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Global Convergence of Subspace Correction Methods for Convex Optimization Problems

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Abstract

A general technique based on space decomposition and subspace correction is used to solve nonlinear convex minimization problems. The differential of the minimization functional is required to satisfy some growth conditions that are weaker than Lipschitz continuity and strong monotonicity. Optimal rate of convergence is proved. If the differential is Lipschitz continuous and strongly monotone, then the algorithms have uniform rate of convergence. The algorithms can be used for domain decomposition and multigrid type of techniques. Applications to linear elliptic and some nonlinear degenerated partial differential equation are considered.

1 Introduction

Domain decomposition and multigrid methods have been intensively studied for linear partial differential equations. Recent research, see for example [31], reveals that domain decomposition and multigrid methods can be analysed using a same framework, see also [3], [13], [23], [17]. The present work uses this framework to analyse the convergence of two algorithms for convex optimization problems. However, our emphasis is on nonlinear problems instead of linear

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problems. The algorithms reduce to the standard additive and multiplicative Schwarz methods when used for linear partial differential equations.

Researches for domain decomposition and multigrid methods have been mostly concentrating on linear elliptic and parabolic partial differential equations. Extension to more difficult problems have been considered by some recent works. In this work, a general nonlinear convex minimization problems is considered. The proposed algorithms can be used for nonlinear partial differential equations, optimal control problems related to partial differential equations and eigenvalue problems [8] [22]. The space decomposition can be a domain decomposition method, a multigrid method or some other decomposition techniques.

Domain decomposition methods and multigrid methods have been studied for nonlinear partial differential equations by some earlier works, see [1], [2], [4], [5], [12], [15], [18], [20], [29], [24], [25], [26], [27], [28], [32], [33], etc. In comparison with the existing works, our approach has several features. For example, the proposed algorithms can be used for certain degenerated or singular nonlinear diffusion problems, i.e the nonlinear diffusion coefficient can be zero or infinity and our approach do not need extra assumption on the smoothness of the solutions. The methods work for natural domain decomposition and multigrid meshes. Moreover, only small size nonlinear problems need to be solved on the decomposed subspaces. We also emphasis that our approach is valid for general space decomposition techniques. So the applications is not restricted to domain decomposition and multigrid methods. Other space decomposition techniques can also be considered, see [10], [23, p.169-184], [34].

The two algorithms given in this work were first proposed in [24], see also [25], [27], [28] and [29], where the qualitative convergence of the algorithms was proved, but the uniform rate of convergence was not given there.

2 Optimization problems and subspace correction methods

Consider the nonlinear optimization problem

$$\min_{v\in V} F(v) \ . \tag{1}$$

Here V is a reflexive Banach space and $F : V \to R$ is a convex functional. This problem has different applications, see §6

We shall use a space decomposition method to solve (1). A space decomposition method refers to a method that decomposes the space V into a sum of subspaces, i.e. there are subspaces V_i, $i = 1, 2, \cdots, m$, such that

$$V = V_1 + V_2 + \cdots + V_m \ . \tag{2}$$

This means that for any v, there exists $v_i \in V_i$ such that $v = \sum_{i=1}^m v_i$. Following the framework of [31] for linear problems, we consider two types of subspace correction methods based on (2), namely the parallel subspace correction (PSC) method and the successive subspace correction (SSC) method.

Algorithm 2.1 *[A parallel subspace correction method].*

1. *Choose initial value* $u^0 \in V$ *and relaxation parameters* $\alpha_i > 0$ *such that* $\sum_{i=1}^{m} \alpha_i \leq 1$.

2. *For* $n \geq 0$, *if* $u^n \in V$ *is defined, then find* $e_i^n \in V_i$ *in parallel for* $i = 1, 2, \cdots, m$ *such that*

$$F\left(u^n + e_i^n\right) \leq F\left(u^n + v_i\right) , \quad \forall v_i \in V_i . \tag{3}$$

3. *Set*

$$u^{n+1} = u^n + \sum_{i=1}^{m} \alpha_i e_i^n , \tag{4}$$

and go to the next iteration.

Algorithm 2.2 *[A successive subspace correction method].*

1. *Choose initial values* $u_i^0 = u^0 \in V$.

2. *For* $n \geq 0$, *if* $u^n \in V$ *is defined, find* $u^{n+i/m} = u^{n+(i-1)/m} + e_i^n$ *with* $e_i^n \in V_i$ *sequentially for* $i = 1, 2, \cdots, m$ *such that*

$$F\left(u^{n+(i-1)/m} + e_i^n\right) \leq F\left(u^{n+(i-1)/m} + v_i\right) , \quad \forall v_i \in V_i . \tag{5}$$

3. *Go to the next iteration.*

3 Global convergence of the algorithms

3.1 Assumptions on F

In the following, the notation $\langle \cdot, \cdot \rangle$ is used to denote the duality pairing between V and V', here V' is the dual space of V. The functional F is assumed to be Gateaux differentiable (see [7]) and there are constants $K,\ L > 0$, $p \geq q > 1$ such that

$$\begin{aligned} \langle F'(w) - F'(v), w - v \rangle &\geq K \|w - v\|_V^p, \quad \forall w, v \in V , \\ \|F'(w) - F'(v)\|_{V'} &\leq L \|w - v\|_V^{q-1}, \quad \forall w, v \in V , \end{aligned} \tag{6}$$

and from which it is easy to deduce that

$$K\|w - v\|_V^p \leq \langle F'(w) - F'(v), w - v \rangle \leq L \|w - v\|_V^q \ , \forall w, v \in V . \tag{7}$$

Under assumption (6), problem (1) and subproblems (3) and (5) have unique solutions, see [14, p. 35]. For some nonlinear problems, the constants K and L depend on v and w. However, just under the condition that F is strictly convex, it has been proved in [24] and [28] that the iterative solutions of Algorithm 2.1 and Algorithm 2.2 converge to the true solution. Thus, one can assume that the computed solutions are in a neighbourhood of the true solution and so the constants K and L can be assumed to be independent of v and w. In case that the functional F is only locally convex in a neighbourhood of the true solution, by choosing the initial value close enough to the true solution, it can be proved that the computed solutions stay always inside the neighbourhood that the functional F is convex (the essential techniques of the proof is contained in the proof of Lemma 4.2 and 4.3 of [19]), and so the results given in this work are also applicable to this kind of problems.

For simplicity, we define

$$\sigma = \frac{p}{p-q+1}, \quad \sigma' = \frac{p}{q-1}, \quad \text{which satisfy} \quad \frac{1}{\sigma} + \frac{1}{\sigma'} = 1.$$

Note that $\sigma \le p$ and by Hölder inequality

$$\sum_{i=1}^{m} |a_i|^{q-1}|b_i| \le \left(\sum_{i=1}^{m} |a_i|^p \right)^{\frac{q-1}{p}} \left(\sum_{i=1}^{m} |b_i|^\sigma \right)^{\frac{1}{\sigma}}. \tag{8}$$

The following lemma can be proved in a similar way as [14, p. 25], and the proof can be found in [24].

Lemma 3.1 *If the condition (7) is valid, then*

$$F(w) - F(v) \ge \langle F'(v), w-v \rangle + \frac{K}{p}\|w-v\|_V^p \,, \quad \forall v, w \in V \,, \tag{9}$$

$$F(w) - F(v) \le \langle F'(v), w-v \rangle + \frac{L}{q}\|w-v\|_V^q \,, \quad \forall v, w \in V \,. \tag{10}$$

We shall use u to denote the unique solution of (1) which satisfies

$$\langle F'(u), v \rangle = 0, \quad \forall v \in V \,. \tag{11}$$

It is an easy consequence of Lemma 3.1 that

$$\frac{K}{p}\|v-u\|_V^p \le F(v) - F(u) \le \frac{L}{q}\|v-u\|_V^q, \quad \forall v \in V \,. \tag{12}$$

Therefore, in the following, we shall use

$$d_n = F(u^n) - F(u), \ \forall n \ge 0 \,, \tag{13}$$

as a measure of the error between u^n and the true solution u.

3.2 Assumptions on the space decomposition

For the decomposed spaces, we assume that there exits a constant $C_1 > 0$ such that for any $v \in V$, we can find $v_i \in V_i$ to satisfy:

$$v = \sum_{i=1}^{m} v_i , \qquad \text{and} \qquad \left(\sum_{i=1}^{m} \|v_i\|_V^\sigma \right)^{\frac{1}{\sigma}} \le C_1 \|v\|_V . \tag{14}$$

Moreover, assume that there is a $C_2 > 0$ such that there holds

$$\begin{aligned} &\sum_{i=1}^{m} \sum_{j=1}^{m} \langle F'(w_{ij} + u_i) - F'(w_{ij}), v_j \rangle \\ \le\ & C_2 \left(\sum_{i=1}^{m} \|u_i\|_V^p \right)^{\frac{q-1}{p}} \left(\sum_{j=1}^{m} \|v_j\|_V^\sigma \right)^{\frac{1}{\sigma}} , \\ & \forall\ w_{ij} \in V, u_i \in V_i \text{ and } v_j \in V_j . \end{aligned} \tag{15}$$

Domain decomposition methods, multilevel methods and multigrid methods can be viewed as different ways of decomposing finite element spaces into sums of subspaces. See §4 and §5 for some examples of some decompositions of a finite element space and the corresponding estimates for constants C_1 and C_2. If F is strictly convex, then the iterative solutions of the algorithms converges to the true solution, i.e they are in a neighbourhood of the true solution. Therefore, we just need to estimate (15) for w_{ij} from a neighbourhood of the true solution. For linear problems, estimate (15) is a consequence of the well-known strengthened Cauchy-Schwartz inequality, see Xu [31].

3.3 The convergence of the parallel subspace correction method

Before we state our first main result, we state the following elementary result.

Lemma 3.2 *Given $r > 1$ and $\eta > 0$. If $a \in (0, a_0]$ and $b > 0$ satisfy the inequality*

$$b + \eta b^r \le a. \tag{16}$$

Then there exists a constant $\xi_0 = \xi_0(a_0, \eta, r) \in [0, 1]$, depending only on a_0, η and r, such that

$$b \le \left(\eta(r-1)\xi_0 + a^{1-r}\right)^{\frac{1}{1-r}} \le a . \tag{17}$$

We postpone the proof of the above lemma in the appendix. With the help of the above lemma, We can prove the rate of convergence for Algorithm 2.1 as in the following:

Theorem 3.3 *Assume that the space decomposition satisfies (14), (15) and the functional F satisfies (6). Set $r = \frac{p(p-1)}{q(q-1)}$. Then for Algorithm 2.1 and d_n given by (13), we have:*

1. *If $r = 1$ (namely $p = q$),*

$$d_{n+1} \leq \frac{C^*}{1+C^*} d_n, \quad \forall n \geq 1 \ . \tag{18}$$

   Here $C^* > 0$ depending on p, q, K, L, C_1 and C_2, see (27).

2. *If $r > 1$, then there exists an $\xi_0 = \xi_0(d_0, C^*, r) \in [0,1]$ such that*

$$d_{n+1} \leq \left(\frac{r-1}{C^*} \xi_0 + {d_n}^{1-r} \right)^{\frac{1}{1-r}} \leq \left(\frac{r-1}{C^*} (n+1) \xi_0 + d_0^{1-r} \right)^{\frac{1}{1-r}}, \quad \forall n \geq 1 \ . \tag{19}$$

Proof. Using (4), the convexity of F and (9), we get

$$\begin{aligned}
F(u^n) - F(u^{n+1}) &= F(u^n) - F\Big(u^n + \sum_{i=1}^m \alpha_i e_i^n\Big) \\
&= F(u^n) - F\left(\sum_{i=1}^m \alpha_i (u^n + e_i^n) + \Big(1 - \sum_{i=1}^m \alpha_i\Big) u^n\right) \\
&\geq F(u^n) - \sum_{i=1}^m \alpha_i F(u^n + e_i^n) - \Big(1 - \sum_{i=1}^m \alpha_i\Big) F(u^n) \qquad (20) \\
&= \sum_{i=1}^m \alpha_i \Big(F(u^n) - F(u^n + e_i^n) \Big) \\
&\geq -\sum_{i=1}^m \alpha_i \langle F'(u^n + e_i^n), e_i^n \rangle + \frac{K}{p} \sum_{i=1}^m \alpha_i \|e_i^n\|_V^p \\
&= \frac{K}{p} \sum_{i=1}^m \alpha_i \|e_i^n\|_V^p \ .
\end{aligned}$$

For notational simplicity, we introduce for a given i

$$w_j^n = \begin{cases} u^n + \displaystyle\sum_{k=i}^{j+i-1} \alpha_k e_k^n \ , & \forall j \in [1, m-i+1] \ ; \\ u^n + \displaystyle\sum_{k=i}^{m} \alpha_k e_k^n + \sum_{k=1}^{j-m+i-1} \alpha_k e_k^n \ , & \forall j \in [m-i+2, m] \ . \end{cases}$$

It is clear that w_j^n is depending on i. Moreover, we see that

$$
\begin{aligned}
w_1^n &= u^n + \alpha_i e_i^n, \\
w_2^n &= u^n + \alpha_i e_i^n + \alpha_{i+1} e_{i+1}^n, \\
&\cdots \\
w_m^n &= u^n + \sum_{k=1}^m \alpha_k e_k^n.
\end{aligned}
$$

It is easy to see that

$$
F'\Big(u^n + \sum_{j=1}^m \alpha_j e_j^n\Big) - F'(u^n + \alpha_i e_i^n) = \sum_{j=2}^m \big(F'(w_j^n) - F'(w_{j-1}^n)\big) \ . \tag{21}
$$

From the property (14) of the space decomposition, there exists $v_i \in V_i$ such that

$$
u^{n+1} - u = \sum_{i=1}^m v_i \quad \text{and} \quad \left(\sum_{i=1}^m \|v_i\|_V^\sigma\right)^{\frac{1}{\sigma}} \le C_1 \|u^{n+1} - u\|_V \ . \tag{22}
$$

We shall use (11), (20), (21) and (6) to estimate:

$$
\begin{aligned}
&\Big\langle F'(u^{n+1}) - F'(u), u^{n+1} - u \Big\rangle \\
= \quad & \langle F'(u^{n+1}), u^{n+1} - u\rangle = \sum_{i=1}^m \Big\langle F'(u^{n+1}), v_i \Big\rangle \\
= \quad & \sum_{i=1}^m \Big\langle F'(u^{n+1}) - F'(u^n + e_i^n), v_i \Big\rangle \qquad\qquad (23) \\
= \quad & \sum_{i=1}^m \Big\langle F'\Big(u^n + \sum_{j=1}^m \alpha_j e_j^n\Big) - F'(u^n + e_i^n), v_i \Big\rangle \\
= \quad & \sum_{i=1}^m \Big\langle F'\Big(u^n + \sum_{j=1}^m \alpha_j e_j^n\Big) - F'(u^n + \alpha_i e_i^n), v_i \Big\rangle \\
& + \sum_{i=1}^m \Big\langle F'(u^n + \alpha_i e_i^n) - F'(u^n + e_i^n), v_i \Big\rangle \\
= \quad & \sum_{i=1}^m \sum_{j=2}^m \Big\langle F'(w_j^n) - F'(w_{j-1}^n), v_i \Big\rangle + \sum_{i=1}^m \Big\langle F'(u^n + \alpha_i e_i^n) - F'(u^n + e_i^n), v_i \Big\rangle \\
\le \quad & C_2 \Big(\sum_{j=1}^m \|(\alpha_j e_j^n)\|_V^p\Big)^{\frac{q-1}{p}} \Big(\sum_{i=1}^m \|v_i\|_V^\sigma\Big)^{\frac{1}{\sigma}} + C_2 \sum_{i=1}^m (1 - \alpha_i) \|e_i^n\|_V^{q-1} \|v_i\|_V \\
\le \quad & C_2 \Big(\sum_{j=1}^m \alpha_j^p \|e_j^n\|_V^p\Big)^{\frac{q-1}{p}} \Big(\sum_{i=1}^m \|v_i\|_V^\sigma\Big)^{\frac{1}{\sigma}}
\end{aligned}
$$

$$+\frac{C_2}{\min \alpha_i^{\frac{q-1}{p}}}\left(\sum_{i=1}^{m}\alpha_i\|e_i^n\|_V^p\right)^{\frac{q-1}{p}}\left(\sum_{i=1}^{m}\|v_i\|_V^\sigma\right)^{\frac{1}{\sigma}} . \tag{24}$$

$$\leq \quad C_2|\max \alpha_i|^{\frac{(p-1)(q-1)}{p}}\left(\sum_{i=1}^{m}\alpha_i\|e_i^n\|_V^p\right)^{\frac{q-1}{p}} \cdot C_1\|u^{n+1}-u\|_V$$

$$+C_2|\min \alpha_i|^{-\frac{q-1}{p}}\left(\sum_{i=1}^{m}\alpha_i\|e_i^n\|_V^p\right)^{\frac{q-1}{p}} \cdot C_1\ \|u^{n+1}-u\|_V \tag{25}$$

$$\leq \quad C_1C_2\left(\alpha_{max}^{\frac{(p-1)(q-1)}{p}}+\alpha_{min}^{-\frac{q-1}{p}}\right)\left[\frac{p}{K}\left(F(u^n)-F(u^{n+1})\right)\right]^{\frac{q-1}{p}} \cdot \|u^{n+1}-u\|_V .$$

In the above, α_{max} and α_{min} are used to denote

$$\alpha_{max} = \max_{1\leq i\leq m} \alpha_i, \qquad \alpha_{min} = \min_{1\leq i\leq m} \alpha_i.$$

By assumption (6) and relation (12), we have

$$\frac{\langle F'(u^{n+1})-F'(u), u^{n+1}-u\rangle}{\|u^{n+1}-u\|_V}$$

$$\geq \quad K\|u^{n+1}-u\|_V^{p-1} \geq K\left[\frac{q}{L}(F(u^{n+1})-F(u))\right]^{\frac{p-1}{q}} . \tag{26}$$

Defining

$$C^* = \left[\frac{C_1C_2\left(\alpha_{max}^{\frac{(p-1)(q-1)}{p}}+\alpha_{min}^{-\frac{q-1}{p}}\right)}{K}\right]^{\frac{p}{q-1}} \frac{p}{K}\left(\frac{L}{q}\right)^r , \tag{27}$$

one gets from (25) and (26) that

$$(d_{n+1})^r \leq C^*(d_n - d_{n+1}) . \tag{28}$$

Thus $\sum_{n=0}^{\infty}(d_{n+1})^r \leq C^* d_0$ and this implies

$$d_{n+1} \to 0 \text{ as } n \to \infty . \tag{29}$$

If $r=1$, then from (28), we deduce that

$$d_{n+1} = \frac{C^*}{1+C^*}d_n .$$

Next, we consider the case that $r > 1$. If $d_n = 0$ for an $n \geq 1$, then (28) tells that $d_m = 0, \forall m \geq n$. In this case, Theorem 3.3 is correct. Now, let us assume that $d_n > 0$, $\forall n \geq 1$. Relation (28) is equivalent to

$$d_{n+1} + \frac{1}{C^*}(d_{n+1})^r \leq d_n .$$

An application of Lemma 3.2 assures that there is an $\xi_0 = \xi_0(d_0, C^*, r) \in [0,1]$ such that

$$d_{n+1} \le \left(\frac{r-1}{C^*}\xi_0 + d_n^{1-r}\right)^{\frac{1}{1-r}} .$$

By induction, it follows that

$$\begin{aligned} d_{n+1} &\le \left(\frac{r-1}{C^*}2\xi_0 + d_{n-1}^{1-r}\right)^{-\frac{1}{r-1}} \\ &\le \cdots \\ &\le \left(\frac{r-1}{C^*}(n+1)\xi_0 + d_0^{1-r}\right)^{-\frac{1}{r-1}} . \end{aligned}$$

This proves Theorem 3.3. □

The analysis implies that when $r = 1$, the convergence is uniform. In case that $r > 1$, the convergence can be slow, i.e. $d_n = O\left((rn)^{-\frac{1}{r-1}}\right)$. Especially, when r is very big, $\frac{1}{1-r} \approx 0$ and the convergence can be very slow. However, in Theorem 7.1 in the appendix, we shall show that estimate (19) is really sharp. Using that fact that $\sigma \le p$, we see that it is impossible to have $r < 1$. In order to have $r = 1$, we must require $p = q$. The analysis given in [29] and [1] was done for $p = q = 2$.

Remark 3.1 If there is no extra condition on the decomposed spaces, the condition $\sum_{i=1}^m \alpha_i \le 1$ is sufficient and also necessary for the convergence of Algorithm 2.1. In Remark 4.1 of [28, p. 146], an example is given which shows that if $\sum_{i=1}^m \alpha_i > 1$, then Algorithm 2.1 can be divergent. For overlapping domain decomposition with a suitable coloring, condition $\sum_{i=1}^m \alpha_i \le 1$ is nearly optimal. However, for multigrid method as we shall discuss later, the upper bound of $\sum_{i=1}^m \alpha_i$ with which the algorithm is convergent can be much larger than 1. The upper bound depends on matrix $\mathcal{E} = (\epsilon_{ij})$, where ϵ_{ij} satisfies

$$\langle F'(w_{ij} + u_i) - F'(w_{ij}), v_j\rangle \le \epsilon_{ij}\|u_i\|_V^{q-1}\|v_i\|_V \ ,\forall w_{ij} \in V, \forall u_i \in V_i, \forall v_j \in V_j \ .$$

If the decomposed spaces are orthogonal, it is easy to determine the upper bound of $\sum_{i=1}^m \alpha_i$. In computations for general decomposed spaces, a line search to find the value of t such that the following functional:

$$g(t) = F\left(u^n + t\sum_{i=1}^m e_i^n\right)$$

is attaining its minimum value would be appropriate. To find such a t, we do not need to solve any system of equations and it only needs to evaluate the functional values, which is not computationally expensive.

3.4 The convergence of the successive subspace correction method

The convergence of Algorithm 2.2 is similar to Algorithm 2.1.

Theorem 3.4 *Let the space decomposition satisfies (14), (15) and the functional F satisfies (6). Define*

$$r = \frac{p(p-1)}{q(q-1)}, \quad C^* = \left[\frac{C_1 C_2}{K}\right]^{\frac{p}{q-1}} \frac{p}{K}\left(\frac{L}{q}\right)^r . \tag{30}$$

1. *If $r = 1$, we have*

$$d_{n+1} \leq \frac{C^*}{1+C^*} d_n, \quad \forall n \geq 1 . \tag{31}$$

2. *If $r > 1$, then there exists an $\xi_0 = \xi_0(d_0, C^*, r) \in [0, 1]$ such that*

$$d_{n+1} \leq \left(\frac{r-1}{C^*}\xi_0 + d_n{}^{1-r}\right)^{\frac{1}{1-r}} \leq \left(\frac{r-1}{C^*}(n+1)\xi_0 + d_0^{1-r}\right)^{\frac{1}{1-r}}, \quad \forall n \geq 1 . \tag{32}$$

Proof. Notice

$$F(u^n) - F(u^{n+1}) = \sum_{i=1}^m \left[F(u^{n+i/m}) - F(u^{n+(i-1)/m})\right] . \tag{33}$$

As $u^{n+\frac{i}{m}}$ is the minimizer of (3), we get by (9)

$$F(u^{n+(i-1)/m}) - F(u^{n+i/m}) \geq \frac{K}{p}\|e_i^n\|_V^p . \tag{34}$$

Thus, estimates (33) and (34) together tell that

$$F(u^n) \geq F(u^{n+1}), \tag{35}$$

and

$$F(u^n) - F(u^{n+1}) \geq \frac{K}{p}\sum_{i=1}^m \|e_i^n\|_V^p . \tag{36}$$

Similarly to the proofs for (24)-(26), there holds for any $v_i \in V_i$, which satisfies $\sum_{i=1}^m v_i = u^{n+1} - u$, the relation

$$\begin{aligned} &\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\rangle \\ = &\sum_{i=1}^m \left\langle F'(u^{n+1}) - F'(u^{n+(i-1)/m} + e_i^n), v_i\right\rangle \\ = &\sum_{i=1}^m \sum_{j\geq i}^m \left\langle F'(u^{n+j/m}) - F'(u^{n+(j-1)/m}), v_i\right\rangle \\ \leq &\ C_2\left(\sum_{j=1}^m \|e_j^n\|_V^p\right)^{\frac{q-1}{p}} \left(\sum_{i=1}^m \|v_i\|_V^\sigma\right)^{\frac{1}{\sigma}} . \end{aligned} \tag{37}$$

Let v_i be given as in (22) and using estimates (36) and (37) to obtain

$$\begin{aligned}
&\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\rangle \\
\leq\ & C_2\left[\left(\sum_{i=1}^{m} \|e_i^n\|_V^p\right)^{\frac{q-1}{p}}\right]\left(\sum_{i=1}^{m} \|v_i\|_V^\sigma\right)^{\frac{1}{\sigma}} \\
\leq\ & C_1 C_2\left(\sum_{i=1}^{m} \|e_i^n\|_V^p\right)^{\frac{q-1}{p}} \cdot \|u^{n+1} - u\|_V \qquad (38) \\
\leq\ & C_1 C_2\left(\frac{p}{K}[F(u^n) - F(u^{n+1})]\right)^{\frac{q-1}{p}} \cdot \|u^{n+1} - u\|_V
\end{aligned}$$

The rest of the proof is the same as for Theorem 3.3. □

4 Overlapping domain decomposition for $W^{1,p}(\Omega)$

In this section, we show that how we can use overlapping domain decomposition to decompose a finite element space and to guarantee that the constants C_1 and C_2 do not depend on the mesh parameters.

Let $\{\Omega_i\}_{i=1}^M$ be a shape-regular finite element division, or a coarse mesh, of Ω and Ω_i has diameter of order H. For each Ω_i, we further divide it into smaller simplices with diameter of order h. In case that Ω has a curved boundary, we shall also fill the area between $\partial\Omega$ and $\partial\Omega_H$, here $\bar\Omega_H = \cup_{i=1}^M \bar\Omega_i$, with finite elements with diameters of order h. We assume that the resulting elements form a shape regular finite element subdivision of Ω, see Ciarlet [9]. We call this the fine mesh or the h-level subdivision of Ω with mesh parameter h. We denote $\Omega_h = \cup\{\mathcal{T} \in \mathcal{T}_h\}$ as the fine mesh subdivision. Let $S_0^H \subset W_0^{1,p}(\Omega_H)$ and $S_0^h \subset W_0^{1,p}(\Omega_h)$ be the continuous, piecewise r^{th} order polynomial finite element spaces, with zero trace on $\partial\Omega_H$ and $\partial\Omega_h$, over the H-level and h-level subdivisions of Ω respectively. More specifically,

$$S_0^H = \left\{v \in W_0^{1,p}(\Omega_H) \middle|\quad v|_{\Omega_i} \in P_r(\Omega_i), \forall i\right\},$$

$$S_0^h = \left\{v \in W_0^{1,p}(\Omega_h) \middle|\quad v|_{\mathcal{T}} \in P_r(\mathcal{T}), \forall \mathcal{T} \in \mathcal{T}_h\right\}.$$

For each Ω_i, we consider an enlarged subdomain $\Omega_i^\delta = \{\mathcal{T} \in \mathcal{T}_h, dist(\mathcal{T}, \Omega_i) \leq \delta\}$. The union of Ω_i^δ covers $\bar\Omega_h$ with overlaps of size δ. Let us denote the piecewise r^{th} order polynomial finite element space with zero traces on the boundaries $\partial\Omega_i^\delta$ as $S_0^h(\Omega_i^\delta)$. Then one can show that

$$S_0^h = S_0^H + \sum S_0^h(\Omega_i^\delta)\,. \qquad (39)$$

For the overlapping subdomains, assume that there exist m colors such that each subdomain Ω_i^δ can be marked with one color, and the subdomains with the same color will not intersect with each other. For suitable overlaps, one can always choose $m = 2$ if $d = 1; m \le 4$ if $d = 2; m \le 8$ if $d = 3$, see Figure 1. Let Ω_i' be the union of the subdomains with the i^{th} color, and

$$V_i = \{v \in S_0^h | \quad v(x) = 0, \quad x \notin \Omega_i'\} \ .$$

By denoting subspaces $V_0 = S_0^H$, $V = S_0^h$, we find that decomposition (39) means

$$V = V_0 + \sum_{i=1}^{m} V_i, \tag{40}$$

and so the two level method is a way to decompose the finite element space. Similar as in [30], let $\{\theta_i\}_{i=1}^m$ be a partition of unity with respect to $\{\Omega_i'\}_{i=1}^m$, i.e. $\theta_i \in C_0^\infty(\Omega_i' \cap \Omega)$, $\theta_i \ge 0$ and $\sum_{i=1}^m \theta_i = 1$. It can be chosen so that

$$|\nabla\theta_i| \le C/\delta, \quad \theta_i(x) = \begin{cases} 1 & \text{if distance } (x, \partial\Omega_i') \ge \delta \text{ and } x \in \Omega_i', \\ 0 & \text{on } \overline{\Omega \backslash \Omega_i'}. \end{cases} \ .$$

Let I_h be an interpolation operator which uses the function values at the h-level nodes. For any $v \in V$, let $v_0 \in V_0$ be the solution of $(v_0, \phi_H) = (v, \phi_H), \forall \phi_H \in V_0$, and $v_i = I_h(\theta_i(v - v_0))$. They satisfy $v = \sum_{i=0}^m v_i$, and

Lemma 4.1 *For any* $s \ge 1$,

$$\left(\|v_0\|_{1,p}^s + \sum_{i=1}^m \|v_i\|_{1,p}^s \right)^{\frac{1}{s}} \le C(m+1)^{\frac{1}{s}} \left(1 + \left(\frac{H}{\delta}\right)^{\frac{p-1}{p}} \right) \|v\|_{1,p}. \tag{41}$$

Proof of Lemma 4.1. Let us denote $w = v - v_0$, $\Gamma_{i,\delta}$ the union of all the finite elements $\mathcal{T}$ of Ω_i' on which θ_i is not constant, and by $\bar{\theta}_i$ the average of θ_i on a given finite element. By construction, we have on any given $\mathcal{T}$

$$v_i = \bar{\theta}_i w + I_h((\theta_i - \bar{\theta}_i)w),$$

$$\int_{\mathcal{T}} |\nabla v_i|^p dxdy \le 2^{p-1} \int_{\mathcal{T}} |\nabla(\bar{\theta}_i w)|^p dxdy + 2^{p-1} \int_{\mathcal{T}} |\nabla I_h((\theta_i - \bar{\theta}_i)w)|^p dxdy \ . \tag{42}$$

The last term is zero if $\mathcal{T}$ does not belong to $\Gamma_{i,\delta}$. It is easy to show that

$$\left\| \nabla I_h\Big((\theta_i - \bar{\theta}_i)w\Big) \right\|_{0,p,\mathcal{T}}^p \le \frac{C}{\delta^p} \|w\|_{0,p,\mathcal{T}}^p \ .$$

Therefore, by summing over all elements $\mathcal{T}$ of Ω_i', we get

$$\int_{\Omega_i'} |\nabla v_i|^p dxdy \le 2^{p-1} \int_{\Omega_i'} |\nabla w|^p dxdy + \frac{C}{\delta^p} \int_{\Gamma_{i,\delta}} |w|^p dx \ . \tag{43}$$

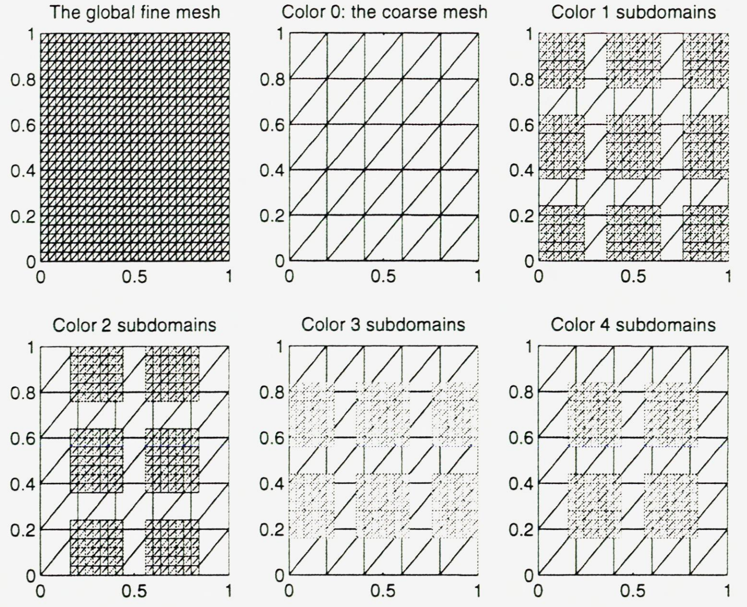

Figure 1: The coloring of the subdomains and the coarse mesh grid

But $\Gamma_{i,\delta}$ is a subdomain of Ω_i', whose width is at most δ. For simplicity, we assume that $\Omega_i' = [0, H] \times [0, H]$ and $\Gamma_{i,\delta} = [0, H] \times [0, \delta]$. The modifications necessary for the case of arbitrary shaped subdomains and higher dimensional problems are routine. Writing

$$w(x, y) = w(x, 0) + \int_0^y \frac{\partial w}{\partial y} dy \tag{44}$$

Using Cauchy Schwartz inequality, we have therefore,

$$H \int_0^H |w(x,0)|^p dx \le 2^{p-1} \int_0^H \int_0^H |w|^p dx dy + 2^{p-1} H^p \int_0^H \int_0^H |\nabla w|^p dx dy.$$

This shows that

$$\int_{\partial \Omega_i'} |w|^p ds \le C 2^{p-1} H^{-1} \int_{\Omega_i'} |w|^p dx dy + C 2^{p-1} H^{p-1} \int_{\Omega_i'} |\nabla w|^p dx dy \ . \tag{45}$$

Considering the integral over $\Gamma_{i,\delta}$ and using (44), we obtain

$$\int_{\Gamma_{i,\delta}} |w|^p dx dy \le 2^{p-1} \delta^p \int_{\Omega_i'} |\nabla w|^p dx dy + 2^{p-1} \delta \int_{\partial \Omega_i'} |w|^p ds. \tag{46}$$

Combining (43), (45) and (46), we see that

$$\int_{\Omega_i'} |\nabla v_i|^p dxdy \le C\left(1+\frac{H^{p-1}}{\delta^{p-1}}\right)\int_{\Omega_i'} |\nabla w|^p dxdy + \frac{C}{H\delta^{p-1}}\int_{\Omega_i'} |w|^p dxdy \quad (47)$$

Taking into account that $w = v - v_0$, we get from (47) that

$$\int_{\Omega} |\nabla v_i|^p dxdy \le C\left(1+\frac{H^{p-1}}{\delta^{p-1}}\right)\int_{\Omega} |\nabla w|^p dxdy. \quad (48)$$

It is true that

$$\int_{\Omega} |\nabla v_0|^p dxdy \le C\int_{\Omega} |\nabla v|^p dxdy. \quad (49)$$

Therefore, relations (48) and (49) imply that

$$\|v_0\|_{1,p} \le C\|v\|_{1,p}, \quad \|v_i\|_{1,p} \le C\left(1+\frac{H^{p-1}}{\delta^{p-1}}\right)^{\frac{1}{p}}\|v\|_{1,p}. \quad (50)$$

Lemma 4.1 follows from (50) and the inequality

$$(1+x^{p-1})^{\frac{1}{p}} \le 1 + x^{\frac{p-1}{p}}, \quad \forall x \ge 0, \quad p > 1.$$

□

Using the Cauchy-Schwarz inequality, it is easy to prove:

Lemma 4.2 *Under condition (6), we have for any $s > 1$:*

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\langle F'(w_{ij}+u_i) - F'(w_{ij}), v_j\rangle \le \sum_{i=1}^{m}\sum_{j=1}^{m} L\|u_i\|_V^{q-1}\|v_i\|_V$$

$$\le \; Lm^{\frac{s-1}{s}}\left(\sum_{i=1}^{m}\|v_i\|_V^s\right)^{\frac{1}{s}} m^{\frac{1}{\sigma}}\left(\sum_{i=1}^{m}\|u_i\|_V^{\frac{p}{s}}\right)^{\frac{q-1}{p}} \quad (51)$$

$$\forall u_i \in V_i, \; w_{ij} \in V \text{ and } v_j \in V_j \; .$$

Estimates (41) and (51) show that for overlapping domain decomposition, the constants in (14) and (15) are

$$C_1 = C(m)\left(1+\left(\frac{H}{\delta}\right)^{\frac{p-1}{p}}\right), \qquad C_2 = C(m).$$

By requiring $\delta = c_0 H$, where c_0 is a given constant, we have that C_1 and C_2 are independent of the mesh parameters h and H, the number of subdomains, and estimate (41), (51) are also valid for 3D problems. So if the proposed algorithms are used, their error reductions per step are independent of these parameters.

5 Multilevel decomposition for $W^{1,p}(\Omega)$

In this section, we discuss the application of our theory to multigrid methods. From the space decomposition point of view, a multigrid algorithm is built upon the subspaces that are defined on a nested sequence of finite element partitions.

We assume that the finite element partition $\mathcal{T}$ is constructed by a successive refinement process. More precisely, $\mathcal{T} = \mathcal{T}_J$ for some $J > 1$, and $\mathcal{T}_j$ for $j \leq J$ are a nested sequence of quasi-uniform finite element partitions, i.e. $\mathcal{T}_j$ consist of finite elements $\mathcal{T}_j = \{\tau_j^i\}$ of size h_j such that $\Omega = \cup_i \tau_j^i$ for which the quasi-uniformity constants are independent of j (cf. [9]) and τ_{j-1}^l is a union of elements of $\{\tau_j^i\}$. We further assume that there is a constant $\gamma < 1$, independent of j, such that h_j is proportional to γ^{2j}.

As an example, in the two dimensional case, a finer grid is obtained by connecting the midpoints of the edges of the triangles of the coarser grid, with $\mathcal{T}_1$ being the given coarsest initial triangulation, which is quasi-uniform. In this example, $\gamma = 1/\sqrt{2}$.

Corresponding to each finite element partition $\mathcal{T}_j$, a finite element space $\mathcal{M}_j$ can be defined by

$$\mathcal{M}_j = \{v \in W^{1,p}(\Omega) : v|_\tau \in \mathcal{P}_1(\tau), \quad \forall\, \tau \in \mathcal{T}_j\}.$$

Each finite element space $\mathcal{M}_j$ is associated with a nodal basis, denoted by $\{\phi_j^i\}_{i=1}^{n_j}$ satisfying

$$\phi_j^i(x_j^k) = \delta_{ik}$$

where $\{x_j^k\}_{k=1}^{n_j}$ is the set of all nodes of the elements of $\mathcal{T}_j$. Associated with each such a nodal basis function, we define a one dimensional subspace as follows

$$\mathcal{M}_j^i = \text{span } (\phi_j^i).$$

On each level, the nodes can be colored so that the neighboring nodes are always of different colors. The number of colors needed for a regular mesh is always a bounded constant; call it m_c. Let V_j^k, $k = 1, 2, \cdots m_c$ be the sum of the subspaces $\mathcal{M}_j^i$ associated with nodes of the k^{th} color on level j. Letting $V = \mathcal{M}_J$, we have the following trivial space decomposition:

$$V = \sum_{j=1}^{J} \sum_{k=1}^{m_c} V_j^k. \tag{52}$$

Each subspace V_j^k contains some orthogonal one dimensional subspaces $\mathcal{M}_j^i$ and so the minimization problems (3) and (5) for each V_j^k can be done in parallel over the one dimensional subspaces $\mathcal{M}_j^i$.

5.1 Estimation of the constant C_1

For any $j \leq J$, let Q_j be the L^2 project operator to the finite element space $\mathcal{M}_j$ at level j. For any $v \in V$, define $v_j = (Q_j - Q_{j-1})v \in \mathcal{M}_j$. A further decomposition of v_j is given by

$$v_j = \sum_{i=1}^{n_j} \nu_j^i \quad \text{with} \quad \nu_j^i = v_j(x_j^i)\phi_j^i.$$

Let v_j^k, $k = 1, 2, \cdots, m_c$ be the sum of ν_j^i associated with the nodes of the k^{th} color on level j. It is easy to see that

$$v_j = \sum_{k=1}^{m_c} v_j^k = \sum_{i=1}^{n_j} \nu_j^i.$$

Denote Ω_j^k the union of the support sets of the basis functions associated with the k^{th} color nodes on level j. We estimate

$$\sum_{k=1}^{m_c} |v_j^k|_{1,p}^{\sigma} = \sum_{k=1}^{m_c} \Big(\sum_{x_j^i \in \Omega_j^k} |v_j(x_j^i)|^p \, |\phi_j^i|_{1,p}^p \Big)^{\frac{\sigma}{p}} \leq C h_j^{\frac{\sigma(d-p)}{p}} \sum_{k=1}^{m_c} \Big(\sum_{x_j^i \in \Omega_j^k} |v_j(x_j^i)|^p \Big)^{\frac{\sigma}{p}}.$$

In the above, we have assumed that $\Omega \subset R^d, d = 1, 2, 3, \cdots$. Using the inequality

$$\sum_{k=1}^{m_c} |a_k|^{\sigma} \leq \Big(\sum_{k=1}^{m_c} |a_k|^p \Big)^{\frac{\sigma}{p}} \Big(m_c \Big)^{1-\frac{\sigma}{p}},$$

we get that

$$\sum_{k=1}^{m_c} |v_j^k|_{1,p}^{\sigma} \leq C h_j^{\frac{\sigma(d-p)}{p}} (m_c)^{1-\frac{\sigma}{p}} \Big(\sum_{i=1}^{n_j} |v_j(x_j^i)|^p \Big)^{\frac{\sigma}{p}} \leq C h_j^{-\sigma} \|v_j\|_{0,p}^{\sigma}.$$

Here, we have used the fact that, in the finite element space, an L^p norm is equivalent to some discrete L^p norm, namely

$$\|v_j\|_{0,p}^p \cong h_j^d \sum_{i=1}^{n_j} |v_j(x_j^i)|^p.$$

As a consequence,

$$\sum_{j=1}^{J} \sum_{k=1}^{m_c} \|v_j^k\|_{1,p}^{\sigma} \leq C \sum_{j=1}^{J} h_j^{-\sigma} \|v_j\|_{0,p}^{\sigma}$$

$$\leq C \sum_{j=1}^{J} h_j^{-\sigma} \Big\| \Big(Q_j - Q_{j-1}\Big) v \Big\|_{0,p}^{\sigma} \leq C \sum_{j=1}^{J} h_j^{-\sigma} \Big\| Q_j \Big(I - Q_{j-1}\Big) v \Big\|_{0,p}^{\sigma}$$

$$\leq C \sum_{j=1}^{J} h_j^{-\sigma} \left\| \left(I - Q_{j-1}\right) v \right\|_{0,p}^{\sigma} \leq C \sum_{j=1}^{J} h_j^{-\sigma} h_{j-1}^{\sigma} \|v\|_{1,p}^{\sigma} \tag{53}$$
$$\leq C\gamma^{-2\sigma} J \|v\|_{1,p}^{\sigma} .$$

which proves that

$$C_1 \cong J^{\frac{1}{\sigma}} \cong |\log h|^{\frac{1}{\sigma}}.$$

In proving the inequality (53), we have used the stability in L^p of the L^2-projection [11] and the error estimate for L^2-projections, see [9].

5.2 Estimation of the constant C_2

From condition (9), we see that

$$\langle F'(w_{ij} + u_i) - F'(w_{ij}), v_j \rangle \leq L \|u_i\|_V^{q-1} \|v_j\|_V . \tag{54}$$

However, in order to estimate the constant C_2, we need to use a finer estimate than (54). For any $w, u, v \in V$, we need the functional F to satisfy

$$\langle F'(w + u) - F'(w), v \rangle \leq L \|u\|_{1,p,supp(u)\cap supp(v)}^{q-1} \|v\|_{1,p,supp(u)\cap supp(v)} . \tag{55}$$

In the above and also later, *supp* denotes the support set of a function. For any $u \in \mathcal{M}_j^i$ and $v \in \mathcal{M}_l^k$, $j \leq l$, we note that the size of $supp(u) \cap supp(v)$ is at most the size of $supp(v)$. Thus since u is piecewise linear

$$\|u\|_{1,p,supp(u)\cap supp(v)} \leq C\gamma^{\frac{2d}{p}|j-l|} \|u\|_{1,p}, \quad \forall u \in \mathcal{M}_j^i, \forall v \in \mathcal{M}_l^k. \tag{56}$$

Let $w \in V, u \in V_j^i$ and $v \in V_l^k$. We decompose u and v as

$$u = \sum_{\alpha=1}^{n_i} u_\alpha, \quad u_\alpha = u(x_j^\alpha)\phi_j^\alpha, \quad v = \sum_{\beta=1}^{n_l} v_\beta, \quad v_\beta = v(x_l^\beta)\phi_l^\beta,$$

i.e. functions u and v are decomposed into functions from the one dimensional subspaces of the same colors. We shall assume that the following inequality is valid for the above decomposition:

$$\langle F'(w + \sum u_\alpha) - F'(w), \sum v_\beta \rangle \leq \sum_\alpha \sum_\beta \langle F'(w + u_\alpha) - F'(w), v_\beta \rangle. \tag{57}$$

The above inequality is often a consequence of the orthogonality of the one dimensional subspaces of the same color and the fact that u is zero at the nodes that do have the color of u.

From (57), (55), (56) and the orthogonality of the one dimensional subspaces of the same color, it is easy to see that

$$\langle F'(w + u) - F'(w), v \rangle \leq C\gamma^{\frac{2d(q-1)}{p}(l-j)} L \sum_\alpha \sum_\beta \|u_\alpha\|_{1,p}^{q-1} \|v_\beta\|_{1,p}$$
$$= C\gamma^{\frac{2d(q-1)}{p}(l-j)} L \|u\|_{1,p}^{q-1} \|v\|_{1,p}, \quad \forall u \in V_j^i, \ v \in V_l^k, \ j \leq l.$$

For $j > l$, we shall have

$$\langle F'(w+u) - F'(w), v\rangle \le C\gamma^{\frac{2d}{p}(j-l)} L\|u\|_{1,p}^{q-1}\|v\|_{1,p}, \quad \forall u \in V_j^i,\ v \in V_l^k.$$

Denoting $\gamma_0 = \gamma^{\frac{2d}{p}\max(q-1,1)}$, we get from the above two estimates

$$\langle F'(w+u) - F'(w), v\rangle \le C{\gamma_0}^{|j-l|} L\|u\|_{1,p}^{q-1}\|v\|_{1,p}, \quad \forall u \in V_j^i,\ v \in V_l^k. \tag{58}$$

To estimate the constant C_2 we need the next lemma, which extends a result of [23, p.184]:

Lemma 5.1 *Let $A = \{A_{ij}\theta_{ij}\}$ be an $n_1 \times n_2$ matrix. Then*

$$\|Ax\|_{\ell^\sigma} \le \max_j \left(\sum_i |\theta_{ij}|^\sigma\right)^{\frac{1}{\sigma}} \max_i \left(\sum_j |A_{ij}|^{\sigma'}\right)^{\frac{1}{\sigma'}} \|x\|_{\ell^\sigma}.$$

Proof of Lemma 5.1. The Cauchy-Schwarz inequality gives

$$\begin{aligned}
\|Ax\|_{\ell^\sigma}^\sigma &= \sum_i \left|\sum_j A_{ij}\theta_{ij}x_j\right|^\sigma \\
&\le \sum_i \left(\sum_j |A_{ij}|^{\sigma'}\right)^{\frac{\sigma}{\sigma'}} \left(\sum_j |\theta_{ij}|^\sigma |x_j|^\sigma\right) \\
&\le \max_i \left(\sum_j |A_{ij}|^{\sigma'}\right)^{\frac{\sigma}{\sigma'}} \sum_j \left(\sum_i |\theta_{ij}|^\sigma |x_j|^\sigma\right) \\
&\le \max_j \left(\sum_i |\theta_{ij}|^\sigma\right) \max_i \left(\sum_j |A_{ij}|^{\sigma'}\right)^{\frac{\sigma}{\sigma'}} \sum_j |x_j|^\sigma,
\end{aligned}$$

which proves the lemma. □

As a consequence of the above Lemma, we easily get the following corollary which generalizes a well-known result from linear algebra, see [21, p.3-38].

Corollary 5.1 *Let $A = \{A_{ij}\}$ be a symmetric matrix; then*

$$\|Ax\|_{\ell^\sigma} \le \left(\max_i \sum_j |A_{ij}|\right) \|x\|_{\ell^\sigma}.$$

Proof of Corollary 5.1. It is easy to see that

$$|A_{ij}| = |A_{ij}|^{\frac{1}{\sigma'}} |A_{ij}|^{\frac{1}{\sigma}}.$$

The Corollary is an easy consequence of Lemma 5.1 by setting $A_{ij} := |A_{ij}|^{\frac{1}{\sigma'}}$ and $\theta_{ij} := |A_{ij}|^{\frac{1}{\sigma}}$. □

For a given $u_j^i \in V_j^i$ and $v_l^k \in V_l^k$, an application of Corollary 8 gives

$$\sum_{i=1}^{m_c}\sum_{k=1}^{m_c} \|u_j^i\|_{1,p}^{q-1} \|v_l^k\|_{1,p} \le m_c \left(\sum_{i=1}^{m_c} \|u_j^i\|_{1,p}^p\right)^{\frac{q-1}{p}} \left(\sum_{k=1}^{m_c} \|v_l^k\|_{1,p}^\sigma\right)^{\frac{1}{\sigma}}$$

Using the above inequality and Corollary 5.1, we get that

$$\begin{aligned}
&\sum_{j=1}^{J}\sum_{l=1}^{J}\sum_{i=1}^{m_c}\sum_{k=1}^{m_c} \gamma_0^{|j-l|} \|u_j^i\|_{1,p}^{q-1} \|v_l^k\|_{1,p} \\
&\le m_c \sum_{j=1}^{J}\sum_{l=1}^{J} \gamma_0^{|j-l|} \left(\sum_{i=1}^{m_c} \|u_j^i\|_{1,p}^p\right)^{\frac{q-1}{p}} \left(\sum_{k=1}^{m_c} \|v_l^k\|_{1,p}^\sigma\right)^{\frac{1}{\sigma}} \qquad (59)\\
&\le m_c \left(\max_j \sum_{l=1}^{J} \gamma_0^{|j-l|}\right) \left(\sum_{j=1}^{J}\sum_{i=1}^{m_c} \|u_j^i\|_{1,p}^p\right)^{\frac{q-1}{p}} \cdot \left(\sum_{l=1}^{J}\sum_{k=1}^{m_c} \|v_l^k\|_{1,p}^\sigma\right)^{\sigma} \\
&\le \frac{m_c}{1-\gamma_0} \left(\sum_{j=1}^{J}\sum_{i=1}^{m_c} \|u_j^i\|_{1,p}^p\right)^{\frac{q-1}{p}} \cdot \left(\sum_{l=1}^{J}\sum_{k=1}^{m_c} \|v_l^k\|_{1,p}^\sigma\right)^{\sigma}.
\end{aligned}$$

From (15), (58) and (59), we conclude that the constant C_2 is independent of the mesh size h and the number of levels J for decomposition (52).

Remark 5.1 *In case that $p = q$, the estimations we have derived for the constants C_1 and C_2 are also valid for decomposition*

$$V = \sum_{j=1}^{J}\sum_{i=1}^{n_j} \mathcal{M}_j^i, \qquad (60)$$

i.e. the coloring is not necessary for implementing the algorithms.

6 Some Applications

In this section we illustrate some problems that our algorithms are applicable without going into the details of analyses.

6.1 Linear problems

The algorithms can be used for linear second order equation

$$\begin{cases} -\nabla \cdot (a\nabla u) = f \text{ in } \Omega \subset R^d , \\ u = 0 \text{ on } \partial\Omega , \end{cases} \qquad (61)$$

and linear fourth order equation

$$\begin{cases} -\Delta(a\Delta u) = f \text{ in } \Omega \subset R^d \ , \\ u = 0, \ \frac{\partial u}{\partial n} = 0, \ \text{on } \partial\Omega \ . \end{cases}$$

If we use Algorithm 2.2 for a general symmetric positive define linear problem

$$a(u, v) = (f, v), \qquad \forall v \in V.$$

then the implementation can be divided into the following steps:

Algorithm 6.1 *(Application to linear problems)*

1. *Choose initial values* $u^0 \in V$ *and compute the initial residual* r^0 *such that* $(r^0, v) = (f, v) - a(u^0, v), \forall v \in V$.

2. *For* $i = 1, 2, \cdots, m$, *if* $r^{n+\frac{(i-1)}{m}}$ *is known, compute* $e_i^n \in V_i$ *such that*

$$a(e_i^n, v_i) = (r^{n+\frac{(i-1)}{m}}, v_i) \ , \quad \forall v_i \in V_i \ . \tag{62}$$

3. *Update the residual* $r^{n+\frac{i}{m}}$ *such that*

$$(r^{n+\frac{i}{m}}, v) = (r^{n+\frac{(i-1)}{m}}, v) - a(e_i^n, v), \quad \forall v \in V \ . \tag{63}$$

4. *Update the solution as*

$$u^{n+\frac{i}{m}} = u^{n+\frac{(i-1)}{m}} + e_i^n \ . \tag{64}$$

   and go to the next iteration.

The implementation for Algorithm 2.1 is similar. If the subspaces V_i are associated with the overlapping domain decomposition, then equation (62) is the solving of the subdomain problems. Equations (63) and (64) are just the simple updatings of the residual and the solution in the subdomains. If the subspaces V_i are associated with the multigrid method, then equation (62) is to compute the correction value for the nodal bases at different levels. Equations (63) and (64) are the updatings for the residual and solution corresponding to the nodal bases.

For such a kind of symmetrical linear problem, we have $p = q = 2$, and so the decomposition (60) can be used. We can also do V-cycle and W-cycle types of iteration if we just repeat some of the nodal bases in the decomposition of V in (60). It is preferable to use a V-cycle decomposition and then use the conjugate gradient method as an out-iteration to accelerate the convergence.

6.2 Nonlinear elliptic equations

Consider

$$\begin{cases} - & \nabla \cdot (|\nabla u|^{s-2} \nabla u) = f \text{ in } \Omega \subset R^d \ (1 < s < \infty)\ , \\ & u = 0 \text{ on } \partial\Omega\ . \end{cases} \tag{65}$$

For equation (65), we assume $f \in W^{-1,s'}(\Omega)$, $\frac{1}{s} + \frac{1}{s'} = 1$. By standard techniques, it can be shown, see [14], that (65) possesses a unique solution which is the minimizer of

$$\min_{v \in W_0^{1,s}(\Omega)} \left[\frac{1}{s} \int_\Omega |\nabla v|^s - \langle f, v \rangle \right] .$$

Even with very smooth data, the solution u may not be in the space $W_0^{2,s}$, see Ciarlet [9, p. 324]. When s is close to 1 or is very big ($s \gg 2$), it is difficult to solve this problem numerically. Conditions (6) are fulfilled by equation (65), see p. 319 and p. 325 of Ciarlet [9]. More precisely, we have for

$$V = W_0^{1,s}(\Omega), \quad F(v) = \int_\Omega (\frac{1}{s}|\nabla v|^s - fv) dx$$

the following estimates:

$$\langle F'(v) - F'(w), v - w \rangle \geq \|v - w\|_{1,s}^s, \quad \text{if } s \geq 2. \tag{66}$$

$$\langle F'(v) - F'(w), v - w \rangle \geq \alpha \frac{\|v - w\|_{1,s}^2}{(\|v\|_{1,s} + \|w\|_{1,s})^{2-s}}, \quad \text{if } 1 < s \leq 2. \tag{67}$$

$$\|F'(v) - F'(w)\|_{V'} \leq \beta (\|v\|_{1,s} + \|w\|_{1,s})^{s-2} \|v - w\|_{1,s} \text{ if } s \geq 2. \tag{68}$$

$$\|F'(v) - F'(w)\|_{V'} \leq \beta \|v - w\|_{1,s}^{s-1} \text{ if } 1 < s \leq 2. \tag{69}$$

In the above, α and β are independent of v and w and are strictly positive. The proof of (66) and (68) is given in p. 319 of Ciarlet [9]. The proof of (67) and (69) can be found in Glowinski and Marrocco [16]. Corresponding to condition (6), these estimates imply that

$$p = s, \quad q = 2 \quad \text{if } s \geq 2;$$

$$p = 2, \quad q = s \quad \text{if } 1 < s \leq 2.$$

As we explained in §3.1, it is assumed that v and w are in a neighborhood of the true solution. The full potential equation considered in [6] is of a similar type to equation (65).

For more general problem

$$\min_{v \in W_0^{1,s}(\Omega)} \int_\Omega \frac{1}{2} a(|\nabla v|^2) + f(v) \ , \tag{70}$$

we assume that a is strictly convex and f is convex and both are differentiable. If we use Algorithm 2.2 for (70), then we obtain

Algorithm 6.2 *(Application to nonlinear problems)*

1. *Choose initial values* $u^0 \in V$.
2. *For* $i = 1, 2, \cdots, m$, *if* $u^{n+\frac{(i-1)}{m}}$ *is known, compute* $e_i^n \in V_i$ *such that*

$$\int_\Omega \Big[a'(|\nabla(u^{n+\frac{(i-1)}{m}} + e_i^n)|^2)\nabla(u^{n+\frac{(i-1)}{m}} + e_i^n) \cdot \nabla v_i$$
$$+ f'(u^{n+\frac{(i-1)}{m}} + e_i^n) v_i \Big] dx = 0 \ , \forall v_i \in V_i \ . \tag{71}$$

3. *Update the solution as*

$$u^{n+\frac{i}{m}} = u^{n+\frac{(i-1)}{m}} + e_i^n \ . \tag{72}$$

and go to the next iteration.

If V_i are the domain decomposition subspaces, then problem (71) is a nonlinear problem in each subdomain, which has a smaller size than the original problem. For some minimization methods, the convergence and the computing time depend on the size of the problem. Thus by first reducing the problem into smaller size problems and then minimize, we may gain efficiency. If V_i are the multigrid nodal basis subspaces, then (71) is equivalent to some one dimensional nonlinear problems and we can use efficient minimization routines to solve the one dimensional problems.

6.3 Eigenvalue problems

Consider the minimization of the following functional to obtain the smallest eigenvalue and the corresponding eigenvector for a symmetric positive definite matrix A:

$$F(v) = \frac{(Av, v)}{\|v\|_V^2}.$$

This functional is not convex globally, but is convex in the neighbourhood of the true minimizer. See [8] and [22] for some detailed analysis and numerical simulations.

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7 Appendix

Proof of Lemma 3.2. For the given $a > 0, \eta > 0, r > 1$, let $y = y(t)$ be the solution of the ordinary differential equation

$$\left\{ \begin{array}{lll} \frac{dy}{dt} & = -\eta y^r \ , & t > 0 \ , \\ y & = a \quad \text{at} & t = 0 \ . \end{array} \right. \tag{73}$$

It is easy to calculate that its solution is

$$\begin{array}{r} \frac{dy}{y^r} = -\eta dt \ , \\ \frac{1}{1-r}\left(y^{1-r} - a^{1-r}\right) = -\eta \ t \ , \\ y(t) = \left(\eta(r-1)t + a^{1-r}\right)^{\frac{1}{1-r}} \ . \end{array} \tag{74}$$

Next, we show that there exists a $\xi > 0$ such that

$$E(\xi) \equiv y(\xi) + \eta(y(\xi))^r - a \geq 0 \ .$$

The Taylor expansion formula asserts that there exists an $\bar{\xi} \in [0, \xi]$ such that

$$\begin{array}{rl} & E(\xi) = y(\xi) + \eta(y(\xi))^r - a \\ = & y(0) + y'(\bar{\xi})\xi + \eta(y(\xi))^r - a \\ = & y'(\bar{\xi})\xi + \eta(y(\xi))^r \ . \end{array} \tag{75}$$

It is clear that y is a non-increasing function of t, i.e. $y(\bar{\xi}) \leq y(0) = a$. Equation (73) indicates that $\eta(y(t))^r = -y'(t)$, $\forall t$. So (75) implies

$$\begin{array}{rl} E(\xi) = & -\eta(y(\bar{\xi}))^r\xi + \eta(y(\xi))^r \\ \geq & -\eta a^r \xi + \eta(y(\xi))^r \\ = & -\eta a^r \xi - y'(\xi) \ . \end{array} \tag{76}$$

Using (73) we see that

$$y''(\xi) = \frac{d}{d\xi}(-\eta y^r(\xi)) = -\eta r(y(\xi))^{r-1}y'(\xi) = \eta^2 r(y(\xi))^{2r-1} \ .$$

Again, by (73) and using Taylor expansion for $y'(\xi)$ in (76), we know that there exists an $\tilde{\xi} \in [0, \xi]$ such that

$$\begin{array}{rl} E(\xi) \geq & -y'(0) - y''(\tilde{\xi})\xi - \eta a^r \xi \\ = & \eta a^r - \eta^2 r(y(\tilde{\xi}))^{2r-1}\xi - \eta a^r \xi \ . \end{array} \tag{77}$$

It is true that $y(\tilde{\xi}) \leq y(0) = a$. Thus (77) and the fact that $a \leq a_0$ infer that

$$\begin{array}{rl} E(\xi) \geq & \eta a^r - \eta^2 r a^{2r-1}\xi - \eta a^r \xi \\ \geq & \eta a^r - \eta^2 r a^r a_0^{r-1}\xi - \eta a^r \xi \\ = & \eta a^r(1 - \eta r a_0^{r-1}\xi - \xi) \ . \end{array}$$

Thus, let

$$\xi_0 = \frac{1}{\eta r a_0^{r-1} + 1}, \qquad \text{there holds} \qquad E(\xi_0) \geq 0 \ . \tag{78}$$

Clearly, $\xi_0 \in [0,1]$ does not depends on a and b. The inequality of (78) shows that

$$y(\xi_0) + \eta(y(\xi_0))^r \geq a \ . \tag{79}$$

A combination of (16) and (79) tells that

$$b + \eta b^r \leq y(\xi_0) + \eta(y(\xi_0))^r \ ,$$

and from which we see that

$$b \leq y(\xi_0) \ .$$

This proves the lemma.

Next, we show that the estimates of Theorems 3.3 and 3.4 are really sharp.

Theorem 7.1 *Under the conditions of Theorem 3.3, assume that $r > 1$ and*

$$(d_{n+1})^r = C^*(d_n - d_{n+1}) \ , \tag{80}$$

i.e. d_{n+1} is reaching the maximum possible error at each iteration. Then there holds

$$d_{n+1} \geq \left(\frac{r-1}{C^*} + d_n{}^{1-r}\right)^{\frac{1}{1-r}} \geq \left(\frac{r-1}{C^*}(n+1) + d_0^{1-r}\right)^{\frac{1}{1-r}}, \quad \forall n \geq 1 \ . \tag{81}$$

Proof of Theorem 7.1. We define, for $n \geq 1$, $\delta_n > 0$ to be the unique number which satisfies

$$d_{n+1} = \left(\frac{r-1}{C^*} + \delta_n^{1-r}\right)^{\frac{1}{1-r}} \ . \tag{82}$$

In addition, let $y = y_a(t)$ be the solution of the ordinary differential equation (73) with $\eta = \frac{1}{C^*}$, i.e.

$$y_a(t) = \left(\frac{r-1}{C^*}t + a^{1-r}\right)^{\frac{1}{1-r}} \ . \tag{83}$$

From definition (82), it is true that $d_{n+1} = g_{\delta_n}(1)$. Using Taylor expansion, we know that there exists $\xi \in [0,1]$ such that

$$\begin{aligned} d_{n+1} &= y_{\delta_n}(1) = y_{\delta_n}(0) + y'_{\delta_n}(\xi) = \delta_n - \tfrac{1}{C^*} y^r_{\delta_n}(\xi) \\ &= \delta_n - \tfrac{1}{C^*}\left(\tfrac{r-1}{C^*}\xi + \delta_n^{1-r}\right)^{\frac{r}{1-r}} \ . \end{aligned} \tag{84}$$

From which it follows

$$C^*(\delta_n - d_n) = C^*(d_{n+1} - d_n) + \left(\frac{r-1}{C^*}\xi + \delta_n^{1-r}\right)^{\frac{r}{1-r}} \ .$$

From relation (80), this implies

$$\begin{aligned} & C^*(\delta_n - d_n) \\ & = -(d_{n+1})^r + \left(\tfrac{r-1}{C^*}\xi + \delta_n^{1-r}\right)^{\frac{r}{1-r}} \\ & = -\left(\tfrac{r-1}{C^*} + \delta_n^{1-r}\right)^{\frac{r}{1-r}} + \left(\tfrac{r-1}{C^*}\xi + \delta_n^{1-r}\right)^{\frac{r}{1-r}} \\ & \geq 0\ . \end{aligned} \tag{85}$$

Therefore, we get

$$\delta_n \geq d_n\ . \tag{86}$$

From (83), it is easy to check that y_a is an increasing function with respect to a. So from (82) and (86), one concludes

$$d_{n+1} \geq \left(\frac{r-1}{C^*} + d_n^{1-r}\right)^{\frac{1}{1-r}}\ . \tag{87}$$

An induction of (87) proves (81).

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