Department
of
APPLIED MATHEMATICS

Rate of Convergence of a Space Decomposition method and Applications to Linear and Nonlinear Elliptic Problems

by

Magne S. Espedal and Xue-Cheng Tai

Report No. 103

September 1996

UNIVERSITY OF BERGEN
Bergen, Norway

Department of Mathematics
University of Bergen
5007 Bergen
Norway

ISSN 0084-778x

Rate of Convergence of a Space Decomposition method and Applications to Linear and Nonlinear Elliptic Problems

by

Magne S. Espedal and Xue-Cheng Tai

Report No. 103

September 1996

RATE OF CONVERGENCE OF A SPACE DECOMPOSITION METHOD AND APPLICATIONS TO LINEAR AND NONLINEAR ELLIPTIC PROBLEMS

Xue–Cheng Tai and Magne Espedal

Department of Mathematics,
University of Bergen,
Alleg. 55, 5007,
Bergen, Norway.

ABSTRACT. Convergence of a space decomposition method is proved for a general convex programming problem. The space decomposition refers to methods that decompose a space into sums of subspaces, which could be a domain decomposition or a multilevel method for partial differential equations. Two algorithms are proposed. Both can be used for linear as well as nonlinear elliptic problems and they reduce to the standard additive and multiplicative Schwarz methods for linear elliptic problems. In the numerical implementations, two "hybrid" algorithms are also presented. They converge faster than the additive one and have better parallelism than the multiplicative method. Numerical tests with a two level domain decomposition for linear, nonlinear and interface elliptic problems are presented for the proposed algorithms.

CONTENTS

1. INTRODUCTION

This work presents a general space decomposition method for convex programming problems and gives an estimation of the rate of convergence of the method. One intension is to use the method to solve linear and nonlinear elliptic partial differential equations by domain decomposition or multilevel methods. In the applications given in this work, a two level overlapping domain decomposition method is considered.

The essence of the proposed method is to decompose the minimization space into a sum of subspaces and then solve the original minimization problem sequentially or in parallel over each of the subspaces. Due to the fact that the decomposed spaces can be arbitrary, especially since they are not orthogonal to each other, the usual convergence proofs for block relaxation methods cannot be used here to predict the convergence. However, using the experiences from domain decomposition and multigrid methods, we assume that the decomposed spaces satisfy a certain

---

1991 *Mathematics Subject Classification.* 65J10, 65M55, 65Y05.

Key words and phrases. Parallel, domain decomposition, nonlinear, elliptic equation, space decomposition.

The work was supported by by VISTA, a research cooperation between the Norwegian Academy of Science and Letters and Den norske oljeselskap a.s. (Statoil).

"spectral" bound, see constants C_1 and C_2 in (2.8) and (2.9), and then use these constants to estimate the convergence rate of the proposed methods.

The proposed algorithms are given for a convex programming problem. We expect that they could also be used to get efficient algorithms for some optimal control problems related to partial differential equations, see Kunisch and Tai [**26**] and [**27**] for applications.

The two level domain decomposition method can be viewed a space decomposition is inspired by the work of Xu [**36**], where it was observed that domain decomposition methods, multilevel methods and multigrid methods can be viewed in some way as space decomposition techniques and many of the methods proposed in literature for the above mention techniques are in essence similar to the Gauss-Seidel or Jacobi method. an abstract convergence was given for linear self-adjoint and also indefinite

Two schemes are proposed in this work. They could be used both for linear and nonlinear elliptic problems. In the linear case, they reduce to the standard additive and multiplicative Schwarz methods. Therefore, the algorithms generalise the known additive and multiplicative methods to certain nonlinear cases. Due to appearance of the nonlinearity, a modified abstract convergence theory is given. In the numerical implementations, two "hybrid" algorithms are proposed. They converge faster than the additive scheme and have better parallelism than the multiplicative scheme when used for overlapping domain decomposition.

The well-known substructuring BPS (see [**6**], [**7**], [**8**]) and BEPS (see [**5**]) preconditioners use nonoverlapping subdomains, see also [**4**], [**29**]. For a nonoverlapping domain decomposition, a finite element function w can be decomposed as $w = w_p + w_H$, here w_p has zero trace on the interfaces and w_H equals to w on the interfaces and is extended to the interior by harmonic extension. If we use Gauss-Seidel iteration, we get the exact solution in one iteration. However, to get the harmonic extension w_H is equivalent to solving the original problem. The construction of the preconditioners in [**7**]–[**8**] and [**5**] can be regarded as Jacobi iteration with approximate solvers for the harmonic extensions. The methods of [**6**] and [**29**] is a Gauss-Seidel iteration with a further suitable decomposition for w_H. By using a slightly different decomposition, in Espedal and Ewing [**22**, p. 125], a parallel nonoverlapping method was derived for solving a linearised two-phase immiscible flow. We hope that by viewing the construction of nonoverlapping preconditioners as an iterative approximate solving of a space decomposition, an abstract convergence analysis can also be obtained for them for some nonlinear problems.

In the literature, domain decomposition methods, multigrid methods and multilevel methods have been successfully used for different kinds of linear partial differential equations, see [**25**], [**35**], [**36**]. However, the results for using them for nonlinear problems are not as rich as for linear problems. In Cai and Dryja [**11**], a semilinear elliptic equation is first linearised by the Newton's method and then solved by the additive Schwarz scheme. In papers by Xu [**37**], [**38**], a two level method without doing domain decomposition is used for nonlinear elliptic problems. In Axellsson and Kaporin [**1**], a minimum residual adaptive multilevel method is given for some nonlinear problems. In Dawson and Wheeler [**18**], a two level method is used for a nonlinear parabolic equation; The work of Lions [**28**] seems to be the pioneering work for using domain decomposition methods for nonlinear partial differential equations. In Rannacher [**30**], a Newton type algorithm is studied for nonlinear elliptic problems. Multigrid methods for nonlinear problems are studied by Bank [**2**], Brandt [**10**], etc. For some earlier works of the authors related to this one, consult [**31**] and [**32**]–[**34**].

When we apply the methods here for a nonlinear problem, we need to solve many smaller size problems in an iterative way and this iterative procedure convergence as "quickly" as for linear problems. For some nonlinear problems, by reducing the large size problem into many smaller size problems and then linearising the smaller size problems, substantial computational efforts can be saved compared to first linearising and then decomposing the problem, see §5.

2. Statement of the problem and the algorithms

Consider the nonlinear problem

$$\min_{v \in V} F(v) \ . \tag{2.1}$$

Above, the function F is differentiable and convex, the space V is a reflexive Banach space. One

knows that partial differential equations of the type

$$-\sum D_i(a_{ij}D_ju)+bu=f \text{ in } \Omega \ ,$$

and

$$-\nabla\cdot(\rho(|\nabla u|)\nabla u)=f \text{ in } \Omega \ ,$$

with a suitably given ρ, can be solved by (2.1) by defining the function F and space V properly.

We shall use space decomposition methods to solve (2.1). A space decomposition method refers to a method that decomposes the space V into a sum of subspaces, i.e. there are spaces V_i, $i=1,2,\cdots,m$ such that

$$V=V_1+V_2+\cdots+V_m \ . \tag{2.2}$$

The meaning of the above decomposition is that $\forall v$, there exists $v_i\in V_i$ such that $v=\sum_{i=1}^m v_i$ and on the other hand, if $v_i\in V_i$, then $\sum_{i=1}^m v_i\in V$. If the space can be decomposed as in (2.2), then the followings algorithms can be used to solve (2.1).

Algorithm 2.1. (An additive space decomposition method).

Step 1. *Choose initial values* $u_i^0=u^0\in V$ *and relaxation parameters* $\alpha_i>0$ *such that* $\sum_{i=1}^m\alpha_i\le 1$.

Step 2. *For* $n\ge 0$, *find* $u_i^{n+\frac{1}{2}}\in V_i$ *in parallel for* $i=1,2,\cdots,m$ *such that*

$$F\left(\sum_{k=1,k\neq i}^m u_k^n+u_i^{n+\frac{1}{2}}\right)\le F\left(\sum_{k=1,k\neq i}^m u_k^n+v_i\right) \ , \quad \forall v_i\in V_i \ . \tag{2.3}$$

Step 3. *Set*

$$u_i^{n+1}=u_i^n+\alpha_i(u_i^{n+\frac{1}{2}}-u_i^n) \ , \tag{2.4}$$

and go to the next iteration.

Algorithm 2.2. (A multiplicative space decomposition method).

Step 1. *Choose initial values* $u_i^0=u^0\in V$.

Step 2. *For* $n\ge 0$, *find* $u_i^{n+1}\in V_i$ *sequentially for* $i=1,2,\cdots,m$ *such that*

$$\begin{aligned}&F\left(\sum_{1\le k<i}u_k^{n+1}+u_i^{n+1}+\sum_{i<k\le m}u_k^n\right)\\ \le &F\left(\sum_{1\le k<i}u_k^{n+1}+v_i+\sum_{i<k\le m}u_k^n\right) \ , \quad \forall v_i\in V_i \ .\end{aligned} \tag{2.5}$$

Step 3. *Go to the next iteration.*

In the following, the notation $\langle\cdot,\cdot\rangle$ is used to denote the duality pairing between V and V', here V' is the dual space of V. Function F is assumed to be Gateaux differentiable (see [**15**]) and there are constants $K>0$, $L<\infty$ such that

$$\begin{aligned}\langle F'(w)-F'(v),w-v\rangle &\ge K\|w-v\|_V^2 \ , \quad \forall w,v\in V \ ,\\ \|F'(w)-F'(v)\|_{V'} &\le L\|w-v\|_V \ , \quad \forall w,v\in V \ ,\end{aligned} \tag{2.6}$$

and from which, it is easy to deduce that

$$K\|w-v\|_V^2\le\langle F'(w)-F'(v),w-v\rangle\le L\|w-v\|_V^2 \ , \quad \forall w,v\in V \ . \tag{2.7}$$

Under assumption (2.6), problem (2.1) and subproblems (2.5) and (2.3) have unique solutions, see [**21**, p. 35].

know that partial differential equations of the type

$$-\sum_{i,j=1}^{d} D_i(a_{ij}D_j u) + bu = f \text{ in } \Omega ,$$

and

$$-\nabla\cdot(\rho(|\nabla u|)\nabla u) = f \text{ in } \Omega ,$$

with a suitably given ρ, can be solved by (2.1) by defining the function F and space V properly.

We shall use space decomposition methods to solve (2.1). A space decomposition method refers to a method that decomposes the space V into a sum of subspaces, i.e. there are spaces V_i, $i = 1, 2, \cdots, m$ such that

$$V = V_1 + V_2 + \cdots + V_m . \tag{2.2}$$

The meaning of the above decomposition is that $\forall v$, there exists $v_i \in V_i$ such that $v = \sum_{i=1}^{m} v_i$ and on the other hand, if $v_i \in V_i$, then $\sum_{i=1}^{m} v_i \in V$. If the space can be decomposed as in (2.2), then the following algorithms can be used to solve (2.1):

Algorithm 2.1. (An additive space decomposition method)

Step 1. *Choose initial values $u_i^0 = u^0 \in V$ and relaxation parameters $\alpha_i > 0$ such that $\sum_{i=1}^{m} \alpha_i \le 1$.*

Step 2. *For $n \ge 0$, find $u_i^{n+1} \in V_i$ in parallel for $i = 1, 2, \cdots, m$ such that*

$$F\left(\sum_{\substack{j=1\\ j\ne i}}^{m} u_j^n + u_i^{n+1}\right) \le F\left(\sum_{\substack{j=1\\ j\ne i}}^{m} u_j^n + v_i\right) \quad \forall v_i \in V_i \tag{2.3}$$

Step 3. *Set*

$$u_i^{n+1} = u_i^n + \alpha_i(u_i^{n+1} - u_i^n) , \tag{2.4}$$

and go to the next iteration.

Algorithm 2.2. (A multiplicative space decomposition method).

Step 1. *Choose initial values $u_i^0 = u^0 \in V$.*

Step 2. *For $n \ge 0$, find $u_i^{n+1} \in V_i$ sequentially for $i = 1, 2, \cdots, m$ such that*

$$F\left(\sum_{j<i} u_j^{n+1} + u_i^{n+1} + \sum_{j>i} u_j^n\right) \le F\left(\sum_{j<i} u_j^{n+1} + v_i + \sum_{j>i} u_j^n\right) \quad \forall v_i \in V_i \tag{2.5}$$

Step 3. *Go to the next iteration.*

In the following, the notation $\langle\cdot,\cdot\rangle$ is used to denote the duality pairing between V and V', here V' is the dual space of V. Function F is assumed to be Gateaux differentiable (see [15]) and there are constants $K > 0$, $L < \infty$ such that

$$\begin{aligned} &\langle F'(w) - F'(v), w - v\rangle \ge K\|w - v\|_V^2 , \quad \forall w, v \in V , \\ &\|F'(w) - F'(v)\|_{V'} \le L\|w - v\|_V , \quad \forall w, v \in V , \end{aligned} \tag{2.6}$$

and from which, it is easy to deduce that

$$K\|w - v\|_V^2 \le \langle F'(w) - F'(v), w - v\rangle \le L\|w - v\|_V^2 , \quad \forall w, v \in V . \tag{2.7}$$

Under assumption (2.6), problem (2.1) and subproblems (2.3) and (2.5) have unique solutions, see [21, p. 35].

For the decomposed spaces, we assume that there is a constant $C_1 > 0$ such that $\forall v \in V$, we can find $v_i \in V_i$ to satisfy:

$$v = \sum_{i=1}^{m} v_i \ , \ and \qquad \sum_{i=1}^{m} \|v_i\|_V^2 \le C_1^2 \|v\|_V^2 \ . \tag{2.8}$$

Moreover, assume that there is a $C_2 > 0$ such that there holds

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \langle F''(w_{ij})u_i, v_j\rangle \le C_2 \Big(\sum_{i=1}^{m} \|u_i\|_V^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{m} \|v_i\|_V^2\Big)^{\frac{1}{2}} , \tag{2.9}$$
$$\forall w_{ij} \in V, \forall u_i \in V_i, \forall v_j \in V_j \ .$$

Domain decomposition methods, multilevel methods and multigrid methods can be viewed as different ways of decomposing finite element spaces into sums of subspaces. For the estimation of the constants C_1 and C_2 for different type of decomposition of finite element methods for linear problems, one can find the proofs or references in Xu [**36**].

Later, the error reduction factor for the above two algorithms shall be estimated. In the following, we shall use $e^n, n = 0, 1, 2, \cdots$, which is defined as:

$$e^n = |\langle F'(u^n) - F'(u), u^n - u\rangle|^{\frac{1}{2}} \ ,$$

as a measure of the error between u^n and u. Here and later, u stands for the unique solution of (2.1). For convenience, constants α_{min} and α_{max} are defined as $\alpha_{min} = \min_{1\le i\le m} \alpha_i$, $\alpha_{max} = \max_{1\le i\le m} \alpha_i$, and α_i is the relaxation parameters in Algorithm 2.1. Constants C_p and C_s, which are

$$C_p = (\alpha_{min}^{-\frac{1}{2}} L + \alpha_{max}^{\frac{1}{2}} C_2) C_1, \qquad C_s = C_2 C_1, \tag{2.10}$$

will play an important rule in analysing the error reduction factor.

Remark 2.1.

(1) When F is differentiable and if we define

$$w_i^{n+\frac{1}{2}} = \sum_{k=1, k\ne i}^{m} u_k^n + u_i^{n+\frac{1}{2}} \ , \tag{2.11}$$

then (2.3) is equivalent to solving

$$\Big\langle F'(w_i^{n+\frac{1}{2}}), v_i \Big\rangle = 0, \quad \forall v_i \in V_i \ . \tag{2.12}$$

(2) Let

$$u^{n+1} = \sum_{i=1}^{m} u_i^{n+1}, \ n = 0, 1, 2, \cdots \tag{2.13}$$

and $w_i^{n+\frac{1}{2}}$ be defined as in (2.11), then

$$w_i^{n+\frac{1}{2}} = u^n + u_i^{n+\frac{1}{2}} - u_i^n$$

and the value of u^{n+1} corresponding to (2.4) can be obtained by

$$\begin{aligned} u^{n+1} &= \sum_{i=1}^{m} u_i^n + \sum_{i=1}^{m} \alpha_i (u_i^{n+\frac{1}{2}} - u_i^n) \\ &= u^n + \sum_{i=1}^{m} \alpha_i (u_i^{n+\frac{1}{2}} - u_i^n) \\ &= \sum_{i=1}^{m} \alpha_i (u^n + u_i^{n+\frac{1}{2}} - u_i^n) + (1 - \sum_{i=1}^{m} \alpha_i) u^n \\ &= \sum_{i=1}^{m} \alpha_i w_i^{n+\frac{1}{2}} + (1 - \sum_{i=1}^{m} \alpha_i) u^n. \end{aligned} \tag{2.14}$$

For the decomposed spaces, we assume that there is a constant $C_1 > 0$ such that $\forall v \in V$, we can find $v_i \in V_i$ to satisfy:

$$v = \sum_{i=1}^{m} v_i, \text{ and } \sum_{i=1}^{m} \|v_i\|_a^2 \le C_1^2 \|v\|_a^2. \tag{2.8}$$

Moreover, assume that there is a $C_2 > 0$ such that there holds

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \langle F''(w_{ij}) u_i, v_j \rangle \le C_2 \left(\sum_{i=1}^{m} \|u_i\|_a^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{m} \|v_j\|_a^2 \right)^{\frac{1}{2}}, \tag{2.9}$$

$$\forall w_{ij} \in V, \forall u_i \in V_i, \forall v_j \in V_j.$$

Domain decomposition methods, multilevel methods and multigrid methods can be viewed as different ways of decomposing finite element spaces into sums of subspaces. For the estimation of the constants C_1 and C_2 for different type of decomposition of finite element methods for linear problems, one can find the proofs or references in Xu [40].

Later, the error reduction factor for the above two algorithms shall be estimated. In the following, we shall use $e^n, n = 0, 1, 2, \cdots$, which is defined as

$$e^n = \langle F'(u^n) - F'(u), u^n - u \rangle$$

as a measure of the error between u^n and u. Here and later, u stands for the unique solution of (2.1). For convenience, constants ω_{min} and ω_{max} are defined as $\omega_{min} = \min_{1 \le i \le m} \omega_i$, $\omega_{max} = \max_{1 \le i \le m} \omega_i$, and ω_i is the relaxation parameters in Algorithm 2.2. Constants C_p and C_a, which are

$$C_p = (\omega_{min}^{-1} L + \omega_{max} C_2) C_1^2, \qquad C_a = (2C_1 \ldots) \tag{2.10}$$

will play an important role in analyzing the error reduction factor.

Remark 2.1.

(1) When F is differentiable and if we define

$$w_i^{n+1} = \sum_{k=1, k \ne i}^{m} u_k^n + u_i^{n+1}, \tag{2.11}$$

then (2.3) is equivalent to solving

$$\left\langle F'(w_i^{n+1}), v_i \right\rangle = 0, \quad \forall v_i \in V_i. \tag{2.12}$$

(2) Let

$$u^{n+1} = \sum_{i=1}^{m} u_i^{n+1}, \quad n = 0, 1, 2, \cdots \tag{2.13}$$

and w_i^{n+1} be defined as in (2.11), then

$$w_i^{n+1} = u^n + u_i^{n+1} - u_i^n$$

and the value of u^{n+1} corresponding to (2.4) can be obtained by

$$u^{n+1} = \sum_{i=1}^{m} u_i^n + \sum_{i=1}^{m} \alpha_i (u_i^{n+1} - u_i^n)$$

$$= u^n + \sum_{i=1}^{m} \alpha_i (u_i^{n+1} - u_i^n)$$

$$= \sum_{i=1}^{m} \alpha_i (u^n + u_i^{n+1} - u_i^n) + (1 - \sum_{i=1}^{m} \alpha_i) u^n$$

$$= \sum_{i=1}^{m} \alpha_i w_i^{n+1} + (1 - \sum_{i=1}^{m} \alpha_i) u^n. \tag{2.14}$$

In the applications of §5, with a two level domain decomposition method, only the values of u^{n+1} and the coarse mesh problem are needed for the next iteration, and u^{n+1} is updated by the above formula after the computations of each of subdomain problems.

(3) For Algorithm 2.2, if we define

$$w_i^{n+1} = \sum_{k<i} u_k^{n+1} + u_i^{n+1} + \sum_{k>i} u_k^n , \tag{2.15}$$

then it satisfies

$$\langle F'(w_i^{n+1}), v_i \rangle = 0, \quad \forall v_i \in V_i . \tag{2.16}$$

and after the solving of w_i^{n+1} from (2.16) for each i, we only need to set $u^{n+1} = w_m^{n+1}$.

(4) Intuitively, one may think that the algorithms need rather large amount of memory. However, in the implementation later for a two level domain decomposition, we only need to store the value of u^{n+1} and one of the $w_i^{n+\frac{1}{2}}$ (the coarse mesh solution) in the memory.

Remark 2.2. Algorithm 2.2 solves the minimization problems sequentially over each subspace. Algorithm 2.1 solves the minimizations in parallel over each of the subspaces. In applications, by suitably decomposing the minimization space, the minimization problem over each subspace can be done by many parallel processors, and so both algorithms are suitable for parallel machines, see §5. Moreover, with a suitable decomposition, the constant C_1 can be made to be independent of the size of the problem, and so the convergence of the above two algorithms also does not depend on the size of the problem.

3. The convergence of the additive algorithm

We first give the rate of convergence for Algorithm 2.1.

Theorem 3.1. *If the space decomposition satisfies (2.8),(2.9) and the function satisfies (2.6), then for Algorithm 2.1 we have:*

(a). *If F is quadratic with respect to v and the norm of V is chosen as $\|v\|_V = \langle F'(v), v\rangle$, then there holds*

$$|e^{n+1}|^2 \le \frac{C_p^2}{1+C_p^2}|e^n|^2 \ , \forall n \ge 0 . \tag{3.1}$$

(b). *If F is third order continuously differentiable, then*

$$|e^{n+1}| \to 0 \ as \ n \to \infty \ , \ and \quad |e^{n+1}|^2 \le \beta_n |e^n|^2 \ , \quad \forall n \ge 0 .$$

For n sufficiently large, we have $0 < \beta_n < 1$. In fact

$$\lim_{n\to\infty} \beta_n = \frac{C}{1+C} < 1 \quad and \quad C = \frac{C_p{}^2}{K^2} . \tag{3.2}$$

Before we go to the proof of the theorem, we first present a lemma which is needed in the proof. The lemma can be proved in a similar way as [**21**, p. 25], and the proof can be found in [**32**].

Lemma 3.2. *If condition (2.7) is valid, then we have for function F:*

$$F(w) - F(v) \ge \langle F'(v), w-v\rangle + \frac{K}{2}\|w-v\|_V^2 \ , \quad \forall v, w \in V \ , \tag{3.3}$$

$$F(w) - F(v) \le \langle F'(v), w-v\rangle + \frac{L}{2}\|w-v\|_V^2 \ , \quad \forall v, w \in V \ . \tag{3.4}$$

Proof of Theorem 3.1. Let u^{n+1} and $w_i^{n+\frac{1}{2}}$ be defined as in (2.13) and (2.11). As F is a convex function, by using (2.4), (2.12), (2.14) and (3.3), one obtains

$$F(u^n) - F(u^{n+1}) = F(u^n) - F\Big(\sum_{i=1}^m u_i^{n+1}\Big)$$

In the applications of §5, with a two level domain decomposition method, only the values of u^{n+1} and the coarse mesh problem are needed for the next iteration, and u^{n+1} is updated by the above formula after the computations of each of subdomain problems.

(3) For Algorithm 2.2, if we define

$$w_i^{n+1} = \sum_{j<i} u_j^{n+1} + u_i^{n+1} + \sum_{j>i} u_j^n, \qquad (2.15)$$

then it satisfies

$$(F'(w_i^{n+1}), v_i) = 0, \quad \forall v_i \in V_i, \qquad (2.16)$$

and after the solving of u_i^{n+1} from (2.10) for each i, we only need to set $u^{n+1} = w_m^{n+1}$.

(4) Intuitively, one may think that the algorithms need rather large amount of memory. However, in the implementation later for a two level domain decomposition, we only need to store the value of u^{n+1} and one of the u_i^{n+1} (the coarse mesh solution) in the memory.

Remark 2.2. Algorithm 2.2 solves the minimization problems sequentially over each subspace. Algorithm 2.1 solves the minimizations in parallel over each of the subspaces. In applications, by suitably decomposing the minimization space, the minimization problem over each subspace can be done by many parallel processors, and so both algorithms are suitable for parallel machines, see §5. Moreover, with a suitable decomposition, the constant C_1 can be made to be independent of the size of the problem, and so the convergence of the above two algorithms also does not depend on the size of the problem.

3. THE CONVERGENCE OF THE ADDITIVE ALGORITHM

We first give the rate of convergence for Algorithm 2.1.

Theorem 3.1. *If the space decomposition satisfies (2.3) (2.4) and the function satisfies (2.9), then for Algorithm 2.1 we have:*

(a). *If F is quadratic with respect to v and the norm of V is chosen as $\|v\|_a = (F''(v)v, v)$, then there holds*

$$e^{n+1} \le \frac{C}{1+C} e^n, \quad \forall n \ge 0. \qquad (3.1)$$

(b). *If F is third order continuously differentiable, then*

$$|e^{n+1}| \to 0 \text{ as } n \to \infty, \text{ and } |e^{n+1}| \le \theta_n |e^n|, \quad \forall n \ge 0.$$

For n sufficiently large, we have $0 < \theta_n < 1$. In fact

$$\lim_{n\to\infty} \theta_n = \frac{C}{1+C} < 1, \quad C = \frac{C_1 L}{\alpha}. \qquad (3.2)$$

Before we go to the proof of the theorem, we first present a lemma which is needed in the proof. The lemma can be proved in a similar way as [13, p. 27], and the proof can be found in [27].

Lemma 3.2. *If condition (2.9) is valid, then we have for function F,*

$$F(u) - F(v) \ge (F'(v), u - v) + \frac{\alpha}{2}\|u - v\|_a^2, \quad \forall u, v \in V, \qquad (3.3)$$

$$F(u) - F(v) \le (F'(v), u - v) + \frac{L}{2}\|u - v\|_a^2, \quad \forall u, v \in V. \qquad (3.4)$$

Proof of Theorem 3.1. Let u^{n+1} and u_i^{n+1} be defined as in (2.12) and (2.11). As F is a convex function, by using (2.4), (2.12), (2.14) and (2.3), one obtains

$$F(u^n) - F(u^{n+1}) = F(u^n) - F\Big(\sum_{i=1}^{m} u_i^{n+1}\Big)$$

$$
\begin{aligned}
&= F(u^n) - F\Big(\sum_{i=1}^m \alpha_i w_i^{n+\frac12} + (1-\sum_{i=1}^m \alpha_i)u^n\Big) \qquad (3.5)\\
&\ge F(u^n) - \sum_{i=1}^m \alpha_i F\left(w_i^{n+\frac12}\right) - \Big(1-\sum_{i=1}^m \alpha_i\Big)F(u^n)\\
&= \sum_{i=1}^m \alpha_i F(u^n) - \sum_{i=1}^m \alpha_i F(w_i^{n+\frac12})\\
&\ge \sum_{i=1}^m \alpha_i \langle F'(w_i^{n+\frac12}), u_i^n - u_i^{n+\frac12}\rangle + \frac{K}{2}\sum_{i=1}^m \alpha_i \|u_i^n - u_i^{n+\frac12}\|_V^2\\
&= \frac{K}{2}\sum_{i=1}^m \alpha_i \|u_i^n - u_i^{n+\frac12}\|_V^2 \,.
\end{aligned}
$$

As u is the solution of (2.1), it satisfies $\langle F'(u), v\rangle = 0, \quad \forall v \in V$. For any $v_i \in V_i$, $i = 1,2,\cdots,m$ such that $\sum v_i = u$, we shall use (2.12), and (2.6) to estimate:

$$
\begin{aligned}
&\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\rangle\\
&= \langle F'(u^{n+1}), u^{n+1} - u\rangle = \sum_{i=1}^m \left\langle F'(u^{n+1}), u_i^{n+1} - v_i\right\rangle\\
&= \sum_{i=1}^m \left\langle F'(u^{n+1}) - F'(u^n + u_i^{n+\frac12} - u_i^n), u_i^{n+1} - v_i\right\rangle\\
&= \sum_{i=1}^m \left\langle F''(\theta_i^{n+1})(u^{n+1} - u^n), u_i^{n+1} - v_i\right\rangle - \sum_{i=1}^m \left\langle F''(\theta_i^{n+1})(u_i^{n+\frac12} - u_i^n), u_i^{n+1} - v_i\right\rangle\\
&\qquad (\ \theta_i^{n+1} = \theta u^{n+1} + (1-\theta) w_i^{n+1},\ \theta \in [0,1]\ )\\
&= \sum_{i=1}^m\sum_{j=1}^m \left\langle F''(\theta_i^{n+1})(u_j^{n+1} - u_j^n), u_i^{n+1} - v_i\right\rangle - \sum_{i=1}^m \left\langle F''(\theta_i^{n+1})(u_i^{n+\frac12} - u_i^n), u_i^{n+1} - v_i\right\rangle\\
&\le C_2\Big(\sum_{i=1}^m \|u_i^{n+1} - u_i^n\|_V^2\Big)^{\frac12}\Big(\sum_{i=1}^m \|u_i^{n+1} - v_i\|_V^2\Big)^{\frac12} + L\sum_{i=1}^m \|u_i^n - u_i^{n+\frac12}\|_V \|u_i^{n+1} - v_i\|_V \qquad (3.6)\\
&\le C_2\Big(\sum_{i=1}^m \alpha_i^2 \|u_i^{n+\frac12} - u_i^n\|_V^2\Big)^{\frac12}\Big(\sum_{i=1}^m \|u_i^{n+1} - v_i\|_V^2\Big)^{\frac12}\\
&\quad + \frac{L}{\min\sqrt{\alpha_i}}\Big(\sum_{i=1}^m \alpha_i \|u_i^n - u_i^{n+\frac12}\|_V^2\Big)^{\frac12}\Big(\sum_{i=1}^m \|u_i^{n+1} - v_i\|_V^2\Big)^{\frac12}\Big] \,.
\end{aligned}
$$

From the property of the space decomposition (2.8), there exists $\phi_i \in V_i$ such that $u^{n+1} - u = \sum_{i=1}^m \phi_i$, and $\sum_{i=1}^m \|\phi_i\|_V^2 \le C_1^2\|u^{n+1} - u\|_V^2$, So we take $v_i = u_i^{n+1} - \phi_i$ and see that

$$
\sum_{i=1}^m \|u_i^{n+1} - v_i\|_V^2 = \sum_{i=1}^m \|\phi_i\|_V^2 \le C_1^2\|u^{n+1} - u\|_V^2 \,. \qquad (3.7)
$$

By combining (3.5)–(3.7), there comes

$$
\begin{aligned}
&\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\rangle\\
&\le C_2\sqrt{\max\alpha_i}\Big(\sum_{i=1}^m \alpha_i \|u_i^{n+\frac12} - u_i^n\|_V^2\Big)^{\frac12} \cdot C_1\|u^{n+1} - u\|_V\\
&\quad + L\alpha_{min}^{-\frac12}\Big(\sum_{i=1}^m \alpha_i \|u_i^{n+\frac12} - u_i^n\|_V^2\Big)^{\frac12} \cdot C_1\ \|u^{n+1} - u\|_V \qquad (3.8)\\
&\le C_1\left(\alpha_{max}^{\frac12} C_2 + \alpha_{min}^{-\frac12} L\right)\left[\frac{2}{K}\left(F(u^n) - F(u^{n+1})\right)\right]^{\frac12} \cdot \|u^{n+1} - u\|_V \,.
\end{aligned}
$$

Let us note that

$$\|u^{n+1}-u\|_V^2 \le K^{-1}\langle F'(u^{n+1})-F'(u), u^{n+1}-u\rangle ,$$

therefor there follows from (3.8) that

$$\langle F'(u^{n+1})-F'(u), u^{n+1}-u\rangle \le C_p\left[\frac{2}{K}\left(F(u^n)-F(u^{n+1})\right)\right]^{\frac{1}{2}}$$
$$\cdot K^{-1/2}\sqrt{\langle F'(u^{n+1})-F'(u), u^{n+1}-u\rangle} ,$$

and so

$$\begin{aligned} &K^2\langle F'(u^{n+1})-F'(u), u^{n+1}-u\rangle \\ \le& 2C_p^2[F(u^n)-F(u^{n+1})] \\ =& 2C_p^2\left[F(u^n)-F(u)+F(u)-F(u^{n+1})\right] \end{aligned} \tag{3.9}$$

Summing (3.9) for $n=0,1,2,\cdots,N$, we find that

$$\sum_{n=0}^{N}|e^{n+1}|^2 \le 2{C_p}^2/K^2[F(u^0)-F(u^{N+1})] \le 2{C_p}^2/K^2[F(u^0)-F(u)] ,$$

and so

$$|e^{n+1}| \to 0, \text{ as } n\to\infty . \tag{3.10}$$

We shall first prove (a) and then prove (b). From relations (3.3)–(3.4), there holds

$$\begin{aligned} F(u^n)-F(u) \le& \langle F'(u), u^n-u\rangle + \frac{L}{2}\|u^n-u\|_V^2 \\ =& \frac{L}{2}\|u^n-u\|_V^2 , \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} F(u)-F(u^{n+1}) \le& -\langle F'(u), u^{n+1}-u\rangle - \frac{K}{2}\|u^{n+1}-u\|_V^2 \\ =& -\frac{K}{2}\|u^{n+1}-u\|_V^2 . \end{aligned} \tag{3.12}$$

Substituting (3.11) and (3.12) to (3.9) and using (2.7), it gives

$$\begin{aligned} &K^2|e^{n+1}|^2 \\ \le& 2{C_p}^2\left(\frac{L}{2}\|u^n-u\|_V^2 - \frac{K}{2}\|u^{n+1}-u\|_V^2\right) \\ \le& 2{C_p}^2\left(\frac{L}{2K}|e^n|^2 - \frac{K}{2L}|e^{n+1}|^2\right) , \end{aligned}$$

which shows that

$$|e^{n+1}|^2 \le \frac{LK^{-1}{C_p}^2}{K^2+KL^{-1}{C_p}^2}|e^n|^2 . \tag{3.13}$$

As F is quadratic and satisfies (2.7), then $\sqrt{\langle F'(v), v\rangle}$ defines a norm for V and $F'(v)$ is linear with respect to v. In the proof given above, if we choose the norm of V to be

$$\|v\|_V = \sqrt{\langle F'(v), v\rangle} , \quad \forall v\in V ,$$

then we have $K=L=1$ in (2.7), moreover

$$|e^n|^2 = |\langle F'(u^n)-F'(u), u^n-u\rangle| = |\langle F'(u^n-u), u^n-u\rangle| = \|e^n\|_V^2$$

and so (3.13) implies (3.1).

Next, we prove (b). First, we note that $|e^n| \to 0$ as $n \to \infty$, so there exists a ball $B(u,\delta)$ which is centred at u and is with a radius of δ such that $u^n \in B(u,\delta)$, $\forall n$. As F is third order continuously differentiable, one can assume that there is a constant $C(u)$ such that

$$|F'''(\xi)\cdot(v,v,v)| \le C(u)\|v\|_V^3, \quad \forall \xi \in B(u,\delta), \forall v \in V .$$

We use the Taylor's formula (see Cea [**15**, Chap.2]) to get:

$$\begin{aligned} F(u^n) - F(u) = {} & \langle F'(u), u^n - u\rangle + \frac{1}{2}F''(u)\cdot(u^n-u)^2 \\ & + \frac{1}{6}F'''(u+\theta^n(u^n-u))\cdot(u^n-u)^3 ; \end{aligned} \tag{3.14}$$

$$\begin{aligned} F(u) - F(u^{n+1}) = {} & -\langle F'(u), u^{n+1} - u\rangle - \frac{1}{2}F''(u)\cdot(u^{n+1}-u)^2 \\ & - \frac{1}{6}F'''(u+\theta^{n+1}(u^{n+1}-u))\cdot(u^{n+1}-u)^3 . \end{aligned} \tag{3.15}$$

Above, $\theta^n, \theta^{n+1} \in [0,1]$. Summing (3.14) and (3.15), and using (2.7) and the property that $\langle F'(u), v\rangle = 0$, $\forall v \in V$, it follows that:

$$F(u^n) - F(u^{n+1}) = \frac{1}{2}|e^n|^2 - \frac{1}{2}|e^{n+1}|^2 + I_1 + I_2 + I_3 + I_4 , \tag{3.16}$$

where

$$\begin{aligned} I_1 = {} & \frac{1}{2}F''(u)\cdot(u^n-u)^2 - \frac{1}{2}\langle F'(u^n) - F'(u), u^n - u\rangle , \\ \le {} & C(u)\|u^n - u\|_V^3 \le \frac{C(u)}{K^{\frac{3}{2}}}|e^n|^3 , \\ I_2 = {} & -\frac{1}{2}F''(u)\cdot(u^{n+1}-u)^2 + \frac{1}{2}\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\rangle , \\ \le {} & C(u)\|u^{n+1} - u\|_V^3 \le \frac{C(u)}{K^{\frac{3}{2}}}|e^{n+1}|^3 , \\ I_3 = {} & \frac{1}{6}F'''(u+\theta^n(u^n-u))\cdot(u^n-u)^3 \\ \le {} & C(u)\|u^n - u\|_V^3 \le \frac{C(u)}{K^{\frac{3}{2}}}|e^n|^3 , \\ I_4 = {} & -\frac{1}{6}F'''(u+\theta^{n+1}(u^{n+1}-u))\cdot(u^{n+1}-u)^3 \\ \le {} & C(u)\|u^{n+1} - u\|_V^3 \le \frac{C(u)}{K^{\frac{3}{2}}}|e^{n+1}|^3 , \end{aligned} \tag{3.17}$$

Let $C^* = \frac{4C_p^2C(u)}{K^{\frac{3}{2}}}$. From relations (3.9) and (3.16)–(3.17), there follows

$$\begin{aligned} & \left(K^2 + {C_p}^2\right)|e^{n+1}|^2 \\ \le {} & {C_p}^2|e^n|^2 + C^*|e^n|^3 + C^*|e^{n+1}|^3 . \end{aligned}$$

and so

$$|e^{n+1}|^2 \le \frac{C_p^2 + C^*|e^n|}{K^2 + C_p^2 - C^*|e^{n+1}|}|e^n|^2 .$$

From (3.10), we see $|e^n| \to 0$, and so for n large enough if

$$|e^{n+1}| \le \frac{K^2}{2C^*}, \quad |e^n| \le \frac{K^2}{2C^*} , \tag{3.18}$$

then

$$\beta_n = \frac{C_p^2 + C^*|e^n|}{K^2 + C_p^2 - C^*|e^{n+1}|} < 1 .$$

Moreover

$$\lim_{n\to\infty}\beta_n = \frac{C}{1+C} < 1, \text{ and } C = \frac{C_p^2}{K^2} .$$

4. The convergence of the multiplicative algorithm

The convergence of Algorithm 2.2 is similar as Algorithm 2.1.

Theorem 4.1. *Let the space decomposition satisfies (2.8) and the function satisfies (2.7), then for Algorithm 2.2 we have:*

(a). *If F is quadratic with respect to v and the norm of V is taken as $\|v\|_V = \langle F'(v), v\rangle$, then there holds*

$$|e^{n+1}|^2 \leq \frac{{C_s}^2}{1+{C_s}^2}|e^n|^2 \ , \forall n \geq 0 \ . \tag{4.1}$$

(b). *If F is third order continuously differentiable, then*

$$|e^{n+1}| \to 0 \ \text{as} \ n \to \infty \ , \ \text{and} \quad |e^{n+1}|^2 \leq \beta_n |e^n|^2 \ , \quad \forall n \geq 0 \ .$$

For n sufficiently large, we have $0 < \beta_n < 1$. In fact

$$\lim_{n\to\infty} \beta_n = \frac{C}{1+C} < 1 \quad \text{and} \quad C = \frac{{C_s}^2}{K^2} \ . \tag{4.2}$$

Proof of Theorem 4.1. Let u^{n+1} and w_i^{n+1} be defined as in (2.13) and (2.15). We see that $u^{n+1} = w_m^{n+1}$. If we also define $w_0^{n+1} = u^n$, we observe that

$$\begin{aligned} F(u^n) - F(u^{n+1}) &= \sum_{i=1}^m \left(F(w_{i-1}^{n+1}) - F(w_i^{n+1})\right) \\ &\geq \sum_{i=1}^m \langle F'(w_i^{n+1}), u_i^n - u_i^{n+1}\rangle + \frac{K}{2}\sum_{i=1}^m \|u_i^n - u_i^{n+1}\|_V^2 \\ &= \frac{K}{2}\sum_{i=1}^m \|u_i^n - u_i^{n+1}\|_V^2 \ . \end{aligned} \tag{4.3}$$

Similar as in the proof of (3.6), there exists $v_i \in V_i$, $i = 1, 2, \cdots, m$ such that $\sum v_i = u$. Using (2.16), and (2.7) to estimate:

$$\begin{aligned} &\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\rangle \\ &= \langle F'(u^{n+1}), u^{n+1} - u\rangle = \sum_{i=1}^m \left\langle F'(u^{n+1}), u_i^{n+1} - v_i\right\rangle \\ &= \sum_{i=1}^m \left\langle F'(u^{n+1}) - F'(w_i^{n+1}), u_i^{n+1} - v_i\right\rangle \\ &= \sum_{i=1}^m \left\langle F''(\theta_i^{n+1})(u^{n+1} - w_i^{n+1}), u_i^{n+1} - v_i\right\rangle \qquad (\ \theta_i^{n+1} = \theta u^{n+1} + (1-\theta)w_i^{n+1}, \ \theta \in [0,1]\ ) \\ &= \sum_{i=1}^m \sum_{j>i} \left\langle F''(\theta_i^{n+1})(u_j^{n+1} - u_j^n), u_i^{n+1} - v_i\right\rangle \\ &\leq C_2 \left(\sum_{i=1}^m \|u_i^{n+1} - u_i^n\|_V^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^m \|u_i^{n+1} - v_i\|_V^2\right)^{\frac{1}{2}} \ . \end{aligned} \tag{4.4}$$

Take v_i such that (3.7) is valid. By combining (4.3)–(4.4), there comes

$$\begin{aligned} &\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\rangle \\ &\leq C_2 \left(\sum_{i=1}^m \|u_i^{n+1} - u_i^n\|_V^2\right)^{\frac{1}{2}} \cdot C_1 \|u^{n+1} - u\|_V \\ &\leq C_1 C_2 \left[\frac{2}{K}\left(F(u^n) - F(u^{n+1})\right)\right]^{\frac{1}{2}} \cdot \|u^{n+1} - u\|_V \ . \end{aligned} \tag{4.5}$$

Similar as in getting (3.9), one deduces from (4.5)

$$\begin{aligned}
&K^2\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\rangle \\
\leq &2{C_s}^2[F(u^n) - F(u^{n+1})] \\
= &2{C_s}^2\left[F(u^n) - F(u) + F(u) - F(u^{n+1})\right] .
\end{aligned}$$

The rest of the proof is the same as for Theorem 3.1.

5. Applications to linear and nonlinear elliptic problems

In this section, the space decomposition algorithms will be applied to linear problem:

$$\begin{cases} -\nabla\cdot(a\nabla u) = f \text{ in } \Omega \subset R^2 , \\ u = 0 \text{ on } \partial\Omega . \end{cases} \tag{5.1}$$

and to nonlinear elliptic problem

$$\begin{cases} -\nabla\cdot(|\nabla u|^{s-2}\nabla u) = f \text{ in } \Omega \subset R^2 \ (1 < s < \infty) , \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{5.2}$$

Defining

$$V = H_0^1(\Omega), \quad F(v) = \int_\Omega \left(\frac{1}{2}|\nabla v|^2 - fv\right) dx ,$$

it is known that problem (5.1) is equivalent to solving (2.1). More general boundary conditions for (5.1) can also be considered. Correspondingly, we just need to modify the definitions of space V and function F.

For equation (5.2), we assume $f \in W^{-1,s'}(\Omega)$, $\frac{1}{s} + \frac{1}{s'} = 1$. By standard techniques, it can be shown, see [**21**], that (5.2) possesses a unique solution which is the minimizer of problem

$$\min_{v\in W_0^{1,s}(\Omega)} \left[\frac{1}{s}\int_\Omega |\nabla v|^s - \langle f, v\rangle\right] .$$

This problem appears in certain mathematical models describing the mechanical deformation of ice (see for example [**23**] [**24**]). Even with very smooth data, the solution u may not be in the space $W_0^{2,s}$, see Ciarlet [**16**, p.324]. When s is close to 1 or is very big ($s \gg 2$), it is difficult to solve this problem numerically.

We see that the algorithms can also be used to compute the full potential equation for the velocity potential of fluid flows:

$$-\nabla\cdot(\rho(|\nabla u|)\nabla u) = 0 ,$$

where the density is given in terms of the potential

$$\rho(q) = \rho_\infty\left(1 + \frac{r-1}{2}M_\infty^2(1 - \frac{q^2}{q_\infty^2})\right)^{\frac{1}{r-1}} .$$

For the derivation of this equation and for the meaning of the parameters, consult [**13**] and [**17**]. Suitable boundary conditions should be supplied. If the flow is everywhere subsonic, this problem fits into our frame work. This equation has important applications in aerospace industry. For recent numerical results by domain decomposition for this equation, see [**12**] and [**13**].

In the following, we shall use a two level domain decomposition as a space decomposition method to solve these problems. Numerical experiments show very good convergence properties. For problems like (5.2), the method is especially appealing. Normally, one uses a linearization method or other type iterative methods to solve (5.2), see [**24**], [**23**]. These methods converge slowly when the size of the problem is big. The proposed algorithms here reduce problem (5.2) into many smaller size problems of the same kind. These smaller size problems can be solved more efficiently than the original large size problem. Moreover, we solve these subproblems iteratively. The analysis tells that this iterative procedure produces a solution that converges to the true solution with a rate that is independent of the size of the original problem. For the kind of problems as in (5.2), by first decomposing the problem and then linearising, we gain efficiency compared to first linearising and then decomposing the problem.

5.1. Decomposition of the finite element space .

As the multigrid method [**25**], multilevel method, domain decomposition method [**35**] can be viewed as different ways of decomposing the finite element space, the proposed algorithms of this paper can use them to solve the above problems. The analysis indicates that the convergence does not depend on the regularity of the solution, it only depends on the lower and upper bound of the differential operator, i.e. the constants L and K. In this section, a two-level domain decomposition, i.e, an overlapping domain decomposition with a coarse mesh shall be used. For the two level method, let $\{\Omega_i\}_{i=1}^M$ be a shape-regular finite element division, or a coarse mesh, of Ω and Ω_i has diameter of order H. For each Ω_i, we further divide it into smaller simplices with diameter of order h. In case that Ω has a curved boundary, we shall also fill the area between $\partial\Omega$ and $\partial\Omega_H$, here $\bar{\Omega}_H = \cup_{i=1}^M \bar{\Omega}_i$, with finite elements with diameters of order h. We assume that the resulting elements form a shape regular finite element subdivision of Ω, see Ciarlet [**16**]. We call this the fine mesh or the h-level subdivision of Ω with mesh parameter h. We denote $\Omega_h = \cup\{T \in \mathcal{T}_h\}$ as the fine mesh subdivision. Let $S_0^H \subset H_0^1(\Omega)$ and $S_0^h \subset H_0^1(\Omega)$ be the continuous, piecewise linear function spaces, with zero trace on $\partial\Omega_H$ and $\partial\Omega_h$, over the H-level and h-level subdivisions of Ω respectively.

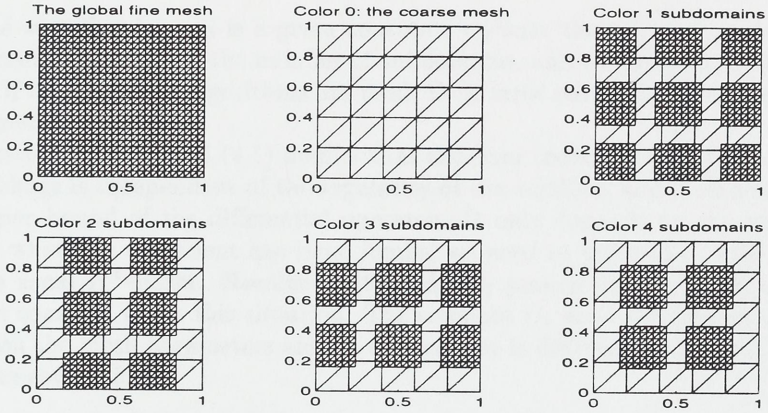

Figure 5.1. The coloring of the subdomains and the coarse mesh grid

For each Ω_i, we consider an enlarged subdomain $\Omega_i^\delta = \{T \in \mathcal{T}_h, dist(T, \Omega_i) \le \delta\}$. The union of Ω_i^δ covers $\bar{\Omega}_h$ with overlaps of size δ. Let us denote the piecewise linear finite element space with zero traces on the boundaries $\partial\Omega_i^\delta$ as $S_0^h(\Omega_i^\delta)$. Then one can show that

$$S_0^h = S_0^H + \sum S_0^h(\Omega_i^\delta)\ . \tag{5.3}$$

For the overlapping subdomains, assume that there are m colors such that each subdomain Ω_i^δ can be marked with one color, and the subdomains with the same color will not intersect with each other. For suitable overlaps, one can always choose $m = 2$ if $d = 1; m \le 4$ if $d = 2; m \le 6$ if $d = 3$, see Figure 5.1. Let Ω_i' be the union of the subdomains with the i^{th} color, and

$$V_i = \{v \in S_0^h | \quad v(x) = 0, \quad x \notin \Omega_i'\}\ .$$

By denoting subspaces $V_0 = S_0^H$, $V = S_0^h$, we find that decomposition (5.3) means

$$V = V_0 + \sum_{i=1}^m V_i, \tag{5.4}$$

and so the two level method is a way to decompose the finite element space. Let $\{\theta_i\}_{i=1}^m$ be a partition of unity with respect to $\{\Omega_i'\}_{i=1}^m$, i.e. $\theta_i \in C_0^\infty(\Omega_i' \cap \Omega)$ and $\sum_i^m \theta_i = 1$. It can be chosen so that $|\nabla\theta_i| \le C/\delta$. Let I_h be an interpolation operator which uses the function values at the

h-level nodes. For any $v \in V$, let $v_0 \in V_0$ be the solution of $(v_0, \phi_H) = (v, \phi_H), \forall \phi_H \in V_0$, and $v_i = I_h(\theta_i(v - v_0))$. They satisfy $v = \sum_i^m v_i$, and

$$\|v_0\|^2_{H^1(\Omega)} + \sum_i^m \|v_i\|^2_{H^1(\Omega_i)} \leq \frac{CH^2}{\delta^2}\|v\|^2_{L^2(\Omega)} + C\|\nabla v\|^2_{L^2(\Omega)} \leq C(1 + \frac{H^2}{\delta^2})\|v\|^2_{H^1(\Omega)}. \tag{5.5}$$

The proof of (5.5) can be found in different places, we refer to Xu [**36**, p. 608]. Moreover, using the Cauchy-Schwarz inequality, it is easy to see that

$$\left|\sum_{i=1}^m \sum_{j=1}^m (a\nabla u_i, \nabla v_j)\right| \leq C\sum_{i=1}^m \|u_i\|_{H^1} \sum_{j=1}^m \|v_j\|_{H^1} \leq Cm\Big(\sum_{i=1}^m \|u_i\|^2_{H^1}\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^m \|v_j\|^2_{H^1}\Big)^{\frac{1}{2}} \tag{5.6}$$

Estimates (5.5) and (5.6) show that for overlapping domain decomposition, the constants in (2.8) and (2.9) are

$$C_1 = C\sqrt{1 + \frac{H^2}{\delta^2}}, \qquad C_2 = Cm.$$

By requiring $\delta = c_0 H$, where c_0 is a given constant, we have that C_1 and C_2 are independent of the mesh parameters h and H, the number of subdomains, and estimate (5.5) is also valid for 3D problems. So if the proposed algorithms are used, their error reductions per step are independent of these parameters.

The error estimates (3.1) and (4.1) predict that the error reduction per step in the *energy norm* for linear problems is independent of the regularity of the solution, and is also independent of the lower and upper bound of the differential operator. It only depends on the parameters C_1 and C_2. However, when the coefficient has large jumps, we need to prove (5.5) with the energy norm instead of the usual H^1–norm. Results by [**9**] and [**39**] present some elementary techniques for estimating the constant C_1 in this situation. The constant C_1 does not depend on the jumps, but depends now on the mesh parameters and its dependency is different for 2D and 3D problems, see [**19**], [**3**] for the estimations.

5.2. Applications to linear elliptic equations .

As was shown above, the two level method is a space decomposition method. With the coarse mesh, the number of the subspaces is $m = 5$, see Figure 5.1. For Algorithm 2.1, by defining $w_i^{n+\frac{1}{2}}$ as in (2.11), the subproblems that need to be solved in each subdomain is

$$\begin{cases} (a\nabla w_i^{n+\frac{1}{2}}, \nabla v_i) = (f, v_i), & \forall v_i \in S_0^h(\Omega_i') \ , \\ w_i^{n+\frac{1}{2}} = u^n \text{ on } \partial\Omega_i' \ , \end{cases} \tag{5.7}$$

and $w_i^{n+\frac{1}{2}} = u^n$ in $\Omega \backslash \Omega_i'$. If we define $w_H^{n+\frac{1}{2}} = u_0^{n+\frac{1}{2}} - u_0^n \in S_0^H(\Omega)$, then the coarse mesh problem is

$$(a\nabla(u^n + w_H^{n+\frac{1}{2}}), \nabla v_H) = (f, v_H) \ , \quad \forall v_H \in S_0^H(\Omega) \ . \tag{5.8}$$

For Algorithm 2.2, for the coarse mesh problem, if we let $w_H^{n+1} = u_0^{n+1} - u_0^n$, then it satisfies

$$(a\nabla(u^n + w_H^{n+1}), \nabla v_H) = (f, v_H) \ , \quad \forall v_H \in S_0^H(\Omega) \ . \tag{5.9}$$

After solving the coarse problem, let w_i^{n+1} be defined as in (2.15), then the subdomain problems are

$$\begin{cases} (a\nabla w_i^{n+1}, \nabla v_i) = (f, v_i), & \forall v_i \in S_0^h(\Omega_i') \ , \\ \text{if } i > 1, \ w_i^{n+1} = w_{i-1}^{n+1} \text{ on } \partial\Omega_i' \ , \\ \text{if } i = 1, \ w_i^{n+1} = u^n + w_H^{n+1} \text{ on } \partial\Omega_i' \ , \end{cases} \tag{5.10}$$

and for each i

$$w_i^{n+1} = \begin{cases} u^n + w_H^{n+1} \ , \ if \ i = 1, \\ w_{i-1}^{n+1} \ , \ if \ i > 1, \end{cases} in \ \Omega \backslash \Omega_i' \ . \tag{5.11}$$

After computation of the subdomain problems and the coarse mesh problem, for Algorithm 2.1, solution u^{n+1} is updated by

$$u^{n+1} = \alpha_0(u^n + w_0^{n+\frac{1}{2}}) + \sum_{i=1}^{4} \alpha_i w_i^{n+\frac{1}{2}} + (1 - \sum_{i=0}^{4} \alpha_i) u^n$$

and this is needed for the boundary values for the subdomain problems for the next iteration and for computing the residual for the coarse mesh problem. For Algorithm 2.2, one simply sets

$$u^{n+1} = w_4^{n+1} \ .$$

For each i, domain Ω_i' is the union of the disjoint subdomains of the same color. Thus, the computations of (5.7) and (5.10) can be done in parallel in each of the subdomains of the i^{th} color. For Algorithm 2.1, the computations for different i can again be done in parallel.

For the linear problem, the above formulation shows that Algorithm 2.1 and Algorithm 2.2 reduce to the standard additive and multiplicative Schwarz algorithms, see [**35**, Chap. 5]. In literature, the condition number of the matrices for the additive and multiplicative methods are estimated for different types of space decomposition, see [**36**], and then the conjugate gradient method is applied to accelerate the convergence. However, it may not be possible to use conjugate gradient method to accelerate the convergence for nonlinear and nonsymmetric problems.

Example 5.1. In this example, Algorithm 2.2 is tested for the case that $a = e^{xy}$, $u = \sin(3\pi x)\sin(3\pi y)$, $\Omega = [0,1] \times [0,1]$. Uniform mesh is used both for the coarse mesh problem and fine mesh problem. For a given N, the coarse mesh size is taken as $H = Hx = Hy = \frac{1}{N}$. The fine mesh is then taken as $h = hx = hy = \frac{1}{N^2}$. Each subdomain is extended by M elements to get overlaps. In Table 5.2, the initial guess is taken as the coarse mesh solution. In Table 5.4, the initial guess is taken as

$$u^0 = \begin{cases} 10^4 & \text{in the interior nodes of } \Omega \ , \\ 0 & \text{on the boundary of } \Omega \ . \end{cases} \tag{5.12}$$

Figure 5.3 is the computed solution and errors. One sees that it is reducing the maximum error from 10^4 to 10^{-3} in about 9 steps. The same kind of convergence was also observed with tests of other smooth solutions.

Iteration	max-error	reduction
0	3.0552e-01	
1	0.0729	0.24
2	0.0166	0.23
3	0.0036	0.22
4	0.0017	0.46
5	0.0015	0.88

Table 5.2. Maximum error with H=1/10, h=1/100, M=2.

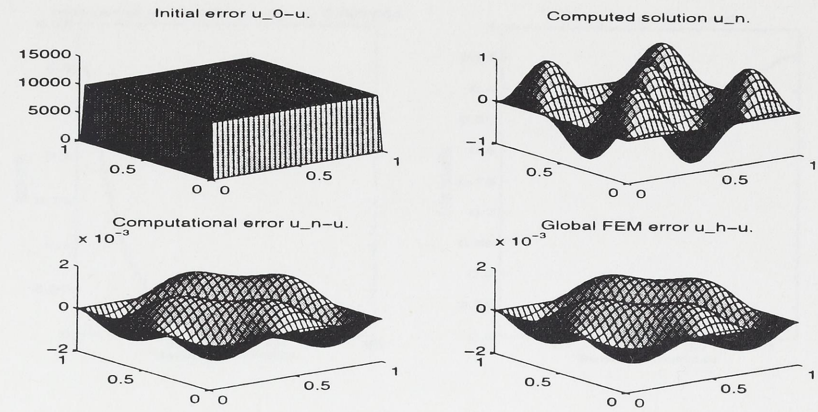

Figure 5.3. The computational for $H = 1/10, h = 1/100, M = 2, u^0$ as in (5.12).

Iteration	max-error	reduction
0	10000.9995	
1	3248.4466	0.32
2	760.5149	0.23
3	93.3381	0.12
4	8.0150	0.09
5	1.5830	0.20
6	0.2481	0.16
7	0.0346	0.14
8	0.0034	0.10
9	0.0015	0.43
10	0.0014	1.00

Table 5.4. Maximum error with $H = 1/10, h = 1/100, M = 2, u^0$ as in (5.12).

Example 5.2. In this example, Algorithm 2.1 is used to compute the same problem of the last example. As $m = 5$, the relaxation parameters are taken as $\alpha_i = \frac{1}{5}$. Algorithm 2.1 can use more processors, but the error reduction per step is not as good as Algorithm 2.2. With overlap size $\delta = \frac{H}{5}$, the error reduction per step is nearly always around 0.85 for different mesh sizes. Figure 5.5 demonstrates the relation between the error and iteration number for initial guess $u^0 = u_H$, where u_H is the coarse mesh solution. Figure 5.6 shows the errors and error reduction with u_0 taken as in (5.12).

Different tests were done for Algorithm 2.1 and the convergences are similar. For a given initial guess, it always reduces the error by a factor ≈ 0.86. When the computed solution is getting closer to the global FEM solution, the error reduction is getting closer and closer to 1. Nearly in all tests, we find that if the error reduction factor is bigger than 0.95, then the computational error is always less than two times the global FEM solution error. However, the error reduction factor depends on the overlapping size. If we decrease it ($\delta < \frac{H}{5}$), then the reduction number becomes bigger. If we increase it, the reduction number becomes smaller.

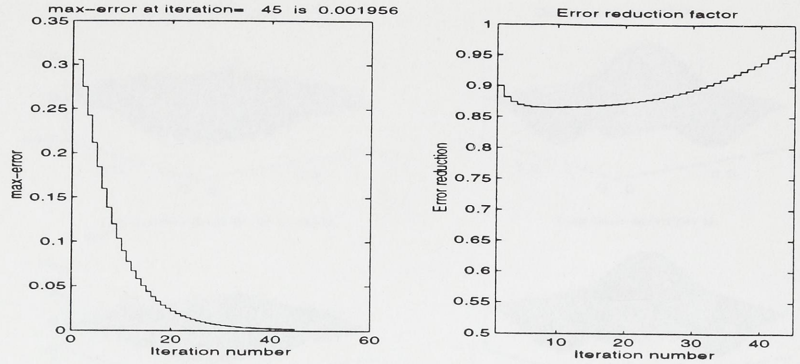

Figure 5.5. The computational results with $H = 1/10, h = 1/100, M = 2$, $u^0 = u_H$, error reduction ≈ 0.86.

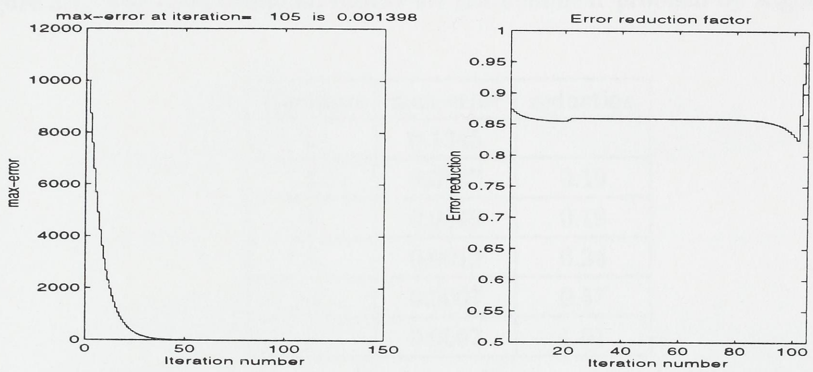

Figure 5.6. The computational results with $H = 1/10, h = 1/100, M = 2$, u^0 as in (5.12), error reduction ≈ 0.86.

5.3. Applications to nonlinear elliptic problems.

The Gauss-Newton method (Matlab subroutine fiminu) is used to solve the minimization problems (2.3) and (2.5). Without using the domain decomposition, the original problem is simply too large and costly to be solved. With 500 grid points, we are already run out of memory and it takes days to compute the global problems. With the domain decomposition, we can compute the problem with 10^5 unknowns.

Example 5.3. We use an analytical solution $u = \sin(2\pi x)\sin(2\pi y)$ on $\Omega = [0,1] \times [0,1]$ to test the Algorithm 2.2. Figure 5.7 and Table 5.8 show the computational results with fine mesh $hx = hy = \frac{1}{100}$, and coarse mesh $Hx = Hy = \frac{1}{10}$. Each subdomain is extended by 2 elements to get overlaps. The initial guess is the coarse mesh solution. The value of s is 3. For this test problem, $a(u) = |\nabla u|^{s-2} = 0$ at some points. This violates the normal assumption $a(u) > c > 0$. Our error analysis is still valid for this problem, however, the numerical results in Figure 5.7 shows that the computational error is bigger near the points that $a(u) = 0$.

Tests with different overlapping sizes were also done. The error reduction number is smaller with bigger overlapping size and becomes bigger if we decreases the overlapping size. According to the error analysis, for the nonlinear problem, the convergence may be slow in the beginning if the initial guess is not good enough. However, numerical tests show that the algorithms converge for arbitrary initial guess and the error reduction does not depend on the initial guess, see Table 5.9.

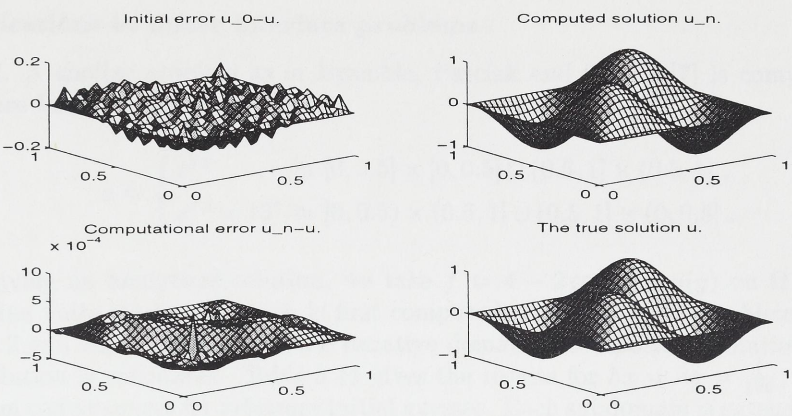

Figure 5.7. The computational results for the nonlinear problem by Algorithm 2.2.

Iteration	max-error	reduction
0	0.1345	
1	0.0257	0.19
2	0.0049	0.19
3	0.0012	0.24
4	0.0007	0.57
5	0.0007	1.01

Table 5.8. Maximum error for the nonlinear problem by Algorithm 2.2.

Iteration	max-error	reduction
1	10000.9961	
2	4308.0546	0.43
3	1291.3744	0.30
4	244.0996	0.19
5	48.5262	0.20
6	10.7099	0.22
7	2.7714	0.26
8	0.7627	0.28
9	0.1026	0.13
10	0.0139	0.14
11	0.0073	0.53
12	0.0062	0.85
13	0.0061	0.98

Table 5.9. Maximum error for the nonlinear problem by Algorithm 2.2 with initial guess as in (5.12), $H = 1/5, h = 1/25, M = 1$.

5.4. Applications to linear interface problems .

Example 5.4. A similar problem as in Bramble, Pasciak and Schatz [7] is computed here. The coefficients are taken as (see Figure 5.10)

$$a = \begin{cases} e^{xy} & \text{in } [0,0.5]\times[0,0.5]\cup(0.5,1]\times(0.5,1]\ , \\ e^{xy}\times 10^4 & \text{in } [0,0.5)\times(0.5,1]\cup(0.5,1]\times(0,0.5]\ . \end{cases}$$

Instead of giving an analytical solution, we take $f = 4 - 2\cos(x)\exp(y)$ on $\Omega = [0,1]\times[0,1]$. The global fine finite element solution is first computed. After that, the problem is computed by Algorithm 2.2 and the error between the iterative domain decomposition solution and the global fine mesh solution is calculated. Table 5.11 gives the results for $hx = hy = \frac{1}{100}$, $Hx = Hy = \frac{1}{10}$. The algorithm convergences for arbitrary initial guesses. Each subdomain is extended by 2 elements to get overlap. Table 5.11 shows the error reduction in the energy A–norm, i.e. the energy norm $(\int_\Omega a\nabla(u^n - u_h)dx)^{\frac{1}{2}}$ and here u_h is the global fine mesh solution. Tests with different number of subdomains and different overlap sizes were also done. The convergence is similar as for smooth problems.

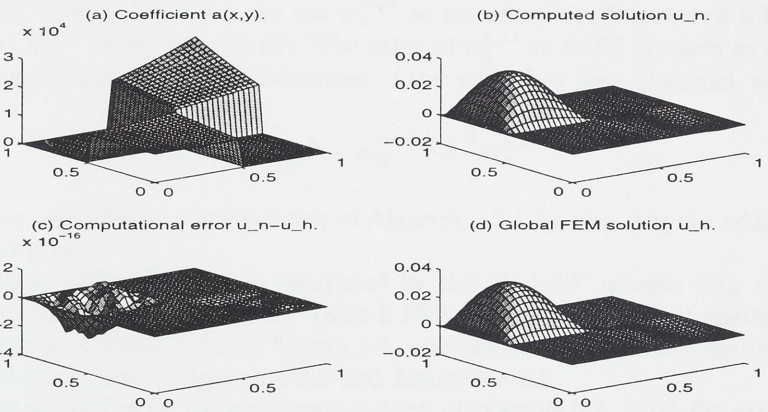

Figure 5.10. Computational results for a linear interface problem.

Iteration	A-error	reduction
0	1.7594e+05	
1	2.3242e+04	0.13
3	4.2870e+02	0.13
5	6.5544e+00	0.13
7	1.7773e-01	0.18
9	1.1470e-02	0.27
11	8.7597e-04	0.28
13	6.7270e-05	0.28
15	5.1710e-06	0.28
...	...	...
34	8.8572e-16	0.88

Table 5.11. A-error for a linear interface problem.

5.5. Two hybrid algorithms .

When combined with the two level method, both Algorithm 2.2 and Algorithm 2.1 can be used with parallel processors. For Algorithm 2.2, each processor need to take care of 4 neighbouring subdomains, but all the processors must wait for the coarse mesh processor after their computations. For Algorithm 2.1, each subdomain problem is computed by one processor in parallel and the coarse mesh problem is also computed by a processor in parallel with the subdomain problems. The good point of Algorithm 2.2 is that it converges faster. The weak point is that it uses less processors. Moreover, the coarse problem is becoming a "bottle neck" in the iterative procedure. In order to over come this difficulty, we shall propose some "hybrid" algorithms, i.e, we shall computed the coarse mesh problem in parallel with the computation of the 4 color subdomain problems.

Example 5.5. We compute the coarse mesh problem in parallel with the subdomain problems. However, the subdomain problems of the 4 colors are computed sequentially. More specifically, let u^n be known for iteration n, then we use u^n to form the residual vector to solve the coarse mesh problem (5.8). In solving the subdomain problems, instead of solving (5.7), we solve (5.10) sequentially for the 4 color subdomains, but for the color 1 subdomains, instead of using $u^n + w_H^{n+1}$ as boundary condition and set $w_1^{n+1} = u^n + w_H^{n+1}$ *in* $\Omega\backslash\Omega_1'$, we use u^n as boundary conditions and set $w_1^{n+1} = u^n$ *in* $\Omega\backslash\Omega_1'$, so it can be done in parallel with the coarse mesh problem. After the computation for w_1^{n+1} has been done, we use w_1^{n+1} as boundary conditions and solve the color 2 subdomains to get w_2^{n+1} in Ω_2' from (5.10). The value of w_2^{n+1} in $\Omega\backslash\Omega_2'$ is taken as $w_2^{n+1} = w_1^{n+1}$, and do so for the color 3 and color 4 subdomains. After w_4^{n+1} has been obtained, we set

$$u^{n+1} = \frac{1}{2}(u^n + w_H^{n+\frac{1}{2}}) + \frac{1}{2}w_4^{n+1} .$$

This scheme can be regarded as a special case of Algorithm 2.1 by choosing the relaxation parameters α_i in a suitable way.

The same problem as Example 5.1 is computed by this "hybrid" scheme with the same mesh sizes, same overlap and same initial guess. Table 5.12 displays the computed results. Comparing them with the results of Table 5.2 and Figure 5.5, one obviously finds that the error reduction number here is between that of Algorithm 2.2 and Algorithm 2.1.

We also test this algorithm for the nonlinear problem of example 5.3. With the same parameters as in Example 5.3, the computational results are given in Table 5.13.

Iteration	max-error	reduction
0	0.3055	
1	0.2221	0.73
3	0.0981	0.67
5	0.0445	0.67
7	0.0197	0.67
9	0.0090	0.68
11	0.0045	0.72
13	0.0026	0.79
15	0.0019	0.87
17	0.0016	0.94
18	0.0015	0.96

Table 5.12. Maximum error by the hybrid algorithm of Example 5.5 for the linear problem of Example 5.1.

Example 5.6. In this example, we test another way of dealing with the coarse mesh problem. In each iteration, the coarse mesh problem is first computed in parallel with the color 1 subdomains problems. When both of them have done their computations, we update the solution by using the color 1 subdomain solution and the coarse mesh solution. After that we compute the coarse

Iteration	max-error	reduction
1	0.7718	
3	0.2668	0.63
5	0.1137	0.65
7	0.0495	0.66
9	0.0229	0.69
11	0.0127	0.77
13	0.0087	0.84
15	0.0069	0.91

Table 5.13. Maximum error by the hybrid algorithm of Example 5.5 for the nonlinear problem of Example 5.3, see also Table 5.9.

mesh problem again with the color 2 subdomain problems and when finished, update the solution by using both of them. Then compute the coarse mesh problem in parallel with the color 3 subdomain problems and update, and do so for color 4 subdomains. Then goto the next iteration. More specifically, let u^n be known, then we compute the coarse mesh problem (5.8) to obtain $w_H^{n+\frac{1}{2}}$ and solve (5.7) to obtain $w_1^{n+\frac{1}{2}}$ in Ω_1'. The value of $w_1^{n+\frac{1}{2}}$ in $\Omega\backslash\Omega_i'$ is $w_1^{n+\frac{1}{2}} = u^n$. After the computation of $w_H^{n+\frac{1}{2}}$ and $w_1^{n+\frac{1}{2}}$, we update u^n by

$$u^n \leftarrow \frac{1}{2}(u^n + w_H^{n+\frac{1}{2}}) + \frac{1}{2}w_1^{n+\frac{1}{2}} .$$

With this updated u^n, we solve again (5.8) to get a new coarse mesh solution $w_H^{n+\frac{1}{2}}$ and solve (5.7) for the color 2 subdomains to obtain $w_2^{n+\frac{1}{2}}$ and similarly set $w_2^{n+\frac{1}{2}} = u^n$ in $\Omega\backslash\Omega_2'$ (Here u^n is the newly updated one). Continue in this way for color 3 and color 4 subdomains. After we have done this for the 4^{th} color subdomains, we take the newly updated solution as u^{n+1}, i.e..

$$u^{n+1} \leftarrow u^n ,$$

and go to the next iteration. This scheme is also a special case of the Algorithm 2.1.

As in the previous example, in the computation, only the coarse mesh solution and the computed solution from the previous updating need to be stored. Table 5.14 is the computational results with this new hybrid scheme. The functions and needed parameters are the same as Example 5.1. The error reduction is better than the previous hybrid algorithm, but still not as good as Algorithm 2.2.

5.6. Special caution for the coarse mesh problem .

In using the proposed algorithms, special care must be applied for the coarse mesh problem. For example, when we are solving the coarse mesh problems (5.8) and (5.9), we need to use an integration to form the needed matrices and vectors. If we just use the cell centre of the coarse mesh as the integration point to form the matrices and vectors for (5.8) and (5.9), then an error of $O(H^2)$ will be carried with $w_H^{n+\frac{1}{2}}$ and w_H^{n+1}, and this error will pollute the computed solutions globally, because it affects $w_i^{n+\frac{1}{2}}$ and w_i^{n+1} through the boundary conditions in (5.7) and (5.10). This is confirmed by numerical results, see Table 5.15 and Figure 5.16. So, we should use as many integration points as for the fine mesh when we are forming the vector (f, v_H). This observation is obvious for linear problems and the vector (f, v_H) is often implicitly calculated by using the fine mesh elements when one is getting the residual vectors. However, for nonlinear problems, special caution must be paid. In the following two examples, we use the procedure given in §5.2 to solve the linear problem and show the effect of the integration error.

Example 5.7. In this example, the cell centres of the coarse mesh elements are used as the integration points for assembling matrix $A_H = (a\nabla\phi_i^H, \nabla\phi_j^H)$, and vector (f, ϕ_i^H), here ϕ_i^H are the FEM basis for $S_0^H(\Omega)$ and A_H is the stiffness matrix for (5.8). Table 5.15 and Figure 5.16 show the computational results for the same problem of Example 5.1. The iterative solution stops to converge

Iteration	max-error	reduction
0	0.3055	
1	0.1537	0.50
2	0.0787	0.51
3	0.0429	0.55
4	0.0234	0.55
5	0.0131	0.56
6	0.0077	0.59
7	0.0047	0.62
8	0.0032	0.68
9	0.0024	0.74
10	0.0019	0.81
11	0.0017	0.87
12	0.0015	0.93
13	0.0015	0.97

Table 5.14. Maximum error by the hybrid algorithm of Example 5.6 for the linear problem of Example 5.1, see also Table 5.2 and Table 5.12.

to the global fine mesh solution when it reaches an accuracy which is nearly the same as the coarse mesh solution accuracy $\|u_H - u\|_\infty = 0.12$. The global fine mesh error is $\|u_h - u\|_\infty = 0.0014$. Similar thing was observed for the nonlinear problem (5.2). In all the numerical tests of Example 5.1–Example 5.6, the integration that is needed for the coarse mesh problem is done with an accuracy that is the same as for the fine mesh problem.

Iteration	max-error	reduction
0	10000.9995	
1	3247.6241	0.32
2	758.6012	0.23
3	93.1200	0.12
4	7.9555	0.09
5	1.6210	0.20
6	0.2855	0.18
7	0.0692	0.24
8	0.0636	0.92

Table 5.15. Iterative solution stops to converge to the fine mesh solution when it reaches the coarse mesh solution accuracy

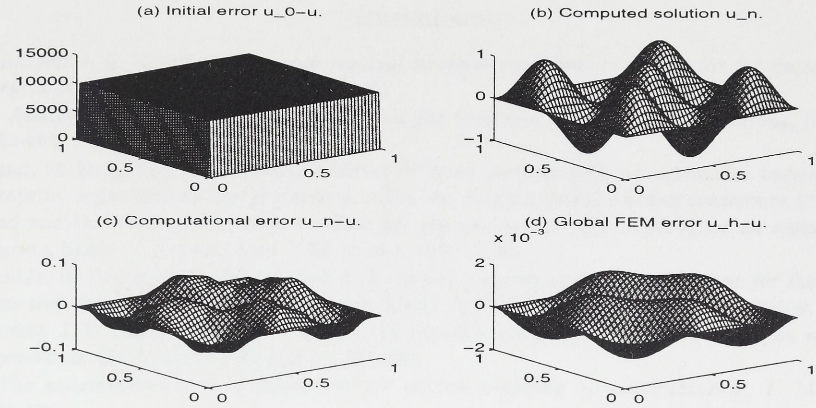

Figure 5.16. Computational results for the linear problem of Example 5.1

Example 5.8. In this example, the cell centres of the coarse mesh elements are used as the integration points for assembling matrix $A_H = (a\nabla\phi_i^H, \nabla\phi_j^H)$, but the integration for assembling the vector (f, ϕ_i^H) is done by using the fine mesh cell centres. Table 5.15 shows the computational result for the same problem of Example 5.1. The convergence is the same. This shows that the integration error for the stiffness matrix does not effect the convergence, but the integration error for vector (f, v_H) can pollute the iterative solution.

Iteration	max-error	reduction
0	10000.9995	
1	3247.6315	0.32
2	758.6414	0.23
3	93.1445	0.12
4	7.9947	0.09
5	1.5787	0.20
6	0.2480	0.16
7	0.0344	0.14
8	0.0033	0.10
9	0.0015	0.44
10	0.0014	1.00

Table 5.17. Same convergence as Example 5.1.

6. Conclusion

Using the observation that the domain decomposition and multilevel methods are space decomposition techniques, a convergence analysis is given for a general convex programming problem. The applications here are given for a two level domain decomposition, but the algorithms can be equally applied for decompositions of multigrid type if a suitable approximate solver is used for the subproblems.

Acknowledgement. The authors like to thank J. Xu and P. Bjorstad. Many insightful comments from J. Xu during the process of the work helped us to improve some of the results and the presentation of the paper. Some discussions with and valuable comments by P. Bjorstad clarify the relationship of our methods with the literature results.

References

1. O. Axellsson and I. E. Kaporin, *Minimum residual adaptive multilevel procedure for the finite element solution of nonlinear stationary problems*, Preprint, 1995.
2. R. Bank, *Analysis of a multilevel iterative method for nonlinear finite element equations*, Math. Comp. . **39** (1982), 453-465.
3. P. Bjorstaad, M. Dryja and E. Vainikko, *Additive Schwarz methods with no subdomain overlap and new coarse spaces*, Preprint, submitted to the proceeding of the 8th domain decomposition conference (1995).
4. P. Bjorstad and O. Widlund, *Iterative methods for the solution of elliptic problems on regions portioned into substructuring*, SIAM J. Numer. Anal. . **23** (1986), 1097–1120.
5. J. H. Bramble, R. Ewing, J. E. Pasciak and A. H. Schatz, *A preconditioning technique for the efficient solution of problems with local grid refinement*, Comput. Meth. Appl. Mech. Eng. **67** (1988), 149-159.
6. J. H. Bramble, J. E. Pasciak and A. H. Schatz, *An iterative method for elliptic problems on regions partitioned into substructures*, Math. Comp. **46** (1986), 361–369.
7. ———, *The construction of preconditioners for elliptic problems by substructuring. I.*, Math. Comp. . **47** (1986), 103-134.
8. ———, *The construction of preconditioners for elliptic problems by substructuring. IV*, Math. Comp. **53** (1989), 1–24.
9. J. H. Bramble and J. Xu, *Some estimates for a weighted L^2 projection*, Math. Comp. **56** (1991), 463-476.
10. A. Brandt, *Multilevel adaptive solutions to boundary value problems*, Math. Comp. . **31** (1977), 333-309.
11. X.-C. Cai and M. Dryja, *Domain decomposition methods for monotone nonlinear elliptic problems*, Proceeding of the seventh international conference on domain decomposition methods in Science and scientific computing (Penn. state Univ.) (D. E. Keyes and Xu, eds.), AMS, Providence, 1994, pp. 21–28.
12. X.-C. Cai, W. D. Groop, D. E. Keyes and M. D. Tidriri, *Parallel implicit methods for aerodynamics*, Proceeding of the seventh international conference on domain decomposition methods in Science and scientific computing (Penn. state Univ.) (D. E. Keyes and Xu, eds.), AMS, Providence, 1994, pp. 21–28.
13. X.-C. Cai, W. D. Groop, D. E. Keyes and M. D. Tidriri, *Newton-Krylov-Schwarz methods in CFD*, Preprint (1995).
14. X. C. Cai and O. B. Widlund, *Multiplicative Schwarz algorithms for some nonsymmetric and indefinite problems*, SIAM J. Numer. Anal. **30** (1993), 936-952.
15. J. Cea, *Optimisation – thérie et algorithmes*, Dunod, 1971.
16. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
17. R. Courant and K. O. Friedrichs, *Supersonic flows and shock waves*, Springer, New York, 1976.
18. C. N. Dawson and M. F. Wheeler, *Two-grid methods for mixed finite element approximations of nonlinear parabolic equations*, Seventh international conference on domain decomposition methods in Science and scientific computing (Penn. state Univ.) (D. E. Keyes and Xu, eds.), AMS, Providence, 1994, pp. 191–203.
19. M. Dryja, B. Smith and O. Widlund, *Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions*, SIAM J. Numer. Anal. . **31** (1994), 1662-1694.
20. M. Dryja and O. B. Widlund, *Domain decomposition algorithms with small overlap*, SIAM J. Sci. Comp. **15** (1994), 604-620.
21. I. Ekeland and R. Temam, *Convex analysis and variational problems*, North–Holland, Amsterdam, 1976.
22. M. Espedal and R. Ewing, *Characteristic Petrov-Galerkin subdomain methods for two phase immiscible flow*, Comput. Methods in Appl. Mech. and Engig. . **64** (1987), 113-135.
23. Fortin m. and Glowinski. R. (eds), *Augmented Lagrangian methods: Applications to the numerical solution of boundary value problems*, North-Holland, Amsterdam, 1983.
24. R. Glowinski and A. Marrocco, *Sur l'approximation par éléments finis d'ordre un, et lan résolution par pénalisation-dualité, d'une classe de problémes de Dirichlet non linéaires*, Rev. Fr. Autom. Inf. Rech. Oper. Anal. Numér. R-2 (1975), 41–76.
25. W. Hackbush, *Iterative methods for large sparse linear systems*, Springer, Heidelberg, 1993.
26. K. Kunisch and X.-C. Tai, *Sequential and parallel splitting methods for bilinear control problems in Hilbert spaces*, To appear in SIAM J. Numer. Anal. (Preprint, Technische Universität Graz, July 1993).
27. ———, *Domain decomposition methods for elliptic parameter estimation problems*, preprint (1995).
28. P. L. Lions, *On the Schwarz alternating method II,*, Domain decomposition methods for partial differential equations,II (T. F. Chan, R. Glowinski, J. Periaux and O. B. Widlund, eds.), SIAM, Philadelphia, 1989, pp. 47-70.
29. L. D. Marini and A. Quarteroni, *A relaxation procedure for domain decomposition methods using finite elements*, Numer. Math. **55** (1989), 575–598.
30. R. Rannacher, *On the convergence of the Newton-Raphson method for strongly nonlinear finite element equations*, Nonlinear computational mechanics (P. Wriggers and W. Wagner, eds.), Springer-Verlag, 1991.
31. X.-C. Tai and M. Espedal, *Rate of convergence of a two level domain decomposition method*, Submitted to the proceeding of the 8th domain decomposition conference (1995).
32. X.–C. Tai, *Parallel function decomposition and space decomposition methods with applications to optimisation, splitting and domain decomposition*, Preprint No. 231–1992, Institut für Mathematik, Technische Universität Graz (1992).
33. ———, *Parallel function and space decomposition methods.*, Finite element methods, fifty years of the courant element (P. Neittaanmäki, eds.), Lecture notes in pure and applied mathematics, vol. 164, Marcel Dekker inc, 1994, pp. 421-432.

REFERENCES

1. O. Axelsson and I. E. Kaporin, *Minimum residual adaptive multilevel procedure for the finite element solution of nonlinear stationary problems*, Preprint, 1998.
2. R. Bank, *Analysis of a multilevel iterative method for nonlinear finite element equations*, Math. Comp., 39 (1982), 453-465.
3. P. Bjørstad, M. Dryja and E. Vainikko, *Additive Schwarz methods with no subdomain overlap and new coarse spaces*, Preprint, submitted to the proceeding of the 8th domain decomposition conference (1995).
4. P. Bjørstad and O. Widlund, *Iterative methods for the solution of elliptic problems on regions partitioned into substructures*, SIAM J. Numer. Anal., 23 (1986), 1097-1120.
5. J. H. Bramble, R. Ewing, J. E. Pasciak and A. H. Schatz, *A preconditioning technique for the efficient solution of problems with local grid refinement*, Comput. Meth. Appl. Mech. Eng, 67 (1988), 149-159.
6. J. H. Bramble, J. E. Pasciak and A. H. Schatz, *An iterative method for elliptic problems on regions partitioned into substructures*, Math. Comp. 46 (1986), 361-369.
7. ———, *The construction of preconditioners for elliptic problems by substructuring, I*, Math. Comp., 47 (1986), 103-134.
8. ———, *The construction of preconditioners for elliptic problems by substructuring, IV*, Math. Comp. 53 (1989), 1-24.
9. J. H. Bramble and J. Xu, *Some estimates for a weighted L^2 projection*, Math. Comp. 56 (1991), 463-476.
10. A. Brandt, *Multilevel adaptive solutions to boundary value problems*, Math. Comp., 31 (1977), 333-390.
11. X.-C. Cai and M. Dryja, *Domain decomposition methods for monotone nonlinear elliptic problems*, Proceeding of the seventh international conference on domain decomposition methods in Science and scientific computing (Penn. state Univ.) (D. E. Keyes and Xu, eds.), AMS, Providence, 1994, pp. 21-28.
12. X.-C. Cai, W. D. Gropp, D. E. Keyes and M. D. Tidriri, *Newton-Krylov-Schwarz methods in CFD*, Proceeding of the seventh international conference on domain decomposition methods in Science and scientific computing (Penn. state Univ.) (D. E. Keyes and Xu, eds.), AMS, Providence, 1994, pp. 17-28.
13. X.-C. Cai, W. D. Gropp, D. E. Keyes and M. D. Tidriri, *Newton-Krylov-Schwarz methods in CFD*, Preprint (1995).
14. X.-C. Cai and O. B. Widlund, *Multiplicative Schwarz algorithms for some nonsymmetric and indefinite problems*, SIAM J. Numer. Anal. 30 (1993), 936-952.
15. J. Céa, *Optimisation - théorie et algorithmes*, Dunod, 1971.
16. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
17. R. Courant and K. O. Friedrichs, *Supersonic flows and shock waves*, Springer, New York, 1976.
18. C. N. Dawson and M. F. Wheeler, *Two-grid methods for mixed finite element approximations of nonlinear parabolic equations*, Seventh international conference on domain decomposition methods in Science and scientific computing (Penn. state Univ.) (D. E. Keyes and Xu, eds.), AMS, Providence, 1994, pp. 191-203.
19. M. Dryja, B. Smith and O. Widlund, *Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions*, SIAM J. Numer. Anal., 31 (1994), 1662-1694.
20. M. Dryja and O. B. Widlund, *Domain decomposition algorithms with small overlap*, SIAM J. Sci. Comp. 15 (1994), 604-620.
21. I. Ekeland and R. Temam, *Convex analysis and variational problems*, North-Holland, Amsterdam, 1976.
22. M. Espedal and R. Ewing, *Characteristic Petrov-Galerkin subdomain methods for two phase immiscible flow*, Comput. Methods in Appl. Mech. and Engrg., 64 (1987), 113-135.
23. Fortin m. and Glowinski, R. (eds), *Augmented Lagrangian methods: Applications to the numerical solution of boundary value problems*, North-Holland, Amsterdam, 1983.
24. R. Glowinski and A. Marrocco, *Sur l'approximation, par éléments finis d'ordre un, et la résolution par pénalisation-dualité, d'une classe de problèmes de Dirichlet non linéaires*, Rev. Fr. Autom. Inf. Rech. Opér. Anal. Numér. R-2 (1975), 41-76.
25. W. Hackbusch, *Iterative methods for large sparse linear systems*, Springer, Heidelberg, 1993.
26. K. Kunisch and X.-C. Tai, *Sequential and parallel splitting methods for bilinear control problems in Hilbert spaces*, To appear in SIAM J. Numer. Anal. (Preprint, Technische Universität Graz, July 1995).
27. ———, *Domain decomposition methods for elliptic parameter estimation problems*, preprint (1995).
28. P. L. Lions, *On the Schwarz alternating method II*, Domain decomposition methods for partial differential equations,II (T. F. Chan, R. Glowinski, J. Periaux and O. B. Widlund, eds.), SIAM, Philadelphia, 1989, pp. 47-70.
29. L. D. Marini and A. Quarteroni, *A relaxation procedure for domain decomposition methods using finite elements*, Numer. Math. 55 (1989), 575-598.
30. R. Rannacher, *On the convergence of the Newton-Raphson method for strongly nonlinear finite element equations*, Nonlinear computational mechanics (P. Wriggers and W. Wagner, eds.), Springer-Verlag, 1991.
31. X.-C. Tai and M. Espedal, *Rate of convergence of a two level domain decomposition method*, Submitted to the proceeding of the 8th domain decomposition conference (1995).
32. X.-C. Tai, *Parallel function decomposition and space decomposition methods with applications to optimisation, splitting and domain decomposition*, Preprint No. 231-1992, Institut für Mathematik, Technische Universität Graz (1992).
33. ———, *Parallel function and space decomposition methods*, Finite element methods, fifty years of the Courant element (P. Neittaanmäki, eds.), Lecture notes in pure and applied mathematics, vol 164, Marcel Dekker inc., 1994, pp. 421-432.

34. ———, *Domain decomposition for linear and nonlinear elliptic problems via function or space decomposition*, Domain decomposition methods in scientific and engineering computing (Proc. of the 7th international conference on domain decomposition, Penn. State University, 1993) (D. Keyes and J. Xu, eds.), American Mathematical Society, 1995, pp. 355-360.
35. B. F. Smith, P. E. Bjorstad and W. D. Gropp, *Domain decomposition: Parallel multilevel algorithms for elliptic partial differential equations*, Combridge Univ. Press, Cambridge, 1995 (toappear).
36. J. C. Xu, *Iteration methods by space decomposition and subspace correction*, SIAM Rev. **34** (1992), 581-613.
37. ———, *A novel two-grid method for semilinear elliptic equations*, SIAM J. Sci. Comp. . **15** (1994), 231-237.
38. ———, *Two-grid discretization techniques for linear and nonlinear PDE*, Tech. report AM105, Depart. of Math., Penn. State University, University park, July, 1992.
39. ———, *Counter examples concerning a weighted L^2 projection*, Math. Comp. **57** (1991), 563-569.
40. X. Zhang, *Multilevel Schwarz methods*, Numer. Math. . **63** (1992), 521-539,.