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SOLUTION OF A STATIONARY FOKKER-PLANCK EQUATION.

by

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Report No. 29 June 1971

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Abstract.

We solve a stationary, linearized and inhomogeneous Fokker-Planck equation describing the electrons of a weakly coupled and weakly inhomogenous plasma in a magnetic field at times large compared to the effective electron - electron collision time.

1. Introduction.

We propose ourselves to solve a stationary Fokker-Planck equation. Motivation for this study is to be found in reference [1], [2], where evolution of a weakly coupled and weakly inhomogeneous plasma in a magnetic field is studied by the multiple-time-scale method. The electron-ion mass ratio and a weak inhomogenity parameter being introduced as small parameters, kinetic equations for electrons and ions are obtained at different orders of approximation in these parameters. These equations appear as non-secularity conditions in the multiple-time-scale expansion, and they are valid at times which are large compared to the effective electron-electron collision time. As is always the case when applying the multiple-time-scale method to kinetic theory, some assumptions are made, which are difficult to give a strict justification. Therefore it is of some importance to show that equations obtained as non-secularity conditions do have solutions which are physically reasonable.

The kinetic equations for electrons obtained in reference[2]contain, in addition to the linearized Fokker-Planck operator, a diffusion term which is due to the fact that electrons have a greater velocity than ions, and a magnetic field term. The kinetic equations for ions [2] do contain only the Fokker-Planck operator. We will here concentrate on the equation for electrons. However, similar results are easily obtained for the equation for ions (by making $\tilde{\gamma}=0$, B=0). As to the possibility of extending the results to equations where the right hand

sides are of more general form than the ones we consider, see comments in section [4].

In section 2, we expand the distribution function in series of surface spherical harmonics. Thus we reduce the problem to get the solution of an infinite set of ordinary integro-differential equations. The number of inhomogeneous equations is determined by the order of anisotropy of the right-hand side. In our case this order is finite. It seems to be very important to make use of this property. Two parameters, α and γ, are introduced; the choice $\alpha = \frac{1}{2}$ and $\gamma = 0$ corresponds to the case treated by Su [3] and McLeod and Ong [4].

In section 3 we show that if the relations $2\alpha = \gamma + 1$ and $\gamma < 1$ hold, we have: The obtained integral operators $e^{c^2} L_{2\ell}$, are symmetric and completely continuous, and the second-order differential operators, $e^{\gamma c^2} L_{1\ell}$, are self-adjoint. Thus $e^{\gamma c^2} L_{1\ell}$ and $e^{\gamma c^2}(L_{1\ell} + L_{2\ell})$ have the same essential spectrum. The choice $\gamma = 0$ gives an essential spectrum ranging from $-\infty$ to 0, while $0 < \gamma < 1$ gives a negative, discrete spectrum.

In section 4 we localize the spectrum to obtain the necessary information on the inverse operators. We conclude that the solutions of the integro-differential equations under consideration, do exist, under suitable conditions.

We also touch upon a corresponding plasma model where ions are neglected, and we give results in this case.

In section 5 we show that the solutions are twice differentiable. We also give results concerning the asymptotic behaviour of the solutions in different cases.

2. Reformulation of the problem.

The kinetic equation to be solved is [2], Eq.(2.50)

$$FP_{11}\left[f^{o}_{1M}(\underline{C}_1)f^{1}_{1M}(\underline{C}'_1) + f^{o}_{1M}(\underline{C}'_1)f^{1}_{1M}(\underline{C}_1)\right] -$$

$$- \frac{e_1}{m_1}\underline{C}_1 \times \underline{B} \cdot \frac{\partial f^{1}_{1M}}{\partial \underline{C}_1} + D_1(f^{1}_{1M}(\underline{C}_1)) = f^{o}_{1M}(\underline{C}_1)\left[\left(\frac{m_1 C_1^2}{2kT_1^o} - \frac{5}{2}\right)\frac{1}{T_1^o}\frac{\partial T_1^o}{\partial \underline{r}} + \right. \tag{1}$$

$$\left. + \frac{e_1}{kT_1^o}\left(\frac{m_1}{e_1}\underline{F}_1 + \underline{c}_o^o \times \underline{B} + \frac{kT_1^o}{e_1}\frac{\partial}{\partial \underline{r}} \ln p_1^o\right)\right] \cdot \underline{C}_1 \ .$$

As long as nothing else is indicated, index 1 refers to electrons and 2 to ions. $\underline{C}_1$, e_1, m_1 are the peculiar velocity, charge and mass of electrons, n_1^o, p_1^o, T_1^o are the density, pressure and temperature of electrons, and $\underline{c}_o^o$ is the total mass transport velocity at zeroth order of approximation. f^{o}_{1M} is the Maxwell distribution function at density n_1^o and temperature T_1^o, $f^{o}_{1M} + f^{1}_{1M}$ is the distribution function of electron velocity at first order of approximation. $\underline{B}$ is an external magnetic field, $\underline{F}_1$ an external force. The Fokker-Planck operator FP_{11} describing electron-electron interactions and the diffusion operator D_1 of electrons by heavy ions are defined by

$$FP_{11} = \frac{1}{m_1^2}\frac{\partial}{\partial \underline{C}_1} \cdot \int d\underline{C}'_1 \, \underset{\sim}{\Phi}^{11}(\underline{C}_1 - \underline{C}'_1) \cdot \left(\frac{\partial}{\partial \underline{C}_1} - \frac{\partial}{\partial \underline{C}'_1}\right)$$

$$D_1 = \frac{n_2^o}{m_1}\frac{\partial}{\partial \underline{C}_1} \cdot \left(\underset{\sim}{\Phi}^{12}(\underline{C}_1) \cdot \frac{\partial}{\partial \underline{C}_1}\right) \tag{2}$$

Tensors $\underset{\sim}{\Phi}^{11}$ and $\underset{\sim}{\Phi}^{12}$ are given by

$$\underset{\sim}{\Phi}^{ij}(\underline{w}) = \int dx_{ij} \frac{\partial\varphi_{ij}}{\partial\underline{x}_j} \int_0^\infty d\tau \, \frac{\partial\varphi_{ij}}{\partial\underline{x}_{ij}}[\underline{x}'_{ij} = \underline{x}_{ij} - \underline{w}\tau], \; i,j = 1, 2 \, .$$

Specifying φ_{ij} to the Colomb potential and making the appropiate cutoffs, we obtain, see for instance [3],

$$\underset{\sim}{\Phi}^{ij}(\underline{w}) = 2\pi e_i^2 e_j^2 \ln \Lambda \, \frac{w^2 \underset{\sim}{I} - \underline{w}\,\underline{w}}{w^3}$$

where $\underset{\sim}{I}$ is the unit tensor and

$$\Lambda = \frac{3\lambda_D kT_1^o}{2e_1^2} \qquad\qquad \lambda_D^{-2} = 4\pi \, \frac{e_1^2 n_1^o + e_2^2 n_2^o}{kT_1^o}$$

A new unknown function Φ is defined by $f_{1M}^1 = f_{1M}^o \, \Phi$ and the non-dimensional velocity $\underline{c} = (\frac{m_1}{2kT_1^o})^{\frac{1}{2}} \underline{C}_1$ is introduced. After some calculations the following forms are obtained for FP_{11}, [3], and D_1:

$$FP_{11}\left[f_{1M}^o(C_1) f_{1M}^o(C'_1)(\Phi(\underline{C}_1) + \Phi(\underline{C}'_1)) \right] = \frac{8\pi^2 m_1 n_1^{o2} e_1^4 \ln \Lambda}{(2\pi kT_1^o)^3} \, e^{-c^2} FP'_{11}(\Phi) \tag{3}$$

$$D_1\left[f_{1M}^o(C_1)\Phi(\underline{C}_1) \right] = \frac{8\pi^2 m_1 n_1^{o2} e_1^4 \ln \Lambda}{(2\pi kT_1^o)^3} \, e^{-c^2} D'_1(\Phi) \tag{4}$$

$$FP'_{11}(\Phi) = \left\{\left[\frac{\mathrm{erf}(c)}{2c^3}(2c^2-1) + \frac{e^{-c^2}}{2c^2}\right]\underset{\sim}{I} - \left[\frac{\mathrm{erf}(c)}{2c}(2c^2-3) + \right.\right.$$

$$\left.\left. + \frac{3e^{-c^2}}{2c^2}\right]\frac{\underline{c}\,\underline{c}}{c^2}\right\} : \left[\tfrac{1}{2}\,\frac{\partial^2\Phi}{\partial\underline{c}\partial\underline{c}} - \underline{c}\,\frac{\partial\Phi}{\partial\underline{c}}\right] + \tag{5}$$

$$+ \left(\frac{e^{-c^2}}{c^2} - \frac{\mathrm{erf}(c)}{c^3}\right)\underline{c}\cdot\frac{\partial\Phi}{\partial\underline{c}} + 2e^{-c^2}\Phi(\underline{c}) +$$

$$+ \frac{1}{\pi}\int d\underline{c}_1 e^{-c_1^2}\,\frac{(c^2-1)g^2 - \underline{c}\cdot\underline{g}}{g^3}\,\Phi(\underline{c}_1)$$

$$D'_1(\Phi) = \tilde{\gamma} e^{c^2}\frac{\partial}{\partial\underline{c}}\cdot\left[e^{-c^2}\,\frac{c^2\underset{\sim}{I} - \underline{c}\,\underline{c}}{c^3}\cdot\frac{\partial\Phi}{\partial\underline{c}}\right] \tag{6}$$

$$\tilde{\gamma} = \frac{\sqrt{\pi}\; n_2^o e_2^2}{4n_1^o e_1^2}$$

$$\mathrm{erf}(c) = \int_0^c e^{-x^2}dx$$

Thus Eq.(1) writes

$$FP'_{11}(\Phi) + D'_1(\Phi) - \underline{c}\times\underline{B}'\cdot\frac{\partial\Phi}{\partial\underline{c}} = \underline{h}\cdot\underline{c} \tag{7}$$

where $\underline{B}' = \dfrac{(2\pi kT_1^o)^{\frac{3}{2}}}{8\pi^2\sqrt{m_1}\,n_1^o \ln\Lambda}\,\underline{B}$

and $\underline{h}\cdot\underline{c}$ represents the right-hand side of Eq.(1) divided by $8\pi^2 m_1 n_1^{o2} e_1^4 \ln\Lambda\,(2\,kT_1^o)^{-3} e^{-c^2}$. Thus it is a known quantity. In order to solve Eq.(7) with respect to $\Phi(\underline{c})$, we introduce spherical polar coordinates c, θ, χ, the fixed

polar axis being directed along the magnetic field $\underline{B}$. We may formally expand Φ in series of surface spherical harmonics [3], [4]

$$\Phi(\underline{c}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{c} e^{\alpha c^2} \Phi_\ell^m(c) Y_\ell^m(\theta,\chi) \tag{8}$$

The factor $c^{-1}e^{\alpha c^2}$, where α is a constant, unspecified so far, is introduced for mathematical purposes. Let us assume for the moment that the summation and the differential operators in Eq.(7) do commute. Using Eqs.(7), (8), Parseval's theorem and Eqs.(9), (10), we obtain after some calculations the set of equations Eqs.(11), (12), (13), (14), (15)

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}Y_\ell^m\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\chi^2}Y_\ell^m = -\ell(\ell+1)Y_\ell^m \tag{9}$$

$$\frac{\partial}{\partial\chi} Y_\ell^m = imY_\ell^m \tag{10}$$

$$e^{\gamma c^2}(L_{1o} + L_{2o})\Phi_o^o = 0 \tag{11}$$

$$e^{\gamma c^2}(L_{11}+L_{21}-i2B')\Phi_1^{-1} = -(h_1-ih_2)\sqrt{\frac{2\pi}{3}}\, 2c^2 e^{(\gamma-\alpha)c^2} \tag{12}$$

$$e^{\gamma c^2}(L_{11}+L_{21})\Phi_1^o = h_3\sqrt{\frac{4\pi}{3}}\, 2c^2 e^{(\gamma-\alpha)c^2} \tag{13}$$

$$e^{\gamma c^2}(L_{11}+L_{21}+i2B')\Phi_1^1 = -(h_1+ih_2)\sqrt{\frac{2\pi}{3}}\, 2c^2 e^{(\gamma-\alpha)c^2} \tag{14}$$

$$e^{\gamma c^2}(L_{1\ell}+L_{2\ell}+i2mB')\Phi_\ell^m = 0 \;, \; -\ell \leq m \leq \ell,\; \ell=2,3,4,\ldots \tag{15}$$

Here the functions $h_i(c)$, $i = 1,2,3$, are the projections of $\underline{h}(c)$ along three orthogonal vectors $\underline{e}_i$, $i = 1,2,3$, $\underline{e}_3$ being parallel to the polar axis, $\underline{e}_1$ and $\underline{e}_2$ corresponding to the $\theta = 0$ and $\theta = \frac{\pi}{2}$ directions. The $L_{1\ell}$ and $L_{2\ell}$ operators are defined by

$$\begin{aligned} L_{1\ell}\psi = {} & \left(\frac{\operatorname{erf}(c)}{c^3} - \frac{e^{-c^2}}{c^2}\right)\frac{d^2\psi}{dc^2} + \Bigg[3\frac{e^{-c^2}}{c^3} - 3\frac{\operatorname{erf}(c)}{c^4} + \\ & + (4\alpha-2)\frac{\operatorname{erf}(c)}{c^2} + 4(1-\alpha)\frac{e^{-c^2}}{c}\Bigg]\frac{d\psi}{dc} + \Bigg[3\,\frac{\operatorname{erf}(c)}{c^5} - 3\frac{e^{-c^2}}{c^4} + \\ & + 2(1-2\alpha)\frac{\operatorname{erf}(c)}{c^3} - 4\alpha(1-\alpha)\frac{\operatorname{erf}(c)}{c} - 4(1-\alpha)\frac{e^{-c^2}}{c^2} + \\ & + 4(1+\alpha(2-\alpha))e^{-c^2} - \frac{\ell(\ell+1)}{2c^4}\left(\frac{\operatorname{erf}(c)}{c}(2c^2-1)+e^{-c^2}\right) - \frac{2\tilde{\gamma}\ell(\ell+1)}{c^3}\Bigg]\psi \end{aligned} \tag{16}$$

$$L_{2\ell}\psi = 4\int_0^\infty cc_1 e^{-(1-\alpha)c_1^2-\alpha c^2} K_\ell(c,c_1)\psi(c_1)dc_1$$

$K_\ell(c,c_1)$ is a symmetric kernel defined by [3], [4]

$$K_\ell(c,c_1) = \frac{2}{2\ell+1}\left\{\frac{c_1^\ell}{c^{\ell+1}}\left[\frac{(\ell+1)(\ell+2)}{2\ell+3}\,c_1^2 - 1\right] - \frac{\ell(\ell+1)}{2\ell-1}\,\frac{c_1^\ell}{c^{\ell-1}}\right\}$$

for $c_1 < c$.

$L_{2\ell}\psi$ comes from the integral part of FP'_{11} , Eq.(5). The term with coefficient $\tilde{\gamma}$ in $L_{1\ell}$ comes from the diffusion operator D'_1, Eq.(6), while the remaining part of $L_{1\ell}$ is provided by the differential part of FP'_{11}, Eq.(5). As it can be seen, both sides of Eqs.(11)-(15)

have been multiplied by $e^{\gamma c^2}$, where γ is a constant, unspecified so far. This is done for mathematical convenience and helps only to invert Eq.(13). As it does not complicate arguments, we keep this factor also in the other equations in order to unify the notations. We will now proceed to the solution of Eqs(11)-(15).

3. Properties of the operators.

We find by inspection that the right-hand sides in the equations all belong to $L^2(o,\infty)$ when $\alpha > \gamma$. Thus we investigate $e^{\gamma c^2} L_{1\ell}$ and $e^{\gamma c^2} L_{2\ell}$ in this space. It is easily seen that the kernel $H_\ell(c,c_1)$ of $e^{\gamma c^2} L_{2\ell}$ is symmetric ans satisfies

$$\int_o^\infty \int_o^\infty H_\ell^2(c,c_1)dc\, dc_1 < \infty$$

if

$$2\alpha = \gamma + 1 \quad \text{and} \quad \gamma < 1 \tag{17}$$

We will throughout assume that α and γ satisfy this relation. It follows that $e^{\gamma c^2} L_{2\ell}$ is selfadjoint and completely continous in $L^2(0,\infty)$. Symmetry of $e^{\gamma c^2} L_{1\ell}$ follows from Eq.(17), and selfadjointmess when $\ell \neq 0$ is shown in appendix 1. Thus $e^{\gamma c^2} L_{1\ell}$ and $e^{\gamma c^2}(L_{1\ell}+L_{2\ell})$, $\ell \neq 0$, will have the same essential spectrum. A study of

the differential operator $e^{\gamma c^2} L_{1\ell}$, $\ell \neq 0$, shows that the essential spectrum is void when $0 < \gamma < 1$ and is the negative semi-axis when $\gamma \leqq 0$, see appendix 2 for the proof. Finally, when $0 < \gamma < 1$ $e^{\gamma c^2} (L_{1\ell}+L_{2\ell})$, $\ell \neq 0$, shows to be negative, as expected for physical reasons, and such that every eigenvalue λ satisfies

$$\lambda < -2\tilde{\gamma}\ \ell(\ell+1)M \tag{18}$$

where M is a strictly positive constant, see appendix 3. This result justifies the introduction of the factor $e^{\gamma c^2}$: it is only when $0 < \gamma < 1$ that it is possible to invert Eq.(13), as well as those of the following equations corresponding to $\ell > 1$ and $m = 0$.

4. Existence of solutions.

It follows directly from Eq.(A.3.2) that Eq.(11) has only solutions of the form

$$\Phi_o^o = ce^{-\alpha c^2}(k_1+k_2c^2) \tag{19}$$

where k_1, k_2 are constants. They correspond to conservation of mass and energy in the interactions which are considered. The only solution of Eq.(15) is zero: where $m \neq 0$, this follows from the fact that the operator is negative definite. Accordingly, there are only a finite number of terms in

expansion Eq. (8). It was easy to predict this result due to the fact that the right-hand side in Eq. (1) has an order of anisotropy equal to one, and since interaction- and magneto-field operators do conserve the order of anisotropy for physical reasons. Thus the problem can be reduced by studying the restriction of the operator in Eq. (1) to subspaces of functions with zeroth and first order of anisotropy. If the right-hand side of Eq. (1) had not a finite order of anisotropy, an infinite number of Eq. (15) would be inhomogeneous. As we have seen, it is always possible to solve such an equation provided the right-ahnd side belongs to $L^2(0,\infty)$. However, one would have to prove the convergence in some meaning, of the corresponding series in Eq. (8) to achieve the solution of Eq. (1). This problem has not been treated so far.

The spectrum of $e^{\gamma c^2}(L_{11}+L_{21})$ being real, as shown in section 3, the left-hand side of Eqs(12) and (14) can be inverted and these equations have a unique solution in $L^2(0,+\infty)$. Zero is a regular value for the operator in Eq. (13) where one chooses $0 < \gamma < 1$ and when $\tilde{\gamma}$ is different from zero (see Eq. (18)) i.e. provided election-ion interactions are taken into account. Thus Eq. (13) has a unique solution in $L^2(0,+\infty)$. When $\tilde{\gamma} = 0$, and $0 < \gamma < 1$, zero is an isolated eigenvalue of $e^{\gamma c^2}(L_{11}+L_{21})$, see appendix 3. Then Eq. (13) has solutions in $L^2(0,+\ )$ if and only if the right-hand side is ort ogonal to $c^2e^{-\alpha c^2}$; then solutions are determined to a $c^2e^{-\alpha c^2}$ near. To check up that the orthogonality condition indeed is fullfilled, one has to turn back to the original eqiation in [2] since making $\tilde{\gamma} = 0$ modifies the form of the right-hand side (see [2], p.38

for an analogue).

We have thus shown that Eq.(1) has solutions depending on arbitrary constants k_i (two if $\tilde{\gamma} \neq 0$, three if $\tilde{\gamma} = 0$) given by

$$f^1_{1M}(\underline{c}_1) = h^o_1\left(\frac{m_1}{2\pi kT^o_1}\right)^{\frac{3}{2}}\left\{\frac{1}{c}e^{\frac{\gamma-1}{2}c^2}(\Phi^o_1(c)\cos\theta + \Phi^{-1}_1(c)\sin\theta\sin\chi + \Phi^1_1(c)\sin\theta\cos\chi) + e^{-c^2}\left[k_1+k_2c^2+(1-\delta_{\tilde{\gamma},o})k_3 c\cos\theta\right]\right\} \tag{20}$$

$\underline{c}$ is the nondimensional velocity, $\underline{c} = (\frac{m_1}{2kT^o_1})^{\frac{1}{2}}\underline{C}_1$. Φ^{-1}_1, Φ^o_1 and Φ^1_1 are solutions of Eqs.(12), (13), (14), respectively, in $L^2(0,+\infty)$. (c,θ,χ) are the spherical polar coordinates of $\underline{c}$, the polar axis being directed along $\underline{B}$. Thus the solutions, Eq.(20) are such that

$$\int d\underline{c}\; e^{(1-\gamma)c^2}|f^1_{1M}(\underline{C}_1)|^2 < \infty\,, \quad 0 < \gamma < 1.$$

5. Properties of solutions.

Further properties of solutions of Eq.(1) are obtained. First we establish differentiality of solutions of Eqs(11)-(15). We see by inspection that $e^{\gamma c^2}L_{2\ell}\psi(c)$ is continous in $(0,+\infty)$. Further a straight forward analysis gives a rough estimation of $e^{\gamma c^2}L_{2\ell}\psi(c)$:

$$|e^{\gamma c^2}L_{2\ell}\psi(c)| < (Nc^{\frac{3}{2}}+Pc^3)e^{\frac{\gamma-1}{2}c^2}\left[\int_o^\infty|\psi|^2dc\right]^{\frac{1}{2}} \tag{21}$$

for all c in $(0,+\infty)$, where N and P are independent of c and ψ. Defining $(e^{\gamma c^2} L_{2\ell}\psi)_{c=0} = 0$, we get that $e^{\gamma c^2} L_{2\ell}\psi(c)$ is continuous everywhere when ψ belongs to $L^2(0,+\infty)$; since the right-hand sides of Eqs.(11)-(15) are continuous everywhere, and zero is the only singular point of $e^{\gamma c^2} L_{1\ell}$ at finite distance, it follows that the solution of these equations (which we know belong to $L^2(0,+\infty)$) are twice continuously differentiable on $(0,+\infty)$. Continuity and differentiality of solutions at $c = 0$ follow from a study of asymptotic properties.

Using Eq.(21), we estimate the non-differential terms in Eqs.(11)-(15) to be of order $O(c^{\frac{3}{2}})$ as $c \to 0$. For $\ell = 1$, relevant equations in the neighbourhood of zero are

$$y'' - \frac{6}{5}cy' - \frac{6\tilde{\gamma}}{c^3}y - \frac{2}{c^2}y + 2imB'y = C_1 c^{\frac{3}{2}}$$

$$m = 0, \pm 1 \; ; \; C_1 \text{ constant}$$

Asymptotic solutions of this equation are obtained by using the method of variation of coefficients and asymptotic expansions Eq.(A.1.4) of the solutions of the corresponding homogeneous differential equations. We get $\Phi_1^m(c) = O(c^{\frac{9}{2}})$ when $\tilde{\gamma} > 0$ and $\Phi_1^m(c) = O(c^2)$ when $\tilde{\gamma} = 0$ as $c \to 0$.

Using the same method, asymptotic behaviour of solutions may be obtained for large c. Relevant equations in the neighbourhood of $c = \infty$ are

$$y'' + [2(2\alpha-1)c-3c^{-1}]y' + [4\alpha(\alpha-1)c^2-imBc^3]y = C_2c^7e^{-\alpha c^2}$$

$$m = 0, \pm 1; \; C_2 \text{ constant.}$$

When $B \neq 0$, we find that $\Phi_1^o(c) = O(c^7 e^{-\alpha c^2})$ and $\Phi_1^{\pm 1}(c) =$ $= O(c^4 e^{-\alpha c^2})$. When $B = 0$, we find that $\Phi_1^m(c) = O(c^7 e^{-\alpha c^2})$, $m = 0, \pm 1$.

Summarizing the results, we have shown that Eq.(1) has solutions given by Eq.(20). These solutions are twice continuously differentiable everywhere and are such that

$$\int d\underline{c}\ e^{(1-\gamma)c^2} |f^1_{1M}(\underline{C}_1)|^2 < \infty\ , \quad 0 < \gamma < 1\ .$$

When $c \to \infty$, they tend towards zero at least as fast as $c^6 e^{-c^2}$ ($c^3 e^{-c^2}$ if the magnetic field and/or gradients parallel to the magnetic field are equal to zero). When $c \to 0$, the part of the solution arising from the inhomogeneous term tend towards zero at least as fast as $c^{\frac{7}{2}}$ (c if election-ion interactions are not taken into account).

Appendix 1.

We determine the number of boundary values needed to make $e^{\gamma c^2} L_{1\ell}$ selfadjoint,[5], p.1306, i.e. we study the equation

$$e^{\gamma c^2} L_{1\ell}\psi = -\lambda\psi \ , \quad \mathrm{Im}\lambda \neq 0 \ , \quad \ell \neq 0. \qquad \text{(A.1.1)}$$

We show that Eq.(A.1.1) has exactly one solution which is square integrable near $c = 0$ and one near $c = \infty$. Hence the operator is selfadjoint with no boundary condition imposed.

Equation (A.1.1) becomes near $c = 0$

$$y'' - \frac{6}{5} cy' - \left[3\tilde{\gamma}\ell(\ell+1)\frac{1}{c^3} + \ell(\ell+1)\frac{1}{c^2}\right]y = -\frac{3}{2}\lambda e^{-\gamma c^2}\psi \qquad \text{(A.1.2)}$$

We consider the following two cases $\tilde{\gamma} = 0$ or $\tilde{\gamma} > 0$.

(i) $\tilde{\gamma} = 0$. We get,[3], [4],two linearly independent solutions of (A.1.2) which for small c behave as $c^{\ell+1}$ and $c^{-\ell}$. Hence only one of them is square integrable near $c = 0$.

(ii) $\tilde{\gamma} > 0$. We substitute

$$z = a\, c^{-1} \ , \quad y = u\, z^{-1} \qquad \text{(A.1.3)}$$

where a is a constant, $a = 3\tilde{\gamma}\ell(\ell+1)$. For large z, Eq.(A.1.1) becomes

$$u'' - z^{-1}u = 0 \ .$$

This equation has subnormal solutions,[6]. For small c, the two linearly independant solutions of Eq.(A.1.1) are

asymptotically represented by

$$\psi \cong \left(\frac{c}{a}\right)^{\frac{3}{4}} \exp\left(\pm 2\sqrt{\frac{a}{c}}\right) \sum_{k=0}^{\infty} c_k \left(\frac{c}{a}\right)^{\frac{k}{2}}, \quad c_o \neq 0 \qquad \text{(A.1.4)}$$

Only one of them is square integrable in the neighbourhood of $c = 0$.

Investigating the solutions of Eq.(A.1.1) for large c, we find that the relevant equation is

$$y'' + [2(2\alpha-1)c - 3c^{-1}]y' + 4\alpha(\alpha-1)c^2 y = 0$$

Assuming $\alpha > \frac{1}{2}$ and substituting

$$y = uv \quad u = \exp[-(\alpha-\tfrac{1}{2})c^2]$$

the relevant equation for v near $c = \infty$ is

$$v'' - c^2 = 0$$

Asymptotic solutions of this equation may be obtained [7] in the form

$$v(c) = e^{\pm\frac{c^2}{2}} c^{-\frac{1}{2}} \sum_{k=o}^{\infty} c_k c^{-k} \quad c_o \neq 0$$

Since $\gamma < 1$, we find that only one of the solutions of Eq.(A.1.2) hence of Eq.(A.1.1), is square integrable near $c = \infty$. The case $\alpha = \frac{1}{2}$ is treated similarly with the same result. We have thus shown that $e^{\gamma c^2} L_{1\ell}$, $\ell \neq 0$, is self-adjoint.

Appendix 2.

We write

$$e^{\gamma c^2} L_{1\ell} = \frac{d}{dc}(p(c)\frac{d}{dc}) + q_\ell(c)$$

where p and q_ℓ are defined by this relation and Eq.(16). In order to study the spectrum of $e^{\gamma c^2} L_{1\ell}$ we examine the eigenvalue problem of the Sturm-Liouville equation

$$e^{\gamma c^2} L_{1\ell}\psi = -\lambda\ \psi\ .$$

To reach standard form we apply the Liouville transformation

$$U_\ell = [p(c)]^{\frac{1}{4}}\psi\,;\ x = \int_0^c [p(\xi)]^{-\frac{1}{2}} d\xi$$

or

$$x = \int_0^c e^{-\frac{\gamma}{2}\xi^2}\left(\frac{\mathrm{erf}(\xi)}{\xi^3} - \frac{e^{-\xi^2}}{\xi^2}\right)^{-\frac{1}{2}} d\xi$$

to get the following equation

$$\frac{d^2}{dx^2}U_\ell + [\lambda - Q(x)]U_\ell = 0 \qquad x \in (0,A)$$

$$e^{-\gamma c^2}Q(x) = \frac{\mathrm{erf}(c)}{c}\left(1 - \frac{\gamma^2}{4}\right) + e^{-c^2}\left(\frac{\gamma^2}{4} - \frac{\gamma}{2} - 8\right) + \frac{\gamma\,\mathrm{erf}(c)}{4c^3} -$$

$$- \frac{\gamma e^{-c^2}}{4c^2} - \frac{\ell(\ell+1)}{2c^4}\left(\frac{\mathrm{erf}(c)}{c}(2c^2-1) + e^{-c^2}\right) - \frac{2\tilde{\gamma}\ell(\ell+1)}{c^3} -$$

$$- \frac{c}{16}\,\frac{\left(3\,\frac{e^{-c^2}}{c^2} - 3\,\frac{\mathrm{erf}(c)}{c^3} + 2e^{-c^2}\right)^2}{\left(\mathrm{erf}(c) - ce^{-c^2}\right)}\ .$$

We have two cases

(i) when $\gamma > 0$, then $\lim\limits_{c\to\infty} x = A < \infty$

(ii) when $\gamma \leqq 0$, then $\lim\limits_{c\to\infty} x = \infty$

(i) When $x \to A$ the dominant term in $Q(x)$ is $(1 - \frac{\gamma^2}{4})\frac{1}{c}e^{\gamma c^2}$ and $\lim\limits_{x\to A} Q(x) = +\infty$ when $0 < \gamma < 2$. On the other side

$$\lim_{x\to 0} Q(x) \cong \lim_{c\to 0} \frac{2\ell(\ell+1)}{3c^2} + 2\gamma \frac{\ell(\ell+1)}{c^3} = +\infty .$$

Noticing that the essential spectrum of $e^{\gamma c^2} L_{1\ell}$ is the union of the essenstial spectrum of $e^{\gamma c^2} L_{1\ell}$ on $(0,A_0]$ and $[A_0,A)$, $A_0 < A$, we get (see [5], p.1594 and p.1599) that the essential spectrum of $e^{\gamma c^2} L_{1\ell}$ is void when $0 < \gamma < 1$. Hence the operator has only a discrete spectrum in this case.

(ii) We have still $\lim\limits_{x\to 0} Q(x) = +\infty$ while $\lim\limits_{x\to\infty} Q(x) = 0$. For the same reasons as before we have now that the essential spectrum of $e^{\gamma c^2} L_{1\ell}$ is the negative semi-axis $(-\infty,0]$.

Appendix 3.

We study now the sign of $e^{\gamma c^2}(L_{1\ell}+L_{2\ell})$. To do so we examine the quantity

$$I = \int d\underline{c}\Psi(\underline{c})FP_{11}[f^o_{1M}(\underline{c})f^o_{1M}(\underline{c}_1)(\Psi(\underline{c}) + \Psi(\underline{c}_1))] \qquad (A.3.1)$$

It is standard work to show that

$$I \leq 0 \qquad (A.3.2)$$

for all Ψ which are bounded, $\frac{\partial\Psi}{\partial\underline{c}}$ bounded and $\frac{\partial^2\Psi}{\partial\underline{c}^2}$ continuous (see [8] for an analogue) and that $I = 0$ if and only if

$$\Psi(\underline{c}) = \rho + \quad c^2 + \underline{k}\cdot\underline{c}$$

where ρ,δ are constants and $\underline{k}$ a constant vector.

Further, let us specify $\Psi(\underline{c}) = c^{-1}e^{\gamma c^2}\psi(c)Y^m_\ell$. Using Eqs(3), (4), (5), (6), (16) and (A.3.1) and assuming $0 < \gamma < 1$ we get

$$\int_0^\infty dc\bar{\psi}e^{\gamma c^2}(L_{1\ell} + L_{2\ell})\psi \leq -2\tilde{\gamma}\ell(\ell+1)\int_0^\infty \frac{e^{\gamma c^2}}{c^3}|\psi|^2dc$$

$$\leq -2\tilde{\gamma}\ell(\ell+1)M\int_0^\infty |\psi|^2dc$$

where $M = \inf \frac{e^{\gamma c^2}}{c^3}$, $c \in [0,+\infty)$. Thus the operator is semibounded on a set of functions ψ which is dense in $L^2(0,\infty)$, and it can be extended [9] to an operator which is selfadjoint and semibounded with the same bound. Thus

$e^{\gamma c^2}(L_{1\ell}+L_{2\ell})$ is negative definite when $\tilde{\gamma} \neq 0$ and $\ell \neq 0$. When $\tilde{\gamma} = 0$, i.e. when interactions with ions are not taken into account, total momentum of electrons is conserved during electron-electron interactions, and zero is an eigenvalue for $\ell = 1$. $e^{\gamma c^2}(L_{11}+L_{21})$ is thus negative. Since $0 < \gamma < 1$ is assumed, the essential spectrum is void, see appendix 2, and zero is isolated. It is thus possible to invert $e^{\gamma c^2}(L_{11}+L_{21})$ on the subspace of element which are orthogonal to $c^2 e^{-\alpha c^2}$ (corresponding to $\Psi(\underline{c}) = \underline{k}\cdot\underline{c}$, $\underline{k}$ constant vector). When $\ell = 0$, $e^{\gamma c^2}(L_{1o}+L_{2o})$ also is negative, independent of the value of $\tilde{\gamma}$ and B. Indeed zero is eigenvalue as can be seen from Eqs(A.3.1), (3), (4), (5) and (6). This corresponds to conservation of mass and total kinetic energy during electron-electron interactions, to conservation of mass and energy of electrons during electrons-heavy ions interactions (see Eq. (6)) together with the fact that the magnetic field operator $-\frac{e_1}{m_1}\underline{c}_1 \times \underline{B}\cdot\frac{\partial}{\partial\underline{c}_1}$ is a differential rotation operator.

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