Department of APPLIED MATHEMATICS

Rate of Convergence of a Space Decomposition method and Applications to Linear and Nonlinear Elliptic Problems

by

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Report No. 103

September 1996



UNIVERSITY OF BERGEN Bergen, Norway



ISSN 0084-778x

Department of Mathematics University of Bergen 5007 Bergen Norway

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RATE OF CONVERGENCE OF A SPACE DECOMPOSITION METHOD AND APPLICATIONS TO LINEAR AND NONLINEAR ELLIPTIC PROBLEMS

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ABSTRACT. Convergence of a space decomposition method is proved for a general convex programming problem. The space decomposition refers to methods that decompose a space into sums of subspaces, which could be a domain decomposition or a multilevel method for partial differential equations. Two algorithms are proposed. Both can be used for linear as well as nonlinear elliptic problems and they reduce to the standard additive and multiplicative Schwarz methods for linear elliptic problems. In the numerical implementations, two "hybrid" algorithms are also presented. They converge faster than the additive one and have better parallelism than the multiplicative method. Numerical tests with a two level domain decomposition for linear, nonlinear and interface elliptic problems are presented for the proposed algorithms.

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1. INTRODUCTION

This work presents a general space decomposition method for convex programming problems and gives an estimation of the rate of convergence of the method. One intension is to use the method to solve linear and nonlinear elliptic partial differential equations by domain decomposition or multilevel methods. In the applications given in this work, a two level overlapping domain decomposition method is considered.

The essence of the proposed method is to decompose the minimization space into a sum of subspaces and then solve the original minimization problem sequentially or in parallel over each of the subspaces. Due to the fact that the decomposed spaces can be arbitrary, especially since they are not orthogonal to each other, the usual convergence proofs for block relaxation methods cannot be used here to predict the convergence. However, using the experiences from domain decomposition and multigrid methods, we assume that the decomposed spaces satisfy a certain

¹⁹⁹¹ Mathematics Subject Classification. 65J10, 65M55, 65Y05.

Key words and phrases. Parallel, domain decomposition, nonlinear, elliptic equation, space decomposition.

The work was supported by by VISTA, a research cooperation between the Norwegian Academy of Science and Letters and Den norske oljeselskap a.s. (Statoil).

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The proposed algorithms are given for a convex programming problem. We expect that they could also be used to get efficient algorithms for some optimal control problems related to partial differential equations, see Kunisch and Tai [26] and [27] for applications.

The two level domain decomposition method can be viewed a space decomposition is inspired by the work of Xu [36], where it was observed that domain decomposition methods, multilevel methods and multigrid methods can be viewed in some way as space decomposition techniques and many of the methods proposed in literature for the above mention techniques are in essence similar to the Gauss-Seidel or Jacobi method. an abstract convergence was given for linear selfadjoint and also indefinite

Two schemes are proposed in this work. They could be used both for linear and nonlinear elliptic problems. In the linear case, they reduce to the standard additive and multiplicative Schwarz methods. Therefore, the algorithms generalise the known additive and multiplicative methods to certain nonlinear cases. Due to appearance of the nonlinearity, a modified abstract convergence theory is given. In the numerical implementations, two "hybrid" algorithms are proposed. They converge faster than the additive scheme and have better parallelism than the multiplicative scheme when used for overlapping domain decomposition.

The well-known substructuring BPS (see [6], [7], [8]) and BEPS (see [5]) preconditioners use nonoverlapping subdomains, see also [4], [29]. For a nonoverlapping domain decomposition, a finite element function w can be decomposed as $w = w_p + w_H$, here w_p has zero trace on the interfaces and w_H equals to w on the interfaces and is extended to the interior by harmonic extension. If we use Gauss-Seidel iteration, we get the exact solution in one iteration. However, to get the harmonic extension w_H is equivalent to solving the original problem. The construction of the preconditioners in [7]–[8] and [5] can be regarded as Jacobi iteration with approximate solvers for the harmonic extensions. The methods of [6] and [29] is a Gauss-Seidel iteration with a further suitable decomposition for w_H . By using a slightly different decomposition, in Espedal and Ewing [22, p. 125], a parallel nonoverlapping method was derived for solving a linearised two-phase immiscible flow. We hope that by viewing the construction of nonoverlapping preconditioners as an iterative approximate solving of a space decomposition, an abstract convergence analysis can also be obtained for them for some nonlinear problems.

In the literature, domain decomposition methods, multigrid methods and multilevel methods have been successfully used for different kinds of linear partial differential equations, see [25], [35], [36]. However, the results for using them for nonlinear problems are not as rich as for linear problems. In Cai and Dryja [11], a semilinear elliptic equation is first linearised by the Newton's method and then solved by the additive Schwarz scheme. In papers by Xu [37], [38], a two level method without doing domain decomposition is used for nonlinear elliptic problems. In Axellsson and Kaporin [1], a minimum residual adaptive multilevel method is given for some nonlinear problems. In Dawson and Wheeler [18], a two level method is used for a nonlinear parabolic equation; The work of Lions [28] seems to be the pioneering work for using domain decomposition methods for nonlinear partial differential equations. In Rannacher [30], a Newton type algorithm is studied for nonlinear elliptic problems. Multigrid methods for nonlinear problems are studied by Bank [2], Brandt [10], etc. For some earlier works of the authors related to this one, consult [31] and [32]–[34].

When we apply the methods here for a nonlinear problem, we need to solve many smaller size problems in an iterative way and this iterative procedure convergence as "quickly" as for linear problems. For some nonlinear problems, by reducing the large size problem into many smaller size problems and then linearising the smaller size problems, substantial computational efforts can be saved compared to first linearising and then decomposing the problem, see §5.

2. Statement of the problem and the algorithms

Consider the nonlinear problem

$$\min_{v \in V} F(v) . \tag{2.1}$$

Above, the function F is differentiable and convex, the space V is a reflexive Banach space. One

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Consider the nonlinear problem.

Above, 456 function & us aidercasts the and everyor, the space V is a whether Banach crites. One

knows that partial differential equations of the type

$$-\sum D_i(a_{ij}D_ju) + bu = f \text{ in } \Omega ,$$

and

$$-\nabla \cdot (\rho(|\nabla u|)\nabla u) = f \text{ in } \Omega ,$$

with a suitably given ρ , can be solved by (2.1) by defining the function F and space V properly.

We shall use space decomposition methods to solve (2.1). A space decomposition method refers to a method that decomposes the space V into a sum of subspaces, i.e. there are spaces V_i , $i = 1, 2, \dots, m$ such that

$$V = V_1 + V_2 + \dots + V_m . (2.2)$$

The meaning of the above decomposition is that $\forall v$, there exists $v_i \in V_i$ such that $v = \sum_{i=1}^m v_i$ and on the other hand, if $v_i \in V_i$, then $\sum_{i=1}^m v_i \in V$. If the space can be decomposed as in (2.2), then the followings algorithms can be used to solve (2.1).

Algorithm 2.1. (An additive space decomposition method).

Step 1. Choose initial values $u_i^0 = u^0 \in V$ and relaxation parameters $\alpha_i > 0$ such that $\sum_{i=1}^m \alpha_i \leq 1$. Step 2. For $n \geq 0$, find $u_i^{n+\frac{1}{2}} \in V_i$ in parallel for $i = 1, 2, \cdots, m$ such that

$$F\left(\sum_{k=1,k\neq i}^{m} u_k^n + u_i^{n+\frac{1}{2}}\right) \le F\left(\sum_{k=1,k\neq i}^{m} u_k^n + v_i\right) , \quad \forall v_i \in V_i .$$

$$(2.3)$$

Step 3. Set

$$u_i^{n+1} = u_i^n + \alpha_i (u_i^{n+\frac{1}{2}} - u_i^n) , \qquad (2.4)$$

and go to the next iteration.

Algorithm 2.2. (A multiplicative space decomposition method).

Step 1. Choose initial values $u_i^0 = u^0 \in V$.

Step 2. For $n \ge 0$, find $u_i^{n+1} \in V_i$ sequentially for $i = 1, 2, \cdots, m$ such that

$$F\left(\sum_{1 \le k < i} u_k^{n+1} + u_i^{n+1} + \sum_{i < k \le m} u_k^n\right)$$

$$\leq F\left(\sum_{1 \le k < i} u_k^{n+1} + v_i + \sum_{i < k \le m} u_k^n\right) , \quad \forall v_i \in V_i .$$

$$(2.5)$$

Step 3. Go to the next iteration.

In the following, the notation $\langle \cdot, \cdot \rangle$ is used to denote the duality pairing between V and V', here V' is the dual space of V. Function F is assumed to be Gateaux differentiable (see [15]) and there are constants $K > 0, L < \infty$ such that

$$|F'(w) - F'(v), w - v| \ge K ||w - v||_V^2 , \quad \forall w, v \in V , ||F'(w) - F'(v)||_{V'} \le L ||w - v||_V , \quad \forall w, v \in V ,$$
(2.6)

and from which, it is easy to deduce that

$$K \|w - v\|_{V}^{2} \le \langle F'(w) - F'(v), w - v \rangle \le L \|w - v\|_{V}^{2} , \quad \forall w, v \in V .$$
(2.7)

Under assumption (2.6), problem (2.1) and subproblems (2.5) and (2.3) have unique solutions, see [21, p. 35].

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The meaning of the above decomposition is that by there exists $u_i \in V_i$ such that $u \in \Sigma^m_i$, u_i and on the other hand, if $u \in V_i$, then $\sum_{i=1}^{m} u_i \in V_i$ this space can be decomposed as in 2.22, then the following algorithms can be used networks (2.1):

Algorithms 2.1. (An enteritive space detailmost or method A

Step 1. Choose with a convert $f = u^{1} \in V$ and extended products productions g_{1} , $g_{1} \leq 1$. Step 2. For $u \geq 0$, find $u^{1} \in V$, in parallel for d = 1/2, and when that

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In the following, the optation (...) is used to existing she dealing pairing between N and N , have N" is the dual space of N. Foundier, P is assumed to be Grussary defensation (see [15]) and there are constants R , b. C. e. or even this.

$$(2^{2} (a) - 2^{2} (a), a - 2) \ge 63[a - 2]a, a + 2a, a \in \mathbb{N},$$

$$(2.6)$$

$$(2.6)$$

and from which, it is easy to deduce that

$$K[w \to w] \le (F^*(w) \to F^*(w), w \to w] \le C_1[w \to w] \le C_2[w \to w] \le C_2[$$

Under esemption (26% problem (2.1) and subproblems (2.5) and (2.3) have unique solutions, sec [21, p. 35]. For the decomposed spaces, we assume that there is a constant $C_1 > 0$ such that $\forall v \in V$, we can find $v_i \in V_i$ to satisfy:

$$v = \sum_{i=1}^{m} v_i$$
, and $\sum_{i=1}^{m} \|v_i\|_V^2 \le C_1^2 \|v\|_V^2$. (2.8)

Moreover, assume that there is a $C_2 > 0$ such that there holds

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \langle F''(w_{ij})u_i, v_j \rangle \le C_2 \left(\sum_{i=1}^{m} \|u_i\|_V^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \|v_i\|_V^2 \right)^{\frac{1}{2}},$$

$$\forall w_{ij} \in V, \forall u_i \in V_i, \forall v_j \in V_j.$$
(2.9)

Domain decomposition methods, multilevel methods and multigrid methods can be viewed as different ways of decomposing finite element spaces into sums of subspaces. For the estimation of the constants C_1 and C_2 for different type of decomposition of finite element methods for linear problems, one can find the proofs or references in Xu [36].

Later, the error reduction factor for the above two algorithms shall be estimated. In the following, we shall use e^n , $n = 0, 1, 2, \cdots$, which is defined as:

$$e^{n} = |\langle F'(u^{n}) - F'(u), u^{n} - u \rangle|^{\frac{1}{2}},$$

as a measure of the error between u^n and u. Here and later, u stands for the unique solution of (2.1). For convenience, constants α_{min} and α_{max} are defined as $\alpha_{min} = \min_{1 \le i \le m} \alpha_i$, $\alpha_{max} = \max_{1 \le i \le m} \alpha_i$, and α_i is the relaxation parameters in Algorithm 2.1. Constants C_p and C_s , which are

$$C_p = (\alpha_{min}^{-\frac{1}{2}}L + \alpha_{max}^{\frac{1}{2}}C_2)C_1, \qquad C_s = C_2C_1,$$
(2.10)

will play an important rule in analysing the error reduction factor.

Remark 2.1.

(1) When F is differentiable and if we define

$$w_i^{n+\frac{1}{2}} = \sum_{k=1,k\neq i}^m u_k^n + u_i^{n+\frac{1}{2}} , \qquad (2.11)$$

then (2.3) is equivalent to solving

$$\left\langle F'(w_i^{n+\frac{1}{2}}), v_i \right\rangle = 0, \quad \forall v_i \in V_i .$$
 (2.12)

(2) Let

$$u^{n+1} = \sum_{i=1}^{m} u_i^{n+1}, \ n = 0, 1, 2, \cdots$$
 (2.13)

and $w_i^{n+\frac{1}{2}}$ be defined as in (2.11), then

$$w_i^{n+\frac{1}{2}} = u^n + u_i^{n+\frac{1}{2}} - u_i^n$$

and the value of u^{n+1} corresponding to (2.4) can be obtained by

$$u^{n+1} = \sum_{i=1}^{m} u_i^n + \sum_{i=1}^{m} \alpha_i (u_i^{n+\frac{1}{2}} - u_i^n)$$

= $u^n + \sum_{i=1}^{m} \alpha_i (u_i^{n+\frac{1}{2}} - u_i^n)$
= $\sum_{i=1}^{m} \alpha_i (u^n + u_i^{n+\frac{1}{2}} - u_i^n) + (1 - \sum_{i=1}^{m} \alpha_i) u^n$
= $\sum_{i=1}^{m} \alpha_i w_i^{n+\frac{1}{2}} + (1 - \sum_{i=1}^{m} \alpha_i) u^n.$ (2.14)

for the docomposed species we assure that there is a constant (5 > 0 ench that yo 6 N, we can find w 6 N, to estimic

$$u = \sum_{i=1}^{n} u_i, \text{ and } \sum_{i=1}^{n} \|u_i\|_{1}^{2} \leq G^{2} \|u_i\|_{1}^{2}$$
(2.3)

Moreover, assume that there is a Cr > R and that these balks

Domain decomposition multiple, intellected multiplies and multiplie methods can be viewed as different whole of convergencing furthe element spaces into turns of end-concess from the editeration of the constants C₁ and C₂ for defenses type of the only gath is of station distance method. For finear problems, one can find this produce reference in Xu (id)

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In the applications of §5, with a two level domain decomposition method, only the values of u^{n+1} and the coarse mesh problem are needed for the next iteration, and u^{n+1} is updated by the above formula after the computations of each of subdomain problems.

(3) For Algorithm 2.2, if we define

$$w_i^{n+1} = \sum_{k < i} u_k^{n+1} + u_i^{n+1} + \sum_{k > i} u_k^n , \qquad (2.15)$$

then it satisfies

$$\langle F'(w_i^{n+1}), v_i \rangle = 0, \quad \forall v_i \in V_i$$
 (2.16)

and after the solving of w_i^{n+1} from (2.16) for each *i*, we only need to set $u^{n+1} = w_m^{n+1}$.

(4) Intuitively, one may think that the algorithms need rather large amount of memory. However, in the implementation later for a two level domain decomposition, we only need to store the value of uⁿ⁺¹ and one of the w_i^{n+1/2} (the coarse mesh solution) in the memory.

Remark 2.2. Algorithm 2.2 solves the minimization problems sequentially over each subspace. Algorithm 2.1 solves the minimizations in parallel over each of the subspaces. In applications, by suitably decomposing the minimization space, the minimization problem over each subspace can be done by many parallel processors, and so both algorithms are suitable for parallel machines, see §5. Moreover, with a suitable decomposition, the constant C_1 can be made to be independent of the size of the problem, and so the convergence of the above two algorithms also does not depend on the size of the problem.

3. The convergence of the additive algorithm

We first give the rate of convergence for Algorithm 2.1.

Theorem 3.1. If the space decomposition satisfies (2.8), (2.9) and the function satisfies (2.6), then for Algorithm 2.1 we have:

(a). If F is quadratic with respect to v and the norm of V is chosen as $||v||_V = \langle F'(v), v \rangle$, then there holds

$$|e^{n+1}|^2 \le \frac{C_p^2}{1+C_p^2} |e^n|^2 , \forall n \ge 0 .$$
 (3.1)

(b). If F is third order continuously differentiable, then

$$|e^{n+1}| \to 0 \text{ as } n \to \infty \text{ , and } |e^{n+1}|^2 \leq \beta_n |e^n|^2 \text{ , } \forall n \geq 0 \text{ .}$$

For n sufficiently large, we have $0 < \beta_n < 1$. In fact

$$\lim_{n \to \infty} \beta_n = \frac{C}{1+C} < 1 \quad and \quad C = \frac{C_p^{-2}}{K^2} .$$
 (3.2)

Before we go to the proof of the theorem, we first present a lemma which is needed in the proof. The lemma can be proved in a similar way as [21, p. 25], and the proof can be found in [32].

Lemma 3.2. If condition (2.7) is valid, then we have for function F:

$$F(w) - F(v) \ge \langle F'(v), w - v \rangle + \frac{K}{2} ||w - v||_V^2 , \quad \forall v, w \in V ,$$
(3.3)

$$F(w) - F(v) \le \langle F'(v), w - v \rangle + \frac{L}{2} ||w - v||_V^2 , \quad \forall v, w \in V .$$
(3.4)

Proof of Theorem 3.1. Let u^{n+1} and $w_i^{n+\frac{1}{2}}$ be defined as in (2.13) and (2.11). As F is a convex function, by using (2.4), (2.12), (2.14) and (3.3), one obtains

$$F(u^{n}) - F(u^{n+1}) = F(u^{n}) - F\left(\sum_{i=1}^{m} u_{i}^{n+1}\right)$$

In the applications of §5, with a two level domain descrappolition missione, only the volues of of ²⁷ and the course mesh problem, are needed for the regi lightetion, and w²7 is reputated by the above formula after the computations of and of mission problems.

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$$a_{1}^{(n)} = \sum_{i=1}^{n} a_{1}^{(n)} + a_{2}^{(n)} + \sum_{i=1}^{n} a_{1}^{(n)}$$
 (2.15)

then it satisfies

$$(F'(a)^{-1}) = 0, \quad \forall a \in \mathcal{U}_{i}, \quad (2.13)$$

and after the solution of "" ween (2.16) for each 4, set on " word to set a "" ... and (4) Infutively one way think that the electrifices are stated a formation of memory. How . . ever, in the implementation have for a two lives formation becomposition, we only a set to state the completence at an of the state for a formation of the state of

Remark 2.2. Algorithm 3.2 solves the momentum problems reprinted to over mark without Algorithm 2.1 alrest the set of the later of the rest of the rest constraints of the parts of a support saitably decomposing the automateria, and without the relationships (by even with automate by be done by many problem therees as, and without the station rest of the relation of the 3. Moreover, when a sub-site documentation of the station of the station of the state of the indicated as of the size of the problem, and to the constraints of the state being reactions are the indicated as of on the size of the problem, and to the constraints of the state being reactions are to declard on the size of the problem.

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Theorem 3.1. If the more tecomposition addition (2.3) (20) and the Interious particles (2.3). then for Algorithm 3.1 we have

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(b) If F is their order continuously define made. Also, ".

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Before we go to the proof of the threasant, we drat present a brance which is needed in the proof. The lemma can be proved in a visible way as juit, or 37, and the proof out is feared in [13]. I among 3.2

Proof of Theorem 3.1. Let a¹⁷¹ and a¹77 his defined as a \$2.135 and \$2.111. At 37 h a conserv function, by value \$2.45, (2.121, 52.145) and (2.27, from oblection

$$= F(u^{n}) - F\left(\sum_{i=1}^{m} \alpha_{i} w_{i}^{n+\frac{1}{2}} + (1 - \sum_{i=1}^{m} \alpha_{i}) u^{n}\right)$$

$$\geq F(u^{n}) - \sum_{i=1}^{m} \alpha_{i} F\left(w_{i}^{n+\frac{1}{2}}\right) - \left(1 - \sum_{i=1}^{m} \alpha_{i}\right) F(u^{n})$$

$$= \sum_{i=1}^{m} \alpha_{i} F(u^{n}) - \sum_{i=1}^{m} \alpha_{i} F(w_{i}^{n+\frac{1}{2}})$$

$$\geq \sum_{i=1}^{m} \alpha_{i} \langle F'(w_{i}^{n+\frac{1}{2}}), u_{i}^{n} - u_{i}^{n+\frac{1}{2}} \rangle + \frac{K}{2} \sum_{i=1}^{m} \alpha_{i} ||u_{i}^{n} - u_{i}^{n+\frac{1}{2}}||_{V}^{2}$$

$$= \frac{K}{2} \sum_{i=1}^{m} \alpha_{i} ||u_{i}^{n} - u_{i}^{n+\frac{1}{2}}||_{V}^{2} .$$
(3.5)

As u is the solution of (2.1), it satisfies $\langle F'(u), v \rangle = 0$, $\forall v \in V$. For any $v_i \in V_i$, $i = 1, 2, \dots, m$ such that $\sum v_i = u$, we shall use (2.12), and (2.6) to estimate:

$$\langle F'(u^{n+1}) - F'(u), u^{n+1} - u \rangle$$

$$= \langle F'(u^{n+1}), u^{n+1} - u \rangle = \sum_{i=1}^{m} \langle F'(u^{n+1}), u^{n+1}_{i} - v_{i} \rangle$$

$$= \sum_{i=1}^{m} \langle F'(u^{n+1}) - F'(u^{n} + u^{n+\frac{1}{2}}_{i} - u^{n}_{i}), u^{n+1}_{i} - v_{i} \rangle$$

$$= \sum_{i=1}^{m} \langle F''(\theta^{n+1}_{i})(u^{n+1} - u^{n}), u^{n+1}_{i} - v_{i} \rangle - \sum_{i=1}^{m} \langle F''(\theta^{n+1}_{i})(u^{n+\frac{1}{2}}_{i} - u^{n}_{i}), u^{n+1}_{i} - v_{i} \rangle$$

$$(\theta^{n+1}_{i} = \theta u^{n+1} + (1 - \theta) w^{n+1}_{i}, \theta \in [0, 1])$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \langle F''(\theta^{n+1}_{i})(u^{n+1}_{j} - u^{n}_{j}), u^{n+1}_{i} - v_{i} \rangle - \sum_{i=1}^{m} \langle F''(\theta^{n+1}_{i})(u^{n+\frac{1}{2}}_{i} - u^{n}_{i}), u^{n+1}_{i} - v_{i} \rangle$$

$$\leq C_{2} \left(\sum_{i=1}^{m} \|u^{n+1}_{i} - u^{n}_{i}\|_{V}^{2} \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \|u^{n+1}_{i} - v_{i}\|_{V}^{2} \right)^{\frac{1}{2}} + L \sum_{i=1}^{m} \|u^{n}_{i} - u^{n+\frac{1}{2}}_{i}\|_{V} \|u^{n+1}_{i} - v_{i}\|_{V}$$

$$\leq C_{2} \left(\sum_{i=1}^{m} \alpha^{2}_{i}\|u^{n+\frac{1}{2}}_{i} - u^{n}_{i}\|_{V}^{2} \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \|u^{n+1}_{i} - v_{i}\|_{V}^{2} \right)^{\frac{1}{2}}$$

$$+ \frac{L}{\min \sqrt{\alpha_{i}}} \left(\sum_{i=1}^{m} \alpha_{i}\|u^{n}_{i} - u^{n+\frac{1}{2}}_{i}\|_{V}^{2} \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \|u^{n+1}_{i} - v_{i}\|_{V}^{2} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

From the property of the space decomposition (2.8), there exists $\phi_i \in V_i$ such that $u^{n+1} - u = \sum_{i=1}^{m} \phi_i$, and $\sum_{i=1}^{m} \|\phi_i\|_V^2 \leq C_1^2 \|u^{n+1} - u\|_V^2$, So we take $v_i = u_i^{n+1} - \phi_i$ and see that

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$$\sum_{i=1}^{m} \|u_i^{n+1} - v_i\|_V^2 = \sum_{i=1}^{m} \|\phi_i\|_V^2 \le C_1^2 \|u^{n+1} - u\|_V^2 .$$
(3.7)

By combining (3.5)-(3.7), there comes

$$\langle F'(u^{n+1}) - F'(u), u^{n+1} - u \rangle$$

$$\leq C_2 \sqrt{\max \alpha_i} \left(\sum_{i=1}^m \alpha_i \| u_i^{n+\frac{1}{2}} - u_i^n \|_V^2 \right)^{\frac{1}{2}} \cdot C_1 \| u^{n+1} - u \|_V$$

$$+ L \alpha_{min}^{-\frac{1}{2}} \left(\sum_{i=1}^m \alpha_i \| u_i^{n+\frac{1}{2}} - u_i^n \|_V^2 \right)^{\frac{1}{2}} \cdot C_1 \| u^{n+1} - u \|_V$$

$$\leq C_1 \left(\alpha_{max}^{\frac{1}{2}} C_2 + \alpha_{min}^{-\frac{1}{2}} L \right) \left[\frac{2}{K} \left(F(u^n) - F(u^{n+1}) \right) \right]^{\frac{1}{2}} \cdot \| u^{n+1} - u \|_V .$$

$$(3.8)$$

From the property of the space decomposition (3.8.), thus, exists by e^{-N} such that $e^{N+1} + a = \sum_{i=1}^{n} a_{ii}$ and $\sum_{i=1}^{n} A_{ii} R_{ii}$ (b) we take $a = e^{N+1} - a_{ii}$ and $\sum_{i=1}^{n} A_{ii}$ (b) we take $a = e^{N+1} - a_{ii}$ (b)

By combining (3.5)-(3.7), there comes

Let us note that

$$||u^{n+1} - u||_V^2 \le K^{-1} \langle F'(u^{n+1}) - F'(u), u^{n+1} - u \rangle$$
,

therefor there follows from (3.8) that

$$F'(u^{n+1}) - F'(u), u^{n+1} - u) \le C_p \left[\frac{2}{K} \left(F(u^n) - F(u^{n+1})\right)\right]^{\frac{1}{2}} \cdot K^{-1/2} \sqrt{\langle F'(u^{n+1}) - F'(u), u^{n+1} - u \rangle},$$

and so

$$K^{2}\langle F'(u^{n+1}) - F'(u), u^{n+1} - u \rangle$$

$$\leq 2C_{p}^{2}[F(u^{n}) - F(u^{n+1})]$$

$$= 2C_{p}^{2}[F(u^{n}) - F(u) + F(u) - F(u^{n+1})]$$
(3.9)

Summing (3.9) for $n = 0, 1, 2, \dots, N$, we find that

$$\sum_{n=0}^{N} |e^{n+1}|^2 \le 2C_p^2 / K^2 [F(u^0) - F(u^{N+1})] \le 2C_p^2 / K^2 [F(u^0) - F(u)] ,$$

and so

$$|e^{n+1}| \to 0$$
, as $n \to \infty$. (3.10)

We shall first prove (a) and then prove (b). From relations (3.3)-(3.4), there holds

$$F(u^{n}) - F(u) \leq \langle F'(u), u^{n} - u \rangle + \frac{L}{2} ||u^{n} - u||_{V}^{2}$$

= $\frac{L}{2} ||u^{n} - u||_{V}^{2}$, (3.11)

and

$$F(u) - F(u^{n+1}) \leq -\langle F'(u), u^{n+1} - u \rangle - \frac{K}{2} ||u^{n+1} - u||_{V}^{2}$$

= $-\frac{K}{2} ||u^{n+1} - u||_{V}^{2}$. (3.12)

Substituting (3.11) and (3.12) to (3.9) and using (2.7), it gives

$$K^{2}|e^{n+1}|^{2} \leq 2C_{p}^{2} \left(\frac{L}{2}||u^{n}-u||_{V}^{2}-\frac{K}{2}||u^{n+1}-u||_{V}^{2}\right) \leq 2C_{p}^{2} \left(\frac{L}{2K}|e^{n}|^{2}-\frac{K}{2L}|e^{n+1}|^{2}\right) ,$$

which shows that

$$|e^{n+1}|^2 \le \frac{LK^{-1}C_p^2}{K^2 + KL^{-1}C_p^2} |e^n|^2 .$$
(3.13)

As F is quadratic and satisfies (2.7), then $\sqrt{\langle F'(v), v \rangle}$ defines a norm for V and F'(v) is linear with respect to v. In the proof given above, if we choose the norm of V to be

$$||v||_V = \sqrt{\langle F'(v), v \rangle} , \quad \forall v \in V ,$$

then we have K = L = 1 in (2.7), moreover

$$|e^{n}|^{2} = |\langle F'(u^{n}) - F'(u), u^{n} - u \rangle| = |\langle F'(u^{n} - u), u^{n} - u \rangle| = ||e^{n}||_{V}^{2}$$

and so (3.13) implies (3.1).

Let us which that

therefor there ballows from (3.8) that.

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Samming (3.9) for n = 0, 1, 2, ..., M. we find they

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We shall had prove (b) and then prove (b). From remember (a.f.) (0.f.), there had

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Substituting (3.11) and (3.12) to (24) and asing (2.2), a syna

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As P is quadratic and standers (2.7), then $Q^{(2)}(r)$ is distance a norm, for V and N(q) is linear with respect to r_{i} in the proof gives above, W we choose the mean of V to be

then we have K = L = 1 in (2.7), inner yer

(114) and (cl.2) on bas

Next, we prove (b). First, we note that $|e^n| \to 0$ as $n \to \infty$, so there exists a ball $B(u, \delta)$ which is centred at u and is with a radius of δ such that $u^n \in B(u, \delta)$, $\forall n$. As F is third order continuously differentiable, one can assume that there is a constant C(u) such that

$$|F'''(\xi) \cdot (v, v, v)| \le C(u) ||v||_V^3, \quad \forall \xi \in B(u, \delta), \forall v \in V .$$

We use the Taylor's formula (see Cea [15, Chap.2]) to get:

$$F(u^{n}) - F(u) = \langle F'(u), u^{n} - u \rangle + \frac{1}{2} F''(u) \cdot (u^{n} - u)^{2} + \frac{1}{6} F'''(u + \theta^{n}(u^{n} - u)) \cdot (u^{n} - u)^{3} ;$$

$$F(u) - F(u^{n+1}) = -\langle F'(u), u^{n+1} - u \rangle - \frac{1}{2} F''(u) \cdot (u^{n+1} - u)^{2} - \frac{1}{6} F'''(u + \theta^{n+1}(u^{n+1} - u)) \cdot (u^{n+1} - u)^{3} .$$
(3.14)
$$(3.15)$$

Above, $\theta^n, \theta^{n+1} \in [0, 1]$. Summing (3.14) and (3.15), and using (2.7) and the property that $\langle F'(u), v \rangle = 0, \forall v \in V$, it follows that:

$$F(u^{n}) - F(u^{n+1}) = \frac{1}{2}|e^{n}|^{2} - \frac{1}{2}|e^{n+1}|^{2} + I_{1} + I_{2} + I_{3} + I_{4} , \qquad (3.16)$$

where

$$\begin{split} I_{1} &= \frac{1}{2} F''(u) \cdot (u^{n} - u)^{2} - \frac{1}{2} \langle F'(u^{n}) - F'(u), u^{n} - u \rangle ,\\ &\leq C(u) ||u^{n} - u||_{V}^{3} \leq \frac{C(u)}{K^{\frac{3}{2}}} |e^{n}|^{3} ,\\ I_{2} &= -\frac{1}{2} F''(u) \cdot (u^{n+1} - u)^{2} + \frac{1}{2} \langle F'(u^{n+1}) - F'(u), u^{n+1} - u \rangle ,\\ &\leq C(u) ||u^{n+1} - u||_{V}^{3} \leq \frac{C(u)}{K^{\frac{3}{2}}} |e^{n+1}|^{3} ,\\ I_{3} &= \frac{1}{6} F'''(u + \theta^{n}(u^{n} - u)) \cdot (u^{n} - u)^{3} \\ &\leq C(u) ||u^{n} - u||_{V}^{3} \leq \frac{C(u)}{K^{\frac{3}{2}}} |e^{n}|^{3} ,\\ I_{4} &= -\frac{1}{6} F'''(u + \theta^{n+1}(u^{n+1} - u)) \cdot (u^{n+1} - u)^{3} \\ &\leq C(u) ||u^{n+1} - u||_{V}^{3} \leq \frac{C(u)}{K^{\frac{3}{2}}} |e^{n+1}|^{3} , \end{split}$$

$$(3.17)$$

Let $C^* = \frac{4C_p^2 C(u)}{K^{\frac{3}{2}}}$. From relations (3.9) and (3.16)–(3.17), there follows

$$(K^2 + C_p^2) |e^{n+1}|^2 \leq C_p^2 |e^n|^2 + C^* |e^n|^3 + C^* |e^{n+1}|^3 .$$

and so

$$|e^{n+1}|^2 \le \frac{C_p^2 + C^* |e^n|}{K^2 + C_p^2 - C^* |e^{n+1}|} |e^n|^2$$
.

From (3.10), we see $|e^n| \to 0$, and so for n large enough if

$$|e^{n+1}| \le \frac{K^2}{2C^*}, \quad |e^n| \le \frac{K^2}{2C^*},$$
 (3.18)

then

$$\beta_n = \frac{C_p^2 + C^* |e^n|}{K^2 + C_p^2 - C^* |e^{n+1}|} < 1 \; .$$

Moreover

$$\lim_{n \to \infty} \beta_n = \frac{C}{1+C} < 1, \text{ and } C = \frac{C_p^2}{K^2} .$$

Next, we prove (b). First, we note that $|\sigma^2| \rightarrow 0$ as $\sigma \rightarrow co$, to these scate, a ball lifered, which is control at σ and is with a tudies of σ such that $a^2 \in E(a, \delta)$. Ver, As F(a, t) denoted other continuously differentiable, one can assume that there is a constant G(a) such that

$$\mathcal{F}^{m}(\mathfrak{g}) = \{u, v, v\} \in \mathcal{G}(\mathfrak{g}) \| \mathfrak{g} \|_{L^{\infty}} = \mathcal{F}(\mathfrak{g}, \mathcal{G}) = \mathcal{F}(\mathfrak{G}) = \mathcal{F}(\mathfrak{G$$

We not the Taylor's formula (see Con [15, Chap.2]) to get:

$$F(a^{-1}) - F(a) = (a^{-1}(a)) - (a^{-1}(a$$

$$(21, 5) \qquad (3, 4)$$

Above, δ^{n} , $\delta^{n+1} \in [0,1]$. Summing (3.14) and 15.15), and noise (3.5) and the property that (F'(u), v) = 0, $\forall v \in V$. It follows that:

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Let $O^* = \frac{1}{1+1} + \frac{1}{2}$. From matrices (a.2) and (2.16) + (3.17) there follows

from (3.10), we use $|n^{+}| \rightarrow 0$, and so for a large margin θ

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4. The convergence of the multiplicative algorithm

The convergence of Algorithm 2.2 is similar as Algorithm 2.1.

Theorem 4.1. Let the space decomposition satisfies (2.8) and the function satisfies (2.7), then for Algorithm 2.2 we have:

(a). If F is quadratic with respect to v and the norm of V is taken as $||v||_V = \langle F'(v), v \rangle$, then there holds

$$|e^{n+1}|^2 \le \frac{C_s^2}{1+C_s^2} |e^n|^2 , \forall n \ge 0 .$$
(4.1)

(b). If F is third order continuously differentiable, then

$$|e^{n+1}| \to 0 \text{ as } n \to \infty \text{ , and } |e^{n+1}|^2 \le \beta_n |e^n|^2 \text{ , } \forall n \ge 0 \text{ .}$$

For n sufficiently large, we have $0 < \beta_n < 1$. In fact

$$\lim_{n \to \infty} \beta_n = \frac{C}{1+C} < 1 \quad and \quad C = \frac{C_s^2}{K^2} .$$
(4.2)

Proof of Theorem 4.1. Let u^{n+1} and w_i^{n+1} be defined as in (2.13) and (2.15). We see that $u^{n+1} = w_m^{n+1}$. If we also define $w_0^{n+1} = u^n$, we observe that

$$F(u^{n}) - F(u^{n+1}) = \sum_{i=1}^{m} \left(F(w_{i-1}^{n+1}) - F(w_{i}^{n+1}) \right)$$

$$\geq \sum_{i=1}^{m} \langle F'(w_{i}^{n+1}), u_{i}^{n} - u_{i}^{n+1} \rangle + \frac{K}{2} \sum_{i=1}^{m} \|u_{i}^{n} - u_{i}^{n+1}\|_{V}^{2} \qquad (4.3)$$

$$= \frac{K}{2} \sum_{i=1}^{m} \|u_{i}^{n} - u_{i}^{n+1}\|_{V}^{2} .$$

Similar as in the proof of (3.6), there exists $v_i \in V_i$, $i = 1, 2, \dots, m$ such that $\sum v_i = u$. Using (2.16), and (2.7) to estimate:

$$\langle F'(u^{n+1}) - F'(u), u^{n+1} - u \rangle$$

$$= \langle F'(u^{n+1}), u^{n+1} - u \rangle = \sum_{i=1}^{m} \langle F'(u^{n+1}), u^{n+1}_{i} - v_{i} \rangle$$

$$= \sum_{i=1}^{m} \langle F'(u^{n+1}) - F'(w^{n+1}_{i}), u^{n+1}_{i} - v_{i} \rangle$$

$$= \sum_{i=1}^{m} \langle F''(\theta^{n+1}_{i})(u^{n+1} - w^{n+1}_{i}), u^{n+1}_{i} - v_{i} \rangle \qquad (\theta^{n+1}_{i} = \theta u^{n+1} + (1 - \theta)w^{n+1}_{i}, \theta \in [0, 1])$$

$$= \sum_{i=1}^{m} \sum_{j>i} \langle F''(\theta^{n+1}_{i})(u^{n+1}_{j} - u^{n}_{j}), u^{n+1}_{i} - v_{i} \rangle$$

$$\leq C_{2} \left(\sum_{i=1}^{m} \|u^{n+1}_{i} - u^{n}_{i}\|_{V}^{2} \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \|u^{n+1}_{i} - v_{i}\|_{V}^{2} \right)^{\frac{1}{2}} .$$

$$(4.4)$$

Take v_i such that (3.7) is valid. By combining (4.3)–(4.4), there comes

$$\langle F'(u^{n+1}) - F'(u), u^{n+1} - u \rangle$$

$$\leq C_2 \left(\sum_{i=1}^m \|u_i^{n+1} - u_i^n\|_V^2 \right)^{\frac{1}{2}} \cdot C_1 \|u^{n+1} - u\|_V$$

$$\leq C_1 C_2 \left[\frac{2}{K} \left(F(u^n) - F(u^{n+1}) \right) \right]^{\frac{1}{2}} \cdot \|u^{n+1} - u\|_V .$$

$$(4.5)$$

The convergence of Algorithm 2.2 is scalar as Algorithm 2.1.

Theorem 4.1. (A the space decomposition actuality (2.6) and the function putation (2.7), then for Algorithm 2.2 we asso

(a). If f' is geometric with respect to a and the source of i' is follow as (a) as (i')(i', i, i), then it are holds

$$(1.8)$$
 $0 \le a^{-1}, b^{-1} \le b^{-1} = b^{-1}$

(b). If F is third order contractionally differentiality, these

For a sufficiently length are have 0 < 3, < 1. In fact

$$\frac{2}{10} = \frac{1}{10} < 1 \quad \text{and} \quad C = \frac{1}{10}$$

Proof of Theorem 1.1. Let w^ath and of ¹⁴ be defined as is (2.13) and (2.45). We are then a¹⁴⁴ = a w₂⁴⁴. If we also define w^{ath} = a², we observe that

Similar as in the proof of (2.6), there exists up 5 N₂ is = 1.2, ..., margin that $\sum p_i = 0$. Simpler (2.16), and (2.7) to estimate:

Take of such that (3.7) is vilid. By combinize (4.3)- (4.4), there enter

Similar as in getting (3.9), one deduces from (4.5)

$$K^{2}\langle F'(u^{n+1}) - F'(u), u^{n+1} - u \rangle$$

$$\leq 2C_{s}^{2}[F(u^{n}) - F(u^{n+1})]$$

$$= 2C_{s}^{2}[F(u^{n}) - F(u) + F(u) - F(u^{n+1})]$$

The rest of the proof is the same as for Theorem 3.1.

5. Applications to linear and nonlinear elliptic problems

In this section, the space decomposition algorithms will be applied to linear problem:

$$\begin{cases} -\nabla \cdot (a\nabla u) = f \text{ in } \Omega \subset R^2 ,\\ u = 0 \text{ on } \partial\Omega . \end{cases}$$
(5.1)

and to nonlinear elliptic problem

$$\begin{cases} -\nabla \cdot (|\nabla u|^{s-2} \nabla u) = f \text{ in } \Omega \subset R^2 \ (1 < s < \infty) \ ,\\ u = 0 \text{ on } \partial \Omega \end{cases}$$
(5.2)

Defining

$$V = H_0^1(\Omega), \quad F(v) = \int_\Omega \left(rac{1}{2}|
abla v|^2 - fv
ight) dx \; ,$$

it is known that problem (5.1) is equivalent to solving (2.1). More general boundary conditions for (5.1) can also be considered. Correspondingly, we just need to modify the definitions of space V and function F.

For equation (5.2), we assume $f \in W^{-1,s'}(\Omega)$, $\frac{1}{s} + \frac{1}{s'} = 1$. By standard techniques, it can be shown, see [21], that (5.2) possesses a unique solution which is the minimizer of problem

$$\min_{v \in W_0^{1,s}(\Omega)} \left[\frac{1}{s} \int_{\Omega} |\nabla v|^s - \langle f, v \rangle \right]$$

This problem appears in certain mathematical models describing the mechanical deformation of ice (see for example [23] [24]). Even with very smooth data, the solution u may not be in the space $W_0^{2,s}$, see Ciarlet [16, p.324]. When s is close to 1 or is very big ($s \gg 2$), it is difficult to solve this problem numerically.

We see that the algorithms can also be used to compute the full potential equation for the velocity potential of fluid flows:

$$-\nabla \cdot (\rho(|\nabla u|)\nabla u) = 0 ,$$

where the density is given in terms of the potential

$$\rho(q) = \rho_{\infty} \left(1 + \frac{r-1}{2} M_{\infty}^2 \left(1 - \frac{q^2}{q_{\infty}^2} \right) \right)^{\frac{1}{r-1}} .$$

For the derivation of this equation and for the meaning of the parameters, consult [13] and [17]. Suitable boundary conditions should be supplied. If the flow is everywhere subsonic, this problem fits into our frame work. This equation has important applications in aerospace industry. For recent numerical results by domain decomposition for this equation, see [12] and [13].

In the following, we shall use a two level domain decomposition as a space decomposition method to solve these problems. Numerical experiments show very good convergence properties. For problems like (5.2), the method is especially appealing. Normally, one uses a linearization method or other type iterative methods to solve (5.2), see [24], [23]. These methods converge slowly when the size of the problem is big. The proposed algorithms here reduce problem (5.2) into many smaller size problems of the same kind. These smaller size problems can be solved more efficiently than the original large size problem. Moreover, we solve these subproblems iteratively. The analysis tells that this iterative procedure produces a solution that converges to the true solution with a rate that is independent of the size of the original problem. For the kind of problems as in (5.2), by first decomposing the problem and then linearising, we gain efficiency compared to first linearising and then decomposing the problem.

Similar as in cetting (3.9), the sedness from (4.5)

 $[(1,1,2)]_{ij}^{ij} = [(0)]_{ij}^{ij} + [(0)]_{ij}^{ij} = ((0))_{ij}^{ij} = (((0)))_{ij}^{ij} = ((()))_{ij}^{ij} = ((()))_{ij}^{ij} = ((()))_{ij}^{ij} = ((()))_{ij}^{ij} = ((()))_{ij}^{ij} = ((()))_{ij}^{ij} = (()))_{ij}^{ij}$

The rest of the proof is the same as for Thinness 3.1.

5. APPERATES TO LINUAR AND ROMINIAN SLEPTIC PAUSIANS

An this acction, the space decomposition she will be applied to these pro-

and to polyimose elliptic problem

$$\left\{ -7 - 0(7m)^{-2}(n) = 0^{-2}(n)^{2} C R^{2} \left(1 + 0 + 2m) \right\},$$

$$\left\{ -7 - 0 + 0 + 2m R^{2} \right\}$$

$$\left\{ -7 - 0 + 0 + 2m R^{2} \right\}$$

$$\left\{ -7 - 0 + 2m R^{2} \right\}$$

Defining

$$= a_{1}(\alpha), \quad b(\alpha) = \int_{\Omega} \left(\frac{1}{2} [\nabla \alpha]^{2} - \langle \alpha \rangle \right) d\alpha = 0$$

it is logent that problem (5.1) is equivalent to adving (2.1). Mure crusted benedary consistents for (5.1) can also be considered. Consequentingly, we just used to aboutly the definitions of quare V and function V

For equation (5.2), we are analy $f \in \mathbb{N}^{2n+1}(n)$, $f \in \frac{1}{2}$ with the constant methodown is can be shown, see [21]; that (3.3) possibles a unique solution which is the minimizer of closing

This problem appears in vertient mathematical tradels describing the talebaserial differences of the tradels describing the talebaserial differences and the talebaserial of the talebaser of talebaser of talebaserial (talebaserial differences of talebaserial differences of talebaserial differences of talebaserial differences of the talebaserial differences of talebaserial differe

The are that the algorithms can also be used to complete the fail potential equation for the

where the density is given in recup of the parential

For the derivation of this canation and for the meaning of the parameters, zeneich (3.2), and (3.2). Saittible boundary conditions about his supplied. If the fort is everywhere then it in produces the into our frame work. This reputitor, but important, cycloations in annotative tains (1.5) recent museched menter by domain decomposition for the version on (3.5) and (3.5).

in the bilineous, we wind use a two basis dependences along an a space bide position and hod to solve these problem. Namenical experiments along very good convergence properities for problems like (5.2), the method is expectally appealing. Should by one very a branched method or other type iterative methods to erive (5.2), we (5.4), [33]. These methods convergence alowly when the true of the problem is big. The properiod approximation (5.2) into many smaller size of the problem is big. The properiod approximation to relate the solution into many smaller size of the problem of the same inter. These samples is a problem and the solution (5.2) affected is than the original large size problem. Measured, we aslow these subgradues is relatively efficiently than the original large size problem. Measured, we aslow these subgradues is relatively the analysis tells that this intractive provident products a related data to be solved with a problems as in (5.2), by first data the structure the structure to data original or the time true problems as in (5.2). The first decomposing the problem and the data of the data converges to the time true problems as in (5.2). The first data the structure the structure the true of the true true problems as in (5.2).

5.1. Decomposition of the finite element space .

As the multigrid method [25], multilevel method, domain decomposition method [35] can be viewed as different ways of decomposing the finite element space, the proposed algorithms of this paper can use them to solve the above problems. The analysis indicates that the convergence does not depend on the regularity of the solution, it only depends on the lower and upper bound of the differential operator, i.e. the constants L and K. In this section, a two-level domain decomposition, i.e., an overlapping domain decomposition with a coarse mesh shall be used. For the two level method, let $\{\Omega_i\}_{i=1}^M$ be a shape-regular finite element division, or a coarse mesh, of Ω and Ω_i has diameter of order H. For each Ω_i , we further divide it into smaller simplices with diameter of order h. In case that Ω has a curved boundary, we shall also fill the area between $\partial\Omega$ and $\partial\Omega_H$, here $\bar{\Omega}_H = \bigcup_{i=1}^M \bar{\Omega}_i$, with finite element subdivision of Ω , see Ciarlet [16]. We call this the fine mesh or the h-level subdivision of Ω with mesh parameter h. We denote $\Omega_h = \bigcup \{T \in \mathcal{T}_h\}$ as the fine mesh subdivision. Let $S_0^H \subset H_0^1(\Omega)$ and $S_0^h \subset H_0^1(\Omega)$ be the continuous, piecewise linear function spaces, with zero trace on $\partial\Omega_H$ and $\partial\Omega_h$, over the H-level and h-level subdivisions of Ω



Figure 5.1. The coloring of the subdomains and the coarse mesh grid

For each Ω_i , we consider an enlarged subdomain $\Omega_i^{\delta} = \{T \in \mathcal{T}_h, dist(T, \Omega_i) \leq \delta\}$. The union of Ω_i^{δ} covers $\overline{\Omega}_h$ with overlaps of size δ . Let us denote the piecewise linear finite element space with zero traces on the boundaries $\partial \Omega_i^{\delta}$ as $S_0^h(\Omega_i^{\delta})$. Then one can show that

$$S_0^h = S_0^H + \sum S_0^h(\Omega_i^\delta) .$$
 (5.3)

For the overlapping subdomains, assume that there are m colors such that each subdomain Ω_i^{δ} can be marked with one color, and the subdomains with the same color will not intersect with each other. For suitable overlaps, one can always choose m = 2 if d = 1; $m \leq 4$ if d = 2; $m \leq 6$ if d = 3, see Figure 5.1. Let Ω'_i be the union of the subdomains with the i^{th} color, and

$$V_i = \{ v \in S_0^h | \quad v(x) = 0, \quad x \notin \Omega_i' \} .$$

By denoting subspaces $V_0 = S_0^H$, $V = S_0^h$, we find that decomposition (5.3) means

$$V = V_0 + \sum_{i=1}^{m} V_i,$$
(5.4)

and so the two level method is a way to decompose the finite element space. Let $\{\theta_i\}_{i=1}^m$ be a partition of unity with respect to $\{\Omega'_i\}_{i=1}^m$, i.e. $\theta_i \in C_0^{\infty}(\Omega'_i \cap \Omega)$ and $\sum_i^m \theta_i = 1$. It can be chosen so that $|\nabla \theta_i| \leq C/\delta$. Let I_h be an interpolation operator which uses the function values at the

5.1. Decomposition of the finite element space .

As the multiplied multiplies multiplies, realizioned and halfs demand decomposition multipli [35], can be viewed as different areas of invariantial to the fails demand quees, the proposed significant of the add depend on the resultation of the convertes 1 and K. In this method, a multiplier demands of the differential operator, i.e. the convertes 1 and K. In this method, a non-local data and down a down of the second of the mathod, for $\{0,1\}^2$, be a shape-regular band K. In this method, a second data and down a down mathod, for $\{0,1\}^2$, be a shape-regular band K. In this method, and is could down a down mathod, for $\{0,1\}^2$, be a shape-regular band K. In this method, and is a subdimense of order K. For a shape-regular band K is the method K of the down K of the mathod, for $\{0,1\}^2$, be a shape-regular band K is the method K of the down K of the dimense of order K. The math K is the method of the second K of the down K of the dimense of order K. The math K is the method K of the dimense of the second K of the dimense of the dimense of K is case that K has a shape-regular band K is the method K of the dimense of the dimense 0 of the second K is the method K of K of K of K of K of the dimense of K of the dimension of the second K is the second K of K of K of K of K of the dimense of K of the dimension of the second K of Kof K and K is the dimension of K of the dimense of K of



Figure 5.1. The coloring of the solution and the costee mesh and

For each Ω_{i} , we consider an entropy subdecentral $\Omega_{i} = \{T \in \mathcal{T}_{i}, dist(T, \Omega_{i}) \leq S\}$. This indice of Ω_{i}^{i} covers Ω_{i} with overlaps of size δ . Let us denote the preserves have finite element operation areas traces on the boundation $\delta\Omega_{i}^{i}$ as $S_{i}^{i}(\Omega_{i}^{i})$. From our can show that

For the occuracy and the second many there are no bolors are in the cash and the cash made to an C_1 can be marked with one color, and the standomains with the state color with an intersect with duch with respectively. For suitable overlaps, one can always choose at = 2 if d = 1, $m \leq 4$ if d = 2, $m \leq 5$ if d = 1, $m \leq 5$ if d = 1.

By demoting references $V_{0} = S_{0}^{\mu}$, $V = S_{0}^{\mu}$, we find that decomposition (5.61 contra

and so the two level method is a way to decompose the filler dependentions A by $\{A_i\}_{i=1}^{n}$ be a partition of unity with respect to $\{B_i\}_{i=1}^{n}$ be $B_i \in G_i^{n}(B_i)$ B_i and $\sum_{i=1}^{n} b_{i+1}$ and A be an an observation of that $|\nabla B_i| \leq C/A$. Let B_i be an intervaluence operator which uses any function values at the

h-level nodes. For any $v \in V$, let $v_0 \in V_0$ be the solution of $(v_0, \phi_H) = (v, \phi_H), \forall \phi_H \in V_0$, and $v_i = I_h(\theta_i(v - v_0))$. They satisfy $v = \sum_i^m v_i$, and

$$\|v_0\|_{H^1(\Omega)}^2 + \sum_{i}^m \|v_i\|_{H^1(\Omega_i)}^2 \le \frac{CH^2}{\delta^2} \|v\|_{L^2(\Omega)}^2 + C\|\nabla v\|_{L^2(\Omega)}^2 \le C(1 + \frac{H^2}{\delta^2}) \|v\|_{H^1(\Omega)}^2.$$
(5.5)

The proof of (5.5) can be found in different places, we refer to Xu [36, p. 608]. Moreover, using the Cauchy-Schwarz inequality, it is easy to see that

$$\left|\sum_{i=1}^{m}\sum_{j=1}^{m}(a\nabla u_{i},\nabla v_{j})\right| \leq C\sum_{i=1}^{m}\|u_{i}\|_{H^{1}}\sum_{j=1}^{m}\|v_{j}\|_{H^{1}} \leq Cm\left(\sum_{i=1}^{m}\|u_{i}\|_{H^{1}}^{2}\right)^{\frac{1}{2}}\left(\sum_{j=1}^{m}\|v_{j}\|_{H^{1}}^{2}\right)^{\frac{1}{2}}$$
(5.6)

Estimates (5.5) and (5.6) show that for overlapping domain decomposition, the constants in (2.8) and (2.9) are

$$C_1 = C\sqrt{1 + \frac{H^2}{\delta^2}}, \qquad C_2 = Cm.$$

By requiring $\delta = c_0 H$, where c_0 is a given constant, we have that C_1 and C_2 are independent of the mesh parameters h and H, the number of subdomains, and estimate (5.5) is also valid for 3D problems. So if the proposed algorithms are used, their error reductions per step are independent of these parameters.

The error estimates (3.1) and (4.1) predict that the error reduction per step in the energy norm for linear problems is independent of the regularity of the solution, and is also independent of the lower and upper bound of the differential operator. It only depends on the parameters C_1 and C_2 . However, when the coefficient has large jumps, we need to prove (5.5) with the energy norm instead of the usual H^1 -norm. Results by [9] and [39] present some elementary techniques for estimating the constant C_1 in this situation. The constant C_1 does not depend on the jumps, but depends now on the mesh parameters and its dependency is different for 2D and 3D problems, see [19], [3] for the estimations.

5.2. Applications to linear elliptic equations .

As was shown above, the two level method is a space decomposition method. With the coarse mesh, the number of the subspaces is m = 5, see Figure 5.1. For Algorithm 2.1, by defining $w_i^{n+\frac{1}{2}}$ as in (2.11), the subproblems that need to be solved in each subdomain is

$$\begin{cases} (a\nabla w_i^{n+\frac{1}{2}}, \nabla v_i) = (f, v_i), \quad \forall v_i \in S_0^h(\Omega_i') ,\\ w_i^{n+\frac{1}{2}} = u^n \text{ on } \partial \Omega_i' , \end{cases}$$
(5.7)

and $w_i^{n+\frac{1}{2}} = u^n$ in $\Omega \setminus \Omega'_i$. If we define $w_H^{n+\frac{1}{2}} = u_0^{n+\frac{1}{2}} - u_0^n \in S_0^H(\Omega)$, then the coarse mesh problem is

$$(a\nabla(u^{n} + w_{H}^{n+\frac{1}{2}}), \nabla v_{H}) = (f, v_{H}), \quad \forall v_{H} \in S_{0}^{H}(\Omega).$$
(5.8)

For Algorithm 2.2, for the coarse mesh problem, if we let $w_H^{n+1} = u_0^{n+1} - u_0^n$, then it satisfies

$$(a\nabla(u^n + w_H^{n+1}), \nabla v_H) = (f, v_H) , \quad \forall v_H \in S_0^H(\Omega) .$$

$$(5.9)$$

After solving the coarse problem, let w_i^{n+1} be defined as in (2.15), then the subdomain problems are

$$\begin{cases} (a\nabla w_i^{n+1}, \nabla v_i) = (f, v_i), \quad \forall v_i \in S_0^n(\Omega_i') ,\\ \text{if } i > 1, \ w_i^{n+1} = w_{i-1}^{n+1} \text{ on } \partial \Omega_i' ,\\ \text{if } i = 1, \ w_i^{n+1} = u^n + w_H^{n+1} \text{ on } \partial \Omega_i' , \end{cases}$$
(5.10)

and for each i

$$w_i^{n+1} = \begin{cases} u^n + w_H^{n+1} , & \text{if } i = 1, \\ w_{i-1}^{n+1} , & \text{if } i > 1, \end{cases} \text{ in } \Omega \backslash \Omega_i' .$$
(5.11)

i-level codes. For any $v \in V$, her $v_0 \in V_0$ be the solution of $(v_0, \phi_B) = (v, \phi_B)$, $\forall \phi_B \in V_0$, and $u_0 = f_0(\theta_0(v - v_0))$. They eathly $v = \sum_{i=1}^{n} v_0$ and

The proved of (5.5) can be found in different phones, we return to Na [36, p. 506]. Accessen, using the Cauchy Schwarz inequality, it is seen to see that

batristes (5.5) and (5.6) show that for overlapping domain deep manifold, the constraint (2.5) and (2.9) are

By requiring 6 wears a new or is a given conjunt, we have that 6, and 65 are independent of the mesh parameters is and 21, the number of subdottsma, and statutate (5,5) is this valid for 50 problems. So if the proposed elgenitates are first; these error estations per step ere indipendent of these parameters.

The error expresses (3.1) and (4.1) presses that the end which as provided in the comparation for linear problems is independent of the regularity of the contract and react references into lower and upper bound of the fifterential operator. If only declarity of the contract of and G. However, when the continuent has large maps are real contract (5.5) with the contract of a instead of the small R² - norm. However, it and (60) present your (5.5) with the contract of the estimating the contract of the interaction is the first and (60) present your definition the first of the dependences for a the track of this structure is found (60) present your definition is an interact to depend of the structure of the interaction of the contract of the structure of the dependences for the contract of the interaction of the contract of the interaction of the dependences for the contract of the interaction of the contract of the interaction of the dependences for the contract of the interaction of the contract of the interaction of the dependences for the contract of the interaction of the contract of the interaction of the contract of the structure of the dependence of the contract of the structure of the interaction of the interaction of the structure of the dependence of the contract of the structure of the structure of the structure of the structure of the dependence of the structure of th

5.2, Applications in Roser alliptic equations

As was shown above, the two level method is a space decomposition mathod. With the creater total, the trember of the subspaces if m = 5, see Figure 5.1. For fill writh: 2.1. by defining $m^{21/2}$ as in (2.11), the subgrabients this steed to be adved in each withdomain to

and many a market is an define and a said - mark fill, that the cases math perible d

$$(33) = (33) = (37)^{1/2} = (3)^{1/2} = (33$$

For Alvantair 2.2. for this course much moblem. If we let off?" south " - off duties is sateling ?"

$$(12) = (12)^{-1} (12)^{-$$

After solving the coarse problem, let wi¹⁴³ be defined as in (2.15), and the subformit problems

$$\left\{ \left\{ \left\{ e^{\nabla u_{i}}^{2}, \nabla u_{i} \right\} = \left\{ \left\{ i \neq u \right\}, \forall u \in \mathcal{A}_{i}^{2}(0) \right\} \right\}$$

$$\left\{ \left\{ i \neq u \}, u \notin^{2} = u_{i}^{2} \left\{ u = \partial u_{i}^{2} \right\}, u \in \mathcal{A}_{i}^{2}(0) \right\}$$

$$\left\{ i \neq u \}, u \notin^{2} = u_{i}^{2} \left\{ u = \partial u_{i}^{2} \right\}, u \in \mathcal{A}_{i}^{2}(0)$$

$$\left\{ i \neq u \}, u \notin^{2} = u_{i}^{2} \left\{ u = \partial u_{i}^{2} \right\}, u \in \mathcal{A}_{i}^{2}(0) \right\}$$

and for each i

After computation of the subdomain problems and the coarse mesh problem, for Algorithm 2.1, solution u^{n+1} is updated by

$$u^{n+1} = \alpha_0(u^n + w_0^{n+\frac{1}{2}}) + \sum_{i=1}^4 \alpha_i w_i^{n+\frac{1}{2}} + (1 - \sum_{i=0}^4 \alpha_i)u^n$$

and this is needed for the boundary values for the subdomain problems for the next iteration and for computing the residual for the coarse mesh problem. For Algorithm 2.2, one simply sets

$$u^{n+1} = w_{4}^{n+1}$$
.

For each i, domain Ω'_i is the union of the disjoint subdomains of the same color. Thus, the computations of (5.7) and (5.10) can be done in parallel in each of the subdomains of the i^{th} color. For Algorithm 2.1, the computations for different i can again be done in parallel.

For the linear problem, the above formulation shows that Algorithm 2.1 and Algorithm 2.2 reduce to the standard additive and multiplicative Schwarz algorithms, see [35, Chap. 5]. In literature, the condition number of the matrices for the additive and multiplicative methods are estimated for different types of space decomposition, see [36], and then the conjugate gradient method is applied to accelerate the convergence. However, it may not be possible to use conjugate gradient method to accelerate the convergence for nonlinear and nonsymmetric problems.

Example 5.1. In this example, Algorithm 2.2 is tested for the case that $a = e^{xy}$, $u = \sin(3\pi x)\sin(3\pi y)$, $\Omega = [0, 1] \times [0, 1]$. Uniform mesh is used both for the coarse mesh problem and fine mesh problem. For a given N, the coarse mesh size is taken as $H = Hx = Hy = \frac{1}{N}$. The fine mesh is then taken as $h = hx = hy = \frac{1}{N^2}$. Each subdomain is extended by M elements to get overlaps. In Table 5.2, the initial guess is taken as the coarse mesh solution. In Table 5.4, the initial guess is taken as

Figure 5.3 is the computed solution and errors. One sees that it is reducing the maximum error from 10^4 to 10^{-3} in about 9 steps. The same kind of convergence was also observed with tests of other smooth solutions.

u

Iteration	max-error	reduction
0	3.0552e-01	
1	0.0729	0.24
2	0.0166	0.23
3	0.0036	0.22
4	0.0017	0.46
5	0.0015	0.88

Table 5.2. Maximum error with H=1/10, h=1/100, M=2.

After computation of the subdownin problems and the course such problem, for Argonishra 2.1. solution u^{e+1} is updated by

$$S^{n+1} = c_0(u^n + v_0^{n+1}) + \sum_{i=1}^{n} c_i c_i^{n+1} + (i + \sum_{i=1}^{n} c_i c_i^{n+1}) = b^{n+1}$$

and this is moded for the houndary values for the infolments problems for the most than the new transform and for comparing the residual for the colors must need for Algorithm 2.2, and finally ners

For each 4, domain 11, in the polest of the depolet subdemains of the sense color. Thus, the computations of (5.7) and (5.10) and be denote parallel to each of the abdumnation (2.0m)? color. For Algorithm 2.1, the computations for defend 4 are again be and a parallel.

For the threat problem the above to real actual stores which that Alershine will and Alershine 2.2 relates to it a standard within and polyanoshive Schwerz alershine are (20. Chap. 5). In Star office, the conduitor transferred the transferrers for thread thirty and matchin active realburk are materialed for different types of areast the transferrers for thread thirty and then the sector transferrer are material a sprint to succeedence and the transferrers for the sector and the sector and the material a sprint to succeedence and the transferrers for the sector and the sector and the gradient method to accelerate the transferrer for transferrer is the sector and by the gradient method to accelerate the transferrer for transferrer is the sector and by the sector and the gradient method to accelerate the transferrer for transferrer is the sector and by the sector of the sector.

Example 5.1 In this exempts, Algorithms 2.2 is bound for the case that $a = e^{-2}$ to $a = 400(3 \pm 2) \sin(3 \pm 2)$ $\Omega = [0, 1] \times [0, 1]$. Further, much is then both for the contex much problem and the much confidenfor a given N, the course mean way is taken by $M = M = M = M = \frac{1}{2}$. The fact return is then taken as $h = hc = \frac{1}{2}$. Roth exterior are is entropic of N meaning in the set of the line of the line taken to be the lattice area is taken as the course mean is entropic. In fact, we want the lattice is the line taken to be the lattice area is taken as the course mean when the lattice is $\frac{1}{2}$. In the lattice is the lattice is the lattice of the mean in the lattice is the lattice of the lattice is the lattice

Figure 5.3 in the compared solution and errors. One sees that it is reducing the malanizer or or or from 10° and 10° is about 5 steps. The same had of our regences was also class bed with bests of other amount solvetons.

Table 5.2 Marinette with Marine 1.19.5 State



Figure 5.3. The computational for $H = 1/10, h = 1/100, M = 2, u^0$ as in (5.12).

Iteration	max-error	reduction
0	10000.9995	
1	3248.4466	0.32
2	760.5149	0.23
3	93.3381	0.12
4	8.0150	0.09
5	1.5830	0.20
6	0.2481	0.16
7	0.0346	0.14
8	0.0034	0.10
9	0.0015	0.43
10	0.0014	1.00

Table 5.4. Maximum error with $H = 1/10, h = 1/100, M = 2, u^0$ as in (5.12).

Example 5.2. In this example, Algorithm 2.1 is used to compute the same problem of the last example. As m = 5, the relaxation parameters are taken as $\alpha_i = \frac{1}{5}$. Algorithm 2.1 can use more processors, but the error reduction per step is not as good as Algorithm 2.2. With overlap size $\delta = \frac{H}{5}$, the error reduction per step is nearly always around 0.85 for different mesh sizes. Figure 5.5 demonstrates the relation between the error and iteration number for initial guess $u^0 = u_H$, where u_H is the coarse mesh solution. Figure 5.6 shows the errors and error reduction with u_0 taken as in (5.12).

Different tests were done for Algorithm 2.1 and the convergences are similar. For a given initial guess, it always reduces the error by a factor ≈ 0.86 . When the computed solution is getting closer to the global FEM solution, the error reduction is getting closer and closer to 1. Nearly in all tests, we find that if the error reduction factor is bigger than 0.95, then the computational error is always less than two times the global FEM solution error. However, the error reduction factor depends on the overlapping size. If we decrease it $(\delta < \frac{H}{5})$, then the reduction number becomes bigger. If we increase it, the reduction number becomes smaller.



Figure 5.5. The computational for H = 1/10, h = 1/100 M = 2. e^{0} , h in (5.13).

Table 5.4. Mexican erect with B = 1/10.2 = 1/10.2 = 2, 2 = 2, 1

Different tests were done for Algorithm 2.1 and the convergences are singles. For a down-initial guine, it always reluces the error by a factor w 0.56. When theorem and all statements of the convergences are singled with a guine of the convergences are singled and the convergence of the convergence of the convergences are singled and the convergences are singled and the convergence of t

Example, 5.3. In this mainple, Algorithm 2.4 is used to compute the same mobilish is the last example. As m = 5, the control of parametric are taken as $m = \frac{1}{2}$. Algorithm 2.1 can use more processors, but the error relation per step is not as good as Algorithm 2.2. With control are $h = \frac{1}{2}$, the error relation per step is nearly size a good as Algorithm 2.2. With control are b.5 dynamications and the last step is nearly size and as good as Algorithm 2.2. With control are b.5 dynamications and the second step is nearly size and an end of the filterest medicates. Figure b.5 dynamications are the main and the step and an area areas and the areas area and an area where u_{ij} is the course final metric of Figure b.5 shows the errors and step minimum of 1.3. Taken as in (0.13).



Figure 5.5. The computational results with H = 1/10, h = 1/100, M = 2, $u^0 = u_H$, error reduction ≈ 0.86 .



Figure 5.6. The computational results with $H = 1/10, h = 1/100, M = 2, u^0$ as in (5.12), error reduction ≈ 0.86 .

5.3. Applications to nonlinear elliptic problems.

The Gauss-Newton method (Matlab subroutine fiminu) is used to solve the minimization problems (2.3) and (2.5). Without using the domain decomposition, the original problem is simply too large and costly to be solved. With 500 grid points, we are already run out of memory and it takes days to compute the global problems. With the domain decomposition, we can compute the problem with 10^5 unknowns.

Example 5.3. We use an analytical solution $u = \sin(2\pi x) \sin(2\pi y)$ on $\Omega = [0,1] \times [0,1]$ to test the Algorithm 2.2. Figure 5.7 and Table 5.8 show the computational results with fine mesh hx = $hy = \frac{1}{100}$, and coarse mesh $Hx = Hy = \frac{1}{10}$. Each subdomain is extended by 2 elements to get overlaps. The initial guess is the coarse mesh solution. The value of s is 3. For this test problem, $a(u) = |\nabla u|^{s-2} = 0$ at some points. This violates the normal assumption a(u) > c > 0. Our error analysis is still valid for this problem, however, the numerical results in Figure 5.7 shows that the computational error is bigger near the points that a(u) = 0.

Tests with different overlapping sizes were also done. The error reduction number is smaller with bigger overlapping size and becomes bigger if we decreases the overlapping size. According to the error analysis, for the nonlinear problem, the convergence may be slow in the beginning if the initial guess is not good enough. However, numerical tests show that the algorithms converge for arbitrary initial guess and the error reduction does not depend on the initial guess, see Table 5.9.



Figure 5.5. The appropriational results with H=1/10, h=1/100, M=2





5.3. Applications to nonlinear elliptic problem.

The Gauss-Merrica method (Madah referenting farma) is used to gobe the administration problangs (2.3) and (2.5). Without using the densate decomposition, its function in the barrier public to a large and cardy to be solved. With 500 grid politic, we are sites strendy out up of memory and its takes days to compute the global problem. With the demice decomposition with 10⁵ rulenovas.

Example 5.2 We use an analytical valuence u + marking products on $U = \{0, 1\}$ is $\{0, 1\}$ and the Algorithm 2.2. Figure 5.3 and Fable 5.5 choos the company interval marks with the avel $hg = g_{0}$, and coarse much file $u = d_0$ u_{0} . Such attributence u is seconded by 3 direction to u_{0} and u_{0} a

Tests with different overlapping sizes were also done. The error reduction semifies is multiwith bigger overlapping due and become bigger if we donnesses the overlapping tax. Separating to the error analysis, for the nonlinear problem the correspondence may be show in the backming of deinitial group is not good enough. Elevenne, sumerical next phow that the elevation overcept do whitney initial group is not good should be error reduction does out depend on the time test, for this with the orbitancy initial event and the error reduction does out depend on the multi-prove test, for this figure, for the error reduction does out depend on the ratio of the back the second secon



Figure 5.7. The computational results for the nonlinear problem by Algorithm 2.2.

Iteration	max-error	reduction
0	0.1345	
1	0.0257	0.19
2	0.0049	0.19
3	0.0012	0.24
4	0.0007	0.57
5	0.0007	1.01

Table 5.8. Maximum error for the nonlinear problem by Algorithm 2.2.

Iteration	max-error	reduction
1	10000.9961	
2	4308.0546	0.43
3	1291.3744	0.30
4	244.0996	0.19
5	48.5262	0.20
6	10.7099	0.22
7	2.7714	0.26
8	0.7627	0.28
9	0.1026	0.13
10	0.0139	0.14
11	0.0073	0.53
12	0.0062	0.85
13	0.0061	0.98

Table 5.9. Maximum error for the nonlinear problem by Algorithm 2.2 with initial guess as in (5.12), H = 1/5, h = 1/25, M = 1.



Flore 5.7. The computational results souths multiples problem by Algorithm 2.4.

Duble 5.8. Maximina erres for the standardar product by Altrock [statics]

Table 5.9. Maximum arrow for the nonlinear profilers by Magnithan 2.2. with initial grade as in (5.12). If = 1.15, k = 1.155.4 = 1.

5.4. Applications to linear interface problems.

Example 5.4. A similar problem as in Bramble, Pasciak and Schatz [7] is computed here. The coefficients are taken as (see Figure 5.10)

$$a = \begin{cases} e^{xy} & \text{in } [0, 0.5] \times [0, 0.5] \cup (0.5, 1] \times (0.5, 1] ,\\ e^{xy} \times 10^4 & \text{in } [0, 0.5) \times (0.5, 1] \cup (0.5, 1] \times (0, 0.5] . \end{cases}$$

Instead of giving an analytical solution, we take $f = 4 - 2\cos(x)\exp(y)$ on $\Omega = [0,1] \times [0,1]$. The global fine finite element solution is first computed. After that, the problem is computed by Algorithm 2.2 and the error between the iterative domain decomposition solution and the global fine mesh solution is calculated. Table 5.11 gives the results for $hx = hy = \frac{1}{100}$, $Hx = Hy = \frac{1}{10}$. The algorithm convergences for arbitrary initial guesses. Each subdomain is extended by 2 elements to get overlap. Table 5.11 shows the error reduction in the energy *A*-norm, i.e. the energy norm $(\int_{\Omega} a\nabla(u^n - u_h)dx)^{\frac{1}{2}}$ and here u_h is the global fine mesh solution. Tests with different number of subdomains and different overlap sizes were also done. The convergence is similar as for smooth problems.



Figure 5.10. Computational results for a linear interface problem.

Iteration	A-error	reduction
0	1.7594e + 05	
1	2.3242e + 04	0.13
3	4.2870e + 02	0.13
5	6.5544e + 00	0.13
7	1.7773e-01	0.18
9	1.1470e-02	0.27
11	8.7597e-04	0.28
13	6.7270e-05	0.28
15	5.1710e-06	0.28
34	8.8572e-16	0.88

Table 5.11. A-error for a linear interface problem.

5.5. Two hybrid algorithms .

5.4. Applications to inservinteriace problems .

Ecompte 5.4. À aimilier problem as lo Frambia, Fracial and Schutz [7] is computed here: The conflictence as the mark 10).

$$= \left\{ \begin{array}{l} e^{-\alpha} & \\ e^{-\alpha} \times 10^{3} \mbox{ in } [0,0.5] \times [0,0.5] (1,0.5,1] \times [0.5,1] \times [0.5,1] \\ e^{-\alpha} \times 10^{3} \mbox{ in } [0,0.5] \times (0.5,1] \cup [0.5,1] \times [0.5,1] \\ \end{array} \right.$$

Instead of giving as analytical solution, we take f = 4 - 2 costal marked is = [0, 1] is [0, 1]. The global frac fairs demonstrations to first compared all After that the problem is the powerd by Algorithm 2.2 and the error between the increase demon descence data is set in the problem is the global fine mesh solution is calculated. Take 2.1.1 gives the contra far have $[0, - \gamma_0]$. We is $[1, - \gamma_0]$ The algorithm correspondent in a strong unital success. Each subdoment is introduced by 2 demonster to get even by. Take 5.11 shows the strong unital success Each subdoment is introduced by 2 demonster $(\int_{0} a \nabla (a^{n} - \alpha_{0})^{2})^{2}$ and here α_{0} is the strong in the strategy of some set and the contract subdomains and different overlap sizes we cannot done. The contraction is such as a first success and domains and different overlap sizes we cannot done. The contraction is such as a strong of anotherms.



Figure 5.10. Competizizioni resolte for a lingua interlase peddente

Table 5.11. A-cetar for a linear lasedate arousen.

5.6. Two hybrid algorithmas.

When combined with the two level method, both Algorithm 2.2 and Algorithm 2.1 can be used with parallel processors. For Algorithm 2.2, each processor need to take care of 4 neighbouring subdomains, but all the processors must wait for the coarse mesh processor after their computations. For Algorithm 2.1, each subdomain problem is computed by one processor in parallel and the coarse mesh problem is also computed by a processor in parallel with the subdomain problems. The good point of Algorithm 2.2 is that it converges faster. The weak point is that it uses less processors. Moreover, the coarse problem is becoming a "bottle neck" in the iterative procedure. In order to over come this difficulty, we shall propose some "hybrid" algorithms, i.e, we shall computed the coarse mesh problem in parallel with the computation of the 4 color subdomain problems.

Example 5.5. We compute the coarse mesh problem in parallel with the subdomain problems. However, the subdomain problems of the 4 colors are computed sequentially. More specifically, let u^n be known for iteration n, then we use u^n to form the residual vector to solve the coarse mesh problem (5.8). In solving the subdomain problems, instead of solving (5.7), we solve (5.10) sequentially for the 4 color subdomains, but for the color 1 subdomains, instead of using $u^n + w_{H}^{n+1}$ as boundary condition and set $w_1^{n+1} = u^n + w_H^{n+1}$ in $\Omega \setminus \Omega'_1$, we use u^n as boundary conditions and set $w_1^{n+1} = u^n + w_H^{n+1}$ in $\Omega \setminus \Omega'_1$, we use u^n as boundary conditions and set $w_1^{n+1} = u^n + w_H^{n+1}$ in $\Omega \setminus \Omega'_1$, so it can be done in parallel with the coarse mesh problem. After the computation for w_1^{n+1} has been done, we use w_1^{n+1} as boundary conditions and solve the color 2 subdomains to get w_2^{n+1} in Ω'_2 from (5.10). The value of w_2^{n+1} in $\Omega \setminus \Omega'_2$ is taken as $w_2^{n+1} = w_1^{n+1}$, and do so for the color 3 and color 4 subdomains. After w_4^{n+1} has been obtained, we set

$$u^{n+1} = \frac{1}{2}(u^n + w_H^{n+\frac{1}{2}}) + \frac{1}{2}w_4^{n+1}$$
.

This scheme can be regarded as a special case of Algorithm 2.1 by choosing the relaxation parameters α_i in a suitable way.

The same problem as Example 5.1 is computed by this "hybrid" scheme with the same mesh sizes, same overlap and same initial guess. Table 5.12 displays the computed results. Comparing them with the results of Table 5.2 and Figure 5.5, one obviously finds that the error reduction number here is between that of Algorithm 2.2 and Algorithm 2.1.

We also test this algorithm for the nonlinear problem of example 5.3. With the same parameters as in Example 5.3, the computational results are given in Table 5.13.

provide the second seco		
Iteration	max-error	reduction
0	0.3055	
1	0.2221	0.73
3	0.0981	0.67
5	0.0445	0.67
7	0.0197	0.67
9	0.0090	0.68
11	0.0045	0.72
13	0.0026	0.79
15	0.0019	0.87
17	0.0016	0.94
18	0.0015	0.96

Table 5.12. Maximum error by the hybrid algorithm of Example 5.5for the linear problem of Example 5.1.

Example 5.6. In this example, we test another way of dealing with the coarse mesh problem. In each iteration, the coarse mesh problem is first computed in parallel with the color 1 subdomains problems. When both of them have done their computations, we update the solution by using the color 1 subdomain solution and the coarse mesh solution. After that we compute the coarse

When combined with the two icoust methods, both Algorithm 7.2 and Algorithm 2.1 cm he used with parallel processors. For Algorithm 3.2, each processor and to take care of 4 quickboarder albdomains, but all the processors areas with in the commenced progress after their comparetions. For Algorithm 2.1, each explains in products is parallel with the compared after the coarea used problem is also continued by a processor in parallel with the compared and the coarea used problem is also continued by a processor in parallel with the context of the second The good point of Algorithm 2.2 is that is convergentened. She weak point is like its days less processors. Moreover, the coarea problem is hermaning a burdened, in the transition procedum processors. Moreover, the coarea problem is hermaning a burdened, in the transition procedum in order to over some time difficulty, we shall with the corporation of the transition is weak in order to over some time difficulty, we shall with the composition of the second state weak compared the coarea mech problem to parallel with the comparallel of the second state of problem.

Example 5.5 We compute the course train problem in parallel with the relation in the test 1 and 1 and 1 is the value of the test 1 and 1 is the test 1 and 1 is the test 1 is

$$[m_{1}^{2}m_{2}^{2}+(\frac{1}{2}m_{1}^{2}m_{2}+m_{3})\frac{1}{2}+(\frac{1}{2}m_{3}^{2}m_{3}^{2})$$

This scheme can be regarded as a special case of Algorithm O.I.F. etc. fing the viscotion paramy etcre of the viscotion paramy.

The same mobilem as Example 5.4 is compared an time "Exhibit exhence weighting variant in a frame and a second sec

We also rest this appoint to for the eminant product of exempte 5.5. Will's the same parameters in at in Example 5.3, the cemputational results are circa in Table 5.1.

Table 5.12. Maximum arrow by the hybrid algorithmich Examples for the lungar method of Frankishes []

Example 5.6. In this example, we test smoother way of results with the cales marked problem. In each iteration, the coarse much problem is first computed in parallel with the color 1 subscientific problems. When both of them have from cheir computations, white the color 1 subscientific the color 1 subdomain solution and the coarse much relation. After that he coupled the course

Iteration	max-error	reduction
1	0.7718	
3	0.2668	0.63
5	0.1137	0.65
7	0.0495	0.66
9	0.0229	0.69
11	0.0127	0.77
13	0.0087	0.84
15	0.0069	0.91

Table 5.13. Maximum error by the hybrid algorithm of Example 5.5 for the nonlinear problem of Example 5.3, see also Table 5.9.

mesh problem again with the color 2 subdomain problems and when finished, update the solution by using both of them. Then compute the coarse mesh problem in parallel with the color 3 subdomain problems and update, and do so for color 4 subdomains. Then go to the next iteration. More specifically, let u^n be known, then we compute the coarse mesh problem (5.8) to obtain $w_H^{n+\frac{1}{2}}$ and solve (5.7) to obtain $w_1^{n+\frac{1}{2}}$ in Ω'_1 . The value of $w_1^{n+\frac{1}{2}}$ in $\Omega \setminus \Omega'_i$ is $w_1^{n+\frac{1}{2}} = u^n$. After the computation of $w_H^{n+\frac{1}{2}}$ and $w_1^{n+\frac{1}{2}}$, we update u^n by

$$u^n \leftarrow \frac{1}{2}(u^n + w_H^{n+\frac{1}{2}}) + \frac{1}{2}w_1^{n+\frac{1}{2}}$$
.

With this updated u^n , we solve again (5.8) to get a new coarse mesh solution $w_H^{n+\frac{1}{2}}$ and solve (5.7) for the color 2 subdomains to obtain $w_2^{n+\frac{1}{2}}$ and similarly set $w_2^{n+\frac{1}{2}} = u^n$ in $\Omega \setminus \Omega'_2$ (Here u^n is the newly updated one). Continue in this way for color 3 and color 4 subdomains. After we have done this for the 4th color subdomains, we take the newly updated solution as u^{n+1} , i.e.,

$$u^{n+1} \leftarrow u^n$$
,

and go to the next iteration. This scheme is also a special case of the Algorithm 2.1.

As in the previous example, in the computation, only the coarse mesh solution and the computed solution from the previous updating need to be stored. Table 5.14 is the computational results with this new hybrid scheme. The functions and needed parameters are the same as Example 5.1. The error reduction is better than the previous hybrid algorithm, but still not as good as Algorithm 2.2.

5.6. Special caution for the coarse mesh problem .

In using the proposed algorithms, special care must be applied for the coarse mesh problem. For example, when we are solving the coarse mesh problems (5.8) and (5.9), we need to use an integration to form the needed matrices and vectors. If we just use the cell centre of the coarse mesh as the integration point to form the matrices and vectors for (5.8) and (5.9), then an error of $O(H^2)$ will be carried with $w_H^{n+\frac{1}{2}}$ and w_H^{n+1} , and this error will pollute the computed solutions globally, because it affects $w_i^{n+\frac{1}{2}}$ and w_H^{n+1} through the boundary conditions in (5.7) and (5.10). This is confirmed by numerical results, see Table 5.15 and Figure 5.16. So, we should use as many integration points as for the fine mesh when we are forming the vector (f, v_H) . This observation is obvious for linear problems and the vector (f, v_H) is often implicitly calculated by using the fine mesh elements when one is getting the residual vectors. However, for nonlinear problems, special caution must be paid. In the following two examples, we use the procedure given in §5.2 to solve the linear problem and show the effect of the integration error.

Example 5.7. In this example, the cell centres of the coarse mesh elements are used as the integration points for assembling matrix $A_H = (a \nabla \phi_i^H, \nabla \phi_j^H)$, and vector (f, ϕ_i^H) , here ϕ_i^H are the FEM basis for $S_0^H(\Omega)$ and A_H is the stiffness matrix for (5.8). Table 5.15 and Figure 5.16 show the computational results for the same problem of Example 5.1. The iterative solution stops to converge

Table 5.13. Maximum error by the hybrid alcounties of Example 6.5 for the nonlinear problem of Strampin 5.5, see the Table 5.5.

used problem again with the color 2 subdomain problems and which functed, update the robulus by using both of them. Then compute the robust problem an encoded with the robust of subdomain problems and update, and do so for this to be detect which are problem and update the robust of the robust α with t

$$\mathbf{n}^{n} \gets \frac{1}{2} (\mathbf{n}^{n} + \mathbf{n}^{n+1}) + \frac{1}{2} \mathbf{n}^{n+1}$$

With this spectrul u^{*}, we saive again (5.8) to get a new types a methodomic of ^{*} and new (5.7) for the color 2-sthetemins to obtain w^{***} and similarly get by ^{**} or w^{*} in (1/2) ((fore we's in the newly updated one). Continue in this way for color 1 and color (reddition when the dependence this for the c^{**} velor subdomning we take the newly optated solution of a ^{**}, i.e.

and go to the next invition. This scheme is also a megal there of the Averithm 2.1

As in the previous example, in the outputction, only the owner with solution and the columned solution from the provious updating used to be stored. Totale 2014 in the computational research via this new hybrid scheme. The functions and needed permission are the area of future (1. Chr. error reduction is better then the previous lightid algorithm, but still not as good as Algorithm. 19.

5.6. Special caution for the course meak thebien in

As any let J.X. In this example, the call contrast of the transmosterious destribute to the interact of the points for an example J.X. In this example, the call contrast (J, qf') is a second interact (J, qf') is a second interact black of the points for J.X. In this exact, (J, qf') is a second interact black of the points for J.X and J.X. In this exact, J.X are the difference quarter for (J, qf') is a difference of the points for J.X and J.X. This is a difference of the point J.X and J.X and J.X are the difference quarter for (J, qf'). For J.X and J.X are the difference quarter for (J, qf'). This is difference of the point J.X and J.X are the point of the point of the point of the point J.X and J.X are the difference quarter for (J, qf'). The point J.X are the difference of the point J.X and J.X are the difference quarter for J.X and J.X are the point J.X and J.X are the difference quarter for (J, qf'). The point J.X are the difference quarter for J.X and J.X are the difference of J.X are the difference of J.X are th

Iteration	max-error	reduction
0	0.3055	
1	0.1537	0.50
2	0.0787	0.51
3	0.0429	0.55
4	0.0234	0.55
5	0.0131	0.56
6	0.0077	0.59
7	0.0047	0.62
8	0.0032	0.68
9	0.0024	0.74
10	0.0019	0.81
11	0.0017	0.87
12	0.0015	0.93
13	0.0015	0.97

Table 5.14. Maximum error by the hybrid algorithm of Example 5.6 for the linear problem of Example 5.1, see also Table 5.2 and Table 5.12.

to the global fine mesh solution when it reaches an accuracy which is nearly the same as the coarse mesh solution accuracy $||u_H - u||_{\infty} = 0.12$. The global fine mesh error is $||u_h - u||_{\infty} = 0.0014$. Similar thing was observed for the nonlinear problem (5.2). In all the numerical tests of Example 5.1–Example 5.6, the integration that is needed for the coarse mesh problem is done with an accuracy that is the same as for the fine mesh problem.

Iteration	max-error	reduction
0	10000.9995	
1	3247.6241	0.32
2	758.6012	0.23
3	93.1200	0.12
4	7.9555	0.09
5	1.6210	0.20
6	0.2855	0.18
7	0.0692	0.24
8	0.0636	0.92

Table 5.15. Iterative solution stops to converge to the fine mesh solution when it reaches the coarse mesh solution accuracy

able 5.14. Maximum error by the hybrid elignithm of Brannyle fablics. he inver problem of Example 5.1, see also Incils 5.5 and Incils 5.12.

to the global for mesh solution when it reaches an advance which is nearly the starting the dataset mesh solution accuracy $[[u, v] - u]_{0,2} \approx 0.12$. File globas fire much among the starting ϵ 0.1014. Similar this closered for the confineer problem (0.1). In all the mereoded tests of Scanple 5.1-Example 5.6, the integration that is reacted for the corrections problem is down with an accuracy that is the start as for the fire mesh problem.

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Figure 5.16. Computational results for the linear problem of Example 5.1

Example 5.8. In this example, the cell centres of the coarse mesh elements are used as the integration points for assembling matrix $A_H = (a \nabla \phi_i^H, \nabla \phi_j^H)$, but the integration for assembling the vector (f, ϕ_i^H) is done by using the fine mesh cell centres. Table 5.15 shows the computational result for the same problem of Example 5.1. The convergence is the same. This shows that the integration error for the stiffness matrix does not effect the convergence, but the integration error for vector (f, v_H) can pollute the iterative solution.

Iteration	max-error	reduction
0	10000.9995	
1	3247.6315	0.32
2	758.6414	0.23
3	93.1445	0.12
4	7.9947	0.09
5	1.5787	0.20
6	0.2480	0.16
7	0.0344	0.14
8	0.0033	0.10
9	0.0015	0.44
10	0.0014	1.00

Table 5.17. Same convergence as Example 5.1.

6. CONCLUSION

Using the observation that the domain decomposition and multilevel methods are space decomposition techniques, a convergence analysis is given for a general convex programming problem. The applications here are given for a two level domain decomposition, but the algorithms can be equally applied for decompositions of multigrid type if a suitable approximate solver is used for the subproblems.

Acknowledgement. The authors like to thank J. Xu and P. Bjorstad. Many insightful comments from J. Xu during the process of the work helped us to improve some of the results and the presentation of the paper. Some discussions with and valuable comments by P. Bjorstad clarify the relationship of our methods with the literature results.



Frence 3.16. Computational results for the length problem of Example 5.1

Example 1.6 in this example, the cell centres of the corrections have react and the reaction and as the integration points for essenthing matrix $A_{ij} = (45\%^2, 75\%)$, but the integration (presentabling the vector (f, ϕ_i^2)) is done by using the fine much cell centres. Unlike this choice the despitability ment for the same problem of Example 5.1. The convergence is the choice this define that the fraggation error for the stiffness matrix does not effect the convergence is the choice that the fraggation error for the stiffness matrix does not effect the convergence is the choice that the fraggation error for the stiffness matrix does not effect the convergence, but the integration error for vector f, we have polynes the iterative solution.

Table 5.17. Same convergence on Education 5.1

B. Concuestors

Using the observation that the domain ceramparties a grant multilever multiple for space search position verbuigges, a convergence analysis is given by a grant convers programming content. The applications here are given for a two level strain decampaction, lett the algorithmican to equally applied for decompositions of multigrid type if a cutable spectricities afron is used fit

Acknowledgement. The authors like to fiend J. Na sed P. Diomak. Many mathelia comments from J. Na during the process of the work helped as to impose some of the contine and the presentation of the paper. Bong discupaires with and subship-respective for P. Diverted classic the relationship Effort methods with the interactor results.

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