Department of APPLIED MATHEMATICS

On the mass transport induced by timedependent oscillations of finite amplitude in a nonhomogeneous fluid

I General results for a perfect gas

by

Jacqueline Naze Tjøtta and Sigve Tjøtta

Report No. 42

May 1973



UNIVERSITY OF BERGEN Bergen, Norway



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Abstract

The mass transport induced by time-dependent oscillations of finite amplitude in a nonhomogeneous fluid, is considered. The mass transport is given by the Lagrangian mean velocity, calculated to the second order in the Mach number of the oscillations. We find no mass transport in a non-dissipative model. Taking into account dissipation, however, the theory leads to non-zero vertical drift and horizontal flow. The vertical drift becomes zero in an incompressible model.



Introduction.

In the present paper a general theory is outlined for the mass transport induced by time-dependent oscillations of finite amplitude in a nonhomogeneous, one-component fluid. The theory is motivated in the possible applications within astrophysics and geophysics, where non-linear effects like the one studied probably influence the mean currents and the flux of energy. (See, for example, Munk and Moore (1968), Longuet-Higgins (1970) for studies of ocean currents).

The mass transport is given by the Lagrangian mean velocity, calculated to the second order in the Mach number of the oscillations. We compute the vertical component of this mean velocity, the divergence and the vertical component of the vorticity, whereby determining the flow field. The fluid is a viscous, heat conductive and heat radiative perfect gas. Compressibility effects are fully accounted for. Different equilibrium models are considered.

There is no mass transport in a non-dissipative model. Taking into account dissipation, however, the theory leads in general to a non-zero effect. A vertical drift is obtained in addition to a flow in the horizontal plane. The vertical drift becomes zero in an incompressible model. *).

There are indications that flows of this type occur in experiments on standing acoustic waves (See Schaaffs and Haun (1968), Schaaffs (1973), Hobæk (1973) who have observed the formation of periodic density stratification, the scale of which is the half wave length).

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^{*)} This general result for an incompressible fluid explains why a zero mass transport is found by Kildal (1969), in his detailed analysis of a viscous boundary layer model.

Introduction

La tra present paper a general tracory is continued for the mass traceport induced by time-dependent continues of finite amplifude in a noniversectore, one-component finite. The theory is motiveted in the possible appliestions within attractore and geophysics, where non-linear effects like the one sindeed probably influence the near partents and the finan of energy. (See, for transle, Wrate and Hoore (1968), forghet-Highlan (1970)

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2. Basic equations.

After elimination of the entropy, the equations of hydrodynamics can be written as follows:

- (2.1) $\int \frac{DV}{DE} + \nabla p + \frac{1}{2}gg = F^{\vee}$
- $(2.2) \quad \frac{Dg}{Dt} + g \overline{V} \cdot V = 0$
- $(2.3) \frac{DP}{DE} + gc^2 \nabla \cdot V = \Phi^T + \Phi^R + \Phi^V .$

 $\forall, \beta, \beta, \lambda, \beta, \zeta$ are respectively the Eulerian velocity, density, pressure, unit vector in upward direction, acceleration of gravity and sound speed. Further, F' is the viscous force, $\Phi^{T}/\gamma-1$ the accession of heat due to conduction, $\Phi^{R}/\gamma-1$ the accession of heat due to radiation, Φ^{V} the viscous dissipation function; γ is the ratio c_{p}/c_{v} of the specific heats.

 $(2.4) \stackrel{\mathsf{F}}{=} \nabla \left[\left(\frac{\mu}{3} + \mu_{\mathsf{B}} \right) \nabla \cdot \nabla + \nabla \cdot \nabla \mu \right] + \mu \nabla^{2} \nabla - \nabla \nabla^{2} \mu + \nabla x \left(\nabla x \nabla \mu \right)$ $(2.5) \stackrel{\mathsf{\Phi}}{=} (\gamma - 1) \nabla \cdot (\sigma \nabla T)$

$$(2.6) \Phi^{V} = (\chi - 1) \left[\mu \left(\nabla V : \nabla V + \nabla V : \left(\nabla V \right)^{*} - \frac{2}{3} \left(\nabla . V \right)^{2} \right) + \mu_{B} \left(\nabla . V \right)^{2} \right],$$

where $(\nabla \underline{V})^*$ denotes the conjugate of $\nabla \underline{V}$. Further, we put

(2.7)a
$$\Phi^{R} = (\gamma - 1) \nabla \cdot (\sigma_{R} \nabla T)$$

for the optically thick case, and

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(2.7)b $\Phi^{R} = -(\gamma - 1)c_{\gamma}g(T - T_{o})$ (Newtons cooling law)

for the optically thin case.

The temperature is given by the equation of state

(2.8)
$$p = R_{g}T$$
, $R = c_{p} - c_{r}$.

Thus the radiative effects are accounted for only in the energy equation (2.3).

 χ and R are taken constant. The dissipative coefficients $\sigma, \mu, \mu_b, \sigma_R$ and q depend on the temperature (mainly) and the density. How, for example, σ and μ vary with these variables is known from classical kinetic theory. If the binary collision model with central force proportional to $r^{-\mu}$ is adopted, σ and μ are found to be independent of g but proportional to $T^{\frac{1}{2}+\frac{2}{V-T}}$, (Chapman & Cowling (1958)).

The following equation of evolution is obtained for \mathcal{T} by eliminating β , β between (2.2), (2.3) and (2.8), and inserting $c^2 = \beta \beta / \beta$:

(2.9)
$$\frac{DT}{Dt} + (\gamma - i) T \nabla V = \frac{i}{R_{f}} \left(\phi^{T} + \phi^{R} + \phi^{V} \right) .$$

At equilibrium the velocity is supposed to be equal to zero. The other variables β_{\circ} , β_{\circ} , T_{\circ} must then satisfy the following equations:

(2.10)
$$\begin{cases} \nabla p_{o} + \frac{\lambda}{2}gg_{o} = 0 \\ \nabla \cdot (\sigma_{o} \nabla T_{o}) = 0 \\ p_{o} = Rg_{o}T_{o} \end{cases}$$

for the optically thin case, or the analogous equations where σ_{e} is replaced by $\sigma_{e} + \sigma_{Re}$ for the optically thick case.

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for the optically this case. The temperature is given by the equation of at

Thus the redistive effects are eccounted for only in the energy equation (2.3).

See hours and a are taken constant. The dissipative coefficients See hours, and a depend on the temperature (mainly) and the density, and for example, s and r with these versales is known from elemental binetic theory. If the himser collicits and he are found be helependent of s but proportional to and he are found be helependent of s but proportional to T the

The following equation of evolution is obtained for T by eliminating P., folloween (2.2), (2.3) and (2.8), and inserting of Frif

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At squilibrium the velocity is supposed to be equal to zero. The other veriables P. . S. . T. must then satisfy the following equations:

We suppose that ρ_0, β_0, T_0 depend only on the altitude z, so that we have

(2.11)
$$\frac{g'_{o}}{g_{o}} = -\frac{g}{RT_{o}} - \frac{T'_{o}}{T_{o}}$$
,

where the prime denotes the $\frac{2}{2}$ -derivative.

Of special interest is the isothermal equilibrium (2.12) $T_o = \text{constant}, \quad p_o = p_s e^{-\frac{2}{H}}, \quad f_o = f_s e^{-\frac{2}{H}}$

where β_s , β_s are the values of β , β at z = o and H is the scale heigth. Here $c_o^2 = \gamma \beta_o / \beta_o = \gamma R T_o = \gamma g H$.

Other possible equilibria are obtained by solving for instance, the equation $(\sigma(\tau_o) \tau'_o)'=o$. When σ is an increasing function of T, ρ_o , g_o , T_o are power functions of \gtrless , and T_o increases with altitude.

M being the Mach number of the perturbation, we suppose that \bigvee , \flat , \Im , \top can be developed in powers of M, at least up to the second order

(2.13)
$$\frac{V}{P} = \frac{V_1 + V_2 + \cdots}{P = P_0 + P_1 + P_2 + \cdots}$$

Substituting these expressions in (2.1)-(2.3) and (2.8), we obtain to the first order the linearized equations, where $w = \frac{V.2}{2}$,

$$(2.14) \quad g_{0} \frac{\partial V_{1}}{\partial t} + \nabla p_{1} + \frac{\partial}{\partial} g_{1} = F^{V}$$

$$(2.15) \quad \frac{\partial f_1}{\partial t} + f'_0 w_1 + f_0 \overline{V} \cdot V_1 = 0$$

(2.16)
$$\frac{\partial p_1}{\partial t} + p'_0 w_1 + \gamma p_0 \overline{P} \cdot V_1 = \Phi_1^T + \Phi_1^R$$

(2.17) $p_1 = R(g_0 T_1 + g_1 T_0)$.

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We suppose that p., f. depend only on the siticude , so that we have

$$(2.17) \quad \int_{0}^{1} = -\frac{8}{87} - \frac{1}{57} + \int_{0}^{1} + \frac{1}{57} + \frac{1}{57}$$

where the prime denotes the g -derivative.

Of epecial interest is the isothermal aquilibrant

where beight. Here the values of p. s at is a and H is the scale beight. Here the planes of p. s at is a and H is the other possible equilibris are obtained by solving for instance, the equation (c(T.)T.) so when a is an increasing function of T. E. S. are power functions of s is and T.

It being the Mach number of the perturbation, we suppose that Y, p, r, f, T can be developed in powers of M, at least ap to the second order

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1 = 1285 + 19 + 1 2 (41.5)

9 = 12 2 + 1 2 + 13 6 (er.s)

2.76) <u>114</u> + 11 w + 19, R.Y. = 41 +

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From (2.9) (or from (2.15)-(2.17)) we obtain

(2.18)
$$\frac{\partial T_1}{\partial t} + T'_0 w_1 + (\xi - 1) T_0 V_1 = \frac{1}{Rg_0} (\phi_1^T + \phi_1^R)$$

Another consequence of the first order equations is obtained by taking the curl of (2.14)

$$(2.19) \frac{3}{3t} \nabla_{X} V_{1} = \frac{3}{2} \times \nabla \left(\frac{g_{o}' \not b_{1} - p_{o}' g_{1}}{g_{o}^{2}} \right) + \nabla_{X} \frac{F_{1}'}{g_{o}}$$

An alternative form is obtained by taking the time derivative of (2.19) and by using (2.15), (2.16)

$$(2.20) \frac{\partial^2}{\partial t^2} \nabla_X V_1 = \frac{c_o^2 N^2}{g} \frac{1}{2} \times \nabla \nabla \nabla_1 V_1 + \frac{g'_o}{g_o^2} \frac{1}{2} \times \nabla (\phi_1^T + \phi_1^R) + \frac{\partial}{\partial t} \nabla_X \frac{F_1^V}{g_o}$$

Here $N^2 = -g(\frac{g'_o}{g_o} + \frac{g}{c_o^2})$ is the Väisälä frequency, see later, §.4.1. Identifying terms of order M we obtain the second order equations

$$(2.21) \quad S_0 \frac{\partial V_2}{\partial t} + P p_2 + \partial g S_2 = F_2 - \left(S_1 \frac{\partial V_1}{\partial t} + S_0 \frac{V}{V}, PV_1\right)$$

$$(2.22) \quad \frac{\partial S_2}{\partial t} + S_0' \frac{W_1}{W_1} + S_0 \frac{V}{V}, V = -\overline{V}, \left(S_1 \frac{\partial V_1}{\partial t} + S_0 \frac{V}{V}, PV_1\right)$$

OF

 $(2.23) \frac{\partial P_2}{\partial t} + p'_0 W_2 + \gamma P_0 \overline{V} V_2 = \Phi_2^T + \Phi_2^R + \Phi_2^V - (V_1 \cdot \overline{V} P_1 + \gamma P_1 \overline{V} V_1)$ $(2.24) p_2 = R(g_0 T_2 + g_1 T_1 + g_2 T_0).$

Equation (2.9) (or (2.22)-(2.24)) gives now
(2.25)
$$\frac{2}{2}\frac{T_2}{t} + T'_0 w_2 + (\chi_{-1})T_0 \overline{V} \cdot V_2 = \frac{1}{R_{so}} \left(\phi_2^T + \phi_2^R + \phi_2^V \right) - \left(\frac{V_1 \cdot \nabla T_1}{1 + (\chi_{-1})T_1 \overline{V} \cdot V_1} - \frac{S_1}{R_{so}^2} \left(\phi_1^T + \phi_1^R \right) \right)$$

Irom (2.9) (or from (2.15)-(2.17)) we obtain

(2.18)
$$\frac{3}{6L} + T$$
, $w_1 + G - 0T$, $RX = \frac{1}{8\Omega} (\Phi_1^{T} + \Phi_1^{T})$.
Another consequence of the first order equations is obtained
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An alternative form is obtained by paring the time derivative of (2.19) and by using (2.15), (2.16)

Here $N^2 = -g(\frac{f_1}{f_2} + \frac{f_2}{q_2})$ is the Väisälä frequency, see later, 5.4.1. Identifying terms of order N we obtain the second order equations

 F_{4}^{ν} , ϕ_{4}^{τ} , ϕ_{4}^{κ} are the first order terms in the expansion of F^{ν} , ϕ^{τ} , ϕ^{κ} in powers of M (ϕ_{4}^{ν} is equal to zero). They are linear combinations of first order variables and their gradients; the coefficients of these linear combinations depend on the dissipative coefficients and the first derivatives of these coefficients with respect to T, taken at $T = T_{\rho}(z)$. For example,

$$(2.26) \quad \Phi_1^T = (\gamma - 1) \nabla \cdot (\sigma_0 \nabla T_1 + (\frac{d\sigma}{dT})_0 T_1 \nabla T_0) ,$$

where σ_{o} , $(\frac{d\sigma}{d\tau})_{o}$ stand for $\sigma(T_{o})$, $(\frac{d\sigma}{d\tau})(T_{o})$ and vary with 2 in general. F_{2}^{V} , Φ_{2}^{T} , Φ_{2}^{R} , Φ_{2}^{V} are the second order terms in the expansion of F^{V} , Φ^{T} , Φ^{R} , Φ^{V} . Φ_{2}^{V} is a quadratic form of V_{1} and its gradient, with coefficients of the same form as above. In addition to similar quadratic forms of the first order variables, F_{2}^{V} contains terms in V_{2} and Φ_{2}^{T} , Φ_{2}^{R} terms in T_{2} . For example,

$$(2.27) \quad \Phi_{2}^{T} = (\chi - 1) \nabla \cdot \left(\sigma_{o} \nabla T_{2} + \sigma_{1} \nabla T_{1} + \sigma_{2} \nabla T_{o} \right) \\ = (\chi - 1) \nabla \cdot \left[\left(\sigma_{o} \nabla T_{2} + \left(\frac{d\sigma}{dT} \right)_{o} T_{2} \nabla T_{o} \right) + \left(\left(\frac{d\sigma}{dT} \right)_{o} T_{1} \nabla T_{1} + \left(\frac{d^{2}\sigma}{dT^{2}} \right)_{o} \frac{T_{1}^{2}}{2} \nabla T_{o} \right) \right]$$

 $[f', \Phi', \Phi', \Phi' = 0$ are the first order terms in the expension of $f', \Phi', \Phi' = 0$ in powers of $M(\Phi' = 0$ a spal to zero). They are dients combinations of first order veriables and their gradients; the coefficients of these lipear combinations depend on the dissipative coefficients and the first derivetives of these coefficients with respect to T, taken at $T_{c}T_{c}(t)$. For example,

(2.27) \$ = (1-1) \$. (5. \$ I; + 5, \$ I; + 5, \$ I; + 5, \$ I;)

3. Lagrangian velocity. Time averages. Assumptions.

The Lagrangian velocity $V_L(\underline{x}_o, t)$ is now introduced, \underline{x}_o being the position at t = o of a particle, the position of which is \underline{x} at t:

(3.1)
$$V_{L}(x_{o},t) = V(x_{o} + \int_{0}^{t} V_{L}(x_{o},t) dt, t)$$

(3.2) $x(x_0,t) = x_0 + \int_0^t V_L(x_0,t) dt$

Assuming that V_{L} may be expanded in powers of M up to the second order, $V_{L} = V_{L1} + V_{L2} + \cdots$, and that $\int_{0}^{t} V_{Li}(x_{o},z) dz$ is of the same order in M as $V_{Li}(x_{o},t)$, we obtain

$$V_{L_{1}}(\underline{x}_{o}, t) = V_{1}(\underline{x}_{o}, t)$$

$$V_{L_{2}}(\underline{x}_{o}, t) = V_{2}(\underline{x}_{o}, t) + (\int_{0}^{t} V_{1}(\underline{x}_{o}, t) dt) \cdot \nabla V_{1}(\underline{x}_{o}, t) \cdot$$

We note that the last two equations hold for every Σ_{\circ} , and therefore

(3.3)
$$V_{L1}(x,t) = V_1(x,t)$$

(3.4)
$$V_{L2}(x,t) = V_2(x,t) + (\int_0^t V_1(x,t) dt) \cdot \nabla V_1(x,t)$$

for every χ . (We build here on the assumption that every χ in the volume occupied by the fluid may be regarded as an initial position, i.e. that the transformation (3.2) is invertible in this volume). 5. Lagrangian valoality. Time sverages. Assumptions.
The Lagrangian valoality N (5.1) is now introduced, t.
being the position at t = g of a particle, the position of which is T at t :

- (3.1) V.(3.1) = V(3.1) (1.5) (1.5) = (3.1) (3.1) (3.1) (3.1) (3.1) (3.1)
- Assuming that Y_{i} may be expended in powers of M_{i} in to the second order, $Y_{i} = Y_{i} + Y_{i} + \cdots$, and that $\int Y_{i}(x_{i}, x_{i}) dx_{i}$ is of the same order in M as $Y_{i}(x_{i}, x_{i})$, we obtain $Y_{i}(x_{i}, x_{i}) = Y_{i}(x_{i}, x_{i})$
 - $Y_{12} (z_{0}, b) = Y_{1} (z_{0}, b) + (\int Y_{1}(z_{0}, c) ds) \cdot \nabla Y_{1}(z_{0}, b)$

end therefore (3.3) V, (2, 5) = V, (2, 5)

(J.z) XV. (3, 5) + ((S, 5) + ((S, 5) 2) (J. (S, 5) 2) (J. (S, 6))

for every 3. (We build here on the assumption that every 3 in the volume cocupied by the fluid may be regarded as an initial position, i.e. that the transformation (3.2) is invertible in this volume). The last term in (3.4) is the Stokes drift velocity, (see Longuet-Higgins (1969) for its interpretation in geophysics). It is important to introduce the Lagrangian velocity when looking for mass transport; in fact, in the case of a finite amplitude oscillation, V_{L2} may have a non zero mean value, even if V_2 has a zero mean value.

Let now V_1 , p_1 , f_1 be a solution of (2.14) - (2.17) which is sinusoidal in time with period ω . The mass transport velocity may in principle be calculated by solving (2.21) -(2.24) and by using (3.4). As the solution of (2.21) - (2.24)is not readily obtained, we replace these equations by averaged equations. Here the time average $a(\underline{x})$ of a function $a(\underline{x}, t)$ is defined by

(3.5)
$$\overline{\alpha(x)} = \frac{\omega}{2\pi} \int_{t}^{t+\frac{2\pi}{\omega}} \alpha(x, t) dt$$

We then obtain a system of equations which contains the unknowns $\overline{V_2}$ (or V_{L_2}), $\overline{P_2}$, $\overline{f_2}$, the averages of quadratic forms of the first order variables, and the averages of time derivatives of second order variables, as $\frac{\partial f_2}{\partial t}$ for example. We <u>suppose</u> that the last type of averages are equal to zero, i.e.

(3.6)
$$\frac{\partial V_2}{\partial t} = \frac{\partial p_2}{\partial t} = \frac{\partial p_2}{\partial t} = 0$$
.

This assumption is plausible since the source terms in (2.21) - (2.24) are periodic functions. However it has to be justified. To do this, we should prove, for instance, that V_2 , P_2 , S_2 are not secular in time, which in turn would make

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(3.5)
$$+ \overline{a}(\underline{x}) = \frac{\omega}{2\pi} \Big|_{\underline{x}}^{\underline{b} + \frac{\omega}{2}}$$

We then obtain a system of equations which contain the unianowas \overline{X} (or \overline{X}_1), \overline{P}_1 , \overline{f}_2 , the averages of quadratic forms of the first order variables, and the averages of time derivetives of second order variables, as $\frac{3}{24}$ for example. We avance that the last type of averages are equal to zero.

$$9 = \frac{316}{36} = \frac{346}{36} = \frac{316}{36}$$
 (5.8)

This assumption is plausible since the source terms in (2.23) -- (2.24) are periodic functions. However it has to be justified. To do this, we should prover for instance, that a V. P. S. are not secular in time, which in turn would make it necessary to define boundary conditions. This problem will not be treated here.

Further, we note the following:

(3.7) $\frac{\partial b}{\partial t} = 0$ whenever b is periodic in time with period ω .

Taking the divergence and the average of (3.4), noting that

$$\left(\int_{0}^{t} \nabla V_{1}\right): \nabla V_{1} = \frac{1}{2} \frac{\partial}{\partial t} \left[\left(\int_{0}^{t} \nabla V_{1}\right): \int_{0}^{t} \nabla V_{1} \right]$$

is periodic in time with period ω , we obtain the identity

$$(3.8) \quad \overline{\nabla V}_{L2} = \overline{\nabla V}_2 + (\int_0^t V_1) \cdot \overline{\nabla \nabla V}_1 .$$

it necessary to define boundary conditions. This problem will not be treated here. Further, we note the following:

 $(3.7) \quad \frac{35}{3t} = 0 \qquad \text{whenever } b \text{ is periodic in time}$

 $\left(\int_{0}^{n} \nabla y_{n}\right) : \nabla y = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2} \operatorname{sc}\left[\left(\int_{0}^{n} \nabla y_{n}\right) : \int_{0}^{n} \nabla y_{n} = \frac{1}{2}$

is pariedle is time with period (), we obtain the identity

(3.8) PY22 = 9.4 + ((X). PP.4.

4. Study of the mass transport velocity.

We now proceed to the solution of the averaged equations (2.21)-(2.24), where $\overline{V_{L2}}$, $\overline{\beta_2}$, $\overline{f_2}$ ($\overline{T_2}$) are the unknowns after \underline{V}_4 has been eliminated by using (3.4). Let w_{Li} be the vertical component of \underline{V}_{Li} , i=1,2,

and let index 1 indicate the horizontal projection, such that

$$\Delta = \Delta^{T} + \overline{y} \frac{\sqrt{y^{2}}}{\sqrt{y}}$$

$$\Delta = \overline{A}^{T} + \overline{y} M^{T}$$

4.1. First results for W_{L2} and $\nabla \cdot V_{L2}$.

We take the average of the second order equation of continuity (2.22) and equation of energy (2.23) and obtain respectively (4.1) and (4.2). To get these equations we have used (3.4) and the identity (3.8), and have made the assumption (3.6).

$$(4.1) \quad g'_{\circ} \overline{W_{L2}} + g_{\circ} \overline{\nabla} \overline{V_{L2}} = \circ = \overline{\nabla} \cdot \left(g_{\circ} \overline{V_{L2}}\right)$$

$$(4.2) \quad \varphi'_{\circ} \overline{W_{L2}} + g_{\circ} \overline{\nabla} \overline{V_{L2}} = \overline{\Phi_{2}^{T}} + \overline{\Phi_{2}^{R}} + \overline{\Phi_{2}^{V}} - (g_{-1})(\overline{\nabla} \overline{V_{1}}) \int_{0}^{t} (\Phi_{1}^{T} + \Phi_{1}^{R}) - \overline{\nabla} \cdot \left(\overline{V_{1}} \int_{0}^{t} (\Phi_{1}^{T} + \Phi_{1}^{R})\right).$$

(4.1)-(4.2) may be considered as a system of equations giving \overline{W}_{L2} and $\overline{\nabla}.\overline{V}_{L2}$. The determinant is

$$(4.3) \quad \gamma p_{\circ} g'_{\circ} - p'_{\circ} g_{\circ} = -g_{\circ}^{2} ((\gamma - 1)g + \gamma R T'_{\circ}) = -\frac{c_{\circ}^{2} g_{\circ}^{2}}{g} N^{2} ,$$

where N is the Väisälä frequency, see Eckart (1960). For the

4. Study of the mass transport velocity,

We now proceed to the solution of the averaged equations (2.21)-(2.24), where V_{11} , P_{1} , P_{2} , P_{2} , P_{3} , P_{4} , P_{1} , P_{2} , P_{3} , P_{4} , P

and let index 4 indicate the horizontal projection, such that

4.1. Birch results for who and F.Y.o.

We take the average of the second order equation of continuity (2.22) and equation of energy (2.23) and obtain respectively (4.1) and (4.2). To get these equations we have used (3.4) and the identity (3.8), and have made the assump-. tion (3.6).

$$(a,a) \quad f_{a} = f_{a} = f_{a} = f_{a} + f_{a$$

(4.1)-(4.2) may be considered as a system of equations giving \overline{w}_{12} and $\overline{V.V}_{12}$. The determinant is

$$(4.3) \quad (10.5) = -10.((1-1)) + 10.70) = -\frac{2}{2} N^{2} N^{2}$$

isothermal equilibrium $N^2 = (\gamma - 1)g / \gamma H$. Solving (4.1)-(4.2) we obtain

$$(4.4) \quad \overline{W}_{L2} = \frac{9}{g_{o}c_{o}^{2}N^{2}} \left[\overline{\Phi_{2}^{T} + \Phi_{2}^{R} + \Phi_{2}^{V} - (\gamma - i)(\overline{v}, \underline{V}_{1})} \int_{0}^{t} (\Phi_{1}^{T} + \Phi_{n}^{R}) - \frac{\overline{v}}{\sqrt{1-1}} \left[\overline{\nabla_{v}} \left[\overline{\nabla_{v}} \int_{0}^{t} (\Phi_{1}^{T} + \Phi_{n}^{R}) \right] \right] \right]$$

$$(4.5) \quad \overline{\overline{v}} \cdot \underline{V}_{-L2} = \frac{9(9 + RT_{0}')}{\frac{1}{9}c_{o}^{2}N^{2}} \left[\overline{\Phi_{2}^{T} + \Phi_{2}^{R} + \Phi_{2}^{V} - (\gamma - i)(\overline{v}, \underline{V}_{n})} \int_{0}^{t} (\Phi_{1}^{T} + \Phi_{n}^{R}) - \overline{\overline{v}} \cdot \left(\overline{V}_{-1} \int_{0}^{t} (\Phi_{1}^{T} + \Phi_{n}^{R}) \right) \right]$$

$$- \quad \overline{\overline{v}} \cdot \left(\overline{V}_{-1} \int_{0}^{t} (\Phi_{1}^{T} + \Phi_{n}^{R}) \right) \right] \cdot \frac{\overline{v}}{\sqrt{1-1}} \left[\overline{\Phi_{1}} \cdot \overline{\Phi_{1}} + \overline{\Phi_{1}} \right]$$

In fact $\overline{W_{L2}}$ and $\overline{V}.\underline{V}_{L1}$ are completely determined only when $\overline{T_2}$, which appears through the second order thermal dissipation term $\overline{\Phi_2^R}$, is computed. Before we show how this can be done, we make some comments.

(i) It follows from (4.1)-(4.2) that a vertical mass transport ($\overline{w_{L2}} \neq 0$) is impossible in absence of dissipation, i.e. when $\mu = \mu_B = \sigma = 0$, $\sigma_R = 0$ or q = 0. This result still holds in the case of a rotating but non-dissipative fluid, since the Coriolis force only appears in the equation of motion, which has not been used in deriving (4.1)-(4.2).

(ii) (4.1) and the identity (3.8) show that a vertical mass transport is impossible for the incompressible model ($\nabla . \forall = \circ$). Again, the result still holds in the case of a rotating fluid.

4.2. Computation of P2 and T2.

Averaging the second order equation of motion (2.21) and equation of state (2.24) and eliminating $\overline{f_2}$ and $\overline{F_2}$ between these equations, we obtain successively taothermal equilibrium N = (7+0)9/17H . Solving (4.7)-(4.2)

In fact we and V_{i} , are completely determined only when \overline{T}_{i} , which appears through the second order thermal distipather term \overline{V}_{i} and radiative dissipation term \overline{V}_{i} , to computed. Before we show how this can be done, we make space commute. (i) it follows from (4, 1) - (4, 2) that a varifical mask here when $\mu = \mu_{i} = x \times 0$, $\overline{\tau}_{i} = 0$ or i = 0. This result still here the cortalis force only appears is the equation of show the base of a rotating but non-dissipation of (4, 7) and the identity (3.8) abov that a varifical (4, 7), (4, 7) and the identity (3.8) abov that a varifical (7, 7) = 0. The result still holds in the case of a (7, 7) = 0. The result still holds in the case of a rotating flaid.

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eraging the second order equati

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(4.6)
$$\overline{V_{\perp}} \,\overline{P_{2}} = -\underline{C_{\perp}} + \overline{F_{2\perp}}^{V}$$

(4.7) $\overline{V_{\perp}} \,\overline{T_{2}} = \frac{R \,\overline{T_{0}}^{2}}{\vartheta} \Big[\overline{V_{\perp}} \Big(\frac{C_{2} - \overline{F_{2,2}}^{V}}{P_{0}} \Big) - \frac{\vartheta}{\vartheta_{z}} \Big(\frac{C_{\perp} - \overline{F_{2,\perp}}^{V}}{P_{0}} \Big) \Big] .$

Taking the curl of the averaged horizontal projection of the equation of motion and introducing for $\underline{F}_2^{\ v}$ an expansion similar to (2.27), we obtain an equation for the vertical component of $\overline{\forall} \times \overline{\underline{V}}_2$:

$$(4.8) \nabla \cdot \left(\mu_{\circ} \nabla (\nabla \times \overline{V}_{2})_{2}\right) = \left\{\nabla \times \left[\underline{C} - \nabla \cdot (\overline{\mu_{1}} \nabla \underline{V}_{1} - \underline{V}_{1} \nabla \mu_{1})\right]\right\}_{2}$$

Here C is defined by

$$(4.9) \ \underline{C} = \beta_1 \frac{\partial V_1}{\partial t} + \beta_0 \frac{\nabla V_1}{\nabla t} - \frac{\partial g}{T_0} \frac{g_1 T_1}{T_0} \ .$$

By using the first order equations (2.14)-(2.19), we obtain after some computation (see Appendix I)

$$(4.10) \quad \underline{C} = \underline{P}_{o} \nabla \left\{ \frac{1}{\underline{P}_{o}} \left[\underline{S}_{o} \frac{\underline{V}_{1}^{2}}{2} - \frac{\underline{P}_{1}^{2}}{2\,\underline{\gamma}\,\underline{P}_{o}} - \frac{\underline{g}^{2} \left(\underline{P}_{1} - \underline{C}_{o}^{2} \underline{g}_{1} \right)^{2}}{2\,\underline{g}_{o} \, \underline{C}_{o}^{4} \, N^{2}} \right] \right\} - \\ - \frac{2}{\underline{N}} \left[\underline{g} \, \frac{\underline{g}_{o}}{\underline{T}_{o}} \left(\int_{0}^{\underline{t}} \underline{V}_{1} \right) \cdot \nabla T_{1} + \underline{g}_{o} \frac{\underline{T}_{o}^{\prime}}{\underline{T}_{o}} \frac{\underline{V}_{1}^{2}}{2} + \frac{1}{2} \frac{\underline{T}_{o}^{\prime\prime}}{\underline{T}_{o}} \frac{\underline{g}^{3} \left(\underline{P}_{1} - \underline{C}_{o}^{2} \underline{g}_{1} \right)^{2}}{\underline{g}_{o} \, \underline{C}_{o}^{4} \, N^{4}} \right] + \\ + \frac{\underline{g}}{\underline{g}_{o} \, \underline{C}_{o}^{2} \, N^{2}} \left(\int_{0}^{\underline{t}} \left(\underline{\Phi}_{1}^{T} + \underline{\Phi}_{1}^{R} \right) \right) \left[\underline{g} \, \frac{\underline{g}_{o}}{\underline{T}_{o}} \nabla T_{1} + \frac{\underline{T}_{o}^{\prime}}{\underline{T}_{o}} \left(\nabla \underline{P}_{1} + \underline{2} \, \underline{g}_{1} \right) \right] + \\ + \left[\frac{\underline{g}_{1}}{\underline{g}_{o} \, \underline{C}_{o}^{2} \, N^{2}} + \underline{g}_{o} \left(\int_{0}^{\underline{t}} \underline{V}_{1} \right) \times \overline{\nabla} \times \frac{\underline{F}_{1}^{\prime}}{\underline{f}_{o}} - \underline{2} \, \frac{\underline{T}_{o}^{\prime\prime}}{\underline{T}_{o}} \left(\int_{0}^{\underline{t}} \underline{V}_{1} \right) \cdot \underline{F}_{1}^{\prime\prime}} \right] .$$

The first two terms in (4.10) do not depend explicitly on the dissipative coefficients, while the last two terms do.

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Now let S_M be a number which gives the order of magnitude of the dissipative terms. For instance, when the scale heigth is H and the wave length λ , with $\lambda \lesssim H$, S_M is the modified Stokes number

(4.11)
$$S_{M} = \frac{3\delta\omega}{4c_{o}^{2}}$$
, $\delta = \frac{1}{f_{o}} \left[\frac{\gamma-1}{c_{p}} (\sigma_{o} + \sigma_{Ro}) + \frac{\gamma-1}{\gamma} (\frac{\lambda}{2\pi})^{2} f_{o} q_{o} + \frac{4}{3} \mu_{o} + \mu_{Bo} \right]$,
where δ is the diffusivity of sound.

If terms depending explicitly on S_M are neglected in (4.10), ζ reduces to the sum of the first two terms. We thus obtain from (4.6)-(4.7) the following approximated expressions for \overline{p}_2 and \overline{T}_2 :

$$(4.12) \overline{P}_{2} = \left[-g_{0} \frac{V_{1}^{2}}{2} + \frac{P_{1}^{2}}{2g_{0}c_{0}^{2}} + \frac{g^{2}(P_{1} - c_{0}^{2}g_{1})^{2}}{2g_{0}c_{0}^{4}N^{2}} \right] (1 + O(S_{M})) + \Pi(2)$$

$$(4.13) \overline{T}_{2} = -\left[(\int_{0}^{t} V_{1}) \cdot \nabla T_{1} + \frac{T_{0}'}{g} \frac{V_{1}^{2}}{2} + \frac{g^{2} T_{0}''(P_{1} - c_{0}^{2}g_{1})^{2}}{2g_{0}^{2}c_{0}^{4}N^{4}} \right] (1 + O(S_{M})) + \theta(2).$$

Here n and θ are arbitrary functions of ξ , which in general depend on the boundary conditions. Substituting (4.13) in (4.4) for example, we obtain $\overline{w_{L2}}$ correct to the first order in S_M. It should be noted, however, that this result may not be correct in other cases than the one mentioned above. In the boundary layer, for instance, it may be necessary to keep the dissipative terms in (4.10) in order to obtain $\underline{\zeta}$, and therefore $\overline{w_{L2}}$, with the same accuracy.

Before we look further at the result obtained for \overline{W}_{L2} , we give some interpretations of the expression (4.12) for \overline{P}_2 . $\mathcal{P}_0 \overline{V_1^2}/2$ is the second order kinetic energy, $\overline{p_1^2}/2\mathcal{P}_0 c_0^2$ is the elastic energy, while the third term may be written as

-13-

$$\left[(a_{1}, a_{1}) + \frac{1}{2} + \frac{1}$$

where a is the diffusivity of sound.

If terms depending explicitly on S_{M} are neglected in (4.10), (reduces to the sum of the first two terms. We thus obtain from (4.6)-(4.2) the following approximated expressions for F_{c} and T_{c} :

Hare a and 6 are arbitrary functions of 2 , which in geneand depend on the boundary conditions. Substituting (4.13) in (6.1) for exemple, we obtain w_i, correct to the first order in S_g. If should be moted, however, that this result may not be correct in other cases then the one mentioned shows. In the boundary invor, for instance, it may be necessary to heap the dissipative terms in (4.10) in order to obtain (. and therefore W., with the sume acountry.

Before we look further at the result obtained for W_{44} , we give some interpretations of the expression (4.42) for \overline{P}_{4} . To $\overline{V_{4}}$ is the second order kinetic merces, \overline{V}_{4} is the classified on ergy, while the third term may be written as Here $\gamma_1 = \frac{\gamma_1 - c_o^2 \beta_1}{(\gamma - 1) \beta_o T_o}$ is the first order perturbation of entropy, and γ_o is the entropy at equilibrium, with

$$\gamma'_{o} = -c_{p}\left(\frac{g'_{o}}{g_{o}} + \frac{g}{c_{o}^{2}}\right) = c_{p}\left(\frac{g-1}{c_{o}^{2}}g + \frac{T'_{o}}{T_{o}}\right) = c_{p}\frac{N^{2}}{g}$$

This term is discussed by Eckart (1960), p.53, and called the thermobaric energy. It is a part of what he defines as "external energy density" - a quadratic form he derives from the linearized equations.

4.3. Computation of WI2.

Substituting (4.13) in (4.4), we notice that the vertical mass transport velocity may be decomposed in the sum of three expressions

$$(4.14) \quad \overline{W}_{L2} = \overline{W}_{L2}^{T} + \overline{W}_{L2}^{R} + \overline{W}_{L2}^{V} + \ell(2)$$

which describe respectively the mean vertical drift due to thermal, radiative and viscous dissipation, correct to the first order in S_M . Here $\ell(z)$ is the sum of the terms depending on the arbitrary function $\theta(z)$ which appears in (4.13).

For the viscous vertical drift we get

$$(4.15) \quad \overline{W}_{L2} = \frac{(8-1)g}{g_{o}c_{o}^{2}N^{2}} \left[\mu_{o} \left(\nabla V_{1} : \nabla V_{1} + \nabla V_{1} : (\nabla V_{1})^{*} - \frac{2}{3} (\nabla V_{1})^{2} \right) + \mu_{Bo} (\nabla V_{1})^{2} \right] (1 + O(S_{m}))$$

This expression is always positive.

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 $\frac{g_{s}}{2c_{p}}\frac{\overline{\gamma_{i}^{2}}}{\gamma_{o}^{\prime}}$

Here $y_i = \frac{y_i - c_i^2 y_i}{(i-1) \int c_i^2 x_i}$ is the first order perturbation of entropy, and y_i is the entropy at equilibrium, with

This term is discussed by Eduart (1960), p.53, and called the thermobaric emergy. It is a part of what he defines as "external emergy density" - a guadratic form he derives from th himseland countions.

Supertructing (4.13) in (4.4), we notice that the vertical mass turnsport velocity may be decomposed in the sum of three expressions

(4.14) TY - Will + Will + Will + 2(8)

which describe respectively the mean vertical drift due to thermal, radiative and viscous dissipation, correct to the first order in S_{ij} . Here l(i) is the sum of the terms depending on the arbitrary function P(i) which appears in (4, 15).

For the viscous vertical drift we get

In the same way we get

$$(4.16) \ \overline{W}_{L2}^{T} = \frac{g}{\int_{0}^{c} c_{0}^{2} N^{2}} \left[\Phi_{2}^{T} - (\chi - 1) (\nabla, \underline{V}_{1}) \int_{0}^{t} \Phi_{1}^{T} - \nabla \cdot (\underline{V}_{1} \int_{0}^{t} \Phi_{1}^{T}) \right] (1 + O(S_{M})),$$

where Φ_1^T , Φ_2^T are given by (2.26)-(2.27).

As an example, let us consider the special case of <u>isothermal equilibrium</u>, where we shall first neglect the variation of σ with temperature. Then Φ_1^T , Φ_2^T are simplified, and we obtain (see Naze Tjötta and Tjötta (1972))

$$(4.17) \ \overline{W_{L2}} = \frac{(Y-1)g}{g_{o}c_{o}^{2}N^{2}} \sigma_{o} \left\{ \frac{(\nabla T_{1})^{2}}{T_{o}} + \nabla \left[\nabla \left(-\left(\int_{0}^{k} V_{1}\right) \cdot \nabla T_{1} - \frac{T_{1}^{2}}{2T_{o}} \right) + \left(\int_{0}^{k} V_{1}\right) \nabla^{2} T_{1} \right] \right\} (1 + O(S_{M})).$$

When the variation of σ is taken into account, the divergence term in (4.17) is modified by a term $\frac{1}{\sigma_0} \left(\frac{d\sigma}{dT}\right)_0 \overline{T_1 \nabla T_1}$.

For plane vertical progressive or standing waves where the first order variables only depend on 2, t, we get

$$(4.18) \ \overline{W}_{L2}^{T} = \frac{(\chi-1)g}{g_{o}c_{o}^{2}N^{2}} \frac{\sigma_{o}}{T_{o}} \left[\left(\frac{\partial T_{1}}{\partial z}\right)^{2} - \frac{\chi-2}{\chi-1} \frac{\partial}{\partial z} \left(T_{1} \frac{\partial T_{1}}{\partial z}\right) \right] (1 + O(S_{M})).$$

Let us consider a vertical progressive beam

(4.19)
$$T_{1} = F_{A} e^{-d z} \sin(k z + \omega t)$$
,

where $F_A(\alpha)$ is equal to a constant A inside the beam $\left(|\alpha| \leqslant \frac{\mathcal{D}}{2}\right)$ and is equal to zero outside the beam $\left(|\alpha| > \frac{\mathcal{D}}{2}\right)$. (α, k, ω) have to satisfy the dispersion relation associated with the linearized equations (2.14)-(2.17), i.e. in the non-dissipative approximation

In the same way we get

 $\frac{1+1}{1+1+1}$ and is equal to zero outside the bass (is) $\frac{1}{1+1+1}$ $\frac{1+1+1}{1+1+1}$ have to satisfy the dispersion relation according $\frac{1}{1+1+1}$ have to satisfy the dispersion relation to according $\frac{1}{1+1+1}$ the linearized equations (2.14)-(2.17), i.e. in the non distinctive corrections on

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$$(4.20) \begin{cases} \alpha = -\frac{1}{2H} , \quad k^2 = \Delta , \quad \text{if } \Delta \geqslant 0 \\ d^2 + \frac{\alpha}{H} + \Delta + \frac{1}{4H^2} = 0 , \quad k = 0 , \quad \text{if } \Delta < 0 , \end{cases}$$

where $\Delta = (\omega^2/\gamma HH) - 1/4H^2$. Equations (4.18) and (4.14) give

$$(4.21) \quad \overline{W}_{L2} \approx \frac{\sigma_{o}}{\int_{S} g T_{o}} \frac{F_{A}^{2}}{2} \left(k^{2} + \frac{3-\delta}{\delta-1} \alpha^{2} \right) e^{\left(\frac{1}{H} - 2\alpha\right) \frac{\lambda}{2}} + \ell(2),$$

where for simplicity we have not taken into account the effects of radiation and viscosity.

If we suppose that the flow is two-dimensional, the horizontal component \overline{u}_{L2} of the mass transport velocity is determined by (4.1)

(4.22)
$$\overline{u}_{L2} = -\int_{0}^{\infty} \frac{1}{\beta_{0}} \frac{\partial}{\partial z} \left(\beta_{0} \overline{w}_{L2} \right) dx + m(z),$$

where M is an arbitrary function. If we suppose the mass transport velocity field to be non-secular in x , we must choose

$$l(z) = a e^{\frac{z}{H}},$$

where a is an arbitrary constant. Also, if we require \overline{u}_{L2} to be antisymmetric in x, m (2) has to be equal to zero.

Choosing for α the root of (4.19) which is not bigger than $-\frac{1}{2H}$, we may neglect the $\alpha e^{\frac{2}{H}H}$ term in $\overline{w_{L2}}$ inside the beam, thus obtaining

where $\Delta = (\omega^2/3HH) - J/4H^2$. Equations (4,18) and (4.14) give

$$(4,24) = \frac{1}{2} \left(\frac{1}{2} + \frac{1}$$

where for simplicity we have not taken into account the '

If we evapore that the flow is two-dimensional, the horizontal component Res of the mass transport velocity is determined by (0.1)

where h is an arbitrary function. If we suppose the mass transport velocity field to be non-secular in x, we must choose

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where a is an explanary constant. Also, if we require it, to be equilarmetric.in a , m.(2) , bis to be secal to rere it, about the root. of (4, 19) which is not street than - it, we may neglect the a s' term in . We increase the beam, thus obtained

(4.23)
$$\overline{W}_{L2} \approx \frac{\overline{\sigma}_{o}}{\beta_{s}gT_{o}} \frac{A^{2}}{2} \left(k^{2} + \frac{3-\gamma}{\gamma-1}\alpha^{2}\right) e^{\left(\frac{1}{H}-2\alpha\right)2}$$

This shows that an upward directed material flow takes place inside the beam.

Now let us consider the case of a vertical standing beam

$$(4.24) T_1 = F_A e^{-\alpha \xi} \cos \omega \xi \cos \omega t,$$

where F_A , α , k, ω are the same as above. Requiring for \overline{u}_{LL} non-secularity and antisymmetry in ∞ , we find for the vertical component of the mass transport velocity inside the beam

$$(4.25) \quad \overline{W_{L2}} \approx \frac{\overline{\sigma_{o}}}{g_{s}gT_{o}} \quad \frac{A^{2}}{2(\gamma-1)} e \qquad \left\{ (3-\gamma)dk \sin 2kz + (3-\gamma$$

It can be shown that \overline{W}_{L2} given by (4.25) changes sign twice on a half wave length. However, when integrating this expression of a half wave length $\left(n\frac{\pi}{k}, (n+i)\frac{\pi}{k}\right)$, we see that the first two terms do not contribute. The last term in (4.25) being positive, we conclude that this integrated vertical drift is directed upward inside the beam.

When taking into account the variation of σ with T, a term $\frac{T_o}{\sigma_c} \left(\frac{d\sigma}{dT}\right)_c \frac{2}{\partial \xi} \left(\overline{T_1 \frac{2T_1}{\partial \xi}}\right)$ has to be added in the parenthesis This shows that an upward directed material flow takes place inside the beam.

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where F_{1} , σ_{1} is a set the same as above. Requiring for F_{1} , non-secularity and antisymmetry in Σ , we find for the vertical component of the mass transport velocity inside the beam

 $\left\{ \frac{1}{2} \left(\frac{1}{2} \left(\frac{1}{2} - \frac{1}{2} \right) + \frac{1}{2} \left(\frac{1}{2} - \frac{1$

of (4.18). It is easily shown that this effect does not change qualitatively the above results for most of the interaction potentials mentioned in § 2. It is of special interest to note that in the case of a progressive wave, this effect always increases the value of $\overline{w_{L2}}$ given by (4.23).

Looking at the contribution from radiation in (4.4) and using (4.13), we obtain \overline{w}_{L1}^{R} . In the optically thick case, where Φ^{R} is given by (2.7)a, \overline{w}_{L2}^{R} has the same form as \overline{w}_{L1}^{T} given by (4.10), only σ being replaced by σ_{R} . In the optically thin case, where Φ^{R} is given by (2.7)b, we have

$$\Phi_{\lambda}^{R} = -(\chi - 1) c_{\nu} g_{0} q_{0} T_{1}$$

$$\Phi_{2}^{R} = -(\chi - 1) c_{\nu} \left[g_{0} q_{0} T_{2} + T_{1} \left(q_{0} g_{1} + g_{0} \left(\frac{dq}{dT} \right)_{0} T_{1} \right) \right]$$

Substituting these expressions in the relevant part of (4.4), and using (2.15), (2.18), we obtain

$$(4.26) \quad \overline{W}_{L2}^{R} = -\frac{(\gamma-1)c_{v}g}{c_{o}^{2}N^{2}} q_{o} \left[T_{2} + \left(\int_{0}^{t} V_{1}\right) \cdot \nabla T_{1} + (\gamma-1)\left(1 - \frac{T_{o}}{q_{o}}\left(\frac{dq}{dT}\right)_{o}\right) T_{1} \int_{0}^{t} \nabla V_{1} \right]$$

Using (4.13), we get finally

$$(4.27) \overline{W}_{L2}^{R} = \frac{(\aleph-1)^{2} c_{\nu} g}{c_{\nu}^{2} N^{2}} q_{0} \left[\left(\frac{T_{0}}{q_{0}} \left(\frac{dq}{dT} \right)_{0} - 1 \right) T_{1} \int_{0}^{t} \overline{\nabla} \cdot V_{1} + \frac{T_{0}' V_{1}^{2}}{2(\aleph-1) g} + \frac{T_{0}'' g^{2} y_{1}^{2}}{2(\aleph-1) c_{p}^{2} N^{4}} \right] (1 + O(S_{M}))^{2}$$

The last two terms are equal to zero in the case of isothermal

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of (4.18). It is easily shown that this effect does not change qualitatively the shows results for most of the interaction potentials montioned in § 2. It is of special interval to note that in the case of a progressive mave, this effect always increases the value of W₁, given by (4.29). Looking at the contribution from rediction in (4.4) and using (4.13), we obtain W₁. In the optically thick ones, where Ψ is given by (2.7)s, W₁, has the agens form as the optically thin case, where Ψ is given by (2.7)b, we have

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Substituting these expressions in the relevant part of (4.4).

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Caing (4. 13), we get Finally

rest two terms are equal to zero in the case of isother

$$(4.28) \quad \overline{W}_{L2} = \frac{(\gamma-1)c_{v}q}{c_{o}^{2}N^{2}} \frac{q_{o}}{T_{o}} \left(1 - \frac{T_{o}}{q_{o}} \left(\frac{dq}{dT}\right)_{o}\right) T_{1}^{2} \left(1 + O(S_{M})\right),$$

showing that the direction of this vertical drift may be very sensitive to the variation of q(T). For instance, if q is proportional to T^n , $\overline{W_{L2}}^R$ has the sigh of I-h.

It is easily seen that the results obtained for $\overline{W_{L2}}^{R}$ (as well as for $(\nabla x \overline{V}_{L2})_{2}$, see § 4.4) at the present order of approximation in S_{M} , are not modified if we add a radiative pressure term ∇p_{R} in the equation of motion.

4.4 <u>Computation of</u> $(\nabla x \overline{V}_{L2})_2$.

$$(4.29) \mu_{0}\overline{\nabla} \mathcal{I} + \mu_{0}^{\prime} \frac{\partial}{\partial \xi} \mathcal{I} = \overline{2} \cdot \overline{\nabla}_{1} \times \left[\underline{C}_{1} - \overline{\nabla} \cdot \left(\mu_{1} \nabla \underline{\nabla}_{11} - \underline{\nabla}_{1} \nabla \underline{\Gamma}_{11} - \mu_{0} \nabla \left(\left(\int_{0}^{E} \underline{\nabla}_{1} \right) \cdot \nabla \underline{\nabla}_{11} \right) \right]$$

equilibrium. For other equilibria mentioned in § 2, where T. increases with altitude as a power of 2, these terms are positive and give an upward drift. In the case of isothermal equilibrium, w. 1 is given by

((1~2)0+1) ((1+1) 王-1) - (1-20(-1)) - 二元 (10-1)

about that the direction of this vertical drift may be very sensitive to the variation of q(T). For instance, if q is propositional to T. W. has the sign of 1-6. It is senily seen that the results obtained for W. (as well as so (V.Y.), see § 1.4) at the present order of approximation in S., are not modified if ve add

The mass transport velocity field is determined when we know $W_{1,1} = V_{1,1}$ and the vertical component of the vol ticity, $A_{1} = (V_{1}V_{1})_{1} = 2 \cdot V_{1} \times V_{11}$. We obtain an equation for A_{1} by substituting (4.10) in (4.8):

2] 117 - ARY-1, YV A) V-12] . V & = I = 14 - R V.4 (es.4

The last term is the contribution of the Stokes velocity. Since this equation is linear, we may again decompose $\overline{\mathcal{A}}$ into the sum of three expressions

$$(4.30) \quad \overline{\Lambda} = \overline{\Lambda}^{T} + \overline{\Lambda}^{R} + \overline{\Lambda}^{V}.$$

We have

$$(4.31) \mu_{o} \nabla_{-}^{2} \overline{\Lambda}^{T} + \mu_{o}' \frac{\partial}{\partial z} \overline{\Lambda}^{T} = \frac{\vartheta}{\vartheta_{o} c_{o}^{2} N^{2}} \overline{\Lambda} \cdot \nabla_{\perp} x \left[\left(\frac{\vartheta \beta_{o}}{T_{o}} \nabla_{\perp}^{T} T_{1} + \frac{T_{o}'}{T_{o}} \nabla_{\perp} p_{1} \right) \int_{0}^{t} \Phi_{1}^{T} \right]$$

and a similar equation for $\overline{\Omega}^R$, Φ_1^T being replaced by Φ_1^R . For $\overline{\Omega}^V$ we get

$$(4.32) \mu_{\circ} \nabla^{2} \overline{\Omega}^{\vee} + \mu_{\circ}^{\vee} \frac{2}{32} \overline{\Omega}^{\vee} = \overline{2} \cdot \nabla_{I} x \left[\frac{g_{1}}{g_{\circ}} \overline{F}_{11}^{\vee} + g_{\circ}^{\vee} (\int_{0}^{t} \nabla_{I}) x \nabla x \frac{\overline{F}_{1}}{g_{\circ}} \right] - \nabla_{\circ} (\mu_{1} \nabla V_{11} - V_{1} \nabla_{I} \mu_{1} - \mu_{\circ} \nabla ((\int_{0}^{t} \nabla_{I}) \cdot \nabla V_{11})]$$

F^V₁ is given by (2.4) where μ, μ_B are replaced by μ_o, μ_{Bo}. Expressions for Ω^T, Ω^R, Ω^V correct to the zeroth order in S_M are obtained by substituting in the source terms of (4.31)-(4.32) the solution of the non-dissipative linearized equations (see §.4.2 for the validity of such an approximation).

For waves where the first order variables depend on the horizontal coordinates $\underline{\gamma}_1 = (x, y)$ only through $\underline{k}_1, \underline{\gamma}_1$, it is easily seen that the source terms of (4.31)-(4.32) are equal to zero. However, a lateral confinement in the horizontal plane of the waves, leads in general to non-zero source terms in (4.31)-(4.32), and thus to a strong horizontal circulation (see below):

For the sake of simplicity, let us now suppose that the equilibrium is isothermal. Using (4.31), (4.32) and the non-dissipative first order equations, we obtain

$$(4.33) \quad \nabla^{2} \overline{D}^{T} = \underline{\lambda} \cdot \nabla_{\underline{I}} \times \left[\frac{1}{c_{p}} \frac{\sigma_{o}}{\mu_{o}} \frac{g^{2}}{T_{o}^{2} N^{2}} \nabla_{\underline{I}} T_{A} \int_{o}^{t} \nabla^{2} T_{A} \right] (1 + O(S_{M}))$$

$$(4.34) \quad \nabla^{2} \overline{D}^{V} = \underline{\lambda} \cdot \nabla_{\underline{I}} \times \left\{ -\left(\frac{1}{3} + \frac{\mu_{Bo}}{\mu_{o}}\right) \left(\int_{o}^{t} \nabla \cdot \underline{V}_{A}\right) \nabla_{\underline{I}} \nabla \cdot \underline{V}_{A} + (\vartheta - 1) \frac{T_{o}}{\mu_{o}} \left(\frac{d\mu}{dT}\right)_{o} \nabla \cdot \left[\left(\int_{o}^{t} \nabla \cdot \underline{V}_{A}\right) \nabla \underline{V}_{A} - \frac{V_{A} \nabla_{\underline{I}} \int_{o}^{t} \nabla \cdot \underline{V}_{A}}{\left(\int_{o}^{t} \nabla \cdot \underline{V}_{A}\right) \nabla^{2} \nabla \underline{V}_{A}} + \left[\left(\int_{o}^{t} \nabla_{\underline{V}} \underline{V}_{A} + \nabla^{2} \left(\left(\int_{o}^{t} \nabla_{\underline{I}}\right) \cdot \nabla \underline{V}_{A}\right)\right]\right] (1 + O(S_{M}))$$

In the optically thick case, the equation for $\overline{\mathcal{M}}^{R}$ is obtained by replacing σ_{c} with σ_{Ro} in (4.33). In the optically thin case, we obtain

$$(4.35) \nabla^2 \widehat{\Pi}^R = \frac{g^2 g_o}{g T_o^2 N^2} \frac{q_o}{\mu_o} \overline{\chi} \cdot \overline{\nabla_1 T_1 \times \nabla_2 \int_0^E T_1 (1 + o(S_M))}.$$

It is easily, seen that the source term in (4.35) is equal to zero in the case of a standing wave.

In the incompressible approximation, where $V \cdot V_1 \rightarrow o$, the last two terms in (4.34) will give the dominant part of $\overline{-2}$.

In the non-dissipative case, (4.33)-(4.34) are no longer valid, since the hydrodynamical equations are singular at $\mu = 0$. We then obtain \mathcal{N} by taking the curl of (2.21) and by taking into account the other second order equations. We obtain

$$\frac{\partial^2 \mathcal{L}}{\partial t^2} = 0 \quad .$$

Because of the non secularity condition (see §.3), \mathcal{A} is equal to its initial value. Together with the result obtained in §.4.1, (i), this shows that a finite amplitude oscillation generates no mass transport in the non-dissipative case.

It is of interest to note that the source terms in the equations for $\overline{\Lambda}^{\mathsf{T}}$ and $\overline{\Lambda}^{\mathsf{R}}$ in the optically thick case, for example, contain terms which are proportional to $\frac{1}{c_{\mathsf{p}}} \frac{\sigma_{\mathsf{e}}}{\mu_{\circ}}, \frac{1}{c_{\mathsf{p}}} \frac{\sigma_{\mathsf{R}\circ}}{\mu_{\circ}}$



respectively. In the same way, the source term in the equation for $\overline{\Lambda}^{\vee}$ contains terms proportional to $\frac{\mu_{Bo}}{\mu_{o}}$ and others independent of viscosity.

For a finite amplitude oscillation such that at least one of these source terms is non zero, the horizontal mass circulation is thus of zeroth order in S_M . The vertical mass transport, on the other hand, is of the first order in S_M . For such cases, the horizontal mass transport may dominate over the vertical drift, especially if $\frac{1}{c_p} \frac{\sigma_0 + \sigma_{RO}}{\mu_0}$ is large, which normally is the case in the lower solar atmosphere.

respectively. In the same way, the source term in the equation for $\overline{\Lambda}^*$ contains serms propertional to $\frac{\mu_{0,0}}{\mu_{0}}$ and others interpendent of viscosity.

For a finite applitude confliction such that at least one of these source terms to non zero, the horizontal mass diredlation is that of zeroth order in 5, The vertical mass transport, on the ether hand, is of the first order in 5, for such cases, inc horizontal mass transport mby dominate over the vertical draft, especially if <u>the first</u> is large, which normally is the sets in the lower colar sincephere.

5. Concluding remarks.

We have shown that it is theoretically possible for finite amplitude oscillations in a dissipative, nonhomogeneous fluid to sustain a mass transport streaming. The fluid was a perfect gas, but generalisations to other models are possible. The theory, we believe , is applicable in the studies of transport phenomenon in the solar atmosphere. For instance, it provides a mechanism for associating vertical drift, and the observed large scale horizontal flow in the supergranules, with oscillatory motions in the atmosphere. The mass transport also influences the energy transport.

In oceanography, it has been suggested by Munk and Moore (1968) that the Cromwell current, which flows eastward in a narrow zone along the equator, with maximum velocities (in the Pacific) of the order 1 m/sec at 100 m depths, is associated with confined internal planetary waves. Their model, however, being non-dissipative, fails to predict the necessary transport of mass, a fact which was later pointed out by Moore (1969). It is suggested by our theory that mean currents in the ocean may be produced by such internal waves if dissipative effects are accounted for (cf. §.4.4). Further, including the effect of compressibility, it is also suggested that a vertical drift may be attributed to the same waves. An upward vertical drift up to order 1 m/day, i.e. of order 10^{-5} compared to the maximum horizontal flow, has been estimated at the equator (Knauss (1966)).

An experimental test of the results obtained should be possible. For instance, in a vertically directed standing

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wave the theory predicts microstreaming, the scale of which in the vertical direction is the half wave length. We find that this streaming should be strong enough to be observable in an ultrasonic wave, and it may be the mechanism behind the formation of periodic density stratification observed by Schaaffs and Haun (1968), Schaaffs (1973) and Hobæk (1973). The last author's similar observation in a horizontally directed wave, can also be explained qualitatively by the vertical drift in our theory. However, further observations are needed before definite conclusion can be drawn.

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We wish to thank Mr. H. Hobæk at the Department of Physics for making the results of his observations available prior to publication, and for stimulating discussions. We are gratefull to Mr. J. Ekern and Mr. S. Mykkeltveit for checking several of the computations. have the theory predicts microstronning, the scale of which in the verifical direction is the half wave length. We did that this streaming should be strong shough to be observants in an ultrescale wave, and it may be the scenarios pained and formation of periodic density stratification observed by Reheales and Hear (1963), Schoolfs (1973) and Hobek (1973). The last michae's similar observation in a horizontally directs in our theory. However, further observation to show (1973).

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Appendix I

Let <u>C</u> be defined by (4.9). Using (2.14) and (2.17) together with the identity $V_1 \cdot \nabla V_1 = \nabla \frac{V_1^2}{2} - \frac{V_1 \times \nabla \times V_1}{2}$, we obtain

$$(A.1) \quad \underline{C} = \int_{0}^{1} \frac{\nabla V_{1}^{2}}{2} - \frac{g_{1}}{g_{0}} \nabla P_{1} - \frac{\gamma}{2} g \frac{g_{1} P_{1}}{P_{0}} - g_{0} \frac{V_{1} \times \nabla \times V_{1}}{g_{0}} + \frac{g_{1}}{g_{0}} \frac{F_{1} \vee V_{1}}{g_{0}} + \frac{g_{1}}{g_{0}} + \frac$$

The fourth term in the above expression is computed by integrating (2.19) with respect to t and by noting that the integration constant does not contribute to \underline{C} , since \underline{V}_1 is periodic in t. We obtain

$$(A.2) - g_{\circ} \overline{V_{x} \nabla x V_{1}} = \overline{2} \left[-g \frac{g_{\circ}}{T_{\circ}} \nabla T_{1} \cdot \int_{\circ}^{t} V_{1} + \frac{T_{\circ}'}{T_{\circ}} V_{1} \cdot \int_{\circ}^{t} \nabla P_{1} \right] + \left[g \frac{g_{\circ}}{T_{\circ}} \nabla T_{1} + \frac{T_{\circ}'}{T_{\circ}} \nabla P_{1} \right] \int_{\circ}^{t} w_{1} + g_{\circ} \left(\int_{\circ}^{t} V_{1} \right) \times \nabla x \frac{F_{1}'}{g_{\circ}} \right]$$

Integrating (2.14) with respect to t and multiplying by ∇p_1 we get in the same way

(A.3)
$$V_{A} \cdot \int_{0}^{t} \nabla p_{1} = -g_{0} V_{1}^{2} + gg_{1} \int_{0}^{t} w_{1} - F_{1}^{v} \cdot \int_{0}^{t} V_{1}$$

Elimination of $V. \frac{V}{1}$ between (2.15) and (2.16), and integration with respect to t gives

(A.4)
$$\frac{c_0^2 p_0^2 N^2}{g} \int_{0}^{t} W_1 = f_0(c_0^2 g_1 - \dot{P}_1) + f_0 \int_{0}^{t} (\phi_1^T + \phi_1^R) + \text{const.}$$

Using (A.2)-(A.4) to transform (A.1) we obtain finally

$$(A.5) \quad \underline{C} = p_{\circ} \nabla \left(\frac{g_{\circ} V_{1}^{2}}{2 p_{\circ}} \right) + \underline{A} + \frac{1}{2 p_{\circ}} + \frac{1}{2 p_{$$

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Let \underline{C} ba defined by (4.9). Using (2.14) and (2.17) to constant together with the identity \underline{V} , \underline{V}

The fourth term in the above expression is computed by intograting (2.19) with respect to t and by noting that the integration constant does not contribute to Q., since V, is periodic is t. We obtain

Integratiag (2.74) with respect to t and multiplying by Vp.

Elimination of V(, between (2.15) and (2.16), and integration vith rampost to t gives

$$(A,A) = \frac{a^{2}e^{2}e^{2}}{2} \left[e_{A} = g_{A}(a^{2}g_{A} - e_{A}) + g_{A}(e^{2}g_{A} - e^{2}g_{A}) + g_{A}(e^{2}g_{A} - e^{2}g_{A}) \right]$$

Using (A.2)-(A.4) to transform (A.4) we obtain firelly

$$(A,S) = P, P\left(\frac{SN^2}{2P_0}\right) + A +$$

$$+\frac{g}{g_{0}c_{0}^{2}N^{2}}\left[g\frac{f_{0}}{T_{0}}\nabla T_{1}+\frac{T_{0}'}{T_{0}}\nabla p_{1}+\frac{\gamma}{T_{0}}g\frac{T_{0}'}{T_{0}}g_{1}\right]\int_{0}^{t}(\Phi_{1}^{T}+\Phi_{1}^{R}) + \frac{g}{g}\frac{T_{0}'}{T_{0}}g_{1}\int_{0}^{t}(\Phi_{1}^{T}+\Phi_{1}^{R}) + \frac{g}{g}\frac{T_{0}'}{T_{0}}g_{1}+\frac{g}{g}\frac{T_{$$

where \underline{A} is defined by $\underline{A} = -\frac{g_1}{g_0} \nabla p_1 + \frac{g}{g_0 C_0^2 N^2} (c_0^2 g_1 - p_1) (g \frac{g_0}{T_0} \nabla T_1 - \frac{T_0'}{T_0} \nabla p_1).$

Using (2.17) we also have

$$(A.6) \quad \underline{A} = -\underline{P}_{o} \nabla \left[\frac{\underline{g}^{2} (\underline{P}_{1} - \underline{c}_{o}^{2} \underline{g}_{1})^{2}}{2 \underline{P}_{o} \underline{g}_{o} c_{o}^{4} N^{2}} + \frac{\underline{P}_{1}^{2}}{2 \underline{g}_{o} \underline{g}_{o}^{2}} \right] + \\ + \underline{n} \left[\frac{\underline{g}^{2}}{c_{o}^{2} N^{2}} \frac{\underline{T}_{o}'}{\underline{T}_{o}} (\underline{P}_{1} - \underline{c}_{o}^{2} \underline{g}_{1}) \frac{\underline{g}_{1}}{\underline{g}_{o}} + \frac{\underline{g}_{1} \underline{p}_{1} \underline{g}_{1}}{\underline{P}_{o}} + \frac{\underline{T}_{o}''}{\underline{T}_{o}} \frac{\underline{g}^{3}}{\underline{g}_{o} c_{o}^{4} N^{4}} (\underline{P}_{1} - \underline{c}_{o}^{2} \underline{g}_{1})^{2} \right].$$

Substituting this expression for A in (A.5) we obtain (4.10).

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