Department of APPLIED MATHEMATICS

APPLICATION OF THE GALERKIN'S METHOD ON THE PROBLEM OF CELLULAR CONVECTION INDUCED BY SURFACE TENSION GRADIENTS.

by

Einar Mæland

Report No.30

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UNIVERSITY OF BERGEN Bergen, Norway



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Abstract.

We consider the problem of cellular convection induced by surface tension gradients. The solutions are expanded in Fourier series, and we use a modification of the Galerkin's method for the case of natural boundary conditions to determine the Fourier coefficients. The problem is reduced to a system of ordinary differential equations, and some explicit calculations are carried out to illustrate the nonlinear effects. APPLICATION OF THE GALERALMENTS RAFERED ON THE FRONTEN OF GENERAL CONVERTING

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1. Introduction.

We study the cellular convection in a horizontal fluid layer heated from below, and allow for surface tension gradients resulting from temperature variations at a free surface. The fluid is assumed to be infinite in horizontal extent.

- 1 -

We assume that the Fourier decompositions of the convective motions in the horizontal (x,y) directions can be represented by the wave numbers ia₁ and ja₂,i,j = 1,2,3,... Choosing the decomposition in the vertical (z) direction at will, the solutions are sought as (truncated) Fourier series. We use a modification of the Galerkin's method for the case of natural boundary conditions to determine the Fourier coefficients. - Since the explicit calculations required for a non-linear analysis of this problem can be very long, we have used as our model the limiting case of an infinite Prandtl number.

In this way we obtain the equations (14a,b,c) which govern the Fourier coefficients (the amplitude equations). - In what follows, we use a decomposition of the convective motions in the z-direction given by the eigenvalue problems (15a,b,c). We consider a state which differs slightly from the onset of convection, and carry out some preliminary calculations to illustrate the non-linear problem. We obtain the amplitude equations(20a,b) which have the same form as those discussed by Segel and Stuart (1962). For the situation to which our analysis applies, hexagonal convection cells may be attributed to surface tension gradients.

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2. Formulation of the problem.

Referring to Cartesian coordinates (x,y,z) with the z-axis taken to be in the vertical direction, we consider a horizontal fluid layer of infinite horizontal extent bounded by a rigid plane at z = -h and a non-deformable free surface at z = 0. The lower boundary is taken to be a solid body at rest which has a constant temperature T_1 . At the free surface we consider the fluid to be in contact with an inviscid atmosphere which has a constant atmospheric pressure p_a . It is assumed that the fluid density and the surface tension are the only physical properties which vary with temperature.

Neglecting the dissipation of energy due to viscosity, the governing equations in the Boussinesq approximation are

(1a)
$$\rho_{\rm m}(\partial \underline{u}/\partial t + \underline{u} \cdot \nabla \underline{u}) = -\nabla p + \mu \nabla^2 \underline{u} - \rho_{\rm m}(1 - \alpha (T - T_{\rm m})) \underline{gk}$$

(1b)
$$\rho_m c_p (\partial T / \partial t + \underline{u} \cdot \nabla T) = \lambda \nabla^2 T$$

(1c)
$$\nabla \cdot \underline{u} = 0$$

where $\underline{u} = (u,v,w)$ is the velocity, p is the pressure, T is the temperature, g is the acceleration of gravity, \underline{k} is the vertical unit vector, ρ_m is a constant reference density and T_m a constant reference temperature. α , c_p , λ and μ are the coefficient of thermal expansion, the specific heat at constant pressure, the coefficient of heat conduction and the coefficient of viscosity, respectively. - (The thermal diffusivity $\lambda/\rho_m c_p$ and the kinematic viscosity μ/ρ_m ,

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Representation to Chineseian coordinations (22,2,2) which which the results of the to be in the certical direction, we consider a portaon of the base of infinite bardsontal esterit bounded by a night pinite of a sector deformable lines autiface at 2 = 0. The base boundary is taken to be it solid body at rost which have a consider the finite in the solid body at rost which consider the finite is its conductive with an insistent at above which also a constants and a solid body at rost which base only physical is the instants the solid body at rost which the base only physical is the the conductive with an insistent tenescol at actioned the the finite defender and the minister tenescol at the only physical of the defender with an insister tenescol at the solid base the finite defender and the solid base of the anticket the solid base the finite defender and the solid base of the anticket the solid base and the defender at the anticket tenescol at the solid base of the defender of the finite at the anticket tenescol at the solid base of the defender of the formation of the anticket tenescol at the solid base of the defender of the formation of the anticket tenescol at the solid base of the defender of the formation at the solid base of the anticket at the solid base the solid base of the finite in the formation of an the solid base of the anticket at the solid base of the solid base of the anticket at the solid base of the anticket at the solid base of the solid base of the anticket at the solid base of the solid bas

where y = (x, x, x) is the velocity of a the oreanie, i is the sist temperature, of is the accelerability of gravity, i is the velocitiest unit vectors of a constant reference detaily and in a constant reference contention of y = 1 and y = 200the constant presence contention of y = 1 and y = 200the constant presence of the restriction and the constant parameter, the constant of both contains and the constant parameter, the constant of both contains and the constant parameter, the constant of both contains and the constant of the constant, respectively. - (the thermal constants of viscontor, respectively - (the thermal will be denoted by χ and ν , respectively).

The kinematic boundary condition at the free surface is that the normal component of the velocity vanishes

(2a)
$$u \cdot n = 0$$
, $(n \equiv k)$ at $z = 0$

The dynamical free-surface condition is imposed by the requirement that the viscous stress on the two sides of the surface can differ only as a result of surface tension

(2b)
$$-(p-p_a)\underline{n} + \underline{P} \cdot \underline{n} = \nabla_s \sigma \quad \text{at} \quad z = 0$$

where \underline{P} is the viscous stress tensor in the fluid, σ is the surface tension and ∇_s is the surface gradient operator.

The transport of heat across the free surface is supposed to be proportional to the temperature difference between the boundary and the adjacent medium. Denoting the temperature in the adjacent medium by T_a , this can be written

(2c)
$$-\lambda n \cdot \nabla T = \kappa (T - T_{2})$$
 at $z = 0$

where κ is the heat-transfer coefficient, assumed constant. At the lower boundary, the boundary conditions are

(2d,e)
$$\underline{u} = 0$$
, $T = T_1$ at $z = -h$

where T1 is a prescribed temperature.

The difficulties in formulating proper boundary

will be denoted by Y and ys remeduely.

le the normal component of the velocity vehicles

$$(2a) = 2 \quad (2 = 2) \quad (2 = 2 \cdot 2) \quad (as)$$

The dynamical free-surface condition is imposed by the requirement that the viscous stress on the two sides of the surface can differ the viscous a result of surface tending.

$$(25) \quad -(p-2_{2}) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \nabla_{0} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2$$

where i is the viscous stress tensor at the little of the the surface tension and V, is the surface gradient operator. The transport of heat across the free surface is supposed to be proportional to the temperature difference between the boundary and the adjusct medium. Manoidary the

(2c)
$$-32 \cdot \nabla T = \kappa (T - T_{g})$$
 as $n = 0$

where of the heat-transfer coefficient, assumed constant.

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conditions as $x, y \rightarrow \pm \infty$ are circumvented by limiting the discussion to solutions which are spatially periodic in these directions. The interval over which this periodicity takes place is, however, not known apriori.

When no motion is present, $\underline{u} \equiv \underline{0}$, the temperature distribution which satisfies (1b) and (2c,e) is

$$T = T_{s}(z) = T_{0} - \beta z$$

where $T_0 = T_s(0) = T_a + \beta \lambda / \kappa$ and $\beta = (T_1 - T_0) / h$. - The pressure distribution $p = p_s(z)$ is not explicitly required, we note, however, that $p_s(0) = p_a$.

The stability problem is now formulated in the usual manner by substituting

$$u = u'$$
, $T = T_{s}(z) + T'$, $p = p_{s}(z) + p'$

into the governing equations (1a-c) and boundary conditions (2a-e). - Assuming that the surface tension can be regarded as a linear function of temperature, this can be written

$$\sigma(\mathbf{T}) = \sigma(\mathbf{T}_{O}) + \gamma(\mathbf{T} - \mathbf{T}_{O})$$

where T_0 is the undisturbed temperature at z = 0, and $\gamma = d\sigma/dT$ evaluated at T_0 . For most fluids the surface tension decreases with increasing temperature, i.e. $\gamma < 0$.

Measuring the velocity <u>u</u>', temperature T', pressure p' and the time and length in units of χ/h , βh , $\mu \chi/h^2$,

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condicions as 2.5 as a are directivented of listons the discussion to colutions which are spatially paylodic is these directions. The interval over which this partodicaty takes place is, however, are more from forieri.

When no motion is present, $\underline{u} = \underline{0}$, the temperature.

where $E_0 = E_0 [0] = E_0 + 0$ is and $0 = (E_1 - E_0)/L_1 - 1$ is pressure distribution $p = p_0(z)$ is not explicitly required, we note, however, that $p_0(0) = p_0$

manner by substituting

 $\underline{a} = \underline{a}^{*}, \quad \underline{a} = \underline{a}^{*} (\underline{a}) + \underline{a}^{*} (\underline{a}) + \underline{b}^{*} (\underline{a}) + \underline{$

into the governing equations (12-0) and boundary conditions (22-4). - Assuming this bas surfice tension can be regarded

o(t) = o(t_ot + v(t)-t_ot)o

where To is the undisturbed temperature at is a 0, and yet dove the evaluated of To. For most flatds the surfate tanaloh dromeases with increding temperature, 1.0. years the Measuring the velocity of , temperature T', pressing

of and the time and length in analis of Time Bh. Dates -

 h^2/χ and h , respectively, and dropping the primes, the governing equations are:

(3a)
$$P^{-1}(\partial \underline{u}/\partial t + \underline{u} \cdot \nabla \underline{u}) = -\nabla p + \nabla^2 \underline{u} + RT\underline{k}$$

(3b) $\partial T/\partial t + \underline{u} \cdot \nabla T = \nabla^2 T + \underline{u} \cdot \underline{k}$

$$(3c) \qquad \nabla \cdot \underline{u} = 0$$

with the boundary conditions

(4a,b) $\underline{u} \cdot \underline{n} = 0$, $-\underline{p}\underline{n} + \underline{P} \cdot \underline{n} = -M\nabla_{s}T$ at z = 0

$$(4c) n \cdot \nabla T + LT = 0 at z = 0$$

(4d,e) u = 0, T = 0 at z = -1

where R, P, M and L are the Rayleigh-, the Prandtl-, the Marangoni- and the Nusselt number, respectively,

$$R = \alpha\beta gh^4/\nu\chi$$
, $P = \nu/\chi$, $M = -\beta\gamma h^2/\mu\chi$, $L = \kappa h/\lambda$.

3. Method of solution.

The method of solving our problem will be as follows. We ask for approximate solutions \underline{u}_n and \underline{T}_n which have the form

 h^2/χ and h , respectively, and dropping the primes, the doverning equations are:

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(5a)
$$\underline{u}_{n}(\underline{x},t) = \sum_{k=1}^{n} A_{kn}(t) \underline{\Phi}_{k}(\underline{x})$$

- 6 --

(5b)
$$T_{n}(\underline{x},t) = \sum_{k=1}^{n} B_{kn}(t) \varphi_{k}(\underline{x})$$

where $\underline{\Phi}_k$ and φ_k are some functions chosen beforehand, and $\nabla \cdot \underline{\Phi}_k = 0$, $k = 1, 2, 3, \ldots, n$. Due to the constraint $\nabla \cdot \underline{\Phi}_k = 0$ the pressure $p_n(\underline{x}, t)$ is not explicitly required. - We can always consider the functions $\underline{\Phi}_k$ to be linearly independent and to represent the first n functions of some set of functions $\{\underline{\Phi}_k\}$, $k = 1, 2, 3, \ldots$, which is complete.^{*)} The same assumptions apply to the functions φ_k . The functions $\underline{\Phi}_k$ and φ_k are moreover taken to be orthonormal

$$(\underline{\Phi}_{i}, \underline{\Phi}_{i}) = \delta_{ij}, \quad (\varphi_{i}, \varphi_{j}) = \delta_{ij}$$

where (,) denotes the spatial average over the layer, or inner product. - Our aim is to determine the functions $A_{kn}(t)$ and $B_{kn}(t)$ so that (5a,b) satisfies the governing equations (3a,b) and the boundary conditions (4a-e) as accurately as possible. To do so, we shall use a modification of Galerkin's method for the case of natural boundary conditions, Mikhlin (1964). In applying this method to our problem, we need not subject the functions Φ_k and ϕ_k to any boundary conditions beforehand, but can choose them at will.

*) In a function space, a set $\{\Phi_k\}$ is complete if any function in the space can be expanded in terms of the $\{\Phi_k\}$. - In this paper, we shall be concerned with the set of infinitely differentiable functions.

$$(\underline{s}_{n}) = \sum_{n=1}^{n} A_{nn}(\underline{s}_{n}) = \sum_{n=1}^{n} A_{nn}(\underline{s}_{n})$$

$$(z_{n})_{n} = \sum_{k=1}^{n} a_{n}(b) \phi_{k}(z)$$

where \underline{a}_{i} and φ_{i} are some functions thosen beforehand, and $\nabla \cdot \underline{a}_{i} = 0$, $i = 1, 2, 3, \dots, n$. Due to the constraint $\nabla \cdot \underline{a}_{i} = 0$ the pressure $p_{n}(\underline{a}, \underline{b})$ is not explicitly required. We can always consider has functions \underline{a}_{i} to be linearly independent and to represent the first a functions of some set of functions (\underline{a}_{i}), beil, 2,3, \dots which is complete.⁽¹⁾ The same assaultions (\underline{a}_{i}), beil, 2,3, \dots which is complete.⁽²⁾ for functions \underline{a}_{i} to the functions \underline{a}_{i} . The same assaultions (\underline{a}_{i}), beil, \underline{a}_{i} , \underline{b}_{i} to be estimated as \underline{a}_{i} .

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where (.) denotes the spatial average over the layer, or inner product. - Our aim is to determine the functions has (t) and D_{em}(t) so that (Daib) estimilas the governing equations (Ba,b) and the boundery conditions (Ba-e) as accurately as possible. To do so, we shall use a modification of Galerkin's method for the case of natural use a modification withlin (1964). In applying this method to our problem, we conditing a beforehand, but can choose them at will accurately as beforehand, but can choose them at will and the siunction space, a set (b, is complete if any and the state if any base is not spin any bandary accurately as beforehand, but can choose them at will any base if a state of a set (b, is complete if any any spin any base is set (b, is complete if any any spin any base is set (b, is complete if any any spin any spin any set (b, is complete if any any spin any spin any spin any spin any spin any any spin any set (b, is complete if any any spin any spin any spin any spin any spin any any spin any spin any spin any spin any spin any spin any any spin any spin any spin any spin any spin any spin any any spin any spin any spin any spin any spin any spin any any spin any spin any spin any spin any spin any spin any any spin any spin any spin any spin any spin any spin any any spin any any spin any any spin any spin

- In this paper, we shall be concerned with the get of

The modified Galerkin's method takes the form:

(6a)
$$(P^{-1}\partial \underline{u}_{n}/\partial t + P^{-1}\underline{u}_{n} \cdot \nabla \underline{u}_{n}, \underline{\Phi}_{\gamma}) = R(T_{n}\underline{k}, \underline{\Phi}_{\gamma}) - (P_{n}, \nabla \underline{\Phi}_{\gamma}) - M(\nabla_{s}T_{n}, \underline{\Phi}_{\gamma})_{s}$$

(6b)
$$(\partial T_n / \partial t + \underline{u}_n \cdot \nabla T_n, \varphi_{\gamma}) = (w_n, \varphi_{\gamma}) - (\nabla T_n, \nabla \varphi_{\gamma}) - L(T_n, \varphi_{\gamma})_s$$

where (,)s denote the horizontal average at the free surface and $\gamma = 1, 2, ..., n$. Substituting the expressions (5a,b) into (6a,b), we obtain a system of ordinary differential equations for the amplitudes $A_{kn}(t)$ and $B_{kn}(t)$, which can be solved when given suitable initial conditions, say

$$A_{kn}(0) = (\underline{u}(\underline{x}, 0), \underline{\Phi}_{k}(\underline{x})), \quad B_{kn}(0) = (T(\underline{x}, 0), \varphi_{k}(\underline{x})).$$

There is a valuable physical interpretation of the approximate method given by (6a,b). If we multiply (6a) by $A_{\gamma n}(t)$ and sum over γ from 1 to n, we obtain the equation which is the balance equation for the kinetic energy of the approximate solution. An analogous relation is obtained if we multiply (6b) by $B_{\gamma n}(t)$ and sum over γ . This relation is often interpreted as an entropy balance equation. - If we take the inner product of (3a) and \underline{u} , of (3b) and T, and use (3c) and (4a-e), we obtain the same integral properties for the exact solutions \underline{u} and T. Then, even though \underline{u} and T are approximated by the forms \underline{u}_n and T_n , the fundamental integral properties are satisfied.

The medicine Galerida's method bakes the Lora:

 $((a)_{a,a}(0,a)_{a}) = ((b)_{a,a}(0,a)_{a}) + ((b)_{a,a}(0,a)_{a})_{a} = ((b)_{a,a}(0,a)_{a})_{a} = ((b)_{a,a}(0,a)_{a})_{a} = ((b)_{a}(a)_{$

The most important and most difficult step is the selection of the functions $\underline{\Phi}_k$ and φ_k , $k = 1, 2, 3, \ldots$, In selecting these, we should carefully insure that the functions are linearly independent and members of a complete set. Violation of these requirements can lead to gross error if n (the degree of approximation) is successively increased.^{*)} - Besides, we should insure that the functions incorporate the most important physical characteristics of the problem, e.g. some (or all) of the boundary conditions.

4. The amplitude equations.

When the solutions are expanded in orthogonal functions, the analysis is in general very long and complicated. To simplify the analysis, we will consider the limiting case of an infinite Prandtl number. It is believed that this simplification gives a good description for fluids which have large Prandtl numbers and provides at least a qualitatively

*) Some convergence proofs are available for certain problems in hydrodynamics. Ladyzhenskaya (1964) uses Galerkin's method to prove the existence of a "generalized solution" to the incompressible Navier-Stokes equation. Chernyakov (1966a) considered the problem of thermal convection in a bounded region, and a related problem with a free surface was considered by Chernyakov (1966b), but he did not allow for surface tension. It is worth noting that the modified Galerkin's method is in agreement with the definition of a "generalized solution".

- 8 -

The most important and nost difficult are is the selection of the functions is and q. K = 1.2.2.2.... in selecting these, is should carefully insure that the functions are linearly independent and nembers of a complete set. Violation of these requirements can load to gross error if a (the degree of approximation) is successively increased. Bestdes, we should insure that the functions incorporate the cost important physical oneracteristics of the problem, e.g.

4. The amplitude squares of

correct description of fluids with Prandtl numbers greater than unity, Scanlon and Segel (1967).

An important consequence in the limit of infinite Prandtl number is that the vertical component of the curl of the velocity, $(\nabla x\underline{u}) \cdot \underline{k} = \zeta$, vanishes. This can be seen from the equations of motion and the boundary conditions, which take the form (cf. Chandrasekhar (1961), Chap. II)

$$\nabla^2 \zeta = 0$$
, $\partial \zeta / \partial z = 0$ at $z = 0$ and $\zeta = 0$ at $z = -1$

which admits only the solution $\zeta \equiv 0$ (assuming boundedness of ζ as $x, y \rightarrow \pm \infty$).

It can be shown from the identity $\nabla x (\nabla x \underline{u}) \equiv \nabla (\nabla \cdot \underline{u}) - \nabla^2 \underline{u}$, that in general

$$\hat{\nabla}^2 \underline{\hat{u}} = \underline{k} x \hat{\nabla} \zeta - \hat{\nabla} (\partial w / \partial z) - \underline{k} \partial (\nabla \cdot \underline{u}) / \partial z$$

where $\underline{\hat{u}} = (u, v, 0)$, $\hat{\nabla} = (\partial/\partial x, \partial/\partial y, 0)$ and $\hat{\nabla}^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$. When $\zeta = 0$ and $\nabla \cdot \underline{u} = 0$, it follows that

(7)
$$\hat{\nabla}^2 \hat{\underline{u}} = -\hat{\nabla} (\partial w / \partial z) .$$

When explicit calculations are required, it is convenient to divide each dependent variable into two parts: one, which depends on z and t alone, is the horizontal average of that variable, the remaining part is then periodic in the horizontal directions. We shall use a horizontal bar to denote the horizontal average, correct description of fluidgrainh Prese. Dumbers granes,

An important consected in the worthest of the component of the curigrand i number is that the worthest of component of the curiof the velocity. (Vz1) is a for variables. This can be asen from the equations of mation and the boundary conditions, which take the form (at Consideration; (1961), Chap. III

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which shalts only the solution f = 0 (assuming nonnoclosic) of f as $x_x y \to z = 0$.

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where $\underline{\hat{u}} = (n, v, 0)$, $\hat{v} = (\partial v \partial n, \partial v \partial v, 0)$, and $\hat{v}^* = \partial^* \partial v \partial v + \partial^* \partial v + \partial^$

(x) (x6) 0 = = 20 (x (x) (x)

and write

$$T(x,y,z,t) \equiv \overline{T}(z,t) + \theta(x,y,z,t) , \quad p(x,y,z,t) \equiv \overline{p}(z,t) + \omega(x,y,z,t)$$

By averaging (3c) and (4a,d), it follows that $\overline{\underline{u}}(z,t) \equiv \underline{0}$. Then, by averaging (3a,b) directly ($P^{-1} = 0!$), we obtain

(8a,b)
$$-\nabla \omega + \nabla^2 \underline{u} + R\theta \underline{k} = 0$$
, $\overline{p}_Z = R\overline{T}$

(8c)
$$\theta_{t} + \underline{u} \cdot \nabla \theta - (\overline{w}\overline{\theta})_{z} + w\overline{T}_{z} = \nabla^{2}\theta + w$$

(8d)
$$\overline{T}_{t} + (\overline{w\theta})_{z} = \overline{T}_{zz}$$

(subscripts z and t denote partial derivatives). By means of the constraint $\nabla \cdot \underline{u} = 0$, the boundary conditions (4a-e) take the form

(9a,b)
$$w = 0$$
, $w_{ZZ} - \hat{\nabla}^2 w = +M\hat{\nabla}^2 \theta$ at $z = 0$

(9c,d)
$$\overline{T}_{z} + L\overline{T} = 0$$
, $\theta_{z} + L\theta = 0$ at $z = 0$

(9e-h)
$$W = 0$$
, $W_{\overline{T}} = 0$, $\overline{T} = 0$, $\theta = 0$ at $z = -1$

Note that \overline{p} does only appear in the equation (8b). Hence this equation is used only to evaluate \overline{p} after the other equations have been solved.

We expand the solutions in Fourier series, and assume that the x,y- and z-dependencies (in each term of the

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2(x, x, z, z) = E(z, z) + O(x, z, z, z) + D(x, z, z, z) = D(z, z, z) + D(z, z) + D(z) + D(z, z) + D(z, z) + D(z) +

By swartsfing (30) and (48,4); it follows that $\underline{B}(2,5) \neq \underline{0}$. Then, by scarzence (30,0) and (48,4); it follows that

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(subcontrol 2. and 5. denote partial derived to 2. by means of the constraint ∇^{*} = 0, the boundary, conditions (i.e.,), trakes the form

We expand the solutions in Fourier rerier, and assand

series) are separable. The x,y dependence which is periodic and wave-like, will be denoted by $f_{i,j}(x,y)$, i,j = 1,2,3,...

(10)
$$\hat{\nabla}^2 f_{ij} + a^2_{ij} f_{ij} = 0$$

where $a_{ij}^2 = (ia_1)^2 + (ja_2)^2$ is the overall wave number of the periodic structure and defines the size (though not the shape) of the cellular pattern. ia_1 and ja_2 , i,j = 1,2,3,..,are the wave numbers in the x and y directions, respectively.

If the velocity w(x,y,z,t) is represented by a truncated Fourier expansion, this can be written

$$w_{n}(x,y,z,t) = \sum_{ij} w_{ij}(z,t) f_{ij}(x,y)$$

(on the cell-walls certain boundary conditions must be satisfied, Appendix A). For the functions $w_{ij}(z,t)$, $i,j = 1,2,3,\ldots$, we write

$$W_{ij}(z,t) = \sum_{k} A_{ijk}(t) W_{k}(z)$$

where the sequence $\{W_k(z)\}, k = 1, 2, 3, ..., \text{ is complete}$ On the interval: $-1 \leq z \leq 0$. In this way we obtain

(11a)
$$W_n(x,y,z,t) = \sum_{ijk} A_{ijk}(t) f_{ij}(x,y) W_k(z).$$

In the same way we obtain the following expressions for $\theta_n(x,y,z,t)$ and $\overline{T}_n(z,t)$:

(11b)
$$\theta_{n}(x,y,z,t) = \sum_{ijk} B_{ijk}(t) f_{ij}(x,y) F_{k}(z)$$

 $(2)_{3}^{3}(2,2)_{1}^{2}(2,2)$

(11c)
$$\overline{T}_n(z,t) = \sum_k C_k(t)T_k(z)$$

and it follows from equations (7), (10) and (11a) that

(11d)
$$\underline{\hat{u}}_{n} = (u_{n}, v_{n}, 0) = \sum A_{ijk} a^{-2}_{ij} \partial W_{k} / \partial z$$

The expressions (11a-d) should be compared with (5a,b). It was necessary to introduce the functions $f_{ij}(x,y)$ to account for the horizontal structure. For reasons of convenience we have omitted the index n in the amplitudes . - We have pointed out that we should choose the sequences $\{W_k\}, \{F_k\}$ and $\{T_k\}$ to form complete sets. While any complete sets may be used it is often convenient to choose the functions W_k , F_k and T_k as eigenfunctions of one or another simple eigenvalue problem, but at the same time related to the problem. For the moment we shall only assume that the following boundary conditions are satisfied

(12a)
$$W_k(0) = W_k(-1) = DW_k(-1) = 0$$

(12b)
$$DF_k(0) + LF_k(0) = 0, F_k(-1) = 0$$

(12c)
$$DT_k(0) + LT_k(0) = 0, T_k(-1) = 0$$

for $k = 1, 2, 3, \ldots$, where $D \equiv \partial/\partial z$.

By means of the boundary conditions (12a,b,c), it is convenient to rewrite the equations which govern the

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$$(z)_{j} = (\sigma_{\alpha} z)_{\alpha} = (\sigma_{\alpha} z)_{\alpha}$$

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$$=6(y_1 + y_2)^{-2} + \frac{1}{2} + \frac{1$$

The expressions (1, 22, -d) simuld to assumed both (1, 22, -d). It was reconcerty to introduce the function $1_{1,1}(0, -d)$ is account the introduce the function of 0 or the constraints in have exclutioned the limits is in the resource of 0 or vertienes in have excluted the limits is in the resources of 0 or - $10_{1,1}(0, -d)$ and (1, -d) to reach any limit encode the sequences (0, -d). (2, -d) and (1, -d) to reach any limit encode the sequences in (0, -d). (2, -d) and (1, -d) to reach any limit encode the sequences is a standard the reaction (-d, -d) to reach any limit encode the sequences it is functioned to the problem of the sector bits at the reaction dreaded to the problem. First the near the set of the limit the three definitions of the reaction of the sector d of the sector readed to the problem. First the near the set of the limit that the set of the formulation d is the near the set of the sector 0 of the reaction (-d) is the reaction of the set of the sector d is the set of the problem. First the near the set of the sector d is the set of the formulation d is the near the set of the sector d is the set of the formulation d is the set of the sector d is the set of the sector d is the set of the sector d is the set of the set of the sector d is the set of the sector d is the set of the sector d is the set of the set of the sector d is the set of the sector d is the set of the set of the sector d is the set of the

$$0 = ((+))_{i} | MI = ((-))_{i} | (= (0)_{i} | (-))_{i} | (ast)_{i}$$

$$= (1-)_{\underline{\alpha}} \mathbf{u} \quad (\alpha)_{\underline{\alpha}} = (\alpha)_{\underline{\alpha}} \mathbf{u} \quad (\alpha)_{\underline{\alpha}} \mathbf$$

$$(p_{\rm star}(0+))_{\rm star} = (p_{\rm star}(0) + m_{\rm star}(0) = (p_{\rm star})$$

amplitudes
$$A_{ijk}$$
, B_{ijk} and C_k as, cf. (6a,b) and (8a,c,d)
(13a) $(\nabla^2 \underline{\hat{u}}_n, a_{\alpha\beta}^{-2} \nabla f_{\alpha\beta} DW_{\gamma}) + (\nabla^2 w_n + R\theta_n, f_{\alpha\beta} W_{\gamma}) - (\frac{\partial}{\partial z} \underline{\hat{u}}_n + M \nabla \theta_n, a_{\alpha\beta}^{-2} \nabla f_{\alpha\beta} DW_{\gamma})_s = 0$

(13b)
$$\left(\frac{\partial \theta_n}{\partial t} + \underline{u}_n \cdot \nabla \theta_n - \frac{\partial}{\partial z} (\overline{w_n \theta_n}) + w_n \frac{\partial t_n}{\partial z} - \nabla^2 \theta_n - w_n, f_{\alpha\beta} F_{\gamma}\right) = 0$$

(13c)
$$\left(\frac{\partial \overline{T}_n}{\partial t} + \frac{\partial}{\partial z} \left(\overline{w_n \theta_n}\right) - \frac{\partial^2 \overline{T}_n}{\partial z^2}, T_\gamma\right) = 0$$

for $\alpha, \beta, \gamma = 1, 2, ..., n$. Substituting the expressions (11a-d) into (13a,b,c) the relations between $A_{ijk}(t)$, $B_{ijk}(t)$ and $C_k(t)$ are explicitly obtained. In doing so, we use the integral relations given in Appendix A, besides, some other simplifications are made by means of the boundary condition (12a).

Suppressing the summation sign, the summation now being indicated by the repeated indices i,j,k, l,m and n, we obtain the following amplitude equations:

(14a)
$$A_{\alpha\beta k}((D^2 - a_{\alpha\beta}^2)^2 W_k, W_{\gamma}) = a_{\alpha\beta}^2 RB_{\alpha\beta k}(F_k, W_{\gamma}) - (A_{\alpha\beta k}D^2 W_k(0) + a_{\alpha\beta}^2 MB_{\alpha\beta k}F_k(0))DW_{\gamma}(0)$$

Suppret tilles the second in state we wanted the second in the second second in the second se

 $= \left(\left(\left(\frac{1}{2} - \frac{1}{2} \right) \right) \right) \left(\frac{1}{2} - \frac{1}{2} \right) \left(\frac{1}{2} - \frac{1}{2}$

$$\frac{dB_{\alpha\beta k}}{dt}(F_{k},F_{\gamma}) = B_{\alpha\beta k}((D^{2}-a_{\alpha\beta}^{2})F_{k},F_{\gamma}) +$$

$$+ A_{\alpha\beta k}(W_{k},F_{\gamma}) - A_{\alpha\beta k}C_{n}(W_{k}DT_{n},F_{\gamma}) +$$

$$(14b)$$

$$+ \frac{1}{2}a_{ij}^{-2}A_{ijk}B_{\ell m n}(a_{ij}^{2} + a_{\ell m}^{2} - a_{\alpha\beta}^{2})(\overline{f_{ij}f_{\ell m}f_{\alpha\beta}})(W_{k}F_{n},DF_{\gamma}) -$$

$$- \frac{1}{2}a_{ij}^{-2}A_{ijk}B_{\ell m n}(a_{ij}^{2} - a_{\ell m}^{2} + a_{\alpha\beta}^{2})(\overline{f_{ij}f_{\ell m}f_{\alpha\beta}})(W_{k}DF_{n},F_{\gamma}).$$

$$(14c) \quad \frac{dC_{k}}{dt}(T_{k},T_{\gamma}) = C_{k}(D^{2}T_{k},T_{\gamma}) + A_{ijk}B_{ijn}(W_{k}DT_{\gamma},F_{n})$$

- 14 -

for
$$\alpha, \beta, \gamma = 1, 2, \ldots, n$$
.

Usually, it is not possible to choose some approximating functions as the "best", but the following eigenvalue problems turn out to be suitable and will be used in the next section.

(15a)
$$(D^2 - a_{\alpha\beta}^2)^2 W_k + \lambda_k^2 (D^2 - a_{\alpha\beta}^2) W_k = 0$$

(15b,c)
$$(D^2 - a_{\alpha\beta}^2)F_k + \mu_k^2F_k = 0$$
, $D^2T_k + \nu_k^2T_k = 0$

k = 1,2,3,..., with the boundary conditions (12a,b,c) together with $D^2W_k(0) = 0$. We verify that (15a,b,c) define denumerable infinite sequences of eigenfunctions, Appendix B. Due to the boundary conditions, the following orthogonality relations are obtained

$$((D^2 - a_{\alpha\beta}^2)^2 W_k, W_{\gamma}) = -\lambda_k^2 ((D^2 - a_{\alpha\beta}^2) W_k, W_{\gamma}) = \lambda_k^2 \delta_{k\gamma}$$

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Figurally, it is not possible to obview cone quartoring the problems functions as the "best", but the following engesvalue problems cure out to be suitedily and will be used in the next restion.

(6.27)

 $((0^{2}-2^{2}))^{2} = (((0^{2}-2^{2}))^{2} = (((0^{2}-2^{2}))^{2})^{2} = ((0^{2}-2^{2})^{2})^{$

$$(D^{2}-a_{\alpha\beta}^{2})F_{k},F_{\gamma}) = -u_{k}^{2}(F_{k},F_{\gamma}) = -\mu_{k}^{2}\delta_{k\gamma}$$
$$(D^{2}T_{k},T_{\gamma}) = -v_{k}^{2}(T_{k},T_{\gamma}) = -v_{k}^{2}\delta_{k\gamma}$$

It is worth noting that eigenvalue problems which are not selfadjoint can also be used. The adjoint problem generally differs from the original, but has the same eigenvalues, and each of its eigenfunctions is orthogonal to every eigenfunction of the original problem except the one belonging to the same eigenvalue. For that reason, we replace W_{γ} , F_{γ} and T_{γ} in (13a,b,c) and (14a,b,c) by the adjoint functions.

5. Solutions of the amplitude equations.

In the present section we will investigate the factors which govern the wave numbers a_1 and a_2 by studying some simplified systems of the amplitude equations. We consider first the linearized problem which gives the growth or decay of small perturbations. At the onset of convection, we assume that stable modes are divided from unstable modes by curves of marginal stability for which $\partial/\partial t \equiv 0$ ^{*)}.

*) The validity of the principle of the exchange of stabilities is not exactly known for this problem. The validity can be verified rigorously if M = 0, Chandrasekhar (1961), but it seems impossible to prove or disprove it analytically if $M \neq 0$. Numerical computations of Vidal and Acrivos (1966) indicate, however, that oscillatory instability does not occur when R = 0.

It is worke adding hast sigenvalue problems which are not solfadioint the class the used. The adjoint problem compressive differe from the original, but has the same elgenvalues, and abon of its elgenfunctioned is orthorogenei the every elgenfunction of the eriginal problem execut the one obtomate to the same elgenvalue. For that reason, he replace bolonding to the same elgenvalue. For that reason, he replace its is and to the same elgenvalue. For that reason, he replace its is and to the same elgenvalue. For that reason, he replace the same same elgenvalue. For that reason, he replace

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* The value of the of the value of the second water of the value of

Assuming that only one overall wave number is present, $(\alpha a_1)^2 + (\beta a_2)^2 = a^2$, say, we obtain in the simplest case $\gamma = 1$

(16a)
$$\lambda_1^2 A_{\alpha\beta1} = a^2 B_{\alpha\beta1} \{ R(F_1, W_1) - MF_1(0) DW_1(0) \}$$

(16b)
$$\mu_1^2 B_{\alpha\beta1} = A_{\alpha\beta1} (W_1, F_1)$$

This is an eigenvalue problem from which M can be found in terms of a^2 , L and R; alternatively, R in terms of a^2 , L and M. Solving the eigenvalue problem (16a,b) and setting $M = M_1$ in the case R = 0, and $R = R_1$ in the case M = 0, we obtain

$$\frac{M}{M_1} + \frac{R}{R_1} = 1$$

where

$$M_{1} = \frac{-\lambda_{1}^{2} \mu_{1}^{2}}{a^{2}(W_{1},F_{1})F_{1}(0)DW_{1}(0)}, \quad R_{1} = \frac{\lambda_{1}^{2} \mu_{1}^{2}}{a^{2}(W_{1},F_{1})^{2}}$$

If we minimise M or R as functions of the wave number a, we obtain the critical values M_c or R_c . Both M and R are proportional to the (static) temperature gradient, and the trivial solutions $\underline{u} \equiv \underline{0}$ and $T \equiv 0$ are stable solutions only if the temperature gradient is so small that $M < M_c$ and $R < R_c$.

In the general case the minimasation process must certainly be done by numerical methods, but some preliminary Assemble that only are overall wave mather to present, $(a_{2})^{2} + (a_{2})^{2} = a^{2}$, say, he obtain in the simplest case

$$(16a) \quad \lambda_1^{2} \alpha_{01} = a^{2} \alpha_{01} (\pi(x_1, y_1) - nx_1 (0) xy_1 (0))$$

This is an eigenvalue protien from which M can be found in terms of a², L and R, alternarively, K in terms of a², L and R. Solving the eigenvalue protiem (165,5); and setsing. Me M, in the case, R = 0, and R = R, to the case w = 0, we obtain



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If we minimise M or R as functions of the wave mustor, a., we obtain the critical values M or R. Both M and R, are proportional to the (shate) temperature gradients, and the trivial solutions are proportional to the facto stable of burlons only if the temperature restlering is so

In the general case that infranction process surt

calculations can easily be made in the simple cases R = 0or M = 0 to get some idea of the accuracy which can be attained. *) Although expansion in other functions may give greater accuracy, we believe that the functions W_{γ} and F_{γ} defined by (15a,b) are useful in solving the problem.

When the temperature gradient becomes large enough to make $M > M_c$ or $R > R_c$, the neglect of the non-linear terms in the amplitude equations (14a,b,c) is no longer justified. We consider the case in which either M_c or R_c are <u>slightly</u> exceeded. We may then assume that the wave numbers that are amplified most according to linear theory, dominate in the nonlinear problem. (The experimental results do not indicate motions which have a continous range of wave numbers.) If so, the linear theory can be useful in the prediction of cell size (that is a^2), since it provides realistic initial conditions for the non-linear problem. Formulated in this way, a non-linear analysis can specify the wave numbers a_1 and a_2 .

In the non-linear equation (14b) it is to be expected that the horizontal average $(f_{ij}f_{\ell m}f_{\alpha\beta})$ deserves particular interest, because when this average vanishes, the modes $f_{\alpha\beta}(x,y)$ do not interact except through the mean temperature profile $T_{\gamma}(z)$. If only modes with wave numbers $a^2 = (ia_1)^2 + (ja_2)^2$ are present, we write

*) The "exact" values of M_c and R_c in these cases are computed by other methods and tabulated by Nield (1964).

"" "ghe" exact " values of Mo sidil Bo ". Luithness sages are

$$ia_1 = a \sin \varphi_k$$
, $ja_2 = a \cos \varphi_k$, $k = 1,2,3,...$
and we consider a particular solution of (10), for

(17)
$$f_{i,i}(x,y) = \cos\{ia_1x + ja_2y\}$$

Setting $x = r \cos \theta$ and $y = r \sin \theta$, this can be written

$$f_{ij}(x,y) \equiv f_k(r,\theta) = \cos\{\arg \sin(\varphi_k + \theta)\}, k = 1,2,3,...$$

It can be shown that the average $(f_1f_2f_3)$ is nonzero only if

$$\varphi_2 = \varphi_1 \pm \frac{\pi}{3}$$
 and $\varphi_3 = \varphi_1 \mp \frac{\pi}{3}$.

or:

 $\varphi_2 = \varphi_1 \pm \frac{2\pi}{3}$ and $\varphi_3 = \varphi_1 \pm \frac{\pi}{3}$

The results follow from the analysis given by Segel $(1965b)^{*}$. For our purpose, it is convenient to consider the modes f_1 and $f_2 - f_3$. Then, by rotation of the frame of reference we can choose $\varphi_1 = 0$, and reverting to our earlier notation, we can write the modes as

*) Segel (1965b) not only allows for cosines in (17), but also sines. The result is the same: modes interact with each other only if they are associated with the wave-number angles φ , $\varphi - \frac{\pi}{3}$ and $\varphi + \frac{\pi}{3}$.

and we consider a particular solution of (10). for

 $(\tau_S at + x_s at | aco = (\tau, x)_s, \tau$ (τ_s)

Setting $x = r \cos \theta$ and $y = r \sin \theta$, this can be written

(x,x) = f((x,0) - dos(ar sin(q,+0)) . K = 1,2,3,...

 $\varphi_2 = \varphi_1 \pm \frac{2\pi}{5} \quad \text{and} \quad \varphi_3 = \varphi_4 \pm \frac{\pi}{5}$

The results follow from the unalysis given by Segel (16650) . For our purpose, it is convenient to consider the modes

 r_1 and $r_2 - r_3$ Then, by robation of the frame of reference we can choose $\phi_1 = 0$, and reversing to our earlier notation, we can write the modes as

b) Second (1963b) not only allows for cosines in (12). Full also these. The result is the same: moles interact with each other only if they are seconded with the wavematter angles of $\phi - \frac{1}{2}$ and $\phi + \frac{1}{2}$.

- 11 ---

(18a,b)
$$f_{02}(x,y) = \sqrt{2} \cos 2a_2 y$$
, $f_{11}(x,y) = 2 \cos a_1 x \cos a_2 y$

where $a_1 = a \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}a$ and $a_2 = a \cos \frac{\pi}{3} = \frac{1}{2}a$. These modes are of particular interest in the investigations of cellular convection, since

 $\sqrt{2} f_{11}(x,y) \pm f_{02}(x,y) = \sqrt{2}(2 \cos a_1 x \cos a_2 y \pm \cos 2a_2 y)$

is the analytical expression for a hexagonal cell pattern, cf. Chandrasekhar (1961), besides, they are of interest since $(f_{11}f_{11}f_{02}) = \frac{\sqrt{2}}{2} \neq 0.$

Since $(\overline{f_{11}f_{11}f_{02}}) \neq 0$, we can carry out some preliminary calculations to illustrate the non-linear effects by studying the amplitude equations for $\gamma = 1,2$ and $(\alpha,\beta) = (0,2)$ and (1,1). However, even this simplification results in extremely complicated sets of equations for the amplitudes, and it seems likely that we have to make some approximations if our aim is to solve the equations in a closed form. To obtain a closed form, we will follow the arguments given by Segel and Stuart (1962) in a related case. It is valid to neglect the time differentials in the equations for the mean field C_{γ} , and all components of the disturbances $B_{\alpha\beta\gamma}$, except in those for the fundamentals $B_{\alpha\beta1}$, <u>provided</u> the purpose is to obtain the dominant part of the non-linear problem.

Then, if we eliminate $A_{\alpha\beta\gamma}$, we believe that and adequate approximation to the problem is:

y = 1.2 and (a, p) = (0, 2) and mplification results in extremelyA state of the stader contract back through the state of the state of

(19a)
$$dB_{021}/dt = \varepsilon_1 B_{021} + \alpha_1 B_{022} + \beta_1 B_{111} B_{112} + \gamma_1 B_{021} C$$

(19b) $dB_{111}/dt = \varepsilon_1 B_{111} + \alpha_1 B_{112} + \beta_1 (B_{111} B_{022} + B_{021} B_{112}) + \gamma_1 B_{111} C_1$

(19c)
$$0 = \varepsilon_2 B_{022} + \alpha_2 B_{021} + \beta_2 B_{111}^2$$

(19d)
$$0 = \varepsilon_2 B_{112} + \alpha_2 B_{111} + 2\beta_2 B_{111} B_{021}$$

(19e)
$$0 = C_1 + \gamma_2 (B_{111}^2 + B_{021}^2)$$

where the coefficients in these equations are functions of a², L, M and R, and are given in Appendix C.

By elimination of B_{022} , B_{112} and C_1 we obtain

(20a)
$$dB_{021}/dt = \epsilon B_{021} - \Gamma B_{111}^2 - Q_1 B_{021}^3 - Q_3 B_{111}^2 B_{021}$$

(20b) $dB_{111}/dt = \epsilon B_{111} - 2\Gamma B_{111}B_{021} - Q_2 B_{111}^3 - Q_3 B_{021}^2 B_{111}$

Q1

$$\epsilon = \frac{\varepsilon_1 \varepsilon_2 - \alpha_1 \alpha_2}{\varepsilon_2}, \quad \Gamma = \frac{\alpha_1 \beta_2 + \alpha_2 \beta_1}{\varepsilon_2}$$
$$= \gamma_1 \gamma_2, \quad Q_2 = \gamma_1 \gamma_2 + \frac{\beta_1 \beta_2}{\varepsilon_2}, \quad Q_3 = \gamma_1 \gamma_2 + \frac{2\beta_1 \beta_1}{\varepsilon_2}$$

Apart from a slightly different notation (due to the normalized modes $f_{i,i}(x,y)$, the equations (20a,b) are identical with those discussed by Segel and Stuart (1962).

We do not quote the general results given by Segel

$$(192) \quad d_{0} = (100) \quad d_{0$$

$$150^{8}111^{8}s^{45}?^{2}rrr^{4}s^{2}rrr^{4}s^{3} = 0$$
 (621)

where the coefficients in thuse equations are functions of a^2 , L. M and R. and are given in Appendix C. We obtain the elements of b_{022} . Equal to b_{022} . Equal to b_{022} .

agart in a alightly different notablen (due to the normalling modes fig((n.g)), the equations (20m,3) are identified with there discussed by Regal and Stuset (1952). and Stuart. However, the most "interesting" steady state solutions of (20a,b) is $B_{111} = \pm \sqrt{2B_{021}}$ (note that $Q_3 = 2Q_2 - Q_1$) which characterizes the (hexagonal) convection cells

$$\{B_{111}f_{11} + B_{021}f_{02}\} = B_{111}\{2\cos a_1x \cos a_2 \pm \cos 2a_2y\}$$

This particular solution may be a <u>stable</u> equilibrium state only if $\Gamma \neq 0$, cf. Segel and Stuart. The coefficient Γ is given explicitly in Appendix C, and we observe that $\Gamma = 0$ if M = 0. The hexagonal convection cells may then be attributed to surface tension gradients (and not to buoyancy). - A different approach to this problem is given by Scanlon and Segel (1967).

6. Final Remarks.

We can of course proceed to study other disturbances $B_{\alpha\beta\gamma}$ than those discussed in the previous section, but the complexity of the amplitude equations becomes even greater, and they may lose their attraction. However, the behaviour of any finite number of modes which have the <u>same</u> overall wave number can be deduced with little further work. The reader is referred to the analysis by Segel (1965a,b) in his analysis of buoyancy driven flow.

Originally, our prupose was to suggest an approximate method to solve the problem of cellular convection subject to natural boundary conditions. The problem is reduced to

- 21 -

and Sources. However, the more "incompluted" events the chain $\mathcal{O}_{1,2}$ and sources in $\mathcal{O}_{1,2}$ and solutions of (208.5) is $\mathcal{B}_{1,1} = \frac{1}{2} \mathcal{O}_{1,2} \mathcal{O}_{1,2} \mathcal{O}_{1,2} \mathcal{O}_{2,2} \mathcal{O}_{2,2}$

only if $f \neq 0$, of. Segel and Stuart. The confidence fis given explicitly in Appandix C. and we observe then $f \in 0$ if X = 0. The hexagonal contraction calls may then be attributed to surface tenator gradients (and not to bigrades): a different sobreach to bhis problem is given to Semiler and Secol (1967).

We can al course proceed to study other distumpances aday than those discussed in the previous section, but the complexity of the samplitude equations becomes even greater, and they way lose their structure on Hovever, the behaviour of any finite number of modes which have the <u>sama</u> overall wave number can be deduced with libble further work. The reacer is releved to the analysis by Segel (1965a.b) in

and the provise was to succeed an energies and energies and an energies and the succeed to be succ

a system of ordinary differential equations, which is the most attractive result. Selecting approximating functions may, however, be crucial in the applications, and is often regarded as a major disadvantage of the method. As far as we know, no rational methods for selecting approximating functions are known, and it remains somewhat dependent on the user's intuition and experience. Nevertheless, we believe that, if only the most important physical characteristics are incorporated in the approximating functions, the qualitative description of the present problem should not be affected.

Acknowledgements.

The author is grateful to the staffmembers of the Department of Applied Mathematics for help with the manuscript.

Appendix A.

Previously we described the solutions of the equation (10) as periodic cell pattern. However, the precise definition requires that on the cell-walls the normal gradient of the vertical velocity vanishes, that is $\hat{\mathbf{n}} \cdot \hat{\nabla} \mathbf{f}_{ij} = 0$, where $\hat{\mathbf{n}}$ is a unit vector normal to the cell-walls, Chandrasekhar (1961).

By means of Green's first identity in the plane

 $\int \int \widehat{\nabla} \varphi \cdot \widehat{\nabla} \psi dx dy + \int \int \varphi \widehat{\nabla}^2 \psi dx dy = \oint \varphi \underline{\widehat{n}} \cdot \widehat{\nabla} \psi ds$

- 22 -

The author is grateful to the stafframbers of the Department of Applied Wathematics for halp with the push-

where ds is a line element on the boundary enclosing the region $-\pi/a_1 \leq x \leq \pi/a_1$ and $-\pi/a_2 \leq y \leq \pi/a_2$, we obtain the following integral relations when φ and ψ are any (orthonormalized) solutions of (10) subject to the boundary conditions given above

$$\overline{\widehat{\nabla}f_{ij}\cdot\widehat{\nabla}f_{\alpha\beta}} = a^2_{ij}\overline{f_{ij}f_{\alpha\beta}} = a^2_{ij}\delta_{i\alpha}\delta_{j\beta}$$

and, setting $\varphi = f_{ij}f_{\ell m}$, $\psi = f_{\alpha\beta}$, permutating the indices and taking sum and difference

$$2(\overline{f_{\alpha\beta}\hat{\nabla}f_{ij}\cdot\hat{\nabla}f_{\ell m}}) = (a^{2}_{ij} + a^{2}_{\ell m} - a^{2}_{\alpha\beta})(\overline{f_{ij}f_{\ell m}f_{\alpha\beta}})$$

Appendix B.

The solutions of the eigenvalue problems (15a,b,c) with the boundary conditions (12a,b,c) and $D^2W_{\gamma}(0) = 0$, are

$$W(z) = C_1 \left\{ \frac{\sinh a_{\alpha\beta} z}{\sinh a_{\alpha\beta}} - \frac{\sin\sqrt{\lambda^2 \gamma - a_{\alpha\beta}^2}}{\sin\sqrt{\lambda^2 \gamma - a_{\alpha\beta}^2}} \right\}$$
$$F_{\gamma}(z) = C_2 \sin\sqrt{\mu^2 \gamma - a_{\alpha\beta}^2}(z+1)$$

 $T_{\gamma}(z) = C_{3} \sin v_{\gamma}(z+1)$

where is is a line classer of the boundary solihooding the $r = \frac{1}{2} \frac{1}{$

al protection and the second second

and, setting $\varphi = f_{1,j} f_{j,\ell}$, $\psi = f_{0,j}$, permissing the incless and taking com and difference

Lopenseix B.

The solutions of the eigenvalue problems (152, $b_{2}c_{1}$) the tree boundary conditions (128, $b_{2}c_{1}$) and $D^{2}b_{2}(0) \neq 0$.

where C_1 , C_2 and C_3 are normalization constants, while the eigenvalues are solutions of

$$\sqrt{\lambda^{2}_{\gamma} - a^{2}_{\alpha\beta} \cdot \tanh \cdot a_{\alpha\beta}} = a_{\alpha\beta} \tan \sqrt{\lambda^{2}_{\gamma} - a^{2}_{\alpha\beta}}$$

$$\sqrt{\mu^{2}_{\gamma} - a^{2}_{\alpha\beta}} = -L \tan \sqrt{\mu^{2}_{\gamma} - a^{2}_{\alpha\beta}}$$

$$v_{\gamma} = -L \tan v_{\gamma}$$

which yields $\lambda_{\gamma} = \lambda_{\gamma}(a_{\alpha\beta}^2), \mu_{\gamma} = \mu_{\gamma}(a_{\alpha\beta}^2, L)$ and $\nu_{\gamma} = \nu_{\gamma}(L)$

Appendix C.

For reasons of convenience, we use the notations

$$p_{k\gamma} \equiv (F_k, W_{\gamma}), \quad q_{k\gamma} \equiv F_k(0) DW_{\gamma}(0), \quad r_{kn\gamma} \equiv (W_k DF_n, F_{\gamma}),$$
$$s_{kn\gamma} \equiv (W_k DT_n, F_{\gamma}).$$

The coefficients in the equations (19a-e) are

$$\varepsilon_{1} = a^{2}R(p^{2}_{11}\lambda_{1}^{-2}+p^{2}_{12}\lambda_{2}^{-2}) - a^{2}M(p_{11}q_{11}\lambda_{1}^{-2}+p_{12}q_{12}\lambda_{2}^{-2}) - \mu^{2}_{1}$$

$$\varepsilon_{2} = a^{2}R(p^{2}_{21}\lambda^{-2}_{1}+p^{2}_{22}\lambda^{-2}_{2}) - a^{2}M(p_{21}q_{21}\lambda^{-2}_{1}+p_{22}q_{22}\lambda^{-2}_{2}) - \mu^{2}_{2}$$

$$\alpha_{1} = a^{2}R(p_{11}p_{21}\lambda_{1}^{-2}+p_{12}p_{22}\lambda_{2}^{-2}) - a^{2}M(p_{11}q_{21}\lambda_{1}^{-2}+p_{12}q_{22}\lambda_{2}^{-2})$$

- 29 -

$$\alpha_{2} = a^{2}R(p_{21}p_{11}\lambda_{1}^{-2} + p_{22}p_{12}\lambda_{2}^{-2}) - a^{2}M(p_{21}q_{11}\lambda_{1}^{-2} + p_{22}q_{12}\lambda_{2}^{-2})$$

$$\beta_{1} = -\beta_{2} = \frac{1}{2}a^{2}\lambda_{1}^{-2} (Rp_{11} - Mq_{11})(r_{112} - r_{121})(\overline{f_{11}f_{11}f_{02}})$$

$$\gamma_{1} = v^{2}_{1}\gamma_{2} = -a^{2}\lambda_{1}^{-2} s_{111}(Rp_{11} - Mq_{11})$$

where λ^2_1 , λ^2_2 , μ^2_1 , μ^2_2 and ν^2_1 are the eigenvalues given in Appendix B.

The coefficient Γ in the equations (20a,b) can now be written,

$$\Gamma = \frac{\alpha_1 \beta_2 + \alpha_2 \beta_1}{\varepsilon_2} = \frac{\beta_2}{\varepsilon_2} (\alpha_1 - \alpha_2)$$

and by the elimination of α_1 and α_2 we obtain

$$\Gamma = a^{2}M \frac{\beta_{2}}{\varepsilon_{2}} \left(p_{21}q_{11} - p_{11}q_{21} \right) \lambda^{-2} + \left(p_{22}q_{12} - p_{12}q_{22} \right) \lambda^{-2}_{2} \right)$$

where λ^2_{i} , λ^2_{i} , μ^2_{i} , μ^2_{i} , μ^2_{i} , and ν^2_{i} , are the eigenvalues given in Appendix B.

The coefficient (in the equations (201, b) can now be written.

and by the elimination of a and a we obtain

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