0.0029	0.0035
h=0.025	0.0045	0.0297	0.0122	0.0165	0.0029	0.0037	0.0014	0.0016
h=0.0125	0.0023	0.0280	0.0082	0.0122	0.0032	0.0036	0.0007	0.0007

Table 1. L^2-error of the global solution and the domain decomposition solution.

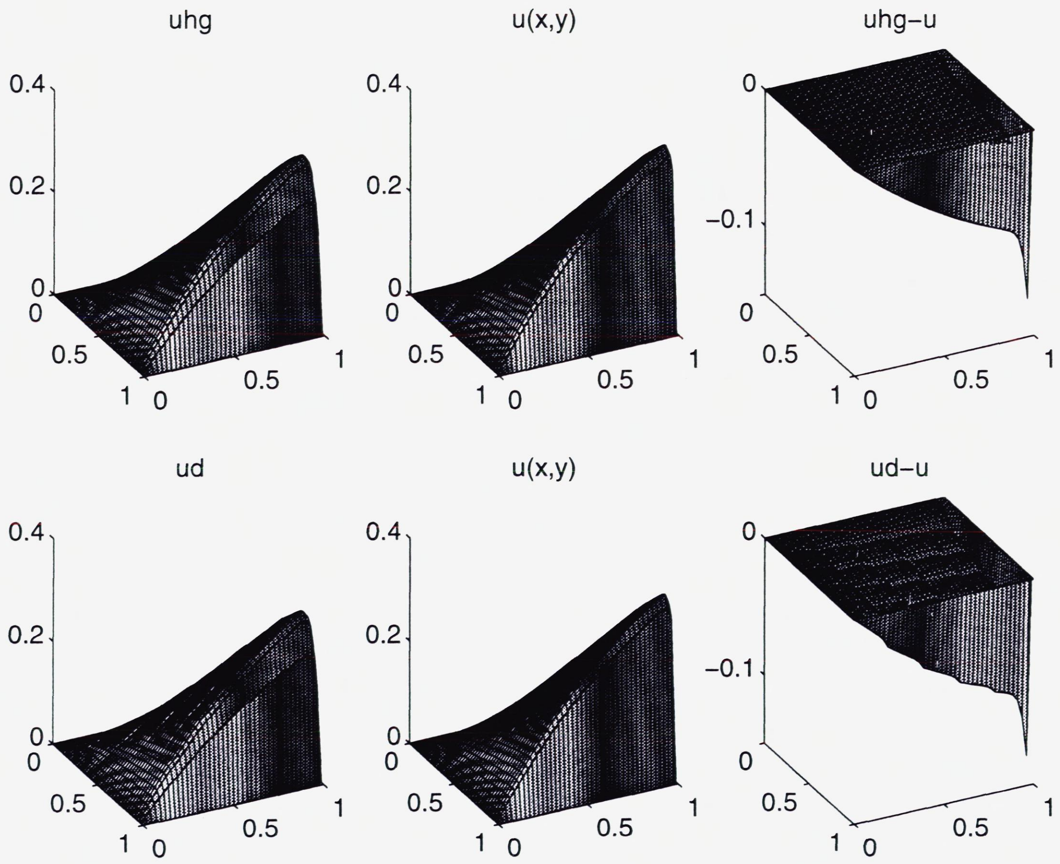

Figure 1: The global FEM solution and the domain decomposition solution for $\epsilon = 0.01$, $h = 0.025$ and the corresponding errors.

6. Conclusion

Both theoretical analysis and numerical tests reveal that the proposed algorithms are suitable for problems with small ϵ. When the diffusion parameter is small, the singular layers are very narrow. In order to resolve the singular layers, the ratio between the mesh size in the singular layers and the mesh size in the part of the domain where the solution is smooth shall be very large. In this case, the error introduced by the domain decomposition algorithms are neglectable in comparison with the errors in the singular layers. However, the domain decomposition algorithms allow easy and efficient grid refinement in the subdomains that contain the singular layers.

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