# Department of APPLIED MATHEMATICS

Perturbation about neutral solutions occurring in shear flows in stratified, incompressible and inviscid fluids.

by

Leif Engevik

eport No. 43

June 1973



# UNIVERSITY OF BERGEN Bergen, Norway



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## Summary.

The unstable solution  $\varphi(y,k^2,c)$  contiguous to the neutral one  $\varphi_s(y,k_s^2,c_s)$  which may occur in shear flows in stratified, incompressible and inviscid fluids, can be expressed as

$$\varphi = \varphi_{s} + \sum_{l=1}^{\infty} \varphi_{l} (c-c_{s})^{l}$$
, where  $k^{2} - k_{s}^{2} = \sum_{l=1}^{\infty} k_{l} (c-c_{s})^{l}$ .

Here k is the wave-number and c the wave-velocity corresponding to the unstable solution, and  $k_s$  and  $c_s$ the wave-number and wave-velocity of the neutral solution. Expressions for  $\varphi_1$  and  $k_1$  are given. Perurostion about nearest solutions occupies a serie of love in straidfled, . incompressions a and inviseld frides...

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## I. Introduction.

In this paper we are concerned with the unstable solutions contiguous to the neutral ones which may occur in shear flows in stratified, incompressible and inviscid fluids. In this connection we have to find solutions of the equation:

(1.1) 
$$\varphi'' + \left\{ \frac{\beta g}{(U-c)^2} - \frac{U''}{U-c} - k^2 \right\} \varphi = 0$$
.

Here U(y) denotes the unperturbed flow velocity, and  $\beta(y) = -\rho'(y)/\rho(y)$ , where  $\rho(y)$  is the unperturbed density field. The prime denotes differentiation with respect to y. U(y) and  $\rho(y)$  vary in the y-direction perpendicular to the flow direction, which is taken to be the x-direction. This basic state is perturbed, and the perturbation stream function is  $\Psi(x,y,t) = \operatorname{Re}\{\varphi(y) e^{ik(x-ct)}\}$ , where k is the wavenumber (real), and  $c = c_r + ic_i$  is the wave-velocity (which may be complex,  $c_i \neq 0$ ). Re{...} means the real part of the quantity within the brackets. Eq.(1.1) is the equation for the amplitude function  $\varphi(y)$ . The fluid is supposed to be confined between two rigid horizontal planes at  $y = y_1$  and  $y = y_2$ . The boundary conditions to be satisfied are therefore:

(1.2)  $\phi = 0$  at  $y = y_1, y_2$ .

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In this paper we are corcerned when the unabable solutions contiguous to the neutral orde which may occur in shear flows in stratified, incomercealble and inviscinfluids. In this connection we bays, the find, solutions of the equation:

$$0 = -\frac{1}{2} \left\{ \frac{3}{2} - \frac{1}{2} \frac{1}{2} - \frac{3}{2} \frac{3}{2} + \frac{3}{2} \frac{1}{2} \frac{1}{2$$

Here U(y) denotes the upserputped figurvelocity, and P(x) = -P((y)/p(y)), where, P((x)) is the upserturbed denentry field. The pring dangers, D((y) is the y-direction perpendicular to tag flog dangers, <math>D((y), very to the y-direction perpendicular to tag flog dangers, L(y), very to the y-direction to the x-direction. This bester state is perpended, and the perturbation stream functions is

(5.1)

In this paper we assume that U(y) and  $\beta(y)$  are analytic functions of y on the interval  $I = \{y | y_1 \le y \le y_2\}$  of the real axis. Then U(y) and  $\beta(y)$  are analytic in some region in the complex plane close to this interval. Further it is assumed that  $U'(y) \ne 0$  on I. Let us take U' > 0 on I. The case U' < 0 can be treated in an analogous way.

We consider the case when the fluid is statically stable. In this case there may exist singular neutral solutions, i.e. solutions which are located on the stability boundary in a wave number - Richardson number plane, see for instance [1]. The singular neutral solution  $\varphi_s$  with the wave velocity  $c_s$  and the wave number  $k_s$ satisfies eq.(1.1) with  $c = c_s$  and  $k = k_s$  and the boundary conditions eq.(1.2). Since  $U' \neq 0$  for  $y \in [y_1, y_2], \varphi_s$  must be of the form, see [1]:

(1.3) 
$$\phi_{s} = (U-c_{s})^{\frac{1}{2}+\mu} Y_{s}$$
, where  $\mu \in [-\frac{1}{2}, \frac{1}{2}]$ .

 $\mu = \left(\frac{1}{4} - R_{\rm s}\right)^{\frac{1}{2}}, \text{ where } R_{\rm s} = g\beta/(U')^2 \text{ is the Richardson}$  number at the critical layer defined by  $U(y_{\rm s}) - c_{\rm s} = 0.$  $Y_{\rm s}$  is analytic on I since U and  $\beta$  are assumed to be analytic there. We define  $\arg(U-c_{\rm s})$  in the following manner:  $\arg(U-c_{\rm s}) = 0$  when  $U - c_{\rm s} > 0$ , and  $\arg(U-c_{\rm s}) = -\pi$  when  $U - c_{\rm s} < 0$ . If  $\arg(U-c_{\rm s})$  is defined in this way, we have shown in [2] that  $\phi_{\rm s}$  coincides almost everywhere with the viscous solution within the limit of zero viscosity. Also with this definition of  $\arg(U-c_{\rm s}) \phi_{\rm s}$  will

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we consider the case when the first is statically atable. In this, case there may axist singular nectral solutions, i.e. selections which and loaded on the stability boundary inge mays manager - Richny dator manager. plane, see fee higherer (1), The singular neutrick solution of with the mass velocity, of and the vers number satisfies eq.(\*.1) with the since view of the vers number boundary conditions eq.(1.2). Since View can be for

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 $H_{12}$   $(\frac{1}{2}, \frac{1}{2})^2$ , where  $H_{0} = eH(10)^2$  is the Hithardson number at the critical layer defined by  $U(r_{0})$ ,  $H_{0} = 0$  $Y_{0}$  is analytic on 1 since U and H are assumed to be anguetic there. We define  $\arg(0-r_{0})$  in the following memory arc(H < 0) when  $U = r_{0} < 0$ . When  $U = r_{0} > 0$ , and  $\arg(V < r_{0})$  is defined in this way when  $U = r_{0} < 0$ . If  $\arg(U < r_{0})$  is defined in this way at the isotoge solution within the limit of defined  $\frac{1}{2}$  and  $\frac{1}{2}$  a

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be the limit when  $c_1 \to 0^+$  of the unstable solution. Let L be the contour shown in fig.1.  $\phi_s$  given by the

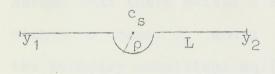


Fig.1

L if  $\rho$  is made small enough. (Note that U' > 0 for  $y \in [y_1, y_2]$ ). We also observe that an

eq.(1.3) is analytic along

unstable solution is analytic on I, and will therefore also be analytic along L if  $\rho$  is made small enough. We also see that  $-\pi \leq \arg(U-c) \leq 0$  when  $c_i \geq 0$  both when  $y \in I$  and  $y \in L$ .

#### II. Perturbation about the neutral solution.

We assume that there exists a neutral solution  $\varphi_s$ as defined in eq.(1.3). As mentioned this solution is analytic along L if  $\rho$  is made snall enough. Let  $|c-c_s| \leq \rho_1$ , where  $\rho_1 < \rho$ . Further let us define  $\arg(U-c)$  in the following manner:  $-\pi - \epsilon_2 < \arg(U-c) < \epsilon_1$ , where  $\epsilon_1 > 0$  and  $\epsilon_2 > 0$ . Our choice of  $\epsilon_1$  and  $\epsilon_2$ will depend on  $\rho_1$ . We see that if  $c_i \geq 0$ ,  $-\pi \leq \arg(U-c) \leq 0$ ; and if  $c_i < 0$ ,  $-\pi - \epsilon_2 < \arg(U-c) < \epsilon_1$ . A solution  $\varphi$  of eq.(1.1) is an analytic function of  $y \in L$ ,  $c \in \left\{ c | |c-c_s| \leq \rho_1 \right\}$  and  $k^2$ , with  $\arg(U-c)$ defined as above. This solution can therefore be expanded in a series:

(2.1)  $\varphi = \varphi_0 + \left(\frac{\partial \varphi}{\partial c}\right)_s (c-c_s) + \left(\frac{\partial \varphi}{\partial k^2}\right)_s (k^2 - k_s^2) + \dots,$ 

 $v_{g_1}$  the limit when  $v_1 \rightarrow 0^{-1}$  of the unstable colution. Let  $I_1$  be the contour shown in fig.1.  $v_{g_1}$  given by the equation (1,3) is scalarly the shifts:

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unatable solution is any yels on I, and whill therefore also be analytic along, I; if  $p_{1}$  is made install snough. We also see that  $-\pi > any (U_{2})_{y} < 0$  when  $c_{1} \geq 0^{-1}$  but when y s I and y  $c_{1}$ .

 $Y \in \mathbf{L}_{E}$ ,  $e \in \{o | f_{e}-e_{e}| \in o, \}$  and  $\mathbf{L}^{2}$ , with arg( $(f_{e}-e)$ ) defined as above. This solution can therefore be where  $(...)_{s}$  means that the quantity within the brackets is calculated at  $c = c_{s}$  and  $k^{2} = k_{s}^{2}$ . Assume that there exists a solution  $\varphi$  of eq.(1.1), with a c near  $c_{s}$  and a  $k^{2}$  near  $k_{s}^{2}$ , that satisfies the boundary conditions eq.(1.2). By introducing eq.(2.1) into eq.(1.1) and eq.(1.2) we get within the limit when  $c \rightarrow c_{s}$  and  $k^{2} \rightarrow k_{s}^{2}$ :

(2.2) 
$$\phi_0'' + \left\{ \frac{\beta g}{(U - c_s)^2} - \frac{U''}{U - c_s} - k_s^2 \right\} \phi_0 = 0$$
, and  $\phi_0 = 0$   
for  $y = y_1, y_2$ .

Wee see that  $\varphi_0$  must satisfy the same equations as  $\varphi_s$ , and we get that  $\varphi_0 = A_0 \varphi_s$ , where  $A_0$  is a constant. By using the equations which govern  $\varphi$  and  $\varphi_s$ , we get:

(2.3) 
$$\int_{L} \left[ \left( \frac{\beta g}{(U-c)^{2}} - \frac{U''}{U-c} - k^{2} \right) - \left( \frac{\beta g}{(U-c_{s})^{2}} - \frac{U''}{U-c_{s}} - k_{s}^{2} \right) \right] \phi \phi_{s} \, dy = 0.$$

Let us denote by E the expression  $\frac{\beta g}{(U-c)^2} - \frac{U''}{U-c} - k^2$ , and by E<sub>s</sub> the same expression with c<sub>s</sub> and k<sub>s</sub><sup>2</sup> instead of c and k<sup>2</sup>. By introducing eq.(2.1) into (2.3) we get:

Tet us denote by  $B_{1}$  the expression  $\frac{B_{1}}{(u-e)^{2}} - \frac{U_{1}}{U_{1}} - \frac{U_{1}}{(u-e)^{2}}$ end, by  $B_{2}$  the same expression wigh  $e_{1}$  and  $k_{2}$  insides: of e and  $k_{1}$  by introducting eq.  $(e_{1}, 1)$ ; into (E, S) we get

$$(2.4) \left\{ \begin{array}{c} \left(k^{2}-k_{s}^{2}\right)\int_{L}\left[\frac{\partial}{\partial k^{2}}\left\{\left(E-E_{s}\right)\phi\right]\right]_{s}\phi_{s} dy_{-}+ \\ + \left(c-c_{s}\right)\int_{L}\left[\frac{\partial}{\partial c}\left\{\left(E-E_{s}\right)\phi\right\}\right]_{s}\phi_{s} dy_{-}+ \\ \frac{1}{1!}\int_{L}\left[\left\{\left(c-c_{s}\right)\frac{\partial}{\partial c}+\left(k^{2}-k_{s}^{2}\right)\frac{\partial}{\partial k^{2}}\right]^{1}\left\{\left(E-E_{s}\right)\phi\right\}\right]_{s}\phi_{s}dy_{-}+ \\ \dots = 0, \end{array} \right.$$

where  $[\ldots]_{s}$  means that the expression within the brackets is calculated at  $c = c_{s}$  and  $k^{2} = k_{s}^{2}$ . The integrals in eq.(2.4) will exist because  $\varphi$  is an analytic function on L.

We assume that the coefficient of  $(k^2-k_s^2)$  in eq.(2.4),

1.e.  $\int_{L} \frac{\left[\frac{\partial}{\partial k^2} \left\{ (E - E_s) \phi \right\} \right]_s}{\varphi_s dy} = - \int_{L} \phi_s^2 dy$ , is not equal to L zero. Then if eq.(2.4) is to be satisfied within the limit when  $c \rightarrow c_s$ , we must have that:

(2.5) 
$$k^2 - k_s^2 = k_1 (c - c_s) + k_2 (c - c_s)^2 + \dots + k_1 (c - c_s)^1 + \dots$$

where  $k_1$ , l = 1, 2, ... are constants. When U is an odd and  $\beta$  is an even function of y,  $y_1 = -y_2$ , and  $\varphi_s$  is a singular neutral solution with wave velocity  $c_s = 0$ , the coefficient of  $(k^2 - k_s^2)$  in eq.(2.4) is equal to  $(e^{-i2\pi\mu} - 1)\int_{1}^{y_2} U^{1+2\mu} Y_s^2 dy$ , where it has been assumed that U' > 0. This expression is not equal to zero when  $|\mu| \in (0, \frac{1}{2}]$ . In section III we have considered an example of this type. When  $\mu = 0$ , i.e.  $R_s = \frac{1}{4}$ , the coefficient of  $(k^2 - k_s^2)$ 

$$(2.5) \quad k^{2} + k_{g}^{(2)} = k_{f}(c - c_{g})^{2} + k_{g}(c - c_{g})^{2} + \dots + k_{g}(c - c_{g})^{2} + \dots$$

is zero. However, to find the unstable solution close to the neutral one on this point of the stability boundary. we should expand  $\varphi$  and  $(c-c_s)$  in a series in  $(R-R_s)$ keeping  $k^2 = k_s^2$  fixed, rather than expanding  $\varphi$  and  $(k^2 - k_s^2)$  in a series in  $(c - c_s)$  keeping  $R = R_s$  fixed, as is done in this paper. In [1] we have found the formula for  $\left(\frac{\partial c}{\partial R}\right)_k$  , i.e. we have found the first term in the series for  $(c-c_s)$  in powers of  $(R-R_s)$ , keeping  $k^2 = k_s^2$  fixed. The coefficient of  $(c-c_s)$  in eq.(2.4) may be zero, (see the example in section III), and that is the reason why we have expanded  $(k^2 - k_s^2)$  in a series of  $(c - c_s)$ . If the coefficient of  $(c-c_s)$  is not equal to zero,  $k_1 \neq 0$ , and  $(k^2 - k_s^2)$  behaves as  $k_1(c-c_s)$  for c close to c<sub>s</sub>. If this coefficient is equal to zero,  $k_1 = 0$ , and  $(k^2 - k_s^2)$  behaves as  $k_2(c - c_s)^2$  for c close to  $c_s$ . This shows that if we had expanded  $(c-c_s)$  in a series in  $(k^2-k_s^2)$ , we would have had to treat these two cases separately.

We have assumed that there exists a solution  $\varphi$  for a c near c<sub>s</sub> and for a k<sup>2</sup> near k<sub>s</sub><sup>2</sup>, which satisfies the boundary conditions eq.(1.2), i.e.:

(2.6)  $\varphi(y_1, k^2, c) = 0$  and  $\varphi(y_2, k^2, c) = 0$ .

We have assumed that there exists a solution  $\phi$  for a. c. heat,  $c_{\rm s}$  and for a  $k^2$  heat  $k_{\rm s}^{2}$ , which exists the the boundary conditions eq. (1.2), i.e.:

 $(2.61) \quad \varphi(x_{ij} + \hat{\psi}_{ij}) = 0 \quad \text{and} \quad \varphi(x_{ij} + \hat{\psi}_{ij}) = 0$ 

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The functions in the eqs.(2.6) are analytic functions of  $\mathbf{c} \in \{\mathbf{c} \mid |\mathbf{c}-\mathbf{c}_{\mathbf{s}}| \leq \rho_1 \}$  and  $\mathbf{k}^2$ , so that all the derivatives with respect to  $\mathbf{c}$  and  $\mathbf{k}^2$  exist. Derivating the eqs. (2.6) with respect to  $\mathbf{c}$ , yields:

$$(2.7) \quad \frac{\partial \varphi}{\partial c} + \frac{\partial \varphi}{\partial k^2} \frac{dk^2}{dc} = 0,$$

where the value of y is either  $y_1$  or  $y_2$ .

 $\begin{pmatrix} \Phi \\ \partial \kappa^2 \end{pmatrix}_s \neq 0$  because of the assumption above that the coefficient of  $(k^2 - k_s^2)$  in eq.(2.4) is not equal to zero. It then follows from eq.(2.7) that  $\frac{dk^2}{dc}$  exists in some region around  $c_s$ . Therefore  $(k^2 - k_s^2)$  is analytic in this region  $|c-c_s| < \rho_2$ , and can be expanded in a power series, which is valid for  $|c-c_s| < \rho_2$ . Consequently the series given by eq.(2.5) will converge whithin this region.

Taking into account eq.(2.5), we find that eq.(2.4) is satisfied if:

$$(2.8) \begin{cases} \int \left[ \frac{d}{dc} \left\{ (E - E_s) \phi \right\} \right]_s \phi_s \, dy = 0 \\ \vdots \\ \int \left[ \frac{d^2}{dc^2} \left\{ (E - E_s) \phi \right\} \right]_s \phi_s \, dy = 0 \\ \vdots \end{cases}$$

where  $\frac{d^{1}}{dc^{1}} = \left(\frac{\partial}{\partial c} + \frac{dk^{2}}{dc}\frac{\partial}{\partial k^{2}}\right)^{1}$ , and the derivatives of  $k^{2}$  with respect to c at  $c=c_{s}$  are given by eq.(2.5). By introducing eq.(2.5) into eq.(2.1), the solution  $\varphi$  can be written as:

(2.9) 
$$\varphi = \varphi_0 + \varphi_1(c-c_s) + \dots \frac{1}{1!} \varphi_1(c-c_s)^1 + \dots$$
,  
where  $\varphi_1 = \left(\frac{d^2\varphi}{dc^1}\right)^2$ 

From the above it follows that this series is valid in some region  $|c-c_s| < \rho_3$  for all  $y \in L$ .

From the first of the eqs.(2.8) we find that we have to know  $\varphi_0$  in order to find  $k_1$ . But  $\varphi_0 = A_0 \varphi_s$ , where  $\varphi_s$  is known. We find that  $k_1$  is independent of the value of  $A_0$ .  $A_0$  must of course not be equal to zero. In the expression for  $k_1$ , we may therefore put  $A_0=1$ , which is done. To find  $k_1$  we have to know  $\varphi_0, \dots, \varphi_{l-1}$  and  $k_1, \dots, k_{l-1}$ . We observe that  $k_1$ does not depend on  $\varphi_1$ , which follows from the fact that

$$\left[\frac{d^{\perp}}{dc^{\perp}}\left\{(E-E_{s})\phi\right\}\right]_{s} = \left[(E-E_{s})\frac{d^{\perp}}{dc^{\perp}}\phi\right]_{s} + \left[\frac{d^{\perp-1}}{dc^{\perp-1}}\phi\frac{d}{dc}(E-E_{s})\right]_{s} + \dots$$

 $+ \left[ \varphi \frac{d^{\perp}}{dc^{\perp}} (E-E_s) \right]_s$ , where the first term on the right hand side of this expression is equal to zero. The equation for  $\varphi_{\perp}$  is obtained by differentiating eq(1.1) 1 times with respect to c. We write:

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where  $\frac{d_1}{d_0} = \left(\frac{1}{2c} + \frac{dk^2}{dc}\right)$ ; and the derivatives of k with respect to a st  $\frac{d_1}{dc}$  are given by eq. (2.3). By introducing eq. (2.3) into eq. (2.1); the solution e

From the above it follows that this Series is valid in

(2.10) 
$$\left\{\begin{array}{c} -\left[\frac{d^{1}}{dc^{1}}\left\{(E-E_{s})\phi\right\}\right]_{s},\\ \phi_{1}=0 \quad \text{for} \quad y=y_{1},y_{2}.\end{array}\right.$$

We observe that the expression on the right hand side of eq.(2.10) is known if  $\varphi_0, \ldots, \varphi_{l-1}$  and  $k_1, \ldots, k_l$  are known.

The homogeneous equation corresponding to eq.(2.10) has the two linearly independent solutions  $\varphi_s$  and  $\theta_s$ .  $\theta_s \neq 0$  for  $y = y_1, y_2$ , since  $\varphi_s=0$  for  $y = y_1, y_2$ . The general solution of eq.(2.10) is easily found by the method of variation of parameters:

$$\begin{split} \varphi_1 &= A_1 \varphi_s + B_1 \theta_s + \varphi_s \int_{y_1}^{y} \frac{J_1 \theta_s}{W} \, dt + \theta_s \int_{y}^{y_2} \frac{J_1 \varphi_s}{W} \, dt \;, \\ \text{where } J_1 &= -\left[\frac{d^1}{dc^1} \left\{ (E - E_s) \varphi \right\} \right]_s. \text{ The integration is along} \\ \text{the contour L. } A_1 \; \text{and } B_1 \; \text{are constants, and} \\ \text{W} &= \varphi_s' \theta_s - \varphi_s \theta_s' \; \text{ is the Wronskian which is a constant} \\ \text{in this case because } \varphi' \; \text{does not appear in eq.(2.10).} \\ \varphi_1 \; \text{satisfies the boundary conditions if } B_1 = 0, \; \text{because} \\ y_2 \; \int_{y_1}^{y_2} J_1 \varphi_s \; dt = -\int_{y_1}^{y_2} \left[ \frac{d^1}{dc^1} \left\{ (E - E_s) \varphi \right\} \right]_s \varphi_s dt = 0 \;, \; \text{where the} \\ \text{integration is along L. This follows from (2.8).} \quad \varphi_1 \end{split}$$

which satisfies the boundary conditions, is therefore :



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Was obvisance that the explosion on the right hand alle of eq. (2110) is bnown if  $\psi_0$  ...,  $\psi_{1-1}$  and  $k_1$  ...,  $k_1$ arecknown.

 $\sum_{i=1}^{n} \frac{1}{2} = \frac{$ 

...

(2.11) 
$$\varphi_{1} = A_{1}\varphi_{s} + \varphi_{s}\int_{y_{1}}^{y} \frac{J_{1}\theta_{s}}{W} dt + \theta_{s}\int_{y}^{y_{2}} \frac{J_{1}\varphi_{s}}{W} dt$$

where the integration is along L. We see that if  $\varphi_0, \dots, \varphi_{l-1}$  and  $k_1, \dots, k_l$  are known,  $\varphi_l$  can be determined except for the constant  $A_l$ .

It has been mentioned previously that  $k_1$  is independent of the value of  $A_0$ , except that  $A_0$  shall not be equal to zero. From the second of the equations in (2.8) it is easily found that  $k_2$  is independent of both  $A_0$  ( $A_0 \neq 0$ ) and  $A_1$ . We may therefore put  $A_0 = 1$  and  $A_1 = 0$  when calculating  $k_1$ , and this is done. Generally  $k_1$  must be independent of  $A_0$  ( $A_0 \neq 0$ ),  $A_1, \dots, A_{1-1}$ . This is equivalent to saying that the value of ( $k^2 - k_s^2$ ) for a given c close to  $c_s$  is independent of the choice of the constants  $A_0$  ( $A_0 \neq 0$ ),  $A_1$   $1 = 1, 2, \dots$ . Let us show this. Let  $\psi_1$  be the solution given by eq.(2.9) when the constants are chosen to be  $A_0 = C_0 \neq 0$ ,  $A_1 = C_1$   $1 = 1, 2, \dots$ . Let  $\psi_2$ be the solution when the constants are  $A_0 = D_0 \neq 0$ ,  $A_1 = D_1$  $1 = 1, 2, \dots$ . The wave number for a given c close to  $c_s$  which corresponds to  $\psi_1$  and  $\psi_2$  is  $\kappa_1$  and  $\kappa_2$  respectively.  $\psi_1$  and  $\psi_2$  satisfy the equation:

$$\varphi'' + E_{g}\varphi = - (E - E_{g})\varphi$$

and the boundary conditions eq.(1.2). Note that in the expression for E we have to put  $k^2 = \kappa_1^2$  when  $\varphi = \psi_1$ , and  $k^2 = \kappa_2^2$ when  $\varphi = \psi_2$ . By using the equation for  $\psi_1$ , the equation for  $\psi_2$  and the boundary conditions, it is easily obtained that  $(\kappa_1^2 - \kappa_2^2) \int_L \psi_1 \psi_2 \, dy = 0$ . We have assumed previously that  $\int_L \varphi_s^2 \, dy \neq 0$ , from which it follows that  $\int_L \psi_1 \psi_2 \, dy \neq 0$  in some region close to  $c_s$ . But then it follows that  $\kappa_1^2 = \kappa_2^2$  in that

It has been mentioned providualy that  $x_{i}$  is independent of the value of  $A_{i}$ , except that  $A_{i}$  shall not be equal to zero; into the second of the equations in (2.8) it is consist found that  $A_{i}$  is independent of both  $A_{i}$   $(A_{i} \neq 0)$  and  $A_{i}$ . We may therefore but  $A_{i} = 1$  and  $A_{i} = 0$  when calculating  $u_{i}$  and this (a done; four  $A_{i} = 1$  and  $A_{i} = 0$  when calculating  $u_{i}$  and this (a done; done;  $A_{i}$  = 1 and  $A_{i} = 0$  when calculating  $u_{i}$  and this (a done;  $A_{i}$  = 1 and  $A_{i} = 0$  when calculating  $u_{i}$  and this (a done;  $A_{i}$  = 1 and  $A_{i} = 0$  when calculating  $u_{i}$  and this (a done;  $A_{i}$  = 1 and  $A_{i} = 0$  when calculating  $u_{i}$  and this (a done;  $A_{i}$  = 1 and  $A_{i} = 0$  when calculating  $u_{i}$  = 1 be the equivalent to zering that the value of  $(u^{2} - u^{2})$  for a constant  $e_{i}$  is interval on  $e_{i}$  is interval on the constant is  $v_{i}$  to the solution gliven by  $e_{i}$  (2.9) when the constant is  $v_{i}$  to the solution  $A_{i} = 0$   $A_{i} = 0$ is the constant  $a_{i}$  = 1  $a_{i$ 

> $\varphi^{n} + E_{g} \varphi = -(E - E_{g})\varphi$ and the boundary conditions  $\frac{2C_{g}}{2}$  (1.

for E we have to put  $k^2 = k^2$  when  $k = k^2$ , and  $k^2 = k^2$ when  $p = k_2$ . By using the equation for  $k_1$ , the equation for  $k_2$  and the boundary conditions, it is easily obtained that  $k^2 = k^2$ ,  $k_1 k_2$  by -0. We have assumed previously that  $k_1 = k^2$ ,  $k_1 k_2$  by -0. We have assumed previously that

region close to  $c_{1}$ . But then it follows that  $x_{1}^{2} = x_{2}^{2}$  in that

region, which means that the series for  $(k^2 - k_s^2)$  is independent of the choice of the constants, except that  $A_0 \neq 0$ .

 $\psi_1$  and  $\psi_2$  satisfy the same differential equation and the boundary conditions eq.(1.2). The Wronskian  $\psi_1\psi'_2 - \psi'_1\psi_2$  is zero, and  $\psi_1$  and  $\psi_2$  are therefore linearly dependent, i.e.  $\psi_1 = A(c)\psi_2$ , where A(c) is a function of c. This can also be shown directly by using the expressions for  $\psi_1$  and  $\psi_2$ , and then A(c) is also found. This means that the solutions, eq.(2.9), which are obtained by different choices of the constants, are linearly dependent solutions.

Above we have shown that if there exists a solution  $\varphi$  of eq.(1.1) which satisfies the boundary conditions eq.(1.2), and which tends to  $\varphi_s$  given by eq.(1.3) when  $c \rightarrow c_s$  and  $k^2 \rightarrow k_s^2$ , it must be given by the eq.(2.9) with  $\varphi_o = \varphi_s$ ,  $\varphi_1$  given by eq.(2.11) and  $(k^2 - k_s^2)$  by eq.(2.5).

Now, if there exists a singular neutral solution  $\varphi_s$ , there will always exist a solution  $\varphi$  close to  $\varphi_s$  with a c close to  $c_s$  and a  $k^2$  close to  $k_s^2$  which satisfies the eq.(1.1) and the boundary conditions eq.(1.2). This solution  $\varphi$  tends to  $\varphi_s$  and  $k^2$  tends to  $k_s^2$  when  $c \rightarrow c_s$ . This follows from the fact that the solutions of eq.(1.1) are analytic functions of  $c \in \{c \mid |c-c_s| < \rho\}$  and of  $k^2$  especially for

 $k^{2} \in \{k^{2} \mid |k^{2} - k_{s}^{2}| < \gamma\}$  for all  $y \in L$ . From the analysis above it follows that  $\varphi$  is given by the eq.(2.9) with  $\varphi_{0} = \varphi_{s}$ ,  $\varphi_{1}$  (1 = 1,2,...) given by the eq.(2.11) and  $(k^{2} - k_{s}^{2})$  given by eq.(2.5). We see that both  $\varphi_{1}$  and  $k_{1}$  are given when  $\varphi_{s}$  and  $\theta_{s}$  are known, so that  $\varphi$  and  $(k^{2} - k_{s}^{2})$  can be found. region, which means that the soules for  $(t - t_0)$  is independent of the choice of the constants, except that  $A_0 \neq 0$ .  $\psi_1$  and  $\psi_2$  satisfy the same differential equation and the bolindary conditions eq. (1,2). The isomektan  $\psi_1\psi_2 - \psi_1\psi_1$  is zero, and  $\psi_1$  and  $\psi_2$  are therefore linearly dependent, i.e.  $\psi_1 - A(b)\psi_2$ , where A(t) is a function of t. This can also be shown diffectly by value this expressions for  $\psi_1$  and  $\psi_2$  and  $\psi_3$  and then A(t) is also found. This means that he constitutes, are then A(t) is also found. This means that he constitutes, are

Abbye we have shown that if there exists a solution  $\varphi$  of eq.(1.1) which settiftes the boundary conditions eq.(1.2), and which tends to  $\varphi_{\alpha}$  given by Eq.(1.3) when  $\varphi \rightarrow \varphi_{\alpha}$  and  $k^{2} \rightarrow k^{2}_{\alpha}$ it must be given by the eq.(2.9) with  $\varphi_{\alpha} = \varphi_{\alpha}$ ,  $\varphi_{1} \ge g$  and by eq.(2.1) and ( $k^{2} - k^{2}_{\alpha}$ ) by eq.(2.5).

Now, is there exters a singular Fouriel solution  $\phi_{0}$ , there will always exter a solution  $\phi_{1}$  close to  $\phi_{1}$  which shiftelies the  $\phi_{1}$  (i.i) and to  $\phi_{1}$  and a k<sup>2</sup> close to  $\phi_{1}$  which shiftelies the  $\phi_{1}$  (i.i) and the boundary conditions eq.(i.2). This solution is tende to  $\phi_{2}$ and  $k^{2}$  tends to  $k^{2}$  when  $e = \phi_{1}$ . This solution is tende to  $\phi_{3}$ that the solutions of eq.(i.i) are similar to how from the fact that the solutions of eq.(i.i) are similar to how from the fact

 $k^2 \in \{k^2 \mid | k^2 + k_0^2 \mid < v\}$  for all  $y \in L$ . From the analysts above is fullows that y is given by the eq. (2.9) with  $v_0 - v_0$ .  $v_1 \in 1 + 2 + \cdots$ ) stren by the eq. (2.11) and  $(k^2 - k_0^2)$  given by eq. (2.5)). We see that both  $v_1$  and  $v_0$  are given when  $v_0 = v_0$ ,  $v_0 = v_0$  that both  $v_1$  and  $v_0$  are given when  $v_0 = v_0$ .

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It is important to be aware of the following. @ given by eq.(2.9) is valid in some region  $|c - c_s| < \rho_3$ for all  $y \in L$ . However, it is only the solution with  $c_1 > 0$  for real values of  $(k^2 - k_s^2)$  which is relevant to the stability problem of shear flows in stratified, incompressible and inviscid fluids. This unstable solution with  $\varphi_0 = \varphi_s$  tends, when  $c \rightarrow c_s$ ,  $c_i \rightarrow 0^+$ , to the singular neutral solution  $\varphi_{s}$  defined in eq.(1.3), where  $\arg(U - c_s) = 0$  when  $(U - c_s) > 0$ , and  $\arg(U - c_s) = -\pi$ when  $(U - c_s) < 0$  . The solution with  $c_i < 0$  for real values of  $(k^2 - k_s^2)$  which is obtained from eq.(2.9), has no relevance to our stability problem. This solution with  $\phi_0 = \phi_s$  will also tend, when  $c \rightarrow c_s$ ,  $c_i \rightarrow 0^-$ , to  $\varphi_{\rm S}$  given by eq.(1.3), with the definition of  $\arg(U - c_s)$  given above. The stable solution  $(c_i < 0)$ which has relevance to our stability problem, is the one which is obtained by taking the complex conjugate of the unstable solution, and this stable solution will tend, when  $c \rightarrow c_s$ ,  $c_i \rightarrow 0^-$ , to  $(U - c_s)^{\frac{1}{2} + \mu} Y_s$  where  $\arg(U - c_s) = 0$  when  $(U - c_s) > 0$ , and  $\arg(U - c_s) = \pi$ when  $(U - c_{1}) < 0$ .

From eq.(2.5) we find for what real values of  $k^2$  close to  $k_s^2$  there is instability. It is for those real values which make  $c_i > 0$ .

Note that  $\varphi_s$  and  $\theta_s$  in general have singularities at  $c = c_s$ . Then also  $\varphi_0$ , ...,  $\varphi_1$ , ... have singularities

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at  $c = c_s$ , Let  $L_r$  be a contour of the same kind as L, but with the radius r of the small semicircle instead of  $\rho \cdot \phi_0 \cdots \phi_1 \cdots$  will be analytic on  $L_r$  for every r such that  $0 < r \le \rho$ , and for the integrals in eq.(2.8) we therefore have :

$$\int (\dots) dy = \lim_{r \to 0} \int_{L_r} (\dots) dy$$

We may use this when evaluating the constants  $k_1$  , ...  $k_1$  ...

#### III An example.

In this section we will use  $k_1$  and  $k_2$ , and let us therefore write out the explicit expressions for  $k_1$  and  $k_2$ . From (2.8) we get :

(3.1) 
$$\lim_{h \to 0} \frac{\int \left\{ \frac{2\beta g}{(U-c_s)^3} - \frac{U''}{(U-c_s)^2} \right\} \phi_s^2 dy }{\lim_{\rho \to 0} \int_{L} \phi_s^2 dy }$$

which is the inverse of the expression for  $\left(\frac{\partial c}{\partial k^2}\right)_{R_s}$  obtained in [1] .

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in this section was divided in and ky such intination to the side the sector of the se

$$(3.2) \quad k_{2} = \frac{\lim_{\rho \to 0} \int_{L} \left\{ \frac{2\beta g}{(U-c_{s})^{3}} - \frac{U''}{(U-c_{s})^{2}} - k_{1} \right\} \varphi_{1} \varphi_{s} dy}{\lim_{\rho \to 0} \int_{L} \varphi_{s}^{2} dy} + \frac{\lim_{\rho \to 0} \int_{L} \varphi_{s}^{2} dy}{\lim_{\rho \to 0} \int_{L} \left\{ \frac{3\beta g}{(U-c_{s})^{4}} - \frac{U''}{(U-c_{s})^{3}} \right\} \varphi_{s}^{2} dy}{\lim_{\rho \to 0} \int_{L} \varphi_{s}^{2} dy}$$

where

$$(3.3) \quad \varphi_{1} = -\frac{\varphi_{s}}{W} \int_{y_{1}}^{y} \left\{ \frac{2\beta g}{(U-c_{s})^{3}} - \frac{U''}{(U-c_{s})^{2}} - k_{1} \right\} \varphi_{s} \theta_{s} dy - \frac{\theta_{s}}{W} \int_{y}^{y_{2}} \left\{ \frac{2\beta g}{(U-c_{s})^{3}} - \frac{U''}{(U-c_{s})^{2}} - k_{1} \right\} \varphi_{s}^{2} dy ,$$

where the integration is along L .

Let us consider the case :

(3.4) 
$$U = y$$
,  $\beta g = Qy^2 + R_0$ , where  $Q \ge 0$  and  $R_0 \ge 0$ .

The horizontal rigid planes are at  $y_1 = -1$  and  $y_2 = 1$ . This case has been studied in [1], where

. . . .

where

$$=\frac{1}{2}\left[\frac{2\pi}{2}\right]\left[\frac{2\pi}{2}-\frac{2\pi}{2}\right]=\frac{2\pi}{2}\left[\frac{2\pi}{2}-\frac{2\pi}{2}\right]\left[\frac{2\pi}{2}-\frac{2\pi}{2}\right]$$

where the integration is along h.

Let us consider the case :

(3.4) U = Y . BS = Qy<sup>2</sup> + B<sub>0</sub> . where Q = 0 and B<sub>0</sub> = Q :

The hogh control statis planes are at  $\pi_1 = 1$ , and  $\pi_2 = 1$ . This gase has been studied in [1], where

we have found that there may exist singular neutral solutions with  $c_s = 0$  when  $R_o \le \frac{1}{4}$ . These singular neutral solutions are:

(3.5)  $\varphi_{s} = y^{\frac{1}{2}} J_{\mu}(\lambda_{j,\mu} y)$  j = 1, 2, ..., n, where  $|\mu| = (\frac{1}{4} - R_{o})^{\frac{1}{2}}$  and  $|\mu| \in [0, \frac{1}{2}] \cdot \lambda_{j,\mu}$  is the  $j^{\frac{t}{h}}$ zero of the Besselfunction  $J_{\mu}$ . The wave number corresponding to the  $j^{\frac{t}{h}}$  singular neutral solution is :

(3.6) 
$$k_{j,\mu}^2 = Q - \lambda_{j,\mu}^2$$
.

The number of solutions is given by the number n which satisfies Q -  $\lambda_{n,\mu}^2 \ge 0$  , but Q -  $\lambda_{n+1,\mu}^2 < 0$  .

When  $\mu = -\frac{1}{2}$ ,  $\varphi_s$  is proportional to  $\cos \lambda_{j,-\frac{1}{2}} = \cos \left(\frac{2j-1}{2}\pi\right) y$ , and when  $\mu = \frac{1}{2}$ ,  $\varphi_s$  is proportional to  $\sin \lambda_{j,\frac{1}{2}}y = \sin (j\pi)y$ .

In the following we will discuss the case when Q = 15. In this case  $\pi^2 < Q < (\frac{3}{2}\pi)^2$ , which means that the number n in the eq.(3.5) is equal to 1, and that  $|\mu| \in [0, \frac{1}{2}]$ .

Let us consider the cases  $|\mu| \in (0, \frac{1}{2})$ ,  $\mu = -\frac{1}{2}$ ,  $\mu = \frac{1}{2}$ . 1) When  $|\mu| \in (0, \frac{1}{2})$ , the singular neutral solution is we have found Chitichiere "Sign exist singular neutral colutions with 's,' = 0 'Sign R,'s'', These singular neutral colutions with

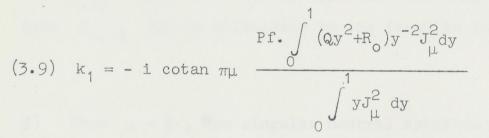
The maker of solutions is given by the number n which

(3,7)  $\begin{cases} \varphi_{s} = y^{\frac{1}{2}} J_{\mu}(\lambda_{1,\mu} y) , \text{ and the corresponding wave} \\ \text{number is:} \\ k_{1,\mu}^{2} = Q - \lambda_{1,\mu}^{2} \quad (Q = 15). \end{cases}$ 

The function  $\theta_{s}$  defined in section II is :

(3.8) 
$$\theta_{s} = y^{\frac{1}{2}} J_{-\mu}(\lambda_{1,\mu} y)$$
.

By introducing eq. (3.4), eq. (3.7) and  $c_s = 0$  into eq. (3.1) we get:



where Pf. in front of the integral sign means the finite part.

We see from eq.(3.9) that  $k_1$  is purely imaginary. In [1] we have shown that the integrals in eq.(3.9) are positive when  $|\mu| \in (0, \frac{1}{2})$ , so that  $k_1$  changes sign with cotan  $\pi\mu$ . Taking into account the expression for  $k_1$ , we get from eq.(2.5) that there is instability  $(c_1 > 0)$ for  $k > k_{1,\mu}$  when  $\mu \in (0, \frac{1}{2})$ , and for  $k < k_{1,\mu}$ when  $\mu \in (-\frac{1}{2}, 0), (k \ge 0)$ . Both  $\varphi_s$  and  $\theta_s$  are known, and by using the formulae in section II we can calculate the unstable solution for a given k in the vicinity of  $k_{1,\mu}$ . From eq.(2.5) we can find c which corresponds to a given k near  $k_{1,\mu}$ .

The function 6, defined, is subtion II is

$$(3.8)$$
,  $\theta_{a} = g^{2} J_{-\mu} (\lambda_{1a\mu} y)$ .

By introducing sq.(3.1), sq.(3.7) and og = 0 into, eq.(3.1) we get:



where Pf. in front of the integral sign means the

We see from eq. (3.9) that is noreignate in eq. (3.9) and in [1] we have shown that the intergrate in eq. (3.9) and possibles where  $[u] \in (0, 2)$ , so that k, obsides sign with cotan m. Maine into account the expression ten k, we get from eq. (2.5)) thet there is instability. ( $v_{1} > 0$ ) for  $k > k_{1,1}$ , when  $\mu \in (0, 2)$ , and for  $k < k_{1,1}$ , then  $\mu \in ((-1, 0))$  ( $k \in 0$ ). Body  $q_{1}$  and for  $k < k_{1,1}$ , when  $\mu \in ((-1, 0))$  ( $k \in 0$ ). Body  $q_{2}$  and  $\theta_{3}$  are known to  $(v_{1}, 0)$  ( $k \in 0$ ). Body  $q_{3}$  and for  $k < k_{1,1}$ , the unstable solution for a given k in the visit  $y_{1}$ of  $k_{1,1}$ . From eq. (2.5) we can find c which corresponds.

2) When 
$$\mu = -\frac{1}{2}$$
, the singular neutral solution is:

(3.10) 
$$\varphi_{s} = \cos \frac{\pi}{2} y$$
, and  $k_{1,-\frac{1}{2}}^{2} = Q - (\frac{\pi}{2})^{2}$  (Q = 15).  
 $\theta_{s} = \sin \frac{\pi}{2} y$ .

By introducing eq.(3.10) together with eq.(3.4) and  $c_s = 0$ , into eq.(3.1), we get:  $k_1 = 2i\pi Q$ , which together with the eq.(2.5) yields instability for  $k < k_{1,-\frac{1}{2}}$ . Again the unstable solution for a given k near  $k_{1,-\frac{1}{2}}$  can be calculated by the formulae in section II.

3) When 
$$\mu = \frac{1}{2}$$
, the singular neutral solution is :

(3.11) 
$$\varphi_s = \sin \pi y$$
, and  $k_{1,\frac{1}{2}}^2 = Q - \pi^2$  (Q = 15),  
 $\theta_s = \cos \pi y$ .

We find that  $k_1 = 0$  in this case. From eq. (3.2) we get:

$$(3.12) \quad k_{2} = \frac{\lim_{\rho \to 0} \left[ \int_{L} \frac{2Q}{y} \phi_{1} \phi_{s} dy + \int_{L} \frac{3Q}{y^{2}} \phi_{s}^{2} dy \right]}{\lim_{\rho \to 0} \int_{L} \phi_{s}^{2} dy}$$

where  $\varphi_s$  is given by eq.(3.11) and  $\varphi_1$  by eq.(3.3), i.e.:

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Events the reader of (3, 10) together with eq. (3, 4) and  $c_0 = 0$ , hoto set (3, 1), we get:  $k_1 = 24\pi 0$ , which the gether with the eq. (2, 3) yields instability for  $k < k_1 = 1$ ,  $k_2$ ,  $k_3$ ,  $k_4$ ,  $k_5$ ,  $k_4$ ,  $k_4$ ,  $k_4$ ,  $k_4$ ,  $k_4$ ,  $k_4$ ,  $k_5$ ,  $k_5$ ,  $k_5$ ,  $k_5$ ,  $k_6$ , k

3.) Meen µ + 2 , Oke singular neuteni sulution is :

 $(2n \rightarrow 0) \qquad \gamma_{\rm H} = 0 = \frac{1}{6} \int du du du du du du = \frac{1}{6} \int (11 - 2)$ 

We that this  $k_{ij} \neq 0$  in this case. From eq.((5.2)) we get

(3...12) Kg = (3...12) Kg = (3...12) Kg = (3...12) (3...2) (3...2)

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(3.13) 
$$\varphi_1 = -\frac{Q}{\pi} \sin \pi y \int_{-1}^{y} \frac{\sin 2\pi t}{t} dt - \frac{Q}{\pi} \cos \pi y \int_{y}^{1} \frac{1 - \cos 2\pi t}{t} dt$$
.

Introducing eq. (3.11) and eq. (3.13) into eq. (3.12), we get:

(3.14) 
$$k_2 = 6Q\pi \int_{0}^{1} \frac{\sin 2\pi t}{t} dt - \frac{2Q^2}{\pi} \int_{0}^{1} \frac{1 - \cos 2\pi y}{y} dy \int_{0}^{y} \frac{\sin 2\pi t}{t} dt +$$

$$-\frac{2Q^2}{\pi}\int_{0}^{1}\frac{\sin 2\pi y}{y}dy\int_{0}^{y}\frac{1-\cos 2\pi t}{t}dt -$$

$$-\frac{2Q^2}{\pi}\int_{0}^{1}\frac{\sin 2\pi y}{y}dy\int_{0}^{1}\frac{1-\cos 2\pi t}{t}dt$$

where we have used that

$$\int_{0}^{1} \frac{(\sin \pi t)^{2}}{t^{2}} dt = \pi \int_{0}^{1} \frac{\sin 2\pi t}{t} dt$$

In the case Q = 15,  $\varphi_s = \sin \pi y$  is the only neutral solution with  $c_s = 0$  when  $\mu = \frac{1}{2}$ . In the general case when the value of Q is such that  $\sin n\pi y$  is a neutral solution, we also find that  $k_1 = 0$ , and that  $k_2$  is given by :

This of Reality and (S. 13) and the (S. 13) anto eq. (S. 12), we get:

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The base values Q = 15,  $Q_{0} = 4$  and  $Q_{1}$  is the only natural addition with  $Q_{0} = 0$  when  $\mu = 2$ . In the general dase when the value of Q is such that all any is simplified solution, we disorting that  $Q_{1} = 0$ , and that  $Q_{2}$  is given by :

(3.15) 
$$k_2 = 6Qn\pi \int_{0}^{1} \frac{\sin 2n\pi t}{t} dt - \frac{2Q^2}{n\pi} \int_{0}^{1} \frac{1 - \cos 2n\pi y}{y} dy \int_{0}^{y} \frac{\sin 2n\pi t}{t} dt$$

$$+\frac{2q^2}{n\pi}\int_{0}^{1}\frac{\sin 2n\pi y}{y}dy\int_{0}^{y}\frac{1-\cos 2n\pi t}{t}dt -$$

$$-\frac{2Q^2}{n\pi}\int_{0}^{1}\frac{\sin 2n\pi y}{y}dy\int_{0}^{1}\frac{1-\cos 2n\pi t}{t}dt$$

Using the result from Appendix I, we find that this expression for  $k_2$  is equivalent to the one found in [3] by a less general method.

When Q = 15 we have shown in Appendix II that  $k_2$  given by eq.(3.14) is negative. From eq.(2.5) we find that  $(k^2 - k_{1,\frac{1}{2}}^2) = k_2c^2 + \cdots$ , and we see that there is instability for  $k > k_{1,\frac{1}{2}}$ . Again the unstable solution for a given k near  $k_{1,\frac{1}{2}}$  can be found since  $\varphi_s$  and  $\theta_s$  are known.

The case  $\mu = 0$  remains. In this case  $\varphi_s = y^{\frac{1}{2}} J_o(\lambda_{1,o} y)$  and  $k_{1,o}^2 = Q - \lambda_{1,o}^2$  (Q = 15). We have shown in [1] that  $\left(\frac{\partial c}{\partial k^2}\right)_{R_o}$ , which is equal to  $k_1^{-1}$ , is equal to zero in this case. We have also shown that  $\left(\frac{\partial c_1}{\partial R}\right)_{k_{1,o}} < 0$ , so that there is instability



Using the result from Appendix I, we find that this expression for kg is equivalent to the one found is.

for  $R < \frac{1}{4}$ . The curve in the  $k - R_o$  - plane on which  $c_s = 0$ , has a maximum  $R_o = \frac{1}{4}$  at  $k = k_{1,o}$ , see [1], and therefore we should expand  $\varphi$  and c in a series of powers of  $(R - \frac{1}{4})$  keeping  $k = k_{1,o}$  fixed in order to find an unstable solution close to this point on the curve. We would not find any unstable solution by expanding  $\varphi$  and  $(k^2 - k_{1,o}^2)$  in a series of powers of c, keeping  $R = \frac{1}{4}$  fixed, which is the method used in this paper.

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## Appendix I.

We will show that:

(A1.1) 
$$\begin{cases} \frac{2}{n\pi} \left[ \int_{0}^{1} \frac{1 - \cos 2n\pi y}{y} \, dy \int_{0}^{y} \frac{\sin 2n\pi t}{t} \, dt \right] \\ - \int_{0}^{1} \frac{\sin 2n\pi y}{y} \, dy \int_{0}^{y} \frac{1 - \cos 2n\pi t}{t} \, dt \\ = \int_{0}^{1} \cos 2n\pi t \, \log^{2}(\frac{t}{1-t}) \, dt \, . \end{cases}$$

0

Proof.

It is easily found that:

(A1.2)  
$$\begin{cases} \int_{0}^{1} \frac{1 - \cos 2n\pi y}{y} \, dy \int_{0}^{y} \frac{\sin 2n\pi t}{t} \, dt \\ - \int_{0}^{1} \frac{\sin 2n\pi y}{y} \, dy \int_{0}^{y} \frac{1 - \cos 2n\pi t}{t} \, dt \\ = 2n\pi \left[ \int_{0}^{1} \cos 2n\pi y \, \log^{2} y \, dy - \int_{0}^{1} \log^{2} y \, dy \right] \\ + (2n\pi)^{2} \left[ \int_{0}^{1} \sin 2n\pi y \, \log y \, dy \int_{0}^{y} \cos 2n\pi t \, \log t dt \right] \\ - \int_{0}^{1} \cos 2n\pi y \, \log y \, dy \int_{0}^{y} \sin 2n\pi t \, \log t \, dt \right].$$

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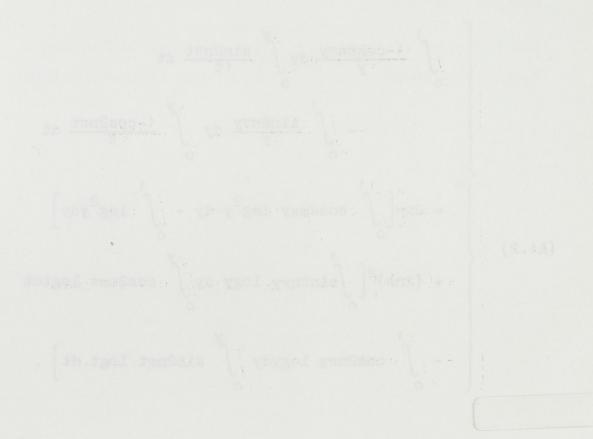
## The second second

We will show that:



Proof.

It is easily found that:



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Put:

$$(A1.3) \begin{cases} I(\alpha) = \int_{0}^{1} \sin \alpha y \log y \, dy \int_{0}^{y} \cos \alpha t \log t \, dt \\ - \int_{0}^{1} \cos \alpha y \log y \, dy \int_{0}^{y} \sin \alpha t \log t \, dt \\ 0 & 0 & 0 \end{cases}$$

The expression on the right hand side of eq.(A1.2) is then equal to:

$$2n\pi \left[\int_{0}^{1} \cos 2n\pi y \, \log^{2} y \, dy - \int_{0}^{1} \log^{2} y \, dy\right] + (2n\pi)^{2} I(2n\pi) .$$

From eq.(A1.3) it follows that:

$$(A1.4)$$
 I(0) = 0.

We differentiate eq.(A1.3) with respect to  $\alpha$  , and find that:

(A1.5) 
$$\begin{cases} \frac{\mathrm{dI}}{\mathrm{d\alpha}} + \frac{2}{\alpha} \mathrm{I} = \frac{1}{\alpha^2} \int_{0}^{1} \cos\alpha y \log y \, \mathrm{dy} - \frac{2}{\alpha^2} \int_{0}^{1} \log y \, \mathrm{dy} \\ + \frac{1}{\alpha^2} \int_{0}^{1} \cos\alpha (1-y) \log y \, \mathrm{dy}. \end{cases}$$

The solution of this equation which satisfies the condition eq.(A1.4), is:

(A1.6) 
$$I(\alpha) = \frac{1}{\alpha^2} \left[ -2\alpha \int_0^1 \log y \, dy + \int_0^1 \frac{\sin \alpha y}{y} \log y \, dy + \int_0^1 \frac{\sin \alpha (1-y)}{1-y} \log y \, dy \right]$$

Put

$$I(\vec{u}) = \int dt dt dt \int dt dt dt dt$$

The expression on the right hand side of eq. (A1.2) is then equal to:

$$2n\pi \left[ \int \cos^2 t \, dy \, \log^2 y \, dy \, - \int \int de^2 y \, dy \right] + (2n\pi)^2 \mathbf{I} (2n\pi)$$

From aq. (Af. 3) it follows that the

$$(X)^{*}, (A) = 0$$

We'differentiate eq. (A(.3) with respect to a , and find that.

$$\left\{ \frac{41}{62} + \frac{2}{6} + \frac{1}{6} +$$

The solution of this equativity which satisfies the con-

$$\frac{(a.r.a)}{(a.r.a)} = \frac{1}{2} \left[ -2a \int 108y \, dy + \int \frac{84m^2}{2} 108y \, dy \\ \frac{64m^2}{2} \int 108y \, dy + \int \frac{1}{2} \frac{1$$

Now:

$$\int_{0}^{1} \frac{\sin 2n\pi (1-y)}{1-y} \log y \, dy = \int_{0}^{1} \frac{\sin 2n\pi (1-y)}{y} \log (1-y) \, dy - 2n\pi \int_{0}^{1} \cos 2n\pi (1-y) \log y \log (1-y) \, dy = -2n\pi \int_{0}^{1} \frac{\sin 2n\pi (1-y)}{1-y} \log y \, dy - 2n\pi \int_{0}^{1} \cos 2n\pi y \log (1-y) \, dy,$$

from which it follows that:

(A1.7) 
$$\int_{0}^{1} \frac{\sin 2n\pi (1-y)}{1-y} \log y \, dy = -n\pi \int_{0}^{1} \cos 2n\pi y \, \log (1-y) \, dy.$$

Further:

(A1.8) 
$$\int_{0}^{1} \frac{\sin \alpha y}{y} \log y \, dy = -\frac{\alpha}{2} \int_{0}^{1} \cos \alpha y \, \log^{2} y \, dy ,$$
  
and 
$$\int_{0}^{1} \log y \, dy = -1.$$

Taking into account eq.(A1.7) and eq.(A1.8), we find that:

(A1.9) 
$$\begin{cases} I(2n\pi) = \frac{1}{(2n\pi)^2} \left[ 4n\pi - n\pi \int_{0}^{1} \cos 2n\pi y \log^2 y \, dy - \int_{0}^{1} \sin 2\pi y \log (1-y) \, dy \right] \\ - n\pi \int_{0}^{1} \cos 2n\pi y \log y \log(1-y) \, dy \end{bmatrix}.$$

Now:

(A1.10) 
$$\begin{cases} \int_{0}^{1} \cos 2n\pi y \, \log^{2} y \, dy = \int_{0}^{1} \cos 2n\pi y \, \log^{2}(1-y) \, dy, \\ 0 & 0 \\ and \int_{0}^{1} \log^{2} y \, dy = 2. \end{cases}$$

Introducing eq.(A1.9) into eq.(A1.2) and using eq.(A1.10), we get:

$$\int_{0}^{1} \frac{1 - \cos 2n\pi y}{y} \, dy \int_{0}^{y} \frac{\sin 2n\pi t}{t} \, dt - \int_{0}^{1} \frac{\sin 2n\pi y}{y} \, dy \int_{0}^{y} \frac{1 - \cos 2n\pi t}{t} \, dt = \frac{1}{2} n\pi \int_{0}^{1} \cos 2n\pi y \Big[ \log^{2} y + \log^{2} (1 - y) - 2\log y \log(1 - y) \Big] dy ,$$

which is equivalent to (A1.1).

## Appendix II.

By using the result from Appendix I,  $k_2$  given by eq.(3.14) can be written as:

(A2.1) 
$$k_{2} = 6Q\pi \operatorname{Si}(2\pi) - Q^{2} \int_{0}^{1} \cos 2\pi t \, \log^{2}(\frac{t}{1-t}) dt - \frac{2Q^{2}}{\pi} \operatorname{Si}(2\pi) \operatorname{Cin}(2\pi),$$
where  $\operatorname{Si}(x) = \int_{0}^{x} \frac{\sin t}{t} dt$ ,  $\operatorname{Cin}(x) = \int_{0}^{x} \frac{1-\cos t}{t} dt$ .

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By using the result rich appendix i. By Elven by

 $(AQ_{-1})$   $R_{2} = 60^{4} Si(2^{4}) = c^{2} \int coa2mb \log^{2}(\frac{1}{1-4}) dt - c^{2}$ 

. (ms) nto (ms) is 208 -

It will be shown that  $k_2$  given by eq.(A2.1) is negative. Q = 15. Cin( $\pi x$ ) is tabulated in [4], and we find that Cin( $2\pi$ ) = 2,44 approximately. From this it follows that  $6Q\pi$  Si( $2\pi$ )  $< \frac{2Q^2}{\pi}$  Si( $2\pi$ ) Cin( $2\pi$ ). Further:

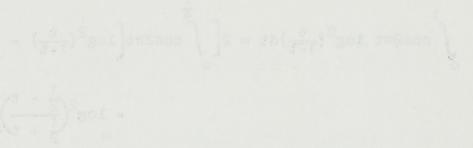
$$\int_{0}^{1} \cos 2\pi t \, \log^2\left(\frac{t}{1-t}\right) dt = 2 \left[ \int_{0}^{1} \cos 2\pi t \left\{ \log^2\left(\frac{t}{1-t}\right) - \right\} \right]$$

$$-\log^{2}\left(\frac{\frac{1}{2}-t}{\frac{1}{2}+t}\right) dt > 0,$$

since  $\log^2(\frac{t}{1-t}) \ge \log^2(\frac{\frac{1}{2}-t}{\frac{1}{2}+t})$  when  $t \in (0,\frac{1}{4}]$ .

From the above it follows that  $k_{2} < 0$ .

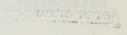
It will be shown that  $k_{e}$  given by eq.(A2.1) is negative. 9 = 15.  $\operatorname{Ota}(\pi x)$  is tabulated in [4], and we find that  $\operatorname{Ota}(2\pi) = 2,44$  approximately. From this it follows that  $\operatorname{Ota}(2\pi) < \frac{32^{2}}{24}$  si(2\pi) Ota(2=). Furthers





From the above it follows that  $k_{\rm g} < 0$ .

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