Department
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On the Stability of a Magnetized Plasma
with a Continuous Density Gradient in
a Uniform External Force Field

by

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Report 2. June 1964.

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Abstract.

Starting from the equations describing the two-fluid model, the stability of a low-β plasma supported by a magnetic field against an external force field, is studied.

Finite gyroradius effect is taken into account. The stability analysis is based on localized perturbations in a plane normal to the magnetic field. Even if the effect of a finite gyroradius is to stabilize the system for perturbations normal to the density gradient, it seems to be overstable for perturbations in other directions.

Introduction.

The first papers dealing with the problem of a low-pressure plasma supported by a strong magnetic field against a gravity force were written by Kruscal and Swarzchild [1], Longmire and Rosenbluth [2]. The conclusions from these works are that a plasma in this situation is unstable at the boundary if the gravity force points away from the plasma, and that it is stable if the force points in the opposite direction. The authors of [1] use the usual MHD-equations in their analysis of the problem. The authors of [2] start with an analysis in terms of virtual displacement and energy considerations. But as this procedure gives no suffisient condition for the occurrence of instability, one has also to show that the plasma actually is able to perform the required displacement, and this is done from a microscopic point of vew by studying drift velocities.

Later investigations on this subject, espescially by B. Lehnert [3], [4], [5], [6]; M.N. Rosenbluth, N.A. Krall, N. Rostoker [7] and K.V. Roberts and T.B. Taylor [8] indicate that the situation is so serious as it appears to be from the conclusions of [1] and [2]. The new effects taken into account are, first the effect of a continuous density gradient, introduced by B. Lehnert [4], and second the effect of finite radius of gyration for ions, studied by Rosenbluth et. al. [7]. But in all the papers known to the author, where both the finite radius of gyration and the continuous density gradient are taken into account, perturbations in one direction, only, are studied; namely, the direction normal to $\underline{B}$, (the magnetic field vector) and normal to ∇n, (the density gradient vector). We shall in this report first state the assumptions

and approximations, and then derive a consistent set of equations which allow perturbations in a plane normal to $\underline{B}$, and especially in the direction parallel to the density gradient vector. Afterwards we analyse these perturbations in normal modes and derive the dispersion relation for the system. Finally, the dispersion relation is discussed and solved numerically for some parameter values. The results are compared with results given in earlier papers.

2. Assumptions and approximations.

We shall here study a low β plasma, which means $\beta \ll 1$, where β as usual is the ratio between the particle pressure and the magnetic "pressure". As we shall justify in Eq. (3.30) we can then neglect the modifications in the magnetic field due to currents in the plasma caused by a small macroscopic electron velocity in equilibrium and by the first order electron and ion velocities due to perturbations. This also allows us to write $\underline{\tilde{E}} = - \nabla\varphi$, i.e., the perturbed electric field is the gradient of a scalar function.

Our stability analysis will be based on the method of localized perturbations, which means that we study the situation in a local region of the plasma and assume the plasma to extend to infinity in all directions, in order to make effects from boundaries negligible. Otherwise one has usually to solve an eigenvalueproblem, which is much more complicated, but, of course, also much more satisfactory from a physical point of view. But we still hope that the method of localized perturbations as a first step, will give some results of physical relevance.

We shall assume the temperature to be constant in space and approximations, and then derive a consistent set of equations which allow perturbations in a plane normal to $\underline{B}$, and especially in the direction parallel to the density gradient vector. Afterwards we analyse these perturbations in normal modes and derive the dispersion relation for the system. Finally, the dispersion relation is discussed and solved numerically for some parameter values. The results are compared with results given in earlier papers.

2. Assumptions and approximations.

We shall here study a low β plasma, which means $\beta \ll 1$, where β as usual is the ratio between the particle pressure and the magnetic "pressure". As we shall justify in Sec. 7 (2.26) we can then neglect the modifications in the magnetic field due to currents in the plasma caused by a small macroscopic electron velocity in equilibrium and by the first order electron and ion velocities due to perturbations. This also allows us to write $\underline{E} = -\nabla\phi$, i.e., the perturbed electric field is the gradient of a scalar function.

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We shall assume the temperature to be constant in space and time, both in equilibrium and in the perturbed state. It seems perhaps somewhat curious that the plasma should be isothermal in the perturbed state, since we have neglected collisions. This assumption, however, can be justified by the assumption of constant magnetic moment for each particle. A consequence of these assumptions is that the pressure gradient is proportional to the density gradient.

We now introduce the mean ion speed $W_\perp$ transverse to $\underline{B}$, and have by definition that the mean radius of gyration for ions is given by $a = \frac{W_\perp}{\omega_i}$, where ω_i is the gyrationfrequency for ions.

We shall now introduce a local cartesian frame of reference as our coordinate system. $\underline{B}$ points in the z-direction and the gravity force, or an equivalent force, is in the y-direction. The density gradient in equilibrium also points in the y-direction. See fig. 1.

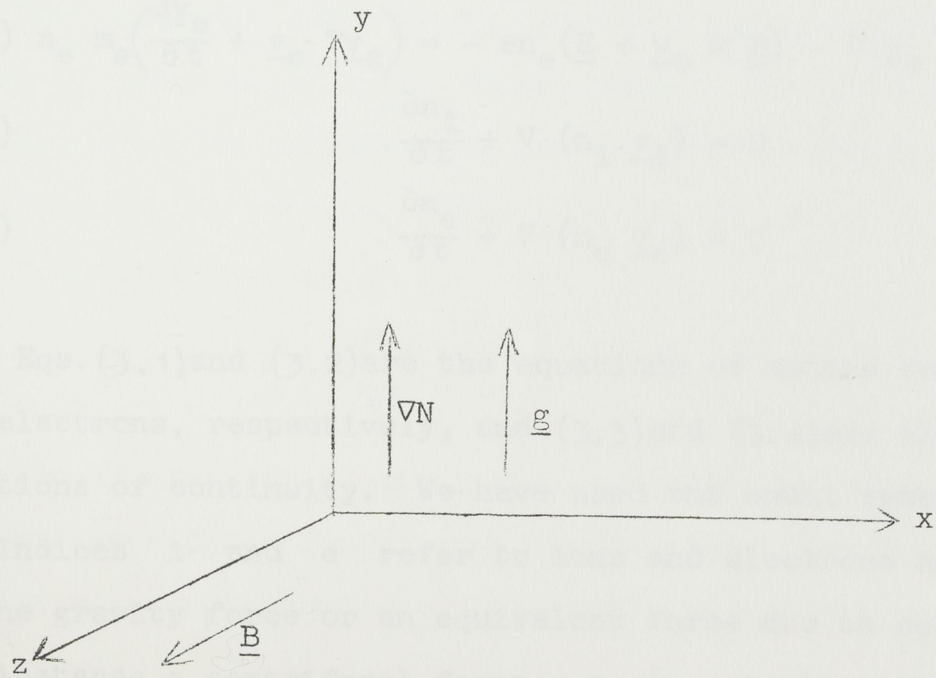

Fig. 1.

All perturbed quantities are assumed to be dependent only on x, y and t, where t as usual denotes the time. As for the z-dependence it is of less interest, since in the low β approximation the flute type instability is the most probable, where matter moves across the magnetic field leaving the latter practically undisturbed. That is because the plasma does not contain enough energy to make perturbations in the magnetic field probable.

In order to obtain a system of equations which is possible to solve without any other restriction than linearization in the perturbed quantities, we choose the density distribution given by Eq. (3.7). We also use the approximation of quasineutrality.

3. Basic equations.

We shall start with the equations for the two-fluid model, where we neglect the interaction terms due to ion-electron collisions. The equations are as follows:

$$(3.1) \quad n_i \, m_i\left(\frac{\partial \underline{v}_i}{\partial t} + \underline{v}_i \cdot \nabla \underline{v}_i\right) = e n_i (\underline{E} + \underline{v}_i \times \underline{B}) - \nabla \cdot \underset{\sim}{\pi}_i + n_i \, m_i \, \underline{g}$$

$$(3.2) \quad n_e \, m_e\left(\frac{\partial \underline{v}_e}{\partial t} + \underline{v}_e \cdot \nabla \underline{v}_e\right) = - e n_e (\underline{E} + \underline{v}_e \times \underline{B}) - \nabla \cdot \underset{\sim}{\pi}_e + n_e \, m_e \, \underline{g}$$

$$(3.3) \quad \frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \, \underline{v}_i) = 0$$

$$(3.4) \quad \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \, \underline{v}_e) = 0$$

Eqs. (3.1) and (3.2) are the equations of motion for ions and electrons, respectively, and (3.3) and (3.4) are the equations of continuity. We have used the usual symbols where the indices i and e refer to ions and electrons and $\underline{g}$ is the gravity force or an equivalent force due to accelerations, for instance a centrifugal force. $\underset{\sim}{\pi}_\nu$ $(\nu = i,e)$ is the

pressure tensor which we shall now specify. We use an expression for the pressure tensor first salculated by Chapman and Cowling [9], and also calculated in a simpler way in the collision-free case by Thompson [10]. This expression for the pressure tensor is also used by Roberts and Taylor [8] and Lehnert [6]. We assume the radius of gyration for electrons to be infinitely small and we can therefore approximate the pressure tensor for electrons by a transverse isotropic pressure $p_{e\perp}$. The relevant parts of the ion pressure tensor in the collisionfree case are then

$$\pi_{xx} = p_{i\perp} - \frac{m_i n_i a^2 \omega_i}{4} \left(\frac{\partial v_{iy}}{\partial x} + \frac{\partial v_{ix}}{\partial y} \right)$$

$$\pi_{yy} = p_{i\perp} + \frac{m_i n_i a^2 \omega_i}{4} \left(\frac{\partial v_{iy}}{\partial x} + \frac{\partial v_{ix}}{\partial y} \right) \tag{3.5}$$

$$\pi_{xy} = \pi_{yx} = \frac{m_i n_i a^2 \omega_i}{4} \left(\frac{\partial v_{ix}}{\partial x} - \frac{\partial v_{iy}}{\partial y} \right)$$

Here a is the radius of gyration for ions. We see from Eqs.(3.5) that the transport terms in the pressure tensor containing the radius of gyration play the formal role of kinematic viscosity. But they have, of course, not exactly the same physical significance, as pointed out by Roberts and Taylor [8]. If we define a parameter α by

$$\frac{a^2 \omega_i}{4} = \alpha \tag{3.6}$$

then α is constant in our problem, according to the assumptions made in Section 2. As mentioned earlier we choose, for the sake of convenience, the following density distribution

$$(3.7) \qquad n_e \approx n_i = \left\{N + n(x,y,t)\right\} \exp\left(\frac{y}{L}\right)$$

where $N \exp\left(\frac{y}{L}\right)$ is the assumed density distribution in equilibrium, with N constant.

We can now calculate the components of the pressure tensor divergence, and get from Eqs. (3.5)

$$(\nabla \cdot \underset{\sim}{\pi}_i)_x = \frac{\partial p_{i\perp}}{\partial x} - \alpha\, m_i \left\{ \frac{\partial}{\partial x}\left(n_i \frac{\partial v_{iy}}{\partial x} + n_i \frac{\partial v_i}{\partial y}\right) + \frac{\partial}{\partial y}\left(n_i \frac{\partial v_{iy}}{\partial y} - n_i \frac{\partial v_{ix}}{\partial x}\right)\right\}$$

where the index x refers to the x-direction and α is given by Eq. (3.6). When we linearize and use Eq. (3.7), $(\nabla \cdot \underset{\sim}{\pi}_i)_x$ can be written

$$(3.8) \quad (\nabla \cdot \underset{\sim}{\pi}_i)_x = \frac{\partial p_{i\perp}}{\partial x} - \alpha\, m_i\, N e^{\frac{y}{L}} \left\{\frac{\partial^2 v_{iy}}{\partial x^2} + \frac{1}{L}\left(\frac{\partial v_{iy}}{\partial y} - \frac{\partial v_{ix}}{\partial x}\right) + \frac{\partial^2 v_{iy}}{\partial y^2}\right\} .$$

In the same way we find

$$(\nabla \cdot \underset{\sim}{\pi}_i)_y = \frac{\partial p_{i\perp}}{\partial y} + \alpha\, m_i \left\{\frac{\partial}{\partial y}\left(n_i \frac{\partial v_{iy}}{\partial x} + n_i \frac{\partial v_{ix}}{\partial y}\right) + \frac{\partial}{\partial x}\left(n_i \frac{\partial v_{ix}}{\partial x} - n_i \frac{\partial v_{iy}}{\partial y}\right)\right\}$$

which gives

$$(3.9) \quad (\nabla \cdot \underset{\sim}{\pi}_i)_y = \frac{\partial p_{i\perp}}{\partial y} + \alpha\, m_i\, N e^{\frac{y}{L}} \left\{\frac{\partial^2 v_{ix}}{\partial y^2} + \frac{\partial^2 v_{ix}}{\partial x^2} + \frac{1}{L}\left(\frac{\partial v_{iy}}{\partial x} + \frac{\partial v_{ix}}{\partial y}\right)\right\} .$$

By the help of Eqs. (3.8) and (3.9) the vectorequation Eq. (3.1) can be written as two ordinary equations, one for the x-direction and one for the y-direction, thus

$$(3.10) \quad \frac{\partial v_{ix}}{\partial t} = \omega_i \frac{E_x}{B} + \omega_i v_{iy} - \frac{\theta}{N}\frac{\partial n}{\partial x} + \alpha \left\{\frac{\partial^2 v_{iy}}{\partial x^2} + \frac{\partial^2 v_{iy}}{\partial y^2} + \frac{1}{L}\left(\frac{\partial v_{iy}}{\partial y} - \frac{\partial v_{ix}}{\partial x}\right)\right\}$$

$$(3.7) \qquad n_e = n_i = \left\{ N + n(x,y,t) \right\} \exp\left(\frac{y}{L}\right)$$

where $N \exp\left(\frac{y}{L}\right)$ is the assumed density distribution in equilibrium, with N = constant.

We can now calculate the components of the pressure tensor divergence, and get from Eqs. (3.5)

$$(\nabla\cdot\pi_1)_x = \frac{\partial p_1}{\partial x} - \alpha m_i \left\{ \frac{\partial}{\partial x}\left(n_i \frac{\partial v_y}{\partial x} + n_i \frac{\partial v_x}{\partial y} \right) + \frac{\partial}{\partial y}\left(n_i \frac{\partial v_y}{\partial y} - n_i \frac{\partial v_x}{\partial x} \right) \right\}$$

where the index x refers to the x-direction and α is given by Eq. (3.6). When we linearize and use Eq. (3.7), $(\nabla\cdot\pi_1)_x$ can be written

$$(3.8) \quad (\nabla\cdot\pi_1)_x = \frac{\partial p_1}{\partial x} - \alpha m_i N e^{\frac{y}{L}} \left\{ \frac{\partial^2 v_x}{\partial x^2} + \frac{1}{L}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) + \frac{\partial^2 v_x}{\partial y^2} \right\}$$

In the same way we find

$$(\nabla\cdot\pi_1)_y = \frac{\partial p_1}{\partial y} + \alpha m_i \left\{ \frac{\partial}{\partial y}\left(n_i \frac{\partial v_x}{\partial x} + n_i \frac{\partial v_y}{\partial y} \right) + \frac{\partial}{\partial x}\left(n_i \frac{\partial v_x}{\partial x} - n_i \frac{\partial v_y}{\partial y} \right) \right\}$$

which gives

$$(3.9) \quad (\nabla\cdot\pi_1)_y = \frac{\partial p_1}{\partial y} + \alpha m_i N e^{\frac{y}{L}} \left\{ \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{1}{L}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) \right\}$$

By the help of Eqs. (3.8) and (3.9) the vector equation Eq. (3.1) can be written as two ordinary equations, one for the x-direction and one for the y-direction, thus

$$(3.10) \quad \frac{\partial v_{ix}}{\partial t} = \omega_i \frac{E_x}{B} + \omega_i v_{iy} - \frac{\partial n}{N\partial x}$$

$$+ \alpha \left\{ \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{1}{L}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \right\}$$

$$(3.11) \qquad \frac{\partial v_{iy}}{\partial t} = \omega_i \frac{E_y}{B} - \omega_i v_{ix} + g - \frac{\theta}{N}\frac{\partial n}{\partial y} - \frac{\theta}{L}$$
$$- \alpha\left\{\frac{\partial^2 v_{ix}}{\partial y^2} + \frac{\partial^2 v_{ix}}{\partial x^2} + \frac{1}{L}\left(\frac{\partial v_{iy}}{\partial x} + \frac{\partial v_{ix}}{\partial y}\right)\right\}$$

Here ω_i is the gyrofrequency for ions and θ is defined by the ideal gas equation

$$p_{i\perp} = m_i (N + n) e^{\frac{y}{L}} \theta$$

which gives

$$(3.12) \qquad \theta = \frac{1}{2} a^2 \omega_i^2$$

At the time $t = 0$ the system is unperturbed and we have by definition

$$E_x = 0 \quad , \quad n = 0 \quad , \quad v_{ix} = v_{iy} = 0$$

(3.3),

We see that these equilibrium values are solutions of Eqs. (3.4) and (3.10). And from Eq. (3.11) we get

$$(3.13) \qquad \frac{\omega_i E_{yo}}{B} + g - \frac{\theta}{L} = 0$$

where E_{yo} is the y-component of the electric field at $t = 0$. The equilibrium conditions $v_{ix} = v_{iy} = 0$ at $t = 0$ may also be considered as a definition of our frame of reference. From Eq. (3.13) we see that in equilibrium the y-component of the electric field must be different from zero in order to balance the pressure gradient and the g-force. According to our assumptions (Sec. 2) we can now write

$$(3.14) \qquad E_x = - \frac{\partial \varphi}{\partial x} \qquad E_y = E_{yo} - \frac{\partial \varphi}{\partial y}$$

and finally we obtain

$$(3.15)\quad \frac{\partial v_{ix}}{\partial t} + \frac{\omega_i}{B}\frac{\partial \varphi}{\partial x} - \omega_i\, v_{iy} + \frac{\theta}{N}\frac{\partial n}{\partial x} - \alpha\left\{\frac{\partial^2 v_{iy}}{\partial x^2} + \frac{\partial^2 v_{iy}}{\partial y^2} + \frac{1}{L}\left(\frac{\partial v_{iy}}{\partial y} - \frac{\partial v_{ix}}{\partial x}\right)\right\} = 0$$

$$(3.16)\quad \frac{\partial v_{iy}}{\partial t} + \frac{\omega_i}{B}\frac{\partial \varphi}{\partial y} + \omega_i\, v_{ix} + \frac{\theta}{N}\frac{\partial n}{\partial y} + \alpha\left\{\frac{\partial^2 v_{ix}}{\partial y^2} + \frac{\partial^2 v_{ix}}{\partial x^2} + \frac{1}{L}\left(\frac{\partial v_{iy}}{\partial x} + \frac{\partial v_{ix}}{\partial y}\right)\right\} = 0$$

In the equation of motion for electrons we shall neglect the inertia of electrons, i.e., we put the electron mass equal to zero. If we consider a series expansion in the parameter $\frac{\omega}{\omega_e}$ where ω is the characteristic frequency of the system and ω_e is the gyration frequency for electrons, our approximation is a zero-order approximation in this parameter, and we get from Eq. (3.2)

$$(3.17)\quad - n_e e\,(E_x + v_{ey}\, B) - \frac{\partial p_e}{\partial x} = 0$$

$$(3.18)\quad - n_e e\,(E_y - v_{ex}\, B) - \frac{\partial p_e}{\partial y} = 0 \; .$$

We can solve these equations with respect to $n_e\, v_{ey}$ and $n_e\, v_{ex}$ and get

$$(3.19)\quad n_e\, v_{ey} = - \frac{1}{eB}\frac{\partial p_e}{\partial x} - \frac{n_e}{B} E_x$$

$$(3.20)\quad n_e\, v_{ex} = \frac{1}{eB}\frac{\partial p_e}{\partial y} + \frac{n_e}{B} E_y \; .$$

If we put these expressions into the equation of continuity for electrons Eq. (3.4), we can easily eliminate v_{ex} and v_{ey}. We also observe that the pressure terms cansel out, and we are left with the equation

$$(3.21) \qquad \frac{\partial n_e}{\partial t} + \frac{1}{B}\left\{\frac{\partial}{\partial x}(n_e E_y) - \frac{\partial}{\partial y}(n_e E_x)\right\} = 0 \quad .$$

Using Eqs. (3.14) and (3.7) we obtain from Eq. (3.21) the following linearized equation

$$(3.22) \qquad \frac{\partial n}{\partial t} + \frac{N}{LB}\frac{\partial \varphi}{\partial x} + \left(\frac{\theta}{L\omega_i} - \frac{g}{\omega_i}\right)\frac{\partial n}{\partial x} = 0 \quad .$$

The equation of continuity for ions Eq. (1.3) can be written as

$$\frac{\partial n}{\partial t} e^{\frac{y}{L}} + \frac{\partial}{\partial x}\left\{v_{ix}(N + n)e^{\frac{y}{L}}\right\} + \frac{\partial}{\partial y}\left\{v_{iy}(N + n)e^{\frac{y}{L}}\right\} = 0$$

which yields

$$(3.23) \qquad \frac{\partial n}{\partial t} + N\frac{\partial v_{ix}}{\partial x} + \frac{N}{L}v_{iy} + N\frac{\partial v_{iy}}{\partial y} = 0 \quad .$$

Eqs. (3.15), (3.16), (3.22) and (3.23) are now our basic equations which we shall analyse further in the next section. But first we want to make a somewhat closer examination of the low β approxiamtion in connection with our assumption of a constant magnetic field.

From Eq. (3.19) we can see that $v_{ey} = 0$ in equilibrium. From Eq. (3.20) we get in the same case

$$v_{ex} = \frac{a_e}{L}\omega_e a_e + \frac{E_{yo}}{B} \quad .$$

Using the expression for $\frac{E_{yo}}{B}$ which we obtain from Eq. (3.13) and Eq. (3.12) for θ, we obtain (index e refers to electrons)

$$v_{ex} = \frac{a_e}{2L}\omega_e a_e + \frac{\theta}{L\omega_i} - \frac{g}{\omega_i} \quad ,$$

$$(3.24) \qquad v_{ex} = \frac{1}{2}\frac{a_e}{L}\omega_e a_e + \frac{1}{2}\frac{a}{L}\omega_i a - \frac{g}{\omega_i}$$

$$= \frac{1}{2}\frac{a^2}{L}\omega_i\left(\frac{T_e}{T_i} + 1 - \gamma\right) \quad .$$

Here we have introduced a new parameter γ defined by

$$(3.25) \qquad \gamma = \frac{gL}{\frac{1}{2}a^2\omega_i^2} = \frac{m_i gL}{\frac{1}{2}m_i a^2\omega_i^2}$$

which physically can be interpreted as the ratio between the potential energy associated with the characteristic length L in the gravity field, and the energy associated with the gyration-motion. We have also introduced T_e and T_i as the transverse electron and ion temperature, respectively. We now examine the space dependence in $\underline{B}$ from Maxwell's equation

$$(3.26) \qquad \nabla \times \underline{B} = \mu_o \underline{J}$$

where $\underline{J}$ is the current density and μ_o the permeability for vacuum.

In equilibrium this current density is given by one component in the x-direction

$$(3.27) \qquad J_x = n_e e v_{ex} \quad .$$

From (3.26) we can now calculate the characteristic length L_{CB} for B in the y-direction since

$$(3.28) \qquad \frac{B}{L_{CB}} \approx -\mu_o n_e e \frac{\frac{1}{2}a^2\omega_i^2}{L\omega_i}\left(\frac{T_e}{T_i} + 1 - \gamma\right)$$

Eq. (3.28) can be reduced to give

$$(3.29) \qquad L_{CB} = -L \frac{1}{\beta\left(\frac{T_e}{T_i} + 1 - \gamma\right)} \quad .$$

This shows that if T_e does not exceed T_i too much and β is small enough we have

(3.30) $L_{CB} >> L$

As for the perturbed velocities we assume these to interact very weakly with the magnetic field, since we are interested in the flute type instability as already mentioned in Section 2.

4. The dispersion relation.

From our basic equations we shall now derive a dispersion-relation. We observe, that when we neglect the y-dependence of of the magnetic field, all the coefficients in Eqs. (3.15), (3.16), (3.22) and (3.23) are independent of x, y and t. This was also the reason why we chose the particular density distribution given in (3.7). Since our system of equations is dependent on x, y and t only through the perturbed quantities, and not through the coefficients, we try to find normal mode solutions of the form $\exp\left\{ i(\omega t + xk_x + yk_y)\right\}$. Assuming all perturbed quantities to depend exponentially in this way on x, y and t, we can readily evaluate all derivatives appearing in the equations, and this results in a set of four linear homogenous equations in the four unknowing n, v_{ix}, v_{iy} and φ given below

(4.1) $$i(\omega + \frac{\alpha}{L} k_s)v_{ix} - (\omega_i - \alpha(k_x^2 + k_y^2) + i \frac{\alpha k_y}{L}) v_{iy} + \frac{ik_x \theta}{N} n + i \frac{\omega_i}{B} k_x \varphi = 0$$

(4.2) $$(\omega_i - \alpha(k_x^2 + k_y^2) + i \frac{\alpha}{L} k_y)v_{ix} + i(\omega + \frac{\alpha}{L} k_x)v_{iy} + i \frac{\theta}{N} k_y n + i \frac{\omega_i}{B} k_y \varphi = 0$$

$$(4.3)\qquad i\left\{\omega + \left(\frac{\theta}{L\omega_i} - \frac{g}{\omega_i}\right) k_x\right\}n + i\,\frac{N}{LB}\,k_x\,\varphi = 0$$

$$(4.4)\qquad i\,k_x\,N\,v_{ix} + (i\,N\,k_y + \frac{N}{L})v_{iy} + i\omega n = 0$$

A necessary and sufficient condition that the linear set of equations Eqs. (4.1) - (4.4) shall have a nontrivial solution, is that the determinant of the coefficient matrix should be zero. This condition gives us the dispersion relation, which we shall now calculate. The determinant is

$$(4.5)\qquad \begin{vmatrix} i(\omega + \frac{\alpha}{L}k_x) & -A & \frac{i\,k_x\,\theta}{N} & i\,\frac{\omega_i}{B}\,k_x \\ A & i(\omega + \frac{\alpha}{L}k_x) & \frac{i\,k_y\,\theta}{N} & i\,\frac{\omega_i k_y}{B} \\ 0 & 0 & i\left\{\omega + \frac{\theta}{L\omega_i} - \frac{g}{\omega_i}\,k_x\right\} & \frac{iN}{LB}k_x \\ ik_x\,N & \left(iNk_y + \frac{N}{L}\right) & i\omega & 0 \end{vmatrix} = 0$$

where

$$A = \omega_i - \alpha(k_x^{\,2} + k_y^{\,2}) + i\,\frac{\alpha}{L}\,k_y$$

When we expand this determinant after the third row we obtain

$$i\left\{\omega + \left(\frac{\theta}{L\omega_i} - \frac{g}{\omega_i}\right) k_x\right\} \begin{vmatrix} i(\omega + \frac{\alpha}{L}\,k_x) & -A & \frac{i\omega_i k_x}{B} \\ A & i(\omega + \frac{\alpha}{L}\,k_x) & \frac{i\omega_i k_y}{B} \\ ik_x\,N & iNk_y + \frac{N}{L} & 0 \end{vmatrix}$$

$$-\,\frac{iN}{LB}k_x \begin{vmatrix} i(\omega + \frac{\alpha}{L}\,k_x) & -A & \frac{ik_x\,\theta}{N} \\ A & i(\omega + \frac{\alpha}{L}\,k_x) & \frac{ik_y\,\theta}{N} \\ ik_xN & iNk_y + \frac{N}{L} & i\omega \end{vmatrix} = 0$$

$$(4.3) \qquad i\left\{\omega + \left(\frac{\theta}{L\omega_i} - \frac{g}{\omega_i}\right)k_x\right\} n + i\,\frac{\Omega}{B}\,k_x\,\varphi = 0$$

$$(4.4) \qquad i\,k_x\,N\,v_{1x} + \left(i\,N\,k_y + \frac{N}{L}\right)v_{1y} + i\omega n = 0$$

A necessary and sufficient condition that the linear set of equations Eqs. (4.1) - (4.4) shall have a nontrivial solution, is that the determinant of the coefficient matrix should be zero. This condition gives us the dispersion relation, which we shall now calculate. The determinant is

$$(4.5) \qquad \begin{vmatrix} i\frac{\Omega}{B}k_x & \frac{ik_x\theta}{N} & -A & i(\omega + \frac{\alpha}{L}k_x) \\ i\frac{\omega_i k_y}{B} & \frac{ik_y\theta}{N} & i(\omega + \frac{\alpha}{L}k_x) & A \\ \frac{iN}{B}k_x & i\left\{\omega + \left(\frac{\theta}{L\omega_i} - \frac{g}{\omega_i}\right)k_x\right\} & 0 & 0 \\ 0 & i\omega & \left(iNk_y + \frac{N}{L}\right) & ik_x N \end{vmatrix} = 0$$

where

$$A = \omega_i - a(k_x^2 + k_y^2) + i\,\frac{g}{L}\,k_y$$

When we expand this determinant after the third row we obtain

$$i\left\{\omega + \left(\frac{\theta}{L\omega_i} - \frac{g}{\omega_i}\right)k_x\right\} \begin{vmatrix} \frac{i\Omega k_x}{B} & -A & i(\omega + \frac{\alpha}{L}k_x) \\ \frac{i\omega_i k_y}{B} & i(\omega + \frac{\alpha}{L}k_x) & A \\ 0 & iNk_y + \frac{N}{L} & ik_x N \end{vmatrix}$$

$$-\frac{iN}{B}k_x \begin{vmatrix} \frac{ik_x\theta}{N} & -A & i(\omega + \frac{\alpha}{L}k_x) \\ \frac{ik_y\theta}{N} & i(\omega + \frac{\alpha}{L}k_x) & A \\ i\omega & iNk_y + \frac{N}{L} & ik_x N \end{vmatrix} = 0$$

The first determinant above we call D_1 and the second we call D_2. We then have the relationship

$$(4.5) \qquad i\left\{\omega + \left(\frac{\theta}{L\omega_i} - \frac{g}{\omega_i}\right) k_x\right\} D_1 - \frac{iNk_x}{LB} D_2 = 0 \quad .$$

We now calculate D_1 and D_2.

$$(4.6) \qquad D_1 = \frac{\omega_i k_y}{B}\left(\omega + \frac{\alpha}{L} k_x\right)\left(iNk_y + \frac{N}{L}\right) + \frac{\omega_i k_y k_x N}{B} A$$

$$+ \frac{ik_x{}^2\omega_i N}{B}\left(\omega + \frac{\alpha}{L} k_x\right) + i\,\frac{\omega_i k_x}{B}\left(iNk_y + \frac{N}{L}\right) A \quad .$$

And when we put in the expression for A this simplifies to give

$$(4.7) \qquad D_1 = \frac{N\omega_i{}^2 k_x}{LB}\left\{\frac{k_y}{k_x}\frac{\omega}{\omega_i} + i\left[\left(\frac{k_y{}^2 L}{k_x} + Lk_x\right)\frac{\omega}{\omega_i} + 1\right]\right\} \quad .$$

For D_2 we find the expression

$$(4.8) \qquad D_2 = i\left\{-\omega^3 - 2\,\frac{\alpha k_x}{L}\,\omega^2 - \frac{\alpha^2 k_x{}^2}{L^2}\,\omega\right\}$$

$$+ \frac{k_y\theta}{N}\left(\omega + \frac{\alpha}{L} k_x\right)\left(iNk_y + \frac{N}{L}\right)$$

$$+ k_x k_y\,\theta\left\{\omega_i - \alpha\left(k_x{}^2 + k_y{}^2\right) + i\,\frac{\alpha k_y}{L}\right\}$$

$$+ i\left(\omega_i - \alpha(k_x{}^2 + k_y{}^2) + i\,\frac{\alpha}{L} k_y\right)^2 \omega$$

$$+ ik_x\,\theta\left(\frac{1}{L} + ik_y\right)A + ik_x{}^2\,\theta\left(\omega + \frac{\alpha k_x}{L}\right).$$

From Eq. (3.6) and Eq. (3.12) we have

$$\alpha = \frac{1}{4}\,a^2\omega_i \qquad \text{and} \qquad \theta = \frac{1}{2}\,a^2\omega_i{}^2$$

By using these expressions for α and θ Eq. (4.8) can be simplified to give

$$(4.9) \qquad D_2 = i\omega^3\Big\{ -\left(\frac{\omega}{\omega_i}\right)^3 - \frac{a^2k_x}{2L}\left(\frac{\omega}{\omega_i}\right)^2 + \frac{\omega}{\omega_i} \\ \frac{1}{4}\left(\frac{a^2k^2}{2}\right)^2\left(1 - \frac{1}{k^2L^2} - i\frac{k_y}{k^2L}\right)\frac{\omega}{\omega_i} + \frac{k_x a^2}{2L}\Big\} .$$

When we divide Eq. (4.5) by $i\,\frac{\omega_i{}^3 k_x N}{LB}$ and substitute for D_1 and D_2, by Eqs. (4.7) and (4.9) we obtain

$$(4.10) \quad \left\{\frac{\omega}{\omega_i} + \frac{a^2k_x}{2L} - \frac{gk_x}{\omega_i{}^2}\right\}\left\{\frac{k_y}{k_x}\frac{\omega}{\omega_i} + i\left[\frac{(k_x{}^2 + k_y{}^2)L}{k_x}\frac{\omega}{\omega_i} + 1\right]\right\} \\ - i\left[-\left(\frac{\omega}{\omega_i}\right)^3 - \frac{a^2k_x}{2L}\left(\frac{\omega}{\omega_i}\right)^2 + \frac{\omega}{\omega_i} + \frac{1}{4}\left(\frac{a^2k^2}{2}\right)^2\left\{1 - \frac{1}{k^2L^2} - i\frac{k_y}{k^2L}\right\}\frac{\omega}{\omega_i} + \frac{k_x a^2}{2L}\right] = 0.$$

After performing the multiplication in (4.10) and multiplying by (-i), we obtain after collecting terms of the same power in $\frac{\omega}{\omega_i}$

$$(4.11) \quad \left(\frac{\omega}{\omega_i}\right)^3 + \left\{\frac{a^2k_x}{2L} + \frac{k_x{}^2 + k_y{}^2}{k_x}L - i\frac{k_y}{k_x}\right\}\left(\frac{\omega}{\omega_i}\right)^2 \\ + \left\{-Lk^2B + \frac{1}{4}\left(\frac{a^2k^2}{2}\right)^2\left(\frac{1}{k^2L^2} - 1\right) + ik_yB + \frac{i}{4}\left(\frac{a^2k^2}{2}\right)^2\frac{k_y}{k^2L}\right\}\frac{\omega}{\omega_i} - \frac{gk_x}{\omega_i{}^2} = 0$$

Where $B = \frac{g}{\omega_i{}^2} - \frac{a^2}{2L}$, and $k^2 = k_x{}^2 + k_y{}^2$.

We have here in Eq. (4.11) finally obtained the dispersion relation we were looking for, and which we want to discuss further in the next section.

5. Discussion and conclusions.

(i) In order to simplify the discussion of Eq. (4.11), we shall now define some new parameters. We put

$$k_x = k \cos \Theta \ , \quad k_y = k \sin \Theta \ , \tag{5.1}$$

where Θ is the angle between the vector $\underline{k}$ (Eq. (5.1)) and the x-axis, this gives

$$k_x^2 + k_y^2 = k^2 \ . \tag{5.2}$$

Further we put

$$P = \frac{1}{2} a^2 k^2 \ , \tag{5.3}$$

$$Q = kL \ . \tag{5.4}$$

When we use these parameters together with γ defined by Eq. (3.25), we can rewrite Eq. (4.11) and get

$$\left(\frac{\omega}{\omega_i}\right)^3 + \left\{\frac{P}{Q}\cos\Theta + \frac{Q}{\cos \Theta} - i \tan \Theta\right\}\left(\frac{\omega}{\omega_i}\right)^2 + \left\{P(1 - \gamma) + \frac{1}{4}\left[\left(\frac{P}{Q}\right)^2 - P^2\right] + i\ (\gamma - 1 + \frac{1}{4}P)\ \frac{P}{Q} \sin \Theta\right\}\frac{\omega}{\omega_i} - \frac{P}{Q}\, \gamma \cos \Theta = 0 \ . \tag{5.5}$$

We see that for $\Theta = 0$ Eq. (5.5) simplifies to a cubic equation with real coefficients. Since $\frac{P}{Q}$ is small compared to Q, we can as a good approximation neglect this term in the coefficient of $(\frac{\omega}{\omega_i})^2$. If we also perform a simple rotation of our coordinate system ($y \rightarrow x$, $x \rightarrow - y$, $z \rightarrow z$), and replace $\frac{1}{L}$ by $- \eta$ and α by ν, it turns out that Eq. (5.5) becomes identical with Eq. (11) of reference [8]. This equation is discussed by the authors of [8] and is found to give the same

results as Eq. (2.13) in reference [7]. It is, however, in this case necessary to redefine the average velocity for the ions. The discussion of this equation given in reference [8] is, however, based on the assumption that the cubic term can be neglected. This is valid as an approximation because one of the three roots of this equation is of the same order of magnitude as $Q > 1$, and the two other roots are much less than one. The error introduced by calculating the two small roots approximately by solving the quadratic equation instead of the cubic is therefore quite small.

(ii) According to Eq. (2.14) in the paper written by Rosenbluth, Krall and Rostoker [7] the stability condition given by these authors can be formulated as

$$(ak)\left(\frac{a}{L}\right) > 4\,\frac{\Omega_H}{\omega_i} \quad (5.6)$$

From paper [4], Lehnert's stability condition can be written as

$$a^2k^2\left(\frac{L}{L_{CB}} + \frac{1}{4}\gamma\right) > 1 \quad (5.7)$$

In the inequalities (5.6) and (5.7) we have used our notations. Ω_H is the instability growthrate in the MHD-approximation given by

$$\Omega_H = \sqrt{\frac{gN}{|\nabla N|}} = \sqrt{gL} \quad (5.8)$$

Using our parameters, Eqs. (3.25), (5.3) and (5.4), the stability condition (5.6) can be rewritten as

$$|\gamma| < \frac{P}{4} \text{ (Rosenbluth et.al.)} \quad . \quad (5.9)$$

results as Eq. (2.13) in reference [7]. It is, however, in this case necessary to redefine the average velocity for the ions. The discussion of this equation given in reference [8] is, however, based on the assumption that the cubic term can be neglected. This is valid as an approximation because one of the three roots of this equation is of the same order of magnitude as $\Omega > 1$, and the two other roots are much less than one. The error introduced by calculating the two small roots approximately by solving the quadratic equation instead of the cubic is therefore quite small.

(ii) According to Eq. (2.19) in the paper written by Rosenbluth, Krall and Rostoker [7] the stability condition given by these authors can be formulated as

$$(5.6) \qquad \frac{\Omega_H^2}{\nu^2}(ak)\left(\frac{a}{L}\right) > 4 \quad .$$

From paper [4], Lehnert's stability condition can be written as

$$(5.7) \qquad a^2k^2\left(\frac{1}{\Gamma_{CB}} + \frac{1}{4}\eta\right) > 1$$

In the inequalities (5.6) and (5.7) we have used our notations. Ω_H is the instability growthrate in the MHD-approximation given by

$$(5.8) \qquad \Omega_H = \sqrt{\frac{ag}{L}} = \sqrt{gL}$$

Using our parameters, Eqs. (3.25), (5.3) and (5.4), the stability condition (5.6) can be rewritten as

$$(5.9) \qquad |\gamma| < \frac{F}{4} \text{ (Rosenbluth et.al.)}$$

The stability condition (5.7) can be written

$$(5.10) \qquad |\gamma| > \frac{4}{P} \quad \text{(Lehnert)}.$$

In order to obtain condition (5.10) we have, however, neglected $\frac{L}{L_{CB}}$ compared to $\frac{1}{4}\gamma$. This turns out to be a good approximation, since we in this case must have $|\gamma| \gg 1$. That is because $P = \frac{1}{2} a^2 k^2$ is a small number compared to one. In the first case we get for the same reason, $|\gamma| \ll 1$. We therefore note that these two stabilizing effects can not be simultanously important

(iii). If we in Eq. (5.5) neglect the cubic term in $\frac{\omega}{\omega_i}$ and other terms of the same order, we are left with the quadratic equation.

$$(5.11) \quad \left(\frac{\omega}{\omega_i}\right)^2 + \frac{P(1-\gamma)\cos\Theta}{Q}\frac{\omega}{\omega_i} - \frac{P\gamma\cos^2\Theta}{Q^2(1 - i\frac{\sin\Theta}{Q})} \approx 0$$

Since $\frac{\sin\Theta}{Q} \ll 1$, we can perform a series expansion in $\frac{1}{1 - i\frac{\sin\Theta}{Q}}$. We then get an approximate value of $\frac{\omega}{\omega_i}$ given by

$$(5.12) \quad \frac{\omega}{\omega_i} \approx -\frac{\cos\Theta}{2}\frac{P}{Q}\left\{1 - \gamma \pm \sqrt{(1-\gamma)^2 + \frac{4\gamma}{P}(1 + i\frac{\sin\Theta}{Q})}\right\}.$$

If we put $\gamma \ll 1$ and $\theta = 0$ we get

$$(5.13) \qquad \frac{\omega}{\omega_i} \approx -\frac{P}{2Q}\left(1 \pm \sqrt{1 + \frac{4\gamma}{P}}\right),$$

which gives the stability condition (5.9). If we put $|\gamma| \gg 1$ and $\Theta = 0$ we get

(5.14) $$\frac{\omega}{\omega_i} \approx -\frac{P\gamma}{2Q}\left(1 \pm \sqrt{1 + \frac{4}{P\gamma}}\right) ,$$

which gives the stability condition (5.10)

If we, however, put $\Theta \neq 0$, we see from Eq. (5.12) that $\frac{\omega}{\omega_i}$ always will contain an imaginary part. Since both plus and minus sign are present there will always be a possibility for overstability. This will be true for all perturbations which are not perpendicular ($\Theta \neq 0$) to the gravity force and the density gradient. It is, however, difficult to justify the validity of Eq. (5.12) when $\Theta \neq 0$. We have therefore performed some numerical calculations based on the cubic equation Eq. (5.5). We have worked out a computer program [11] for solutions of cubic equations with real and complex coefficients. The numerical results are calculated on IBM 1620. These results are presented as graphs in Figs. 2, 3 and 4. In Fig. 2 we have plotted the imaginary part of $\frac{\omega}{\omega_i}$, for the unstable root, versue P (Eq. (5.3)) for three different values of Θ: $\Theta = 0$, $\Theta = 0.1$ and $\Theta = 1$. The angle Θ is in radians. The calculated results are in good agreement with the formulae Eq. (5.16) for $1 + \frac{4}{\gamma P} = 0$, which means equality sign in condition (5.9). In Fig. 3 the growth rate is plotted versues the angle Θ. The system was stable for $\Theta = 0$. We see from fig. 3 that the growth rate in this case has a maximum for $\Theta \approx 0.7$, this is in agreement with the angledependence we obtain from Eq. (5.12).

(5.15) $$\text{Im}\left(\frac{\omega}{\omega_i}\right) \alpha \cos \Theta \sqrt{\sin \Theta} = f(\Theta) \quad .$$

In order to obtain Eq. (5.15) we have put $(1 - \gamma)^2 + \frac{4\gamma}{P} = 0$ in

Eq. (5.12). This $f(\Theta)$ has a maximum at $\Theta \sim 0.61$ and is seen to be in good agreement with the curve Fig. 3.

In Fig. 4 we give some curves for the case $-\gamma \gg 1$, similar to the curves in Fig. 2. But now we have plotted $\mathrm{Im}\left(\frac{\omega}{\omega_i}\right)$ versues $-\gamma$. In this case also there exist a stable region only for $\Theta = 0$. And when $\Theta \neq 0$ the instability is only reduced in the region which is stable for $\Theta = 0$.

The growth rate dependence of the angle Θ in this case is about the same as that presented in Fig. 3 for the case $|\gamma| \ll 1$.

Calculations also shows that when $\Theta = 0$ $\mathrm{Im}\left(\frac{\omega}{\omega_i}\right)$ decreases with an increasing Q. That means that the system becomes more unstable for steeper density gradients, in both cases $|\gamma| \ll 1$ and $|\gamma| \gg 1$.

The result seems therefore to be that for $\Theta \neq 0$ there exist no range in the parameter space for which the system is stable, i. e., it is always overstable.

It is,however, difficult to know how serious this overstability is, since the method we have used, gives little information about the dynamics behinde this fenomena. It may be so that when the system is perturbed, the disturbances start to move with the y-component of $\underline{k}$ in the opposite direction of the density gradient and oscillates with a growing amplitude. The physics behinde may be that fewer and fewer particles participate in the oscillations, and we have a state of convective instability. This can give growing oscillations even if the energy associated with the perturbations is constant. If this is true, and there is no energy contained in the plasma which is converted into growing oscillations, this overstability is probably not dangerous.

There is also another argument against this result, perhaps boundary conditions in the y-direction will change it completely when $\Theta \neq 0$.

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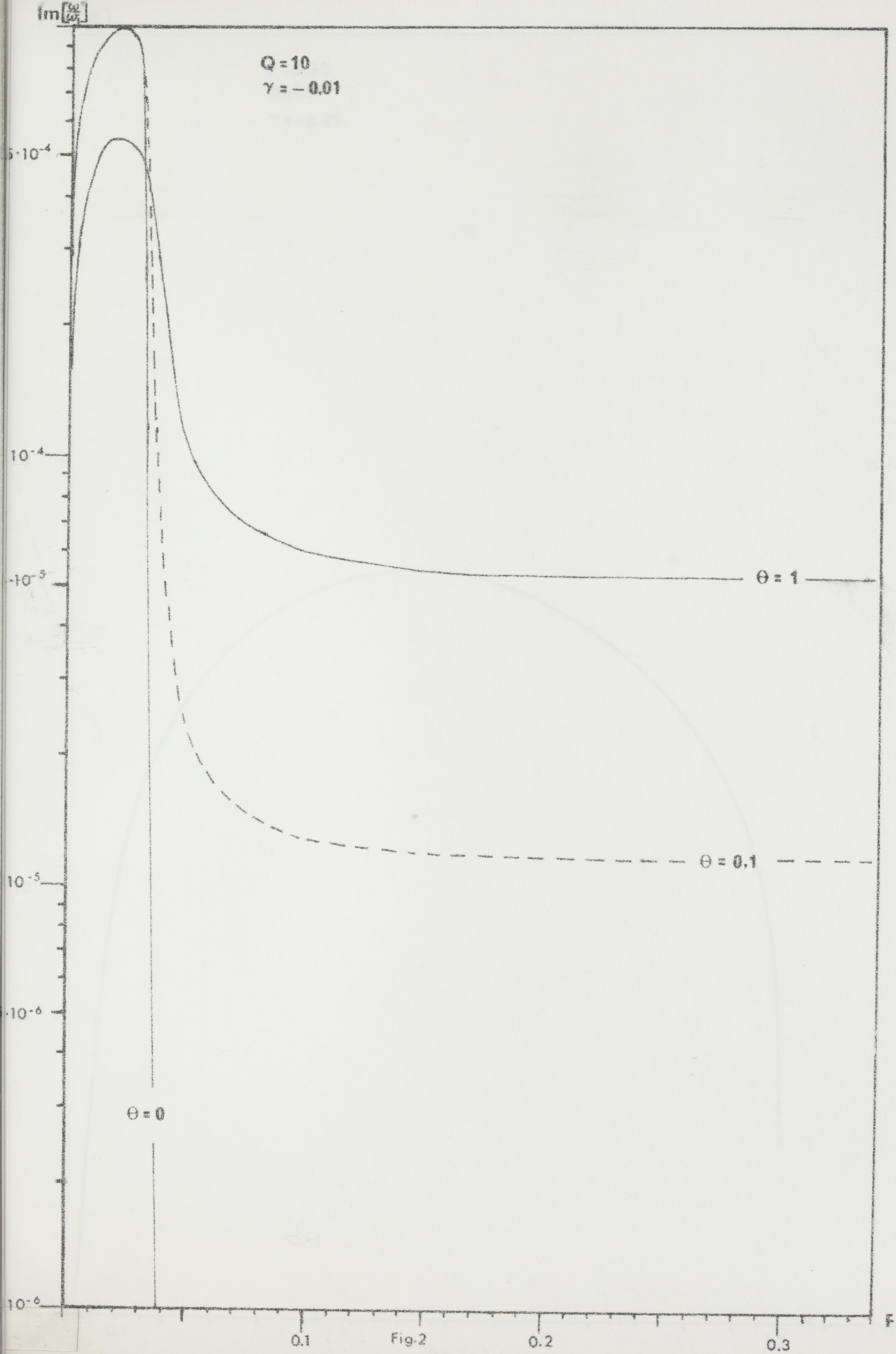
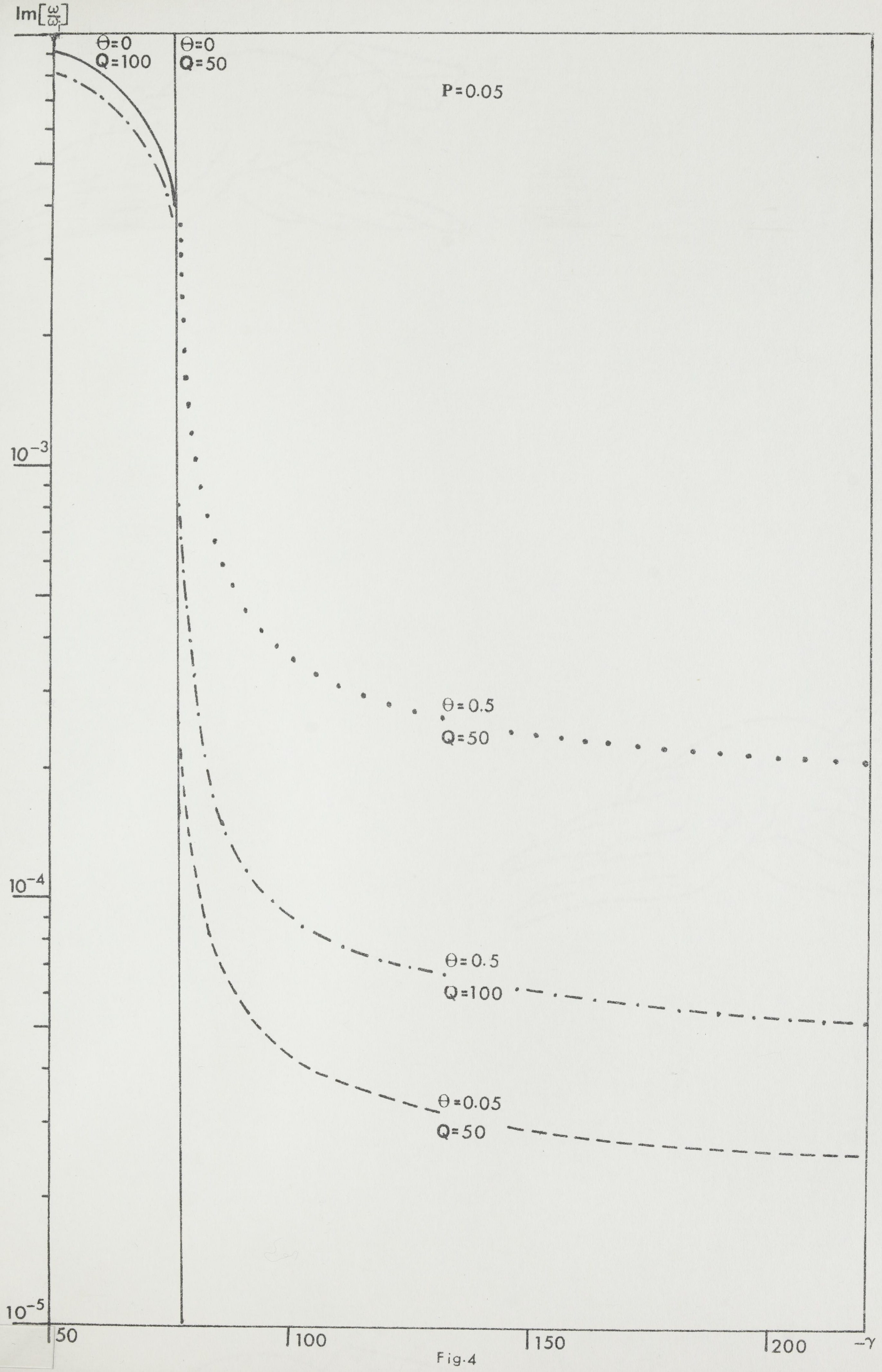


Fig.2

Fig.3



Im[ω/ωi]
θ=0
Q=100
θ=0
Q=50
P=0.05
10⁻³
θ=0.5
Q=50
10⁻⁴
θ=0.5
Q=100
θ=0.05
Q=50
10⁻⁵
50
100
150
200
−γ

Fig.4