Department of APPLIED MATHEMATICS

A strongly degenerate convectio-diffusion Problem modeling centrifugation of flocculated Suspensions.

by

Raimund Bürger^A and Kenneth Hvistendahl Karlsen^B

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Department of Mathematics University of Bergen 5008 Bergen Norway

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A STRONGLY DEGENERATE CONVECTION-DIFFUSION PROBLEM MODELING CENTRIFUGATION OF FLOCCULATED SUSPENSIONS

RAIMUND BÜRGER^A AND KENNETH HVISTENDAHL KARLSEN^B

ABSTRACT. We prove existence and uniqueness of BV entropy solutions of a strongly degenerate convection-diffusion problem modeling centrifugation of flocculated suspensions. A modification of the generalized upwind method is employed to solve the initial-boundary value problem numerically, i.e., to simulate the centrifugation process.

1. INTRODUCTION

We consider the quasilinear strongly degenerate convection-diffusion equation

$$u_t + f(u, x)_x = A(u)_{xx} + g(u, x), \ A(u) := \int_0^u a(s)ds; \ (x, t) \in Q_T := (x_1, x_2) \times (0, T), \ (1)$$

where we assume that $f \ge 0$, $\sup f(\cdot, x) \subset [0, 1]$, a(u) = 0 for $u \le u_c$ and $u \ge 1$, and a(u) > 0 otherwise, i.e., equation (1) is of hyperbolic type for $u \le u_c$ and $u \ge 1$ and of parabolic type for $u_c < u < 1$. We assume that $f(\cdot, x)$ is continuous and piecewise differentiable with $\|\partial_u f(\cdot, x)\| \le M$ and that $A(\cdot)$ and $g(\cdot, x)$ are Lipschitz continuous uniformly in x, and that $f(u, \cdot), g(u, \cdot) \in C^1(x_1, x_2)$ uniformly in u. In particular, the diffusion coefficient $a(\cdot)$ is allowed to be discontinuous. We consider the initial and boundary conditions

$$u(x,0) = u_0(x), \ 0 \le u_0(x) \le 1, \ x \in [x_1, x_2],$$
(2)

$$f(u, x_{\rm b}) - A(u(x_{\rm b}, t))_x = 0, \ x_{\rm b} \in \{x_1, x_2\}, \ t \in (0, T],$$
(3)

and assume that the initial function satisfies

$$u_0 \in \left\{ u \in BV(x_1, x_2) : u(x) \in [0, 1]; \; \exists M_0 > 0 : \; \forall \varepsilon > 0 : \mathrm{TV}_{(x_1, x_2)} \big(\partial_x A^{\varepsilon}(u) \big) < M_0 \right\},$$
(4)

where A^{ε} is defined in terms of a standard C^{∞} mollifier ω_{ε} with supp $\omega_{\varepsilon} \subset (-\varepsilon, \varepsilon)$ via

$$a^{\varepsilon}(u) := \left((a + \varepsilon) * \omega_{\varepsilon} \right)(u), \quad A^{\varepsilon}(u) := \int_{0}^{u} a^{\varepsilon}(s) \, ds.$$
(5)

Remark 1. If $A \in C^1$, then it is sufficient to assume that $TV_{(x_1,x_2)}(u'_0) < \infty$. Moreover, the regularity assumption on $f(u, \cdot)$ used this paper can be weakened, see [14] for details.

^aInstitute of Mathematics A, University of Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany. E-mail: buerger@mathematik.uni-stuttgart.de.

^bDepartment of Mathematics, University of Bergen, Johs. Brunsgt. 12, N-5008 Bergen, Norway. E-mail: kennethk@mi.uib.no.

A STRONGLY BEGENERATE CONVECTION DIFUSION FROBLEM MODELING CENTRIFUGATION OF FLOCCULATED SUBPENSIONS

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where we assume that $f \geq 0$, sup $f(-x) \in [0, 1]$, f(x) = 0 for $x \leq x$, and $x \in 1$, and $y \in 1$. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0, and (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0, and (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0, and (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0, and (-x) = 0. The elements of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0 of (-x) = 0. The elements of (-x) = 0 of (-x) = 0 of (-x) = 0 of (-x) = 0

$$(a, 0) = a_0(a), 0 \leq a_0(a) \leq b_1 \in \mathbb{R}$$

$$f(z,z_{1}) - f(u(z_{1},z))_{*} = 0, z_{1} \in \{f_{1}, z_{2}\} : 1 \in \{0, 1\}, \dots, \{0, 1\}$$

and assume that the initial function satisfies

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Remark 1. (1.4.4.6.6) then it is adhered to exhibit heit 1.7.(1.5., (0.1.4.5.5)). Its replacing meanmatch in flue.) used this prove can be matched, and help for details.

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BÜRGER AND KARLSEN

The study of strongly degenerate hyperbolic-parabolic equations such as (1) is in part motivated by a recent theory of sedimentation-consolidation processes [8]. While the paper [4] is concerned with the case of settling in a gravitational field, which leads to an equation similar to (1) but without a source term, we here focus on the application of this theory to a centrifugal field in a rotating frame of reference [3]. It is the purpose of this contribution to briefly outline how the analysis of strongly degenerate convection-diffusion problems presented in detail in [4, 14] (see also [6]) can be extended to the initial-boundary value problem (IBVP) (1)–(3), and to draw attention to a new application of strongly degenerate equations. In addition, we utilize an adaptation of the generalized upwind finite difference method presented in [5] to solve the IBVP numerically, that is to simulate the centrifugation process. For an overview of the mathematical and numerical theory of (strongly) degenerate parabolic equations, we refer to the lecture notes by Espedal and Karlsen [11].

This paper is organized as follows. In Section 2 we state and comment the definition of entropy solutions of the IBVP. We then outline in Sections 3 and 4 the existence and uniqueness proofs of entropy solutions, following the vanishing viscosity method and recent ideas by Carrillo [9] and Karlsen and Risebro [14], respectively. We come back to the application to centrifugation in Section 5, in which we present a working numerical algorithm and numerical solutions for the IBVP.

2. Entropy solutions

Due to both the degeneracy of the diffusion coefficient $a(\cdot)$ and the nonlinearity of $f(\cdot, x)$, solutions of (1) are discontinuous and have to be defined as entropy solutions.

Definition 1. A function $u \in L^{\infty}(Q_T) \cap BV(Q_T)$ is an entropy solution of the IBVP (1)-(3) if (a) $A(u)_x \in L^2(Q_T)$, (b) for almost all $t \in (0,T)$, $\gamma_{x_b}(f(u,\cdot) - A(u)_x) = 0$, $x_b \in \{x_1, x_2\}$, (c) $\lim_{t \downarrow 0} u(x, t) = u_0(x)$ for almost all $x \in (x_1, x_2)$, and (d)

$$\forall \varphi \in C_0^{\infty}(Q_T), \ \varphi \ge 0, \ \forall k \in \mathbb{R} : \iint_{Q_T} \left\{ |u - k|\varphi_t + \operatorname{sgn}(u - k) \left[\left[f(u, x) - f(k, x) - A(u)_x \right] \varphi_x - \left[f_x(k, x) - g(x, u) \right] \varphi \right] \right\} dt dx \ge 0.$$
(6)

Here γ_{x_1} and γ_{x_2} denote the traces with respect to $x \downarrow x_1$ and $x \uparrow x_2$, respectively. Entropy inequalities such as (6) go back to Kružkov [16] and Vol'pert [17] for first order equations and to Vol'pert and Hudjaev [18] for second order equations.

Remark 2. The $BV(Q_T)$ assumption in Definition 1 is only used to ensure the existence of the traces γ_{x_1} and γ_{x_2} . Moreover, note that we can require that the boundary conditions are satisfied in a pointwise sense almost everywhere, whereas Dirichlet boundary conditions such as those stated in [4] (which are not considered here) have to be treated as entropy boundary conditions [7]. Finally, we point out that it is at present not known whether jump conditions for hyperbolic-parabolic equations such as those by Wu and Yin [20] (see also [7]) are valid here, since these jump conditions rely on stronger regularity properties (for example, Lipschitz continuity) of the diffusion coefficient $a(\cdot)$ than is stipulated here.

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motivated by a recent theory of sciencenetics conventiations processes [2]. While the part [4] is concerned with the case of sciencenetics conventiations india, which lends to up equivansimilar to [1] but without a second terms, we have facts at the application distribution of a centrifugal field in a rotating frame of reference [2]. It is the purpose of this scont back to briefly outline how the analysis of strongly repetitive to the second distribution problems problem (IBVF) (1)-(3), and to the material of the scont of the generative contents to the distribution equations. In addition, we unlike an advect of schemes of the generative contents of the distribution problem (IBVF) (1)-(3), and to the material of the schemes of the generative distribution to the equations. In addition, we unlike an advect of schemes calls the generative distribution to the recent distribution of the multice an advect of the generative content and the schemes problem (IBVF) (1)-(3), and to the multice an advection of the generative distribution that distribution equations. In addition, we unlike an advection of the generative distribution of the distribution problem (IBVF) (1)-(3), and to the multice an advection of the generative distribution the distribution recently advecting to [3] to active the JBVF ender calls the schemes distribution of the schemes distribution problem (IBVF) (1)-(3), and to the multice and distribution of the generative distribution of the schemes distribution of the schemes distribution of the generative distribution distribution of the schemes distribution of the schemes distribution of the generative distribution of the schemes distribution of

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Due to both the degeneracy of the diffusion coefficient of 0 and the nonlinearity of f(-z), solutions of (1) are discontinuous and have to be reflued as entropy solutions. **Definition 1**. A Junction $u \in L^{\infty}(Q_{1})$, $(-BV(Q_{2})) = BV(Q_{2})$, is on entropy solution of the 1847 (1)-(3) ψ (a) $d(u)_{2} \in L^{2}(Q_{2})$. (b) for absend all $i \in (0,2)$, $z_{2}(f(u)_{1}) = d(u)_{2}$, i = 0, $z_{3} \in \{x_{1}, x_{2}\}$, (c) limits $u(x, 0) = u_{0}(x)$ for alreast all $i \in (0,2)$, $z_{2}(f(u)_{1}) = d(u)_{2}$, i = 0.

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 $(0) \quad 0 \leq abib \left\{ \left[e[(u, v) - k] e[(u, v)] - 2e[a(u)] - 2e[u(u)] - (v, u)] e \right] \right\} dda \geq 0, \quad (0)$

Here we and ye denote the trace with respect to c. 1. Scand by T. S. respectively. Extropy mequalities such as (0), go fack to Krushov [16] and Vel'part [17]; for first order equations and to Vel'part and Rudidev [18] for second order equations

CENTRIFUGATION OF FLOCCULATED SUSPENSIONS

3. EXISTENCE OF ENTROPY SOLUTIONS

Existence of entropy solutions is shown by the vanishing viscosity method. To this end, consider the regularized uniformly parabolic IBVP, in which the functions f, g and u_0 have been replaced by smooth approximations that ensure compatibility conditions and existence of smooth solutions for fixed $\varepsilon > 0$, and where A^{ε} is defined in (5):

$$u_t^{\varepsilon} + f^{\varepsilon}(u, x)_x = A^{\varepsilon}(u^{\varepsilon})_{xx} + g^{\varepsilon}(u^{\varepsilon}, x), \quad (x, t) \in Q_T,$$
(7a)

$$u^{\varepsilon}(x,0) = u_0^{\varepsilon}(x), \ x \in (x_1, x_2); \ f^{\varepsilon}(u^{\varepsilon}, x_{\rm b}) - A^{\varepsilon}(u^{\varepsilon}(x_{\rm b}, t))_x = 0, \ x_{\rm b} \in \{x_1, x_2\}, \ t \in (0, T].$$
(7b)

Lemma 1. The following uniform estimates are valid for solutions u^{ε} of the regularized *IBVP* (7), where the constants M_1 to M_3 are independent of ε :

$$\|u^{\varepsilon}\|_{L^{\infty}(Q_{T})} \leq M_{1}; \ \|u^{\varepsilon}_{x}(\cdot,t)\|_{L^{\infty}(x_{1},x_{2})} \leq M_{2} \text{ and } \|u^{\varepsilon}_{t}(\cdot,t)\|_{L^{\infty}(x_{1},x_{2})} \leq M_{3} \forall t \in (0,T].$$

$$(8)$$

Sketch of proof. The application of the maximum principle to establish the first estimate is standard and not repeated here; we refer to the proof of Lemma 9 in [4] for details.

Defining $\operatorname{sgn}_{\eta}(\tau) := \operatorname{sgn}(\tau)$ if $|\tau| > \eta$, $\operatorname{sgn}_{\eta}(\tau) := \tau/\eta$ if $|\tau| \le \eta$ and $|x|_{\eta} := \int_0^x \operatorname{sgn}_{\eta}(\xi) d\xi$ for $\eta > 0$, we obtain by differentiating (7a) with respect to x, multiplying the result by $\operatorname{sgn}_{\eta}(u_x^{\varepsilon})$, integrating over $Q_{T_0}, 0 < T_0 \le T$, integrating by parts and using (7b)

$$\int_{x_1}^{x_2} \left| u_x^{\varepsilon}(x, T_0) \right|_{\eta} dx \le \int_{x_1}^{x_2} \left| (u_0^{\varepsilon})'(x) \right|_{\eta} dx + \int_0^{T_0} \operatorname{sgn}_{\eta} \left(u_x^{\varepsilon}(x, t) \right) u_t^{\varepsilon}(x, t) \Big|_{x_1}^{x_2} dt + \iint_{Q_{T_0}} \operatorname{sgn}_{\eta}'(u_x^{\varepsilon}) dx + \int_0^{T_0} \operatorname{sgn}_{\eta}'(u_x^{\varepsilon}) dx + \int_0^{T_$$

$$\times u_{xx}^{\varepsilon} \left[f_u^{\varepsilon}(u^{\varepsilon}, x) - \partial_x \left(a^{\varepsilon}(u^{\varepsilon}) \right) \right] u_x^{\varepsilon} dt dx + \iint_{Q_{T_0}} \operatorname{sgn}_{\eta}(u_x^{\varepsilon}) \left(g_u^{\varepsilon}(u^{\varepsilon}, x) u_x^{\varepsilon} + g_x^{\varepsilon}(u^{\varepsilon}, x) \right) dt dx.$$
(9)

To derive the second inequality of Lemma 1 from (9), we repeat the proof of part (a) of Lemma 11 in [4] to estimate the first three integrals of the right-hand side of inequality (9) for $\eta \to 0$. The integrand of the last term can for $\eta \to 0$ be rewritten as $|u_x^{\varepsilon}|g_u^{\varepsilon} + \operatorname{sgn}(u_x^{\varepsilon})g_x^{\varepsilon}$. Since g_u^{ε} and g_x^{ε} are uniformly bounded due to our assumptions on g, the desired estimate on $||u_x^{\varepsilon}(\cdot, t)||_{L^{\infty}(x_1, x_2)}$ can be established by an application of Gronwall's lemma.

The same argument can finally be employed to extend the derivation of the estimate on $||u_t^{\varepsilon}(\cdot, t)||_{L^{\infty}(x_1, x_2)}$ in Lemma 11 of [4] to the present equation with source term.

From the estimates established in Lemma 1 we may conclude that there exists a sequence $\varepsilon = \varepsilon_n \downarrow 0$ such that the sequence of solutions $\{u^{\varepsilon_n}\}$ of the IBVP (7) converges in $L^1(Q_T)$ to a function $u \in L^{\infty}(Q_T) \cap BV(Q_T)$. We now have to show that u is actually an entropy solution of the IBVP (1)–(3). Part (a) of Definition 1 follows from the following lemma, whose (short) proof is a straightforward extension of that of Lemma 10 in [4]:

Lemma 2. The limit function u of solutions u^{ε} of (7) satisfies $A(u)_x \in L^2(Q_T)$.

Finally, repeating the proofs of Lemmas 5 and 12 of [4], we can show

Lemma 3. The viscosity limit function u of solutions u^{ε} of (7) satisfies (6) and the initial and boundary conditions mentioned in Definition 1.

Summarizing, we have:

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Lemma 4. The following applying contraction in a bar for solutions of the regularized IBVP (7), where the constants Mi 10 W. Swir, independent of a

Defining an (r) = an(r) if $(r) = a and (r) = rr/n if <math>(r) \leq n$ and $(r) = f_1$ and $(r) < f_2$.

$$\int_{-\infty}^{\infty} |u_{n}^{*}(x, T_{n})|_{p} dx \leq \int_{-\infty}^{\infty} |(u_{n}^{*}(x)|_{p})|_{p} dx + \int_{-\infty}^{\infty} |u_{n}^{*}(x)|_{p} dx +$$

 $(6) \quad \text{spage} ((1, 1, 2)) = g^{-1} (1, 1, 2) (1,$

To derive the second inequality of baryons i form (9), we repeat the proof of part ($\omega_1 \omega_2$). Lemma 11 is (6) to estimate the first three integrals of the right-hand side of integrality (9) for $\eta \rightarrow 0$. The integrand of the first term coulder $\eta \rightarrow 0$ be constituting $\omega_1(\eta_2)$, replicitly (9) Since η_1 and η_2 are uniformity bounded. She to out statistic be a set bar is the desired value to $\omega_1(\eta_1)$, $\omega_2(\eta_1)$, $\omega_3(\eta_2)$, $\omega_4(\eta_1)$, $\omega_1(\eta_2)$, $\omega_2(\eta_2)$, $\omega_3(\eta_2)$, $\omega_4(\eta_1)$, $\omega_1(\eta_2)$, $\omega_1(\eta_1)$, $\omega_2(\eta_1)$, $\omega_3(\eta_1)$, $\omega_3(\eta_2)$, $\omega_4(\eta_1)$, $\omega_3(\eta_2)$, $\omega_4(\eta_1)$, $\omega_4(\eta_2)$, $\omega_4(\eta_1)$, $\omega_4(\eta_1$

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BÜRGER AND KARLSEN

Theorem 1. The IBVP (1)–(3) admits an entropy solution u.

4. UNIQUENESS OF ENTROPY SOLUTIONS

After the important work by Carrillo [9], the uniqueness proof for entropy solutions of degenerate parabolic equations has become very similar to the "doubling of variables" proof introduced by Kružkov [16] for hyperbolic equations. The key proposition allowing to apply Kružkov's "doubling device" to second order equations is the following version (see Karlsen and Risebro [14] for its proof) of an important lemma from [9], which identifies a certain entropy dissipation term (i.e., the right-hand side of (10) below).

Lemma 4 (Carrillo's lemma). Let u be an entropy solution of the IBVP (1)-(3). Then, for any non-negative $\varphi \in C_0^{\infty}(Q_T)$ and any $k \in (u_c, 1)$, we have

$$\iint_{Q_T} \left(|u - k|\varphi_t + \operatorname{sgn}(u - k) \left[\left[f(u, x) - f(k, x) - A(u)_x \right] \varphi_x - \left[f_x(k, x) - g(x, u) \right] \varphi \right] \right) dt dx = \lim_{\eta \downarrow 0} \iint_{Q_T} \left(A(u)_x \right)^2 \operatorname{sgn}'_{\eta} \left(A(u) - A(k) \right) \varphi dt dx.$$

$$(10)$$

Equipped this lemma, one can prove the following main theorem:

Theorem 2. If v and u are two entropy solutions of the IBVP (1)-(3), then we have for any $\varphi \in C_0^{\infty}(Q_T), \ \varphi \ge 0$:

$$\iint_{Q_T} \left(|v - u| \varphi_t + \operatorname{sgn}(v - u) \left[f(v, x) - f(u, x) - \left(A(v)_x - A(u)_y \right) \right] \varphi_x + \operatorname{sgn}(v - u) \left[g(v, x) - g(x, u) \right] \varphi \right) dt dx \ge 0.$$
(11)

Sketch of proof. The argument given below relies on Lemma 4 and Kružkov's idea of doubling the number of dependent variables together with a penalization procedure. We let $\varphi \in C^{\infty}(Q_T \times Q_T), \varphi \geq 0, \varphi = \varphi(x,t,y,s), v = v(x,t), u = u(y,s)$, and introduce the "hyperbolic" sets $\mathcal{E}_v = \{(x,t) \in Q_T : v(x,t) \leq u_c \text{ or } v(x,t) \geq 1\}$ associated with v and $\mathcal{E}_u = \{(y,s) \in Q_T : u(y,s) \leq u_c \text{ or } u(y,s) \geq 1\}$ associated with u.

From the entropy inequality for v(x,t) (with k = u(y,t)), the entropy inequality for u(y,s) (with k = v(x,t)), and Lemma 4, the following inequality was derived in [14]:

$$\iiint_{Q_T \times Q_T} \left(|v - u| (\varphi_t + \varphi_s) + \operatorname{sgn}(v - u) [f(v, x) - f(u, y) - (A(v)_x - A(u)_y)] (\varphi_x + \varphi_y) + \operatorname{sgn}(v - u) [g(v, x) - g(u, y)] \varphi \right) dt dx ds dy + E_{\operatorname{Conv}} \\
\geq \lim_{\eta \downarrow 0} \iiint_{(Q_T \setminus \mathcal{E}_u) \times (Q_T \setminus \mathcal{E}_v)} (A(v)_x - A(u)_y)^2 \operatorname{sgn}'_{\eta} (A(v) - A(u)) \varphi dt dx ds dy \ge 0, \quad (12)$$

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Theorem 1. The 187P (1)-(3) admits an entrony solution u.

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After the important work by Carrilio [9], the intercenses press for entropy solutions of degenerate purchashs equations has become very similar to the "doublink of variables" proof introduced by Krustor [16] for hyperiodic texchinas. The key purposition allowing to apply firstiny's "doublest destor" to scend order entrations at the following temposition of Karisen and Risebre [14], for its proof) of an innormal lemma limit [9], when its danking a

Lemma 4 (Cardio's lemma). Let u be an entropy mannel of the 1019 (1) (3). Then, for any non-mounties of C (C) and my kie (a, 2) we have

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quipped this leaded, one can prove the following main theorem.

Theorem 3. If u and u are two entropy additions of the BUPP (1)-(2), then us inversion and $u \in C_{\mathbb{C}}^{\infty}(Q_{P})$, $u \geq 0$:

$$\int_{A_{T}} \left(|u - u|_{A_{T}} + u_{C}(u - u)|_{f(u, x)} - f(u, x) - f(u, x) - u_{C}(u)|_{f(u, x)} - u_{C}(u)|_{f(u)} - u_{C}(u)|_{f(u, x)} - u_{C}(u)|_{f$$

Shelch of point. The arguments given below relies can bound a real Kinishov's then all doubling the number of degradent tatables (optime with a penalistic of botchine. We let $\phi \in C^{\infty}(Q_{T} \times Q_{T}), \phi \geq 0, \phi = \phi(r, t, y, s), y = \phi(r, t), y = \psi(r, s), and intraduct the$ $"hyperbolic" sets <math>\mathcal{L}_{r} = \{t, r, t\} \in Q_{T}: \psi(r, t) \leq u_{r}$ or which \mathcal{L} is an dimension with r and $\mathcal{L}_{r} = \{t, r, s\} \in Q_{T}: u(r, s), t = \psi(r, t) > 1\}$ respectence with \mathcal{L}_{r} .

From the entropy inequality for q(x,t) (with t = u(x,t)), the entropy inequality for u(y, c) (with t = u(x,t)), and Lemma 4, the following inequality was derived in [14]

$$\iint_{x_{n-1}} \left((x_{n-1}) + (y_{n-1}) + (y_{n-1})$$

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where the "error term" E_{Conv} takes the form

$$E_{\text{Conv}} = \iiint_{Q_T \times Q_T} \operatorname{sgn}(v - u) \left(\left[\left(f(u, y) - f(u, x) \right) \varphi \right]_x - \left[\left(f(v, x) - f(v, y) \right) \varphi \right]_y \right) dt dx ds dy.$$

We are now on familiar ground [16] and introduce in (12) the test function

$$\varphi(x,t,y,s) = \varphi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_h\left(\frac{x-y}{2}\right)\delta_h\left(\frac{t-s}{2}\right),$$

where $\{\delta_h\}_{h>0}$ is a standard regularizing sequence in \mathbb{R} . Observe that

$$\varphi_t + \varphi_t = \partial_t \varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) \delta_h\left(\frac{x-y}{2}\right), \quad \varphi_x + \varphi_y = \partial_x \varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) \delta_h\left(\frac{x-y}{2}\right) \delta_h\left(\frac{t-s}{2}\right).$$

Following [14] (see also [16] since $f(u, \cdot)$ is smooth), one can show that $\lim_{h \downarrow 0} E_{\text{Conv}} = 0$. Consequently, by sending $h \downarrow 0$ in (12), we get (11).

Define for sufficiently small h > 0 the functions

$$\rho_h(x) := \int_{-\infty}^{\infty} \delta_h(\xi) \, d\xi, \ \ \mu_h(x) := 1 - \rho_h(x - x_1 - 2h), \ \ \nu_h(x) := \rho_h\big(x - (x_2 - 2h)\big), \ (13)$$

where $\{\delta_h\}_{h>0}$ is again the standard regularizing sequence in \mathbb{R} . Concerning these functions, we have the following lemma (whose proof is easy):

Lemma 5. Let $u \in L^1(0,T; L^{\infty}(x_1,x_2))$. If the traces $\gamma_{x_1}u := (\gamma u)(x_1,t)$ and $\gamma_{x_2}u := (\gamma u)(x_2,t)$ exist a.e. in (0,T), then we have for $\varphi \in C^{\infty}(Q_T)$

$$\lim_{h \downarrow 0} \iint_{Q_T} \partial_x \Big(\varphi(x,t) \big(1 - \mu_h(x) - \nu_h(x) \big) \Big) u(x,t) \, dt dx = \int_0^T \Big(\varphi(x_1,t) \gamma_{x_1} u - \varphi(x_2,t) \gamma_{x_2} u \Big) \, dt.$$

We are now in a position to deduce from (11) the following uniqueness result:

Corollary 1. Let v, u be entropy solutions of the IBVP (1)–(3) with initial data v_0, u_0 , respectively. Then for all $t \in (0,T)$, $||v(\cdot,t)-u(\cdot,t)||_{L^1(x_1,x_2)} \leq \exp(t||g||_{\text{Lip}})||v_0-u_0||_{L^1(x_1,x_2)}$. In particular, the IBVP (1)–(3) admits at most one entropy solution.

Proof. In (11), we choose $\varphi(x,t) = ((1 - \mu_h(x) - \nu_h(x))\chi(t)$ with $\chi \in C_0^{\infty}(0,T), \chi \ge 0$, and μ_h and ν_h defined in (13). Note that φ tends to $\chi(t)$ as $h \downarrow 0$. Taking the limit $h \downarrow 0$, we obtain from Lemma 5 and the boundary conditions at x = a, b (see Definition 1):

$$\iint_{Q_T} |u - v|\chi'(t) \, dt dx \ge -\iint_{Q_T} \operatorname{sgn}(v - u) \big[g(v, x) - g(u, x) \big] \chi(t) \, dt dx. \tag{14}$$

From (14), it follows that

ar

$$-\iint_{Q_T} |u-v|\chi'(t) \, dt dx \le ||g||_{\operatorname{Lip}} \iint_{Q_T} |u-v|\chi(t) \, dt dx. \tag{15}$$

Fixing $\tau \in (0, T)$, choosing $\chi(t)$ as $\rho_h(t) - \rho_h(t - \tau)$ in (15), subsequently sending $h \downarrow 0$, and using Gronwall's lemma, we get the L^1 stability estimate $||v(\cdot, \tau) - u(\cdot, \tau)||_{L^1(x_1, x_2)} \leq \exp(\tau ||g||_{\text{Lip}}) ||v(\cdot, 0) - u(\cdot, 0)||_{L^1(x_1, x_2)}$. Since $\tau \in (0, T)$ was arbitrary, we are finished.

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where the "error term" Equilates the form

$$E_{com} = \iiint_{avec} = a([O(azu) - f(a,z))a]_{c} - (O(a)z) - f(a,z))ad_{c}$$

We are now on familian ground [16] and introduce in [12], the test function

where {or have is a standard requiations: someace in R. Observe that

Following [14] (see also [16] since $f(n, \cdot)$ is accordib), one can show that he holds $\beta_{n,n,n} = 0$. Consequently, by sending h 1.0 in (18), we get (21)

Define for sufficiently small h > 0 the functions.

$$p_{n}(z) := \int_{-\infty}^{\infty} h_{n}(z) dz; \quad h_{n}(z) := 1 - p_{n}(z - z) + 2p_{n}(z) h_{n}(z) := p_{n}(z - z_{n}). \quad (3.2)$$

where $\{a_i\}_{i=0}$ is again the standard regularizing sequence to R. Concerning these functions, we have the following lemma (where prior is easy): Lemma 5. Let $u \in L^1(0,T; L^{\infty}(x_1, b_1))$. If the frame $y_0 u := (y_0)(x_1, t)$ and $y_0 u := (y_0)(x_1, t)$

$$\lim_{n \to \infty} \int_{\Omega_{n}} \partial_{n} \left(\varphi(\alpha, \alpha) (1 - \alpha_{n}(\alpha) - \alpha_{n}(\alpha)) \right) \varphi(\alpha, \alpha) d\alpha \alpha = \int_{\Omega_{n}} \left(\varphi(\alpha_{n}, \alpha) \alpha_{n} - \varphi(\alpha_{n}, \alpha) \alpha_{n} \right) d\alpha$$

We are seen in a position to deduce it or (11) the following caling case result. Cordilary 4. Let u, or ic quaranty solutions of the (SNT (1)-(1)-(1) with contained at a second respectively. Then for all t 6 (6/7), shot (1)-all (1) is a second second of the second at the second is perticular. The (BIT? (1)-(3) advants at mind there existing a second second at the second Proof, in (11), we showed u(c.t) = ((1-(1-(1))), (c)) with y c. (27(0, 7)), (y > 0), and ps, and a defined in (13). Note that uncertained y (c) as in (1). Taking the liner to (1) we obtain from 1 corns 5 and the france uncertained process of (c) as in (1).

From (14), it follows that

Fixing $r \in (0, T)$, choosing $\chi(t)$ as $\mu(t) = \mu(t - \tau)$ in (25), subsequently evolving h (0), and using Gronwall's boundar, we get the L¹ chability estimate [bd, τ) - $\mu(\tau T)$][span and $\leq q = 0$ (τ] due to $\tau = 0$. T) was arbitrary, we are finited as

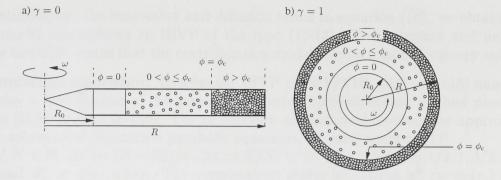


FIGURE 1. (a) Rotating tube, (b) rotating axisymmetric cylinder.

Remark 3. Following [10, 13, 14], it is possible to prove that the unique entropy solution of the IBVP (1)-(3) depends continuously on the nonlinearities in the problem.

5. BATCH CENTRIFUGATION OF FLOCCULATED SUSPENSIONS

5.1. Introduction. Extending the arguments developed in [8] for a purely gravitational force to rotating systems, Bürger and Concha [3] show that the settling of a flocculated suspension in a tube with constant cross section ($\gamma = 0$, Fig. 1 (a)) or in a rotating axisymmetric cylinder ($\gamma = 1$, Fig. 1 (b)) is governed by the field equation

$$\partial_t \phi + r^{-\gamma} \partial_r \left(f_{\rm ck}(\phi) \omega^2 r^{1+\gamma} \right) = r^{-\gamma} \partial_r \left(r^{\gamma} \partial_r A(\phi) \right), \tag{16}$$

in which ϕ is the sought volumetric solids concentration, r the radius, $f_{\rm ck}(\cdot)$ the centrifugal Kynch flux density function, ω the angular velocity and $A(\cdot)$ is defined as in (1). The function $f_{\rm ck}(\cdot)$ is a nonnegative Lipschitz continuous function with support in [0, 1], and the diffusion coefficient is defined by $a(\phi) := f_{\rm ck}(\phi)\sigma'_{\rm e}(\phi)/(\Delta\rho\phi)$, where $\Delta\rho > 0$ is the solid-fluid mass density difference and $\sigma'_{\rm e}(\cdot)$ is the derivative of the effective solid stress function. We mention that equation (1) is based on the neglection of both the gravitational and Coriolis forces compared to the centrifugal force and refer to [3] for details on its derivation.

Equation (16) inherits its degeneracy from the constitutive assumption that $\sigma_{\rm e}(\cdot)$ vanishes while the solid flocs are not in touch with each other, i.e., while the local concentration ϕ does not exceed a critical value $\phi_{\rm c}$, and that $\sigma'_{\rm e}(\phi) > 0$ for $\phi_{\rm c} < \phi \leq 1$. Assuming for simplicity that supp $f_{\rm ck} = (0, 1)$, we see that $a(\phi) = 0$ and hence (16) is of hyperbolic type for $\phi \leq \phi_{\rm c}$ and $\phi \geq 1$ and that otherwise $a(\phi) > 0$, i.e., (16) is of parabolic type. Consequently, (16) is indeed strongly degenerate. The special case $a \equiv 0$ is included in our discussion and corresponds to the equation studied by Anestis and Schneider [1].

We assume that r varies between an inner and outer radii $R_0 > 0$ and $R > R_0$. The solids phase velocity vanishes at $r = R_0$ and r = R, which implies the boundary conditions

$$(f_{\rm ck}(\phi)\omega^2 R_{\rm b} + \partial_r A(\phi))(R_{\rm b}, t) = 0, \quad t > 0, \ R_{\rm b} \in \{R, R_0\}.$$
(17)

The initial condition is

$$\phi(r,0) = \phi_0(r), \quad R_0 \le r \le R.$$
(18)



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5.1. Instroctastion. Extending the regiments developed in [8] for a purely gravitational force to rotating systems, Bilryer and Conclus [3] show this the relating of a forcinisted surplemator in a rotating surplemator in a rotating system (r = 0, Fig. 1 (a.), or in a rotating system (r = 0, Fig. 1 (a.), or in a rotating system (r = 0, Fig. 1 (a.), or in a rotating system (r = 0, Fig. 1 (a.), or in a rotating system (r = 0, Fig. 1 (a.), or in a rotating system (r = 0, Fig. 1 (a.), or in a rotating system (r = 0, Fig. 1 (a.), or in a rotating system (r = 0, Fig. 1 (a.), or in a rotating (r = 0, Fig. 1 (a.), or in a rotating (r = 0).

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In which q is the sought which with solid sailing concentration, r the radius $g \in \{-1, the centrifush <math>K$ and K_{1} as the sought which q is the centrifush K and K_{2} and $g \in \{-1, the centrifush <math>K$ and K_{2} and K_{3} as the centrifush K and the concentration $(g, \{-1, the centrifush <math>K$ and K_{3} and K_{4} and K

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We assume that r verice between as inner and ever, radii $R_{1} > 0$ and $R > R_{2}$. The solids phase velocity variebes at $r = R_{1}$ and $r = R_{2}$ which implies the boundary conditions

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Differentiating out the convection and diffusion terms in equation (16), we obtain in view of the model assumptions an IBVP of the type (1)–(3). The existence and uniqueness analysis therefore states that the centrifugation model admits a unique entropy solution ϕ .

5.2. Numerical algorithm. To solve the IBVP given by (16), (17) and (18) numerically, we employ a modification of the generalized upwind finite difference method presented in detail in [5] for gravity settling. For an overview of numerical methods for approximating entropy solutions of degenerate parabolic equations, we refer again to [11].

Let $J, N \in \mathbb{N}$, $\Delta r := (R - R_0)/J$, $\Delta t := T/N$, $r_j := R_0 + j\Delta r$, $j = 1/2, 1, 3/2, \ldots, J - 1/2, J$ and $\phi_j^n \approx \phi(r_j, n\Delta t)$. The computation starts by setting $\phi_j^0 := \phi_0(r_j)$ for $j = 0, \ldots, J$. Assume then that values $\phi_j^n, j = 0, \ldots, J$ at time level $t_n := n\Delta t$ are known. To compute the values ϕ_j^{n+1} , we first compute the extrapolated values $\phi_j^{\text{L}} := \phi_j^n - (\Delta r/2)s_j^n$ and $\phi_j^{\text{R}} := \phi_j^n + (\Delta r/2)s_j^n$ for $j = 1, \ldots, J - 1$, where the slopes s_j^n can be calculated, for example, by the minmod limiter function $M(\cdot, \cdot, \cdot)$ in the following way:

$$s_j^n = \mathrm{MM}\left(\phi_j^n - \phi_{j-1}^n, (\phi_{j+1}^n - \phi_{j-1}^n)/2, (\phi_{j+1}^n - \phi_j^n)\right)/\Delta r, \quad j = 2, \dots, J-2.$$
(19)

where $MM(a, b, c) = \min\{a, b, c\}$ if a, b, c > 0, $MM(a, b, c) = \max\{a, b, c\}$ if a, b, c < 0 and MM(a, b, c) = 0 otherwise. Moreover we set $s_0^n = s_1^n = s_{J-1}^n = s_J^n = 0$.

The extrapolated values $\phi_j^{\rm L}$ and $\phi_j^{\rm R}$ appear as arguments of the numerical centrifugal Kynch flux density function $f_{\rm ck}^{\rm EO}(\cdot, \cdot)$ which, according to the Engquist-Osher scheme, is defined by $f_{\rm ck}^{\rm EO}(u, v) := f_{\rm ck}^+(u) + f_{\rm ck}^-(v)$, where $f_{\rm ck}^+(u) := f_{\rm ck}(0) + \int_0^u \max\{f_{\rm ck}'(s), 0\} ds$ and $f_{\rm ck}^-(v) := \int_0^v \min\{f_{\rm ck}'(s), 0\} ds$. The interior scheme, which approximates the field equation (16) and from which the values $\phi_1^n, \ldots, \phi_{J-1}^n$ are calculated, can then be formulated as

$$\phi_{j}^{n+1} = \phi_{j}^{n} - \frac{\omega^{2} \Delta t}{r_{j}^{\gamma} \Delta r} \Big[r_{j+1/2}^{1+\gamma} f_{ck}^{EO}(\phi_{j}^{R}, \phi_{j+1}^{L}) - r_{j-1/2}^{1+\gamma} f_{ck}^{EO}(\phi_{j-1}^{R}, \phi_{j}^{L}) \Big] + \frac{\Delta t}{r_{j}^{\gamma} \Delta r^{2}} \Big[r_{j+1/2}^{\gamma} \Big(A(\phi_{j+1}^{n}) - A(\phi_{j}^{n}) \Big) - r_{j-1/2}^{\gamma} \Big(A(\phi_{j}^{n}) - A(\phi_{j-1}^{n}) \Big) \Big], \ j = 1, \dots, J-1.$$

$$(20)$$

The boundary formulas follow by considering (20) for j = 0 and j = J and inserting the discrete versions of the boundary conditions (17). This leads to

$$\phi_0^n = \phi_0^{n-1} - \frac{\omega^2 \Delta t}{R_0^{\gamma} \Delta r} r_{1/2}^{1+\gamma} f_{\rm ck}^{\rm EO}(\phi_0^n, \phi_1^n) + \frac{\Delta t}{R_0^{\gamma} \Delta r^2} r_{1/2}^{\gamma} \left(A(\phi_1^n) - A(\phi_0^n) \right), \tag{21}$$

$$\phi_J^n = \phi_J^{n-1} + \frac{\omega^2 \Delta t}{R^\gamma \Delta r} r_{J-1/2}^{1+\gamma} f_{\rm ck}^{\rm EO}(\phi_{J-1}^n, \phi_J^n) - \frac{\Delta t}{R^\gamma \Delta r^2} r_{J-1/2}^\gamma \left(A(\phi_J^n) - A(\phi_{J-1}^n) \right).$$
(22)

To ensure convergence of the numerical scheme to the entropy weak solution of the IBVP, the CFL stability condition $R\omega^2 \max_{\phi} |f'_{ck}(\phi)| (\Delta t/\Delta r) + 2 \max_{\phi} a(\phi)(\Delta t/\Delta r^2) \leq 1$ must be satisfied. In this work, this condition was ensured by selecting Δr freely and determining Δt appropriately. The accuracy was J = 400. For more details about the upwind method and its convergence analysis, we refer to [12, 15].

5.3. Numerical example. Sambuichi *et al.* [19] published centrifugation experiments with three different flocculent aqueous suspensions, namely of limestone, yeast, and clay, using a cylindrical centrifuge. For each material, the measured gravitational settling rates led to a function $f_{ck}(\phi)$, and compression data determined a unique effective solid stress function $\sigma_{e}(\phi)$ for each material. In this paper, we choose the published data referring

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Differentiating out vite convection and diffusion terms in optation (19), we obtain in view of the model assumptions an HEVP of the type (1)-12). The function and viriqueness analysis therefore states that the centribugation model admits a unique entries volunion d

5.2 Numerical algorithm, 'Ib solve the HNT givin by (15), (17) and (18) numerically we employ a modification of the generated sparned finite difference method presented in detail in (5) for gravity settling. For an overview of sectors we have been overview of sectors and the sector of the sector settling. For an overview of sectors we have been overview of sectors and the sector of the sector settling. For an overview of sectors we have been overview of sectors and the sector overview of sectors and sectors and the sectors of sectors and the sectors and the sectors of sectors and the sectors of sectors and the sectors and sectors and the sectors and the sectors and sectors and the sectors of sectors and the sectors are sectors and the sectors and the sectors and the sectors are sectors and the sectors and the sectors are sectors and the sectors and the sectors are sectors and the sectors are sectors are set to set to sectors are set to set to sectors are set to set to set to set to sectors are set to set to

Let $I_i N \in \mathbb{N}$, $\Delta r := (R - R_0)/J$, $\Delta I := T/N$, $r := R_0 + r \Delta r_1 r = 1/2$, $I_i N/2$, r = 1/2, $I_i N/2$, $I_i N/2$,

 $= MM(\phi_{1}^{*} - \phi_{1-1}^{*}, (\phi_{1+1}^{*} - \phi_{1-1}^{*})/2, (\phi_{1+1}^{*} - \phi_{1+1}^{*})/2, (\phi_{1}^{*} - \phi_{1-1}^{*})/2, (\phi_{1}^{*$

where MM(a, a, c) = min(a, b, c) if a, b, c > 0, MM(a, b, c) = man(c, b, c) if a, b, c < 0 and MM(a, b, c) = 0 otherwise Maximum (a, b, c) = 0 and MM(a, b, c) = 0.

Kynch Bux density function $\langle \beta^{(1)}_{1}(\alpha) \rangle$ which, according to the Engquist Chier educit, is defined by $f_{2}^{(0)}(\alpha, \gamma) := f_{2}^{(1)}(\alpha) - f_{2}^{(1)$

The boundary formulas follow by constituting (20) for j = 0 and j = 1/3 and j, interview the discrete versions of the boundary conditions (17). This fields to

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5.3. Permetrical researches statebuilds at al. [19] published cuitchingshorn experiments with three different florening aquivale properties and distribute of florestance resist, and dist neargy a cylindrical constillance. For such material, the menoided gravitationed and state led to a function (2, (a), and construction data description in florestar relative solid stress fluction c (b) for raid material. In this pape, we decee the validation data description of the state of stress

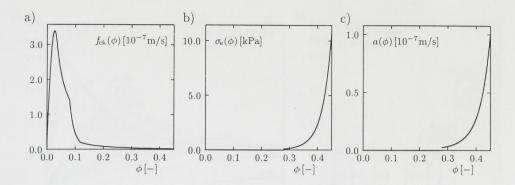


FIGURE 2. The functions (a) $f_{ck}(\phi)$, (b) $\sigma_{e}(\phi)$ and (c) the resulting function $a(\phi)$ used for simulating centrifugation of a limestone suspension.

to a limestone suspension (see [3] for the case of a clay suspension). Sambuichi *et al.* [19] approximated the measured gravity settling rates for various initial concentrations by three different connecting straight segments in a logarithmic plot, which yields the function

$$f_{\rm ck}(\phi) = \begin{cases} (-47.923\phi^2 + 2.5474\phi) \times 10^{-5} \,\mathrm{m/s} & \text{for } 0 \le \phi \le 0.035, \\ 1.3580 \times 10^{-8}\phi^{-0.92775} \,\mathrm{m/s} & \text{for } 0.035 < \phi \le 0.08, \\ 5.6319 \times 10^{-13}\phi^{-4.9228} \,\mathrm{m/s} & \text{for } 0.08 < \phi \le 0.119, \\ 5.9735 \times 10^{-10}\phi^{-1.65} \,\mathrm{m/s} & \text{for } 0.119 < \phi \le \phi_{\rm max} = 0.45, \\ 0 & \text{otherwise.} \end{cases}$$
(23)

It should be pointed out that being cut at ϕ_{\max} , this function does *not* satisfy all assumptions on f stated in Sect. 1. However, due to the presence of the diffusion term, solution values are bounded away from ϕ_{\max} , so the numerical results presented below would look the same if the jump of the function f_{ck} at $\phi = \phi_{\max}$ had been smoothed out in order to produce an example in which these assumptions are precisely satisfied.

The solid pressure relationship suggested in [19] can be converted into the function

$$\sigma_{\rm e}(\phi) = 0 \text{ for } \phi \le \phi_{\rm c} := 0.28, \quad \sigma_{\rm e}(\phi) = 0.30184(1-\phi)^{-17.544} \text{ Pa for } \phi > \phi_{\rm c}.$$
 (24)

The density difference for this material was $\Delta \rho = 1710 \text{ kg/m}^3$. The functions f_{ck} and σ_e given by (23) and (24) and the resulting diffusion coefficient $a(\cdot)$ are plotted in Figure 2.

Figure 3 shows numerical solutions of the phenomenological model calculated with the functions (23) and (24) in the case of a rotating cylindrical vessel ($\gamma = 1$). The left column of Figure 3 shows numerical settling plots, i.e., diagrams of iso-concentration lines for selected values of ϕ , and the right column displays concentration profiles at selected times. The parameters and the data that differ in the three cases considered, viz. $\phi_0 = 0.111$ and $\omega = 146.4 \text{ rad/s}; \phi_0 = 0.138$ and $\omega = 146.4 \text{ rad/s}; \phi_0 = 0.138$ and $\omega = 104.9 \text{ rad/s},$ were chosen in such a way that the simulated supernate-suspension interfaces could be compared with measurements by Sambuichi *et al.* [19], which are shown as open circles (\circ). Figure 3 thus illustrates the different effects of initial concentration and angular velocity on the dynamics of the centrifugation process.



FIGURE 2. The functions (a) (g(d), (b) or a) and (c) the resulting inaction: a(d) used for simulating contribution of a lineators and manyor.

to a litrastone suspension (ese. [3] for the case of a cley magnifical, Samblind) or al. [10] approximated the measured gravity settling materia in which which which valids the finance of different connecting straight segments in a logarithmic plot, which valids the insertion

It should be pointed out that being cut us space, this function done not set inty all assumption tions on f stated in Sect. 1. However, the to the frequence of the diffusion varies polytoin values are bounded away from prace as the measured if texales presented before would look the same if the jump of the function for all of = they, had been star-outed act in order to produce as example in which these issues that are precisely rationed.

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 $\sigma_{i}(\phi) = 0$ for $\phi \leq \phi_{i}$ as 0.28, $-\alpha_{i}(\phi) = 0.30188(1 - \phi_{i})^{-1/2} \log (\alpha_{i} + \beta_{i}) \cos \phi \leq 13.02$

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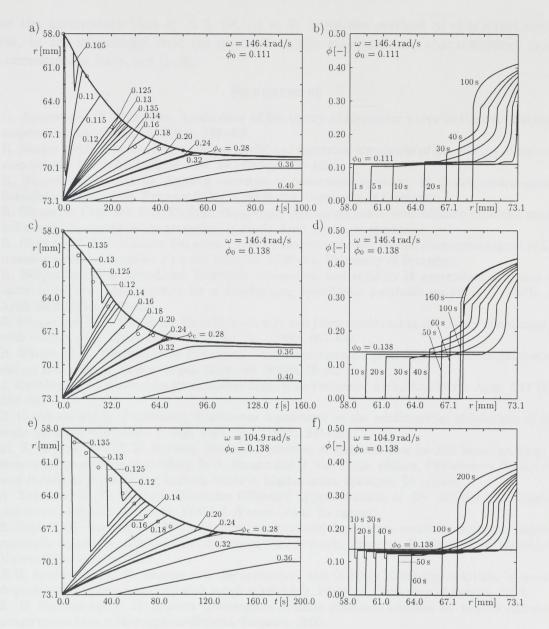


FIGURE 3. Numerical simulation of the centrifugation of a flocculated suspension.

While in the compression zone, where $\phi > \phi_c$ is valid and hence (16) is parabolic, the solutions are similar to those of the pure gravity case [2, 5], there are some distinctive features visible in the hindered settling zone ($\phi \le \phi_c$) where (16) is hyperbolic, due to the rotating frame of reference. Most notably, the vertical iso-concentration lines indicate that the concentration of the bulk suspension is a (decreasing) function of time, and the supernatesuspension interface has a curved trajectory. These properties have previously been found by Anestis and Schneider [1], who determined exact solutions to the centrifugation model

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PIOURS 3. Numerical simulation of the Countlings of a force later introduction

While in the consponsion same, where d is wild and hence (16) is parabolic, the tointions are similar to these of the pluse gravity case [2, 5], there are seen distinctive fraction visible in the hindered softling to us [6, 5] of 1 where (19) is investigation, due to the rotating frame of reference. Must monoton is a (descarsion) function of time, and the supersult concentration of the hull many same transcence is a (descarsion) function of time, and the supersult superstant interface has a function of a (descarsion) function of time, and the supersult superstant interface has a function transcence files of the superstant in the sume, and the supersult by Anastia and Schoelder [1] whet supersuited transcence supersulted base provided been included by Anastia and Schoelder [1] whet dimension is a constant solution in the the contribution would be an an supersulted to the supersult of the supersulted base of the contract of the supersulted by Anastia and Schoelder [1] whet dimension is a contract of the supersulted base of the contribution to the

under the assumption that $\sigma_{\rm e} \equiv 0$, i.e. $A \equiv 0$, using the method of characteristics. Of course, in the centrifugal case (in contrast to the gravitational) characteristics are *not* iso-concentration lines, see [1, 3].

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under the assumption that $\sigma_i \approx 0$ i.e. $A \approx 0$, using the method of characteristics (0) course, in the contributing case (in constant to the gravitational) therefore the interaction in i_i iso-concentration lines are [1, 3].

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