# *Department of APPLIED MATHEMATICS*

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with local Substitution and habit formation

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# A NOTE ON PORTFOLIO OPTIMIZATION IN A LEVY MARKET WITH LOCAL SUBSTITUTION AND HABIT FORMATION

FRED ESPEN BENTH, KENNETH HVISTENDAHL KARLSEN, AND KRISTIN REIKVAM

ABSTRACT. We have in previous papers [2, 3] studied an optimal portfolio-consumption model which takes into account the notion of local substitution and allow the stock price to be governed by a general Lévy (jump-diffusion) process. In this note, we discuss a generalization of this model which includes the effect of habit formation. The resulting portfolio-consumption model is discussed within the framework of dynamic programming and the theory of viscosity solutions. The associated Hamilton-Jacobi-Bellman equation is a second order degenerate elliptic integro-differential variational inequality. We also review various economical interpretations as well as results given by Hindy, Huang, and Zhu [13, 14] for the portfolio-consumption model in the geometric Brownian case.

## 1. INTRODUCTION

In this paper we will present and discuss an optimal portfolio-consumption problem in a Lévy (jump-diffusion) market. A feature of this portfolio-consumption problem is the inclusion of local substitution and habit formation. More specifically, the utility of the investor will not be derived from present consumption directly but from averages over past consumption. The stochastic optimization problem is a generalization of the problem studied in Benth, Karlsen, and Reikvam [2], which does not take into account the effect of habit formation. In [2], we characterized the value function of the portfolio-consumption model as the unique constrained viscosity solution of the associated Hamilton-Jacobi-Bellman equation in the case of a pure-jump market. In the companion paper [3], we calculated explicit consumption and portfolio selection plans for power utility functions when the risky asset follows a geometric Lévy process (see also [5] for numerical examples in real markets). Although we will not discuss it here, a related portfolio-consumption model which also takes into account proportional transaction costs is analyzed in [4].

In this paper, we remark that the viscosity solution characterization of the value function proved in [2] is valid also if we include the effect of habit formation in our model. Although we state and discuss the results leading up to this characterization, the proofs are only sketched since the details will appear elsewhere in connection with numerical studies. In

**Key words:** Portfolio choice, local substitution, habit formation, singular stochastic control, dynamic programming, integro-differential variational inequality, viscosity solution.

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addition to a viscosity solution treatment of the portfolio-consumption model, we discuss various economical interpretations as well as reviewing the results given by Hindy, Huang, and Zhu [13, 14] for this model in the case of geometric Brownian motion. For an overview of papers dealing with control problems related to the one that we study herein, we refer to the discussions and citations in [2, 3, 4], see also the review paper by Zariphopoulou [19]. The reader is also encouraged to consult these papers for references to relevant papers dealing with the theory of viscosity solutions.

One of the main motivation for analyzing our portfolio-consumption model within the framework of viscosity solutions is that such analysis provides the first step in a numerical treatment of the model. When the notion of habit formation is included, it is hard (if possible?) to find explicit consumption and investment plans. If the risky asset follows a geometric Brownian motion, Hindy, Huang, and Zhu [13, 14] conclude from numerical solutions that, for instance, the optimal portfolio selection plan behaves quite differently from the case with no habit formation. It is of interest to generalize their analysis to assets that follow geometric Lévy processes, opening up for a much more realistic modeling of the stock price dynamics. Since we cannot expect to find solutions by analytical means, a natural approach is to attack the problem with a so-called Markov chain approximation method, as was done by Hindy, Huang, and Zhu [13, 14] in the geometric Brownian case. We refer to Kushner and Dupuis [16] for a general introduction to the Markov chain approximation method. The construction and analysis of numerical methods is outside the scope of this paper and will instead be the topic of future work. In fact, we will in future work present a Markov chain approximation method for computing the value function as well as the optimal policies (see [8] for preliminary work in this direction). As is well known by now, the viscosity solution theory provides a very flexible and powerful framework for proving convergence of numerical methods. However, to take advantage of this framework, the analytical results found in the present paper are necessary. In particular, the characterization of the value function as the unique constrained viscosity solution of an integro-differential variational inequality is of fundamental importance for the convergence analysis of a large class of (monotone, stable, and consistent) numerical methods for the portfolio-consumption model studied herein.

An outline of the paper is as follows: In Section 2, we formulate the portfolio-consumption problem and state the basic assumptions. In Section 3, we discuss the economical interpretations of the model, while in Section 4 we study the portfolio selection problem within a viscosity solution framework. The results of Hindy, Huang, and Zhu [13, 14] for geometric Brownian motion are briefly presented and discussed in Section 5.

## 2. The stochastic control problem

Let $(\Omega, \mathcal{P}, \mathcal{F})$ be a probability space and $(\mathcal{F}_t)$ a given filtration satisfying the usual hypotheses. Consider an investor operating in a financial market consisting of a risky asset (e.g., a stock) and a bond. The value of the risky asset is assumed to follow the stochastic process

$$S_t = S_0 e^{L_t}, \tag{2.1}$$

In (2.1), $L_t$ is a Lévy process with Lévy-Khintchine decomposition

$$L_t = \mu t + \sigma W_t + \int_0^t \int_{|\alpha|<1} \alpha \, \tilde{N}(ds, d\alpha) + \int_0^t \int_{|\alpha|\geq 1} \alpha \, N(ds, d\alpha),$$

where $\mu, \sigma$ are constants and $W_t$ is a standard Brownian motion. Furthermore, $N(dt, d\alpha)$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $dt \times n(d\alpha)$, $n(d\alpha)$ is a $\sigma$-finite Borel measure on $\mathbb{R}\backslash\{0\}$ called the Lévy measure, and $\tilde{N}(dt, d\alpha) = N(dt, d\alpha) - dt \times n(d\alpha)$ is the compensated Poisson random measure. We assume that $W_t$ and $N(dt, d\alpha)$ are independent stochastic processes. From now on we shall use the unique càdlàg version of $L_t$, which is also denoted by $L_t$.

We recall that the Lévy measure has the property

$$\int_{\mathbb{R}\backslash\{0\}} \min(1, \alpha^2) \, n(d\alpha) < \infty. \tag{2.2}$$

Under the following additional integrability condition on the Lévy measure

$$\int_{|\alpha|\geq 1} \left|e^\alpha - 1\right| n(d\alpha) < \infty, \tag{2.3}$$

we can write the differential of the stock price dynamics as (using Itô's formula for Lévy processes, see, e.g., [15])

$$dS_t = \hat{\mu} S_t \, dt + \sigma S_t \, dW_t + S_{t-} \int_{\mathbb{R}\backslash\{0\}} \left(e^\alpha - 1\right) \tilde{N}(dt, d\alpha). \tag{2.4}$$

Here we have introduced the short-hand notation

$$\hat{\mu} = \mu + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}\backslash\{0\}} \left(e^\alpha - 1 - \alpha \mathbf{1}_{|\alpha|<1}\right) n(d\alpha). \tag{2.5}$$

The bond dynamics is

$$dB_t = r B_t \, dt,$$

with $r > 0$ being the interest rate. We make the basic assumption that $r < \hat{\mu}$. Hence, the expected rate of return from an investment in the risky asset is greater than the return of the bond, giving potential investors a risk premium $\hat{\mu} - r$.

The investor wants to allocate her wealth in the asset and the bond and consume so as to maximize her utility. Let $\pi_t \in [0, 1]$ be the fraction of wealth invested in the asset at time $t$. If we denote the cumulative consumption up to time $t$ by $C_t$, we have the wealth process $X_t^{\pi,C}$ given as

$$X_t^{\pi,C} = x - C_t + \int_0^t \left(r + (\hat{\mu} - r)\pi_s\right) X_s^{\pi,C} \, ds + \int_0^t \sigma \pi_s X_s^{\pi,C} \, dW_s$$
$$+ \int_0^t \pi_{s-} X_{s-}^{\pi,C} \int_{\mathbb{R}\backslash\{0\}} \left(e^\alpha - 1\right) \tilde{N}(ds, d\alpha),$$

where $x$ is the investor's initial wealth. The market is supposed to be free of any transaction costs (see [4] for the case of transaction costs).

The investor derives utility from the two processes

$$
(2.6)\qquad
\begin{aligned}
Y_t^{\pi,C} &= ye^{-\beta t} + \beta e^{-\beta t}\int_{[0,t]} e^{\beta s}\,dC_s,\\
Z_t^{\pi,C} &= ze^{-\lambda t} + \lambda e^{-\lambda t}\int_{[0,t]} e^{\lambda s}\,dC_s,
\end{aligned}
$$

where $y, z > 0$ and $\beta, \lambda$ are positive weighting factors. The integrals with respect to $C_t$ are interpreted pathwise in a Lebesgue-Stieltjes sense. When there is no risk of confusion, we shall frequently write $X_t, Y_t, Z_t$ instead of $X_t^{\pi,C}, Y_t^{\pi,C}, Z_t^{\pi,C}$, respectively. Note that the differential forms of $Y_t$ and $Z_t$ are

$$
\begin{aligned}
dY_t &= -\beta Y_t\,dt + \beta\,dC_t,\\
dZ_t &= -\lambda Z_t\,dt + \lambda\,dC_t.
\end{aligned}
$$

The economical background for these two processes are discussed in the next section.

Denote by $\mathcal{A}_{x,y,z}$ the set of all admissible controls and let

$$
\mathcal{D} = \Big\{(x,y,z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0\Big\}.
$$

We say that a pair of controls is admissible for $(x,y,z) \in \overline{\mathcal{D}}$ and write $\pi, C \in \mathcal{A}_{x,y,z}$ if:

(**c.1**) $C_t$ is an adapted process that is right continuous with left-hand limits (càdlàg), nondecreasing, with initial value $C_{0-} = 0$ (to allow an initial jump when $C_0 > 0$), and satisfies $\mathbb{E}[C_t] < \infty$ for all $t \geq 0$.

(**c.2**) $\pi_t$ is an adapted càdlàg process with values in $[0,1]$.

(**c.3**) $X_t^{\pi,C} \geq 0$ almost surely for every $t \geq 0$.

Condition (**c.3**) is a state-space constraint, restricting the set of admissible consumption patterns to those avoiding negative wealth.

The objective of the investor is to find an allocation process $\pi_t^*$ and a consumption pattern $C_t^*$ which optimizes the expected discounted utility derived from $Y_t^{\pi^*,C^*}$ and $Z_t^{\pi^*,C^*}$ over an infinite investment horizon. The value function is defined as

$$
(2.7)\qquad V(x,y,z) = \sup_{\pi,C\in\mathcal{A}_{x,y,z}} \mathbb{E}\Big[\int_0^\infty e^{-\delta t}U(Y_t^{\pi,C}, Z_t^{\pi,C})\,dt\Big],
$$

where $\delta > 0$ is the discount factor. The utility function $U : [0,\infty)^2 \to [0,\infty)$ is assumed to have the following properties:

(**u.1**) $U$ is nondecreasing, concave, and continuous in each variable.

(**u.2**) There exist constants $K > 0$ and $\gamma \in (0,1)$ such that $\delta > k(\gamma)$ and

$$
U(y,z) \leq K(1 + y + z)^\gamma,
$$

for all nonnegative $y, z$, where

$$k(\gamma) = \max_{\pi\in[0,1]}\Big[\gamma(r+(\hat{\mu}-r)\pi) - \gamma(1-\gamma)\frac{\sigma^2}{2}\pi^2 + \int_{\mathbb{R}\setminus\{0\}}\Big((1+\pi(e^\alpha-1))^\gamma - 1 - \gamma\pi(e^\alpha-1)\Big)\,n(d\alpha)\Big]. \tag{2.8}$$

A Taylor expansion shows that the integral term in $k(\gamma)$ is finite since (2.2) and (2.3) hold.

We next recall a fundamental property of the value function that goes back to Bellman. Namely, we will assume throughout this paper that the dynamic programming principle holds, that is, for any stopping time $\tau$ and $t \geq 0$,

$$V(x,y,z) = \sup_{\pi,C\in\mathcal{A}_{x,y,z}} \mathbb{E}\Big[\int_0^{t\wedge\tau} e^{-\delta s}U(Y_s^{\pi,C}, Z_s^{\pi,C})\,ds + e^{-\delta(t\wedge\tau)}V(X_{t\wedge\tau}^{\pi,C}, Y_{t\wedge\tau}^{\pi,C}, Z_{t\wedge\tau}^{\pi,C})\Big], \tag{2.9}$$

where $a\wedge b = \min(a,b)$. The infinitesimal version of the dynamic programming principle (2.9) is the Hamilton-Jacobi-Bellman equation. In the our context, this equation is a nonlinear second order degenerate elliptic integro-differential equation subject to a gradient constraint (i.e., an integro-differential variational inequality). If we let $\mathcal{A}$ denote the second order degenerate elliptic integro-differential operator defined as

$$\mathcal{A}v(x,y,z) = -\beta y v_y - \lambda z v_z + \max_{\pi\in[0,1]}\Big[(r+(\hat{\mu}-r)\pi)xv_x + \frac{1}{2}\sigma^2\pi^2x^2v_{xx} + \int_{\mathbb{R}\setminus\{0\}}\Big(v(x+\pi x(e^\alpha-1),y,z) - v(x,y,z) - \pi x v_x(x,y,z)(e^\alpha-1)\Big)\,n(d\alpha)\Big],$$

the Hamilton-Jacobi-Bellman equation takes the form

$$\max\Big\{\beta v_y + \lambda v_z - v_x; U(y,z) - \delta v + \mathcal{A}v\Big\} = 0 \text{ in } \mathcal{D}. \tag{2.10}$$

Note that we have $x + \pi x(e^\alpha - 1) \geq 0$ for all $x \geq 0$ and $\alpha\in\mathbb{R}$. If $v$ is $C^2$ and sublinearly growing, it can be proven that (2.10) is well-defined (see, e.g., [2]). Moreover, if the value function $V$ defined in (2.7) satisfies these conditions, then, by using the dynamic programming principle (2.9) and Itô's formula, one can easily prove that $V$ solves (2.10). However, since it is hard in general to prove that $V$ is sufficiently regular, we shall in Section 4 interpret (2.10) in the sense of viscosity solutions. More precisely, due to the state-space constraint (**c.3**), we shall consider constrained viscosity solutions of (2.10).

## 3. Economical interpretations

In this section we discuss economical interpretations of the optimal portfolio-consumption problem described in Section 2. Contrary to most non-time-additive utility maximization problems, the investor does not derive her utility directly from present consumption but from averages over past consumption (through the processes $Y_t$ and $Z_t$ defined in (2.6)). This structure has many desirable interpretations from an economical point of view. Hindy, Huang, and Zhu [14] suggest three possible interpretations of the two processes $Y_t$ and $Z_t$. In the first, they describe the notions of local substitution and habit formation. Secondly,

they model the service flow from a durable good, and in the final interpretation, $Y_t$ and $Z_t$ describe the utility derived from a composite commodity. We next discuss the three interpretations in more detail.

The notion that consumption at one date reduces marginal utility at nearby dates and consumption at adjacent dates are complementary is called *local substitution*. If we have lunch at noon, the marginal utility of eating again shortly after will be lower since we are not hungry (provided we had a satisfactory lunch, of course). At dinner time we are again hungry (provided the lunch was not too satisfactory), so the marginal utility to eat then is complementary to lunch. If the mathematical model is able to catch the notion of local substitution, it should be optimal to consume (i.e., to eat in our example) periodically, or in gulps. Hindy and Huang [10] show that investors deriving utility from $Y_t$ instead of $C_t$ directly will consume in gulps (see discussion below). The process $Z_t$ models the notion of habit formation. Agents develop habits from earlier consumption and a high standard of living increases the appetite for present consumption. If you are used to a delicious Botswana beef for supper, you will probably be very disappointed being offered a Norwegian beef as a substitute. When changing your old Mercedes car, you will probably want to buy a new and perhaps better Mercedes to keep up with your expectations of what a car should be like. DeTemple and Zapatero [7] suggest to model the mechanism of habit formation by $dZ_t = \rho dC_t - \lambda Z_t\, dt$[1]. The constants $\lambda$ and $\rho$ describe the relative importance of consumption history to inherited standard of living. Furthermore, $\rho$ is the intensity of consumption habitats, while $\lambda$ is the persistence of past consumption. A low $\lambda$ means a high persistence, while increasing $\rho$ places more emphasis on the history of consumption. We choose to follow Hindy, Huang, and Zhu [13, 14] and let $\rho = \lambda$. Investigating this control problem is important in order to improve the understanding of the mechanisms driving security returns.

In the second interpretation of the model, $C_t$ is the total purchase of a durable good up to time $t$. The durable good may be clothing, computers, cars, and even holidays. The process $Y_t$ describes the service flow from the durable good. For instance, buying a car will provide the agent with a mean of transport. However, as long as you use the car, it will deteriorate, and after a while the service flow will start to decrease as long as you do not buy a new one. The standard of living of the agent is reflected through past consumption, and modeled by $Z_t$. Also $Z_t$ will decrease as long as new goods are not purchased, however, at a slower rate. A natural condition from an economic point of view is to assume $\beta > \lambda$. Good quality and fashionable clothes will for instance provide you with a high standard of living (that is, high $Z_t$), while the service of the clothes will be to keep you warm and dry (one may of course argue that fashion changes faster than deterioration of clothes, so perhaps $\beta < \lambda$ instead).

The final interpretation mentioned by Hindy, Huang, and Zhu [14] is composite commodities. Many commodities may give the agent two (or more) utilities. The new portable computers from Macintosh provide you with a high quality computer, but at the same time with style (at least they try to advertise it like that). A bicycle gives you exercise

[1]They use in fact absolute continuous consumption plans $dC_t = c_t\, dt$.

(increasing your health) as well as being a mean of transport. Food, for instance, provide you with vitamins and energy, both important for your well being. The utility derived from such dual purpose commodities are modeled through $Y_t$ and $Z_t$. A natural condition of the utility function would be $\partial^2 U/\partial y \partial z > 0$, meaning that marginal utility of transport is increased at a higher level of health, if you think of bicycles. Of course we may think of commodities with more than two purposes. We will not include that generality here, since it is a straightforward extension mathematically.

## 4. Viscosity solutions

Our analysis of the portfolio-consumption model described in Section 2 is based on the dynamic programming method and the newly developed theory of viscosity solutions of Hamilton-Jacobi-Bellman equations. For a general overview of the viscosity solution theory, we refer to the survey paper by Crandall, Ishii, and Lions [6] and the book by Fleming and Soner [9]. For an overview of the use of viscosity solutions in the area of portfolio management and derivative pricing, we refer to the review paper by Zariphopoulou [19].

As it turns out, the Hamilton-Jacobi-Bellman equation is a direct consequence of the dynamic programming principle and one expects the value function to satisfy this equation. However, due to degeneracy as well as market imperfections such as trading constraints (see (**c.3**)) and transaction costs, to mention only a few, the value function might not satisfy the Hamilton-Jacobi-Bellman equation in the classical sense, that is, the value function might not possess all the continuous derivatives occurring in the Hamilton-Jacobi-Bellman equation and thus not satisfy this equation pointwise everywhere. It therefore becomes important to relax the notion of classical solution of Hamilton-Jacobi-Bellman equations so as to allow functions that are not necessarily smooth as (generalized) solutions. This has been achieved successfully by the introduction of the notion of viscosity solutions, which allows merely continuous functions to be solutions of fully nonlinear first and second order partial differential equations.

As already mentioned in (2.10), the Hamilton-Jacobi-Bellman equation associated with our singular control problem is a second order integro-differential variational inequality which contains a non-local (integral) operator with a highly singular Lévy measure $n(d\alpha)$. If we insist on interpreting (2.10) in the classical sense, we have to consider twice continuously differentiable functions because of the second order differential operator part of (2.10) as well as the (singular) Lévy measure $n(d\alpha)$. We point out that it is *not* easy to show directly that the value function (2.7) is twice continuously differentiable, although we can prove quite easily that it is continuous and sublinearly growing (see below). However, if we interpret (2.10) in the viscosity sense, it is sufficient to consider continuous functions, and one can indeed show that the value function (2.7) is a viscosity solution of (2.10) (see below). Moreover, one can prove that there exists only one viscosity solution (the value function!) of the integro-differential variational inequality (2.10) which is continuous and sublinearly growing (see below).

The (constrained) viscosity solution framework presented below is a straightforward adaption of the framework developed in [2, 3, 4] for first and second order integro-differential

variational inequalities. Due to strong similarities with [2, 3, 4], we are very brief in this section and refer to [2, 3, 4] for details not found herein. Also, we refer to [2, 3, 4] for an overview of the literature dealing with viscosity solutions of integro-differential equations.

As already indicated, the ultimate goal of this section is to characterize the value function (2.7) as the unique constrained viscosity solution of the associated Hamilton-Jacobi-Bellman equation (2.10). To this end, we first verify as in [2, 3, 4] that the value function $V$ is well defined, non-negative, non-decreasing, and concave. The arguments needed to establish these properties are standard (see, e.g., Zariphopoulou [18, 19]).

Next, one can show $V$ is uniformly continuous on $\overline{\mathcal{D}}$ by following the arguments used in the proof of Theorem 3.1 in [2] (see also [1, 3, 4]). In fact, one can even show that $V$ is Hölder continuous if $U$ is Hölder continuous and some extra conditions on $k(\gamma)$ are fulfilled. This was first observed by Alvarez [1] in the Brownian case and later generalized to the Lévy case in [2] (see also [3]). In addition, the value function has sublinear growth of the same order as the utility function, see [1, 2, 3, 4] different proofs of this fact. More precisely, there exists a positive constant $K$ such that

$$V(x,y,z) \le K(1+x+y+z)^{\gamma} \qquad \forall x,y,z \in \overline{\mathcal{D}}. \tag{4.1}$$

In view of (4.1) and for later use, we introduce the set

$$C_{\ell}(\overline{\mathcal{D}}) = \Big\{\phi \in C(\overline{\mathcal{D}}) \,:\, \sup_{\overline{\mathcal{D}}} \frac{|\phi(x,y,z)|}{(1+x+y+z)^{\ell}} < \infty\Big\}, \qquad \ell \ge 0.$$

In particular, we have

$$V \in C_{\gamma}(\overline{\mathcal{D}}).$$

Later we prove that the characterization of $V$ as a constrained viscosity solution is unique at least within the class of continuous and sublinearily ($\gamma < 1$) growing solutions.

Before we introduce the notion of (constrained) viscosity solutions, let us introduce the following short-hand notations: $X = (x,y,z) \in \mathbb{R}^3$, $D_X v$ is the gradient of $v$ with respect to $X$, $D_X^2 v$ is the Hessian of $v$ with respect to $X$, and $G(D_X v) = \beta v_y + \lambda v_z - v_x$. Furthermore, introduce the non-local operator

$$\mathcal{B}^{\pi}(X,v) = \int_{\mathbb{R}\setminus\{0\}} \Big(v(x+\pi x(e^{\alpha}-1),y,z) - v(X) - \pi x v_x(X)(e^{\alpha}-1)\Big)\, n(d\alpha),$$

and the operator

$$\begin{aligned} &F(X,v,D_X v, D_X^2 v, \mathcal{B}^{\pi}(X,v)) \\ &\quad = U(y,z) - \delta v - \beta y v_y - \lambda z v_z + \max_{\pi\in[0,1]}\Big[(r+(\hat{\mu}-r)\pi)x v_x + \frac{1}{2}\sigma^2\pi^2x^2 v_{xx} + \mathcal{B}^{\pi}(X,v)\Big]. \end{aligned}$$

We can now write (2.10) as

$$\max\Big(G(D_X v); F(X,v,D_X v, D_X^2 v, \mathcal{B}^{\pi}(X,v))\Big) = 0 \text{ in } \mathcal{D}. \tag{4.2}$$

A constrained viscosity solution of (4.2) is defined as follows:

**Definition 4.1.** (i) Let $\mathcal{O} \subset \overline{\mathcal{D}}$. Any $v \in C(\overline{\mathcal{D}})$ is a viscosity subsolution (supersolution) of (4.2) in $\mathcal{O}$ if and only if we have, for every $X \in \mathcal{O}$ and $\phi \in C^2(\overline{\mathcal{D}}) \cap C_1(\overline{\mathcal{D}})$ such that $X$ is a global maximum (minimum) relative to $\mathcal{O}$ of $v - \phi$,

$$\max\Big(G(D_X\phi); F(X, v, D_X\phi, D_X^2\phi, \mathcal{B}^\pi(X, \phi))\Big) \geq 0\,(\leq 0).$$

(ii) Any $v \in C(\overline{\mathcal{D}})$ is a constrained viscosity solution of (4.2) if and only if $v$ is a viscosity subsolution of (4.2) in $\overline{\mathcal{D}}$ and $v$ is a viscosity supersolution of (4.2) in $\mathcal{D}$.

Following closely the proof of Theorem 4.1 in [2], we can show that the constrained viscosity property of the value function holds.

**Theorem 4.1** (Existence). *The value function $V(x, y, z)$ defined in* (2.7) *is a constrained viscosity solution of the integro-differential variational inequality* (2.10).

To prove this result, we first show that $V$ is a viscosity supersolution directly by using the dynamic programming principle (2.9) and Itô's formula for Lévy processes. To prove the viscosity subsolution property, we argue by contradiction. Introducing stopping times such that we can control the jumps coming from the Lévy process and consumption, we are able to construct estimates on the value function $V$ which contradict the dynamical programming principle. We refer to [2] for details.

To guarantee that the characterization in Theorem 4.1 is unique, a comparison result is needed. In a numerical treatment of the control problem, one approximates the state variables by Markov chains and consider instead the related discrete-time optimization problem. To ensure convergence of the discretized problem to the *correct* continuous-time problem, we need also in this context a comparison principle for (4.2) (see, e.g., [19] for this type of application).

We have the following theorem:

**Theorem 4.2** (Uniqueness). *Let $\gamma' > 0$ be such that $\delta > k(\gamma')$. Assume $\underline{v} \in C_{\gamma'}(\overline{\mathcal{D}})$ is a subsolution of* (2.10) *in $\overline{\mathcal{D}}$ and $\overline{v} \in C_{\gamma'}(\overline{\mathcal{D}})$ is a supersolution of* (2.10) *in $\mathcal{D}$. Then $\underline{v} \leq \overline{v}$ in $\overline{\mathcal{D}}$. Consequently, in the class of sublinearly growing solutions, the Hamilton-Jacobi-Bellman equation* (2.10) *admits at most one constrained viscosity solution.*

Theorem 4.2 can be proven in the same spirit as the comparison principles in [2, 3, 4]. The proof uses the classical "doubling of variables" device together with the maximum principle for semicontinuous functions (see Crandall, Ishii, and Lions [6]). Since our problem contains a second order differential operator, the proof requires that we use the maximum principle for semicontinuous functions and hence we need an alternative formulation of viscosity solutions based on the notion of sub- and superjets. We will not go into details about this formulation, but refer the reader instead to [3, 4]. We refer to [2] for the comparison proof in the case of a first order differential operator (pure-jump market), which does not require the jet formulation and the maximum principle for semicontinuous functions.

We mention that the treatment of the singular non-local operator $\mathcal{B}^\pi$ is rather involved. Among other things, we need to distinguish the singularities at zero and infinity in the integral operator, which is thus split into two parts $\mathcal{B}^{\pi,\kappa}$ and $\mathcal{B}^\pi_\kappa$. For any $\kappa \in (0, 1)$, we

define

$$\mathcal{B}^{\pi,\kappa}(X,v,D_Xv) = \int_{|\alpha|>\kappa} \Big(v(x+\pi x(e^\alpha-1),y,z) - v(X) - \pi x v_x(X)(e^\alpha-1)\Big)\, n(d\alpha),$$

$$\mathcal{B}^{\pi}_{\kappa}(X,v) = \int_{|\alpha|\leq\kappa} \Big(v(x+\pi x(e^\alpha-1),y,z) - v(X) - \pi x v_x(X)(e^\alpha-1)\Big)\, n(d\alpha).$$

It can be shown (see, e.g., [2]) that $\mathcal{B}^{\pi,\kappa}(X,v,D_Xv)$ is well defined for for $v \in C^1(\overline{\mathcal{D}})\cap C_1(\overline{\mathcal{D}})$ while $\mathcal{B}^{\pi}_{\kappa}(X,v)$ is well defined for $v \in C^2(\overline{\mathcal{D}})$. The splitting of the integral operator $\mathcal{B}^\pi$ is taken into account in the jet formulation of viscosity solutions and is essential for carrying out the comparison proof when the Lévy measure $n(d\alpha)$ is singular (see [3, 4] for details).

Our problem involves a gradient constraint as well as a state constraint boundary condition. To treat the gradient constraint, we construct strict supersolutions that are close to the supersolution being compared. Following closely the proof of Lemma 4.3 in [2], by choosing $K > 0$ and $\overline{\gamma} \in (0,1)$ properly it is easily seen that

$$w = K + \chi^{\overline{\gamma}}, \qquad \chi(X) = \big(1 + x + \tfrac{y}{4\beta} + \tfrac{z}{4\lambda}\big)$$

is a strict supersolution of (2.10). When applying the maximum principle for semicontinuous functions, we choose a test function so that the minimum associated with the supersolution cannot be on the boundary (in the spirit of Soner [17]), we are able to handle the state constraint boundary condition. Similar treatments of gradient and state constraints have been given in [18, 19] (see also [13]) for a related portfolio-consumption model in a geometric Brownian market. Finally, let us mention that the strict supersolutions are also used to "localize" the proof of the comparison principle to a bounded domain (which is convenient). We refer to [2, 3, 4] for further details about the comparison principle.

## 5. Discussion of the geometric Brownian motion case

We will in this section recall the conclusions made by Hindy, Huang, and Zhu [14], which were based on a numerical treatment of portfolio-consumption model in the geometric Brownian motion case. Their results indicate the type of results that we may expect from a study of the Lévy case.

From the portfolio-consumption problem with utility of HARA (Hyperbolic Absolute Risk Aversion) type and local substitution (or durability) but without habit formation[2], the investor optimally keeps a constant fraction of wealth in the stock. Consumption takes place only when the wealth reaches a certain barrier, leading to a periodic consumption pattern (or in more popular terms "consumption in gulps"). Reaching the optimal consumption barrier, the investor consumes a small amount only enough to prevent the state variables from leaving the barrier (i.e., increase $Y_t$ while decreasing wealth $X_t$ through consumption in a "local-time" fashion). Optimal consumption takes place only when the ratio between wealth and $Y_t$ is equal to a constant $k^*$. Hence the optimal consumption boundary is linear in $Y_t$, as was proven by Hindy and Huang [10] when the stock price follows a Brownian motion. In [3], their conclusions were generalized to a Lévy market.

[2]The investor derives utility only from $Y_t$.

Note the resemblence with the classical Merton problem, where the investor also keeps a constant fraction of the wealth in the risky asset.

For the stochastic control problem with both local substitution and habit formation, Hindy, Huang, and Zhu [14] compute the optimal consumption boundary $X^*(y,z)$ using the Markov chain approximation method for a utility function on the form

$$U(y,z) = y^{\gamma_1} z^{\gamma_2}, \qquad \gamma_1, \gamma_2 \in (0,1).$$

If the current wealth is less than the barrier, the investor refrains from consumption, waiting until the state variables hits the consumption barrier. During a period of no consumption, the standard of living and service flow from the good will decrease. When the current wealth is bigger than $X^*(y,z)$, the investor instantly consumes such that the wealth is reduced and $y, z$ are increased to bring the state variabels to the boundary. This consumption pattern is in accordance with the model in [10], where the investor consumes in gulps, thereby introducing local substitution. However, the special feature of the current problem is that the optimal consumption barrier is cyclic as a function of $y$ and $z$. For a fixed standard of living $y$, $X^*(y,z)$ will increase as a function of $z$, then decrease and then increase again. A similar property holds for $X^*(y,z)$ for fixed standard of living and varying $z$. Another striking feature is the suboptimality of keeping a constant fraction of wealth in the stock. The optimal investment policy $\pi$ will be a cyclic function of wealth, standard of living, and service flow, i.e., $\pi^* = \pi^*(x,y,z)$. The partial derivatives with respect to $y$ and $z$ will change sign periodically as $y$ and $z$ change, respectively.

The cyclic pattern in both consumption and investment is explained by Hindy, Huang, and Zhu [14] as coming from an interaction between durability and habit formation. An additional purchase of the durable good reduces the agent's appetite. This satiation effect is in conflict with the indirect stimulation of increasing the agent's appetite for a higher standard of living. When satiation dominates, the agent will tolerate high losses, thus investing a higher fraction of her wealth in the stock. When stimulation is dominating, the agent is more risk averse and protects her standard of living by reducing the fraction invested in the risky asset.

When generalizing to a more realistic Lévy market model, we expect the same qualitive conclusions to hold. However, the optimal consumption and investment policies will quantitativly look different. We remark that this is in accordance with the case of local substitution with no habit formation, where the optimal policies where qualitativly the same for the geometric model and the Lévy market (see [3]).

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