Department
of
APPLIED MATHEMATICS

On the convergence Rate of Operator splitting for weakly coupled Systems of Hamilton-Jacobi Equations.

by

Espen R. Jakobsen, Kenneth Hvistendahl Karlsen
and Nils Henrik Risebro

Report no. 148 June 2000

UNIVERSITY OF BERGEN
Bergen, Norway

Department of Mathematics
University of Bergen
5008 Bergen
Norway

ISSN 0084-778x

On the convergence Rate of Operator splitting for weakkly coupled Systems of Hamilton-Jacobi Equations.

by

Espen R. Jakkobsen, Kenneth Hvistendahl Karlsen and Nils Henrik Risebro

Report no. 148

June 2000

ON THE CONVERGENCE RATE OF OPERATOR SPLITTING FOR WEAKLY COUPLED SYSTEMS OF HAMILTON-JACOBI EQUATIONS

ESPEN R. JAKOBSEN, KENNETH HVISTENDAHL KARLSEN, AND NILS HENRIK RISEBRO

ABSTRACT. Assuming existence and uniqueness of bounded Lipschitz continuous viscosity solutions to the initial value problem for weakly coupled systems of Hamilton-Jacobi equations, we establish a linear L^∞ convergence rate for a semi-discrete operator splitting. This paper complements our previous work [2] on the convergence rate of operator splitting for scalar Hamilton-Jacobi equations with source term.

1. INTRODUCTION

The purpose of this note is to study the error associated with an operator splitting procedure for weakly coupled systems for Hamilton-Jacobi equations of the form

$$
\begin{aligned}
&\frac{\partial u_i}{\partial t} + H_i(t,x,u_i,Du_i) = G_i(t,x,u) \quad \text{in } Q_T = \mathbb{R}^N \times (0,T), \quad i = 1,\dots,m,\\
&u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N,
\end{aligned}
\tag{1.1}
$$

where the Hamiltonian $H = (H_1,\dots,H_m)$, is such that H_i only depends on u_i and Du_i (and x and t). The equations are only coupled through the source term $G = (G_1,\dots,G_m)$.

We assume that under reasonable conditions the present problem has a unique bounded, Lipschitz continuous viscosity solution, see Crandall, Ishii, and Lions [1] for an up-to-date overview of existence and uniqueness results for fully nonlinear first and second order partial differential equations as well an introduction to the general viscosity solution theory.

Our semi-discrete splitting algorithm consists of alternately solving the "split" problems

$$
\begin{aligned}
&\frac{\partial u_i}{\partial t} + H_i(t,x,u_i,Du_i) = 0, \qquad \text{for } i = 1,\dots,m,\\
&u_t = G(t,x,u), \qquad u = (u_1,\dots,u_m),
\end{aligned}
$$

sequentially for a small time step Δt, using the final data from one equation as initial data for the other. We refer to Section 2 for a precise description of the operator splitting. We prove that the operator splitting solution converges linearly in Δt (when measured in the L^∞ norm) to the exact viscosity solution of (1.1). This is a generalization of the results in [2], where convergence of a splitting algorithm was proved in the scalar case.

Date: June 5, 2000.

Key words and phrases. Hamilton-Jacobi equations, weakly coupled systems, viscosity solution, operator splitting, error estimates.

Jakobsen acknowledges support from the Norwegian Research Council (NFR) under grant no. 121531/410.

Before stating the our results, we start by defining our notation and state the necessary preliminaries, for more background we refer the reader to Souganidis [5], see also [1].

Let $||f|| := \text{ess}\sup_{x\in U}|f(x)|$. By $BUC(X)$, $Lip(X)$, and $Lip_b(X)$ we denote the spaces of bounded uniformly continuous functions, Lipschitz functions, and bounded Lipschitz functions from X to $\mathbb{R}$ respectively. Finally, if $f \in Lip(X)$ for some set $X \subset \mathbb{R}^N$, we denote the Lipschitz constant of f by $||Df||$.

Let $F \in C([0,T]\times\mathbb{R}^N\times\mathbb{R}^m\times\mathbb{R}^N)$ and $u_0 \in BUC(\mathbb{R}^N)$ and consider the following initial value problem

$$u_t+F(t,x,u,Du) = 0 \quad \text{in } Q_T, \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $u_0 \in BUC(\mathbb{R}^N)$.

Definition 1.1 (Viscosity Solution). *1) A function $u \in C(Q_T;\mathbb{R})$ is a viscosity subsolution of (1.2) if for every $\phi \in C^1(Q_T)$, whenever $u-\phi$ attains a local maximum at $(x_0,t_0) \in Q_T$, then*

$$\phi_t(x_0,t_0) + F(t_0,x_0,u,D\phi(x_0,t_0)) \le 0.$$

2) A function $u \in C(Q_T;\mathbb{R})$ is a viscosity supersolution of (1.2) if for every $\phi \in C^1(Q_T)$, whenever $u-\phi$ attains a local minimum at $(x_0,t_0) \in Q_T$, then

$$\phi_t(x_0,t_0) + F(t_0,x_0,u,D\phi(x_0,t_0)) \ge 0.$$

3) A function $u \in C(Q_T;\mathbb{R})$ is a viscosity solution of (1.2) if it is both a viscosity sub- and supersolution of (1.2).

4) A function $u \in C(\bar{Q}_T;\mathbb{R})$ is viscosity solution of the initial value problem (1.2) and (1.3) if u is a viscosity solution of (1.2) and $u(x,0) = u_0(x)$ in $\mathbb{R}^N$.

From this the generalization to viscosity solutions of the system (1.1) is immediate.

In order to have existence and uniqueness of (1.3), we need more conditions on F.

- **(F1)** $F \in C([0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)$ is uniformly continuous on $[0,T]\times\mathbb{R}^N\times[-R,R]\times B_N(0,R)$ for each $R>0$, where $B_N(0,R) = \{x \in \mathbb{R}^N : |x| \le R\}$.
- **(F2)** $\sup_{\bar{Q}_T}|F(t,x,0,0)| < \infty$.
- **(F3)** For each $R>0$ there is a $\gamma_R \in \mathbb{R}$ such that $F(t,x,r,p) - F(t,x,s,p) \ge \gamma_R(r-s)$ for all $x \in \mathbb{R}^N$, $-R \le s \le r \le R$, $t \in [0,T]$, and $p \in \mathbb{R}^N$.
- **(F4)** For each $R>0$ there is a constant $C_R > 0$ such that $|F(t,x,r,p) - F(t,y,r,p)| \le C_R(1+|p|)|x-y|$ for all $t \in [0,T]$, $|r| \le R$, and x, y and $p \in \mathbb{R}^N$.

Under these conditions the following theorems hold:

Theorem 1.1 (Uniqueness). *Let $F : [0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ satisfy (F1), (F3), and (F4). Let $u, v \in BUC(\bar{Q}_T)$ be viscosity solutions of (1.2) with initial data $u_0, v_0 \in BUC(\mathbb{R}^N)$, respectively. Let $R_0 = \max(||u||, ||v||)$ and $\gamma = \gamma_{R_0}$. Then for every $t \in [0,T]$,*

$$||u(\cdot,t) - v(\cdot,t)|| \le e^{-\gamma t}||u_0 - v_0||.$$

Theorem 1.2 (Existence). *Let $F : [0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ satisfy (F1), (F2), (F3), and (F4). For every $u_0 \in BUC(\mathbb{R}^N)$ there is a time $T = T(||u_0||) > 0$ and function*

$u \in BUC(\bar{Q}_T)$ such that u is the unique viscosity solution of (1.2) and (1.3). If, moreover, γ_R in (F3) is independent of R, then (1.2) and (1.3) has a unique viscosity solution on $\bar{Q}_T$ for every $T > 0$.

Proposition 1.1. *Let $F : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (F1), (F2), (F3), and (F4). If $u_0 \in Lip_b(\mathbb{R}^N)$, and $u \in BUC(\mathbb{R}^N)$ is the unique viscosity solution of (1.2) and (1.3) in $\bar{Q}_T$, then $u \in Lip_b(\bar{Q}_T)$.*

2. Operator splitting and main results

We now give conditions on G and H which in the scalar case ($m = 1$) will be sufficient to get existence and uniqueness of a viscosity solution in $Lip_b(\bar{Q}_T)$. Moreover these conditions are strong enough to give a linear convergence rate for the operator splitting.

We assume that H and G satisfy the following conditions:

- **(H1 – H4)** For each i, H_i satisfies conditions (F1) - (F4).
- **(H5)** There is a constant $L^H > 0$ such that $|H_i(t,x,r,p) - H_i(t,x,s,p)| \leq L^H|r - s|$ for $t \in [0,T]$, x, $p \in \mathbb{R}^N$, $r \in \mathbb{R}$, and $i = 1,\dots,m$.
- **(H6)** For each $R > 0$ there is a constant $N_R^H > 0$ such that $|H_i(t,x,r,p) - H_i(\bar{t},x,r,p)| \leq N_R^H(1 + |p|)\,|t - \bar{t}|$ for $t, \bar{t} \in [0,T]$, $|r| \leq R$, x, $p \in \mathbb{R}^N$, and $i = 1,\dots,m$.
- **(H7)** For each $R > 0$ there is a constant $M_R > 0$ such that $|H_i(t,x,r,p) - H_i(t,x,r,q)| \leq M_R|p - q|$ for $t \in [0,T]$, $|r| \leq R$, x, p, $q \in \mathbb{R}^N$ such that $|p|$, $|q| \leq R$, and $i = 1,\dots,m$.
- **(G1)** $G \in C\left([0,T] \times \mathbb{R}^N \times \mathbb{R}^m; \mathbb{R}^m\right)$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times B_m(0,R)$ for each $R > 0$.
- **(G2)** There is a constant $C^G > 0$ such that $C^G = \sup_{\bar{Q}_T} |G(t,x,0)| < \infty$.
- **(G3)** For each $R > 0$ there is a constant $C_R^G > 0$ such that $|G(t,x,r) - G(t,y,r)| \leq C_R^G|x - y|$ for $t \in [0,T]$, $|r| \leq R$, and x, $y \in \mathbb{R}^N$.
- **(G4)** There is a constant $L^G > 0$ such that $|G(t,x,r) - G(t,x,s)| \leq L^G|r - s|$ for $(t,x) \in \bar{Q}_T$ and $r, s \in \mathbb{R}^m$.
- **(G5)** For each $R > 0$ there is a constant $N_R^G > 0$ such that $|G(t,x,r) - G(\bar{t},x,r)| \leq N_R^G\,|t - \bar{t}|$ for t, $\bar{t} \in [0,T]$, $|r| \leq R$, and $x \in \mathbb{R}^N$.

Note that by the conditions (F2) and (G2) we can assume that H_i satisfies $H_i(t,x,0,0) = 0$. If this were not so, we could simply redefine H as $H(t,x,u,p) - H(t,x,0,0)$ and G as $G(t,x,u) - H(t,x,0,0)$.

We shall assume that there exists a unique solution $u \in Lip_b(\bar{Q}_T; \mathbb{R}^m)$ to the initial value problem (1.1) under the assumptions (H1)-(H7), (G1)-(G5), and $u_0 \in Lip_b(\bar{Q}; \mathbb{R}^m)$.

First we will state an error bound for the splitting procedure when the ordinary differential equation is approximated by the explicit Euler method. To define the operator splitting, let

$$E(t,s) : Lip_b(\mathbb{R}^N; \mathbb{R}^m) \to Lip_b(\mathbb{R}^N; \mathbb{R}^m)$$

denote the Euler operator defined by

$$E(t,s)w(x) = w(x) + (t - s)G(s,x,w(x)) \tag{2.1}$$

for $0 \leq s \leq t \leq T$ and $w \in Lip_b(\mathbb{R}^N;\mathbb{R}^m)$. Furthermore, let

$$S_H(t,s) : Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$$

be the solution operator of the scalar Hamilton-Jacobi equation without source term

$$u_t + H(t,x,u,Du) = 0, \qquad u(x,s) = \bar{w}(x), \tag{2.2}$$

i.e., we write the viscosity solution of (2.2) as $S_H(t,s)\bar{w}(x)$.

We let S denote the operator defined by $S(t,s)w = (S_{H_1}(t,s)w_1,\ldots,S_{H_m}(t,s)w_m)$ for any $w = (w_1,\ldots,w_m) \in Lip_b(\mathbb{R}^N;\mathbb{R}^m)$. Now we can define our approximate solutions: Fix $\Delta t > 0$ and set $t_j = j\Delta t$, set $v(x,0) = v_0(x)$ and

$$v(x,t_j) = S(t_j,t_{j-1})E(t_j,t_{j-1})v(\cdot,t_{j-1})(x), \tag{2.3}$$

for $j > 0$. Note that this approximate solution is defined only at discrete t-values. Our first result is that the operator splitting solution, when (2.2) is solved exactly, converges linearly in Δt to the viscosity solution of (1.1).

Theorem 2.1. *Let $u(x,t)$ be the viscosity solution of (1.1) on the time interval $[0,T]$, and $v(x,t_j)$ be defined by (2.3). There exists a constant $K > 0$, depending only on T, $||u_0||$, $||Du_0||$, $||v_0||$, $||Dv_0||$, H, and G, such that for $j = 1,\ldots,n$*

$$||u(\cdot,t_j) - v(\cdot,t_j)|| \leq K(||u_0 - v_0|| + \Delta t).$$

We will prove this theorem in the next section.

Our second theorem gives a convergence rate for operator splitting when the explicit Euler operator E is replaced by the exact solution operator $\bar{E}$. More precisely, let $\bar{E}(t,s) : Lip_b(\mathbb{R}^N;\mathbb{R}^m) \to Lip_b(\mathbb{R}^N;\mathbb{R}^m)$ be the solution operator of the system of ordinary differential equations

$$u_t = G(t,x,u) \qquad u(x,s) = w(x). \tag{2.4}$$

where $w \in Lip_b(\mathbb{R}^N;\mathbb{R}^m)$. Note that x acts only as a parameter in (2.4), and that the assumptions on G ensure that $\bar{E}$ is well defined on the time interval $[s,T]$.

Analogously to (2.3) we define the approximate solution $\{\bar{v}(x,t_j)\}_{j=1}^n$,

$$\bar{v}(x,t_j) = S(t_j,t_{j-1})\bar{E}(t_j,t_{j-1})\bar{v}(\cdot,t_{j-1})(x), \tag{2.5}$$

for $j > 0$ and $\bar{v}(x,t_0) = v_0$. Then we have:

Theorem 2.2. *Let $u(x,t)$ be the viscosity solution of (1.1) on the time interval $[0,T]$ and $\bar{v}(x,t_j)$ be defined by (2.5). Then there exists a constant $\bar{K} > 0$, depending only on T, $||u_0||$, $||Du_0||$, $||v_0||$, $||Dv_0||$, H, and G, such that for $j = 1,\ldots,n$*

$$||u(\cdot,t_j) - \bar{v}(\cdot,t_j)|| \leq \bar{K}(||u_0 - v_0|| + \Delta t).$$

Remark 2.3. *Theorems 2.1 and 2.2 are generalizations of Theorems 3.1 and 3.2 in* [2].

for $0 \leq s \leq t \leq T$ and $w \in Lip(\mathbb{R}^N; \mathbb{R}^m)$. Furthermore, let

$$S_H(t, s) : Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$$

be the solution operator of the scalar Hamilton-Jacobi equation without source term

$$u_t + H(t, x, u, Du) = 0, \qquad u(x, s) = \bar{u}(x), \tag{2.2}$$

i.e., we write the viscosity solution of (2.2) as $S_H(t, s)\bar{u}(x)$.

We let S denote the operator defined by $S(t, s)w = (S_{H_1}(t, s)w_1, \ldots, S_{H_m}(t, s)w_m)$ for any $w = (w_1, \ldots, w_m) \in Lip_b(\mathbb{R}^N; \mathbb{R}^m)$. Now we can define our approximate solutions. Fix $\Delta t > 0$ and set $t_j = j\Delta t$, set $v(x, 0) = v_0(x)$ and

$$v(x, t_j) = S(t_j, t_{j-1})E(t_j, t_{j-1})v(\cdot, t_{j-1})(x), \tag{2.3}$$

for $j > 0$. Note that this approximate solution is defined only at discrete t-values. Our first result is that the operator splitting solution, when (2.2) is solved exactly, converges linearly in Δt to the viscosity solution of (1.1).

Theorem 2.1. *Let $u(x, t)$ be the viscosity solution of (1.1) on the time interval $[0, T]$, and $v(x, t_j)$ be defined by (2.3). There exists a constant $K > 0$ depending only on T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, H, and G such that for $j = 1, \ldots, n$*

$$\|u(\cdot, t_j) - v(\cdot, t_j)\| \leq K(\|u_0 - v_0\| + \Delta t).$$

We will prove this theorem in the next section.

Our second theorem gives a convergence rate for operator splitting when the explicit Euler operator E is replaced by the exact solution operator $\bar{E}$. More precisely, let $\bar{E}(t, s) : Lip_b(\mathbb{R}^N; \mathbb{R}^m) \to Lip_b(\mathbb{R}^N; \mathbb{R}^m)$ be the solution operator of the system of ordinary differential equations

$$u_t = G(t, x, u), \qquad u(x, s) = w(x) \tag{2.4}$$

where $w \in Lip_b(\mathbb{R}^N; \mathbb{R}^m)$. Note that x acts only as a parameter in (2.4), and that the assumptions on G ensure that $\bar{E}$ is well defined on the time interval $[s, T]$.

Analogously to (2.3) we define the approximate solution $\{\bar{v}(x, t_j)\}_{j\geq 0}$

$$\bar{v}(x, t_j) = S(t_j, t_{j-1})\bar{E}(t_j, t_{j-1})\bar{v}(\cdot, t_{j-1})(x), \tag{2.5}$$

for $j > 0$ and $\bar{v}(x, t_0) = v_0$. Then we have:

Theorem 2.2. *Let $u(x, t)$ be the viscosity solution of (1.1) on the time interval $[0, T]$ and $\bar{v}(x, t_j)$ be defined by (2.5). Then there exists a constant $K > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, H, and G, such that for $j = 1, \ldots, n$*

$$\|u(\cdot, t_j) - \bar{v}(\cdot, t_j)\| \leq K(\|u_0 - v_0\| + \Delta t).$$

Remark 2.3. *Theorems 2.1 and 2.2 are generalizations of Theorems 3.1 and 3.2 in [2].*

3. Proofs of Theorems 2.1 and 2.2

We will proceed as follows: First we give some estimates we will need later. Then we introduce an auxiliary approximate solution and prove linear convergence rate for this solution. This proof involves the scalar version of Theorem 2.1. We proceed to show that the operator splitting solution converges to this approximate solution with linear rate. This completes the proof of Theorem 2.1. Finally we give a proof of Theorem 2.2. This proof is similar to the proof of Theorem 3.2 in [2].

We start by stating the relevant estimates on S. Let $w, \tilde{w} \in Lip_b(\mathbb{R}^N)$, $0 \le s \le t \le T$, and $R_1 = \sup_{t,s,i} \|S_i(t,s)w\|$, then

$$\|S_i(t,s)w\| \le e^{L^H(t-s)}\|w\|, \tag{3.1}$$

$$\|D\{S_i(t,s)w\}\| \le e^{(L^H+K(R_1))(t-s)}\{\|Dw\| + (t-s)K(R_1)\}, \tag{3.2}$$

$$\|S_i(t,s)w - S_i(t,s)\tilde{w}\| \le e^{L^H(t-s)}\|w-\tilde{w}\|, \tag{3.3}$$

where $K(R)$ is a constant depending on R but independent of i, t, and s. Estimate (3.3) is a direct consequence of Theorem 1.1. Note that in this case $\gamma = L^H$. Estimates (3.1) and (3.2) correspond to estimates (4.7) and (4.8) in [2].

Regarding the approximation defined by (2.3), $v(\cdot, t_j)$, we have the following estimates:

Lemma 3.1. *There is a constant R independent of Δt such that* $\max_{1\le j\le n} \|v(\cdot,t_j)\| < R$. *Moreover for every $1 \le j \le n$,*

(a) $\|v(\cdot,t_j)\| \le m\, e^{(L^H+mL^G)t_j}(\|v_0\| + t_j C^G)$,

(b) $\|Dv(\cdot,t_j)\| \le m\, e^{(L^H+mL^G+K(R))t_j}\{\|Dv_0\| + t_j(C^G_R + K(R))\}$.

Proof. To prove a) and b), we need (3.1), (3.2), and the definition of the operator E. We only give the proof of a). The proof of b) is similar. By (3.1) we get

$$\left\|S_i(t_j,t_{j-1})\left\{E(t_j,t_{j-1})v(\cdot,t_{j-1})\right\}_i\right\| \le e^{L^H\Delta t}\left\|\left\{E(t_j,t_{j-1})v(\cdot,t_{j-1})\right\}_i\right\|. \tag{3.4}$$

We then use the definition of E (2.1) and (G3), (G4) to get

$$\|\{E(t_j,t_{j-1})v(\cdot,t_{j-1})\}_i\| \le \|v_i(\cdot,t_{j-1})\| + \Delta t\left(C^G + L^G\|v(\cdot,t_{j-1})\|\right).$$

Note that $\|v(\cdot,t_{j-1})\| \le \sum_{i=1}^m \|v_i(\cdot,t_{j-1})\|$. Now using this and summing over i in inequality (3.4), we get

$$\begin{aligned}
&\sum_{i=1}^m \|S_i(t_j,t_{j-1})\{E(t_j,t_{j-1})v(\cdot,t_{j-1})\}_i\| \\
&\le e^{L^H\Delta t}\left\{\left(1+\Delta t m L^G\right)\sum_{i=1}^m \|v_i(\cdot,t_{j-1})\| + mC^G\Delta t\right\} \\
&\le e^{(L^H+mL^G)\Delta t}\left\{\sum_{i=1}^m \|v_i(\cdot,t_{j-1})\| + mC^G\Delta t\right\}.
\end{aligned} \tag{3.5}$$

The result in a) now follows from successive use of (3.5) and an application of the inequalities $|x| \leq \sum_{i=1}^{m} |x_i| \leq m|x|$ for $x \in \mathbb{R}^m$. Replacing t_j by T in a), we see that the existence of R is assured. □

Proof of Theorem 2.1.
Let u denote the solution of (1.1) and define
(3.6)

$$\tilde{G}_i(t,x,r) = G_i\left(t,x,u_1(x,t),\dots,u_{i-1}(x,t),r,u_{i+1}(x,t),\dots,u_m(x,t)\right), \quad i=1,\dots,m.$$

Note that the function $\tilde{G}_i$ satisfies (G1)-(G5) for all $i = 1,\dots,m$.

Using $\tilde{G}_i$, we can rewrite (1.1) as a series of "uncoupled" equations

$$\frac{\partial u_i}{\partial t} + H_i(t,x,u_i,Du_i) = \tilde{G}_i(t,x,u_i), \quad i=1,\dots,m. \tag{3.7}$$

Of course, the viscosity solution of (1.1) u is also the unique viscosity solution of the system of equations (3.7).

Now we want to do (scalar) operator splitting for each equation in (3.7). To this end, for any $x = (x_1,\dots,x_m) \in \mathbb{R}^m$, let $x_{i*} = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_m)$. Now for any $w \in Lip_b(\mathbb{R}^N;\mathbb{R}^m)$ let $E_i(t,s)w_i$ be given by

$$E_i(t,s)w_i = w_i + (t-s)\tilde{G}_i\left(s,x,w_i\right).$$

Now we define the following operator splitting solution $\tilde{v} = (\tilde{v}_1,\dots,\tilde{v}_m)$,

$$\tilde{v}_i(x,t_j) = S_i(t_j,t_{j-1})E_i(t_j,t_{j-1})\tilde{v}_i(x,t_{j-1}), \tag{3.8}$$

for $j > 1$, and $\tilde{v}_i(x,t_0) = u_{0i}(x)$. Note that E_i is the Euler operator for the equation

$$\frac{\partial u_i}{\partial t} = \tilde{G}_i(t,x,u_i).$$

Hence by the results of [2]:

Lemma 3.2. *Let $u(x,t)$ be the viscosity solution of (1.1) on the time interval $[0,T]$ and $\tilde{v}(x,t_j)$ be the operator splitting solution (3.8). There exists a constant $K' > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, H, and G, such that for $j = 1,\dots,n$,*

$$\|u(\cdot,t_j) - \tilde{v}(\cdot,t_j)\| \leq K'\Delta t.$$

Using the above lemma, we wish to estimate $\|\tilde{v}(\cdot,t_j) - v(\cdot,t_j)\|$, and start by using the definition of the operator splitting solutions (2.3) and (3.8) and the estimate (3.3). Then

$$\begin{aligned}
|\tilde{v}_i\left(x,t_j\right) - v_i\left(x,t_j\right)| \leq& \left|S_i(t_j,t_{j-1})E_i(t_j,t_{j-1})\tilde{v}_i(x,t_{j-1})\right.\\
&\left. - S_i(t_j,t_{j-1})\left(E(t_j,t_{j-1})v(x,t_{j-1})\right)_i\right|\\
\leq& e^{L^H\Delta t}\left|E_i(t_j,t_{j-1})\tilde{v}_i(x,t_{j-1}) - \left(E(t_j,t_{j-1})v(x,t_{j-1})\right)_i\right|.
\end{aligned}$$

The result in a) now follows from successive use of (3.5) and an application of the inequalities $|x| \le \sum_{i=1}^{m} |x_i| \le m|x|$ for $x \in \mathbb{R}^m$. Replacing t_j by T in a), we see that the existence of K is assured. □

Proof of Theorem 3.1.
Let u denote the solution of (1.1) and define
(3.6)
$$\tilde{G}_i(t, x, r) = G_i(t, x, u_1(x, t), \dots, u_{i-1}(x, t), r, u_{i+1}(x, t), \dots, u_m(x, t)), \quad i = 1, \dots, m.$$

Note that the function $\tilde{G}_i$ satisfies (G1)-(G5) for all $i = 1, \dots, m$.

Using $\tilde{G}_i$, we can rewrite (1.1) as a series of "uncoupled" equations

$$\frac{\partial u_i}{\partial t} + H_i(t, x, u_i, Du_i) = \tilde{G}_i(t, x, u_i), \quad i = 1, \dots, m. \tag{3.7}$$

Of course, the viscosity solution of (1.1) u is also the unique viscosity solution of the system of equations (3.7).

Now we want to do (scalar) operator splitting for each equation in (3.7). To this end, for any $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, let $x_{i^c} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$. Now for any $w \in Lip_b(\mathbb{R}^N; \mathbb{R}^m)$ let $\tilde{E}_i(t, s)w_i$ be given by

$$\tilde{E}_i(t, s)w_i = w_i + (t - s)\tilde{G}_i(s, x, w_i).$$

Now we define the following operator splitting solution $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_m)$,

$$\tilde{v}_i(x, t_j) = \tilde{S}_i(t_j, t_{j-1})\tilde{E}_i(t_j, t_{j-1})\tilde{v}_i(x, t_{j-1}) \tag{3.8}$$

for $j > 1$, and $\tilde{v}_i(x, t_0) = u_{i,0}(x)$. Note that $\tilde{E}_i$ is the Euler operator for the equation

$$\frac{\partial w_i}{\partial t} = \tilde{G}_i(t, x, w_i).$$

Hence by the results of [2]:

Lemma 3.2. *Let $u(x, t)$ be the viscosity solution of (1.1) on the time interval $[0, T]$ and $\tilde{v}(x, t_j)$ be the operator splitting solution (3.8). There exists a constant $K > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, H, and G, such that for $j = 1, \dots, n$,*

$$\|u(\cdot, t_j) - \tilde{v}(\cdot, t_j)\| \le K\sqrt{\Delta t}.$$

Using the above lemma, we wish to estimate $\|v(\cdot, t_j) - \tilde{v}(\cdot, t_j)\|$, and start by using the definition of the operator splitting solutions (2.3) and (3.8) and the estimate (3.3). Then

$$\begin{aligned}
|\tilde{v}_i(x, t_j) - v_i(x, t_j)| &\le \big|S_i(t_j, t_{j-1})E_i(t_j, t_{j-1})v_i(x, t_{j-1}) \\
&\quad - \tilde{S}_i(t_j, t_{j-1})\tilde{E}_i(t_j, t_{j-1})\tilde{v}_i(x, t_{j-1})\big| \\
&\le e^{C\Delta t}\big|E_i(t_j, t_{j-1})v_i(x, t_{j-1}) - \tilde{E}_i(t_j, t_{j-1})\tilde{v}_i(x, t_{j-1})\big|,
\end{aligned}$$

By the Lipschitz continuity of G, we have that

$$\begin{aligned}
&\left|E_i\left(t_j,t_{j-1}\right)\tilde{v}_i\left(x,t_{j-1}\right)-\left(E\left(t_j,t_{j-1}\right)v(\cdot,t_{j-1})\right)_i\right| \\
&\quad\leq \left|\left(\tilde{v}_i-v_i\right)\left(x,t_{j-1}\right)\right|+\Delta t\Big|G_i\left(u_1,\ldots,\tilde{v}_i(x,t_{j-1}),\ldots,u_m\right) \\
&\qquad\qquad -G_i\left(v_1(x,t_{j-1}),\ldots,v_m(x,t_{j-1})\right)\Big| \\
&\quad\leq \left|\left(\tilde{v}_i-v_i\right)\left(x,t_{j-1}\right)\right|+L^G\Delta t\left(\left|(u_{i*}-v_{i*})(x,t_{j-1})\right|+\left|(\tilde{v}_i-v_i)(x,t_{j-1})\right|\right) \\
&\quad\leq \left|\left(\tilde{v}_i-v_i\right)\left(x,t_{j-1}\right)\right|+L^G\Delta t\Big(\left|(u_{i*}-\tilde{v}_{i*})(x,t_{j-1})\right|+\left|(\tilde{v}_{i*}-v_{i*})(x,t_{j-1})\right| \\
&\qquad\qquad +\left|(\tilde{v}_i-v_i)(x,t_{j-1})\right|\Big) \\
&\quad\leq \left|\left(\tilde{v}_i-v_i\right)\left(x,t_{j-1}\right)\right|+L^GK'\Delta t^2+L^G\sqrt{2}\Delta t\left|\tilde{v}(x,t_{j-1})-v(x,t_{j-1})\right|.
\end{aligned}$$

Summing the resulting inequality over i yields

$$\begin{aligned}
&\sum_{i=1}^m\left|\tilde{v}_i(x,t_j)-v_i(x,t_j)\right| \\
&\qquad\leq e^{L^H\Delta t}\left(mK'L^G\Delta t^2+(1+mL^G\sqrt{2}\Delta t)\sum_{i=1}^m\left|\tilde{v}_i(x,t_{j-1})-v_i(x,t_{j-1})\right|\right) \\
&\qquad\leq e^{\left(L^H+m\sqrt{2}K'L^G\right)t_j}\sum_{i=1}^m\left|u_{0,i}(x)-v_{0,i}(x)\right|+mK'L^Gt_j\Delta t
\end{aligned}$$

Hence Theorem 2.1 holds. □

Proof of Theorem 2.2. We end this section by giving the proof of Theorem 2.2. Assume for the moment that

$$\left\|v\left(x,t_j\right)-\bar{v}\left(x,t_j\right)\right\|\leq\bar{C}\Delta t \tag{3.9}$$

for all j, where $\bar{C}$ is a constant depending on G, H, T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, and $\|Dv_0\|$ but not Δt. Using (3.9) and Theorem 2.1, we find

$$\begin{aligned}
\left\|u\left(\cdot,t_j\right)-\bar{v}\left(\cdot,t_j\right)\right\| &\leq \left\|u\left(\cdot,t_j\right)-v\left(\cdot,t_j\right)\right\|+\left\|v\left(\cdot,t_j\right)-\bar{v}\left(\cdot,t_j\right)\right\| \\
&\leq K\left(\|u_0-v_0\|+\Delta t\right)+\bar{C}\Delta t.
\end{aligned}$$

Setting $\bar{K}=K+\bar{C}$, we conclude that Theorem 2.2 holds. It remains to show (3.9). Using the same arguments as when estimating the local truncation error for the Euler method we find that

$$\sum_{i=1}^m\left|\{E(t_{j+1},t_j)v(x,t_j)-\bar{E}(t_{j+1},t_j)\bar{v}(x,t_j)\}_i\right|\leq e^{mL^G\Delta t}\sum_{i=1}^m\left|\{v(x,t_j)-\bar{v}(x,t_j)\}_i\right|+\tilde{C}\Delta t^2,$$

where $\tilde{C}=mL^G(L^G\bar{R}+C^G)+mN^G_{\bar{R}}$. Here $\bar{R}>\max\left(\left\|\bar{E}(t_j,t)\bar{v}(\cdot,t_j)\right\|,\|v(\cdot,t_j)\|\right)$, $\bar{R}$ is finite by arguments similar to those used in the proof of Lemma 3.1. Now using this we

find that

$$
\begin{aligned}
\sum_{i=1}^{m} \|\{v(\cdot, t_{j+1}) - \bar{v}(\cdot, t_{j+1})\}_i\| &= \sum_{i=1}^{m} \Big\| \{S(t_{j+1}, t_j)E(t_{j+1}, t_j)v(\cdot, t_j) \\
&\qquad - S(t_{j+1}, t_j)\bar{E}(t_{j+1}, t_j)\bar{v}(\cdot, t_j)\}_i \Big\| \\
&\le e^{L^H \Delta t} \sum_{i=1}^{m} \|\{E(t_{j+1}, t_j)v(\cdot, t_j) - \bar{E}(t_{j+1}, t_j)\bar{v}(\cdot, t_j)\}_i\| \\
&\le e^{(L^H + mL^G)\Delta t} \Big(\sum_{i=1}^{m} \|\{v(\cdot, t_j) - \bar{v}(\cdot, t_j)\}_i\| + \tilde{C}\Delta t^2 \Big). \qquad (3.10)
\end{aligned}
$$

Since that $\bar{v}(x, 0) = v_0(x)$, repeated use of inequality (3.10) gives (3.9). □

4. A fully discrete splitting method

In this section we present a simple numerical example of the splitting discussed in this paper. For simplicity we shall consider a system of two equations in one space dimension

$$u_t + H(u_x) = f(u, v), \qquad v_t + G(v_x) = g(u, v). \tag{4.1}$$

When testing this numerically, we must replace the exact solution operator S by a numerical method. As most numerical methods for Hamilton-Jacobi equations are have convergence rates of $1/2$ with respect to the time step, we use a front tracking algorithm, which has a linear convergence rate with respect to the time step. This front tracking algorithm is described in [3] and we shall only give a very brief account of front tracking here.

Front tracking uses no fixed grid and the solution is approximated by a piecewise linear function. The discontinuities in the space derivative, the so-called *fronts*, of the approximate solution are tracked in time and interactions between these are resolved. This algorithm works for scalar equations in one space variable of the form

$$u_t + H(u_x) = 0.$$

For equations in several space dimensions, front tracking can be used as a building block in a dimensional splitting method, see [4].

For weakly coupled systems of the form (4.1), the approximate solution operator E depends on both u and v. Therefore, after the action of E, we must add fronts in the approximation of u at the position of the fronts in v and vice versa. In this situation we cannot in general find a global bound on the total number of fronts to track. In order to avoid this problem we use a fixed grid $x_i = i\Delta x$, for $i \in \mathbb{Z}$, and set

$$S := \pi \circ S^{\text{f.t.}}, \tag{4.2}$$

where π is a linear interpolation to the fixed grid and $S^{\text{f.t.}}$ is the front tracking algorithm. Unfortunately, this restricts the order of the overall algorithm to $\mathcal{O}(\Delta x^{1/2})$. Nevertheless, we do not have any inherent relation between Δx and Δt, and we used $\Delta x = \Delta t/10$.

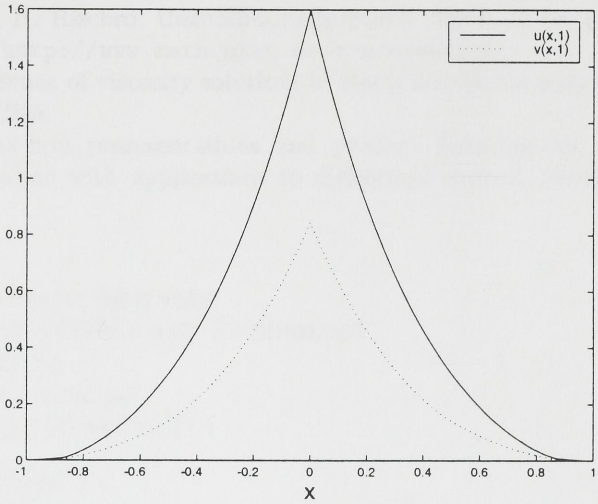

FIGURE 1. $u(x,1)$ and $v(x,1)$

TABLE 1. Δt versus $100 \times L^\infty$ error.

Δt	1	1/2	1/4	1/8	1/16	1/32
Error	32.0	16.5	11.6	8.3	5.1	3.2

We have tested this on the initial value problem

$$\left.\begin{aligned} u_t + \frac{1}{2}(u_x)^2 &= 4v(u+1) \\ v_t + \frac{1}{2}(v_x)^2 &= u^2 + v^2 \end{aligned}\right\} \qquad u(x,0) = v(x,0) = 1 - |x|, \quad \text{for } x \in [-1,1], \tag{4.3}$$

and periodic boundary conditions. In figure 1 we show the approximate solution at $t = 1$ using $\Delta t = 1/8$. To find a "numerical" convergence rate, we compared the splitting solution with a reference solution computed by the Engquist-Osher scheme with $\Delta x = 1/2000$. Table 1 shows the relative supremum error for different values of Δt. These values indicate a numerical convergence rate of roughly 0.63, i.e., error $= \mathcal{O}(\Delta t^{0.63})$, much less than the rate using an exact solution operator for the homogeneous equation.

REFERENCES

[1] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[2] E. R. Jakobsen, K. H. Karlsen and N. H. Risebro. On the convergence rate of operator splitting for Hamilton-Jacobi equations with source terms. Preprint. Preprint 139, Department of Mathematics, University of Bergen. Available at the URL `http://www.math.ntnu.no/conservation`.

[3] K. H. Karlsen and N. H. Risebro. A note on front tracking and the equivalence between viscosity solutions of Hamilton-Jacobi equations and entropy solutions of scalar conservation laws. Available at the URL `http://www.math.ntnu.no/conservation`.

[4] K. H. Karlsen and N. H. Risebro. Unconditionally Stable Methods for Hamilton-Jacobi Equations Available at the URL `http://www.math.ntnu.no/conservation`.

[5] P. E. Souganidis. Existence of viscosity solutions of Hamilton-Jacobi equations. *J. Differential Equations*, 56(3):345–390, 1985.

[6] P. E. Souganidis. Max-min representations and product formulas for the viscosity solutions of Hamilton-Jacobi equations with applications to differential games. *Nonlinear Anal.*, 9(3):217–257, 1985.

(Espen R. Jakobsen)
Department of Mathematical Sciences
Norwegian University of Science and Technology
N–7034 Trondheim, Norway
E-mail address: `erj@math.ntnu.no`
URL: `http://www.math.ntnu.no/~erj/`

(Kenneth Hvistendahl Karlsen)
Department of Mathematics
University of Bergen
Johs. Brunsgt. 12
N–5008 Bergen, Norway
E-mail address: `kenneth.karlsen@mi.uib.no`
URL: `http://www.mi.uib.no/~kennethk/`

(Nils Henrik Risebro)
Department of Mathematics
University of Oslo
P.O. Box 1053, Blindern
N–0316 Oslo, Norway
E-mail address: `nilshr@math.uio.no`
URL: `http://www.math.uio.no/~nilshr/`