Department of APPLIED MATHEMATICS

On the convergence Rate of Operator splitting for weakly coupled Systems of Hamilton-Jacobi Equations.

by

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ON THE CONVERGENCE RATE OF OPERATOR SPLITTING FOR WEAKLY COUPLED SYSTEMS OF HAMILTON-JACOBI EQUATIONS

ESPEN R. JAKOBSEN, KENNETH HVISTENDAHL KARLSEN, AND NILS HENRIK RISEBRO

ABSTRACT. Assuming existence and uniqueness of bounded Lipschitz continuous viscosity solutions to the initial value problem for weakly coupled systems of Hamilton-Jacobi equations, we establish a linear L^{∞} convergence rate for a semi-discrete operator splitting. This paper complements our previous work [2] on the convergence rate of operator splitting for scalar Hamilton-Jacobi equations with source term.

1. INTRODUCTION

The purpose of this note is to study the error associated with an operator splitting procedure for weakly coupled systems for Hamilton-Jacobi equations of the form

(1.1)
$$\frac{\partial u_i}{\partial t} + H_i(t, x, u_i, Du_i) = G_i(t, x, u) \quad \text{in } Q_T = \mathbb{R}^N \times (0, T), \quad i = 1, \dots, m,$$
$$u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N,$$

where the Hamiltonian $H = (H_1, \ldots, H_m)$, is such that H_i only depends on u_i and Du_i (and x and t). The equations are only coupled through the source term $G = (G_1, \ldots, G_m)$.

We assume that under reasonable conditions the present problem has a unique bounded, Lipschitz continuous viscosity solution, see Crandall, Ishii, and Lions [1] for an up-to-date overview of existence and uniqueness results for fully nonlinear first and second order partial differential equations as well an introduction to the general viscosity solution theory.

Our semi-discrete splitting algorithm consists of alternately solving the "split" problems

$$\frac{\partial u_i}{\partial t} + H_i(t, x, u_i, Du_i) = 0, \quad \text{for } i = 1, \dots, m,$$
$$u_t = G(t, x, u), \quad u = (u_1, \dots, u_m),$$

sequentially for a small time step Δt , using the final data from one equation as initial data for the other. We refer to Section 2 for a precise description of the operator splitting. We prove that the operator splitting solution converges linearly in Δt (when measured in the L^{∞} norm) to the exact viscosity solution of (1.1). This is a generalization of the results in [2], where convergence of a splitting algorithm was proved in the scalar case.

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Key words and phrases. Hamilton-Jacobi equations, weakly coupled systems, viscosity solution, operator splitting, error estimates.

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ON THE CONVERGENCE RAFE OF OPERATOR SPELTING FOR WEAKLY COUPLED SYSTEMS OF PAULITON-1 ACOBI EQUATIONS

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Before stating the our results, we start by defining our notation and state the necessary preliminaries, for more background we refer the reader to Souganidis [5], see also [1].

Let $||f|| := \operatorname{ess sup}_{x \in U} |f(x)|$. By BUC(X), Lip(X), and $Lip_b(X)$ we denote the spaces of bounded uniformly continuous functions, Lipschitz functions, and bounded Lipschitz functions from X to \mathbb{R} respectively. Finally, if $f \in Lip(X)$ for some set $X \subset \mathbb{R}^N$, we denote the Lipschitz constant of f by ||Df||.

Let $F \in C([0,T] \times \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^N)$ and $u_0 \in BUC(\mathbb{R}^N)$ and consider the following initial value problem

(1.2)
$$u_t + F(t, x, u, Du) = 0 \quad \text{in } Q_T,$$

(1.3)
$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N,$$

where $u_0 \in BUC(\mathbb{R}^N)$.

Definition 1.1 (Viscosity Solution). 1) A function $u \in C(Q_T; \mathbb{R})$ is a viscosity subsolution of (1.2) if for every $\phi \in C^1(Q_T)$, whenever $u - \phi$ attains a local maximum at $(x_0, t_0) \in Q_T$, then

 $\phi_t(x_0, t_0) + F(t_0, x_0, u, D\phi(x_0, t_0)) \le 0.$

2) A function $u \in C(Q_T; \mathbb{R})$ is a viscosity supersolution of (1.2) if for every $\phi \in C^1(Q_T)$, whenever $u - \phi$ attains a local minimum at $(x_0, t_0) \in Q_T$, then

 $\phi_t(x_0, t_0) + F(t_0, x_0, u, D\phi(x_0, t_0)) \ge 0.$

- 3) A function $u \in C(Q_T; \mathbb{R})$ is a viscosity solution of (1.2) if it is both a viscosity sub- and supersolution of (1.2).
- 4) A function $u \in C(\bar{Q}_T; \mathbb{R})$ is viscosity solution of the initial value problem (1.2) and (1.3) if u is a viscosity solution of (1.2) and $u(x, 0) = u_0(x)$ in \mathbb{R}^N .

From this the generalization to viscosity solutions of the system (1.1) is immediate. In order to have existence and uniqueness of (1.3), we need more conditions on F.

- (F1) $F \in C([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times [-R,R] \times [-R,R]$
 - $B_N(0,R)$ for each R > 0, where $B_N(0,R) = \left\{ x \in \mathbb{R}^N : |x| \le R \right\}.$
- (F2) $\sup_{\bar{Q}_T} |F(t, x, 0, 0)| < \infty.$
- (F3) For each R > 0 there is a $\gamma_R \in \mathbb{R}$ such that $F(t, x, r, p) F(t, x, s, p) \ge \gamma_R(r-s)$ for all $x \in \mathbb{R}^N$, $-R \le s \le r \le R$, $t \in [0, T]$, and $p \in \mathbb{R}^N$.
- (F4) For each R > 0 there is a constant $C_R > 0$ such that $|F(t, x, r, p) F(t, y, r, p)| \le C_R(1+|p|)|x-y|$ for all $t \in [0,T]$, $|r| \le R$, and x, y and $p \in \mathbb{R}^N$.

Under these conditions the following theorems hold:

Theorem 1.1 (Uniqueness). Let $F : [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (F1), (F3), and (F4). Let $u, v \in BUC(\bar{Q}_T)$ be viscosity solutions of (1.2) with initial data $u_0, v_0 \in BUC(\mathbb{R}^N)$, respectively. Let $R_0 = \max(||u||, ||v||)$ and $\gamma = \gamma_{R_0}$. Then for every $t \in [0, T]$,

$$||u(\cdot,t) - v(\cdot,t)|| \le e^{-\gamma t} ||u_0 - v_0||.$$

Theorem 1.2 (Existence). Let $F : [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (F1), (F2), (F3), and (F4). For every $u_0 \in BUC(\mathbb{R}^N)$ there is a time $T = T(||u_0||) > 0$ and function

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Before stating the our readity, we start by defining our notation and state the necessary proliminaries, for more background we relet the reader to Sougastiche [5], are also al.

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A function $w \in C(Q_{1,2})$ is a microsity intervaluation of (1.3). If for an element $G(Q_{2})$, where $G(Q_{2})$, where $G(Q_{2})$ is a dimension of the function $C(Q_{2})$, where $G(Q_{2})$ is the function of the function $G(Q_{2})$.

$(x_i, x_i) \rightarrow E(\phi_i, x_i), \phi_i D\phi(x_i, h_i) \geq 0$

3) A function $u \in C(Q_{12}\mathbb{R})$ is a planning substant of (4.2) if it is finit a spacingly multi-term of (4.2) if it is finit a spacingly

(i) A function $a \in \mathbb{C}(Q_1, Q_2)$ is attributing autilities of the initial value gradient, (i, k), that (i, 3) is a stringenty substant of (1, 2) and $a(x, 0) = a_0(a)$, in \mathbb{R}^2 .

from this the generalization to viscosity solutions of the system. (1.1) is immediate in order to be a set to be a set

(F1) $E \in C([0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ is uniformly continuous on $[0,T] \times \mathbb{R}^n \times [-R, P] \times \mathbb{R}^n$ $B_{\mu}([0,R])$ for each $R \times [0, where <math>E_{\mu}([0,R]) = \{x \in \mathbb{R}^n : |x| \leq R\}$

(1.3) supply (1.1,1,1,1,1,1)

(1.3) For each 4 > 0 there is a problem in the state $(1, n, r, p) - 2(0, r, s, p) \geq 0$ at r = 0 for all r = 0 for all r = 0.

(2.4) For each E > 0 there are consense of p > 0 such that $P(0, \alpha, \gamma, \beta) = P(\alpha, \beta, \beta)$ is $C_n(1 + [p])[p_n - p]$ be all $z \in [0, T] [p_1 \in R$, and $z \in p$ and $p \in \mathbb{R}^n$.

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Theorem 1.1 (Universities), Lech (O T) sile (E. E. E. - E antide (E.), (F.9) and (F.) Let u. v C EUC(Qc) becau terfty substants of (1.2) with findual Euterus, to C EUC(ECS respectively. Let E. = mix(Vul. (w)) and v = v. Theorem story I C E. Th

Theorem 1.2 (Existence), Let $F \to 0$, $X^{0} \times \mathbb{R}^{2} \times \mathbb{R} \times \mathbb{R}^{2} \rightarrow \mathbb{R}$ and $g \in 0$, (F3), (F3), (F3), (F3), (F4).

 $u \in BUC(\bar{Q}_T)$ such that u is the unique viscosity solution of (1.2) and (1.3). If, moreover, γ_R in (F3) is independent of R, then (1.2) and (1.3) has a unique viscosity solution on \bar{Q}_T for every T > 0.

Proposition 1.1. Let $F : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (F1), (F2), (F3), and (F4). If $u_0 \in Lip_b(\mathbb{R}^N)$, and $u \in BUC(\mathbb{R}^N)$ is the unique viscosity solution of (1.2) and (1.3) in \bar{Q}_T , then $u \in Lip_b(\bar{Q}_T)$.

2. Operator splitting and main results

We now give conditions on G and H which in the scalar case (m = 1) will be sufficient to get existence and uniqueness of a viscosity solution in $Lip_b(\bar{Q}_T)$. Moreover these conditions are strong enough to give a linear convergence rate for the operator splitting.

We assume that H and G satisfy the following conditions:

- (H1 H4) For each *i*, H_i satisfies conditions (F1) (F4).
- (H5) There is a constant $L^H > 0$ such that $|H_i(t, x, r, p) H_i(t, x, s, p)| \leq L^H |r s|$ for $t \in [0, T]$, $x, p \in \mathbb{R}^N$, $r \in \mathbb{R}$, and $i = 1, \ldots, m$.
- (H6) For each R > 0 there is a constant $N_R^H > 0$ such that $|H_i(t, x, r, p) H_i(\bar{t}, x, r, p)| \le N_R^H (1 + |p|) |t \bar{t}|$ for $t, \bar{t} \in [0, T], |r| \le R, x, p \in \mathbb{R}^N$, and $i = 1, \ldots, m$.
- (H7) For each R > 0 there is a constant $M_R > 0$ such that $|H_i(t, x, r, p) H_i(t, x, r, q)| \le M_R |p q|$ for $t \in [0, T]$, $|r| \le R$, $x, p, q \in \mathbb{R}^N$ such that |p|, $|q| \le R$, and $i = 1, \ldots, m$.
- (G1) $G \in C([0,T] \times \mathbb{R}^N \times \mathbb{R}^m; \mathbb{R}^m)$ is uniformly continuous on $[0,T] \times \mathbb{R}^N \times B_m(0,R)$ for each R > 0.
- (G2) There is a constant $C^G > 0$ such that $C^G = \sup_{\bar{Q}_T} |G(t, x, 0)| < \infty$.
- (G3) For each R > 0 there is a constant $C_R^G > 0$ such that $|G(t, x, r) G(t, y, r)| \le C_R^G |x y|$ for $t \in [0, T]$, $|r| \le R$, and $x, y \in \mathbb{R}^N$.
- (G4) There is a constant $L^{G} > 0$ such that $|G(t, x, r) G(t, x, s)| \le L^{G}|r s|$ for $(t, x) \in \overline{Q}_{T}$ and $r, s \in \mathbb{R}^{m}$.
- (G5) For each R > 0 there is a constant $N_R^G > 0$ such that $|G(t, x, r) G(\bar{t}, x, r)| \le N_R^G |t \bar{t}|$ for $t, \bar{t} \in [0, T], |r| \le R$, and $x \in \mathbb{R}^N$.

Note that by the conditions (F2) and (G2) we can assume that H_i satisfies $H_i(t, x, 0, 0) = 0$. If this were not so, we could simply redefine H as H(t, x, u, p) - H(t, x, 0, 0) and G as G(t, x, u) - H(t, x, 0, 0).

We shall assume that there exists a unique solution $u \in Lip_b(Q_T; \mathbb{R}^m)$ to the initial value problem (1.1) under the assumptions (H1)-(H7), (G1)-(G5), and $u_0 \in Lip_b(\bar{Q}; \mathbb{R}^m)$.

First we will state an error bound for the splitting procedure when the ordinary differential equation is approximated by the explicit Euler method. To define the operator splitting, let

$$E(t,s): Lip_b(\mathbb{R}^N; \mathbb{R}^m) \to Lip_b(\mathbb{R}^N; \mathbb{R}^m)$$

denote the Euler operator defined by

(2.1)
$$E(t,s)w(x) = w(x) + (t-s)G(s,x,w(x))$$

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 $u \in BUG(Q_2)$ such that u is the unique viscouty solution of (1,2) and (1,3). If indecence, γ_R in (F2) is millipendent of R, then (1,3) and (2,3) has a viscour receiver and Q_2 .

Proposition 1.1. Let $f \in [0, \mathbb{N}] \times \mathbb{R}^{n}$ is $\mathbb{R}^{n} \to \mathbb{R}^{n}$ added (12), (2), (2), and (2), (3) or $f \in Lip_{n}(\mathbb{R}^{n})$, and $u \in BU(C(\mathbb{R}^{n}))$ is the animus in mostly solution of (1.2), and (7.3) so O_{T} , then $u \in Lip_{n}(O_{T})$.

2. OPERATOR SPLITTING STUD MARY RESULTS

We now give conditions on G and Z which make an administrate for a 11 will be minister to get existence and uniqueness of a visited v scholing on Sen (G-1. Measures these conditions are strong anough to give a linear termination rate for the quentor splitting. We assume that Z and C concerving the linear converse matching of the second splitting.

(HI - H4) For each a 10 antiches condutions (D) - (F4).

(H5) There is a constant $L^{(2)} > 0$ such that $|H_1(k, n; r, n) - H_1(k; n; s, n)| \leq L^{(2)}|r - n|$. for $t \in [0, T]$, $t_1 \ge 0$ $\mathbb{R}^{(1)}$, $r \in \mathbb{R}$, and r = 1.

(H6) For each R > 0 three is a constant N(1 > 0 and that $[H_1(t, x, r, b) - H_2(t, x, r, p)] \le N^2(1 + b)! t = 0$ for $t, t \in \{0, T\}$, by $R = R = R^2$, such t = 1.

(H1) For each R > 0 there is a contribution $M_R > 0$ are high $|R|(0, x, x, y) - R_1(0, z, z, y)| \leq N_R|y - y|$ for $t \in [0, T]$. [6] Solve R and R' and R'

(G1) $G \in O'([0, T] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ is uniformly continuous on $[0, T] \times \mathbb{R}^n \times B_{n,i}(0, R]$. for each R > 0

(G2) Then is a constant 0° >0 unit that 0° = spect 100 ± end 0.

(G3) For each R > 0 there is a constant $O_{1}^{2} > 0$ such that $|O(t, x, r)| - O(t, y, r)| \leq O(t_{1} - y_{1}, y_{2}) \leq O(t_{1} - y_{1}, y_{2}) \leq O(t_{1} - y_{2})$

(G4) There is a constant $\mathbb{S}^2 \geq 0$ and that $|G(t, x, t) - G(t, x, n)| \leq t \leq r - \lambda t \log t$. (t, t) $\in O_{\mathbb{C}}$ and t, $n \in \mathbb{R}^n$.

(G5) For each E > 0 times by a constant NE > 0 and think [G(0, x_0) - G(E, b, v)] $\leq NE$ b = B for $b \in E \in [0, T]$ [all $\leq B$, and $v \in E$.

Note that by the conditions (F2) and (G2) volcan a summation B_1 satisfies $M_1(t, \tau, 0, 0) = 0$. If this were not so, we could simply redence H as $h(t, t, \alpha_2) = H(t, \tau, 0, 0)$ and G as $G(t, \tau, u) = H(t, x, 0, 0)$.

We shall assume that there emers a unique solution of Equat(qr.27) to the introl value problem (1.1) under the assumptions (RI), (RI), (RI), (G), (G), and and the Leon (R, E'), First we will state as error bound for the splitting procedure when the onlinerted forential equation is approximated by the explicit Balay method. To define the operator splitting, let

denote the Euler operator defined by

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for $0 \leq s \leq t \leq T$ and $w \in Lip_b(\mathbb{R}^N; \mathbb{R}^m)$. Furthermore, let

$$S_H(t,s): Lip_b(\mathbb{R}^N) \to Lip_b(\mathbb{R}^N)$$

be the solution operator of the scalar Hamilton-Jacobi equation without source term

(2.2)
$$u_t + H(t, x, u, Du) = 0, \quad u(x, s) = \bar{w}(x),$$

i.e., we write the viscosity solution of (2.2) as $S_H(t,s)\overline{w}(x)$.

We let S denote the operator defined by $S(t,s)w = (S_{H_1}(t,s)w_1,\ldots,S_{H_m}(t,s)w_m)$ for any $w = (w_1,\ldots,w_m) \in Lip_b(\mathbb{R}^N;\mathbb{R}^m)$. Now we can define our approximate solutions: Fix $\Delta t > 0$ and set $t_j = j\Delta t$, set $v(x,0) = v_0(x)$ and

(2.3)
$$v(x,t_j) = S(t_j,t_{j-1})E(t_j,t_{j-1})v(\cdot,t_{j-1})(x),$$

for j > 0. Note that this approximate solution is defined only at discrete *t*-values. Our first result is that the operator splitting solution, when (2.2) is solved exactly, converges linearly in Δt to the viscosity solution of (1.1).

Theorem 2.1. Let u(x,t) be the viscosity solution of (1.1) on the time interval [0,T], and $v(x,t_j)$ be defined by (2.3). There exists a constant K > 0, depending only on T, $||u_0||$, $||Du_0||$, $||v_0||$, $||Dv_0||$, H, and G, such that for j = 1, ..., n

$$||u(\cdot, t_j) - v(\cdot, t_j)|| \le K(||u_0 - v_0|| + \Delta t).$$

We will prove this theorem in the next section.

Our second theorem gives a convergence rate for operator splitting when the explicit Euler operator E is replaced by the exact solution operator \overline{E} . More precisely, let $\overline{E}(t,s)$: $Lip_b(\mathbb{R}^N;\mathbb{R}^m) \to Lip_b(\mathbb{R}^N;\mathbb{R}^m)$ be the solution operator of the system of ordinary differential equations

(2.4)
$$u_t = G(t, x, u) \qquad u(x, s) = w(x).$$

where $w \in Lip_b(\mathbb{R}^N; \mathbb{R}^m)$. Note that x acts only as a parameter in (2.4), and that the assumptions on G ensure that \overline{E} is well defined on the time interval [s, T].

Analogously to (2.3) we define the approximate solution $\{\bar{v}(x,t_j)\}_{j=1}^n$,

(2.5)
$$\bar{v}(x,t_j) = S(t_j,t_{j-1})E(t_j,t_{j-1})\bar{v}(\cdot,t_{j-1})(x),$$

for j > 0 and $\bar{v}(x, t_0) = v_0$. Then we have:

Theorem 2.2. Let u(x,t) be the viscosity solution of (1.1) on the time interval [0,T] and $\bar{v}(x,t_j)$ be defined by (2.5). Then there exists a constant $\bar{K} > 0$, depending only on T, $||u_0||$, $||Du_0||$, $||v_0||$, $||Dv_0||$, H, and G, such that for j = 1, ..., n

$$||u(\cdot, t_j) - \bar{v}(\cdot, t_j)|| \le K(||u_0 - v_0|| + \Delta t).$$

Remark 2.3. Theorems 2.1 and 2.2 are generalizations of Theorems 3.1 and 3.2 in [2].

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for $0 \leq i \leq 1 \leq T$ and $i \in E$ Length [12] [12]]. Furthermore, let

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$$(2.2) \qquad (2.2)$$

i.e. we write the vistosity solution of (2.2) as Se(4, a) black

$$(23) = (2,3)$$

for $j \geq 0$. Note that this improvintable solution is defined multiplat discrete t-whites. Our first result in that the operator additions solution, when (2.7) is solved another movely, converges linearly in Δt to the terrestivischment of 0.1.

Theorem 3.4. Let a be the investigity volument of (2, 2) on the time spherics (0, 2), and $u(x, t_i)$ be softned by (x, x). There exists a minimum (x, t_i) be softned by (x, x). There exists a minimum (x, t_i) depending only on T, that: $|Dx_0|$, $|u_0|$, $|Dx_0|$,

$$(22 + 1)n - n(1)n \geq [(22)n - (22)n]$$

We will prove this theorem or the next sections -

Our second theorem gives a convergence rate for operator splitting when the explicit Edict operator C is replaced by the encirch collation operator S. Afore proparity in E(x, z). $Lip_{A}(\mathbf{R}^{+}; \mathbf{R}^{+}) \rightarrow Lip_{A}(\mathbf{R}^{+}; \mathbf{R}^{+})$ is the salution operator of the space of the space ential equations

where $w \in L_{eps}(\mathbb{R}^n \mid \mathbb{R}^n)$. Note that x up to vely as a parsingler in (2, 4), and that the second time interval (x, Y). Assumptions on G evenues that E is well defined on the time interval (x, Y). A microwsky to (2,3) we define the sparredomine solution $(4(x, t_1))$.

$$(2.5) \quad (3.5) \quad (5.7) = \beta(1, 1, 1, 2) + \beta(2, 3, 2) + \beta(3, 2) + \beta(3, 3) + \beta$$

for j > 0 and $\theta(x, b_i) = c_0 \sqrt{1}$ from the basis:

$$||u(z,z)| = ||u(z,z)|| \le ||z|||u_0| = ||u_0|| = ||z|||$$

Remark 2.3. Theorem 2.1 and 2.2 or relendering of Theorems 5.1 and 3.2 or [2]

3. Proofs of Theorems 2.1 and 2.2

We will proceed as follows: First we give some estimates we will need later. Then we introduce an auxiliary approximate solution and prove linear convergence rate for this solution. This proof involves the scalar version of Theorem 2.1. We proceed to show that the operator splitting solution converges to this approximate solution with linear rate. This completes the proof of Theorem 2.1. Finally we give a proof of Theorem 2.2. This proof is similar to the proof of Theorem 3.2 in [2].

We start by stating the relevant estimates on S. Let $w, \tilde{w} \in Lip_b(\mathbb{R}^N), 0 \leq s \leq t \leq T$, and $R_1 = \sup_{t,s,i} ||S_i(t,s)w||$, then

(3.1)
$$||S_i(t,s)w|| \le e^{L^H(t-s)} ||w||,$$

(3.2)
$$\|D\{S_i(t,s)w\}\| \le e^{(L^H + K(R_1))(t-s)}\{\|Dw\| + (t-s)K(R_1)\},$$

(3.3)
$$||S_i(t,s)w - S_i(t,s)\tilde{w}|| \le e^{L^H(t-s)} ||w - \tilde{w}||,$$

where K(R) is a constant depending on R but independent of i, t, and s. Estimate (3.3) is a direct consequence of Theorem 1.1. Note that in this case $\gamma = L^{H}$. Estimates (3.1) and (3.2) correspond to estimates (4.7) and (4.8) in [2].

Regarding the approximation defined by (2.3), $v(\cdot, t_j)$, we have the following estimates: **Lemma 3.1.** There is a constant R independent of Δt such that $\max_{1 \le j \le n} ||v(\cdot, t_j)|| < R$. Moreover for every $1 \le j \le n$,

(a) $||v(\cdot, t_j)|| \le m e^{(L^H + mL^G)t_j} (||v_0|| + t_j C^G),$ (b) $||Dv(\cdot, t_j)|| \le m e^{(L^H + mL^G + K(R))t_j} \{||Dv_0|| + t_j (C_R^G + K(R))\}.$

Proof. To prove a) and b), we need (3.1), (3.2), and the definition of the operator E. We only give the proof of a). The proof of b) is similar. By (3.1) we get

(3.4)
$$\|S_i(t_j, t_{j-1}) \{ E(t_j, t_{j-1}) v(\cdot, t_{j-1}) \}_i \| \le e^{L^H \Delta t} \| \{ E(t_j, t_{j-1}) v(\cdot, t_{j-1}) \}_i \|$$

We then use the definition of E (2.1) and (G3), (G4) to get

$$\|\{E(t_j, t_{j-1})v(\cdot, t_{j-1})\}_i\| \le \|v_i(\cdot, t_{j-1})\| + \Delta t \left(C^G + L^G \|v(\cdot, t_{j-1})\|\right).$$

Note that $||v(\cdot, t_{j-1})|| \leq \sum_{i=1}^{m} ||v_i(\cdot, t_{j-1})||$. Now using this and summing over *i* in inequality (3.4), we get

(3.5)

$$\sum_{i=1}^{m} \|S_{i}(t_{j}, t_{j-1})\{E(t_{j}, t_{j-1})v(\cdot, t_{j-1})\}_{i}\|$$

$$\leq e^{L^{H}\Delta t} \left\{ \left(1 + \Delta tmL^{G}\right)\sum_{i=1}^{m} \|v_{i}(\cdot, t_{j-1})\| + mC^{G}\Delta t \right\}$$

$$\leq e^{\left(L^{H} + mL^{G}\right)\Delta t} \left\{ \sum_{i=1}^{m} \|v_{i}(\cdot, t_{j-1})\| + mC^{G}\Delta t \right\}.$$

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I. PROOPS OF THEOREMS 2.1 AND 2.2

We will proceed as failows. First we give sense estimates we will need lasse. Then we introduce an anelliary approximate solution and prove linear we will need lasse. Then we solution. This proof involves the scales remain of Theorem 2.1. We proceed to shrive that the operator solutions whitting solution concepts to this approximate solution with linear rate. This concepts the operator solution with linear rate. This proof is the operator solution with a solution with linear rate. This concepts the operator solution with linear rate. This concepts the solution of the solution with linear rate. This concepts the operator solution with linear rate.

We start by stating the relevant estimates and here a had brac(\mathbb{R}^n), $0 \le a \le l \le T$, and $R_1 = and \dots (\mathbb{R}^n)$. $0 \le a \le l \le T$.

where K(R) is a verifical differential on R bire independent with k and r. Estimate (3.3) is a direct correspond to Theorem 1.1. Some that in this case r = 4. Estimates (3.4) and (3.2) correspond to estimates (4.7) and (4.3) in [2].

Lemma 3.1. There is a constant R independent of Δt such that $\max_{i \in \mathcal{I}} |w(e, t_i)| < R$. Marcour for every $1 \leq i \leq n$.

- $|\{a\} = \{a\} = \{a, b\} = \{a, b\}$
- $[D_n(\cdot, t_i)] \leq m_n + m_n + m_n + ([D_n]] + i_n (C_n^2 + M(A)))$

Proof. To prove a) and b), we need (2.7), (3.2), upit the definition of the opprace E. We only give the proof of A. The mode of M is same for (3.1) we set.

(3.4) [[5.(6.6.1) [8(6.6.4.1)]] (5.(⁶)]] [[6(6,6.1)]]

Note that $||v(\cdot, t_{i-1})| \leq \sum_{i=1}^{n} ||v(\cdot, t_{i-1})||$. Now using this and semining over (in inequality (0.4), we get

The result in a) now follows from successive use of (3.5) and an application of the inequalities $|x| \leq \sum_{i=1}^{m} |x_i| \leq m|x|$ for $x \in \mathbb{R}^m$. Replacing t_j by T in a), we see that the existence of R is assured.

Proof of Theorem 2.1.

Let u denote the solution of (1.1) and define (3.6)

$$\tilde{G}_i(t,x,r) = G_i(t,x,u_1(x,t),\ldots,u_{i-1}(x,t),r,u_{i+1}(x,t),\ldots,u_m(x,t)), \quad i = 1,\ldots,m$$

Note that the function \tilde{G}_i satisfies (G1)-(G5) for all i = 1, ..., m.

Using \tilde{G}_i , we can rewrite (1.1) as a series of "uncoupled" equations

(3.7)
$$\frac{\partial u_i}{\partial t} + H_i(t, x, u_i, Du_i) = \tilde{G}_i(t, x, u_i), \quad i = 1, \dots, m$$

Of course, the viscosity solution of (1.1) u is also the unique viscosity solution of the system of equations (3.7).

Now we want to do (scalar) operator splitting for each equation in (3.7). To this end, for any $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, let $x_{i*} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$. Now for any $w \in Lip_b(\mathbb{R}^N; \mathbb{R}^m)$ let $E_i(t, s)w_i$ be given by

$$E_i(t,s)w_i = w_i + (t-s)G_i(s,x,w_i).$$

Now we define the following operator splitting solution $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_m)$,

(3.8)
$$\tilde{v}_i(x,t_j) = S_i(t_j,t_{j-1})E_i(t_j,t_{j-1})\tilde{v}_i(x,t_{j-1}),$$

for j > 1, and $\tilde{v}_i(x, t_0) = u_{0i}(x)$. Note that E_i is the Euler operator for the equation

$$\frac{\partial u_i}{\partial t} = \tilde{G}_i(t, x, u_i).$$

Hence by the results of [2]:

Lemma 3.2. Let u(x,t) be the viscosity solution of (1.1) on the time interval [0,T] and $\tilde{v}(x,t_j)$ be the operator splitting solution (3.8). There exists a constant K' > 0, depending only on T, $||u_0||$, $||Du_0||$, H, and G, such that for j = 1, ..., n,

$$||u(\cdot, t_j) - \tilde{v}(\cdot, t_j)|| \le K' \Delta t.$$

Using the above lemma, we wish to estimate $\|\tilde{v}(\cdot, t_j) - v(\cdot, t_j)\|$, and start by using the definition of the operator splitting solutions (2.3) and (3.8) and the estimate (3.3). Then

$$\begin{aligned} |\tilde{v}_{i}(x,t_{j}) - v_{i}(x,t_{j})| &\leq \left| S_{i}(t_{j},t_{j-1}) E_{i}(t_{j},t_{j-1}) \tilde{v}_{i}(x,t_{j-1}) - S_{i}(t_{j},t_{j-1}) (E(t_{j},t_{j-1}) v(x,t_{j-1}))_{i} \right| \\ &\leq e^{L^{H}\Delta t} \left| E_{i}(t_{j},t_{j-1}) \tilde{v}_{i}(x,t_{j-1}) - (E(t_{j},t_{j-1}) v(x,t_{j-1}))_{i} \right|. \end{aligned}$$

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The result in a) now follows have discussive use of (3.5) and an application of the inequalities $|z| \leq \sum_{i=1}^{n} |z_i| \leq m|z|$ for $z \in \mathbb{R}^n$. Replacing (3 by 2 is a), we see that the existence of R is assured.

Froof of Theorems 2.1.

Let u denote the solution of (1.1) and despe

 $G_{i}(1, x, r) = G_{i}(1, x, r) \cdot (x, r) \cdot (x,$

Note that the function of sotistics (G1)-(G5) for all i = 1.....

. Using G., we can reache (1-3) as a series of "uncoupled" equation of the

Of course, the viscosity solarise of (1.1) with size the unique viscosity solution of the system of equations (3.7).

Now we want to do (coular) converse splitting for each equivient in (3.7). It this end, for any $x = (x, y, \dots, x_n) \in \mathbb{R}^n$, let $x_1 = (x_1, \dots, x_n)$, $x_n \in \mathbb{R}_n$, i.e. (3.8). Now for any $x_2 \in \mathbb{R}_n$, \mathbb{R}_n , \mathbb{R}_n , \mathbb{R}_n) is solved by any $x_2 \in \mathbb{R}_n$.

Now we define the following operator soluting solution of # (\$5.12, 16.6).

$$(1.8) = (1.1) + (1.1$$

for j > 1, and $\tilde{w}[x, f_0] = u_0[x]$. Note that E_j is the Euler operator for the equation

Hence by the results of Di-

Lemma 3.2. Let u(x,y) be the overally volution of (1, t) on the line-three 0 (0, T) and $\tilde{u}(x, t_{1})$ be the operator splitting volution $(0, \delta)$. There exists a constant N > 0, depending only on T. [nucl. [Out], δ : and G: each thin for j = 1, ..., n.

$$|\Delta^{2}X \geq ||\langle A, A \rangle|| \leq K^{2}\Delta$$

Using the above leases we rest, to estimate $\|v(\gamma, t_i) - v(\gamma, t_i)\|_2$ and that by using the definition of the operator, splitting efficience (2.3) and (3.8) and the estimate (3.3). These

$$|E_{i}(x, t_{i}) - g_{i}(x_{i}t_{i})| \leq |E_{i}(t_{i})| - |E_{i}(t_{i}, t_{i-1})| \leq |E_{i}(t_{i$$

By the Lipschitz continuity of G, we have that

$$E_{i}(t_{j}, t_{j-1}) \tilde{v}_{i}(x, t_{j-1}) - (E(t_{j}, t_{j-1}) v(\cdot, t_{j-1}))_{i}|$$

$$\leq |(\tilde{v}_{i} - v_{i})(x, t_{j-1})| + \Delta t |G_{i}(u_{1}, \dots, \tilde{v}_{i}(x, t_{j-1}), \dots, u_{m}) - G_{i}(v_{1}(x, t_{j-1}), \dots, v_{m}(x, t_{j-1}))|$$

$$\leq |(\tilde{v}_{i} - v_{i})(x, t_{j-1})| + L^{G} \Delta t (|(u_{i*} - v_{i*})(x, t_{j-1})| + |(\tilde{v}_{i} - v_{i})(x, t_{j-1})|)|$$

$$\leq |(\tilde{v}_{i} - v_{i})(x, t_{j-1})| + L^{G} \Delta t (|(u_{i*} - \tilde{v}_{i*})(x, t_{j-1})| + |(\tilde{v}_{i*} - v_{i*})(x, t_{j-1})|)|$$

$$+ |(\tilde{v}_{i} - v_{i})(x, t_{j-1})|)$$

 $\leq |(\tilde{v}_i - v_i)(x, t_{j-1})| + L^G K' \Delta t^2 + L^G \sqrt{2} \Delta t |\tilde{v}(x, t_{j-1}) - v(x, t_{j-1})|.$

Summing the resulting inequality over i yields

$$\sum_{i=1}^{m} |\tilde{v}_{i}(x,t_{j}) - v_{i}(x,t_{j})|$$

$$\leq e^{L^{H}\Delta t} \left(mK'L^{G}\Delta t^{2} + (1 + mL^{G}\sqrt{2}\Delta t) \sum_{i=1}^{m} |\tilde{v}_{i}(x,t_{j-1}) - v_{i}(x,t_{j-1})| \right)$$

$$\leq e^{\left(L^{H} + m\sqrt{2}K'L^{G}\right)t_{j}} \sum_{i=1}^{m} |u_{0,i}(x) - v_{0,i}(x)| + mK'L^{G}t_{j}\Delta t$$

Hence Theorem 2.1 holds.

Proof of Theorem 2.2. We end this section by giving the proof of Theorem 2.2. Assume for the moment that

$$||v(x,t_i) - \bar{v}(x,t_i)|| \le \bar{C}\Delta t$$

for all j, where \overline{C} is a constant depending on G, H, T, $||u_0||$, $||Du_0||$, $||v_0||$, and $||Dv_0||$ but not Δt . Using (3.9) and Theorem 2.1, we find

$$\begin{aligned} \|u\left(\cdot,t_{j}\right)-\bar{v}\left(\cdot,t_{j}\right)\| &\leq \|u\left(\cdot,t_{j}\right)-v\left(\cdot,t_{j}\right)\|+\|v\left(\cdot,t_{j}\right)-\bar{v}\left(\cdot,t_{j}\right)\|\\ &\leq K\left(\|u_{0}-v_{0}\|+\Delta t\right)+\bar{C}\Delta t. \end{aligned}$$

Setting $\bar{K} = K + \bar{C}$, we conclude that Theorem 2.2 holds. It remains to show (3.9). Using the same arguments as when estimating the local truncation error for the Euler method we find that

$$\sum_{i=1}^{m} \left| \{ E(t_{j+1}, t_j) v(x, t_j) - \bar{E}(t_{j+1}, t_j) \bar{v}(x, t_j) \}_i \right| \le e^{mL^G \Delta t} \sum_{i=1}^{m} \left| \{ v(x, t_j) - \bar{v}(x, t_j) \}_i \right| + \tilde{C} \Delta t^2,$$

where $\tilde{C} = mL^G(L^G\bar{R} + C^G) + mN^G_{\bar{R}}$. Here $\bar{R} > \max\left(\left\|\bar{E}(t_j,t)\bar{v}(\cdot,t_j)\right\|, \|v(\cdot,t_j)\|\right), \bar{R}$ is finite by arguments similar to those used in the proof of Lemma 3.1. Now using this we

Proof of Theorem 3.2. We and this applies by giving the proof of Theorem 2.2. Assumet for the moment that

10(13) 计正正的 (1-13) [[[[[]]]]

for all j, where C is a constant defension on G, B, T, finall, ||Dyall, ||m||, and $||Dv_a||$ but not ΔL . Using (3.9) and Theorem 2.1, we find

$$\|u(\cdot, z_i) - v(\cdot, z_i)\| \le \|u(\cdot, z_i) - v(\cdot, z_i)\| \le \|u(\cdot, z_i) + v(\cdot, z_i)\|$$

Setting R = K + C, we conclude that Theorem 2.2 bolds: It remains to show (3.3). Using the same arguments as when estimating the local transmision error, in the field softwark and the set that re find that

where $C = mL^{2}(L^{0}A + C^{2}) + mV_{2}$, since $R > max_{i}([B(t_{i}, t_{i})], t_{i})]$, let (L)(i), R, isfinite by arguments staticar to these used in (in great of Lemma 3.1. Now using this we find that

$$\sum_{i=1}^{m} \|\{v(\cdot, t_{j+1}) - \bar{v}(\cdot, t_{j+1})\}_{i}\| = \sum_{i=1}^{m} \|\{S(t_{j+1}, t_{j})E(t_{j+1}, t_{j})v(\cdot, t_{j}) - S(t_{j+1}, t_{j})\bar{v}(\cdot, t_{j})\}_{i}\| - S(t_{j+1}, t_{j})\bar{v}(\cdot, t_{j})\}_{i}\|$$

$$\leq e^{L^{H}\Delta t} \sum_{i=1}^{m} \|\{E(t_{j+1}, t_{j})v(\cdot, t_{j}) - \bar{E}(t_{j+1}, t_{j})\bar{v}(\cdot, t_{j})\}_{i}\|$$

$$\leq e^{(L^{H}+mL^{G})\Delta t} \Big(\sum_{i=1}^{m} \|\{v(\cdot, t_{j}) - \bar{v}(\cdot, t_{j})\}_{i}\| + \tilde{C}\Delta t^{2}\Big).$$

Since that $\bar{v}(x,0) = v_0(x)$, repeated use of inequality (3.10) gives (3.9).

4. A FULLY DISCRETE SPLITTING METHOD

In this section we present a simple numerical example of the splitting discussed in this paper. For simplicity we shall consider a system of two equations in one space dimension

(4.1)
$$u_t + H(u_x) = f(u, v), \quad v_t + G(v_x) = g(u, v),$$

When testing this numerically, we must replace the exact solution operator S by a numerical method. As most numerical methods for Hamilton-Jacobi equations are have convergence rates of 1/2 with respect to the time step, we use a front tracking algorithm, which has a linear convergence rate with respect to the time step. This front tracking algorithm is described in [3] and we shall only give a very brief account of front tracking here.

Front tracking uses no fixed grid and the solution is approximated by a piecewise linear function. The discontinuities in the space derivative, the so-called *fronts*, of the approximate solution are tracked in time and interactions between these are resolved. This algorithm works for scalar equations in one space variable of the form

$$u_t + H(u_x) = 0$$

For equations in several space dimensions, front tracking can be used as a building block in a dimensional splitting method, see [4].

For weakly coupled systems of the form (4.1), the approximate solution operator E depends on both u and v. Therefore, after the action of E, we must add fronts in the approximation of u at the position of the fronts in v and vice versa. In this situation we cannot in general find a global bound on the total number of fronts to track. In order to avoid this problem we use a fixed grid $x_i = i\Delta x$, for $i \in \mathbb{Z}$, and set

$$(4.2) S := \pi \circ S^{\text{f.t.}}$$

where π is a linear interpolation to the fixed grid and $S^{\text{f.t.}}$ is the front tracking algorithm. Unfortunately, this restricts the order of the overall algorithm to $\mathcal{O}(\Delta x^{1/2})$. Nevertheless, we do not have any inherent relation between Δx and Δt , and we used $\Delta x = \Delta t/10$.



Since that $\theta(x, 0) = \eta_0(x)$, repeated ago of inequality (3.10) gives (3.1)

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In this section we present a simplication rical example of the splitting discussed in the paper. For simplicity we shall adapted a spinetical of two equations in one space discussion

When testing this runnerically we noted declare the each estimated primeter 5 by a nonversal method. As most surratical methods for Harshtein-Iccold equations and have an averagence rates of 1/2 with respect to the visuation, we use a from tracking algorithm, which has a linear convergence rate with respect to the three ones. This from tracking theories also described in [3] and we abalt only give a search fair scorem of from tracking theories.

From tracking uses no fixed and the solution in approximated by a piectwise linear function. The discentinuities in the space derivative, the co-cilice houts, of the approximate solution are tracked in this and inderactions between these are resolved. This algorithm works for scalar equations in one space wirmble of the form

$u_{i} \in \mathcal{U}(u_{i}) := 0$

For equations in several value simulations, from tracting can be used as a building block in a dimensional rollinities method, associal

For weakly complet appresses of the form (4.1), the approximate solution operator b, depends on both a and c. Therefore, after the action of E, we must add froms in the approximation of c at the position of the neural is t and vice work. In this situation we cannot in general field a global bound on the total manber of fracts to track. In order to word this problem we use a field with c = cE_c . For $c \in E$, and set

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where r is a linear interprisment to the fixed grid and S^{free} is the front studing algorithm. Unfortunately, this president the adder of the desired algorithm to DCA2⁴⁷⁶ h (Kosterhelden), we do not linear the inherent talking determine As and Ad, and we used for - 20110.



FIGURE 1. u(x, 1) and v(x, 1)

TABLE 1. Δt versus $100 \times L^{\infty}$ error.

Δt	1	1/2	1/4	1/8	1/16	1/32
Error	32.0	16.5	11.6	8.3	5.1	3.2

We have tested this on the initial value problem

(4.3)
$$\begin{aligned} u_t + \frac{1}{2} (u_x)^2 &= 4v(u+1) \\ v_t + \frac{1}{2} (v_x)^2 &= u^2 + v^2 \end{aligned} \} \qquad u(x,0) = v(x,0) = 1 - |x|, \quad \text{for } x \in [-1,1], \end{aligned}$$

and periodic boundary conditions. In figure 1 we show the approximate solution at t = 1using $\Delta t = 1/8$. To find a "numerical" convergence rate, we compared the splitting solution with a reference solution computed by the Engquist-Osher scheme with $\Delta x = 1/2000$. Table 1 shows the relative supremum error for different values of Δt . These values indicate a numerical convergence rate of roughly 0.63, i.e., error = $\mathcal{O}(\Delta t^{0.63})$, much less than the rate using an exact solution operator for the homogeneous equation.

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We have tested this op chr fordal value problems,

$$u(z, 0) = u(x, 0) = 1 - |u| - 1$$
 is $z \in [-1, 1]$

and periodic boundary conditions. In ignre 1 we show the approximate solution at t = 1, using $\Delta t = 1/8$. To find a 'reditional' convergence rate, we consignred the eglictic constraint with a reference solution computed by the Engquist-Osher achieve with $\Delta x = 1/2000$. Table 1 shows the relative superstant or rate of roughly 0.013, i.e., error = $O(\Delta t^{0.013})$, much less than the relative superstant of roughly 0.013, i.e., error = $O(\Delta t^{0.013})$, much less than the rate of roughly 0.014 and the relative superstant of roughly 0.015 are the relative superstant of the relative superstant of roughly 0.013, i.e., error = $O(\Delta t^{0.013})$, much less than the rate of roughly 0.013 are the relative superstant of the relative superstant of roughly 0.013 are the relative superstant of the relative superstant of roughly 0.013 are the relative superstant of the relative superstant of roughly 0.013 are the relative superstant of the relative superstant of roughly 0.013 are the relative superstant of roughly 0.015 are the roughly 0.015

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