## Department of APPLIED MATHEMATICS



## UNIVERSITY OF BERGEN <br> Bergen, Norway

# The effects of trapped and untrapped particles on an electrostatic wave packet 

 byMagne S. Espedal

Report No. 46
December 1973

## Abstract.

The propagation of an electrostativ wavepacket in a collissionless plasma is studied. We get a change in amplitude caused by interaction between the packet and particles propagating with velocities near to the group velocity. Also, we get modulation of the plasma in the front of the plasma caused by trapping effects.

## Introduction.

In this paper we shall study the interaction between particles and an electrostatic wave packet. The evolution of a large amplitude wave packet, has been studied earlier by numerical simulation (J. Denavit and R.N. Sudan 1972), but a more complete theory has not been given. As in nonlinear optics and water-wave theory, we shall try to find a waveequation. In collisionless plasmas, the nonlinearity often comes from the trapping of particles in the potential troughs of the waves. Therefore we have to find a procedure which takes care of this effect.

## I. The wave equation.

The equations to govern the onedimensional motion of collisionless plasmas are the Vlasov - Poisson equations:
$\left(\frac{\partial}{\partial t}+v \frac{\partial}{\partial x}\right) f_{j}^{\prime}(x, v, t)+\frac{e_{j}}{m_{j}} E^{\prime}(x, t) \frac{\partial}{\partial v} f_{j}^{\prime}(x, v, t)=0$
$\frac{\partial}{\partial x} E^{\prime}(x, t)=4 \pi \sum_{j} e_{j} \int f_{j}^{\prime}(x, v, t) d v$

The suffix $j$ denotes the species of plasma particles, representing $\ell_{j}=-l, m_{j}=m$ for electrons, and $l_{j}=l$, $m_{j}=M$ for the ions.



20 402










 $(s, i)$
 - Brios grit ro? $=$ 4red

We shall solve eqs. (1.1-2) as an initialvalue problem, where:

$$
\begin{align*}
& f_{j}^{\prime}(x, v, 0)=f_{0}(v)+f_{j}(x, v, 0) e^{i x(x, 0)}  \tag{1.3}\\
& E^{\prime}(x, 0)=E(x, 0) e^{i x(x, 0)} \tag{1.4}
\end{align*}
$$

are given consistently.
In order to solve eqs. (1.1-4), we assume that:

$$
\begin{align*}
& f_{j}^{\prime}(x, v, t)=f_{0}(v)+f_{j}(x, v, t) e^{i x(x, t)}  \tag{1.5}\\
& E^{\prime}(x, t)=E(x, t) e^{i x(x, t)} \tag{1.6}
\end{align*}
$$

Further we define:

$$
\begin{align*}
& \frac{\partial}{\partial t} \chi(x, t)=-w(x, t)  \tag{1.7}\\
& \frac{\partial}{\partial x} \chi(x, t)=k(x, t) \tag{1.8}
\end{align*}
$$

Now, eqs. (1.1-8) gives:

$$
\begin{gather*}
\left(\frac{\partial}{\partial t}+v \frac{\partial}{\partial x}\right) f_{j}(x, v, t)+\frac{e_{j}}{m_{j}} E(x, t) e^{i x(x, t)} \frac{\partial}{\partial v} f_{j}(x, v, t)= \\
\quad i(w-k v) f_{j}(x, v, t)-\frac{e_{j}}{m_{j}} E(x, t) \frac{\partial f_{j}}{\partial v} \tag{1.9}
\end{gather*}
$$

## , mpldorg eulemisittmh fis es $(S-1 . f)$. epo sulor fiacie sW



- VItcostafanos nevig ets



$$
(5 x) x \cdot \frac{1}{4}(3, x) \geq-(5, x)-3
$$


: pritiob ow codtuuly
$(0.1)$

$$
\begin{align*}
& (3 . x) w-(2 x)+x-\frac{6}{3} \\
& (3+x) x^{3} x=(1+x)+x^{\frac{8}{88}} \\
& \text { : epytze (8-1.4) , ape , woll }
\end{align*}
$$

$$
\begin{equation*}
\left(\frac{\partial}{\partial x}+i k\right) E(x, t)=4 \pi \sum_{j} e_{j} \int f_{j}(x, v, t) d v \tag{1.10}
\end{equation*}
$$

Integrating eq. (1.9) along the characteristics we get:

$$
\begin{align*}
& \frac{d t}{d \tau}=1 \\
& \frac{d x_{j}}{d \tau}=V_{j}  \tag{1.11}\\
& \frac{d v_{j}}{d \tau}=\frac{e_{j}}{m_{j}} E e^{i \chi} \\
& \frac{d f_{j}}{d \tau}=i\left(\omega-K v_{j}\right) f_{j}-\frac{e_{j}}{m_{j}} E \frac{\partial f_{0}}{\partial v} \tag{1.12}
\end{align*}
$$

We shall solve eq. (1.11) with the following conditions:

$$
\begin{align*}
& x_{j}(\tau=t)=x \\
& V_{j}(\tau=t)=V \tag{1.13}
\end{align*}
$$

$$
\begin{align*}
& \text { which gives: } \\
& \begin{aligned}
& f_{j}(\tau)= f_{j}(x, v, t) e^{i} \int_{t}^{\tau}\left(\omega-k v_{j}\right) d \tau \\
&-e^{i} \int_{t}^{\tau}\left(\omega-k v_{j}\right) d \tau \\
& \frac{e_{j}}{m_{j}} \int_{t}^{\tau}\left(E(\tau) e^{-i \int_{t}^{\tau}\left(\omega-k v_{j}\right) d s} \frac{\partial f_{0}}{\partial v_{j}}\right) d \tau
\end{aligned}
\end{align*}
$$

$$
-\varepsilon-
$$




$$
1=\frac{46}{\pi b}
$$

（tr，T）

$$
(S 1,1)
$$



$$
x=(k \approx \pi) x
$$

（E「。入）

$$
V=(t+\Gamma) V
$$

（ $4 \mathrm{t}, \mathrm{f}$ ）

$$
\begin{aligned}
& V=\frac{4 x h}{\pi s} \\
& \text { K i } \\
& \square
\end{aligned}
$$

Now, eqs. (1.10) and (1.14) combine to.

$$
\begin{align*}
\frac{\partial E}{\partial x}+i k E= & 4 \pi \sum_{j} e_{j}\left(\int f_{j}(\tau=0) e^{i} \int_{0}^{t}\left(\omega-k v_{j}\right) d \tau\right. \\
& -\sum_{j} \frac{4 \pi e_{j}^{2}}{m_{j}}\left(\int d v \int_{0}^{t} E(\tau) \frac{\partial f_{0}}{\partial v_{j}} l^{-i} \int_{t}^{\tau}\left(\omega-k v_{j}\right) d s\right. \tag{1.15}
\end{align*}
$$

We assume that $f_{j}(x, v, t)$ exists in the sense that the right hand side of eq. (1.15) is finite.

Integrating the last integral in eq. (1.15) by parts,

$$
\begin{align*}
& \text { we get: } \\
& \frac{\partial E}{\partial x}+i k E=4 \pi \sum_{j} e_{j} \int f_{j}(\tau=0) e^{i} \int_{0}^{t}\left(\omega-k v_{j}\right) d \tau d v+ \\
& -\sum_{j} \frac{4 \pi e_{j}^{2}}{m_{j}} \int d v\left[\left.i\left(\frac{E}{\omega-k v_{j}} \frac{\partial f_{0}}{\partial v_{j}} e^{-i} \int_{t}^{\tau}\left(\omega-k v_{j}\right) d s\right)\right|_{0} ^{t}+\right.  \tag{1.16}\\
& +\left.\left(\frac{1}{\omega-k v_{j}} \frac{d}{d \tau}\left(\frac{E}{\omega-k v_{j}} \frac{\partial f_{0}}{\partial v_{j}}\right) e^{-i} \int_{t}^{\tau}\left(\omega-k v_{j}\right) d s\right)\right|_{0} ^{t}- \\
& -\left.i\left(\frac{1}{\omega-k v_{j}} \frac{d}{d \tau}\left(\frac{i}{\omega-k v_{j}} \frac{d}{d \tau}\left(\frac{E}{\omega-k v_{j}} \frac{\partial f_{0}}{\partial v_{j}}\right)\right) e^{-i \int_{t}^{\tau}\left(w-k v_{j}\right) d s}\right)\right|_{0} ^{t} \\
& \left.+i \int_{0}^{t} d \tau\left\{\frac{d}{d \tau}\left(\frac{1}{\omega-k v_{j}} \frac{d}{d \tau}\left(\frac{1}{\omega-k v_{j}} \frac{d}{d \tau}\left(\frac{E}{\omega-k v_{j}} \frac{\partial f_{0}}{\partial v_{j}}\right)\right)\right)_{e}^{-i \int_{t}^{T}\left(\omega-k v_{j}\right) d s}\right\}\right]
\end{align*}
$$

$(\mathrm{Cl}, \mathrm{H})$
 .ottints at (टf, 1) .pe 20 sbla bered trit


$$
(8+.1)
$$

$$
2
$$

Eq. (1.16) is rather complicated, so we want to write it in a more attractive form.

We define:

$$
\begin{align*}
\varepsilon(\omega, k) & =k+\sum_{j} \frac{4 \pi e_{j}^{2}}{m_{j}} \int \frac{\frac{\partial f_{0}}{\partial v}}{\omega-k v} d v  \tag{1.17}\\
V_{g}(x, t) & =-\frac{\partial \varepsilon}{\partial k} / \frac{\partial \varepsilon}{\partial \omega}  \tag{1.18}\\
V_{D}(x, t) & =\left(\frac{\partial \varepsilon}{\partial \omega}\right)^{-1}  \tag{1.19}\\
T(\omega, k) & =\varepsilon(\omega, k)-k  \tag{1.20}\\
\Omega(x, t) & =\frac{1}{2}\left(\frac{\partial}{\partial t} \frac{\partial \Gamma}{\partial \omega}-\frac{\partial}{\partial x} \frac{\partial \Gamma}{\partial k}\right)+ \\
& +\sum_{j} \frac{4 \pi e_{j}^{2}}{m_{j}}\left[i \int \frac{\frac{\partial f}{\partial v}}{\omega-k v} \frac{d}{d t}\left(\frac{1}{\omega-k v} \frac{d}{d t}\left(\frac{1}{\omega-k v}\right)\right) d v+\right. \\
& -\frac{e_{j}}{m_{j}} e^{i \varphi(x, t)}\left(\left[\frac{1}{\omega-k v} \frac{d}{d v}\left(\frac{\frac{\partial f_{0}}{\partial v}}{\omega-k v} \frac{d}{d t}\left(\frac{E}{\omega-k v}\right)\right)+\right.\right.  \tag{1.21}\\
& -i\left(\frac{e_{j}}{m_{j}}\right)^{2}(E)^{2} e^{i 2 \varphi(x, t)}\left(\frac{d v}{\omega-k v} \frac{d}{d v}\left(\frac{1}{\omega-k v} \frac{d}{d v}\left(\frac{\partial f_{0}}{\omega-k v}\right)\right)\right]
\end{align*}
$$


 :antheb evt
(NT, 1)

## $(81.1)$

(er.r)
(0S.7)

$$
\begin{aligned}
& \frac{36}{46} \times \frac{36}{48}-=(5+1) \text { ev } \\
& 1-\left(\frac{38}{46}\right)=(4 x+9) \\
& x+-(2, y, \cos ) 3=(x, \cos )^{T} T \\
& +\left(\frac{56}{x 6} \times \frac{5}{4}-\frac{56}{5}+6\right) \frac{1}{2}=(2, \times) \Omega
\end{aligned}
$$


(18.1)
where $\frac{d}{d t}=\frac{\partial}{\partial t}+v \frac{\partial}{\partial x}$; and

$$
\begin{equation*}
\chi(\tau=t)=\varphi(x, t)+\frac{\pi}{2} \tag{1.22}
\end{equation*}
$$

Using eqs. (1.17-22), eq. (1.16) reduces to:

$$
\begin{aligned}
& \frac{\partial E}{\partial t}+V_{g} \frac{\partial E}{\partial x}-i V_{D} \varepsilon(w, k) E-V_{D} \Omega E+ \\
& -i V_{D} \frac{1}{2}\left(\frac{\partial}{\partial t}\left(\frac{\partial^{2} \Gamma}{\partial w^{2}} \frac{\partial}{\partial t} E-\frac{\partial^{2} \Gamma}{\partial w \partial k} \frac{\partial}{\partial x} E\right)+\frac{\partial}{\partial x}\left(-\frac{\partial^{2} \Gamma}{\partial w \partial k} \frac{\partial}{\partial t} E+\frac{\partial^{2} \Gamma}{\partial k^{2}} \frac{\partial}{\partial x} E\right)\right)=(1.23) \\
& V_{D} \sum_{i=1}^{4} I_{i}(x, t)+v_{D} I_{5}(E, x, t)
\end{aligned}
$$

where:

$$
\begin{align*}
& I_{1}=-4 \pi \sum_{j} \int_{j}(\tau=0) e^{i} \int_{0}^{t}\left(\omega-k v_{j}\right) d s  \tag{1.24}\\
& I_{2}=-i \sum_{j} \frac{4 \pi e_{j}^{2}}{m_{j}} \int\left\{E(\tau) \frac{\frac{\partial f_{0}}{\partial v_{j}}}{\omega-k v_{j}}\right\}_{\tau=0} e^{i \int_{0}^{t}\left(\omega-k v_{j}\right) d s} d v  \tag{1.25}\\
& I_{3}=-\sum_{j} \frac{4 \pi e_{j}^{2}}{m_{j}} \int\left\{\frac{1}{w-k v_{j}} \frac{d}{d \tau}\left(\frac{E}{\omega-k v_{j}}\right)\right\}_{\tau=0} e^{i \int_{0}^{t}\left(\omega-k v_{j}\right) d s} d v  \tag{1.26}\\
& I_{4}=i \sum_{j} \frac{4 \pi l_{j}^{2}}{m_{j}} \int\left\{\frac{1}{\omega-k v_{j}} \frac{d}{d \tau}\left(\frac{1}{\omega-k v_{j}} \frac{d}{d \tau}\left(\frac{E \frac{\partial f_{0}}{\partial v_{j}}}{\omega-k v_{j}}\right)\right)\right\}_{\tau=0}^{e^{i} \int_{0}^{t}\left(\omega-k v_{j}\right) d s} d v  \tag{1.27}\\
& I_{5}=-i \sum_{j} \frac{4 \pi l_{j}^{2}}{m_{j}} \int d v \int_{0}^{t} d \tau\left\{\frac { d } { d \tau } \left(\frac{1}{\omega-k v_{j}} \frac{d}{d \tau}\left(\frac{1}{\omega-k v_{j}} \frac{d}{d \tau}\left(\frac{E}{\omega-k v_{j}} \frac{\partial f_{0}}{\partial v_{j}}\right)\right) e^{\left.-i \int_{t}^{\tau}\left(w-k v_{j}\right) d s\right\}}\right.\right. \tag{1.28}
\end{align*}
$$

(ss, 1)

The left hand side of eq. (1.23) is a nonlinear Schroedinger type of wave equation. But the $I_{5}$ term on the right hand side contains $E(x, t)$, so we still have a integrodifferential equation to solve.
II. The lowest order solution.

In order to solve eq. (1.23), we shall introduce the characteristic time and space scales connected to the problem.

The frequency and wavenumber of the waves making $u p$ the packet are given by $W(x, t)$ and $K(x, t)$, and define the fast timeand space-scales.

If $l$ is the characteristic length for the variation of the amplitude of the wave packet, we may define:

$$
\varepsilon=\frac{1}{k l}
$$

and we shall assume that

$$
\begin{equation*}
\varepsilon \ll 1 \tag{2.1}
\end{equation*}
$$

Therefore we may define the slow space- and timescales by:

$$
\begin{align*}
& x_{1}=\varepsilon x \\
& t_{1}=\varepsilon t \tag{2.2}
\end{align*}
$$

Now, our basic assumtions are that the amplitude of the wave packet, the frequency and the wavenumber vary only on the slow time- and space-scales.


 －ovior oj notさbupe
tmotaulos robxo daevo f ont . II



 palana－anemo hima



$$
\begin{aligned}
& \frac{1}{2 x} x=3 \\
& \text { taniす gmuas ILanla ov bens }
\end{aligned}
$$



$$
\begin{aligned}
& x-3=3, x \\
& 2=-4
\end{aligned}
$$

$$
\begin{aligned}
& \text { sht to gbutiIqme orit tedt 9r.s enoltmueas olaso Two ,woh }
\end{aligned}
$$

$$
\begin{aligned}
& .30 \text { lobe-900ct3 bins -omit wois }
\end{aligned}
$$

$$
\begin{align*}
E(x, t) & =E\left(x_{1}, t_{1}\right) \\
W(x, t) & =W\left(x_{1}, t_{1}\right)  \tag{2.3}\\
K(x, t) & =K\left(x_{1}, t_{1}\right)
\end{align*}
$$

There are two other characteristic timescales which enter into the problem:

$$
\begin{align*}
& \tau_{p}=\frac{2 l}{\left|v_{g}-\frac{w}{k}\right|}  \tag{2.4}\\
& \tau_{t_{r}}^{j}=\frac{2 \pi}{\omega_{B j}}=\left(\left|\frac{e_{j}}{m_{j}} E k\right|\right)^{-\frac{1}{2}}
\end{align*}
$$

$\tau_{p}$ is the typical time which a particle with the velocity $\frac{\omega}{k}$ uses to get through the wave packet. We may note that if we have a very long wave packet, or a finite amplitude wave, $l$ should be taken as the damping or growth scale of the amplitude. $\tau_{t r}^{j}$ is the oscillation time for the trapped particles, and it depends on the particle mass $m_{j}$ :

$$
\begin{equation*}
\tau_{t_{r}}^{i}=\left(\frac{m_{i}}{m_{l}}\right)^{\frac{1}{2}} \tau_{t_{r}}^{e} \tag{2.5}
\end{equation*}
$$

This means that we have to distinguish between electronwaves and ion-waves.

We assume that

$$
\begin{equation*}
\tau_{t_{r}}^{e}>\tau_{p} \tag{2.6}
\end{equation*}
$$

for electron-waves, and

$$
\left(1++r_{0}+\pi\right) \pi=( \pm, x) \pi
$$

$(E, S)$

$$
\begin{aligned}
& (J, \mu x)(t)=(t, x)(N) \\
& ( \pm \ldots x) t=(t, x)) t
\end{aligned}
$$



$$
\text { anintawin antat }-2
$$

(11.9)

$$
\left(\frac{5-(3)}{2^{3}} 1\right)=\frac{2 x^{2}}{13}=2+1
$$





 (ت.S)

> : Irmi pasemf alot trepe oft mo shefociob th

> - a9vew-riol bre asvaw tant emmepe orit


$$
\begin{equation*}
\tau_{t_{m}}^{i}>\tau_{p} \tag{2.7}
\end{equation*}
$$

for ion-waves.
Eq. (2,5-6) means that the trapped electrons make less than one oscillation in the potential well, while the ions feel no trapping effects.

Eqs. (2.5) and (2.7) means that the ions make less than one oscillation in the potential well, in which the electrons may oscillate several times. However, in many cases the electrons behave as an ideal fluid and the electron trapping effects may be neglected.

The phase velocities, $\frac{\omega}{K}$, of the waves making up the wave packet, are given by:

$$
\begin{equation*}
V_{m} \leq \frac{\omega}{k} \leq V_{M} \tag{2.8}
\end{equation*}
$$

and trapping effects will be important in the same range of the velocity-space.

Using eq. (2.3), we notice that $\Omega\left(x, t, x_{1}, t_{1}\right)$ is the only term on the left hand side of eq. (1.23) which depends on the fast time and space scales.

In order to eliminate this dependence, we integrate over the fast variables in the following way:

$$
\begin{equation*}
\bar{A}\left(x_{1}, t_{1}\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi} A\left(\varphi\left(x_{2}, x_{1}, t_{1}\right), x_{1}, t_{1}\right) d \varphi \tag{2.9}
\end{equation*}
$$

where $\varphi$ is defined by eq. (1.22).

$$
\text { Eqs. }(1.7-8),(1.11) \text { and }(1.22) \text { gives }
$$






 ot when etoo990 mitught mortoolo git bms biuls Isobinss be gysiod - Batan「man

: Yol novis ots atslosg פvew
 Manomountinn 「aut

 barano gnamb hava ambt thap

 ( (e.s)

- (SS.T) . po yd bentiob at C eroth


$$
\begin{equation*}
\chi(\tau, t)=\frac{\pi}{2}+\varphi(x, t)-\int_{t}^{\tau}\left(\omega-k v_{j}\right) d s \tag{2.10}
\end{equation*}
$$

Writing eqs. (1.26-28) in a more explicit form, we have:

$$
\begin{aligned}
& I_{3}\left(x_{1}, t_{1}, \varphi\right)=-\sum_{j} \frac{4 \pi e_{j}^{2}}{m_{j}}\left[\varepsilon \int d v\left\{\frac{\frac{\partial f_{0}}{\partial v_{j}}}{\omega-k v_{j}} \frac{d}{d \varepsilon \tau}\left(\frac{E}{\omega-k v_{j}}\right)\right\} \int_{\tau=0} e^{i} \int_{0}^{t}\left(\omega-k v_{j}\right) d s\right. \\
& i e^{i \varphi} \int d v\left\{\frac { E } { \omega - k v _ { j } } \frac { d } { d v _ { j } } \left(\frac{\left.\left.\left.\frac{\partial f_{0}}{\partial-k v_{j}}\right) \frac{e_{j}}{m_{j}} E\right\} e_{\tau=0}^{2 i \int_{0}^{t}\left(\omega-k v_{j}\right) d s}\right] .2(2]}{}\right.\right. \\
& I_{4}\left(x_{1}, t_{1}, \varphi\right)=i \sum_{j} \frac{4 \pi l_{j}^{2}}{m_{j}}\left[\varepsilon^{2} \int d v\left\{\frac{\frac{\partial f_{0}}{\omega v_{j}}}{\omega-k v_{j}} \frac{d}{d \varepsilon \tau}\left(\frac{1}{\omega-k v_{j}} \frac{d}{d \varepsilon \tau}\left(\frac{E}{\omega-k v_{j}}\right)\right)\right]_{\tau=0} e^{i \int_{0}^{t}\left(\omega-k v_{j}\right) d s}+\right. \\
& +\varepsilon i e^{i \varphi}\left(\int d v\left\{\frac{1}{\omega-k v_{j}} \frac{d}{d v_{j}}\left(\frac{\partial f_{V_{j}}}{\omega-k v_{j}} \frac{d}{d \varepsilon T}\left(\frac{E}{\omega-k v_{j}}\right)\right) \frac{e_{j}}{m_{j}} E\right\}_{T=0} e^{i 2 \int_{0}^{t}\left(\omega-k v_{j}\right) d s}+\right. \\
& +\int d v\left\{\frac{1}{\omega-k v_{j}} \frac{d}{d \varepsilon \tau}\left(\frac{E}{\omega-k v_{j}} \frac{d}{d v_{j}}\left(\frac{\frac{\partial f_{0}}{\partial-k v_{j}}}{\omega-k v_{j}}\right) \frac{e_{j}}{m_{j}} E\right\}_{\tau=0} e^{i 2 \int_{0}^{t}\left(\omega-k v_{j}\right) d s}\right)+(2.12)
\end{aligned}
$$

$$
\begin{aligned}
& \left.+e^{i \varphi} \int d v\left\{\frac{E}{\omega-k v_{j}} \frac{d}{d v_{j}}\left(\frac{\frac{\partial f_{0}}{\omega v_{j}}}{\omega-k v_{j}}\right) \frac{e_{j}}{m_{j}} E\right\}_{\tau=0} e^{i 2 \int_{0}^{t}\left(\omega-k v_{j}\right) d s}\right]
\end{aligned}
$$

$$
\left.2 ⿻(x) x-w)^{4}\right) \sin ^{-}(3, x)\left(x+\frac{\pi}{2} \theta(1, x) x^{x}\right.
$$







$$
\begin{aligned}
& I_{5}\left(x_{1, t}, \varphi_{1}, E\right)=-i \sum_{j} \frac{4 \pi \varepsilon_{j}^{2}}{m_{j}} \int d v\left[\varepsilon ^ { 3 } \int _ { 0 } ^ { t } \left\{\frac { d } { d \varepsilon \tau } \left(\frac { \partial f _ { 0 } } { \omega - k v _ { j } } \frac { d } { d \varepsilon \tau } \left(\frac{1}{\omega-k v_{j}} \frac{d}{d \varepsilon \tau}\left(\frac{E}{\omega-k v_{j}}\right) \cdot\right.\right.\right.\right. \\
& \left.e^{-i \int_{z}^{\pi}\left(\omega-k v_{j}\right) d s}\right\} d \tau+ \\
& +i \varepsilon^{2} e^{i \varphi} \int_{0}^{t} d \tau e^{-i 2 \int_{t}^{\tau}\left(\omega-k v_{j}\right) d s}\left\{\frac { d } { d r _ { j } } \left(\frac { 1 } { \omega - k v _ { j } } \frac { d } { d \tau \tau } \left(\frac{1}{\omega-k v_{j}} \frac{d}{d \varepsilon \tau}\left(\frac{E}{\omega-k v_{j}}\right) \frac{\ell_{j}}{m_{j}} E+\right.\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{d}{d v_{j}}\left(\frac{1}{\omega-k v_{j}}\left[\frac{d}{d V_{j}}\left(\frac{1}{\omega-K V_{j}} \frac{d}{d \varepsilon \tau}\left(\frac{E \frac{\partial f}{\partial R_{j}}}{\omega-k v_{j}}\right)\right)+\frac{d}{d \varepsilon T}\left(\frac{1}{\omega-V_{j}} \frac{d}{d v_{j}}\left(\frac{E \frac{\partial f}{\partial-k v_{j}}}{\omega-v_{j}}\right)\right)\right)\left(\frac{l_{j}}{m_{j}} E\right)^{2}\right)+(2.13) \\
& -e^{i \varphi} \int_{0}^{t} d \tau e^{-i 2_{t} \int_{t}^{T}\left(\omega-k v_{j}\right) d s}\left\{\frac{d}{d \varepsilon \tau}\left(\frac{2}{\omega-k v_{j}} \frac{d}{d V_{j}}\left(\frac{E \frac{\partial f_{j}}{\partial-k v_{j}}}{\omega}\right)\right) \frac{e_{j}}{m_{j} j} E+\right. \\
& \left.\left.+\frac{d}{d v_{j}}\left(\frac{1}{w-k v_{j}} \frac{d}{d \varepsilon \tau}\left(\frac{E \frac{\partial f_{j}}{\partial \varepsilon_{i}}}{\omega-k v_{j}}\right)\right) \frac{e_{j}}{m_{j}} E\right\}\right)+ \\
& -i e^{i 3 \varphi} \int_{0}^{t} d \tau e^{-i 4} \int_{t}^{T}\left(\omega-k v_{j}\right) d s\left\{\frac{d}{d v_{j}}\left(\frac{1}{\omega-k v_{j}} \frac{d}{d v_{j}}\left(\frac{1}{\omega-k v_{j}} \cdot \frac{d}{d v_{j}} \cdot\left(\frac{E \frac{\partial \rho_{j}}{\partial u_{j}}}{\omega-k v_{j}}\right)\right)\left(\frac{e_{j}}{m}, E\right)^{3}\right\}+\right.
\end{aligned}
$$

Whan maty

$$
x+x b\{\operatorname{sb(}(2 x-w))^{2}-
$$








$$
0
$$




Assuming that

$$
\begin{equation*}
\frac{w_{t_{r}}^{j}}{w}=O(\varepsilon) \tag{2.14}
\end{equation*}
$$

all the coefficients of $e^{i m \varphi}, n=1.2 .3$. , are slowly varying functions compared to $e^{i m \varphi}$. One should note that the coefficients depend on $\varphi$ through $V_{j}(\tau)$ which is periodic in $\varphi$. Therefore, taking the mean value of eq. (1.23) and eqs. $(2.11-13)$, we get to $O\left(\varepsilon^{3}\right)$ :

$$
\begin{aligned}
& \varepsilon\left(\frac{\partial E}{\partial t_{1}}+V_{g} \frac{\partial E}{\partial x_{1}}\right)-i V_{0} \varepsilon(\omega, k) E-v_{0} \bar{\Omega} E+ \\
& -\frac{1}{2} i V_{0} \varepsilon^{2}\left(\frac{\partial}{\partial t_{1}}\left(\frac{\partial^{2} \Gamma}{\partial \omega^{2}} \frac{\partial}{\partial t_{1}} E-\frac{\partial^{2} \Gamma}{\partial \omega \partial k} \frac{\partial}{\partial x_{1}} E\right)+\frac{\partial}{\partial x_{1}}\left(-\frac{\partial^{2} \Gamma}{\partial \omega \partial k} \frac{\partial}{\partial t_{1}} E+\frac{\partial^{2} \Gamma}{\partial k^{2}} \frac{\partial}{\partial x_{1}} E\right)\right)= \\
& \sum_{i=1}^{4} \bar{I}_{i}
\end{aligned}
$$

where

$$
\begin{align*}
\bar{\Omega} & =\varepsilon \frac{1}{2}\left(\frac{\partial}{\partial t_{1}} \frac{\partial \Gamma}{\partial \omega}-\frac{\partial}{\partial x_{1}} \frac{\partial \Gamma}{\partial k}\right)+\varepsilon^{2} i \sum_{j} \frac{4 \pi l_{j}^{2}}{m_{j}} \Gamma \int \frac{\frac{\partial f}{\partial L}}{\omega-k v} \frac{d}{d t_{1}}\left(\frac{1}{\omega-k v} \frac{d}{d t_{1}}\left(\frac{1}{\omega-k v}\right)\right) d v \\
& =\varepsilon \Omega_{1}+\varepsilon^{2} \Omega_{2} \\
\bar{I}_{1} & =I_{1}(\varphi=0)  \tag{2.17}\\
\bar{I}_{2} & =\bar{I}_{2}(\varphi=0) \tag{2.18}
\end{align*}
$$

$$
(3) 0=\frac{500}{64}
$$




 $:(3) 0$ ot tog ow $\cdot(E 1-11 . S) \cdot$ ape
(er.s)
4. in th

$$
=\frac{x}{3}
$$

$$
\log +\frac{1}{2}
$$

$$
\begin{aligned}
& S_{1} \Omega^{\prime} 9+\Omega_{1} \Omega= \\
& (0=9) I=\bar{I}=
\end{aligned}
$$

$$
\begin{align*}
& \bar{I}_{3}=-\varepsilon \sum_{j} \frac{4 \pi e_{j}^{2}}{m_{j}}\left[\left[\left\{\frac{1}{\omega-k v_{j}} \frac{d}{d \varepsilon \tau}\left(\frac{\frac{\partial f_{j}}{\partial j} E}{\omega-k v_{j}}\right)\right\}_{\tau=0} e^{i \int_{0}^{t}\left(\omega-k v_{j}\right) d s} d v\right]_{\varphi}=0\right.  \tag{2.19}\\
& \bar{I}_{4}=i \varepsilon^{2} \sum_{j} \frac{4 \pi e_{j}^{2}}{m_{j}} \int\left[\left\{\frac{1}{\omega-k v_{j}} \frac{d}{d \varepsilon \tau}\left(\frac{1}{\omega-k v_{j}} \frac{d}{d \varepsilon \tau}\left(\frac{\frac{\partial f_{0}}{\partial v_{j}} E}{\omega-k v_{j}}\right)\right)\right\}_{\tau=0} e^{i \int_{0}^{t}\left(\omega-k v_{j}\right) d s} d v\right]_{\varphi=0} \tag{2.20}
\end{align*}
$$

In eqs. (2.16) and (2.20), we have neglected the self action term (Dysthe 1974), which gives an amplitude dependent frequency shift. In order to solve eq. (2.8), we shall make the following assumptions:

$$
\begin{align*}
R(\omega, k) & =\left(\Gamma_{\omega k}\right)^{2}-\Gamma_{\omega \omega} \Gamma_{k k}>0  \tag{2.21}\\
& >\frac{\Gamma_{\omega k}}{\Gamma_{\omega \omega}}-\frac{\sqrt{R}}{T_{\omega \omega}} \leq v_{g} \leq-\frac{T_{\omega k}}{T_{\omega \omega}}+\frac{\sqrt{R}}{T_{\omega \omega}} \tag{2.21a}
\end{align*}
$$

With the condition (2.21), eq. (2.15) is a hyperbolic type of equation.

Eq. (2.21a) means that the subcharacteristics (J.D. Cole 1968) given by:

$$
\begin{align*}
& \frac{d t_{1}}{d s}=1 \\
& \frac{d x_{1}}{d s}=V_{g}(s) \tag{2.22}
\end{align*}
$$

are timelike, and the initial value problem may be solved. Furthermore, we divide the $\left(x_{1}, t_{1}\right)$ space into two parts according to:
(er.s)
(OS.S)



: 8 mb - ťquyers

(s!S.S )
 Mn*tamen 20


$$
1=\frac{1 d h}{2 h}
$$

(SS.S)

$$
(2), y=\frac{x b}{2 b}
$$




$$
\begin{align*}
& \frac{\partial}{\partial t_{1}}+v_{g} \frac{\partial}{\partial x_{1}} \geq \frac{w}{\varepsilon}  \tag{2.23}\\
& \frac{\partial}{\partial t_{1}}+v_{g} \frac{\partial}{\partial x_{1}}<\frac{\omega}{\varepsilon} \tag{2.24}
\end{align*}
$$

The solution in the region (2.12), we shall call the outer solution, and the other one the inner solution.
III. The outer solution.

In this region it is natural to search for a solution in the form

$$
\begin{equation*}
E\left(x_{1}, t_{1}\right)=\sum_{m=0}^{\infty} \varepsilon^{m} E_{m}^{o n t}\left(x_{1}, t_{1}\right) \tag{3.1}
\end{equation*}
$$

where

$$
\begin{align*}
& E_{0}^{\text {out }}\left(x_{1}, 0\right)=E_{0}\left(x_{1}\right) \\
& E_{i}^{\text {out }}\left(x_{1}, 0\right)=0 \quad i \quad i=1,2,3 \ldots \tag{3.2}
\end{align*}
$$

Eqs. (2.15) and (3.1-2) give:

$$
\begin{equation*}
\left(\frac{\partial}{\partial t_{1}}+v_{g} \frac{\partial}{\partial x_{1}}\right) E_{i}^{\text {out }}-\frac{1}{\varepsilon} i v_{D} \varepsilon(\omega, k) E_{i}^{\text {out }}-v_{D} \Omega, E_{i}^{\text {out }}=\varepsilon^{-i-1} L_{i}\left(x_{1}, t_{1}\right) \tag{3.3}
\end{equation*}
$$

where

$$
\begin{equation*}
L_{0}\left(x_{1}, t_{1}\right)=v_{D} \sum_{i=1}^{3} \bar{I}_{i}\left(x_{1}, t_{1}\right) \tag{3.4}
\end{equation*}
$$

$$
\frac{6}{y} \leq \frac{6}{26} v+\frac{5}{1+6}
$$

#  


．Rolitulos toduo orf？．ITI

arot ョ゙き
（1．ह）

$$
\left(x^{2}, x\right)^{2}+3-3+\frac{3}{3}+m=(, 2, x) 7
$$

$$
(x), 3=(2, x)^{x} \cdot 3
$$

（S．E）

$$
5
$$

* 

$$
(\$, E)
$$

$$
\begin{align*}
& L_{1}\left(x_{1}, t_{1}\right)= V_{D} \bar{I}_{4}\left(x_{1}, t_{1}\right)+\varepsilon^{2} V_{D} \Omega_{2} E_{0}^{o u t}+ \\
&+\frac{1}{2} i \varepsilon^{2} V_{D}\left(\frac{\partial}{\partial t_{1}}\left(T_{\omega \omega} \frac{\partial}{\partial t_{1}}-\Gamma_{\omega k} \frac{\partial}{\partial x_{1}}\right)+\frac{\partial}{\partial x_{1}}\left(-T_{\omega k} \frac{\partial}{\partial t_{1}}+T_{k k} \frac{\partial}{\partial x_{1}}\right)\right) E_{0}^{o u t}  \tag{3.5}\\
& L_{i}\left(x_{1}, t_{1}\right)= \frac{1}{2} i \varepsilon^{2} V_{D}\left(\frac{\partial}{\partial t_{1}}\left(T_{\omega \omega} \frac{\partial}{\partial t_{1}}-T_{\omega k} \frac{\partial}{\partial x_{1}}\right)+\frac{\partial}{\partial x_{1}}\left(-T_{\omega k} \frac{\partial}{\partial t_{1}}+\Gamma_{k k} \frac{\partial}{\partial x_{1}}\right)\right) E_{i-1}^{\text {out }}+ \\
&+\varepsilon^{2} V_{D} \Omega_{2} E_{i-1}^{\text {out }}+V_{D} \bar{I}_{5}\left(E_{i-2}^{\text {out }}, x_{1}, t_{1}\right)  \tag{3.6}\\
& i=2,3,4 \ldots
\end{align*}
$$

The solution of eq. (3.3) is given by:

$$
\begin{align*}
& E_{i}^{\text {out }}\left(x_{1}, t_{1}\right)=\exp \left(i \frac{1}{\varepsilon} \int_{0}^{t_{1}} v_{0}\left(\varepsilon(w, k)+\varepsilon \Omega_{1}\right) d s\right) E_{i}^{\text {out }}\left(x_{1}(s=0), 0\right)+ \\
& \quad+\varepsilon^{-i-1} \int_{0}^{t_{1}} d s\left(\exp \left(-i \frac{1}{\varepsilon} \int_{t_{1}}^{s} v_{D}\left(\varepsilon(w, k)+\varepsilon \Omega_{1}\right) d u\right) L_{i}\left(x_{1}(s), s\right)\right)  \tag{3.7}\\
& i=0,1,2, \ldots
\end{align*}
$$

$$
\frac{d t_{1}}{d s}=1
$$

$$
\begin{equation*}
\frac{d x_{1}}{d s}=v_{g}(s) \tag{3.8}
\end{equation*}
$$

$$
x_{1}\left(t_{1}\right)=x_{1}
$$

(द. ह)
(2. ¿)
:yd movis at (ह.ह) apo to noljuion ont
(r.e.)
(8. ह)

$$
(2) \mathrm{p}=\frac{x b}{2 b}
$$

$$
, x=(\lambda) x
$$

$$
\begin{aligned}
& 5 \cdot 1,0=j \\
& 1=\frac{156}{25}
\end{aligned}
$$

$$
\begin{aligned}
& -4,8 .+=-
\end{aligned}
$$

Because $E_{i}^{\text {out }}\left(x_{1}, t_{1}\right)$ is a slowly varying quantity, $V_{D} \mathcal{E}(\omega, K)$ must be zero to lowest order:

$$
\begin{equation*}
\varepsilon\left(w_{0}, k_{0}\right)=0 \tag{3.9}
\end{equation*}
$$

Expanding $V_{D} \varepsilon(\omega, k)$ in a Taylor-series around $K=K_{0}$, $\omega=\omega_{0}$, we get:

$$
\begin{equation*}
V_{0} \varepsilon(w, k)=\Delta w-\left.V_{g}\right|_{\substack{k=k_{0} \\ \omega=\omega_{0}}} \Delta k+\left.\frac{1}{2} \frac{d V_{g}}{d k}\right|_{\substack{k=k_{0} \\ \omega=\omega_{0}}}(\Delta k)^{2}+O\left((\Delta k)^{3},(\Delta \omega)^{3}\right) \tag{3.10}
\end{equation*}
$$

where $\Delta \omega=\omega-\omega_{0}, \Delta k=K-k_{0}$
We may note that $\Delta \omega-v g \Delta k=O\left((\Delta K)^{2},(\Delta \omega)^{2}\right)$ Because $V_{D} \varepsilon(\omega, k)=O(\varepsilon \omega), V_{D} \Omega_{1}=O(\varepsilon \omega)$, which means that we may meglect this term to lowest order.

This gives:

$$
\begin{align*}
E_{0}^{o u t}\left(x_{1}, t_{1}\right) & =\operatorname{Re}\left[\exp \left(i \frac{1}{\varepsilon} \int_{0}^{t_{1}}\left(\left(\Delta \omega-v_{g}\left(k_{0}, \omega_{0}\right) \Delta k\right)+\left.\frac{d v_{g}}{d k}\right|_{\substack{k=k_{0} \\
w=w_{0}}}(\Delta k)^{2}\right) d s\right) E_{0}\left(x_{1}(s=0), 0\right)+\right. \\
& +\int_{0}^{t_{1}} d s\left(\exp -i \frac{1}{\varepsilon} \int_{t_{1}}^{s}\left(\left(\Delta w-v_{g}\left(k_{0}, \omega_{0}\right) \Delta k\right)+\left.\frac{d v_{0}}{d k}\right|_{\substack{k=k_{0} \\
\omega=w_{0}}}(\Delta k)^{2}\right) d s\right) L_{0}\left(x_{1}(s), s\right) \tag{3.11}
\end{align*}
$$

To get the explicit expressions of eqs. (3.7) and (3.11) we have to solve eqs. (1.11) and (3.8), which will be done in section $V$.

$$
8
$$ :robro jagwoL of oros od jamm $\quad(\mu, \omega) 3_{a} W$

$$
9=(8 x+104) 3
$$

, $X=X$ bnwore מotroe-roIxaT is nt $(x, 4)\}$ Q gnibnegrad

$$
\text { :tog } \rho W: . . W=
$$



-robra teowor at minot etilt toolnom vam ow


IV. The inner solution.

In the inner region, we define:

$$
\begin{align*}
& \frac{\partial}{\partial t_{1}}=-v_{g} \frac{\partial}{\partial x_{1}}+\varepsilon \frac{\partial}{\partial t_{2}}+\varepsilon^{2} \frac{\partial}{\partial t_{3}}+ \\
& \frac{\partial}{\partial x_{1}}=\frac{\partial}{\partial x_{1}}+\varepsilon \frac{\partial}{\partial x_{2}}+\varepsilon^{2} \frac{\partial}{\partial x_{3}}+\cdots \tag{4.1}
\end{align*}
$$

where $t_{i}=\varepsilon^{i} t, \quad x_{i}=\varepsilon^{i} x ; \quad i=1,2,3, \cdots$.
Eq. (4.1) is consistent with the assumption (2.24).
Furthermore, we have from Eqs. (1.7-8):

$$
\begin{align*}
& \frac{\partial \omega}{\partial x_{1}}+\frac{\partial k}{\partial t_{1}}=0 \text {, or }  \tag{4.2}\\
& \frac{\partial \omega}{\partial x_{1}}-v_{g} \frac{\partial k}{\partial x_{1}}=-\left(\varepsilon \frac{\partial}{\partial x_{2}}+\varepsilon^{2} \frac{\partial}{\partial x_{3}}+\cdots\right) \omega-\left(\varepsilon \frac{\partial}{\partial t_{2}}+\varepsilon^{2} \frac{\partial}{\partial t_{3}}+\cdots\right)
\end{align*}
$$

We define:

$$
\begin{equation*}
E^{i m}\left(x_{1}, x_{2}, \cdots, t_{1}, t_{2}, \cdots\right)=\sum_{m=0}^{\infty} \varepsilon^{m} E_{m}^{i n} \tag{4.3}
\end{equation*}
$$

$$
E_{0}^{\text {in }}\left(x_{1}, x_{2}, \ldots, 0,0, \ldots\right)=E_{0}\left(x_{1}\right)
$$

$$
\begin{equation*}
E_{i}^{i n}\left(x_{1}, x_{2}, \ldots, 0,0, \cdots\right)=0 \quad j \quad i=1,2,3, \ldots \tag{4.4}
\end{equation*}
$$

Now, eqs. $(2.15),(4.1-3)$ give to $O\left(\varepsilon^{3}\right)$ :

$$
\begin{aligned}
& \left(\frac{\partial}{\partial t_{2}}+v_{g} \frac{\partial}{\partial x_{2}}\right) E_{0}^{i n}-\frac{1}{\varepsilon^{2}}\left(i v_{0} \varepsilon(\omega, k)+v_{0} \bar{\Omega}\right) E_{0}^{i n}-\frac{1}{2} i\left(\frac{d v_{g}}{d k} \frac{\partial^{2}}{\partial x_{1}^{2}} E_{0}^{i n}+\right. \\
& -\frac{1}{T_{\omega}}\left(\frac{\partial}{\partial x_{1}}\left(T_{\omega} \frac{\partial}{\partial k} v_{g}\right)+v_{g} \frac{\partial}{\partial x_{1}}\left(T_{\omega} \frac{\partial}{\partial \omega} v_{g}\right)\right) \frac{\partial}{\partial x_{1}} E_{0}^{i n}=\varepsilon^{-2} v_{0} \sum_{i=1}^{4} \bar{I}_{i}\left(x_{1}, x_{2}, \cdots, v_{1}, z_{2} \cdot .\right)
\end{aligned}
$$

Using eqs. (1.18), (4.1-2), we have:

$$
\left(\frac{\partial}{\partial x_{1}}+\varepsilon \frac{\partial}{\partial x_{2}}+\cdots\right) \varepsilon(w, k)=-\frac{\partial \varepsilon(w, k)}{\partial \omega}\left(\varepsilon\left(\frac{\partial k}{\partial t_{2}}+v_{g} \frac{\partial k}{\partial x_{2}}\right)+\varepsilon^{2}\left(\frac{\partial k}{\partial t_{3}}+v_{y} \frac{\partial k}{\partial x_{2}}\right)+\cdots\right)(4.6)
$$

which gives: $\frac{\partial \varepsilon(\omega, k)}{\partial x_{1}}=0$

$$
\begin{equation*}
V_{g}=V_{g}\left(x_{2}, x_{3}, \cdots, t_{1}, t_{2}, \cdots\right) \tag{4.8}
\end{equation*}
$$

As in the outer region, $\mathcal{E}(\omega, k)=O(\varepsilon \omega)$. Otherwise
$E(x, t)$ should have a variation on the fast time scale.
Now, eq. (4.5) reduces to:

$$
\begin{align*}
& \frac{d}{d u} E_{0}^{i n}-\left(i \varepsilon^{-2} v_{D} \varepsilon(w, u)+v_{D} \Omega_{2}\right) E_{0}^{i n}-\frac{1}{2} i \frac{d v_{g}}{d k} \frac{\partial^{2}}{\partial x_{1}^{2}} E_{0}^{i n}=\varepsilon^{-2} v_{D} \sum_{i=0}^{4} \overline{I_{i}}  \tag{4.9}\\
& \frac{d t_{2}}{d u}=1 \\
& \frac{d x_{2}}{d u}=v_{g}(u) \quad x_{2}\left(u=t_{2}\right)=x_{2} \tag{4.10}
\end{align*}
$$

(C. 1 )

20

$$
\text { : } 9 \text { verf ow : }(S-1,5),(81,1) \text {, aps gntau }
$$


$(\Gamma, 4)$

$$
\theta=\frac{(x, 04) 36}{1 x 8}: \text { eovig notefw }
$$

$(8 ; 1)$

$$
\begin{aligned}
& s_{3} x=\left(x^{3}=A\right)_{3} x
\end{aligned}
$$

$$
\begin{aligned}
& \left.(-3)^{4}+1^{2}+1+2^{x}+x^{x}\right) 6^{2}=8^{2}
\end{aligned}
$$

$$
\begin{aligned}
& \text { sot soouber (2. } \mathrm{A} \text { ) .pe , woM }
\end{aligned}
$$

We define:

$$
\begin{equation*}
\tilde{E}_{0}^{m}=E_{0}^{i} \exp \left(-\varepsilon^{-2} \int_{t_{2}}^{u}\left(i v_{\Delta} \varepsilon(\omega, k)+\varepsilon^{2} v_{\Delta} \Omega_{2}\right) d u\right) \tag{4.11}
\end{equation*}
$$

noting that $E_{0}^{i}\left(u=t_{1}\right)=E_{0}^{i m}(x, t)$. Eqs. (4.9), (4.11) give:

$$
\begin{align*}
\frac{d}{d u} \tilde{E}_{0}^{i n}-\frac{1}{2} i \frac{d v_{g}}{d k} \frac{\partial^{2}}{\partial x_{1}^{2}} \tilde{E}_{0}^{i n} & =\varepsilon^{-2} v_{D} \exp \left(-\varepsilon^{-2} \int_{t_{2}}^{u}\left(i v_{D} \varepsilon(\omega, k)+\varepsilon^{2} v_{D} \Omega_{2}\right) d u\right) \sum_{i=0}^{3} \bar{I}_{i}  \tag{4.12}\\
& =M\left(x_{1}, t_{1}, \cdots\right)
\end{align*}
$$

Using eqs. (4.7-8), we may fouriertransform eq. (4.12)

$$
\begin{align*}
& \widetilde{E}_{\xi}=(2 \pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{i \xi x_{1}} \tilde{E}_{0}^{i n}\left(x_{1}\right) d x_{1}  \tag{4.13}\\
& \frac{d}{d u} \widetilde{E}_{\xi}+\frac{1}{2} i \frac{d v_{g}}{d k} \xi^{2} \widetilde{E}_{\xi}=M_{\xi}(u)  \tag{4.14}\\
& \tilde{E}_{0}^{i n}(x, t)=(2 \pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty}\left(\widetilde{E}_{\xi}(u=0) e^{-\frac{1}{2} i} \int_{0}^{t_{2}} \frac{d v_{g}}{d k} \xi^{2} d u e^{-i \xi x_{1}} d \xi+\right. \\
& +(2 \pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty}\left(\int_{0}^{t_{2}} e^{\frac{1}{2} i \int_{t_{2}}^{u} \frac{d v v_{2}}{d k} \xi^{2} d u} M_{\xi}(u) d u\right) e^{-i \xi x_{1}} d \xi \tag{4.15}
\end{align*}
$$

$$
\begin{aligned}
& 0-2=0
\end{aligned}
$$

Noting that $e^{i \alpha \xi^{2}}$ is the fouriertransform of for fouriertransforms, we have that:

$$
\begin{gather*}
E_{0}^{i n}(x, t)=\operatorname{Re}\left[\frac { 1 + i } { \sqrt { 2 \pi } } \left[( - \frac { 1 } { 2 } \int _ { 0 } ^ { t _ { 2 } } \frac { d v _ { q } } { d k } d u ) ^ { - \frac { 1 } { 2 } } \int _ { - \infty } ^ { \infty } d \eta \left(E_{0}^{m}(\eta, u=0) .\right.\right.\right. \\
\cdot e^{\left.-i \frac{1}{2}(x-\eta)^{2}\left(-\int_{0}^{t_{2}} \frac{d v_{g}}{d k} d u\right)^{-1}\right)+}  \tag{4.16}\\
\left.\left.\quad+\int_{0}^{t_{2}} d s\left(\left(\int_{t_{2}}^{s} \frac{d v_{g}}{d k} d u\right)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-i \frac{1}{2}(x-\eta)^{2}}\left(\int_{t_{2}}^{s} \frac{d v_{g}}{d k} d u\right)^{-1} M(\eta, s) d \eta\right)\right)\right]
\end{gather*}
$$

Eq. (4.16) describes a diffusion in $X_{1}$-space, during the evolution.

$$
\text { Introducing an intermediate scale } t_{\eta}
$$

$$
\begin{array}{lll}
t_{1}=\eta^{-1} t_{\eta} & j & t_{2}=\eta t_{\eta} \\
t_{1} \rightarrow \infty & t_{2} \rightarrow 0 \\
\eta \rightarrow 0 & \eta \rightarrow 0 \\
t_{\eta}=\text { const } & t_{\eta}=\text { const }
\end{array}
$$

(J.D. Cole 1968).

Choosing $\eta=\varepsilon^{\frac{1}{2}}$, it is easy to show that:

$$
\begin{aligned}
E_{0}^{i n} & \rightarrow E_{0}^{o u t} \\
\eta & \rightarrow 0 \\
t_{\eta} & =\text { const }
\end{aligned}
$$

 monon moldulownoo orlt mink. oa.
 mokturlive


$$
s^{3} x=2^{3} \quad 1 \quad f^{n s} \operatorname{con}^{3}
$$

$$
4 \sin ^{2}=-x^{\frac{2}{3}}
$$

$$
\text { . (8apr orod . } 4.6)
$$



$$
\begin{aligned}
& 0 \times 5 \\
& \text { Lumbl }=\text { mod }
\end{aligned}
$$

$$
\begin{aligned}
& 0.18=\frac{t}{4} \\
& \text { ithin sell wa } \\
& \cos \leftrightarrow 1 \frac{1}{3} \\
& \text { Q. } 4 \mathrm{~m} \\
& 1-2=0 .+
\end{aligned}
$$

As in the outer solution we have to solve eqs. (1.11), (4.10) to get an explicit expression in eq. (4.16).
V. The uniform solution.

To solve eqs. (1.11), (1.13), there are two different effects which have to be taken into account.

Particles with velocities:

$$
\begin{gather*}
\frac{w}{k}-S_{k}<V_{j}<\frac{w}{k}+S_{k}  \tag{5.1}\\
V_{m}<\frac{w}{k}<V_{M}  \tag{2.8}\\
S_{k}=\left(\frac{e_{j}}{m_{j}} E \frac{1}{k}\right)^{\frac{1}{2}} \tag{5.2}
\end{gather*}
$$

are trapped by the waves making up the wave packet, while particles with velocities outside this region, are untrapped.

Furthermore, the wave packet will behave similar to an electrostatic pulse. This means that particles with velocities:
where

$$
\begin{equation*}
V_{g}-S_{g}<V_{j}<V_{g} \tag{5.3}
\end{equation*}
$$

are accellerated by the packet, while particles with velocities

$$
\begin{equation*}
V_{g}<V_{j}<V_{g}+S_{g} \tag{5.5}
\end{equation*}
$$

$$
\begin{aligned}
& \text { (11, 1). apo svioe ot even ow noltuloe rotwo ant at eA }
\end{aligned}
$$

$$
\begin{aligned}
& \text { - notjulos mroftnu sit? , V }
\end{aligned}
$$

: estolooLev fidiw aeloldreq

$$
\begin{equation*}
x<\frac{\omega}{\mu}>\mathrm{L}^{2}>x^{2}-\frac{\omega}{x} \tag{r,c}
\end{equation*}
$$

$$
\begin{equation*}
M>\frac{w}{x}>m \tag{S.S}
\end{equation*}
$$

(s.

$$
\left(\frac{1}{x} 7 \frac{e^{3}}{x^{2 n}}\right)=2
$$





 (E. C)

$$
\beta^{2}+\beta^{2}>L^{2}>
$$

are retarded (M.S. Espedal 1971).
Particles outside the regions (5.1), (5.3) and (5.5)
are passing the wave packet, and get no final change in velocity. They may get a change in phase if the packet is unsymmetrical.

Because of eqs. $(2.6-7)$, the regions (5.1) and (5.3), (5.5) are seperated.

Using eqs. $(1.11),(1.13)$ and (2.10), we get:

$$
\begin{equation*}
\frac{d v_{j}}{d \tau}=\frac{e_{j}}{m_{j}} E(\varepsilon \tau) \sin (\chi(\tau)+\varphi(x, t)) \tag{5.6}
\end{equation*}
$$

$$
\begin{equation*}
\chi(\tau)=-\int_{t}^{\tau}\left(\omega(\varepsilon \tau)-k(\varepsilon \tau) V_{j}(\tau)\right) d \tau \tag{5.7}
\end{equation*}
$$

Eqs. (5.6-7), we shall solve approximately, dividing the velocity space into the following regions:

$$
\begin{align*}
& V<V_{g}-S_{g}  \tag{5.8}\\
& V_{g}-S_{g} \leq V \leq V_{g}+S_{g}  \tag{5.9}\\
& V_{g}+S_{g}<V<V_{m}-S_{k}  \tag{5.10}\\
& V_{m}-S_{k} \leq V \leq V_{M}+S_{K}  \tag{5.11}\\
& V_{M}+S_{K}<V \tag{5.12}
\end{align*}
$$

- (rer Iabogag , 8.M) bobrajex exa (Q.a) (h) mantmom adt ahtotm ontnitmoa



 .botnongegs gats (ב.ट)


$$
((r-x) Q+(r) X) \sin \left((3) \exists \frac{x^{2}}{t^{m}}=\frac{v 6}{T h}\right.
$$



$(8, \mathrm{c})$

$$
b^{2}-x y>v
$$

$$
\begin{equation*}
p^{2}+p^{v}+y=p^{2}-p \tag{C}
\end{equation*}
$$

(or. ᄅ)

$$
x^{2}-x^{2}>v>b^{2}+8 V
$$

$$
(1+. \bar{c})
$$

$$
x^{2}+y y \geq x^{2}-v
$$

$\square$
(St.c)

$$
y>x^{2}+4
$$

Further, we shall approximate $E(\varepsilon \tau)$ to lowest order by:

$$
\begin{equation*}
E_{0}(\varepsilon \tau)=E_{0}(\varepsilon \times(\tau), 0) \tag{5.13}
\end{equation*}
$$

In the regions $(5.8),(5.10)$ and (5.12), the particles are passing the packet. Therefore we take $V_{j}(\tau) \approx V$ to lowest order, which gives:

$$
\begin{equation*}
V_{0 j}^{\prime}(\tau)=v+\frac{e_{j}}{m_{j}} \int_{t}^{\tau} E_{0}(\varepsilon(x+v(\tau-t)), 0) \sin \left(x_{0}^{\prime}(\tau)+\varphi(x, t)\right) d \tau \tag{5.14}
\end{equation*}
$$

where $\quad X_{0}^{\prime}(\tau)=-\int_{t}^{\tau}(\omega(\varepsilon \tau)-k(\varepsilon \tau) v) d \tau$ Now, in the region (5.9), we define:
$V_{j}(\tau)=V_{g}(\varepsilon \tau)+S_{j}(\tau)$
$x_{0}^{2}(\tau)=-\int_{x}^{x(\tau)}\left(\omega-K v_{g}\right) \frac{1}{V_{g}} d x(\tau)$
$x^{\prime}(\tau)=\int_{t}^{\tau} S(\tau) d \tau=x(\tau)-x-\int_{\tau}^{\tau} V_{g} d \tau$

$$
(0 .(T) \times 3))_{a}=(53) \cdot \square
$$


 : eovig nolfiw *Tobro jeozol
:orstzob ow i (e, c) nolger ont nit won

$$
\left(\sin \cos ^{2} \frac{x}{x}\left(g_{x}-\omega\right)\right)^{2}-=(T)^{s} x
$$

Eqs. $(5.6-7)$ and (5.13-14) give:

$$
\begin{equation*}
\frac{d}{d \tau} S_{j}=\frac{e_{j}}{m_{j}} E_{0}(\varepsilon x(\tau), 0) \sin \left(x_{0}^{2}(x(\tau))+\varphi(x, t)\right)-\varepsilon \frac{d v_{g}}{d \varepsilon \tau} \tag{5.16}
\end{equation*}
$$

The lowest order solution of eq. (5.16) is:

$$
\begin{align*}
& S_{j 0}(\tau)= \pm\left(\left(V-V_{g}\right)^{2}+\frac{e_{j}}{m_{j}} \int_{0}^{x^{\prime}(\tau)} E_{0}(\varepsilon x(\tau), 0) \sin \left(x_{0}^{2}+\varphi\right) d x^{\prime}(\tau)\right)^{\frac{1}{2}} \\
& V_{j 0}^{2}(\tau)=V_{g}(\varepsilon \tau)+S_{j_{0}}(\tau) \tag{5.17}
\end{align*}
$$

We should note that particles which have velocities
$V=V_{g} \pm \alpha,|\alpha| \leqq S g$, before the interaction with the packet, get a velocity $V=V_{g} \mp \alpha$ after the interaction.

Similarly, in the region of trapped particles, (5.10), we define:

$$
\begin{equation*}
V_{j}(\tau)=\frac{\omega}{k}(\varepsilon \tau)+u(\tau) \tag{5.18}
\end{equation*}
$$

$$
\begin{align*}
& x_{j}^{\prime \prime}(\tau)=x(\tau)-x-\int_{t}^{\tau} \frac{\omega}{k} d \tau=\int_{t}^{\tau} u(\tau) d \tau  \tag{5.19}\\
& X_{0}^{3}(\tau)=K(\varepsilon \tau) x_{j}^{\prime \prime}(\tau) \tag{5.20}
\end{align*}
$$

So, eq. (5.6) reduces to:

$$
\begin{equation*}
\frac{d}{d T} U_{j}(T)=\frac{e_{j}}{m_{j}} E_{0}(\varepsilon x(\tau), 0) \sin \left(x_{0}^{3}+\varphi\right)-\varepsilon \frac{d}{d \varepsilon T}\left(\frac{w}{k}\right) \tag{5.21}
\end{equation*}
$$

$$
\begin{aligned}
& \text { (85.e) } \\
& (x)^{2}+2^{2}+(x+x)=(x)_{0}^{4} V
\end{aligned}
$$

$$
\begin{aligned}
& \text { : Antash }
\end{aligned}
$$

$$
\begin{aligned}
& \text { (er. } \mathrm{c} \text { ) } \\
& \text { (os.c) } \\
& (7)^{\prime \prime} b^{2}(p,) X=(r)_{a}^{2} x
\end{aligned}
$$

which gives to lowest order:

$$
\begin{align*}
& u_{j 0}(\tau)= \pm\left(\left(v-\frac{w}{k}\right)^{2}+\frac{e_{j}}{m_{j}} \int_{0}^{x_{0}^{\prime \prime}(\tau)} E_{0}(\varepsilon x(\tau), 0) \sin \left(x_{0}^{3}+\varphi\right) d x_{j}^{\prime \prime}(\tau)\right)^{\frac{1}{2}} \\
& v_{j 0}^{3}(\tau)=\frac{w}{k}(\varepsilon \tau)+u_{j 0}(\tau) \tag{5.22}
\end{align*}
$$

If $\tau=p$ is the time when the interaction between the particle and the packet start, and int $^{\text {ant }}$ interaction time, we may write eqs. (5.14), (5.17) and (5.23):

$$
V_{j 0}(\tau)=\left\{\begin{array}{lc}
V & \tau<p \\
V_{j 0}^{i}(\tau) & p \leq \tau \leq p+\tau_{\text {int }} \\
V_{j 0}^{i}\left(\tau=p+\tau_{\text {int }}\right) & p+\tau_{\text {int }}<\tau \\
i=1,2,3 &
\end{array}\right.
$$

To solve eqs. (3.8) and (4.10) to lowest order, we have to approximate Vg . This may be done, using the fact that

$$
\begin{equation*}
\varepsilon(\omega, k)=O(\varepsilon k) \tag{5.24}
\end{equation*}
$$

Therefore eqs. (3.11), (4.16) and (5.23) may be used as a first step in a successive approximation procedure.
VI. The electron plasma wave packet.

In this section, we shall study the podulation of a electron plasma wace packet propagating through a collisionless electron plasma in a background of fixed ions.

We assume that:
$f_{0}(v)=m\left(2 \pi v_{e}^{2}\right)^{-\frac{1}{2}} \exp \left(-\frac{1}{2}\left(\frac{v}{v_{e}}\right)^{2}\right)$

$$
\left((+)^{3} \times k\left(4+{ }^{2} x\right) \text { wid }\left(\sin (\pi x 3), \frac{3}{4}\right)\left(\frac{3}{4}++^{5}\left(\frac{4}{x}-V\right)\right) \leq=(p),(k)\right.
$$

$$
(S S, C)
$$

$$
(r), N+(r g) \frac{w}{x}=(r)_{0}^{\varepsilon} N
$$



 0
$(E S, E)$


$$
E+S, T=I
$$




$$
\text { ) } 110
$$



$$
\begin{align*}
& E_{0}(x, 0)=E_{0} \exp \left(-\frac{1}{2}\left(\frac{x_{1}}{l}\right)^{2}\right)  \tag{6.2}\\
& f_{j}(\tau=0)=0
\end{align*}
$$

$$
\begin{equation*}
V_{e}^{2}=\left(\frac{\partial T_{e}}{m}\right)_{j}^{\frac{1}{2}} \omega_{p e}=\left(\frac{4 \pi m e^{2}}{m}\right)^{\frac{1}{2}} K_{D}=\frac{\omega_{p e}^{2}}{V_{e}^{2}} \tag{6.3}
\end{equation*}
$$

Further, we assume that:

$$
\begin{equation*}
\left(\frac{K}{K_{D}}\right)^{2}=O(\varepsilon) \tag{6.4}
\end{equation*}
$$

Using eq. (5.24), we get:

$$
\begin{equation*}
\omega^{2}(k)=\omega_{p l}^{2}\left(1+3\left(\frac{k}{k_{0}}\right)^{2}+6\left(\frac{k}{k_{D}}\right)^{3}\right)+o\left(\varepsilon^{3} \omega_{p_{0}}^{2}\right) \tag{6.5}
\end{equation*}
$$

Eq. (6.5) gives:

$$
\begin{equation*}
V_{g}=\frac{3 k}{\omega} V_{e}^{2}\left(1+4\left(\frac{k}{k_{0}}\right)^{2}\right)+O\left(\varepsilon^{2} \frac{V_{e}^{2} k}{\omega}\right) \tag{6.6}
\end{equation*}
$$

Also, we assume that:

$$
\begin{equation*}
k=k\left(x_{3}, t_{3}\right) \tag{6.7}
\end{equation*}
$$

To calculate $\bar{I}_{i}, i=1,2,3,4$, we have to estimate the interaction time, $\tau_{\text {int. }}$. In the regions (5.8), (5.10), (5.12), Tint is the passing time:

$$
-08-
$$

$(5, a) \quad\left(5\left(\frac{x}{2}\right) \frac{y}{5}-\right) \operatorname{qux}_{0} \geq=(a, x), \vec{a}$

$$
\theta \quad a \quad(a=r)_{2} 1
$$

$(1,0)$



 : Bevaly (さ. Z ) .pat
$(3.2)$

(1, i)

$$
\left(x^{5}+x^{x}\right) N=A
$$





$$
\tau_{\text {int }} \approx \frac{2 l}{|V-V g|}
$$

In the regions (5.9), $\quad$ int is twice the time needed to accelerate a particle from $V=V g$ to $V=V g+S$. $|S|<S g$. This is approximately:

$$
\begin{equation*}
\tau_{i n t} \approx 2\left(\frac{l}{S_{g}}\right)\left(\left|\frac{E_{0}(x)}{E_{0}}\right|\right)^{\frac{1}{2}} \tag{6.9}
\end{equation*}
$$

The interaction time in the region (5.10), is also the passing time, which is approximately:

$$
\begin{aligned}
& \tau_{\text {int }} \approx \frac{2 l}{\left|V-V_{g}\right|}\left(1-\frac{w-K V}{k S_{0}}\right) \\
& S_{0}=\left(\frac{e}{m} E_{0} \frac{1}{k}\right)^{\frac{1}{2}}
\end{aligned}
$$

A rough figure of the phase-plane is:


$$
\left.\frac{x-s}{\left|g^{y}-v\right|}=\sin \right\rvert\,
$$


 :VIoざsmixorggo al elat?


$$
\begin{aligned}
& \left(\frac{V x-4}{0^{2}+x}-1\right) \frac{38}{|0 y-X|} 2 \tan \rho
\end{aligned}
$$

: al ernilq-9ekig ent 20 stugli flgwor $A$


With these assumptions, the main contribution from the integrals (2.18-20), is:

$$
\begin{aligned}
& \bar{I}_{\text {in }}\left(x_{1,}, t_{1}, t_{2}\right)=\frac{\omega_{p-}^{2}}{\left(2 \pi v_{e}^{2}\right)^{\frac{1}{2}}} E_{0}\left[e^{i\left(\omega-k v_{g}\right)^{3 / 2} l\left(\frac{e}{m} E_{0} v_{g}\right)^{-\frac{1}{2}}} e^{-\frac{1}{2}\left(\frac{x_{1}-v_{g} t_{1}}{e}\right)^{2}} .\right. \\
& \cdot e^{-\frac{1}{2}\left(\frac{V_{g} \pm s_{g}}{V_{e}}\right)^{2}}\left(1-e^{\left. \pm \frac{2 V_{g} s_{g}}{V_{e}^{2}}\right)} \frac{1}{\omega-K V_{g}}\right](-i+ \\
& -\varepsilon \frac{v_{g}}{\omega-k v_{g}}\left(\frac{x_{1}-v_{g} t_{1}}{l^{2}}\right)+ \\
& \left.+i \varepsilon^{2} \frac{v_{g}^{2}}{\left(\omega-k v_{g}\right)^{2} e^{2}}\left(1-\frac{\left(x_{1}-v_{g} t_{1}\right)^{2}}{e^{2}}\right)\right)+O\left(\varepsilon^{3}\right)
\end{aligned}
$$

where the plus-sign should be used for $x_{1}-V_{g} t_{1}>0$ and the minus-sign for $x_{1}-V g t_{1}<0$.

$$
\begin{aligned}
& I_{o u t}\left(x_{1}, t_{1}\right)=\frac{\omega_{p l}^{2}}{\left(2 \pi v_{e}^{2}\right)^{\frac{1}{2}} E_{0}\left[e^{i\left(\frac{k s_{k}}{\omega_{B}} \sin \left(\frac{2 l \omega_{B}}{\left|v-v_{g}\right|}\left(1-\frac{s_{k}}{s_{0}}\right)\right)\right.}\right]\left(\frac{2}{ \pm S_{0}}\right) . . ~ . ~ . ~ . ~} \\
& \cdot e^{-\frac{1}{2}\left(\frac{x_{1}-\frac{\omega}{k} t_{1}}{e^{2}}\right.}\left\{e^{2}-\frac{1}{2}\left(\frac{\frac{w}{k} \pm S_{k}}{V_{e}}\right)^{2}-e^{-\frac{1}{2}\left(\frac{\frac{\omega}{k} \pm S_{k} \cos \left(\frac{2 e \omega_{B}}{1 V-V_{g} \mid}\left(1-\frac{S_{k}}{S_{0}}\right)\right.}{V_{e}}\right)}+(6.12)\right. \\
& +\varepsilon\left(\frac{x_{1}-\frac{\omega}{k} t_{1}}{v_{e}^{2} e^{2}} \int_{ \pm s_{k}}^{ \pm s_{k} \cos \left(\frac{2 e \omega_{B}}{\left|v-v_{g}\right|}\left(1-\frac{s_{k}}{50}\right)\right)^{2}} \frac{1}{s}\left(\frac{\omega}{k}+s\right) e^{-\frac{1}{2}\left(\frac{1}{v_{A}}\left(\frac{\omega}{k}+s\right)^{2}\right)} d s\right\}
\end{aligned}
$$

The plus-sign should be used for $V-\frac{w}{k}>0$ and the minus$\operatorname{sign}$ for $\quad v-\frac{\omega}{k}<0$


$$
\begin{aligned}
& x+\left(\frac{t^{4} \rho^{2}-1 x}{x}\right) p^{2}+x^{2}-\frac{x^{2}}{x} 3 \pi \\
& 2
\end{aligned}
$$






Eqs. $(3.11),(4.16),(6.12-13)$ give:

$$
\begin{aligned}
& E_{0}\left(x_{1}, t_{1}, t_{2}\right)=E_{0}\left(x_{1}+v_{g} t_{1}, 0\right)+R e \int_{0}^{t_{1}} \bar{I}_{o u t}\left(x_{1}+v_{g} s, s\right) d s+ \\
& +R\left(\frac{1+i}{\sqrt{2 \pi}} \int_{0}^{t_{2}} d s\left(\int_{t_{2}}^{s} \frac{d v_{g}}{d k} d u\right)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-i(x-\eta)^{2} \frac{1}{2}\left(\int_{t_{2}}^{s} \frac{d v_{g}}{d k} \frac{d u)^{-1}}{I_{\text {in }}}\left(\eta+v_{g} t_{1}, s\right) d \eta\right)} \quad\right. \text { (6.13) }
\end{aligned}
$$

Eq. (6.13) give that the effect of trapped particles, Iont, propagates as a free streaming effect. Because $\frac{w}{k}-V_{y}>0$, it propagates faster than the packet, and should be observed as a modulation in the front of the packet. (J.N. Denavit and R.N. Sudan, 1972).

The $\bar{I}$ in term in eq. (6.13) takes care of the "pulse effect" (M.S. Espedal, 1971). This interaction effect propagates with the velocity $V_{g}$, and modulates the packet itself.

## VII. Conclusion.

The interaction between particles and an electrostatic wave packet results mainly in two different effects. We get a modulation of the packet caused by particles propagating with velocities near to $V_{g}$. The evolution of these effects is represented by eq. (4.16).

Particles with velocities near to $\frac{\omega}{k}$ get a net change in velocity during the interaction. The evolution of these effects is given by eq. (3.11).

$$
\text { :Qvis }\left(\sum i-S i, \partial\right) \cdot(\partial 1, H)=(11, \varepsilon), \text { Bpad }
$$




- agmanart .jnotto smimeette gout e bs bojansciora ot



$$
\text { lowar mafing in, } a \text { sime +twerian it T.) }
$$










 -(rt.E) . po vó meyly at

We may note that, taking into account wave-wave interaction effects, the average equation is no longer linear. In models where these effects appear, we may get similar equation as those obtained by Y.H.Ichikawa and T.Taniuti.

References.
J.Denavit and R.N.Sudan. Phys. Rev. Letters. Vol. 28 (1972)
pp. 404-407.
K.B.Dysthe. To appear in J. of Plasma Physics (1974). M.S.Espedal. J. Plasma Physics. Vol.5 (1971) pp. 343-355. Y.H.Ichikawa and T.Taniuti. J. of Physical Society of Japan. Vol. 34 (1973) pp. 513-521.




 - 504-40


 . $152-81 己$. पの (8Mer) NE. SOV

