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Abstract.

The propagation of discontinuities for solutions of linear hyperbolic systems of the first order is studied. The transport equations for systems with characteristics of nonuniform multiplicity are found in general. These transport equations are studied in detail in the nonsingular cases, and it is shown how discontinuous initialvalue problems can be solved.

INTRODUCTION.

Discontinuous solutions of hyperbolic partial differential equations have been extensively studied in the literature. The earlier treatments dealt mainly with second order equations. In [2] R.Courant and P.D.Lax extended the theory to first order linear hyperbolic systems with distinct characteristics. In [3] R.M.Lewis extended the theory further to symmetric hyperbolic systems with characteristics of constant multiplicity. D.Ludwig and B.Granoff [5], and J.V.Ralston [6] have considered some problems for hyperbolic systems with characteristics of nonuniform multiplicity.

In this paper we shall try to resolve the problem in general for hyperbolic systems of the first order with characteristics of nonuniform multiplicity. We are forced to restrict the class of problems somewhat, but we hope that the theory covers most of the interesting cases.

The study of propagation of discontinuities is of course important in itself since it gives us information about the solutions. However, since the asymptotic behaviour of solutions of hyperbolic equations is closely related to the propagation of discontinuities, a study of this is more important than one may realize at first glance (for further details on this see Courant-Hilbert [1] and D.Ludwig [4]). The author will consider some of the problems in this connection elsewhere, especially we shall study how the problem of stability is related to the propagation of discontinuities.

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1. ASSUMPTIONS AND FORMULATION OF THE PROBLEM.

We shall study hyperbolic systems of the following form

$$Lu = u_t + \sum_{i=1}^{n} A^i u_{x^i} + Bu = 0 \tag{1.1}$$

where $u = \{u^1, \dots, u^k\}$, while $B, A^i, i = 1, \dots, n$ are $k \times k$ matrices which may depend on the independent variables t and $x = \{x^1, \dots, x^n\}$. The independent variable t (time) is separated from x mostly for practical reasons, but also because this separation is needed in later applications. It is well known that any linear hyperbolic system of the first order can be transformed to a system of the type (1.1) at least locally. Our study will be local in x,t-space. We shall only briefly indicate how the local results can be glued together in hopefully wide classes of problems.

As far as this author knows, a general theory of hyperbolic equations and systems is still not well established. The meaning of the notion hyperbolic above is therefore not clear. In this work we shall by the notion hyperbolic mean that the assumptions later in this section are satisfied, and furthermore that the Cauchy problem for (1.1) is well-posed in "suitable" metric spaces. We shall not give a precise definition of what we mean by a "suitable" metric space, but it will suffice if for instance the solution of (1.1) is in C^N (the space of N-times continuously differentiable functions) when the Cauchy data is in C^{N_o} for some $N_o \geq N$. - Symmetric hyperbolic systems, which we are particular-

ily interested in, and which are covered by a well established theory in the literature, are easily seen to satisfy all our requirements (see [1]).

The characteristic equation associated with (1.1) is

$$\det\left\{-\lambda I + \sum_{i=1}^{n} \xi^i A^i\right\} = 0 \tag{1.2}$$

where I denotes the $k \times k$ unit matrix and $\xi^1, \ldots, \xi^n$ are real numbers which are not all zero simultaneously. Let the roots in the equation (1.2) be given by

$$\lambda = \Omega^\alpha(x,t,\xi^1, \ldots, \xi^n) \ , \ \alpha = 1, \ldots, \gamma \ . \tag{1.3}$$

We shall assume that the functions Ω^α depend on the variables $x,t,\xi^1, \ldots, \xi^n$ sufficiently smooth. The phrase "sufficiently smooth" is chosen here and elsewhere in this paper to mean sufficiently smooth for our later arguments to be valid. The functions Ω^α are obviously homogeneous of degree one with respect to the variables $\xi = \left\{\xi^1, \ldots, \xi^n\right\}$. Except in special cases (weakly coupled hyperbolic systems), some or all of the functions Ω^α will have a branchpoint for $\xi = 0$ and possibly also for other vectors ξ. Since $\xi = 0$ is already excluded above from the smoothness requirements etc., only branchpoints for $\xi \neq 0$ can cause trouble. If branchpoints exist for $\xi \neq 0$, they have to be treated separately; we shall give some comments on such cases in section 5.

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We shall assume that the functions Ω^{α} depend on the variables $x,t,\xi^1, \ldots, \xi^n$ sufficiently smooth. The phrase "sufficiently smooth" is chosen here and elsewhere in this paper to mean sufficiently smooth for our later arguments to be valid. The functions Ω^{α} are obviously homogeneous of degree one with respect to the variables $\xi = \left\{\xi^1, \ldots, \xi^n\right\}$. Except in special cases (weakly coupled hyperbolic systems), some or all of the functions Ω^{α} will have a branchpoint for $\xi = 0$ and possibly also for other vectors ξ. Since $\xi = 0$ is already excluded above from the smoothness requirements etc., only branchpoints for $\xi \neq 0$ can cause trouble. If branchpoints exist for $\xi \neq 0$, they have to be treated separately; we shall give some comments on such cases in section 5.

Let the eigenvectors associated with the eigenvalues (1.3) be given by

$$r^{\alpha\beta}(x,t,\xi) \qquad \begin{matrix} \alpha = 1, \ldots, \gamma \\ \beta = 1, \ldots q^{\alpha} \end{matrix} \qquad l^{\alpha\beta}(x,t,\xi) \tag{1.4}$$

where $r^{\alpha\beta}$ denote the right- and $l^{\alpha\beta}$ the left-eigenvectors. We shall assume that the eigenvectors (1.4) depend sufficiently smooth on x,t,ξ. In general, this smoothness requirement will be only partially valid, because we have to allow discontinuities in the set of eigenvectors $r^{\alpha\beta}$, $l^{\alpha\beta}$. The discontinuities are connected with changes in the multiplicity of a characteristic manifold, and the points x,t,ξ where $r^{\alpha\beta}$, $l^{\alpha\beta}$ are discontinuous have to be treated separately. We shall in section 5 give some comments on how the problem can be handled.

We may without loss of generality assume that $r^{\alpha\beta}$ and $l^{\alpha\beta}$ are normalized by the relations

$$l^{\alpha\beta} \cdot r^{ab} = \delta^{\alpha a}\, \delta^{\beta b} \tag{1.5}$$

$$r^{\alpha\beta} \cdot r^{\alpha b} = \delta^{\beta b} \tag{1.6}$$

We assume furthermore that $r^{\alpha\beta}$ and $l^{\alpha\beta}$ form complete sets, i.e. that $\sum\limits_{\alpha=1}^{\gamma} q^{\alpha} = k$. The relations (1.6) will not be used anywhere in this paper, but will be needed in later applications. By definition we have the following identities

$$\left\{-\Omega^{\alpha} I + \sum_{i=1}^{n} \xi^{i} A^{i}\right\} \cdot r^{\alpha\beta} \equiv 0 \qquad \begin{matrix} \alpha = 1, \ldots, \gamma \\ \beta = 1, \ldots, q^{\alpha} \end{matrix} \qquad l^{\alpha\beta} \cdot \left\{-\Omega^{\alpha} I + \sum_{i=1}^{n} \xi^{i} A^{i}\right\} \equiv 0 \tag{1.7}$$

We assume that all Ω^α, $r^{\alpha\beta}$, $l^{\alpha\beta}$ are real for any real ξ, and that

$$\Omega^\alpha \neq \Omega^a \quad \text{when} \quad \alpha \neq a \tag{1.8}$$

If we for every choice of x,t,ξ have that

$$\Omega^\alpha \neq \Omega^a \quad \text{when} \quad \alpha \neq a \tag{1.9}$$

then the hyperbolic system (1.1) is said to have characteristics of constant multiplicity. If in addition $\gamma = k$ (or equivalently $q^\alpha = 1$ for $\alpha = 1, \ldots, \gamma$), (1.1) is said to have distinct characteristics and the system (1.1) is called totally hyperbolic. In general (1.9) will not be satisfied even locally in x,t, but the multiplicities of the characteristics will be dependent on ξ at every point x,t. There seems to be little known for such systems in the literature; they are, however, not excluded from the discussion in this paper (see also [5] and [6]).

In this paper we want to study propagation of discontinuities for solutions of the hyperbolic system (1.1). Since classical solutions in the strict sense cannot have any discontinuities at all, we have to define what we shall mean by a solution. We shall work within the class of so-called "weak solutions". To define this we introduce the space S of all smooth k-dimensional vector-testfunctions $\eta(x,t)$ with compact support in the region under consideration. We define the adjoint operator M to the operator L in (1.1) by

$$Mv = -v_t - \sum_{i=1}^{n} (A^{i*}v)_{x^i} + B^*v \tag{1.10}$$

where $*$ denotes transposing of the matrix. A measurable function u is defined as being a weak solution of the equation (1.1) if

$$\int u \, M\eta \, dxdt = 0 \quad \forall \, \eta \in S \tag{1.11}$$

By partial integration in (1.11) it is easily seen that a differentiable weak solution of (1.1) is a solution in the strict sense, and that a solution in the strict sense is a weak solution.

The problem of propagation of discontinuities in the whole class of weak solutions is too involved to be studied in detail (some results can be found in [6]). We shall therefore restrict our study to weak solutions which locally are piece-wise smooth. Here, a piece-wise smooth function is defined as being a function for which there exists a finite set of smooth hypersurfaces dividing the domain of definition into a finite set of subdomains in which the function is smooth, and furthermore that the limit of the function and its derivatives exist in every subdomain when we move out to the boundaries. Thus we assume that the discontinuities of the function and its derivatives are everywhere finite, and that locally they are located on a finite set of hypersurfaces.

We are now able to formulate the problem we are going to study in the rest of the paper: Suppose that a solution of (1.1) has a discontinuity at the point x_o, t_o, what then are the equations for the hypersurfaces in a neighbourhood of x_o, t_o where the solution is discontinuous? How are the magnitudes of the discontinuities related on these hypersurfaces?

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2. THE TRANSPORT EQUATIONS.

We shall in this section consider the special case where the discontinuous solution we are considering, u, is smooth everywhere in a neighbourhood of x_o, t_o except on a smooth hypersurface C given by the equation

$$\varphi^o(t,x) = 0 \tag{2.1}$$

where φ^o is nonsingular at x_o, t_o. By our assumptions, u and its derivatives have finite jump discontinuities across C, and the jump discontinuities are smooth functions defined on the manifold C in a neighbourhood of x_o, t_o (which by assumption lies on C).

In a neighbourhood of x_o, t_o we introduce a regular coordinate transformation

$$y^j = \varphi^j(t,x) \; ; \quad j = 0, 1, \; \ldots \; , n \tag{2.2}$$

which utilizes C as a coordinate surface. The equation for C becomes $y^o = 0$. In the new coordinates we have

$$Lu = \sum_{j=o}^{n} H^j u_{y^j} + Bu \tag{2.3}$$

Here we have introduced the matrices H^j defined by

$$H^j = \varphi^j_t I + \sum_{\nu=1}^{n} \varphi^j_{x^\nu} A^\nu \; , \; j = 0, 1, \; \ldots \; , n \tag{2.4}$$

Let D_1 and D_2 denote the regions on either side of C and let $u = u^{I} + u^{II}$ where $u^{I} \equiv 0$ in D_2 and $u^{II} \equiv 0$ in D_1. By Gauss' theorem it easily follows that

$$\int_{D_1 \cup D_2} u^{I} M\eta \, dxdt = \int_{D_1} \eta L u^{I} \, dxdt - \int_{C} \left\{ \eta u^{I} \varphi_t^o + \sum_{\nu=1}^{n} \eta A^{\nu} u^{I} \varphi_{x^{\nu}}^o \right\} \theta dS$$

$$\int_{D_1 \cup D_2} u^{II} M\eta \, dxdt = \int_{D_2} \eta L u^{II} dxdt + \int_{C} \left\{ \eta u^{II} \varphi_t^o + \sum_{\nu=1}^{n} \eta A^{\nu} u^{II} \varphi_{x^{\nu}}^o \right\} \theta dS \tag{2.5}$$

where $\theta = \theta(x,t)$ is a scalarfunction such that $\theta\left\{\varphi_t^o, \varphi_{x^1}^o, \dots, \varphi_{x^n}^o\right\}$ is a unitvector pointing out of the region D_1. Since $Lu = 0$ everywhere except on C, we are only left with the following when we add the equations (2.5) and introduce the notion $[u] = u^{II} - u^{I}$

$$\int_{C} \theta\eta \left\{ \varphi_t^o[u] + \sum_{\nu=1}^{n} \varphi_{x^{\nu}}^o A^{\nu}[u] \right\} dS = 0 \tag{2.6}$$

Since $\theta \neq 0$ everywhere and the components of η are arbitrary, it follows that

$$H^o[u] = 0 \quad \text{on} \quad C \tag{2.7}$$

Here $[u]$ is simply the jump of u across C, thus $[u] \neq 0$ wherever u is discontinuous on C. From equation (2.7) we see that in these points the matrix H^o must be singular, i.e.

$$\det\left(\varphi^o_t I + \sum_{\nu=1}^{n} \varphi^o_{x^\nu} A^\nu\right) = 0 \qquad (2.8)$$

This is the characteristic partial differential equation for the hyperbolic system (1.1). By definition the hypersurface C, given by $\varphi^o = 0$ and satisfying (2.8), is a characteristic manifold for the operator L. Thus we can conclude that if u is discontinuous across a hypersurface C, then C must be characteristic.

The function φ^o which determines the hypersurface C in equation (2.1), need not satisfy the characteristic differential equation (2.8) identically; we only know that φ^o satisfies (2.8) on C, i.e. for $\varphi^o = 0$. At every point x,t on this characteristic manifold C, there is at least one α such that

$$\varphi^o_t + \Omega^\alpha(x,t,\varphi^o_{x^1}, \dots ,\varphi^o_{x^n}) = 0 \qquad (2.9)$$

On the other hand, if a hypersurface $\varphi^o = 0$ satisfies (2.9) for some choice of α at every point, then the hypersurface must be a characteristic manifold. In this sense the family of equations (2.9) is equivalent to the characteristic equation (2.8), we therefore call (2.9), with $\alpha = 1, \dots ,\gamma$, the family of characteristic partial differential equations associated with (1.1).

In general a characteristic manifold may, at some or all points, satisfy more than one of the equations in the family of characteristic partial differential equations (2.9). Furthermore there need not be a single index α such that (2.9) is satisfied

$$\det\left(\varphi^0_t I + \sum_{\nu=1}^{n} \varphi^0_{x_\nu} A^\nu\right) = 0 \qquad (2.8)$$

This is the characteristic partial differential equation for the hyperbolic system (1.1). By definition the hypersurface C, given by $\varphi^0 = 0$ and satisfying (2.8), is a characteristic manifold for the operator L. Thus we can conclude that if u is discontinuous across a hypersurface C, then C must be characteristic.

The function φ^0 which determines the hypersurface C in equation (2.4), need not satisfy the characteristic differential equation (2.8) identically; we only know that φ^0 satisfies (2.8) on C, i.e. for $\varphi^0 = 0$. At every point x,t on this characteristic manifold C, there is at least one α such that

$$\varphi^0_t + c^\alpha(x,t,\varphi^0_{x_1}, \ldots, \varphi^0_{x_n}) = 0 \qquad (2.9)$$

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In general a characteristic manifold may, at some or all points, satisfy more than one of the equations in the family of characteristic partial differential equations (2.9). Furthermore there need not be a single index α such that (2.9) is satisfied at every point on a characteristic manifold. However, if the hyperbolic system (1.1) has characteristics of constant multiplicity, then a characteristic manifold satisfies (2.9) for one choice of α only, and this α is the same all over the manifold.

We shall now study the special case where the hypersurface C in a neighbourhood of x_o,t_c satisfies (2.9) for r different choices of α, say $\alpha_1, \dots ,\alpha_r$, and that nowhere in this neighbourhood C satisfies (2.9) for any other choice of α than $\alpha_1, \dots ,\alpha_r$; we shall later see that the general result can be deduced from this special case. Again the function φ^o is only known to satisfy (2.9) for $\alpha_1, \dots ,\alpha_r$ on the hypersurface C given by (2.1). However, we may here without loss of generality assume that the function φ^o in a neighbourhood of x_o,t_o satisfies (2.9) identically for at least one of the indices $\alpha_1, \dots ,\alpha_r$. In general it will not be possible to get (2.9) satisfied identically for more than one of the indices $\alpha_1, \dots ,\alpha_r$, but φ^o can be chosen such that (2.9) is satisfied identically for any choice of one of these indices (compare with Courant-Hilbert [1]).

If the hypersurface C satisfies (2.9) for only one choice of α at the point x_o,t_o, then, by continuity, there exists a neighbourhood of x_o,t_o where C satisfies (2.9) for this choice of α only. Thus this is a special case of the situation we are considering, namely the case where $r = 1$; this is the only case that arises for hyperbolic systems with characteristics of constant multiplicity. There seems to be little known for the cases $r > 1$ in the literature, some results are obtained in [5] and [6].

From (2.7) we see that [u] is in the right nullspace of H^o. Since we assume that the equation for C satisfies (2.9) for at every point on a characteristic manifold. However, if the hyperbolic system (1.1) has characteristics of constant multiplicity, then a characteristic manifold satisfies (2.9) for one choice of a only, and thus a is the same all over the manifold.

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From (2.7) we see that $[s]$ is in the right nullspace of H^c. Since we assume that the equation for C satisfies (2.9) for

$\alpha_1, \ldots, \alpha_r$, and these only, $[u]$ can be expanded in the following way

$$[u] = \sum_{i=1}^{r} \sum_{\beta=1}^{q^{\alpha_i}} \sigma_\beta^i \, r^{\alpha_i \beta} \tag{2.10}$$

where σ_β^i are scalarfunctions to be determined, and $\varphi^o_{x^\nu}$ are substituted for ξ^ν, $\nu = 1, \ldots, n$, in the expressions for $r^{\alpha_i\beta}$.

Since $Lu = 0$ on both sides of C, (2.3) gives on C

$$\sum_{j=o}^{n} H^j[u_{y^j}] + B[u] = 0 \tag{2.11}$$

For $j \neq 0$, u_{y^j} is a tangential derivative to C, so that $[u_{y^j}] = [u]_{y^j}$. Thus (2.11) may be written as

$$H^o[u_{y^o}] + \sum_{j=1}^{n} H^j[u]_{y^j} + B[u] = 0 \tag{2.12}$$

We multiply (2.12) on the left by $l^{\alpha_\nu \mu}$, $\nu = 1, \ldots, r$ & $\mu = 1, \ldots, q^{\alpha_\nu}$. Since by hypothesis $l^{\alpha_\nu\mu} H^o = 0$, we get

$$\sum_{j=1}^{n} l^{\alpha_\nu\mu} H^j[u]_{y^j} + l^{\alpha_\nu\mu} B[u] = 0 \tag{2.13}$$

In view of (2.10) we see that (2.13) is a system of $k_c = \sum_{i=1}^{r} q^{\alpha_i}$ partial differential equations with respect to the k_c unknown functions σ_β^i. (2.13) is a system of equations on the manifold C, and $[u]$ is only defined there. From this point of view

it is meaningless to treat [u] as a function depending on y_o. However, if we in (2.13) let [u] depend on y_o as a parameter, it does not affect our results as long as we remember that (2.13) has relevance to our problem only for $y_o = 0$. Thus we shall treat [u] as a function also dependent on y_o , because this will simplify our study. Since $l^{\alpha_\nu \mu} H^o = 0$ on C by hypothesis, we may add the following term to (2.13)

$$l^{\alpha_\nu \mu} H^o[u]_{y_o} \tag{2.14}$$

Thus the following system of equations will be equivalent to (2.13) on C

$$\sum_{j=o}^{n} l^{\alpha_\nu \mu} H^j[u]_{y^j} + l^{\alpha_\nu \mu} B[u] = 0 \tag{2.15}$$

$$\nu = 1, \ldots, r \;\&\; \mu = 1, \ldots, q^{\alpha_\nu}$$

If we introduce x, t as independent variables instead of $y^o, \ldots y^n$, (2.15) becomes

$$l^{\alpha_\nu \mu}[u]_t + \sum_{j=1}^{n} l^{\alpha_\nu \mu} A^j[u]_{x^j} + l^{\alpha_\nu \mu} B[u] = 0 \tag{2.16}$$

$$\nu = 1, \ldots, r \;\&\; \mu = 1, \ldots, q^{\alpha_\nu}$$

We substitute (2.10) into (2.16) and get

$$\left(\sigma^{\nu}_{\mu}\right)_t + \sum_{j=1}^{n} \sum_{i=1}^{r} \sum_{\beta=1}^{q^{\alpha_i}} l^{\alpha_\nu \mu} A^j r^{\alpha_i \beta} \left(\sigma^i_\beta\right)_{x^j}$$

$$+ \sum_{i=1}^{r} \sum_{\beta=1}^{q^{\alpha_i}} l^{\alpha_\nu \mu} \left(L r^{\alpha_i \beta}\right) \sigma^i_\beta = 0 \qquad (2.17)$$

$$\nu = 1, \ldots, r \ \& \ \mu = 1, \ldots, q^{\alpha_\nu}$$

We shall call (2.17) <u>the system of transport equations</u> for the hyperbolic system (1.1), it tells us how the discontinuities of u propagate along C. The system of transport equations (2.17) is a hyperbolic system of a very special type. To see this we differentiate (1.7) with respect to ξ^μ and get

$$\left\{-\frac{\partial \Omega^\alpha}{\partial \xi^\mu} I + A^\mu\right\} \cdot r^{\alpha\beta} + \left\{-\Omega^\alpha I + \sum_{i=1}^{n} \xi^i A^i\right\} \cdot \frac{\partial r^{\alpha\beta}}{\partial \xi^\mu} = 0 \qquad (2.18)$$

Multiplication on the left by l^{ab} and using (1.5) and (1.7), gives us

$$l^{ab} A^\mu r^{\alpha\beta} = \frac{\partial \Omega^\alpha}{\partial \xi^\mu} \delta^{\alpha a} \delta^{\beta b} + (\Omega^\alpha - \Omega^a) l^{ab} \cdot \frac{\partial r^{\alpha\beta}}{\partial \xi^\mu} \qquad (2.19)$$

In particular, if $\Omega^\alpha = \Omega^a$ which is the case either if $\alpha = a$ or if we consider a multiple characteristic, (2.19) gives us

$$l^{ab}\, A^{\mu}\, r^{\alpha\beta} = \frac{\partial\Omega^{\alpha}}{\partial\xi^{\mu}}\, \delta^{a\alpha}\, \delta^{b\beta} \tag{2.20}$$

The expressions (2.20) can now be substituted for the coefficients to $(\sigma^{i}_{\beta})_{x^j}$ in (2.17), which gives us on C

$$\left(\sigma^{\nu}_{\mu}\right)_{t} + \sum_{j=1}^{n} \frac{\partial\Omega^{\alpha_\nu}}{\partial\varphi^{o}_{x^j}} \left(\sigma^{\nu}_{\mu}\right)_{x^j} + \sum_{i=1}^{r} \sum_{\beta=1}^{q^{\alpha_i}} l^{\alpha_\nu \mu} \left(L r^{\alpha_i \beta}\right) \sigma^{i}_{\beta} = 0 \tag{2.21}$$

$$\nu = 1, \ldots, r \quad \& \quad \mu = 1, \ldots, q^{\alpha_\nu}$$

We see that the system of transport equations (2.21) is only coupled through the nondifferentiated terms, i.e. (2.21) is a weakly coupled system. Hence the system of transport equations is trivially seen to be a symmetric hyperbolic system; we shall discuss it further in the next section.

In an analogous way we can derive the corresponding equations for the discontinuities in the derivatives of u . From (2.3) we have for $\varkappa = 1, 2, \ldots$

$$\frac{\partial^{\varkappa}}{\partial y^{o\varkappa}}\left(Lu\right) = \sum_{j=o}^{n} H^{j}\, u_{y^{o\varkappa} y^{j}} + B u_{y^{o\varkappa}} \tag{2.22}$$

$$+ \; \varkappa \sum_{j=o}^{n} \left(H^{j}\right)_{y^{o}} u_{y^{o(\varkappa-1)} y^{j}} + N^{(\varkappa-1)}\, u$$

Here the expression $N^{(\varkappa-1)}u$ involves derivatives of u of order at most $\varkappa-1$. Since $Lu \equiv 0$ on both sides of C

$$
\begin{aligned}
H^0\left[u_{y^0(\varkappa+1)}\right] &+ \sum_{j=1}^{n} H^j\left[u_{y^0\varkappa}\right]_{y^j} \\
&+ \left\{B + \varkappa (H^0)_{y^0}\right\}\left[u_{y^0\varkappa}\right] \\
&+ \varkappa \sum_{j=1}^{n} (H^j)_{y^0}\left[u_{y^0(\varkappa-1)}\right]_{y^j} + \left[N^{(\varkappa-1)}u\right] = 0
\end{aligned}
\tag{2.23}
$$

If all the derivatives of u of order less than $\varkappa+1$ are continuous across C, (2.23) shows that $H^0\left[u_{y^0(\varkappa+1)}\right] = 0$. If some $(\varkappa+1)^{st}$ derivative has a non-zero jump, then $\left[u_{y^0(\varkappa+1)}\right] \neq 0$ and hence H^0 must be singular. Thus we may assert that if u or any of its derivatives has a jump discontinuity across C, then C is a characteristic manifold.

Suppose now that $\left[u_{y^0 s}\right]$ is known for $s = 0, 1, \ldots, \varkappa-1$. Then the jumps in all the derivatives of u of order $\varkappa-1$ or less are known. By substituting $\varkappa$ for $\varkappa+1$, we can rewrite (2.23) in the form

$$
H^0\left[u_{y^0\varkappa}\right] + \sum_{i=1}^{n} H^j\left[u_{y^0(\varkappa-1)}\right]_{y^j} + \left[P^{(\varkappa-1)}u\right] = 0 \tag{2.24}
$$

Here $[P^{(\varkappa-1)}u]$ involves derivatives of u of order at most $\varkappa-1$, so it is known. We now expand $[u_{y^0\varkappa}]$ in terms of <u>all</u> the right eigenvectors of H^0

Here the expression $K^{(\kappa-1)}u$ involves derivatives of u of order at most $\kappa-1$. Since $[u] = 0$ on both sides of G

$$M^{\rho}\left[u_{\varphi^{(\kappa+1)}}\right] + \sum_{j=1}^{n} M^{j}\left[u_{\varphi^{\kappa}}\right]_{y^j}$$

$$+ \left\{B + \kappa(M^{\rho})_{\varphi}\right\}\left[u_{\varphi^{\kappa}}\right] \qquad (2.23)$$

$$+ \kappa \sum_{j=1}^{n} (M^{j})_{\varphi}\left[u_{\varphi^{(\kappa-1)}}\right]_{y^j} + \left[K^{(\kappa-1)}u\right] = 0$$

If all the derivatives of u of order less than $\kappa+1$ are continuous across G, (2.23) shows that $M^{\rho}\left[u_{\varphi^{(\kappa+1)}}\right] = 0$. If some $(\kappa+1)^{st}$ derivative has a non-zero jump, then $\left[u_{\varphi^{(\kappa+1)}}\right] \neq 0$ and hence M^{ρ} must be singular. Thus we may assert that if u or any of its derivatives has a jump discontinuity across G, then G is a characteristic manifold.

Suppose now that $\left[u_{\varphi^{s}}\right]$ is known for $s = 0, 1, \ldots, \kappa-1$. Then the jumps in all the derivatives of u of order $\kappa-1$ or less are known. By substituting κ for $\kappa+1$, we can rewrite (2.23) in the form

$$M^{\rho}\left[u_{\varphi^{\kappa}}\right] + \sum_{j=1}^{n} M^{j}\left[u_{\varphi^{(\kappa-1)}}\right]_{y^j} + \left[P^{(\kappa-1)}u\right] = 0 \qquad (2.24)$$

Here $\left[P^{(\kappa-1)}u\right]$ involves derivatives of u of order at most $\kappa-1$, so it is known. We now expand $\left[u_{\varphi^{\kappa}}\right]$ in terms of all the right eigenvectors of M^{ρ}.

$$\left[u_{y\circ\varkappa}\right] = \sum_{\alpha=1}^{\gamma} \sum_{\beta=1}^{q^{\alpha}} {}^{\varkappa}\sigma_{\beta}^{\alpha}\ r^{\alpha\beta} \tag{2.25}$$

Substituting (2.25) into (2.24) gives

$$\begin{aligned} H^{o}\left[u_{y\circ\varkappa}\right] &= -\sum_{\alpha=1}^{\gamma} \sum_{\beta=1}^{q^{\alpha}} \omega_{\beta}^{\alpha}\ {}^{\varkappa}\sigma_{\beta}^{\alpha}\ r^{\alpha\beta} \\ &= -\sum_{j=1}^{n} H^{j}\left[u_{y\circ(\varkappa-1)}\right]_{y^{j}} - \left[P^{(\varkappa-1)}\ u\right] \end{aligned} \tag{2.26}$$

When we make the same assumptions on C as before, i.e. that C satisfies (2.9) for $\alpha_1, \ldots, \alpha_r$ and these only, then we see that

$$\omega_{\beta}^{\alpha} = \Omega^{\alpha j} - \Omega^{\alpha}\ , \quad \alpha = 1,\ldots,\gamma \ \ \&\ \ \beta = 1,\ldots,q^{\alpha} \tag{2.27}$$

In (2.27) j can be chosen to be $1,\ldots,r-1$ or r, and $\overset{o}{\varphi}_{x^{\tau}}$ are substituted for the ξ^{τ}'s in the Ω's. In particular we have that

$$\begin{aligned} &\omega_{\beta}^{\alpha} \equiv 0 \quad \text{for } \alpha = \alpha_i,\ i = 1,\ldots,r \\ &\omega_{\beta}^{\alpha} \neq 0 \quad \text{everywhere for all other } \alpha\text{'s} \ \&\ \beta\text{'s} \end{aligned} \tag{2.28}$$

Multiplication on the left in (2.26) by l^{ab} gives

$${}^{\varkappa}\sigma_{b}^{a} = \frac{1}{\omega_{b}^{a}} \left\{ \sum_{j=1}^{n} l^{ab}\ H^{j}\left[u_{y\circ(\varkappa-1)}\right]_{y^{j}} + l^{ab}\left[P^{(\varkappa-1)}u\right] \right\} \tag{2.29}$$

$$a \neq \alpha_i\ , \quad i = 1,\ldots,r \ \ \&\ \ b = 1,\ldots,q^{a}$$

To obtain the other coefficients in the expansion (2.25), we multiply (2.23) on the left by $1^{a_i b}$

$$\sum_{j=1}^{n} 1^{a_i b} H^j \left[u_{y^o \varkappa}\right]_{y^j} + 1^{a_i b}\left\{B + \varkappa (H^o)_{y^o}\right\}\left[u_{y^o \varkappa}\right] = 1^{a_i b}\left[T^{(\varkappa-1)} u\right] \qquad (2.30)$$

$$i = 1, \ldots, r \quad \& \quad b = 1, \ldots, q^{a_i}$$

Here $\left[T^{(\varkappa-1)} u\right]$ involves only functions we have assumed to be known. The equations (2.30) are valid on the manifold C , and $\left[u_{y^o\varkappa}\right]$ is only defined there. However, we may look at $\left[u_{y^o\varkappa}\right]$ as a function also dependent on y_o , and apply the same arguments as we used in going from (2.13) to (2.16). Thus in the independent variables x,t (2.30) becomes

$$1^{a_i b}\left[u_{y^o\varkappa}\right]_t + \sum_{j=1}^{n} 1^{a_i b} A^j \left[u_{y^o\varkappa}\right]_{x^j} + 1^{a_i b}\left\{B + \varkappa (H^o)_{y^o}\right\}\left[u_{y^o\varkappa}\right] = 1^{a_i b}\left[T^{(\varkappa-1)} u\right] \qquad (2.31)$$

If we substitute (2.25) into (2 31) and use the relations (2.20), the system (2.31) can be written on the following form

$$\left({}^{\varkappa}\sigma_b^{\alpha_i}\right)_t + \sum_{\eta=1}^{n} \frac{\partial \Omega^{\alpha_i}}{\partial \varphi^o_{x^\eta}} \left({}^{\varkappa}\sigma_b^{\alpha_i}\right)_{x^\eta} + \sum_{m=1}^{r} \sum_{\beta=1}^{q^{\alpha_m}} {}^{\varkappa}\tau^{ib}_{m\beta}\, {}^{\varkappa}\sigma_\beta^{\alpha_m} = g^{ib}_{\varkappa} \tag{2.32}$$

$$i = 1, \ldots, r \quad \& \quad b = 1, \ldots, q^{\alpha_i}$$

Here we have introduced the following functions

$${}^{\varkappa}\tau^{ib}_{m\beta} = l^{\alpha_i b}\left(Lr^{\alpha_m \beta}\right) + \varkappa l^{\alpha_i b} (H^o)_{y^o}\, r^{\alpha_m \beta} \tag{2.33}$$

$$g^{ib}_{\varkappa} = l^{\alpha_i b}\left[T^{(\varkappa-1)}u\right] - \sum_{\substack{\alpha=1 \\ \alpha \neq \alpha_\nu \\ \nu=1,\ldots,r}}^{\gamma} \sum_{\beta=1}^{q^\alpha} \left[l^{\alpha_i b} \left\{ L\left({}^{\varkappa}\sigma^\alpha_\beta\, r^{\alpha\beta}\right) - \varkappa (H^o)_{y^o}\, {}^{\varkappa}\sigma^\alpha_\beta\, r^{\alpha\beta} \right\} \right] \tag{2.34}$$

When we have found the equation for C, the expressions (2.33) and (2.34) are known from (2.29) and our assumptions.

The system of equations (2.32) constitutes <u>the transport equations of higher order</u> for the hyperbolic system (1.1). We see that they only differ from the transport equations (2.21) in the nondifferentiated terms.

3. PROPERTIES OF THE TRANSPORT EQUATIONS.

We shall in this section study more closely the transport equations which we obtained in the previous section. We shall restrict ourselves to study only the transport equations of lowest order (2.21), but since the difference between these and the higher order transport equations (2.32) is only in the non-differentiated terms, similar results can be obtained for the higher order transport equations.

The transport equations (2.21) tell us how the discontinuities in u propagate along C. Even though the transport equations (2.21) may be defined in the whole x,t-space, their only relevance to our problem is on the hypersurface C. The hypersurface C, given by the equation (2.1), was in the construction assumed to satisfy (2.9) for $\alpha = \alpha_1, \ldots, \alpha = \alpha_r$, and these choices of α only. Each of the equations (2.9) is a first order partial differential equation with respect to the scalar-function φ^o, and can therefore be solved by the well-known method of characteristics. For α given, the characteristic equations associated with (2.9) are

$$\frac{dt}{ds} = 1 \ , \quad \frac{dx^i}{ds} = \frac{\partial \Omega^\alpha}{\partial \varphi^o_{x^i}} \qquad i = 1,\ldots,n \tag{3.1}$$

$$\frac{d\varphi^o_{x^\mu}}{ds} = - \frac{\partial \Omega^\alpha}{\partial x^\mu} \qquad \mu = 1,\ldots,n \tag{3.2}$$

This closed system of ordinary differential equations is called the bicharacteristic system of equations associated with the hyper-

bolic system (1.1), and the solutions of (3.1 & 3.2) are called the bicharacteristic strips for (1.1). The t,x-components of the bicharacteristic strips are usually called the bicharacteristic curves or simply the bicharacteristics for the hyperbolic system (1.1). There are γ different bicharacteristic systems associated with (1.1), namely one for each $\alpha = 1,\dots,\gamma$, and thus there are γ different families of bicharacteristics.

From (1.8) and the fact that Ω^α is homogeneous of degree 1 with respect to ξ, it is clear that for any pair $\alpha, a = 1, \dots, \gamma$, with $\alpha \neq a$, there is at least one $\mu = 1, \dots, n$ such that

$$\frac{\partial\Omega^\alpha}{\partial\xi^\mu} \neq \frac{\partial\Omega^a}{\partial\xi^\mu} \qquad (3.3)$$

This means that no two of the γ families of bicharacteristics are identical. However, in general it may happen that the n equations

$$\frac{\partial\Omega^\alpha}{\partial\xi^\mu} = \frac{\partial\Omega^a}{\partial\xi^\mu} \quad \mu = 1,\dots,n \qquad (3.4)$$

are all satisfied simultaneously at certain points x,t,ξ for $\alpha \neq a$. At such points the directions of bicharacteristics from two different families are the same. If the equations (3.4) are satisfied at all points on a bicharacteristic strip of the family with index α say, then the families of bicharacteristics with indices α and a must have at least one bicharacteristic in common. If the directions of the bicharacteristics from different families are different at all points, i.e. if the n equations

(3.4) for no points x, t, ξ are simultaneously satisfied, we say that the hyperbolic system (1.1) has bicharacteristics of constant multiplicity. If in addition $\gamma = k$, (1.1) is said to have distinct bicharacteristics.

It is readily seen that a hyperbolic system with characteristics of constant multiplicity also has bicharacteristics of constant multiplicity. The opposite is, however, not true as is easily seen for instance for weakly coupled hyperbolic systems. Thus it is less restrictive to consider the case with bicharacteristics of constant multiplicity than the case with characteristics of constant multiplicity.

We shall now study the transport equations (2.21) in view of the above considerations on bicharacteristics. Let us first consider the special case where on C we have

$$\frac{\partial \Omega^{\alpha_i}}{\partial \varphi^{\circ}_{x^{\eta}}} \equiv \frac{\partial \Omega^{\alpha_j}}{\partial \varphi^{\circ}_{x^{\eta}}} \tag{3.5}$$

for every $i, j = 1, \ldots, r$ and $\eta = 1, \ldots, n$. The equations (3.5) obviously contain no restrictions if $r = 1$. Thus the special case we are considering includes all cases where the hyperbolic system (1.1) has characteristics of constant multiplicity. When $r > 1$, the equations (3.5) means that the bicharacteristics of the r different families with indices $\alpha_1, \ldots, \alpha_r$, are identical on the hypersurface C.

In the transport equations (2.21) we see that the functions σ^{ν}_{μ} are differentiated along the bicharacteristics of the family with index α_{ν}. With the above assumption (3.5), all the functions σ^{ν}_{μ} are in (2.21) differentiated in the same direction, we may

(3.4) for no points x, t, ξ are simultaneously satisfied, we say that the hyperbolic system (1.1) has bicharacteristics of constant multiplicity. If in addition $\gamma = k$, (1.1) is said to have distinct bicharacteristics.

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We shall now study the transport equations (2.21) in view of the above considerations on bicharacteristics. Let us first consider the special case where on C we have

$$\frac{\partial \varphi^{\alpha_i}}{\partial \xi^{\alpha_i}_{\nu}} = \frac{\partial \varphi^{\alpha_j}}{\partial \xi^{\alpha_j}_{\nu}} \tag{3.5}$$

for every $i, j = 1, \ldots, r$ and $\nu = 1, \ldots, n$. The equations (3.5) obviously contain no restrictions if $r = 1$. Thus the special case we are considering includes all cases where the hyperbolic system (1.1) has characteristics of constant multiplicity. When $r > 1$, the equations (3.5) means that the bicharacteristics of the r different families with indices $\alpha_1, \ldots, \alpha_r$ are identical on the hypersurface C.

In the transport equations (2.21) we see that the functions σ^{ν}_{i} are differentiated along the bicharacteristics of the family with index α_i. With the above assumption (3.5), all the functions σ^{ν}_{i} are in (2.21) differentiated in the same direction. We may therefore interpret the system of transport equations (2.21) as ordinary differential equations along the bicharacteristics

$$\frac{d\sigma_\mu^\nu}{ds} = -1^{\alpha_\nu \mu} \left\{ \sum_{i=1}^{r} \sum_{\beta=1}^{q^{\alpha_i}} \left(Lr^{\alpha_i \beta} \right) \sigma_\beta^i \right\} \tag{3.6}$$

$$\nu = 1,\dots,r \quad \& \quad \mu = 1,\dots,q^{\alpha_\nu}$$

We shall now study the expressions on the right hand side in (3.6) a little closer. In general we have for $a = 1,\dots,\gamma$ and $\beta = 1,\dots,q^a$

$$\begin{aligned} Lr^{a\beta} &= \frac{\partial r^{a\beta}}{\partial t} + \sum_{i=1}^{n} A^i \frac{\partial r^{a\beta}}{\partial x^i} + Br^{a\beta} \\ &+ \sum_{\eta=1}^{n} \left\{ \varphi^o_{x^\eta t} + \sum_{i=1}^{n} \varphi^o_{x^i x^\eta} A^i \right\} \frac{\partial r^{a\beta}}{\partial \varphi^o_{x^\eta}} \end{aligned} \tag{3.7}$$

We assume that φ^o satisfies (2.9) identically, by differentiation with respect to x^η we get

$$\varphi^o_{x^\eta t} + \sum_{i=1}^{n} \frac{\partial \Omega^\alpha}{\partial \varphi^o_{x^i}} \varphi^o_{x^i x^\eta} + \frac{\partial \Omega^\alpha}{\partial x^\eta} = 0 \tag{3.8}$$

From (3.8) we see that (3.7) can be written

$$\begin{aligned} Lr^{a\beta} &= \frac{\partial r^{a\beta}}{\partial t} + \sum_{i=1}^{n} A^i \frac{\partial r^{a\beta}}{\partial x^i} + Br^{a\beta} \\ &- \sum_{\eta=1}^{n} \frac{\partial \Omega^\alpha}{\partial x^\eta} \frac{\partial r^{a\beta}}{\partial \varphi^o_{x^\eta}} + \sum_{\eta=1}^{n} \sum_{i=1}^{n} \varphi^o_{x^i x^\eta} \left\{ - \frac{\partial \Omega^\alpha}{\partial \varphi^o_{x^i}} I + A^i \right\} \frac{\partial r^{a\beta}}{\partial \varphi^o_{x^\eta}} \end{aligned} \tag{3.9}$$

If we differentiate (2.18) with respect to ξ^{ν} we get

$$
\begin{aligned}
& - \frac{\partial^2 \Omega^{\alpha}}{\partial \xi^{\mu}\, \partial \xi^{\nu}}\, r^{\alpha\beta} + \left\{ -\Omega^{\alpha} I + \sum_{i=1}^{n} \xi^{i} A^{i} \right\} \cdot \frac{\partial^2 r^{\alpha\beta}}{\partial \xi^{\mu}\, \partial \xi^{\nu}} \\
& + \left\{ - \frac{\partial \Omega^{\alpha}}{\partial \xi^{\mu}} I + A^{\mu} \right\} \frac{\partial r^{\alpha\beta}}{\partial \xi^{\nu}} + \left\{ - \frac{\partial \Omega^{\alpha}}{\partial \xi^{\nu}} I + A^{\nu} \right\} \frac{\partial r^{\alpha\beta}}{\partial \xi^{\mu}} \equiv 0
\end{aligned}
\tag{3.10}
$$

From (3.5), (3.9) and (3.10) we get on C that

$$
\begin{aligned}
l^{\alpha,\mu}(Lr^{\alpha;\beta}) = {} & l^{\alpha,\mu}\left\{ \frac{\partial r^{\alpha;\beta}}{\partial t} + \sum_{j=1}^{n} A^{j} \frac{\partial r^{\alpha;\beta}}{\partial x^{j}} + B r^{\alpha;\beta} \right\} \\
& - \sum_{j=1}^{n} \frac{\partial \Omega^{\alpha}}{\partial x^{j}}\, l^{\alpha,\mu} \cdot \frac{\partial r^{\alpha;\beta}}{\partial \varphi^{o}_{x^{j}}} \\
& + \frac{1}{2} \sum_{j=1}^{n} \sum_{\eta=1}^{n} \varphi^{o}_{x^{j} x^{\eta}} \frac{\partial^2 \Omega^{\alpha;}}{\partial \varphi^{o}_{x^{j}}\, \partial \varphi^{o}_{x^{\eta}}}\, \delta^{\nu}_{i}\ \delta^{\mu}_{\beta}
\end{aligned}
\tag{3.11}
$$

Here α is fixed, and equal to one of the indices $\alpha_1, \cdots, \alpha_r$. Thus we see that in general the right hand side of (3.6) depends on the second derivatives of φ^{o}. However, in (3.6) we only need the second derivatives along the bicharacteristics, there they must satisfy the following equations

$$
\begin{aligned}
\frac{d\varphi^{o}_{x^{i} x^{j}}}{ds} = {} & - \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} \frac{\partial^2 \Omega^{\alpha}}{\partial \varphi^{o}_{x^{\mu}}\, \partial \varphi^{o}_{x^{\nu}}}\, \varphi^{o}_{x^{\nu} x^{j}}\ \varphi^{o}_{x^{\mu} x^{i}} - \frac{\partial^2 \Omega^{\alpha}}{\partial x^{i}\, \partial x^{j}} \\
& - \sum_{\mu=1}^{n} \left\{ \frac{\partial^2 \Omega^{\alpha}}{\partial \varphi^{o}_{x^{\mu}}\, \partial x^{j}}\, \varphi^{o}_{x^{\mu} x^{i}} + \frac{\partial^2 \Omega^{\alpha}}{\partial \varphi^{o}_{x^{\mu}}\, \partial x^{i}}\, \varphi^{o}_{x^{\mu} x^{j}} \right\}
\end{aligned}
\tag{3.12}
$$

In the special case we are considering, we have now obtained a closed system of ordinary differential equations which must be satisfied by the discontinuity functions and the discontinuity surface. In fact, if we denote $\varphi^{o}_{x^i}$ by ξ^i and $\varphi^{o}_{x^i x^j}$ by ξ^i_j, we see that the following closed system of equations must be satisfied:

$$\frac{dt}{ds} = 1 \ , \quad \frac{dx^i}{ds} = \frac{\partial \Omega^\alpha}{\partial \xi^i}$$

$$i = 1, \ldots, n$$

$$\frac{d\xi^i}{ds} = - \frac{\partial \Omega^\alpha}{\partial x^i}$$

$$\frac{d\xi^i_j}{ds} = - \sum_{\nu=1}^{n} \sum_{\mu=1}^{n} \frac{\partial^2 \Omega^\alpha}{\partial \xi^\nu \partial \xi^\mu} \xi^\nu_i \xi^\mu_j - \frac{\partial^2 \Omega^\alpha}{\partial x^i \partial x^j}$$

$$- \sum_{\mu=1}^{n} \left\{ \frac{\partial^2 \Omega^\alpha}{\partial \xi^\mu \partial x^j} \xi^\mu_i + \frac{\partial^2 \Omega^\alpha}{\partial \xi^\mu \partial x^i} \xi^\mu_j \right\}$$

$$\xi^i_j = \xi^j_i \ , \quad i, j = 1, \ldots, n \tag{3.13}$$

$$\frac{d\sigma^\nu_\mu}{ds} = - \sum_{i=1}^{r} \sum_{\beta=1}^{q^{\alpha_i}} l^{\alpha_\nu \mu} \left\{ \frac{\partial r^{\alpha_i \beta}}{\partial t} + \sum_{j=1}^{n} A^j \frac{\partial r^{\alpha_i \beta}}{\partial x^j} + B r^{\alpha_i \beta} \right\} \sigma^i_\beta$$

$$+ \sum_{i=1}^{r} \sum_{\beta=1}^{q^{\alpha_i}} \sum_{j=1}^{n} \frac{\partial \Omega^\alpha}{\partial x^j} l^{\alpha_\nu \mu} \cdot \frac{\partial r^{\alpha_i \beta}}{\partial \xi^j} \sigma^i_\beta$$

$$- \frac{1}{2} \sum_{j=1}^{n} \sum_{\eta=1}^{n} \xi^j_\eta \frac{\partial^2 \Omega^{\alpha_\nu}}{\partial \xi^j \partial \xi^\eta} \sigma^\nu_\mu$$

$$\nu = 1, \ldots, r \ \& \ \mu = 1, \ldots, q^{\alpha_\nu}$$

In (3.13) the index α is fixed, and equal to one of the indices $\alpha_1, \dots, \alpha_r$. From the construction of the equations (3.13) and the general theory of characteristics, we know that the initial-valueproblem for (3.13) with relevant initialvalues, is equivalent to the initialvalueproblem for (2.9) and (2.21). We shall therefore also call (3.13) the system of transport equations. We conclude that the discontinuities propagate along the bicharacteristics when (3.5) is satisfied on C .

When (3.5) is not valid on C , the above conclusions will no longer be true. The discontinuities will no longer propagate along the bicharacteristics, but will spread out on C governed by the transport equations (2.21). Since (2.21) is a symmetric hyperbolic system, there is a well established theory for existence, uniqueness and other properties of solutions, see for instance Courant-Hilbert [1]. Since (2.21) is a weakly coupled system, there is also a more direct approach available. In fact, essentially the same method as that used in [1] for hyperbolic systems with two independent variables can be applied.

In our study which led to the system of transport equations (2.21), we assumed that the jumpdiscontinuity for u across C was a smooth function on C . When u is a piece-wise smooth function, the jumpdiscontinuities of u on the finite number of smooth hypersurfaces will also be piece-wise smooth functions on these surfaces. It is therefore of interest to study how the discontinuities of the jumpdiscontinuities of u propagate. Thus we want to study how discontinuities in the solutions of the system of transport equations (2.21) propagate on C . This problem is of course a special case of the problem we started out with, In (3.13) the index α is fixed, and equal to one of the indices $\alpha_1, \dots, \alpha_k$. From the construction of the equations (3.13) and the general theory of characteristics, we know that the initial-valueproblem for (3.13) with relevant initial-values, is equivalent to the initialvalueproblem for (2.9) and (2.21). We shall therefore also call (3.13) the system of transport equations. We conclude that the discontinuities propagate along the bicharacteristics when (3.5) is satisfied on G.

When (3.5) is not valid on G, the above conclusions will no longer be true. The discontinuities will no longer propagate along the bicharacteristics, but will spread out on G governed by the transport equations (2.21). Since (2.21) is a symmetric hyperbolic system, there is a well established theory for existence, uniqueness and other properties of solutions, see for instance Courant-Hilbert [1]. Since (2.21) is a weakly coupled system, there is also a more direct approach available. In fact, essentially the same method as that used in [1] for hyperbolic systems with two independent variables can be applied.

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we can therefore apply the results we have found so far. Since the system of transport equations (2.21) is weakly coupled and contains one independent variable less than the original problem (because of the restriction to C), the problem we now want to study is considerably simpler than the problem we started out with. The assumptions in section 1 are trivially seen to be satisfied, and the functions corresponding to Ω^α and the eigenvectors corresponding to $r^{\alpha\beta}$ and $l^{\alpha\beta}$ are easily found. In fact, if we let $y^1, \dots, y^{n-1}$, $y^n = t$ be the coordinates on C (since the hyperplanes t = constant are spacelike, there is no loss of generality to take $y^n = t$ as one of the independent variables on C), the system (2.21) can be written on the following form on C

$$(\sigma^\nu_\mu)_t + \sum_{i=1}^{n-1} d^\nu_i \, (\sigma^\nu_\mu)_{y^i} + \sum_{i=1}^{r} \sum_{j=1}^{q^{\alpha_i}} e^{i\nu}_{j\mu} \, \sigma^i_j = 0 \qquad (3.14)$$

$$\nu = 1,\dots,r \quad \& \quad \mu = 1,\dots,q^{\alpha_\nu}$$

where the coefficients d^ν_i, $e^{i\nu}_{j\mu}$ are functions of $t, y^1,\dots,y^{n-1}$. The functions corresponding to Ω^α are

$$\Omega^{*\nu} = \sum_{i=1}^{n-1} \xi^i \, d^\nu_i \, , \; \nu = 1, \dots, r \qquad (3.15)$$

and the eigenvectors corresponding to $r^{\alpha\beta}$ and $l^{\alpha\beta}$ are simply the unitvectors

$$r^{*\nu\mu} = l^{*\nu\mu} = \left\{\delta^{\nu 1}_{\mu 1}, \ldots, \delta^{\nu r}_{\mu q^{\alpha_r}}\right\} \quad (3.16)$$

$$\nu = 1,\ldots,r \;\;\&\;\; \mu = 1,\ldots,q^{\alpha_\nu}$$

In general some of the functions (3.15) may be identical; for instance will all of them be identical when (3.5) is satisfied. If this is the case, one would have to renumber the functions $\Omega^{*\nu}$ and the eigenvectors $r^{*\nu\mu} = l^{*\nu\mu}$ in order to get the assumption corresponding to (1.8) satisfied. Obviously this would complicate the notations, we shall therefore for simplicity restrict our study to the case where the hyperbolic system (1.1) has bicharacteristics of constant multiplicity, since such problems cannot arise in that case. At the end of this section we shall make a few comments on what the differences may be in the general case.

Now, if we study the cases where the hyperbolic system (1.1) has bicharacteristics of constant multiplicity, we know that all the assumptions in section 1 are satisfied for the system of equations (3.14). We can therefore apply the same procedure to (3.14) as we did in section 2 to (1.1) when we wanted to study the propagation of discontinuities. Again, we get that the discontinuities propagate along the characteristic hypersurfaces. If we pull any characteristic surface for (3.14) back to the t,x-space, we get an n-1 dimensional submanifold of C which is generated by a n-2 parameter family of bicharacteristic curves. If we restrict ourselves to characteristic hypersurfaces for (3.14) which satisfy conditions analogous to those imposed on C on page 10, the transport equations will be of the same type

$$r^{*\nu\mu} = l^{*\nu\mu} = \left\{\delta_{\mu 1}^{\nu 1}, \ldots, \delta_{\mu d^*}^{\nu r}\right\} \tag{3.16}$$

$$\nu = 1,\ldots,r \quad \& \quad \mu = 1,\ldots,d^*$$

In general some of the functions (3.15) may be identical; for instance will all of them be identical when (3.5) is satisfied. If this is the case, one would have to renumber the functions $Q^{*\nu}$ and the eigenvectors $r^{*\nu\mu} = l^{*\nu\mu}$ in order to get the assumption corresponding to (1.8) satisfied. Obviously this would complicate the notations, we shall therefore for simplicity restrict our study to the case where the hyperbolic system (1.1) has bicharacteristics of constant multiplicity, since such problems cannot arise in that case. At the end of this section we shall make a few comments on what the differences may be in the general case.

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However, if we apply the same procedure over and over again, i.e. find the transport equations for the transport equations for the transport equations etc. for (1.1), it is clear that sooner or later (i.e. after at most n steps) we will arrive at a stage where these transport equations are of the type (2.21) with $r = 1$.*) As we saw in the beginning of this section, these transport equations will therefore be equivalent to a system of ordinary differential equations of the form (3.13) with $r = 1$. As a result of this we can say that when the hyperbolic system (1.1) has bicharacteristics of constant multiplicity, then the discontinuities of sufficiently high order (i.e. the discontinuities of the discontinuities etc., sufficiently many times) will always propagate along the bicharacteristics and be governed by (3.13) <u>with $r = 1$.</u> In general we do not know a priori the lowest order of the discontinuities that propagate along the bicharacteristics

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*) At each step we have to restrict ourselves to characteristic hypersurfaces which satisfy conditions analogous to those imposed on C on page 10. This will normally require that we restrict ourselves to a sufficiently small neighbourhood of x_o, t_o. To obtain the global behaviour of the discontinuity functions at each step, we have to apply the construction described in section 4.

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*) At each step we have to restrict ourselves to characteristic hypersurfaces which satisfy conditions analogous to those imposed on C on page 10. This will normally require that we restrict ourselves to a sufficiently small neighbourhood of x_0, t_0. To obtain the global behaviour of the discontinuity functions at each step, we have to apply the construction described in section 4.

except in the case where the hyperbolic system (1.1) has characteristics of constant multiplicity, in this case the discontinuities of all orders propagate along the bicharacteristics. Finally we note that in general the lowest order of the discontinuities that propagate along the bicharacteristics depends on the normals of the characteristic and subcharacteristic manifolds, and may also vary from point to point in x,t-space.

In the general case where the hyperbolic system (1.1) does not have bicharacteristics of constant multiplicity, the situation may be much more complicated than above. However, in the non-pathological cases one can also here apply the technique which we are going to describe in the next section, to glue the results together. In short, we can describe the situation as follows: The discontinuities of sufficiently high order will propagate along the bicharacteristics, and locally the transport equations will be of the form (3.13). However, in general we will not have $r = 1$ in (3.13), but we will have that r may vary from point to point on C.

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4. DISCONTINUOUS INITIALVALUE PROBLEMS.

We shall now consider initialvalue problems for the hyperbolic system (1.1). The initialvalues considered

$$u\Big|_{t=t_o} = u_o(x) \tag{4.1}$$

are assumed to be piece-wise smooth functions. Thus u_o is assumed to be smooth everywhere except on a finite number of smooth hypersurfaces; the jumpdiscontinuities of u_o and the derivatives of u_o are assumed to be piece-wise smooth functions on these hypersurfaces.

Since we assume that the initialvalue problem is well-posed when the initialvalues are in C^{N_o} for some N_o, it suffices to study how the initialdiscontinuities of u and its derivatives up to the order N_o propagate. In fact, if this is known the discontinuous initialvalue problem (1.1) & (4.1) can be solved by another initialvalue problem with C^{N_o} initialvalues (for the details on this, see [3]). Furthermore, it suffices to consider the case where u_o is smooth everywhere except on <u>one</u> $n-1$ dimensional smooth manifold Γ. In fact, if u_o for instance is discontinuous along two crossing manifolds, the discontinuity of u_o is smooth everywhere on these $n-1$ dimensional manifolds except on <u>one</u> $n-2$ dimensional submanifold. Thus the initialvalue problem for the discontinuity of the discontinuity function of u_o is of the above type in view of the considerations in the preceding section. If this problem is solved first, the initial value problem for the discontinuities of u_o and its derivatives

can by the same construction as referred to above be transferred to problems where the discontinuities of u_o and its derivatives are smooth on a n-1 dimensional manifold. This corresponds to cases where u_o is smooth everywhere except on one n-1 dimensional manifold. - If several discontinuity manifolds have a submanifold in common we would have to start the construction by considering the discontinuity of sufficiently high order and then successively solve the problems for the lower order discontinuities. Since the discontinuities of sufficiently high order always propagate along the bicharacteristics, the construction will consist of a finite number of steps.

We have thus reduced the problem to the problem of finding out how the initial discontinuities along Γ of u and the derivatives of u propagate. In the following we shall restrict ourselves to the study of how the discontinuities of u itself propagate. The discussion of the propagation of the discontinuities of the derivaties of u is completely analogous and is therefore omitted (see [3] for the construction in the case of symmetric hyperbolic systems with characteristics of constant multiplicity).

In section 2 we found that discontinuities can only propagate along the characteristics. Since the discontinuities are initially located on Γ, we can therefore conclude that the discontinuities must be located on the characteristics going through Γ. If the hyperbolic system (1.1) has characteristics of constant multiplicity, there are exactly γ different characteristic manifolds going through Γ, namely one for each of the characteristic partial differential equations (2.9). In the general case, however, can by the same construction as referred to above be transferred to problems where the discontinuities of u_0 and its derivatives are smooth on a n-1 dimensional manifold. This corresponds to cases where u_0 is smooth everywhere except on one n-1 dimensional manifold. – If several discontinuity manifolds have a submanifold in common we would have to start the construction by considering the discontinuity of sufficiently high order and then successively solve the problems for the lower order discontinuities. Since the discontinuities of sufficiently high order always propagate along the bicharacteristics, the construction will consist of a finite number of steps.

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In section 2 we found that discontinuities can only propagate along the characteristics. Since the discontinuities are initially located on Γ , we can therefore conclude that the discontinuities must be located on the characteristics going through Γ . If the hyperbolic system (1.1) has characteristics of constant multiplicity, there are exactly ν different characteristic manifolds going through Γ , namely one for each of the characteristic partial differential equations (2.7). In the general case, however, there may be a lot more characteristic manifold going through Γ. In fact, a characteristic manifold may in this case satisfy different characteristic equations (2.9) in different regions. Globally there may therefore be an infinite set of characteristic manifolds going through Γ. In the following we shall study what happens locally and briefly indicate the global aspects.

Let $C^{(\alpha)}$ be the characteristic manifold satisfying (2.9) for the index α and going through Γ, $\alpha = 1, \ldots, \gamma$ (some of the characteristics $C^{(\alpha)}$ may partially or completely be equal). We let $[\;]^{\alpha}$, $[\;]^{\Gamma}$ denote jumps across $C^{(\alpha)}$ and Γ respectively. Since the eigenvectors $r^{\alpha\beta}$ form a complete set, the jumps in the initial values of u across Γ have a unique decomposition

$$[u_o]^{\Gamma} = \sum_{a=1}^{\gamma} \sum_{\beta=1}^{q^a} \sigma^{a}_{o\beta} r^{a\beta} \tag{4.2}$$

In the same way we may set

$$[u]^{\alpha} = \sum_{a=1}^{\gamma} \sum_{\beta=1}^{q^a} \sigma^{a}_{\alpha\beta}\, r^{a\beta} \tag{4.3}$$

The initialconditions are

$$\sum_{\alpha=1}^{\gamma} [u]^{\alpha} = [u_o]^{\Gamma} \quad \text{on } \Gamma \tag{4.4}$$

Multiplication by $l^{\mu\nu}$ gives

there may be a lot more characteristic manifold going through Γ. In fact, a characteristic manifold may in this case satisfy different characteristic equations (2.5) in different regions. Generally there may therefore be an infinite set of characteristic manifolds going through Γ. In the following we shall study what happens locally and briefly indicate the global aspects.

Let $C^{(\alpha)}$ be the characteristic manifold satisfying (2.9) for the index α and going through Γ, $\alpha = 1, \ldots, k$ (some of the characteristics $C^{(\alpha)}$ may partially or completely be equal). We let $[\ ]^{\alpha}$, $[\ ]^{\Gamma}$ denote jumps across $C^{(\alpha)}$ and Γ respectively. Since the eigenvectors $r^{\alpha\beta}$ form a complete set, the jumps in the initial values of u across Γ have a unique decomposition

$$[u_0]^{\Gamma} = \sum_{\alpha=1}^{k} \sum_{\beta=1}^{s_\alpha} c^{\alpha}_{0\beta} r^{\alpha\beta} \qquad (4.2)$$

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$$\sum_{\alpha=1}^{k} [u]^{\alpha} = [u_0]^{\Gamma} \quad \text{on } \Gamma \qquad (4.4)$$

Multiplication by $l^{\gamma\delta}$ gives

$$\sum_{\alpha=1}^{\gamma} \sigma^{\mu}_{\alpha\nu} = \sigma^{\mu}_{o\nu} \quad \text{on } \Gamma \tag{4.5}$$

From the discussion in section 2 we know that in every point on $C^{(\alpha)}$ where $C^{(\alpha)}$ only satisfies one of the equations (2.9), $\sigma^{a}_{\alpha\beta} = 0$ if $a \neq \alpha$. We may without loss of generality assume that on Γ this is true everywhere, since we shall see that we are then led to a well defined construction of how the discontinuities propagate.

Now let x_o, t_o be an arbitrary point on Γ. We want to study how the discontinuities in the neighbourhood of x_o, t_o propagate along one of the characteristic manifolds $C^{(\alpha)}$, say $C^{(\alpha_1)}$, near x_o, t_o. By definition $C^{(\alpha_1)}$ satisfies (2.9) for $\alpha = \alpha_1$. If $C^{(\alpha_1)}$ satisfies (2.9) at x_o, t_o for $\alpha = \alpha_1$ only, then as we saw in section 2, the propagation of the discontinuities in a neighbourhood of x_o, t_o on $C^{(\alpha_1)}$ is governed by the transport equations found in section 2. So in this case everything is nice, the discontinuities are propagated along the bicharacteristics and are described by the system of ordinary differential equations (3.13) with $r=1$ as we saw in section 3. The initial conditions for the equations (3.13) are found from the initial conditions given above.

The case above is the "normal" case in the sense that this is the case most frequently met in applications. In general, however, $C^{(\alpha_1)}$ may satisfy (2.9) at x_o, t_o for one or several α's different from α_1. Assume therefore that $C^{(\alpha_1)}$ satisfies

$$\sum_{\alpha=1}^{k} c^{\mu}_{\alpha\nu} = c^{\mu}_{o\nu} \quad \text{on } \Gamma \tag{4.5}$$

From the discussion in section 2 we know that in every point on $G^{(s)}$ where $G^{(q)}$ only satisfies one of the equations (2.9), $c^{\alpha}_{\alpha\beta} = 0$ if $\alpha \neq \beta$. We may without loss of generality assume that on Γ this is true everywhere, since we shall see that we are then led to a well defined construction of how the discontinuities propagate.

Now let x_o, t_o be an arbitrary point on Γ. We want to study how the discontinuities in the neighbourhood of x_o, t_o propagate along one of the characteristic manifolds $G^{(\alpha)}$, say $G^{(\alpha_1)}$, near x_o, t_o. By definition $G^{(\alpha_1)}$ satisfies (2.9) for $\alpha = \alpha_1$. If $G^{(\alpha_1)}$ satisfies (2.9) at x_o, t_o for $\alpha = \alpha_1$ only, then as we saw in section 2, the propagation of the discontinuities in a neighbourhood of x_o, t_o on $G^{(\alpha_1)}$ is governed by the transport equations found in section 2. So in this case everything is nice, the discontinuities are propagated along the bicharacteristics and are described by the system of ordinary differential equations (3.13) with $r=1$ as we saw in section 3. The initial conditions for the equations (3.13) are found from the initial conditions given above.

The case above is the "normal" case in the sense that this is the case most frequently met in applications. In general, however, $G^{(\alpha_1)}$ may satisfy (2.9) at x_o, t_o for one or several α's different from α_1. Assume therefore that $G^{(\alpha_1)}$ satisfies

(2.9) at x_o, t_o for $\alpha_1, \ldots, \alpha_r$. By continuity there is then a neighbourhood of x_o, t_o where $C^{(\alpha_1)}$ does not satisfy (2.9) for any <u>other</u> choice of α than $\alpha_1, \ldots, \alpha_r$. We may without loss of generality assume that the equation for $C^{(\alpha_1)}$ near x_o, t_o is given by

$$\varphi(x,t) \equiv \psi(x) - t = 0 \tag{4.6}$$

Consider now the function

$$\begin{aligned} \Omega(x) \overset{\text{def}}{=} \; & \Omega^{\alpha_1}(x, \psi(x), \psi_{x^1}(x), \ldots, \psi_{x^n}(x)) \\ & - \Omega^{\alpha_2}(x, \psi(x), \psi_{x^1}(x), \ldots, \psi_{x^n}(x)) \end{aligned} \tag{4.7}$$

From the above assumptions we see that $\Omega(x_o) = 0$, furthermore we see that

$$\Omega(x) = 0 \tag{4.8}$$

is the equation for the points on $C^{(\alpha_1)}$ where $C^{(\alpha_1)}$ satisfies (2.9) for α_1 and α_2 in the neighbourhood of x_o, t_o. If $\frac{\partial}{\partial x}\Omega(x_o) \neq 0$, which is the normal case, the solution of (4.8) is an $n-1$ dimensional manifold going through x_o , which defines an $n-1$ dimensional manifold S on $C^{(\alpha_1)}$. In a neighbourhood of $x_o, t_o, C^{(\alpha_1)}$ satisfies in this case (2.9) for $\alpha = \alpha_2$ only on this manifold S going through x_o, t_o . If $\frac{\partial}{\partial x}\Omega(x_o) = 0$ the situation is much more complicated. In this case the function $\Omega(x)$ may at $x = x_o$ either

(2.9) at x_o, t_o for $q_1, \ldots, q_r$. By continuity there is then a neighbourhood of x_o, t_o where $G^{(q_1)}$ does not satisfy (2.9) for any other choice of q than $q_1, \ldots, q_r$. We may without loss of generality assume that the equation for $G^{(q_1)}$ near x_o, t_o is given by

$$\varphi(x,t) \equiv \psi(x) - t = 0 \tag{4.6}$$

Consider now the function

$$\Omega(x) \overset{\text{def}}{=} \Omega^{q_1}(x, \psi(x), \psi_x(x), \ldots, \psi_x(x)) - \Omega^{q_2}(x, \psi(x), \psi_x(x), \ldots, \psi_x(x)) \tag{4.7}$$

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(a) have an extremum,

(b) have a saddlepoint,

(c) $\equiv 0$ in a neighbourhood of x_o, or

(d) be "pathological" in a neighbourhood of x_o.

In (a) we shall by an extremum mean that there is a neighbourhood of x_o where $\Omega(x) \neq 0$ everywhere except at $x = x_o$. In this case there is a neighbourhood of x_o,t_o on $C^{(\alpha_1)}$, where $C^{(\alpha_1)}$ satisfies (2.9) for $\alpha = \alpha_2$ only at x_o,t_o. In (b) we shall by a saddlepoint mean that there is a finite number of manifolds, each of dimension at most $n-1$ and containing x_o, such that (4.8) is satisfied everywhere on these manifolds in a neighbourhood of x_o, and furthermore that in each of the open subregions (we assume that the number of such regions is finite) which these submanifolds divide the neighbourhood of x_o into, either $\Omega \equiv 0$ or $\Omega \neq 0$ everywhere. In this case $C^{(\alpha_1)}$ satisfies (2.9) for $\alpha = \alpha_2$ only on a finite set of submanifolds of $C^{(\alpha_1)}$ of dimension at most $n-1$ and going through x_o,t_o, and in a finite (possibly empty) set of sectors on $C^{(\alpha_1)}$ going out from x_o,t_o. By a sector going out from x_o,t_o we here mean a region bounded by a finite set of $n-1$ dimensional manifolds all containing x_o,t_o. In case (c) $C^{(\alpha_1)}$ satisfies (2.9) for $\alpha = \alpha_2$ in addition to $\alpha = \alpha_1$ everywhere in a neighbourhood of x_o,t_o. Case (d) is by definition all cases which are not contained in (a), (b) or (c). In this case we see that x_o may for instance be an accumulation point for at least one sequence of points, all satisfying (4.8), and such that this sequence of points does not belong to a finite number of connected manifolds where (4.8) is everywhere satisfied. A simple example of the case (d) is given by the function

$$\Omega(x) = \begin{cases} \exp\left(\frac{1}{\cos x - 1}\right) \sin \frac{1}{x} & x \in [-1,0) \cup (0,1] \\ 0 & x = 0 \end{cases} \tag{4.9}$$

$x_o = 0$ is here an accumulation point of the type described above. We are not able to treat case (d) in full generality, and we have not been able to find simple criteria on the coefficients in (1.1) to avoid these cases when (1.1) does not have characteristics of constant multiplicity.

We find the points where $C^{(\alpha_1)}$ in a neighbourhood of x_o, t_o satisfies (2.9) for $\alpha = \alpha_3, \dots, \alpha_r$ by comparing Ω^{α_i}, $i = 3, \dots, r$ with Ω^{α_1} in the same way as we did above with Ω^{α_2}. Thus we may conclude that there is a neighbourhood of x_o, t_o on $C^{(\alpha_1)}$ where $C^{(\alpha_1)}$ is a multiple characteristic on a point set which is a finite union of sets of the above types. If we exclude the pathological possibilities from our discussion, we can therefore summarize the above in the following way: If $C^{(\alpha_1)}$ satisfies (2.9) at x_o, t_o for $\alpha = \alpha_1, \dots, \alpha_r$ then there is a neighbourhood of x_o, t_o on $C^{(\alpha_1)}$ which can be divided into a finite number of subregions with the property that all interior points are of the type we considered in section 2, when we were able to find the transport equations.

From the above discussion we see that locally we know the transport equations for the propagation of discontinuities everywhere on $C^{(\alpha_1)}$, except possibly on a finite set of submanifolds of $C^{(\alpha_1)}$ of dimension at most $n-1$. It should be clear that those of these exceptional submanifolds which have the property

$$\Omega(x) = \begin{cases} \exp\left(\dfrac{1}{\cos x - 1}\right)\sin\dfrac{1}{x}, & x \in [-1,0) \cup (0,1] \\ 0 & x = 0 \end{cases} \qquad (4.9)$$

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We find the points where $C^{(\alpha_1)}$ in a neighbourhood of x_0, t_0 satisfies (2.9) for $\alpha = \alpha_2, \ldots, \alpha_r$ by comparing Ω_i, $i = 3, \ldots, r$ with Ω^{α_1} in the same way as we did above with Ω^{α}. Thus we may conclude that there is a neighbourhood of x_0, t_0 on $C^{(\alpha_1)}$ where $C^{(\alpha_1)}$ is a multiple characteristic on a point set which is a finite union of sets of the above types. If we exclude the pathological possibilities from our discussion, we can therefore summarize the above in the following way: If $C^{(\alpha_1)}$ satisfies (2.9) at x_0, t_0 for $\alpha = \alpha_1, \ldots, \alpha_r$ then there is a neighbourhood of x_0, t_0 on $C^{(\alpha_1)}$ which can be divided into a finite number of subregions with the property that all interior points are of the type we considered in section 2, when we were able to find the transport equations.

From the above discussion we see that locally we know the transport equations for the propagation of discontinuities everywhere on $C^{(\alpha_1)}$, except possibly on a finite set of submanifolds of $C^{(\alpha_1)}$ of dimension at most n-1. It should be clear that those of these exceptional submanifolds which have the property that the multiplicity of $C^{(\alpha_1)}$ is the same *) everywhere in a neighbourhood of the submanifold on $C^{(\alpha_1)}$ (except on the submanifold itself), cannot affect the propagation of the discontinuities on $C^{(\alpha_1)}$ because of the continuity properties the discontinuities necessarily must have. In particular the exceptional manifolds of dimension at most n-2 are all of this type. In exactly the same way we see that those pathological cases which involve a countable set of submanifolds of the above type, such that this set of submanifolds has a finite number of submanifolds as accumulation points, cannot affect the propagation of the discontinuities.

We have thus seen that except in the "most" pathological cases, the exceptional submanifolds which <u>can</u> affect the propagation of the discontinuities, are those n-1 dimensional exceptional submanifolds which are such that $C^{(\alpha_1)}$ has a different multiplicity on either side of the submanifolds. That $C^{(\alpha_1)}$ has a different multiplicity on either side of an n-1 dimensional submanifold, means that there is an α such that $C^{(\alpha_1)}$ satisfies (2.9) for this α on one side of the submanifold and not on the other. We shall call this type of n-1 dimensional exceptional submanifolds on $C^{(\alpha_1)}$ <u>multiplicitychange-manifolds</u>, and we shall now see that such manifolds really affect the propagation of discontinuities in general.

Consider now an isolated multiplicitychange-manifold on $C^{(\alpha_1)}$. We shall restrict our study to the case where the change in multi-

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that the multiplicity of $C^{(G_1)}$ is the same *) everywhere in a neighbourhood of the submanifold on $C^{(G_1)}$ (except on the submanifold itself), cannot affect the propagation of the discontinuities on $C^{(G_1)}$ because of the continuity properties the discontinuities necessarily must have. In particular the exceptional manifolds of dimension at most n-2 are all of this type. In exactly the same way we see that those pathological cases which involve a countable set of submanifolds of the above type, such that this set of submanifolds has a finite number of submanifolds as accumulation points, cannot affect the propagation of the discontinuities.

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*) We say that the multiplicity of a characteristic manifold is the same at two different points, if the characteristic manifold satisfies (2.9) for exactly the same indices q at the two points.

plicity of $C^{(\alpha_1)}$ is exactly one, i.e. we shall suppose that there is one and only one α such that $C^{(\alpha_1)}$ satisfies (2.9) for this α on one side of the multiplicitychange-manifold and not on the other. The general case is then an easy extension of this special case, and will be left to the reader. It is natural to divide the multiplicitychange-manifold into three disjoint sets, namely the sets where $C^{(\alpha_1)}$ 1) loses multiplicity, 2) gains multiplicity, 3) neither loses nor gains multiplicity. We define these concepts in the following way: consider a point on the multiplicitychange-manifold, and consider the bicharacteristic direction for increasing t associated with the equation (2.9) for the exceptional α at that point. If this bicharacteristic direction is tangent to the multiplicitychange-manifold, we say that $C^{(\alpha_1)}$ neither loses nor gains multiplicity at that point. If the bicharacteristic direction is pointing into the region of $C^{(\alpha_1)}$ where $C^{(\alpha_1)}$ satisfies (2.9) for the exceptional α, we say that $C^{(\alpha_1)}$ gains multiplicity at that point. Finally, if the bicharacteristic direction is pointing out of the region of $C^{(\alpha_1)}$ where $C^{(\alpha_1)}$ satisfies (2.9) for the exceptional α, we say that $C^{(\alpha_1)}$ loses multiplicity at that point.

To be able to get a finite process in the following construction, we are also here forced to exclude some "pathological" cases. Namely, we shall assume that locally each of the three types of sets defined above on the multiplicitychange-manifold consists of a finite number of connected sets. Then we may consider each of the three types of pointsets separately, and afterwards glue the results together.

Let now $E^{(\alpha)}$ denote the characteristic manifold which is going through the multiplicitychange-manifold, and which satisfies (2.9) for the exceptional α . From the assumptions we have made above, we see that $E^{(\alpha)}$ is identical with $C^{(\alpha_1)}$ on one side of the multiplicitychange-manifold, while $E^{(\alpha)}$ on the other side of the multiplicitychange-manifold satisfies (2.9) only for the exceptional α (at least locally) if we assume that $C^{(\alpha_1)}$ satisfies (2.9) for no more indices at the multiplicitychange-manifold than in a neighbourhood of this manifold.

From the discussion in section 2 we see that the characteristic manifold $E^{(\alpha)}$ is a possible carrier of discontinuities. In fact, if $C^{(\alpha_1)}$ loses multiplicity everywhere along the multiplicitychange-manifold, there will be a "branching" of the propagation of the discontinuities there; that part of the discontinuity which is associated with the eigenvectors $r^{\alpha\beta}$, for the exceptional α , will follow the characteristic manifold $E^{(\alpha)}$, while the rest will continue to follow $C^{(\alpha_1)}$. We know the transport equations everywhere on these manifolds except on the multiplicitychange-manifold, but there the continuity properties of the discontinuities solve the problem. On the other hand, if $C^{(\alpha_1)}$ gains multiplicity everywhere along the multiplicitychange-manifold, the opposite can happen. Namely, if both $C^{(\alpha_1)}$ and $E^{(\alpha)}$ carry discontinuities before they run together in the multiplicitychange-manifold, the discontinuities will propagate along a single manifold (namely $C^{(\alpha_1)}$ and $E^{(\alpha)}$ which are identical) on the other side of the multiplicitychange-manifold. Here also the transport equations are known everywhere except on the multiplicitychange-manifold, but there again the continuity solves the problem.

Let now $E^{(q)}$ denote the characteristic manifold which is going through the multiplicitychange-manifold, and which satisfies (2.9) for the exceptional q. From the assumptions we have made above, we see that $E^{(q)}$ is identical with $C^{(q_1)}$ on one side of the multiplicitychange-manifold, while $E^{(q)}$ on the other side of the multiplicitychange-manifold satisfies (2.9) only for the exceptional q (at least locally) if we assume that $C^{(q_1)}$ satisfies (2.9) for no more indices at the multiplicitychange-manifold than in a neighbourhood of this manifold.

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Finally, we consider what happens at the points on the multiplicitychange-manifold where $C^{(\alpha_1)}$ neither loses nor gains multiplicity. If these pointsets locally are contained in a finite set of n-2 dimensional submanifolds of $C^{(\alpha_1)}$, the continuity properties of the discontinuities solve the problem. If on the other hand the multiplicitychange-manifold everywhere is such that $C^{(\alpha_1)}$ neither loses nor gains multiplicity, then the situation is entirely different. In fact, $E^{(\alpha)}$ is then not uniquely determined by $C^{(\alpha_1)}$, but in view of the results found in section 3 we know that there is no coupling between the discontinuities carried by $E^{(\alpha)}$ on either side of the multiplicitychange-manifold, and no "information" is carried over this manifold on $E^{(\alpha)}$.

Using the above local results and the continuity properties of the discontinuities, it will in principle be possible to glue the results together, and find out how the discontinuities propagate up to the nearest caustic. In general caustics will exist due to the focussing effects (blow up of $\varphi^{o}_{x^i x^j}$), so the discontinuous solution will in general <u>not</u> exist globally as a piece-wise smooth function. Exceptions to this are the weakly coupled hyperbolic systems, since focussing phenomena do not occur for such systems. The focussing effect will be discussed in a later work on stability for hyperbolic systems.

Finally, we consider what happens at the points on the multiplicity-change-manifold where $C^{(a_1)}$ neither loses nor gains multiplicity. If these points are locally are contained in a finite set of n-2 dimensional submanifolds of $C^{(a_1)}$, the continuity properties of the discontinuities solve the problem. If on the other hand the multiplicity-change-manifold everywhere is such that $C^{(a_1)}$ neither loses nor gains multiplicity, then the situation is entirely different. In fact, $E^{(a)}$ is then not uniquely determined by $C^{(a_1)}$, but in view of the results found in section 3 we know that there is no coupling between the discontinuities carried by $E^{(a)}$ on either side of the multiplicity-change-manifold, and no "information" is carried over this manifold on $E^{(a)}$.

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5. SOME REMARKS.

In the previous sections we have assumed that Ω^α, $r^{\alpha\beta}$, $l^{\alpha\beta}$ are smooth for $\xi \neq 0$. As we mentioned in section 1, this assumption will not be satisfied in general. On the one hand the functions Ω^α may have branchpoints for vectors ξ other than $\xi = 0$, and on the other hand the eigenvectors $r^{\alpha\beta}$, $l^{\alpha\beta}$ may be discontinuous at points x,t,ξ where the multiplicity of a characteristic root changes, since the dimensions of the eigensubspaces change there. The system of equations in ideal magnetohydrodynamics is an example where such problems arise at the points on a characteristic surface where the magnetic field is either tangent or orthogonal to the surface.

The functions Ω^α are solutions of the algebraic equation (2.8) with smooth coefficients. Since we assume that the system of equations (1.1) is hyperbolic, the equation (2.8) belongs to a special class of algebraic equations. This author does not know whether there exist any theory for this class of equations with regard to solvability by root extraction, but in any case it seems likely that the smoothness-requirements etc. we have to impose on the functions Ω^α, essentially will limit our theory to the cases where it is possible to find the functions Ω^α by extraction of roots. Since the coefficients in the equation (2.8) as well as the roots Ω^α are all real, it seems likely that the expressions Ω^α involve square-roots only.

In view of the above, we limit ourselves to cases where the only singularities of the functions Ω^α are branchpoints for square-roots. Since the coefficients in (2.8) are smooth, we see that at least two different Ω^α's become equal at each such branch-

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In view of the above, we limit ourselves to cases where the only singularities of the functions Ω^α are branchpoints for square-roots. Since the coefficients in (2.8) are smooth, we see that at least two different Ω^α's become equal at each such branchpoint. Thus a branchpoint for $\xi \neq 0$ will always involve a change in multiplicity for the characteristics, hence such branchpoints cannot exist for hyperbolic systems with characteristics of constant multiplicity.

The subset of x,t,ξ-space where either $l^{\alpha\beta}$, $r^{\alpha\beta}$ are discontinuous or Ω^{α} have branchpoints, will in the following be referred to as critical points (note that $\xi = 0$ is always excluded). We shall for simplicity assume that the multiplicity for every Ω^{α} is the same in every connected set of critical points in x,t,ξ-space, and thus that $l^{\alpha\beta}$, $r^{\alpha\beta}$ are continuous on such sets. In general there may probably exist double critical points, i.e. points in the set of critical points where the multiplicity of Ω^{α} changes. The following discussion will essentially also cover such cases, because the critical points of different types are always separated on a characteristic surface by multiplicitychange-manifolds.

The continuity properties of the discontinuities are easily seen to imply that critical points cannot affect the propagation of discontinuities along a characteristic surface C, unless there is a domain on C where every point is critical. By the same arguments as those we used in the preceding section, we see that this domain may be divided by multiplicitychange-manifolds into a finite number of subdomains where the assumptions imposed on C on page 10 are satisfied. We shall now study what happens in each of these subdomains, the discussion of what happens at the multiplicitychange-manifolds will then be completely analoguous to the discussion in the preceding section and will therefore be left to the reader. Thus we consider a critical point x_o, t_o on C and point. Thus a branchpoint for $\xi \neq 0$ will always involve a change in multiplicity for the characteristics, hence such branchpoints cannot exist for hyperbolic systems with characteristics of constant multiplicity.

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In domains of critical points therefore, the system of transport equations (2.17) describe the propagation of discontinuities, but this system will in general <u>not</u> be weakly coupled any longer. In fact, in [5] it is shown that the system of transport equations is a strongly coupled hyperbolic system in special cases. At least when (1.1) is symmetric hyperbolic it is clear that the system of transport equations is hyperbolic, thus the initialvalue problem can be solved for it.

From the above considerations we can now conclude that apart from the fact that the transport equations may become a strongly coupled hyperbolic system, the critical points do not change the picture we have given in the earlier sections of the propagation of discontinuities, in any essential way. The qualitative properties for the propagation of discontinuities is described in section 4, while the quantitative properties are given by the system of transport equations (2.17) (which simplifies to (2.21) at noncritical points). Obviously the construction of a global picture may be very tedious in concrete problems, we shall not go into further details of this here.

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Finally, we would like to remark that the discussion in this paper seems to be fairly easy to modify to mixed boundary-initial-value problems for the hyperbolic system (1.1), in the same way as carried out in the cases treated in [3]. Furthermore higher order hyperbolic systems and semilinear hyperbolic systems seem to be fairly easy to study by the same methods which we have used in this paper. We would also like to remark that the discussion in section 3 makesit possible to generalize the WKB method to cover certain hyperbolic systems with characteristics of varying multiplicity. To a certain extent this will be studied in a later work on stability for hyperbolic systems.

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