Department of APPLIED MATHEMATICS

Parameter estimation with the augmented Lagrangian method for a parabolic equation.

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PARAMETER ESTIMATION WITH THE AUGMENTED LAGRANGIAN METHOD FOR A PARABOLIC EQUATION

TRYGVE K. NILSSEN AND XUE-CHENG TAI

ABSTRACT. In this paper, we investigate the numerical identification of the diffusion parameters in parabolic problems. The identification is formulated as a constrained minimization problem. By using the augmented Lagrangian method, the inverse problem is reduced to a coupled nonlinear algebraic system, which can be solved efficiently with the preconditioned conjugate gradient method. Finally, we present some numerical experiments to show the efficiency of the proposed method, even for identifying highly discontinuous parameters.

1. INTRODUCTION

The purpose of this paper is to investigate some numerical methods for efficiently identifying the unknown coefficient q(x) from the following parabolic problem

(1.1)
$$u_t - \nabla \cdot (q(x)\nabla u) = f(x,t) \quad \text{in } \Omega \times (0,T),$$

with the initial-boundary condition

(1.2) $u(x,0) = u_0(x)$ in Ω and u(x,t) = g(x,t) on $\partial \Omega \times (0,T)$.

Here Ω can be any bounded domain in \mathbb{R}^d , $d \geq 1$, with piecewise smooth boundary $\partial\Omega$, and $f(\cdot, t) \in H^{-1}(\Omega)$ $t \in (0, T)$, is a given source term.

The identification process is carried out in a way that the solution u matches its observation data u_d optimally. In many practical applications, it is easier to measure the solution u at various points in the medium than to measure the parameter q(x) itself. In this work we assume that we have available measurements of u at some single points for all time $u_d(x_i, t)$, $i = 1, \ldots, n$; $t \in (0, T)$. The measurements may contain noise.

For the identification problem, the hybrid method of [4, 5, 6, 1] will be used, i.e. both the state variable u and the coefficient q will be regarded as unknown variables and the equation is considered as a constraint. The augmented Lagrangian method will be used to solve the constrained minimization problem. To find a saddle point for the

This work was partially supported by the Research Council of Norway (NFR), under grant 128224/431.

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Here Ω can be sub-bounded domain in K^* , d.g. 1. with the elemetric of locitly boundary $\partial \Omega_{i}$ and $\beta (G, 1) = K^{-1}(\Omega) \cup \in (0, T)$, is a given ended of the locit.

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Lagrangian functional, we need to solve some coupled nonlinear systems. In Chan and Tai [1], the nonlinear systems were solved by the sequential linearization approach of [6]. To solve the linearized equations, it is rather expensive to assemble the system matrices. In this work, we also use the linearization approach of [6] for time dependent problems with sparse point observations. The costs of assembling the matrices in the linearized systems are even more expensive. Thus we have proposed an approach that avoids assembling matrices. Another good point about the approach is that the meshes we use for the state variable u and the coefficient q can be independent of each other. This makes the algorithm very flexible with respect to varying the dimension of the space where q is approximated, which is dependent of the information available from the measurements.

The rest of this paper is organized as follows: The next section presents the numerical scheme that is used to solve the forward problem, and it shows how the numerical parameter estimation problem is formulated and solved with the augmented Lagrangian method. Thereafter we show how the conjugate gradient method can be used to execute the steps in the augmented Lagrangian method in a way which avoids assembling matrices. Finally we give some numerical experiments to show the efficiency of the method.

2. The Augmented Lagrangian method

We shall first present an approximation for the forward problem and then specify the augmented Lagrangian approach in a discrete setting. The forward problem can be discretized in different ways. The Lagrangian functional will be different with different approximations. In this paper, we are going to use a finite element method for spacial approximation and an implicit Euler scheme for the time variable. In real industrial applications, finite difference or finite volume methods may be used for the spacial variables and explicit schemes could be used for the time integration. The Lagrangian functional then need to be modified correspondingly.

2.1. Approximations for the forward problem. For simplicity, assume that $\Omega \subset \mathbb{R}^d$ is a polyhedral domain and \mathcal{T}^h is a regular triangulation of Ω with simplicial elements (cf. Ciarlet [3]). The superscript h denotes the diameter of the largest simplex of the triangulation. Let V_h be the standard piecewise linear finite element space over this triangulation. This is the finite element space where u and f are defined.

To define the space for q, we let \mathcal{T}^H be a similar triangulation of Ω as the one above with either simplicial or rectangular elements. Let W_H denotes the piecewise constant finite element space over this triangulation. Note that the triangulations for V_h and W_H might differ. In practical applications, the dimension for V_h is normally required to be

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2. The aucolented Lagherman Method

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3.1. Approximation of the the derivation problem (for applicit, z_{i} around that $\Omega \in \mathbb{N}^{d}$ is a calculation of the interval and T' is a results to any ibillion of Ω with simplicital dements (c) Γ books is T_{i} is a result of the N_{i} be the standard of the interval (c) Γ books is T_{i} is a subscreation N_{i} be the standard of the interval (c) Γ books is T_{i} and T_{i} is a subscreation N_{i} be the standard of the interval (c) Γ books T_{i} is a subscreation N_{i} be the standard of the interval (c) Γ books T_{i} and T_{i} T_{i} is a subscreation N_{i} be the standard T_{i} be the standard T_{i} be a standard T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i} T_{i}

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much higher than the dimension of W_H . In case that \mathcal{T}^h is a refined mesh of \mathcal{T}^H , the implementation is much simpler.

To fully discretize the parabolic system, we also need a time discretization. To get that, we divide the time interval (0,T) into M equal subintervals by using $t^n = n\tau$, n = 0, ..., M with $\tau = T/M$. We initialize by setting:

$$u_h^0 = I^h(u_0(x)) \in V_h$$

where I^h is the linear interpolation operator into V_h using the nodal point values. The discretized solution u_h^n is then defined recursively by solving:

(2.1)
$$\left(\frac{u_h^n - u_h^{n-1}}{\tau}, v\right) + (q\nabla u_h^n, \nabla v) = (f^n, v), \quad \forall v \in V_h.$$

In the above and also later, (\cdot, \cdot) is used as the inner product of $L^2(\Omega)$ and $f^n = f(x, t_n)$. The equation (2.1) defines

$$u_h = \begin{pmatrix} u_h^0 \\ \vdots \\ u_h^M \end{pmatrix} \in (V_h)^{M+1}.$$

In the rest of the paper we drop the subscript h.

In the augmented Lagrangian method, we need to regard the equation as a constraint. To minimize the equation error, we need to use a proper norm to measure the equation error. In our simulations, we have used the following two inner products to produce different measures for the equation error:

(2.2) a).
$$(u, v)_V = (u, v) + \tau (\nabla u, \nabla v),$$
 b). $(u, v)_V = (u, v).$

The corresponding norm is $\|\cdot\|_{V}^{2} = (\cdot, \cdot)_{V}$. When $\tau = O(h^{2})$, the two norms produced by the two inner products are equivalent with an equivalence constant independent of h and τ for functions from V_{h} . In such cases, we will use the inner product (2.2.b). When τ is big, then we need to use the inner product (2.2.a). In order to evaluate this norm, we need to solve a large linear system. This can be avoided by using equivalent norms produced by multigrid or domain decomposition methods as in [7].

For any $u \in (V_h)^{M+1}$ and $q \in W_H$, the discretized equation error $e = e(q, u) \in (V_h)^M$ is defined

(2.3)
$$(e^n, v)_V = (u^n - u^{n-1}, v) + \tau(q \nabla u^n, \nabla v) - \tau(f^n, v), \quad \forall v \in V_h, \quad \forall n > 0.$$

We see that e^n depends on q, u^n and u^{n-1} . For any given $q \in W_H$ and $u \in (V_h)^{M+1}$, we say that (q, u) satisfies the equation (2.1) if $e^n = 0, \forall n$.

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The corresponding norm is if the important integration of the line two norms produced by the two must product and gravitical with an equivalence constant integration of h and c for file target with an anch cases, we will use the inner product (2.2.b). When c is big then we need to use the inner product (2.2.b). When c is big then anon, we need to use the inner product (2.2.b). When c is big then norm, we need to use the inner product (2.2.b). The one is constant to norm, we need to use the inner product (2.2.b). The one is constant norm, we need to use the inner product (2.2.b). The one is constant to norm is a need to the solution of the solution of the inner to be norm in the solution of the solution of the solution of the inner to be then to be an in the solution of the solution of the solution of the methods as in the

For any $u \in (\mathbb{N}_{+})^{n+1}$ and $g \in \mathbb{N}_{n}$, the discretized equivities prove $g = e(g, u) \in (\mathbb{N}_{+})^{n+1}$ is defined

$$(2.3) \qquad (a^{*}, a)_{\mu} = (a^{*} - a^{*})_{\mu} = (a^{*} - a^{*})_{\mu} = (a^{*}, a^{*})_{\mu} = ($$

We are that e^{α} depends on α is "able of θ ". For any grant $\alpha \in W_{R}$ and $\alpha \in W_{R}$ and $\alpha \in [W_{R}]$

In an explicit form, the equation for e^n in fact can be written in the following form if we use the inner product (2.2.a):

(2.4)
$$e^{n}(q, u) = (I - \tau \Delta_{h})^{-1}(u^{n} - u^{n-1} - \tau \nabla_{h} \cdot (q \nabla_{h} u^{n}) - \tau f^{n}),$$

where the subscript h denotes that we use a discretized version of the operator. The operator $(I - \tau \Delta_h)^{-1}$ can be replaced by some corresponding operators produced by domain decomposition or multigrid methods, see [7]. If we use the inner product (2.2.b), the equation error is:

(2.5)
$$e^n(q,u) = u^n - u^{n-1} - \tau \nabla_h \cdot (q \nabla_h u^n) - \tau f^n.$$

2.2. Discretized minimization. In case that only point observations are available for the state variable u, we shall solve the following minimization problem to find the corresponding coefficient q:

(2.6)
$$\min_{e(q,u)=0} \tau \sum_{n} E(u^{n}) + \beta R(q)$$

subject to $q \in W_H$ and $u \in (V_h)^{M+1}$ satisfying $u^0 = I^h(u_0(x))$. Here

(2.7)
$$E(u^n) = \sum_i \frac{1}{2} |u^n(x_i) - u^n_d(x_i)|^2$$

and R(q) is a regularization functional which will be specified later, and β is a small positive parameter that will be determined by the mesh sizes and the noise level.

2.3. The augmented Lagrangian method. We shall use an augmented Lagrangian method to solve the constrained minimization problem (2.6). The discretized augmented Lagrange functional $L_c: W_H \times (V_h)^{M+1} \times (V_h)^M \to R$ is:

$$L_{c}(q, u, \lambda) = \tau \sum_{n} E(u^{n}) + \beta R(q) + \sum_{n} (\lambda^{n}, e^{n})_{V} + \sum_{n} \frac{c}{2} \|e^{n}\|_{V}^{2} .$$

In the above, c > 0 is a penalization constant which does not need to be very big. In the discrete setting, it is known that L_c has a saddle point and the saddle point is a minimizer for (2.6), see [4, 6, 2].

We will use the following modified Uzawa algorithm to find saddlepoints for this functional. A linear convergence for this algorithm has been proved in [6, 2].

Algorithm 2.1. (The global minimization algorithm)

- (1) Choose initial values for $\lambda_0, u_0 \in V_h$ and set k=1.
- (2) Find q_k from

(2.8)
$$L_c(q_k, u_{k-1}, \lambda_{k-1}) = \min_{q \in W_H} L_c(q, u_{k-1}, \lambda_{k-1}).$$

(3) Set
$$u_k^0 = u^0$$
 and find $u_k = \{u_k^n\}_{n=1}^M$ from

(2.9)
$$L_c(q_k, u_k, \lambda_{k-1}) = \min_{u \in V_h} L_c(q_k, u, \lambda_{k-1}).$$

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2.2. Discretized minimize them. It case that gift the discretized at one are swallable for the state valuable in an abali software hashing mininization problem to buil the loudspanding contents and an are supported at the state of the

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and R(q) is a regularization (continue) which will be specified later, and β is a mual positive paragraph that will be dependent by the mach sizes and the noise level.

2.3. The augmented Exercised as soluted. We shall use an regmented Legrangian method to solve interametricities induced relations, at least lem (2.6). The discretized segmented Legrange functional A₆ : R₂, x (X₁)^{k+1} × (K₁)^M → E^k is

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(4) Update the Lagrange-multiplier as

$$\lambda_k = \lambda_{k-1} + ce(q_k, u_k).$$

If not converged: Set k=k+1 and GOTO (2).

3. Implementation issues with the conjugate gradient method

In this section, we study an efficient method to solve the two subminimization problems in the modified Uzawa algorithm. We will use the notations $L'_c \cdot q = L'_c(q, u, \lambda) \cdot p = \frac{\partial L_c(q, u, \lambda)}{\partial q} \cdot p$ and $L'_c \cdot w = L'_c(q, u, \lambda) \cdot w = \frac{\partial L_c(q, u, \lambda)}{\partial u} \cdot w$ to denote the Gateaux derivatives of the functional $L_c(q, u, \lambda)$. Note that when writing $L'_c \cdot p$, the p indicates that we take the derivative with respect to q in the direction p. Similarly the w in $L'_c \cdot w$ indicates that we take the derivative with respect to u in the direction w. The notations $L''_c(q, u, \lambda) \cdot (p, p) = \frac{\partial^2 L_c(q, u, \lambda)}{\partial q^2} \cdot (p, p)$ and $L''_c(q, u, \lambda) \cdot (w, w) = \frac{\partial^2 L_c(q, u, \lambda)}{\partial u^2} \cdot (w, w)$ are used for the second order derivatives.

The augmented Lagrangian functional $L_c(q, u, \lambda)$ is linear with respect to λ . For fixed (u, λ) , the functional $L_c(q, u, \lambda)$ is quadratic with respect to q, and for fixed (q, λ) , the functional $L_c(q, u, \lambda)$ is quadratic with respect to u. Thus there must exist linear operators $\mathcal{A}(u) : W_H \mapsto W_H$ and $\mathcal{B}(q) : V_h \mapsto V_h$ and functions $\alpha_1(u, \lambda) \in W_H$ and $\alpha_2(q, \lambda) \in V_h$ such that

(3.1)
$$\frac{\partial L_c}{\partial q} = \mathcal{A}(u)q - \alpha_1(u,\lambda),$$

(3.2)
$$\frac{\partial L_c}{\partial u} = \mathcal{B}(q)u - \alpha_2(q,\lambda).$$

Due to the quadratic nature of the augmented Lagrangian functional, it is true that

(3.3)
$$(\mathcal{A}(u)p,p) = L_c''(q,u,\lambda) \cdot (p,p) \qquad \forall p \in V_h,$$

(3.4)
$$(\mathcal{B}(q)w,w) = L_c''(q,u,\lambda) \cdot (w,w) \qquad \forall w \in W_H.$$

In the implementations, $\mathcal{A}(u)$ and $\mathcal{B}(q)$ are matrices depending on uand q respectively, and α_1 and α_2 are vectors depending on (u, λ) and (q, λ) respectively. Thus the subproblems (2.8) and (2.9) are equivalent to solving equations of the following form:

(3.5)
$$\frac{\partial L_c}{\partial q} = \mathcal{A}(u)q - \alpha_1(u,\lambda) = 0,$$

(3.6)
$$\frac{\partial L_c}{\partial u} = \mathcal{B}(q)u - \alpha_2(q,\lambda) = 0.$$

In Chan and Tai [1], the matrices \mathcal{A}, \mathcal{B} and the vectors α_1, α_2 are assembled at each iteration and the linear systems are solved exactly. The cost of the assembling for time dependent problems is getting

$$\frac{\partial L_{0}}{\partial t} = A(a) q - a_{0} a_{0} A(b),$$
(3.3)
$$\frac{\partial L_{0}}{\partial t} = B(a) a - a_{0} a_{0} A(b),$$

$$(3.3) \quad (A(a)p,p) = L^2(q,x,h) \ (p,p) = L_2(q,x,h) \ (p,p) \ (p,p) = L_2(q,x,h) \ (p,p) \$$

$$(3.6) = (a_1 a_2 + B(a) a_1 - a_2(a_1 a_2) = 0,$$

too high. Therefore we look for ways to solve the system without assembling the matrices and the vectors. In the next subsection we will present the preconditioned conjugate gradient method, and then show that this method can be used to solve the equations $\mathcal{A}q = \alpha_1$ and $\mathcal{B}u = \alpha_2$.

3.1. The preconditioned conjugate gradient method. The preconditioned conjugate gradient method solves the equation Ax = bwith a symmetric positive definite preconditioner B. The algorithm is written as:

$$k = 0, x_0 = 0, r_0 = b, z_0 = B^{-1}r_0, p_1 = z_0,$$

while $r_k \neq 0,$

 $\alpha_{k} = r_{k-1}^{T} z_{k-1} / p_{k}^{T} A p_{k},$ $x_{k} = x_{k-1} + \alpha_{k} p_{k},$ $r_{k} = r_{k-1} - \alpha_{k} A p_{k} = b - A x_{k},$ $Solve \quad B z_{k} = r_{k},$ $\beta_{k+1} = r_{k}^{T} z_{k} / r_{k-1}^{T} z_{k-1},$ $p_{k+1} = z_{k} + \beta_{k+1} p_{k},$

end

 $x = x_k$.

In the algorithm, we do not need to form the matrix A. For simulations, we just need subroutines to calculate b - Ax and $p^T Ap$ for given vectors x and p.

Domain decomposition and multigrid methods shall be used for the preconditioner B. When using domain decomposition methods, only very small subproblems defined on the subdomains need to be solved. If multigrid method is used, we do not need to solve any systems of linear algebraic equations. We just need to update the residuals of some equations over all the nodal points from the different levels.

In order to use conjugate gradient methods to solve equation (2.8) and (2.9), it is enough to design some subroutines to calculate (Ap, p), $(\mathcal{B}w, w)$ and the corresponding residuals for the two equations with given p and w.

3.2. Minimization with the conjugate gradient method. All we need to solve equation (3.5), is to calculate $\mathcal{A}q - \alpha_1$ and $p^T \mathcal{A}p$ for given vectors q and p. Similarly, we need to calculate $\mathcal{B}u - \alpha_2$ and $w^T \mathcal{B}w$ for given vectors u and w to solve equation (3.6).

; From the definition of L_c , we get that

(3.7)
$$L'_c \cdot p = \beta R'(q) \cdot p + \sum_n (\frac{\partial e^n}{\partial q} \cdot p, \lambda^n)_V + c \sum_n (\frac{\partial e^n}{\partial q} \cdot p, e^n)_V.$$

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too ingo. Unstators we look for ways to some the restor without assembling the matrices and the vectors. In the next subsection we will present the proceeditioned analogate gradient methods and then abov that this method can be used to solve the equations $A_0 = c_0$ and $E_0 = c_0$.

3.1. The precessitioned conjugate gradient method. The preconditioned conjugate realism method course the searchest Az = 5 with a premetric positive default preconditions of The algorithm is written as

> $k = 0, x_0 = 0, x_0 = 0, x_0 = B^{-1} x_0, x_0 = x_0$ while $x_0 = 0$

In the algorithm, we do not used to form the matrix A. For similar ious, we just need mixedonise to chick and 6 – Az and 9⁷ .55 for given rectors z and p.

Domain decomposition and multiple of metiones shall be used for the preconditioner *B*. When vising domain decomposition methods, cars very small subproblems defined on the subdomains must to be solved. If multiple method is used, we do not used to adve any detraits of inear algebraic equations. We just used to togete the residuals of some equations over all the model points from the different lovels.

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3.2 Minimization with the conjucte gradient method. All as need to solve equation (3.3), is for each equation of a solve equation (3.3), is for each and of a solve equation (3.4), is for each and the solve equation of and p. Scolariy, we need so each integers for each and the solve equation of the solve equation (3.5).

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Thus, we need to calculate d_1^n for all n to get ().

In order to use conjugate gradient method to solve (3.6), we d Define $d_1^n = \frac{\partial e^n}{\partial q} \cdot p$. From the definition of e^n , we see that d_1^n satisfies

(3.8)
$$(d_1^n, v)_V = \tau(p\nabla u^n, \nabla v) \qquad \forall v \in V_h,$$

and thus

(3.9)
$$L'_c \cdot p = \beta R'(q) \cdot p + \sum_n (d_1^n, \lambda^n + ce^n)_V$$
$$= \beta R'(q) \cdot p + \sum_n \tau(p \nabla u^n, \nabla(\lambda^n + ce^n)).$$

Assume that $\{\phi_j\}$ are the basis functions for W_H and $\mathbf{r}_1 = [r_1(j)]$ is the residual vector for equation (3.5), i.e. $r_1(j) = L'_c \cdot \phi_j$. From the calculations above, we see that $r_1(j)$ can be calculated by

$$r_1(j) = \beta R'(q) \cdot \phi_j + \sum_n \tau(\phi_j \nabla u^n, \nabla(\lambda^n + ce^n)).$$

When solving (3.5), u^n and λ^n are known and we only need to compute e^n for each n to get the residual \mathbf{r}_1 from the above formula.

To calculate $(\mathcal{A}p, p)$ for a given $p \in W_H$, we use the quadratic property (3.3) to get that

(3.10)
$$(\mathcal{A}p, p) = L_c''(q, u, \lambda) \cdot (p, p) = \beta R''(q) \cdot (p, p) + c \sum_n (d_1^n, d_1^n)_V$$

Thus, we need to calculate d_1^n for all n to get $(\mathcal{A}p, p)$.

In order to use conjugate gradient method to solve (3.6), we define $d_2^n = \frac{\partial e^n}{\partial u} \cdot w$. We see that d_2^n satisfies

(3.11)
$$(d_2^n, v)_V = (w^n - w^{n-1}, v) + \tau(q\nabla w^n, \nabla v) \qquad \forall v \in V_h.$$

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Thus, we need to calculate of the use to get ().
In order to use conjugate gradient method to solve (3.5), we d Define of the State of the statement of the statement of the statement.

(2.8) = = (17, 2) = = (2.5) / 10 = (2.6)

end then

(2.9) = AR(q) - p+ (q, A" + q) h

Assume that $\{\phi_i\}$ are the basis functions for N_{in} and $r_i = (r, \phi_i)$ is the residual vector for equistion (3, b), i.e. $r_i(i) = i$, ϕ_i , inom the calculations above, we see that $r_i(i)$ can be calculated by

 e^{α} for each α to get the real-factor from the atoms formula: To calculate (.4g, g) for a given $g \in [k_{B}]$, we use the endrate propercy (3.3) to get that

 $3.10(-6.4n,p) = i L_{0}^{2}(q, [q, N] + (p, n) = p M^{2}(q) + (n) M^{2}(q) + c \sum_{i} (q, i) p_{i}$

Thus, we need to calculate of an all a to get (Apro) In order to use conjugate draticate needbad to solve (3.5), we define If = 62, ar We are that all samelies

Similarly, we have the following formulations for calculating the residual for equation (3.6):

$$L'_{c} \cdot w = \sum_{n} \tau E'(u^{n}) \cdot w^{n} + \sum_{n} \left(\frac{\partial e^{n}}{\partial u} \cdot w, \lambda^{n}\right)_{V} + c \sum_{n} \left(\frac{\partial e^{n}}{\partial u} \cdot w, e^{n}\right)_{V}$$

$$= \tau \sum_{n,i} \left(u^{n}(x_{i}) - u^{n}_{d}(x_{i})\right) w^{n}(x_{i}) + \sum_{n} \left(d^{n}_{2}, \lambda^{n} + ce^{n}\right)_{V}$$

$$= \tau \sum_{n,i} \left(u^{n}(x_{i}) - u^{n}_{d}(x_{i})\right) w^{n}(x_{i}) + \sum_{n} \left(w^{n} - w^{n-1}, \lambda^{n} + ce^{n}\right) + \sum_{n} \tau(q \nabla w^{n}, \nabla(\lambda^{n} + ce^{n}))$$

$$(3.12) = \tau \sum_{n,i} \left(u^{n}(x_{i}) - u^{n}_{d}(x_{i})\right) w^{n}(x_{i}) + \sum_{n} \tau(q \nabla w^{n}, (\lambda^{n} + ce^{n}) - (\lambda^{n+1} + ce^{n+1})) + \sum_{n} \tau(q \nabla w^{n}, \nabla(\lambda^{n} + ce^{n})),$$

where we have defined $\lambda^{M+1} = ce^{M+1} = w^0 = 0$ to simplify the notation. For the second order derivative it is true that:

$$(\mathcal{B}w, w) = L_c'' \cdot (w, w) = \tau \sum_n E''(u^n) \cdot (w, w) + c \sum_n (d_2^n, d_2^n)_V$$

(3.13)
$$= \tau \sum_{n,i} (w^n(x_i))^2 + c \sum_n (d_2^n, d_2^n)_V.$$

Assume that $\{\psi_j\}$ are the basis functions for V_h and $\mathbf{r}_2 = [r_2(n, j)]$ is the residual vector for equation (3.6) which contains residuals on all the time levels for all the nodal basis functions. Then $r_2(n, j)$ can be calculated by

$$r_2(n,j) = \tau \sum_i (u^n(x_i) - u^n_d(x_i))\psi_j(x_i)$$
$$+(\psi_j, (\lambda^n + ce^n) - (\lambda^{n+1} + ce^{n+1}))$$
$$+\tau(q\nabla\psi_j, \nabla(\lambda^n + ce^n)).$$

These calculations can be used in the conjugate gradient method to solve the equations $Au = \alpha_1$ and $Bq = \alpha_2$ to execute (3.5) and (3.6) in the modified Uzawa algorithm.

3.3. Efficient minimization algorithms. The most time consuming part of the minimization algorithm is the solving of (3.6), i.e. the minimization of u. This is because u(x,t) is a function of both space and time, and therefore have most degrees of freedom. Usually the dimension of the space V_h is bigger than the dimension of W_H . In this section we suggest two alternative minimization algorithms to the one

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Similariy, we have the following formulations for enterlaying the residual for externion (2.4):

where we have defined $\lambda^{Met} = e^{Met} = w^2 = 0$ to simplify the planetics. For the second order dirivative w is true to at

$$(Bu, u) = E_{i}^{n} \cdot (u, u) = r \sum_{i=1}^{n} E^{n}(u^{n}) \cdot (u, u) + c \sum_{i=1}^{n} (i_{i}^{n}, u_{i}^{n})$$

$$(3.13)$$

Assume that $\{\psi_j\}$ are the hesis functions for W_i and $r_2 = [r_2(r_{k,j})]$ is the residual vector for equation (3.6) where contains residuals in all the time levels for all the model rasis instructs. Thus $r_2(r_i, j)$ can be calculated by

These calculations can be fixed in the conjugate produced method to noive the equations $Au = u_1$ and $By = u_2$ forextruite (3.5) and (3.6) in the modified Graves algorithm.

3.3. Efficient minimization digorithmost The most time constitutes part of the minimization digorithm is the solving of (3.6), i.e., give minimization or u. This is browned with it is a function of both space and time, and therefore have most degrees of finishem. The ally the dimension of the space is is higger than the dimension of N $_{\rm T}$ is this section we suggest two sectors whether inductions algorithms to the finish

PARAMETER ESTIMATION WITH THE AUGMENTED LAGRANGIAN 9

presented in the section above. The new minimization algorithms will not minimize (2.9) exactly as in Algorithm 2.1. Instead, we are trying to use some cheaper and approximate solver for the sub-minimization problem (3.6).

3.3.1. Matching minimization algorithm. For the forward problem (2.1), it is known that u^{n-1} must be computed before we can compute u^n . This is not correct for the sub-minimization problem (3.6). For (3.6), all the u^n are coupled to each other. If we first compute u^1 and then take the computed u^1 to compute u^2 as described in the following and continue, the obtained solution $u = \{u^n\}_{n=0}^M$ is not a minimizer for (3.6), but is a rather good approximation for the minimizer of (3.6).

To be more precise, let us define

$$F(u^{n}, u^{n-1}) = \frac{\tau}{2} E(u^{n}) + (\lambda^{n}, e^{n})_{V} + \frac{c}{2} ||e^{n}||_{V}^{2}.$$

It is clear that $F(u^n, u^{n-1})$ also depends on λ and q. Since we only use this notation for the solving of (3.6), we will omit q and λ in $F(u^n, u^{n-1})$ for notational simplicity. It is easy to see that

$$L_c(q, u, \lambda) = \sum_n F(u^n, u^{n-1}) + \beta R(q).$$

The following algorithm will be used as a replacement for Algorithm 2.1:

Algorithm 3.1. (The matching minimization algorithm)

- (1) Choose initial values for $\lambda_0, u_0 \in V_h$ and set k=1.
- (2) Find q_k from

$$q_k = \arg\min_{q \in W_H} L_c(q, u_{k-1}, \lambda_{k-1}).$$

(3) Set $u_k^0 = u^0$ and find $u_k = \{u_k^n\}_{n=1}^M$ sequentially for $n = 1, 2, \cdots M$ such that

(3.14)
$$u_k^n = \arg\min_{v \in V_b} F(v, u_k^{n-1}).$$

(4) Update the Lagrange-multiplier as

$$\lambda_k = \lambda_{k-1} + ce(q_k, u_k).$$

If not converged: Set k=k+1 and GOTO (2).

When solving (3.14), the newest values for q and λ are used. Step (3) in Algorithm 3.1 defines $u_k = \{u_k^n\}_{n=0}^M$ sequentially for the time steps.

To use the conjugate gradient method to solve this new minimization problem we should do some calculations similar to those in the previous section. The difference is that we now take the Gateaux derivative in the direction of one time level w^n instead of in all time levels.

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presented in the section above. The new multimization appointants will not minimize (2.9) exactly as in Algorithm 2.1. instead, we use trying is use some chapter and approximate solver for the sub-inimizinition problem (3.6).

3.3.1. Matching any matching approximation for the forward promisin (2.1), is is known that e^{-1} must be computed before we can compute e^{-1} . This is not correct for the the ratio minimization problem (3.6). For (3.6), all the e^{0} are completed fortation communication problem (3.6). For (3.6), take the computed of contacts other. If we first compute e^{1} and then take the computed of contacts other. If we first compute e^{1} and then continue, the obtained solution e^{-1} as assoribed in the following and (3.6), but is a rather grant spiritor when a first in the following and To be more precise for the the contact of (3.6).

It is clear that $F(u^{n}, u^{n-1})$ also depends on λ and g. Since we only use this notation for the solving of (3.6), we will omit g and λ in $F(u^{n}, u^{n-1})$ for notational simulation. It is easy to use that

$$L_{\varepsilon}(q,u,\lambda) = \sum_{i=1}^{\infty} F(w^{i}, u^{i-1}) + \mathcal{L} H(q)$$

The following algorithm will be used as a replacement for Algorithm

Ligorithm 3.1. (The mulching iniganization algorithm)

(1) Choose minal usings for he on 6 H and set helt.

(2) Find a firm

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(3) Set $u_1^2 = u^2$ and find $u_n = \{u_1^2\}_{n=1}^n$ sequent any for $u = 1, 2, \cdots, M$ such that

 $L(4) = \{a_1, a_2\} = \{a_2, a_3\} + \{a_4, a_2\} \}.$

(4) Update the Lastance-multiplier as

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When solving (3.13), the actedit values for y and X are used. Step 3) in Algorithm 3.1 defines up + (cf. h., argumentally for the time acre.

To use the conjugate gradient method to raise this new minimizer to problem we should do tonae calculations similar to theore in the previous section. The difference is that we sam only the Catesur, derivative in the direction of one time ferrel of instead of in all three lovels. We define $d_3^n = (e^n)' \cdot w^n = \frac{\partial e^n}{\partial u^n} \cdot w^n$. Then d_3^n satisfies (3.15) $(d_3^n, v)_V = (w^n, v) + \tau(q\nabla w^n, \nabla v) \quad \forall v \in V_h.$

The Gateaux derivative of $F(u^n, u^{n-1})$ in the direction w^n is:

$$F' \cdot w^{n} = \tau E'(u^{n}) \cdot w^{n} + \left(\frac{\partial e^{n}}{\partial u^{n}} \cdot w^{n}, \lambda^{n}\right)_{V} + c\left(\frac{\partial e^{n}}{\partial u^{n}} \cdot w^{n}, e^{n}\right)_{V}$$
$$= \tau \sum_{i} (u^{n}(x_{i}) - u^{n}_{d}(x_{i}))w^{n}(x_{i}) + (d^{n}_{3}, \lambda^{n} + ce^{n})_{V}$$
$$(3.16) \qquad = \tau \sum_{i} (u^{n}(x_{i}) - u^{n}_{d}(x_{i}))w^{n}(x_{i})$$
$$+ (w^{n}, \lambda^{n} + ce^{n}) + \tau(q\nabla w^{n}, \nabla(\lambda^{n} + ce^{n})),$$

and the second order derivative is

(3.17)
$$F'' \cdot (w^n, w^n) = \tau E''(u^n) \cdot (w^n, w^n) + c(d_3^n, d_3^n)_V = \tau \sum_i (w^n(x_i))^2 + c(d_3^n, d_3^n)_V.$$

3.3.2. A Gauss-Seidel algorithm. Another alternative to find an approximate solution for (3.6) is to use the following block Gauss-Seidel algorithm to compute u_k^n :

Algorithm 3.2. (The block Gauss-Seidel minimization algorithm)

- (1) Choose initial values for $\lambda_0, u_0 \in V_h$ and set k=1.
- (2) Find q_k from

$$q_k = \arg\min_{q \in W_H} L_c(q, u_{k-1}, \lambda_{k-1}).$$

(3) Set $\tilde{u}_0 = u_{k-1}$ and m = 1. While $||\mathbf{r}_2|| \ge \epsilon$ do: Set $\tilde{u}_m^0 = u^0$ and find $\tilde{u}_m = {\{\tilde{u}_m^n\}_{n=1}^M}$ sequentially for $n = 1, 2, \cdots M$ such that

3.18)
$$\tilde{u}_m^n = \arg\min_{v \in V_1} (F(v, \tilde{u}_m^{n-1}) + F(\tilde{u}_{m-1}^{n+1}, v)).$$

m = m + 1.End while. Set $u_k = \tilde{u}_m$.

(4) Update the Lagrange-multiplier as

$$\lambda_k = \lambda_{k-1} + ce(q_k, u_k).$$

If not converged: Set k=k+1 and GOTO (2).

Here $\|\mathbf{r}_2\|$ is the L^2 -norm of the residual from the previous section (3.6), and ϵ is the stopping criteria. When solving (3.18), the newest values for q and λ are used.

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We use the same notations as in the previous subsection. The Gateux derivative of $F(u^n, u^{n-1}) + F(u^{n+1}, u^n)$ with respect to u^n in the direction w^n is calculated:

$$F'(u^{n}, u^{n-1}) \cdot w^{n} + F'(u^{n+1}, u^{n}) \cdot w^{n}$$

$$= \tau \sum_{i} (u^{n}(x_{i}) - u^{n}_{d}(x_{i}))w^{n}(x_{i})$$

$$+ (w^{n}, \lambda^{n} + ce^{n}) + \tau(q\nabla w^{n}, \nabla(\lambda^{n} + ce^{n}))$$

$$+ ((e^{n+1})' \cdot w^{n}, ce^{n+1} + \lambda^{n+1})_{V}$$
(3.19)
$$= \tau \sum_{i} (u^{n}(x_{i}) - u^{n}_{d}(x_{i}))w^{n}(x_{i})$$

$$+ (w^{n}, ce^{n} + \lambda^{n}) + \tau(q\nabla w^{n}, \nabla(\lambda^{n} + ce^{n}))$$

$$- (w^{n}, ce^{n+1} + \lambda^{n+1}),$$

and the second order derivative is calculated:

(3.20)
$$F''(u^n, u^{n-1}) \cdot (w^n, w^n) + F''(u^{n+1}, u^n) \cdot (w^n, w^n) = \tau \sum_i (w^n(x_i))^2 + c(d_2^n, d_2^n)_V + c(w^n, w^n)_V.$$

4. NUMERICAL EXPERIMENTS

We now show some numerical experiments on the proposed method for parameter identification. For the tests, we have taken $\Omega = [0, 1] \times$ $[0, 1], T = 0.01, u_0(x) = \sin(\pi x) \cos(\pi y), g(x) = 0$ and q(x) is piecewise constant:

(4.1)
$$q(x) = \begin{cases} q_1, & x \in [0, 0.5] \times [0, 0.5] \\ q_2, & x \in [0, 0.5] \times [0.5, 1] \\ q_3, & x \in [0.5, 1] \times [0, 0.5] \\ q_4, & x \in [0.5, 1] \times [0.5, 1]. \end{cases}$$

In the examples $q_i = i, i = 1, ..., 4$ unless otherwise defined. The source function is

(4.2)
$$f(x) = \sum_{i=1}^{4} \delta(x - x_i) - 4\delta(x - x_5),$$

where x_i for i = 1, ..., 4 are the corners and x_5 is the center of Ω and δ is Diracs delta function.

The domain is triangulated by first dividing it into squares of size $h \times h$. Then each square is divided into two triangles by the diagonal with positive slope to get \mathcal{T}^h . The element functions u(x,t) and f(x,t) are defined over this triangulation with linear finite element functions. The number of time steps is $M = \frac{T}{\Delta t}$. Square meshes \mathcal{T}^H are used for approximating q, and H is used to denote the mesh size. Moreover, the finite element functions for q are piecewise constants over each square.

We use the same set differs is in the previous subsection. The Gaisers defination of $E(u^0, u^{-1}) + E(u^{-1}, u^0)$ with respect to u^0 in the direction u^{-1} is calculated:

and the second order derivative is culculated:

$$(3.20) = -\sum_{i=1}^{n} (\omega^{n}(\omega_{i}))^{2} + c(a_{i}) a^{2} + (a_{i})^{n} + c(a_{i})^{n} + c(a_{$$

We now show some numerical axperiments on the proposed methods for parameter identification. For the close, we have taken $\Omega = [0, 1]$ × [0, 1], T = 0.01, $u_0(x) = sin(\pi x)$, $cos(\pi y)$, g(x) = 0 and g(x) is pictewise constant.

In the examples $q_i = q_i = 1$, ..., and so differniat defined. The source function is

$$(4.2) \qquad \qquad f(x) = \sum f(x - x_1) - 4f(x - x_2).$$

where z_i for i = 1, ..., i are the ormers and z_i is the enter of Ω and δ in Diracs delta function.

The domain is treaspolated by first dividing it into equinve of sim $h \times h$. Then each equate is divided monutorizangles by the disconal with possible along to get T^{λ} . The elegent functions u(x,t) and f(x,t)are defined over this triangulation with linear finite element functions The number of time steps is $h_{T} = \frac{1}{\sqrt{2}}$. Equare methes T^{λ} are used in approximating q, and H is used to define the methes T^{λ} are used in finite element functions for each produce the methes T^{λ} are used in finite elements functions for each produce the methes over each square finite elements functions for each produce the methes over each square. In the implementations, the mesh \mathcal{T}^h that we use for approximating u, is always a refinement of the mesh \mathcal{T}^H we use for q.

With the numerical method described in Section 2, the forward problem can be solved. The solution from this, u, will then be used as a source for the observations that our algorithms will use to recover the permeability, q.

In Example 4.5 we will add normally distributed noise to the observations in a multiplicative way, i.e.:

(4.3)
$$u_d(x_i, t) = u(x_i, t) + \sigma \ u(x_i, t) \ rand(x_i, t).$$

Here rand (x_i, t) is a vector of normally distributed numbers with expectation 0 and standard deviation 1. We refer to σ as the noise level.

In the figures in the following examples we illustrate the convergence rates of the Uzawa algorithms. In all examples we plot $||q_k - q||_{L^2}$ with increasing k-value, where q is the true permeability. In Example 4.1 we also illustrate the convergence rates of u in L^2 -norm, i.e. we plot $\int_0^T ||u_k - u||_{L^2} dt$. In the examples the Uzawa algorithm has been stopped by inspection of these plots.

As an initial value for q, we use q_0 equal to a constant. The constant that is used is the average of the exact permeability, i.e. $q_0 = \frac{1}{|\Omega|} \int_{\Omega} q \, dx$. The initial value for u is the spatial linear interpolation of $u_d(x_i, t)$. The *c*-value is found by trial and error.

In the following examples we have observed convergence when either the inner products (2.2.a) or (2.2.b) are used. We have seen that the conjugate gradient method converges in fewer iterations when (2.2.a)is used, but since (2.2.b) is cheaper this has been preferable in our examples.

For simplicity, we have taken $R(q) = \int_{\Omega} q^2 dx$. For examples without noise we can set the regularization parameter to $\beta = 0$.

As a preconditioner B for the conjugate gradient method we have used domain decomposition to approximate the operator $(I - \tau \Delta)^{-1}$. This is tested with and without coarse grid.

The stopping criteria for the conjugate gradient method is that the relative L^2 -norm of the residual is $\leq \epsilon$. In the following examples we have chosen $\epsilon = 10^{-6}$

In all examples except Example 4.4 we have used $H = \frac{1}{4}$, $h = \frac{1}{8}$, M = 20 and T = 0.01. In the center of each square element of \mathcal{T}^{H} , there is one observation point. That means we have 16 observation points for u_d and 16 parameters representing q.

Example 4.1. In the first example we use the global minimization Algorithm 2.1. In this example the *c*-value was set to $8 \cdot 10^{-5}$. The convergence rate of *q* is shown in Figure 1, and the convergence rate of *u* is shown in Figure 2.

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In the upplementations, the mesh T^{2} shat we use for approximating u, is always a refinement of the mesh T^{2} we use for c_{i}

With the numerical availand described in Section 2, the forward problem, can be solved. The solution from this, a, will then be used as a source for the observations that our significant will ask to problem the permanding, a

In Exclusive 4.5 we will add the tenthily contributed notes to the observations in a multiplicative way. Let

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Here rand (x_0, t) is a vector of normally distributed sumbers with expectation 0 and standard deviation 1. We refer to σ as the exist level.

rates of the Uzera eigenfunns, in all examples we plot have grant with increasing h-value, where q is the true permeability, in Example 4.1 we also illustrate the semicidence rates of u in L^2 -more, i.e. we plot $\int_{0}^{T} ||u_{0} - u||_{L^{2}} dt$. In the transition the Uzera algorithm less been stopped by inspection of these plots.

As an unital value for a vector of equal to a constant. The constant that is used is the averagy of the sector performability, i.e. as = mulo y de The initial value for a is the graftal insert interpolation of ald y, A. The costing is found by trial and error.

in the following examples we have electrical prevergence when sither the inner products (2.2.a) or (2.2.b) are used. We have seen that the conjugate gradient method converges in fever investives when (2.2.a) is used, but more (2.2.b) is cheapet this into term methodie in our examples.

r or simplicity, we have taken $f(q) = \int_{\mathbb{R}} q^2 dz$. For examples we have to get the regularization parameter in g = 0.

As a preconditioner of the the current and address method we have used domain decomposition to approximate the operator (/ -, r, 2); ^. This is tested with and without coarse grid.

The stopping criteria for the conjugate gradient method is that the relative L^2 -norm of the residual is C or in the following examples we have chosen $\epsilon = 10^{-2}$.

In all complete encept leasangle 4.5 we have used $H = \frac{1}{2}$, $h = \frac{1}{2}$, h = 10 and T = 0.01. In the centre of each square demonstrated $T^{(0)}$, there is one observation point. That, means we have 16 circuration points for u_{i} and 16 parameters representing u_{i} .

Example 4.1. In the fifth scample we use the global minipation Algorithm 2.1. In this stample the c-value was set to 3 · 10⁻² The convergence rate of gissinger in Figure 1, and the convergence rate of u is shown in Figure 2.

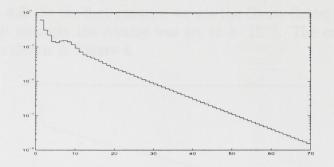


FIGURE 1. $||q_k - q||_{L^2}$ versus k. Logarithmic scale on the vertical axis.

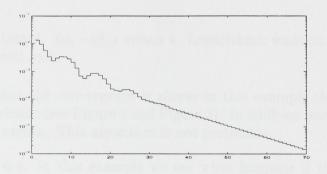


FIGURE 2. $\int_0^T ||u_k - u||_{L^2} dt$ versus k. Logarithmic scale on the vertical axis.

Example 4.2. In this example we use the matching minimization Algorithm 3.1. The *c*-value was set to $8 \cdot 10^{-5}$. The convergence rate of q is shown in Figure 3.

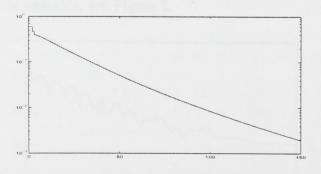
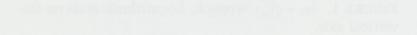


FIGURE 3. $||q_k - q||_{L^2}$ versus k. Logarithmic scale on the vertical axis.

We see that this algorithm takes more iterations to converge (see Figure 3). Since each iteration is much cheaper than in the previous example, this algorithm is much quicker.

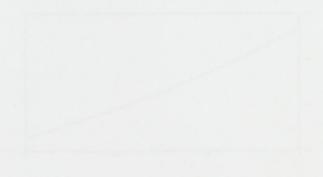






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Example 4.2. In this example we not the matching dualwerken 41. gorithm 3.1. The c-value was set to 3 10 °. The convergence rate of 9 is shown in Figure 3.



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We are that this theoritain being more installing to converge (see Figure-3). Since and installion is much cheaper that in the previous example, this algorithm is much quicker **Example 4.3.** In the first example we use the Gauss-Seidel Algorithm 3.2. In this example the *c*-value was set to $8 \cdot 10^{-5}$. The convergence rate of *q* is shown in Figure 4.

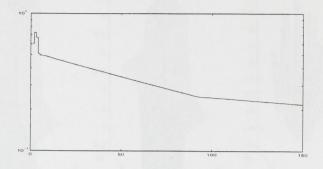


FIGURE 4. $||q_k - q||_{L^2}$ versus k. Logarithmic scale on the vertical axis.

We see that the convergence is slower in this example than for the matching scheme (see Figure 4 and Figure 3). In addition each iteration is more expensive. This algorithm is not preferable.

Example 4.4. In this example we see what happens if the number of parameters representing q is bigger than the number of observation points for u. The example is specified with $H = \frac{1}{8}$, $h = \frac{1}{16}$ M = 10, T = 0.01 and $c = 2 \cdot 10^{-6}$. $H = \frac{1}{8}$ gives 64 parameters representing q. The observation points u_d is taken to be the corners of \mathcal{T}^H in the interior of Ω , i.e. we have 49 uniformly distributed observation points for u. The global minimization Algorithm 2.1 is used to identify q(x). We see that it takes more iterations to converge and that the convergence is unstable, see Figure 5.

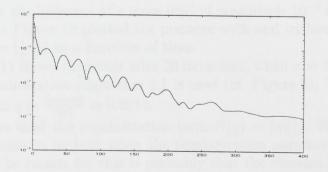


FIGURE 5. $||q_k - q||_{L^2}$ versus k. Logarithmic scale on the vertical axis. We have used 64 parameters representing q and 49 observation points for u.

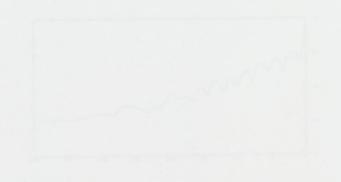
Example 4.3. In the first example we use the Gauss Scielel Algorishm 3.2. In this example the z-value was set to 8 MC². The convergence rate of *a* is shown in Figure 4.



1 GURE 4. [2, -y], a versus a Lagarithmic scale on the

We see that the convergence is slower in this example than for the matching scheme (see France 4 and 1 ignes 3). In addition each iteration is more expensive. This algorithm is not preferable.

Example 4.4. In this example we see what impress if the mather of parameters representing q is began than the number of observation points for u. The aximple is specified with $R = \frac{1}{2}$, $h = \frac{1}{2}$, M = 10, T = 0.01 and $c = 2 \cdot 10^{-6}$. $R = \frac{1}{2}$ gives 54 parameters representing q. The observation points as is being to be the common of T^{0} in the interior of Ω . Let us been detected with we can be the common of T^{0} in points for u. The glassel minimum term is a satisfied with the points for u. The glassel minimum term is a satisfied the 2.1 is used to identify $q(\tau)$. We are that it follow more instants to converge and that the convergence is morable are frequence.



Fround 5. [[eg = 9][2) versies k. Logarithmic male on the vertical axis. We have used 64 parameters representing g and 49 observation points for a

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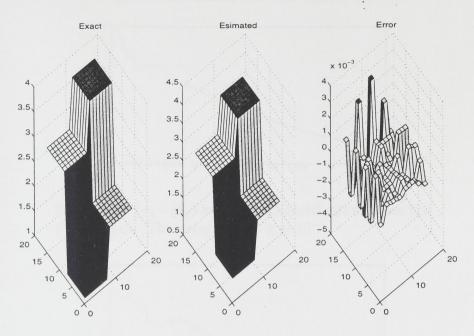


FIGURE 6. The exact- and approximate solution and error in permeability q(x) after 400 iterations.

Example 4.5. We repeat the test in Example 4.1 and Example 4.2, but add noise to the observations (see Equation (4.3)). The convergence rate of q when the noise level is $\sigma = 10^{-3}$ is shown in Figure 7, and the same convergence rate when the noise level is $\sigma = 10^{-2}$ is shown in Figure 8. Here the global minimization Algorithm 2.1 is used to identify q(x).

In Figure 9 we show the convergence rate when the matching minimization Algorithm 3.1 is used, and the noise level is $\sigma = 10^{-2}$. The *c*-value was here set to $2 \cdot 10^{-5}$.

To show the influence of a noise level of magnitude 10^{-2} on the data, we have in Figure 10 plotted the pressure with and without noise in a point $x_1 = (\frac{1}{8}, \frac{1}{8})$ as a function of time.

Figure 11 shows the result after 30 iterations, when $\sigma = 10^{-2}$ and the global minimization Algorithm 2.1 is used (cf. Figure 8). The relative L^2 error in q is $\frac{||q_{30}-q||}{||q||} \approx 0.0014$.

We have used the regularization term $R(q) = ||q||_{L^2}^2$. However, the results seems to be best when the regularization parameter is chosen $\beta = 0$. The reason for this is probably that the dimension on \mathcal{T}^H is relatively small in our examples.

Example 4.6. In oil reservoirs the permeability does often have very large jumps. In this example we try Algorithm 2.1 with permeability as described in (4.1) with $q_i = 10^{i-3}$, i = 1, ..., 4.

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FIGURE 6. The exasts and appropriate colution and arport in permeability of x) sher 400 department.

Example 4.5. We repeat the test in Example 11 and Example 4.5. but add noise to the observations (see Equation (4.3)). The convergence rate of q when the noise level is $c = 10^{-1}$ is shown in Figure 7, and the same convergence rate when the anise level is $c = 10^{-9}$ is shown in Figure 8. Here the global minimum to Agrifian 2.1 is much to identify q(x).

In Figure 9 we show the convergence rate when the matching with inizacion Algorithm 3.1 is used, and the noise level is $\sigma = 10^{-4}$. The c-value was here set to 2 - 10^{-4} .

To show the influence of a noise level of magnitude 10^{-2} on the data, we have in Figure 10 protted the pressure with and without noise in a point $x_1 = (\frac{1}{2}, \frac{1}{2})$ as a function of time.

Figure 11 shows the result after 30 instations, when $\sigma = 10^{-2}$ and the global minimization. Afgurithm 2.1, is used (cf. Figure 3). The relative L^2 error in σ is $\frac{1}{2} \frac{1}{2} \frac{1$

We have used the requisitration term $N(q) = [q]_{q,q}^{2}$. Morever, the results second the requisitration term $Q(q) = [q]_{q,q}^{2}$. Morever, the results second to best when the file regularization parameter is chosen R = 0. The reason for this is probably that the dimension on T^{2} is relatively small in our examples

Example 4.6. In oil reservoirs the pigmentility flock often have very large jumps. In this testingile vertes Algerning 2.1 with points ability as described in (4.1) with a = 100 % I = 1.000

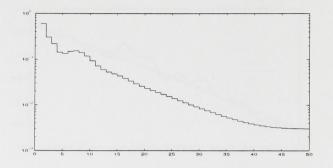


FIGURE 7. $||q_k - q||_{L^2}$ versus k. Logarithmic scale on the vertical axis. The noise level is $\sigma = 10^{-3}$.

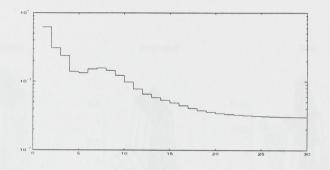


FIGURE 8. $||q_k - q||_{L^2}$ versus k. Logarithmic scale on the vertical axis. The noise level is $\sigma = 10^{-2}$.

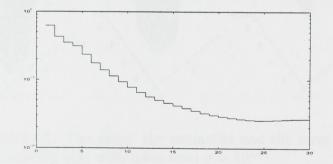


FIGURE 9. $||q_k - q||_{L^2}$ versus k. Logarithmic scale on the vertical axis. The noise level is $\sigma = 10^{-2}$ and the matching minimization Algorithm 3.1 is used.

In this example the c-value was set to $2.6 \cdot 10^{-5}$. The convergence rate of q is shown in Figure 12. We see that the convergence is a little bit slower than in the previous examples. In addition every iteration is about twice as costly as in Example 4.1, because the conjugate gradient method converges slower.



-Floure 7. [[q. - q]g. veste k. Lagarithanic scale on the vertical axis. The noise level is $\sigma = 10^{-4}$



FOURE 3. $||q_k - q||_{2}$ without k. Logarithmic scale on the vertical axis. The paths level is $\sigma = 16^{-2}$.



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In this example the e-value was set to $2.6 \cdot 10^{-6}$. The convergence rate of q is shown in Figure 12. We see that the convergence is a little bit slower than in the provint terminales. In addition every iteration is about twice to easily as in Example 5.1, because the conjugate gradient method, converger slower.

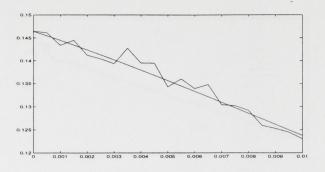


FIGURE 10. The pressure with and without noise in position $x_1 = (\frac{1}{8}, \frac{1}{8})$ i.e. $u(x_1, t)$ and $u_d(x_1, t), t \in (0, T)$. The noise level is $\sigma = 10^{-2}$.

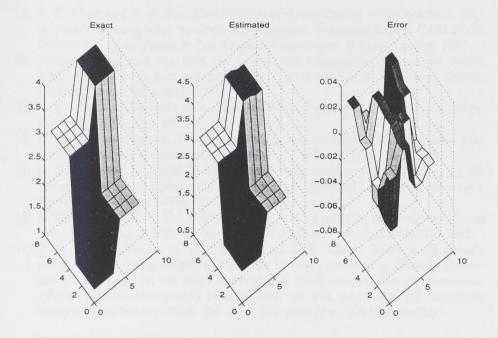


FIGURE 11. The exact, the estimated and the error in permeability q(x) with noise level $\sigma = 10^{-2}$.

5. CONCLUSION

We have seen how the augmented Lagrangian method can be used to solve parameter estimation problems in parabolic PDEs. When using the Uzawa algorithm, the minimization for the pressure is the most time consuming part. In this paper we have suggested three different algorithms for this minimization. The global minimization algorithm performs well in all examples, but the matching scheme is much quicker.



From a $= (\frac{1}{2}, \frac{1}{2})$ i.e. $n(x_{n-1})$ and $x_{n}(x_{n-1})$, $z \in (0, T)$. The noise level is $\alpha = 10^{-2}$



FIGURE 11. The exact, the estimated and the error in permeability of a) with note level $\alpha = 10^{-3}$.

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We have seen how the auginemical Learnington method can be used to solve parameter extination problems in parebolic PDEs. When who the Uzave signifiant the entrimitation for the pressure is the most virte concepting peak. In this remains we have suggested three different algorithms for this minimization. The global minimization algorithm performs well in all strangets, but the mitighting scheme is much quicker.

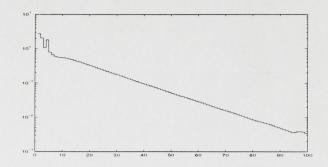


FIGURE 12. $||q_k - q||_{L^2}$ versus k. Logarithmic scale on the vertical axis.

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