Department
of
APPLIED MATHEMATICS

On the stability of plane inviscid Couette flow

by

Leif Engevik.

Report No. 12. November 1966.

UNIVERSITY OF BERGEN
Bergen, Norway

On the stability of plane inviscid

Couette flow

by

Leif Engevik.

Report No. 12.   November 1966.

Abstract.

In the present paper we investigate the stability of plane Couette flow of a homogeneous, incompressible and inviscid fluid. We show how we can find an asymptotic series for the stream function at large values of t (time), and demonstrate the connection between Case's results [1] and ours.

This work has been supported by the Royal Norwegian Council for Scientific and Industrial Research.

I. Introduction.

In a paper on inviscid plane Couette flow, K.M. Case [1] investigated the asymptotic behavior of the x-component of the velocity. His treatment of the question of stability is not satisfactory. He assigned certain initial vorticities and studied the asymptotic behavior at a fixed point in the fluid. Therefore he cannot be sure whether there is convective instability or not.

In this work an symptotic series for the stream function is found, showing how it depends on the initial vorticity. We find that the stream function vanishes as t^{-2} within a fluid particle moving with velocity z.

At the end of section VI the connection between Case's results and ours is shown.

II. Formulation of the problem.

The system to be considered is the following: a horizontal flow of a homogeneous, incompressible and inviscid fluid, confined between two rigid planes, situated at $z = \pm 1$, (see fig. 1). The equations governing this system, are the hydrodynamic equations for motion in the gravity field, viz.:

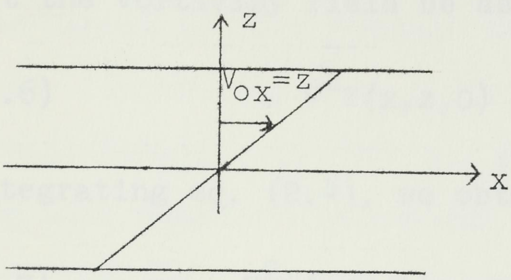

Fig. 1

$$(2.1) \quad \begin{cases} \rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = - \nabla p - \rho g \underline{k} , \\ \nabla \cdot \underline{v} = 0 , \end{cases}$$

where $\underline{k}$ is the unit-vector in the z-direction.
With appropriate choice of velocity units, the basic motion in plane Couette flow is given by:

$$(2.2) \qquad V_{ox} = z , \quad V_{oy} = 0 .$$

The perturbation velocity can be written as:

$$(2.3) \qquad \underline{v}_1 = \nabla \times \Psi(x,z,t)\underline{j} ,$$

where $\underline{j}$ is the unit vector in y-direction, and $\Psi(x,z,t)$ is the stream function.

Linearizing the first of eqs. (2.1), eliminating the pressure, and using eqs. (2.2) and (2.3), we obtain:

$$(2.4) \qquad \left(\frac{\partial}{\partial t} + z \frac{\partial}{\partial x}\right)\nabla^2 \Psi = 0 .$$

The boundary conditions to be imposed on eq. (2.4), are

$$(2.5)\qquad \frac{\partial\Psi}{\partial x}(x,z,t) = 0 \quad \text{at} \quad z = \pm 1 .$$

Let the vorticity field be assigned at $t = 0$.

$$(2.6)\qquad \nabla^2\Psi(x,z,0) = F(x,z) .$$

Integrating eq. (2.4), we obtain:

$$(2.7)\qquad \nabla^2\Psi(x,z,t) = F(x - zt,z) ,$$

which shows that the vorticity is conserved within the fluid particle moving with velocity $V_{ox} = z$.

We assume that ψ depends on x as:

$$(2.8)\qquad \Psi(x,z,t) = \psi(z,t)e^{ikx} ,$$

where $\psi(z,t)$ is the k^{th} Fourier component.
With this assumption eqs. (2.4), (2.5) and (2.6) become:

$$(2.9)\qquad \left(\frac{\partial}{\partial t} + ikz\right)\left(\frac{\partial^2}{\partial z^2} - k^2\right)\psi = 0 ,$$

$$(2.10)\qquad \psi(z,t) = 0 \quad \text{at} \quad z = \pm 1 ,$$

$$(2.11)\qquad \left(\frac{\partial^2}{\partial z^2} - k^2\right)\psi_{t=0} = F_k(z) ,$$

where $F(x,z) = F_k(z)e^{ikx}$.
From eq. (2.7), we obtain:

$$(2.12)\qquad \left(\frac{\partial^2}{\partial z^2} - k^2\right)\psi = F_k(z)e^{-ikzt} ,$$

which is the equation solved by Case [1]. From his solution he tried to find the behavior of $\psi(z,t)$ at large values of t. But his treatment is not satisfactory, because he

assigned certain initial vorticities and studied the time-dependence at a fixed point (x,z) in the fluid. Therefore he cannot be sure whether there is convective instability or not. In the Appendix it is shown how we can find the asymptotic behavior of $\psi(z,t)$, using the theory in [2]. Here we will demonstrate another method, which can be used in order to find the asymptotic behavior of $\psi(z,t)$.

III. Solution to the problem.

In eq. (2.9) we use the Laplace transformation to obtain:

$$(3.1) \qquad \left(\frac{d^2}{dz^2} - k^2\right)\bar{\psi} = \frac{F_k(z)}{ik(z - \zeta)} ,$$

where

$$\bar{\psi} = \int_0^\infty \psi(z,t)e^{-pt}dt ,$$

$$\zeta = i\,\frac{p}{k} .$$

Eq. (3.1) is easily solved. We find:

$$(3.2) \qquad \bar{\psi}(z,\zeta) = \frac{\sinh k(z-1)}{W}\int_{-1}^{z}\frac{F_k(u)}{ik(u-\zeta)}\sinh k(u+1)du$$

$$+ \frac{\sinh k(z+1)}{W}\int_{z}^{1}\frac{F_k(u)}{ik(u-\zeta)}\sinh k(u-1)du ,$$

where $W = W(\sinh k(z+1) , \sinh k(z-1)) = k\sinh 2k$ is the Wronskian for $\sinh k(z+1)$ and $\sinh k(z-1)$.

We observe that

$$(3.3)\qquad \bar{\psi}(1,\zeta) = \bar{\psi}(-1,\zeta) = 0\ ,$$

consistent with our boundary conditions. When $\bar{\psi}(z,\zeta)$ is found, $\psi(z,t)$ is easily obtained by inversion:

$$(3.4)\quad \psi(z,t) = \frac{k}{2\pi}\int_{i\zeta_o-\infty}^{i\zeta_o+\infty} \frac{e^{-ik\zeta t}}{W}\Big[\sinh k(z-1)\int_{-1}^{z}\frac{F_k(u)}{ik(u-\zeta)}\sinh k(u+1)\,du$$

$$+ \sinh k(z+1)\int_{z}^{1}\frac{F_k(u)}{ik(u-\zeta)}\sinh k(u-1)\,du\Big]d\zeta\ .$$

In eq. (3.4) we can change the order of integration, and then obtain the solution of eq. (2.12), given by eq. (A1). In this type of problems this is in general not possible. Generally $\psi(z,t)$ will be written in the form of eq. (3.4). Here we will show how we can find the asymptotic behavior of this function directly from the representation in eq. (3.4).

In order to evaluate the integral in eq. (3.4) we will perform an integration around the contour C , shown in fig. 2 ($k > 0$). The integrand has singularities at $\zeta = 1, z, -1$.*) Therefore we have to make cuts in the complex ζ-plane, as shown. Using Caushy's residue theorem, since the integrand has no poles within the contour C, we obtain:

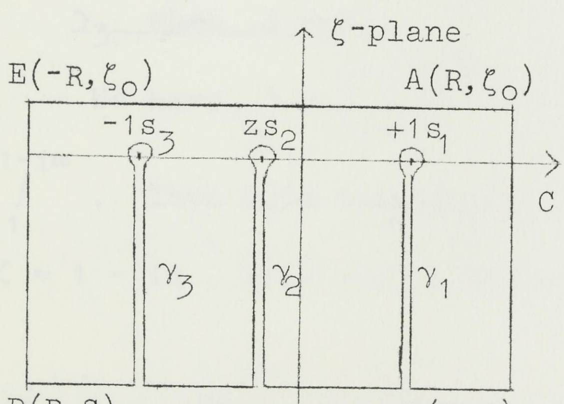

Fig. 2

$$(3.5)\qquad \psi(z,t) = -\int_{\gamma_1}-\int_{\gamma_2}-\int_{\gamma_3}-\int_{ABDE}-\int_{s_1}-\int_{s_2}-\int_{s_3}\ ,$$

*) $F_k(z)$ is assumed to be analytic for $z \in [-1,1]$.

$$(3.3) \qquad \bar{\psi}(1,\zeta) = \bar{\psi}(-1,\zeta) = 0 \;,$$

consistent with our boundary conditions. When $\bar{\psi}(z,\zeta)$ is found, $\psi(z,t)$ is easily obtained by inversion:

$$(3.4) \quad \psi(z,t) = \frac{k}{2\pi}\int_{i\zeta_0-\infty}^{i\zeta_0+\infty} e^{-ik\zeta t}\Big[\sinh k(z-1)\int_{-1}^{z}\frac{F_k(u)}{ik\Delta(\zeta)}\sinh k(u+1)\,du$$

$$+ \sinh k(z+1)\int_{z}^{1}\frac{F_k(u)}{ik\Delta(\zeta)}\sinh k(u-1)\,du\Big]d\zeta \;.$$

In eq. (3.4) we can change the order of integration, and then obtain the solution of eq. (2.12), given by eq. (A1). In this type of problems this is in general not possible. Generally $\psi(z,t)$ will be written in the form of eq. (3.4). Here we will show how we can find the asymptotic behavior of this function directly from the representation in eq. (3.4).

In order to evaluate the integral in eq. (3.4) we will perform an integration around the contour C, shown in Fig. 2 $(k > 0)$. The integrand has singularities at $\zeta = 1, z, -1$.*) Therefore we have to make cuts in the complex ζ-plane, as shown. Using Cauchy's residue theorem, since the integrand has no poles within the contour C, we obtain:

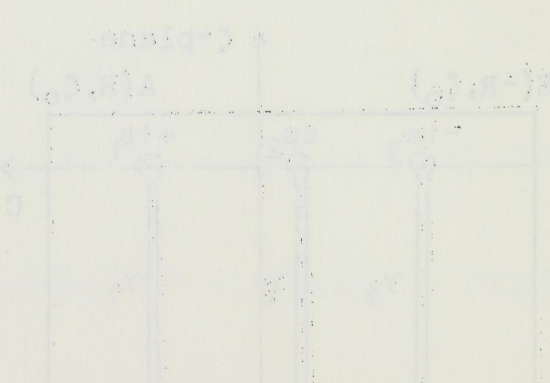

Fig. 2

$$(3.5) \qquad I(z,u) = \int_{\gamma_1} - \int_{\gamma_2} - \int_{\gamma_3} - \int_{ABCDE} - \int_{\varepsilon_1} - \int_{\varepsilon_2} - \int_{\varepsilon_3}$$

*) $F_k(u)$ is assumed to be analytic for $z \in [-1,1]$.

where $\int_{\gamma_1} = \int_{1-iS}^{1-i\rho_1} + \int_{1-i\rho_1}^{1-iS}$. We have analogous expressions for the integrals $\int_{\gamma_2}$ and $\int_{\gamma_3}$. $\int_{s_1}$, $\int_{s_2}$ and $\int_{s_3}$ are the integrals around the small circles s_1, s_2 and s_3, surrounding $\zeta = 1, z, -1$. It is easy to show that $\int_{s_1}$, $\int_{s_2}$ and $\int_{s_3} \to 0$ when the radii ρ_i $(i = 1,2,3)$ of the circles tend to zero. Also $\int_{ABDE} \to 0$, when $R, S \to \infty$.

Therefore:

$$\psi(z,t) = -\int_{\gamma_1} - \int_{\gamma_2} - \int_{\gamma_3}, \tag{3.6}$$

where now $\int_{\gamma_1} = \int_{1-i\infty}^{1} + \int_{1}^{1-i\infty}$. We have analogous expressions for $\int_{\gamma_2}$ and $\int_{\gamma_3}$.

IV. The asymptotic behavior of the integrals along γ_1 and γ_3 when $t \to \infty$.

We have $\int_{\gamma_1} = \int_{1-i\infty}^{1} + \int_{1}^{1-i\infty}$. Let us examine the integral $\int_{1}^{1-i\infty}$. Into this integral we introduce η, given by $\zeta = 1 - i\eta$, as a new variable. We then obtain:

$$P(z,t) = -\frac{ik}{2\pi W}\int_0^{\infty} e^{-ikt-k\eta t}\Big[\sinh k(z-1)\int_{-2+i\eta}^{z-1+i\eta} \frac{F_k(v+1-i\eta)}{ikv}\sinh k(v+2-i\eta)\,dv \tag{4.1}$$

$$+ \sinh k(z+1)\int_{z-1+i\eta}^{i\eta} \frac{F_k(v+1-i\eta)}{ikv}\sinh k(v-i\eta)\,dv\Big]d\eta.$$

The first term in eq. (4.1) will be canceled by the where $\int_{\gamma_1} = \int_{1-i\beta}^{1-i\delta} + \int_{1-i\delta}^{1+i\delta}$. We have analogous expressions for the integrals $\int_{\gamma_2}$ and $\int_{\gamma_3}$. $\int_{s_1}$, $\int_{s_2}$ and $\int_{s_3}$ are the integrals around the small circles s_1, s_2 and s_3, surrounding $\zeta = 1, z, -1$. It is easy to show that $\int_{s_1}$, $\int_{s_2}$ and $\int_{s_3} \to 0$ when the radii ρ_i $(i = 1,2,3)$ of the circles tend to zero. Also $\int \to 0$, when $R, B \to \infty$.

Therefore:

$$(3.6) \qquad \psi(z,t) = -\int_{\gamma_1} - \int_{\gamma_2} - \int_{\gamma_3} ,$$

where now $\int_{\gamma_1} = \int_{1-i\infty}^{1} + \int_{1}^{1+i\infty}$. We have analogous expressions for $\int_{\gamma_2}$ and $\int_{\gamma_3}$.

__IV. The asymptotic behavior of the integrals along γ_1 and γ_3 when $t \to \infty$.__

We have $\int_{\gamma_1} = \int_{1-i\infty}^{1} + \int_{1}^{1+i\infty}$. Let us examine the integral $\int_{1}^{1+i\infty}$. Into this integral we introduce η, given by $\zeta = 1 - i\eta$, as a new variable. We then obtain:

$$(4.1) \quad F(z,t) = -\frac{ik}{2\pi K}\int_0^\infty e^{-ikt-k\eta}\Big[\sinh k(z-1)\int_{-2+i\eta}^{z-1+i\eta} \frac{F_0(\nu+t-i\eta)}{i\kappa\nu}\sinh k(\nu+2-i\eta)\,d\nu$$

$$+ \sinh k(z+1)\int_{z-1+i\eta}^{i\eta} \frac{F_0(\nu+t-i\eta)}{i\kappa\nu}\sinh k(\nu-i\eta)\,d\nu\Big]d\eta .$$

The first term in eq. (4.1) will be canceled by the corresponding term in the integral $\int_{1-i\infty}^{1}$. In the vicinity of $\eta = 0$ the last term can be written as:

$$(4.2)\quad \int_{z-1+i\eta}^{i\eta} \frac{F_k(v+1-i\eta)}{ikv} \sinh k(v - i\eta)dv = \sum_{n=0}^{\infty} a_n\eta^n + \ln\eta \sum_{n=0}^{\infty} b_n\eta^n \text{ *)} .$$

The term $\sum_{n=0}^{\infty} a_n\eta^n$ is of no interest for the same reason as above. b_n is easily found, by differentiating eq. (4.2), and letting η tend to zero. We find:

$$(4.3)\quad \begin{cases} b_o = 0 \\ b_1 = - F_k(1) \\ - - - \end{cases}$$

We are now in a position to find the first term in the asymptotic series for $\int_{\gamma_1}$. Using Watson's lemma [3], we find that the first term is given by:

$$(4.4)\quad -\frac{ik}{2\pi W} e^{-ikt} b_1 \sinh k(z + 1)\left[\int_{\infty}^{0} e^{-\eta kt}\eta\ln\eta d\eta\right.$$

$$\left. + \int_{0}^{\infty} e^{-\eta kt}\eta\{\ln\eta + 2\pi i\}d\eta\right] = -\frac{F_k(1)}{k^2t^2}\frac{\sinh k(z + 1)}{\sinh 2k} e^{-ikt} ,$$

where we have used that $\ln\eta$ is a many-valued function of η.

Taking into account eq. (4.4), we can write:

---

*) We assume that $F_k(z)$ is an analytic function of $z \in [-1,1]$.

$$(4.5)\qquad \int\limits_{\gamma_1} = -\frac{F_k(1)}{k^2t^2}\,\frac{\sinh k(z+1)}{\sinh 2k}\,e^{-ikt} + \ldots\,,\ \text{when}\quad t\to\infty\,.$$

An analogous expression is found for the integral along γ_3. We find:

$$(4.6)\qquad \int\limits_{\gamma_3} = \frac{F_k(-1)}{k^2t^2}\,\frac{\sinh k(z-1)}{\sinh 2k}\,e^{ikt} + \ldots\,,\ \text{when}\quad t\to\infty\,.$$

V. The asymptotic behavior of the integral along γ_2 when $t\to\infty$.

We have $\int\limits_{\gamma_2} = \int\limits_{z-i\infty}^{z} + \int\limits_{z}^{z-i\infty}$. Let us examine the integral $\int\limits_{z}^{z-i\infty}$. Into this integral we introduce η, given by $\zeta = z-i\eta$, as a new variable. We obtain:

$$(5.1)\ P(z,t) = -\frac{ik}{2\pi M}e^{-ikzt}\int\limits_0^{\infty} e^{-k\eta t}d\eta\Big[\sinh k(z-1)\int\limits_{-1-z+i\eta}^{i\eta}\Big\{\frac{F_k(v+z-i\eta)}{ikv}\times \sinh k(v+z+1-i\eta)\Big\}dv+\sinh k(z+1)\int\limits_{i\eta}^{1-z+i\eta}\frac{F_k(v+z-i\eta)}{ikv}\sinh k(v+z-1-i\eta)dv\Big]\,.$$

In the vicinity of $\eta = 0$ we have:

$$(5.2)\quad \begin{cases} \displaystyle\int\limits_{-1-z+i\eta}^{i\eta}\frac{F_k(v+z-i\eta)}{ikv}\sinh k(v+z+1-i\eta)dv = \sum_{n=0}^{\infty} a_{1n}\eta^n + \ln\eta\sum_{n=0}^{\infty} b_{1n}\eta^n \\ \displaystyle\int\limits_{i\eta}^{1-z+i\eta}\frac{F_k(v+z-i\eta)}{ikv}\sinh k(v+z-1-i\eta)dv = \sum_{n=0}^{\infty} a_{2n}\eta^n + \ln\eta\sum_{n=0}^{\infty} b_{2n}\eta^n. \end{cases}$$

The first terms in eqs.(5.2) are of no interest, since they

will be canceled by the corresponding terms in the integral $\int_{z-i\infty}^{z}$. Let us find b_{1n} and b_{2n}. Differentiating eqs.(5.2), and letting $\eta \to 0$, we find:

$$(5.3)\quad \begin{cases} b_{10} = \dfrac{F_k(z)}{ik} \sinh k(z+1) \\ b_{11} = -i \dfrac{\partial}{\partial z}\left[\dfrac{F_k(z)}{ik} \sinh k(z+1)\right] \\ --- \end{cases}$$

$$(5.4)\quad \begin{cases} b_{20} = -\dfrac{F_k(z)}{ik} \sinh k(z-1) \\ b_{21} = i \dfrac{\partial}{\partial z}\left[\dfrac{F_k(z)}{ik} \sinh k(z-1)\right] \\ --- \end{cases}$$

Using Watson's lemma, and taking into account eqs. (5.3) and (5.4), we obtain:

$$(5.5)\quad \int_{\gamma_2} = -\frac{ik}{2\pi W} e^{-ikzt}\Big[\int_{\infty}^{0} e^{-k\eta t} \ln\eta \Big\{\sum_{n=0}^{\infty} b_{1n}\eta^n + \sum_{n=0}^{\infty} b_{2n}\eta^n\Big\} d\eta$$

$$+ \int_0^{\infty} e^{-k\eta t}(\ln\eta + 2\pi i)\Big\{\sum_{n=0}^{\infty} b_{1n}\eta^n + \sum_{n=0}^{\infty} b_{2n}\eta^n\Big\} d\eta\Big]$$

$$= \frac{F_k(z)}{k^2t^2} e^{-ikzt} + \ldots, \quad \text{when} \quad t \to \infty .$$

VI. Some comments.

Taking into account eqs. (3.6), (4.5), (4.6) and (5.5), we obtain:

$$(6.1) \quad \psi(z,t) = -\frac{F_k(z)}{k^2t^2} e^{-ikzt} + \frac{F_k(1)}{k^2t^2} \frac{\sinh k(z+1)}{\sinh 2k} e^{-ikt} - \frac{F_k(-1)}{k^2t^2} \frac{\sinh k(z-1)}{\sinh 2k} e^{ikt} + \dots .$$

From eq. (6.1) we find that the velocity in the x-direction is damped as $\frac{1}{t}$ when $t \to \infty$.

Differentiating eq. (6.1) twice with respect to z, we obtain:

$$\frac{\partial^2\psi}{\partial z^2} = F_k(z)e^{-ikzt} + \dots ,$$

consistent with eq. (2.12).

We observe that if $F_k(z)$ is assigned in such a way that $F_k(z)$ and all its derivatives are equal to zero at $z = \pm 1$, then there is no contribution from the integrals along γ_1 and γ_3.

As to the question of stability of the system, the integral $\int_{\gamma_2}$ is of the greatest interest, since it is only this integral which contributes to the leading term in the asymptotic series for $\frac{\partial\psi}{\partial z}$ and $\frac{\partial^2\psi}{\partial z^2}$.

Let us assume $F_k(1) = F_k(-1) = 0$ *). Then from eq. (6.1), we obtain by inversion: **)

$$(6.2) \qquad \Psi(x,z,t) = \frac{G(x - zt, z)}{t^2} + \dots ,$$

where

---

*) This is analogous to the situation studied by Case.

**) We interpret $\psi(z,t)$ and $F_k(z)$ as the Fourier transforms of $\Psi(x,z,t)$ and $F(x,z)$.

$$(6.1)\qquad \psi(z,t) = -\frac{F_k(z)}{k^2 t} e^{-ikzt} + \frac{F_k(1)}{k^2 t}\frac{\sinh k(z+1)}{\sinh 2k} e^{-ikt}$$

$$- \frac{F_k(-1)}{k^2 t}\frac{\sinh k(z-1)}{\sinh 2k} e^{ikt} + \ldots .$$

From eq. (6.1) we find that the velocity in the z-direction is damped as $\frac{1}{t}$ when $t \to \infty$.

Differentiating eq. (6.1) twice with respect to z, we obtain:

$$\frac{\partial^2 \psi}{\partial z^2} = F_k(z) e^{-ikzt} + \ldots ,$$

consistent with eq. (2.12).

We observe that if $F_k(z)$ is assigned in such a way that $F_k(z)$ and all its derivatives are equal to zero at $z = \pm 1$, then there is no contribution from the integrals along γ_1 and γ_3.

As to the question of stability of the system, the integral $\int_{\gamma_2}$ is of the greatest interest, since it is only this integral which contributes to the leading term in the asymptotic series for $\frac{\partial \psi}{\partial z}$ and $\frac{\partial^2 \psi}{\partial z^2}$.

Let us assume $F_k(1) = F_k(-1) = 0$ *). Then from eq. (6.1), we obtain by inversion: **)

$$(6.2)\qquad \psi(x,z,t) = \frac{G(x - zt, z)}{t^2} + \ldots ,$$

where

---

*) This is analogous to the situation studied by Case.

**) We interpret $\psi(z,t)$ and $F_k(z)$ as the Fourier transforms of $\psi(x,z,t)$ and $F(x,z)$.

(6.3) $$F(x,z) = \frac{\partial^2}{\partial x^2} G(x,z) .$$

Integrating eq. (6.3), we obtain:

(6.4) $$G(x,z) = \int dx \int F(x,z)dx + C_1(z)x + C_2(z) ,$$

where C_1 and C_2 are functions of z only, and $C_1(1) = C_1(-1) = 0$, so that the boundary conditions eq. (2.5) are satisfied. The Fourier transform of $G(x,z)$ exists in the meaning of Fourier transform of a generalised function, see [2].

Let us in conclusion show the connection between Case's results and ours. We consider the two cases:

I. $\left| \int_{-\infty}^{+\infty} F(x,z)dx \right| < \infty$, i.e. the total initial vorticity associated with the perturbation, is finite.

II. $\left| \int_{-\infty}^{+\infty} F(x,z)dx \right| = \infty$, i.e. the total initial vorticity associated with the perturbation, is infinite.

<u>Case I.</u>

For the integral to exist $|F(x,z)|$ must tend to zero at least as fast as $|x|^{-\alpha}$ $(\alpha > 1)$ when $|x| \to \infty$. Then $|G(x,z)| \cong |C_1(z)x|$ when $|x| \to \infty$. And this together with eq. (6.2) shows us that the perturbation at any fixed point (x,z) in the fluid is damped as $\frac{1}{t}$ when $t \to \infty$.

$$(6.3) \qquad F(x,z) = \frac{\partial^2}{\partial x^2}\, G(x,z) \ .$$

Integrating eq. (6.3), we obtain:

$$(6.4) \qquad G(x,z) = \int dx \int F(x,z)dx + C_1(z)x + C_2(z) \ ,$$

where C_1 and C_2 are functions of z only, and $C_1(1) = C_1(-1) = 0$, so that the boundary conditions eq. (2.5) are satisfied. The Fourier transform of $G(x,z)$ exists in the meaning of Fourier transform of a generalised function, see [2].

Let us in conclusion show the connection between Case's results and ours. We consider the two cases:

I. $\left|\int_{-\infty}^{+\infty} F(x,z)dx\right| < \infty$, i.e. the total initial vorticity associated with the perturbation is finite.

II $\left|\int_{-\infty}^{+\infty} F(x,z)dx\right| = \infty$, i.e. the total initial vorticity associated with the perturbation is infinite.

<u>Case I.</u>

For the integral to exist $|F(x,z)|$ must tend to zero at least as fast as $|x|^{-\alpha}$ $(\alpha > 1)$ when $|x| \to \infty$. Then $|G(x,z)| \approx |C_1(z)x|$ when $|x| \to \infty$. And this together with eq. (6.2) shows us that the perturbation at any fixed point (x,z) in the fluid is damped as $\frac{1}{t}$ when $t \to \infty$.

Case II.

It is assumed that $|F(x,z)|$ tends to zero as $|x|^{-\alpha}$ $(0 < \alpha < 1)$ when $|x| \to \infty$. From eq. (6.4) we then obtain:

$|G(x,z)|$ is of order $|x|^{-\alpha+2}$ when $|x| \to \infty$.

Introducing this into eq. (6.2), we find that at any fixed point in the fluid the perturbation will vanish as $t^{-\alpha}$ when $t \to \infty$.

These are the results of Case.

Acknowledgments.

The financial support given by the Royal Norwegian Council for Scientific and Industrial Research is acknowledged.

The author is grateful to the staffmembers of the Department of Applied Mathematics for their interest in this work.

Appendix.

We will show how we can use Theorem 19 in [2] to find the asymptotic behavior of $\psi(z,t)$ in this simple case.

The solution of eq. (2.12) is given by:

$$\text{(A1)} \qquad \psi(z,t) = \frac{\sinh k(z-1)}{W}\int_{-1}^{z} F_k(u)\sinh k(u+1)e^{-ikut}du + \frac{\sinh k(z+1)}{W}\int_{z}^{1} F_k(u)\sinh k(u-1)e^{-ikut}du .$$

The integrals in eq. (A1) can be written in the following way, by means of the Heaviside unit function:

$$\text{(A2)} \begin{cases} \displaystyle\int_{-1}^{z} F_k(u)\sinh k(u+1)e^{-ikut}du \\ \displaystyle\qquad = \int_{-\infty}^{+\infty} F_k(u)\sinh k(u+1)H(u+1)H(z-u)e^{-ikut}du , \\ \displaystyle\int_{z}^{1} F_k(u)\sinh k(u-1)e^{-ikut}du \\ \displaystyle\qquad = \int_{-\infty}^{+\infty} F_k(u)\sinh k(u-1)H(1-u)H(u-z)e^{-ikut}du . \end{cases}$$

The integrand in the first integral in (A2) has singularities at $u = -1$ and $u = z$. Supposing $F_k(z)$ to be analytic at every point $z \in [-1,1]$, we may write:

$$
\text{(A3)} \begin{cases} F_k(u)\sinh k(u+1) = kF_k(-1)(u+1) + \dots \\ \qquad\qquad \text{in vicinity of } u = -1. \\ F_k(u)\sinh k(u+1) = F_k(z)\sinh k(z+1) \\ \qquad + \frac{\partial}{\partial u}[F_k(u)\sinh k(u+1)]_{u=z}(u-z) + \dots \\ \qquad\qquad \text{in vicinity of } u = z\ . \end{cases}
$$

The integrand in the second integral in (A2) has singularities at $u = z$ and $u = 1$. As above we may write:

$$
\text{(A4)} \begin{cases} F_k(u)\sinh k(u-1) = F_k(z)\sinh k(z-1) \\ \qquad + \frac{\partial}{\partial u}[F_k(u)\sinh k(u-1)]_{u=z}(u-z) + \dots \\ \qquad\qquad \text{in vicinity of } u = z\ . \\ F_k(u)\sinh k(u-1) = kF_k(1)(u-1) + \dots \\ \qquad\qquad \text{in vicinity of } u = 1\ . \end{cases}
$$

Let us put:

$$
\text{(A5)} \begin{cases} F_1(u) = kF_k(-1)(u+1)H(u+1)\ , \\ F_2(u) = [F_k(z)\sinh k(z+1) + \frac{\partial}{\partial z}\{F_k(z)\sinh k(z+1)\}(u-z)]H(z-u), \\ F_3(u) = [F_k(z)\sinh k(z-1) + \frac{\partial}{\partial z}\{F_k(z)\sinh k(z-1)\}(u-z)]H(u-z), \\ F_4(u) = kF_k(1)(u-1)H(1-u)\ . \end{cases}
$$

Using Theorem 19 in [2] with $N = 2$, we find:

$$\text{(A6)}\begin{cases}\displaystyle\int_{-1}^{z} F_k(u)\sinh k(u+1)e^{-ikut}du = \sum_{m=1}^{2} \text{F.T.}\{F_m(u)\} + O\left(\frac{1}{|kt|^2}\right), \\ \qquad\qquad \text{when} \quad |kt| \to \infty, \\ \displaystyle\int_{z}^{1} F_k(u)\sinh k(u-1)e^{-ikut}du = \sum_{m=3}^{4} \text{F.T.}\{F_m(u)\} + O\left(\frac{1}{|kt|^2}\right), \\ \qquad\qquad \text{when} \quad |kt| \to \infty,\end{cases}$$

where

$$\text{(A7)}\begin{cases}\displaystyle\text{F.T.}\{F_1(u)\} = \int_{-\infty}^{+\infty} kF_k(-1)(u+1)H(u+1)e^{-ikut}du \\ \displaystyle\qquad = -\,kF_k(-1)\,\frac{e^{ikt}}{k^2t^2}, \quad \text{when} \quad |kt| \to \infty. \\ \displaystyle\text{F.T.}\{F_2(u)\} = e^{-ikzt}\left\{-\frac{F_k(z)\sinh k(z+1)}{ikt} + \frac{\frac{\partial}{\partial z}[F_k(z)\sinh k(z+1)]}{k^2t^2}\right\}, \quad \text{when} \quad |kt| \to \infty. \\ \displaystyle\text{F.T.}\{F_3(u)\} = e^{-ikzt}\left\{\frac{F_k(z)\sinh k(z-1)}{ikt} - \frac{\frac{\partial}{\partial z}[F_k(z)\sinh k(z-1)]}{k^2t^2}\right\}, \quad \text{when} \quad |kt| \to \infty. \\ \displaystyle\text{F.T.}\{F_4(u)\} = kF_k(1)\,\frac{e^{-ikt}}{k^2t^2}, \quad \text{when} \quad |kt| \to \infty.\end{cases}$$

In order to obtain expressions (A7), we have used Table 1 on page 43 in [2]. We are now in a position to determine the asymptotic behavior of $\psi(z,t)$ for large values of t. Taking into account eqs. (A1), (A6) and (A7), we find:

$$\psi(z,t) = -\frac{F_k(z)}{k^2t^2}\,e^{-ikzt} + \frac{F_k(1)}{k^2t^2}\,\frac{\sinh k(z+1)}{\sinh 2k}\,e^{-ikt}$$

$$-\frac{F_k(-1)}{k^2t^2}\,\frac{\sinh k(z-1)}{\sinh 2k}\,e^{ikt} + \ldots ,$$

which is equivalent to eq. (6.1).

References.

[1] Case, K.M.: Stability of Inviscid Plane Couette Flow.
The Physics of Fluids, Vol. 3, (1960), (p. 143).

[2] Lighthill, M.J.: Introduction to Fourier analysis and generalised functions.
Cambridge University Press (1962).

[3] Jeffreys, H.: Asymptotic approximations.
Oxford Clarendon Press (1962), (p. 14).