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ON THE POSTAGE STAMP PROBLEM WITH THREE STAMP DENOMINATIONS, III

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The present paper is an immediate continuation of Selmer [7] and Selmer - Rödne [8]. All references to theorems and formulas from sections 1-13 are automatically to [7] or [8].

14. The sets of h_0 and $(h_0$ - 1)-representable numbers.

Let $A_k' = A_k \cup \{0\}$. The set (1.2) of h-representable numbers (at most h addends) may then in standard terminology be denoted by hA_k' . Our aim in the present section is to determine the sets h_0A_3' and $(h_0 - 1)A_3'$.

We shall rely heavily on the results in Rödseth [6], and use his notation, with one exception: He operates with an integer r, $0 \le r < a_3$. To avoid confusion with our use of r, we shall replace his r by ℓ .

Rödseth's Lemma 4 states that

$$t_{-\ell}^* = x_v(a_3 - 1) + y_v(a_3 - a_2), (x_v, y_v) \in X_v \cup Y_v$$
.

We consider the numbers (all $\equiv \ell \pmod{a_3}$):

$$(14.1) \quad (h_0 - t)a_3 - t_{-\ell}^* = (h_0 - t - x_v - y_v)a_3 + y_va_2 + x_v \ge 0 ,$$

and claim that these belong to $h_0A_3^*$ for $t\ge 0$. This is trivial if h_0 - t - x_V - $y_V\ge 0$, since the coefficient sum \sum = h_0 - $t\le h_0$. It remains to show that the set

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The present paper is an immediate continuation of Selmer [7] and Selmer - Rödne [8]. All references to theorems and formulas from sections 1-15 are automatically to [7] or [8].

14. The sets of ho- and (ho - 1)-representable numbers.

Let $A_k' = A_k \cup \{0\}$. The set (1.2) of h-representable numbers (at most h addends) may then in standard terminology be denoted by hAk. Our aim in the present section is to determine the sets $h_0A_k^2$ and $(h_0 - 1)A_k^4$.

We shall rely heavily on the results in Ködseth [6], and use his notation, with one exception: He operates with an integer r, $0 \le r < a_s$. To avoid confusion with our use of r, we shall replace his r by ℓ .

Rödseth's Lemma 4 states that

 $t_{2}^{*} = x_{V}(a_{3} - 1) + y_{V}(a_{3} - a_{2}), (x_{V}, y_{V}) \in X_{V} \cup Y_{V}$

We consider the numbers (all # % (mod ag)):

 $(14.1) \quad (h_0 - t)a_3 - t = (h_0 - t - x_0 - y_0)a_3 + y_0a_2 + x_0 \ge 0$

and claim that these belong to $h_0A_3^1$ for t 2 0 . This is trivial if $h_0 - t - x_y - y_y \ge 0$, since the coefficient sum $\sum = h_0 - t \le h_0$. It remains to show that the set

$$S_{\ell} = \{\ell, \ \ell + a_3, \ \ell + 2a_3, \ \dots, \ y_{v}a_2 + x_{v} - a_3\} \subset h_0A_3' \ .$$

And this is proved by Rödseth, since S_{ℓ} is just the sequence (4.1) of [6].

On the other hand, the numbers (14.1) do <u>not</u> belong to $h_0A_3^*$ if $t=-t^*<0$. Assume to the contrary that

$$(h_0 + t')a_3 - t_{-\ell}^* = x_3a_3 + x_2a_2 + x_1, \sum x_i = h' \le h_0$$
.

As in section 3, we conclude that

$$t_{-\ell}^* - t'a_3 = (h_0 - h')a_3 + x_1(a_3 - 1) + x_2(a_3 - a_2)$$

has a representation by $\bar{A}_3 = \{a_3 - a_2, a_3 - 1, a_3\}$, cf. (2.15). (Rödseth uses $A_3^* = \bar{A}_3 \cup \{0\}$.) But this is a contradiction, since $t_{-\ell}^*$ is defined as the smallest integer in its residue class (mod a_3) with a representation by \bar{A}_3 .

Letting $(x_V^{},y_V^{})$ run through all lattice points of $X_V^{}\cup Y_V^{}$, we get all residue classes & (mod $a_3^{})$, and have the following

THEOREM 14.1.

$$h_0 A_3^{!} = \bigcup_{\substack{(x_V, y_V) \in X_V \cup Y_V}} \{(h_0 - t - x_V - y_V)a_3 + y_V a_2 + x_V \ge 0, t = 0, 1, ...\} .$$

For use in the next section, we shall also determine $(h_0 - 1)A_3^{\prime}$. Clearly

$$(h_0 - 1)A_3^{\bullet} \subset h_0A_3^{\bullet} - a_3 = \{n - a_3 \mid n \in h_0A_3^{\bullet}\} \ .$$

If A_3 is <u>pleasant</u>, it suffices to use regular representations, and clearly

 $S_{g} = (2, 2 + a_{g}, 2 + 2a_{g}, ..., y_{g} + x_{y} - a_{g}) \in h_{0}\Lambda_{g}^{1}$.

And this is proved by Rödseth, since S, is just the sequence (4.1) of [6].

On the other hand, the numbers (14.1) do not belong to $h_0A_3^1$ if $t=-t^1<0$. Assume to the contrary that

 $(h_0 + t)a_3 - t_2 = x_3a_3 + x_2a_2 + x_1, \Sigma x_1 = h' \le h_0$.

As in section 3, we conclude that

 $t_{2g}^{x} - t_{2g}^{y} = (h_{0} - h^{y})a_{3} + x_{1}(a_{3} - 1) + x_{2}(a_{3} - a_{2})$

has a representation by $\bar{\Lambda}_3' = (a_3 - a_2, a_3 - 1, a_3)$, cf. (2.15). (Rödseth uses $\Lambda_3' = \bar{\Lambda}_3$ V [0] .) But this is a contradiction, since to defined as the smallest integer in its residue class (mod ag) with a representation by $\bar{\Lambda}_3$.

Letting (x_y, y_y) run through all lattice points of $x_y \cup Y_y$ we get all residue classes 4 (mod a_3), and have the following

THEOREM 14.1.

 $h_0A_3^2 = U \quad (0h_0 - t - x_y - y_y)a_3 + y_ya_2 + x_y \ge 0, t = 0, 1, ...)$. $(x_y, y_y) \in X_y \cup Y_y$

For use in the next section, we shall also determine $(h_0 - 1)A_3^2$ Clearly

 $(h_0 - 1)A_3^1 = h_0A_3^1 - a_3 = \{n - a_3 \mid n \in h_0A_3^1\}$.

If A₃ is pleasant, it suffices to use regular representations,
and clearly

$$(h_0 - 1)A_3^{\dagger} = (h_0A_3^{\dagger} - a_3) \cap \mathbb{N}_0$$

(where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$). For non-pleasant \mathbb{A}_3 , however, we get problems with the number \mathbb{n}_0 of (11.13):

$$(14.2) \quad \mathbf{n}_0 = \mathbf{a}_3 - \mathbf{r} - 1 = (\mathbf{f} - 1)\mathbf{a}_2 + \mathbf{a}_2 - 1 = \mathbf{n}_{\mathbf{h}_0 - 1}(\mathbf{A}_3) + 1 \notin (\mathbf{h}_0 - 1)\mathbf{A}_3' ,$$

where $n_0 + a_3 = 2fa_2 + r - 1 \in h_0A_3'$, since $1 \le r \le a_2 - f - 1$ by (4.3). (For pleasant A_3 , it follows from (2.8) that $n_0 + a_3 = n_{h_0}(A_3) + 1 \notin h_0A_3'$.)

We shall show that n_0 is usually the <u>only</u> exception:

THEOREM 14.2. For A_3 non-pleasant, with $r \neq 1$ and $s \neq q$, we have

$$(14.3) \qquad (h_0 - 1)A_3' = (h_0 A_3' - a_3) \cap \mathbb{N}_0 \setminus \{a_3 - r - 1\} .$$

To prove this, we replace h_0 by h_0 - 1 in the arguments leading to Theorem 14.1. The only critical point is whether now $S_{\ell} \subset (h_0 - 1)A_3^{\prime} \ .$

To show that $S_{\ell} \subset h_0 A_3$, Rödseth used his Lemma 5, which states that for $1 \le i \le v$, we have

(14.4)
$$x_{i-1} + y_{i-1} + Q_i - 1 \le h_0$$
 if $P_i \le s_i$

(14.5)
$$x_i + y_i + R_i - 1 \le h_0 \quad \text{if } P_i > s_i$$

If these relations hold with <u>strict inequalities</u>, it follows that $S_{\ell} \subset (h_0 - 1)A_3^{\prime}$.

We note that Rödseth's division algorithm for a_3/a_2 is the same as the one leading to our Theorem 6.1. In particular, we have

(ho - 1)As = (hoAs - as) n No

(where $N_0 = (0, 1, 2, ...)$). For non-pleasant A_3 , however, we get problems with the number n_0 of (11.13):

 $(14.2) \quad n_0 = a_3 - \tau - 1 = (f - 1)a_2 + a_2 - 1 = n_{n_0 - 1}(A_5) + 1 \notin (n_0 - 1)A_5^2 ,$

where $n_0 + a_3 = 2fa_2 + r - 1 \in h_0 A_3^2$, since $1 \le r \le a_2 - f + 1$ by (4.5). (For pleasant A_5 , it follows from (2.8) that $n_0 + a_5 = n_h (A_5) + 1 \notin h_0 A_3^4$.)

We shall show that no is usually the only exception:

THEOREM 14.2. For As non-pleasant, with r + 1 and s + q ,

we have

(14.3) $(h_0 - 1)A_3 = (h_0A_3 - a_3) \cap N_0 \times \{a_3 - x - 1\}$

To prove this, we replace h_0 by h_0 - 1 in the arguments leading to Theorem 14.1. The only critical point is whether now $S_* \subset (h_* - 1) A L$.

To show that $S_g \subset h_0 A_3^1$, Rödseth used his Lemma S, which states that for 1 s i s v , we have

 $x_{i-1} + y_{i-1} + Q_i - 1 \le h_0$ if $P_i \le s_i$

(14.5) $x_1 + y_2 + R_1 - 1 \le h_0$ if $P_1 > s_1$

If these relations hold with strict inequalities, it follows that $S_{\rm s} = (h_{\rm o} - 1) A_{\rm o}^4$.

We note that Rödseth's division algorithm for a_3/a_2 is the same as the one leading to our Theorem 6.1. In particular, we have

 $a_3 = q_1 a_2 - s_1$, hence $q_1 = q$, $s_1 = s$, and v > 0 for a non-pleasant A_3 , when $s \ge q$ by (2.10).

Studying Rödseth's proof of his Lemma 5, we observe the following facts:

1) For i = 1, when $P_1 = q_1 \le s_1$, we have equality in (14.4) if and only if (x_0, y_0) is the upper right corner of Y_0 :

$$(14.6) (x0, y0) = (s0 - 1, P1 - P0 - 1) = (a2 - 1, f - 1).$$

Then $y_0 a_2 + x_0$ is just the number n_0 of (14.2).

2) For i > 1, hence $Q_i > 1$, a necessary condition for equality in (14.4) or (14.5) is $s_i = s_{i-1} - 1$ or $s_{i+1} = s_i - 1$, respectively. But then such a relation must hold from the start:

$$s = s_1 = s_0 - 1 = a_2 - 1$$
, hence $r = 1$

(cf. the recurrence relation $s_{j+1} = q_{j+1}s_j - s_{j-1}, q_{j+1} \ge 2$). If $r \ne 1$, we thus have strict inequalities in (14.4-5) for all i > 1.

In Rödseth's proof of $S_\ell < h_0 A_3^*$, he divides S_ℓ into "subsequences" between $y_{i-1} a_2 + x_{i-1}$ and

$$y_i a_2 + x_i = y_{i-1} a_2 + x_{i-1} + Q_i \left[\frac{x_{i-1}}{s_i} \right] a_3$$
.

We have noted that the case i=1 needs a special treatment. Since $s_1=s$, $Q_1=1$, we must consider the numbers $za_3+y_0a_2+x_0$, $0 \le z < [x_0/s]$. Using the "a_3-transfer" $a_3=qa_2-s$ of section 11, this may be written as

(14.7)
$$za_3 + y_0a_2 + x_0 = (y_0 + zq)a_2 + x_0 - zs$$
,

with positive constant term, and a coefficient sum

$$\sum = x_0 + y_0 - z(s - q) \le x_0 + y_0.$$

 $a_3 = q_1 a_2 - s_1$, hence $q_1 = q_1 s_1 = s_1$, and $q_2 = s_2$, and $q_3 = s_3$, when $q_4 = s_2$ and $q_5 = s_3$, and $q_5 = s_4$, when $q_5 = s_5$ and $q_5 = s_5$, and $q_5 = s_5$.

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 $(14.6) = (x_0, y_0) = (s_0 - 1, p_1 - p_0 - 1) = (a_2 - 1, f - 1)$

Then $y_0a_2 + x_0$ is just the number n_0 of (14.2).

2) For 1 > 1, hence $Q_1 > 1$, a necessary condition for equality in (14.4) or (14.5) is $s_1 = s_{1-1} - 1$ or $s_{1+1} = s_1 - 1$ respectively. But then such a relation must hold from the start:

 $s = s_1 = s_0 - 1 = a_2 - 1$, hence r = 1

(cf. the recurrence relation $s_{j+1} = q_{j+1}s_j = s_{j-1}$, $q_{j+1} \ge 2$). If r + 1, we thus have strict inequalities in (14.4-5) for all i > 1. In Rödseth's proof of $S_c \subset h_0 A_s^1$, he divides S_c into "subsequences" between y_s , $a_s + x_s$, and

 $y_1a_2 + x_1 = y_{1-1}a_2 + x_{1-1} + Q_1\left[\frac{x_{1-1}}{s_1}\right]a_3$

We have noted that the case i=1 needs a special treatment. Since $s_1=s$, $Q_1=1$, we must consider the numbers za_5 , $y_0a_2+x_0$, $0 \le z < [x_0/s]$. Using the "a₅-transfer" $a_5=qa_2-s$ of section 11, this may be written as

 $(14.7) za_3 + y_0a_2 + x_0 = (y_0 + zq)a_2 + x_0 - zs.$

with positive constant term, and a coefficient sum

 $x_0 + y_0 - z(s - q) \le x_0 + y_0$

If $x_0 + y_0 < h_0$, then also $\sum < h_0$ for all z. If $x_0 + y_0 = h_0$, corresponding to the corner (14.6), then $\sum < h_0$ for z > 0 if s > q, but $\sum = h_0$ for all z when s = q.

If s = q , then v = 1 by Theorem 7.1, and the "subsequence" just completed covers the whole of S_{ℓ} . If v > 1 , we have seen that the remaining subsequences yield no problems if r \pm 1 .

This completes the proof of (14.3), and also shows that if s = q, then

$$(14.8) \quad (h_0 - 1)A_3' = (h_0 A_3' - a_3) \cap \mathbb{N}_0 \setminus \left\{ ta_3 - r - 1 \mid t = 1, 2, \dots, \left[\frac{a_2 - 1}{s} \right] \right\} .$$

Here $ta_3 - r - 1 = n_0 + (t - 1)a_3 = n_0 + za_3$, with $0 \le z < [x_0/s] = [(a_2 - 1)/s]$. Note that we may use also $z = [x_0/s]$ in (14.7), but the resulting number is then contained in $h_0A_3^{\dagger}$ but not in $h_0A_3^{\dagger} - a_3$.

We finally treat the case r=1. A modification of Rödseth's method then seems to become rather complicated. However, we can settle the case directly by a more elementary application of a_3 -transfers. With r=1, the only such transfers which may reduce the coefficient sum are of the form

(14.9)
$$ea_3 = (ef + 1)a_2 - (a_2 - e), e = 1, 2, ...$$

As in section 11, we start with the <u>regular</u> representations

(14.10)
$$n = e_3 a_3 + e_2 a_2 + e_1, e_1 \le a_2 - 1, e_2 \le f - 1.$$

For r = 1, it is unnecessary to consider $e_2 = f$, since already $fa_2 + 1$ gives a new a_3 .

If $x_0 + y_0 < h_0$, then also $\sum h_0$ for all $z - k_0 + y_0 = h_0$ corresponding to the corner (14.6), then $\sum h_0$ for z > 0 if s > 0, but $\sum h_0$ for all z when s = 0.

If s=q, then v=1 by Theorem 7.1, and the "subsequence" just completed covers the whole of S_{χ} . If v>1, we have seen that the remaining subsequences yield no problems if $\tau+1$. This completes the proof of (14.3), and also shows that if

s = q , then

 $(14.8) \quad (n_0 - 1)A_3^1 = (n_0 A_3^1 - a_3) \quad (n_0 - \{n_3 - r - 1 \mid r = 1, 2, \dots, \left[\frac{n_2 - 1}{3}\right]\}$

Here $\tan z - r - 1 = n_0 + (t - 1)a_3 = n_0 + aa_3$, with $0 \le z < (x_0/s] = [(a_2 - 1)/s]$. Note that we may use also $z = (x_0/s)$ in (14.7), but the resulting number is then contained in $h_0 A_3^4$ but not in $h_0 A_3^4 - a_3$.

We finally treat the case r = 1. A modification of Hödseth's method then seems to become rather complicated. However, we can settle the case directly by a more elementary application of a_2 -transfers. With r = 1, the only such transfers which may reduce the coefficient sum are of the form

(14.9) es, = (ef + 1)a, - (a, - e), e = 1, 2, ...

As in section 11, we start with the regular representations

(14.10) $n = e_1 a_2 + e_2 a_3 + e_4$, $e_1 \le a_2 - 1$, $e_2 \le f - 1$.

For r=1, it is unnecessary to consider $e_2=f$, since already fa_2+1 gives a new a_2 .

For the n of (14.10), we shall decide if $n \in h_0A_3'$. If $\sum_e = \sum_i e_i \le h_0$, we are finished. If $\sum_e > h_0$, we must try a transfer (14.9) with $e \le e_3$. The transfer is possible only if it yields a non-negative constant term, that is, if $e_1 \ge a_2 - e$.

Similarly, we shall decide if $n' \in (h_0 - 1)A_3'$, where

(14.11)
$$n' = n - a_3 = (e_3 - 1)a_3 + e_2a_2 + e_1 \qquad (e_3 > 0)$$
,

with $\Sigma_e' = \Sigma_e - 1$, hence no problem if $\Sigma_e \le h_0$. If an a_3 -transfer (14.9) is necessary and possible in (14.10), and yields a new $\Sigma \le h_0$, then the <u>same</u> transfer gives $\Sigma' \le h_0 - 1$ in (14.11), provided it is possible, that is, if $e \le e_3 - 1$. It is easily seen that this combination of conditions <u>fails</u> only in the case

(14.12)
$$n = e_3 a_3 + (f - 1)a_2 + a_2 - e_3, \sum = h_0 + 1$$
.

Thus $n' = n - a_3 \notin (h_0 - 1)A_3'$ if $n' = e_3 a_3 - e_3 - 1$. For the n of (14.12), we must use $e = e_3$ in (14.9), and get $n = (e_3 + 1)fa_2$, hence

$$n \in h_0 A_3 \Leftrightarrow (e_3 + 1) f \le h_0 = a_2 + f - 2 \Leftrightarrow e_3 \le \left\lceil \frac{a_2 - 2}{f} \right\rceil$$

We have thus shown that if r = 1, then

$$(14.13) \quad (h_0 - 1)A_3' = (h_0 A_3' - a_3) \cap \mathbb{N}_0 \setminus \left\{ t(a_3 - 1) - 1 \mid t = 1, 2, \dots, \left[\frac{a_2 - 2}{f} \right] \right\}.$$

For t = 1, we get $t(a_3 - 1) - 1 = a_3 - 2 = n_0$.

No problems arise if we have s=q and r=1 simultaneously. Then $s=q=a_2-1$, $f=q-1=a_2-2$, and the "subtrahends" {} in (14.8) and (14.13) both consist of n_0 only.

For the n of (14,10), we shall decide if $n \in h_0 \Lambda_2^1$. If $\sum_e = \sum_{i=1}^n \sum_{j=1}^n \sum_{j=1}^n \sum_{i=1}^n \sum_{j=1}^n \sum_{j=1}$

Similarly, we shall decide if $n' \in (h_0 - 1)\Lambda_+^1$, where

(14.11) $n' = n - a_3 = (e_3 - 1)a_3 + e_2a_2 + e_1 - (e_3 > 0)$,

with $\sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n}$

(14.12) $n = e_3 a_3 + (f - 1)a_2 + a_2 - e_3$, $\Sigma = h_0 + 1$.

Thus n' = n - ag (On - 1)A; if n' = egag - eg - 1 ,

For the n of (14.12), we must use $e = e_3$ in (14.9), and get $n = (e_3 + 1)fa_3$, hence

 $n \in h_0 A_3 \Leftrightarrow (e_3 + 1)f \leq h_0 = a_2 + f - 2 \Leftrightarrow e_3 \leq \left[\frac{a_2 - 2}{f}\right]$

We have thus shown that if r = 1 , then

For t = 1, we get t(ay - 1) - 1 = ay - 2 = ng .

No problems arise if we have s=q and r=1 simultaneously. Then $s=q=a_2-1$, $f=q-1=a_2-2$, and the "subtrahends" i) in (14.8) and (14.13) both consist of n_0 only.

The results (14.3), (14.8) and (14.13) imply that, but for the specified exceptions with t > 1 for r = 1 or s = q, the integers $\stackrel{\geq}{=} a_3$ with a representation in at most h_0 addends from A_3 have such a representation containing a_3 .

In particular, [0, $n_{h_0}(A_3)$] $\subset h_0A_3^*$. It then follows from (14.3) that

$$(14.14) r + 1, s + q \Rightarrow [0, n_{h_0}(A_3) - a_3] \setminus \{a_3 - r - 1\} \subset (h_0 - 1)A_3^{\dagger}.$$

This was first observed numerically for a large number of bases \mbox{A}_3 , and gave the impetus for the investigations in this section.

As in Rödseth [6], let $\Lambda(n)$ denote the number of addends in a \min representation of n by a given basis A_k . We clearly have

$$\Lambda \left(n_h(A_k) - (x+1)a_k + 1 \right) \ge h - x, \Lambda \left(n_h(A_k) - xa_k \right) \ge h - x,$$

since otherwise addition of $(x + 1)a_k$ or xa_k would yield a contradiction. This raises the question whether there are integers x > 0 such that for the interval of length a_k :

We have just seen that this holds with x=1 if k=3, $h=h_0$, A_3 non-pleasant, $r\neq 1$, $s\neq q$. Already for x=2, however, it is easy to find counterexamples:

$$A_3 = \{1, 7, 11\}, h_0 = 6, n_6(A_3) = 48; \Lambda(17) = 5.$$

We have made the interesting observation that for <u>Frobenius-dependent</u> A_3 with r>1, (14.15) holds also with larger x:

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In particular, 10, $n_{h_0}(A_3)$ | $n_{h_0}(A_3)$. It then follows from (14.3) that

(14.14) r + 1, s + q + 10, $n_0 (A_5) - a_5 > (a_5 - r - 1) = (h_0 - 1)A_5$.

This was first observed numerically for a large number of bases λ_3 , and gave the impetus for the investigations in this section.

As in Rödseth [6], let $\Lambda(n)$ denote the number of addends in a minimal representation of n by a given basis Λ_K . We clearly have

 $-\Lambda(n_h(A_h) - (x + 1)a_h + 1) \ge h - x, \Lambda(n_h(A_h) - xa_h) \ge h - x$,

since otherwise addition of $(x + 1)a_{\chi}$ or xa_{χ} would yield a contradiction. This raises the question whether there are integers x > 0 such that for the interval of length a_{χ} :

(14.15) $[n_h(A_k) - (x + 1)a_k + 1, n_h(A_k) - xa_k] \in (n - x)A_k$

We have just seen that this holds with x = 1 if k = 3, $h = h_0$, A_1 non-pleasant, $r \neq 1$, $s \neq q$. Already for x = 2, however, it is easy to find counterexamples:

 $A_{3} = \{1, 7, 11\}, h_{0} = 6, h_{0}(A_{2}) = 48 \ ; h(17) = 5 \ .$

We have made the interesting observation that for <u>Frobenius</u>-dependent A, with r > 1, (14.15) holds also with larger x;

PROPOSITION 14.1. Let A_3 be Frobenius-dependent, with r > 1. In the notation (5.8), let

$$(p-1)a_2 \le n \le n_{h_0}(A_3), x = \left[\frac{n_{h_0}(A_3) - n}{a_3}\right].$$

Then

$$n \in (h_0 - x)A_3^{\dagger}.$$

A proof will be published elsewhere.

15. The cases with $n_h(A_4) = n_h(A_3)$.

In (3.3), we raised the question of <u>basis extensions</u> which do not increase the h-range . We shall solve this question completely in the case

(15.1)
$$n_h(A_4) = n_h(A_3 \cup \{a_4\}) = n_h(A_3), a_4 > a_3$$

Even if $\,{\rm A}_4^{}\,\,$ enters the formulation, the results depend entirely on the properties of $\,{\rm A}_3^{}\,\,.$

We see from (3.4) that the $\underline{regular}$ h-range g_h always increases by a basis extension (assuming admissible bases). The same argument shows that if A_3 is pleasant, then

$$n_h(A_4) \ge g_h(A_4) > g_h(A_3) = n_h(A_3)$$
,

so that we may assume non-pleasant A_3 in (15.1).

If $a_4 > n_{h_0}(A_3) + 1$, then A_4 is <u>not admissible</u> for $h = h_0$ (where $h_0 = a_2 + f - 2$ refers to A_3). If then $h = h_0' > h_0$ is the smallest h for which A_4 is admissible, we trivially have

PROPOSITION 14.1. Let A, be Probenius-dependent, with r > 1
(n the notation (5.8), let

$$(p-1)a_2 \le n \le n_{B_0}(N_3), x = \left[\frac{n_{b_0}(A_3) - n}{a_3}\right]$$

Then

$$n \in (h_0 - x)A_3^t$$

A proof will be published elsewhere.

15. The cases with $n_h(A_4) = n_h(A_3)$

In (3.3), we raised the question of basis extensions which de not increase the h-range. We shall solve this question completely in the case

(15.1)
$$n_h(A_d) = n_h(A_7 \cup \{a_d\}) = n_h(A_7)$$
, $a_d > a_3$.

Even if A_4 enters the formulation, the results depend entirely on the properties of A_4 .

We see from (5.4) that the regular h-range gh always increases by a basis extension (assuming admissible bases). The same argument shows that if A, is pleasant, then

$$n_{\rm H}(A_4) \ge g_{\rm H}(A_4) > g_{\rm H}(A_3) = n_{\rm H}(A_3)$$

so that we may assume non-pleasant A, in (15.1).

If $a_4 > n_{h_0}(A_5) + 1$, then A_4 is not admissible for $h = h_0$ (where $h_0 = a_2 + f - 2$ refers to A_5). If then, $h = h_0^* > h_0$ is the smallest h for which A_4 is admissible, we trivially have

 $n_h(A_4) = n_h(A_3)$ for $h < h_0$. On the other hand, it follows from (2.14) that

$$n_{h_0^{\dagger}}(A_4) \ge a_4 + n_{h_0^{\dagger}-1}(A_3) = a_4 + n_{h_0^{\dagger}}(A_3) - a_3 > n_{h_0^{\dagger}}(A_3)$$
.

Similarly, it follows from from (2.13-14) that

$$n_{h'}(A_4) \ge n_{h'}(A_3), h' \ge h'_0 \Rightarrow n_{h}(A_4) > n_{h}(A_3), h > h'$$
.

We may therefore restrict the problem (15.1) to the case

(15.2)
$$n_{h_0}(A_4) = n_{h_0}(A_3), a_3 < a_4 \le n_{h_0}(A_3) + 1.$$

Note that a similar simplification does not apply to larger bases, since the analogue of (2.14) does not necessarily hold for $k\,>\,3$.

We already know one case of (15.2), resulting from the basis ${\rm A}_{h+2}$ of section 3:

$$(15.3) \quad a_2 = h_0 + 1, \ a_3 = h_0 + 2, \ a_4 = \alpha a_2 + a_3, \ 1 \le \alpha \le h_0 - 1.$$

To solve the general problem, we note that

$$n_{h_0}(A_4) = n_{h_0}(A_3) \Leftrightarrow n_{h_0}(A_3) + 1 \notin h_0A_4^*$$

$$(15.4) \Leftrightarrow n_{h_0}(A_3) + 1 - \delta a_4 \notin (h_0 - \delta)A_3', \delta = 1, 2, ..., h_0.$$

In most cases, it suffices to consider $\delta = 1$. Since

$$N = n_{h_0}(A_3) + 1 - a_4 \in [0, n_{h_0}(A_3) - a_3] \subset (h_0A_3' - a_3) \cap \mathbb{N}_0,$$

(15.4) <u>fails</u> already for δ = 1 if N does not belong to the exceptions in (14.3), (14.8) or (14.13). These cases have the

 $n_h(A_4) = n_h(A_3)$ for $h < h_0^4$. On the other hand, it follows from (2.14) that

 $n_{h_0^{\prime}}(A_4) \ge a_4 + n_{h_0^{\prime}-1}(A_3) = a_4 + n_{h_0^{\prime}}(A_3) - a_5 > n_{h_0^{\prime}}(A_3)$

Similarly, it follows from from (2.13-14) that

 $n_h(\Lambda_A) \ge n_h(\Lambda_S)$, $h' \ge h_0' \Rightarrow n_h(\Lambda_A) > n_h(\Lambda_S)$, h > h'.

We may therefore restrict the problem (15.1) to the case

(15.2) $n_{h_0}(A_4) = n_{h_0}(A_3)$, $a_3 < a_4 \le n_{h_0}(A_3) + 1$

Note that a similar simplification does not apply to larger bases; since the analogue of (2.14) does not necessarily hold for

-We already know one case of (15.2), resulting from the basis An+2 of section 5:

(15.3) $a_2 = h_0 + 1$, $a_3 = h_0 + 2$, $a_4 = aa_2 + a_3$, $1 \le \alpha \le h_0 = 1$

To solve the general problem, we note that

 $n_{h_0}(\Lambda_4) = n_{h_0}(\Lambda_3) \leftrightarrow n_{h_0}(\Lambda_3) \leftrightarrow 1 \notin h_0\Lambda_4$

(15.4) $\approx n_{h_0}(A_3) + 1 - \delta a_4 \notin (h_0 - \delta) A_3^1, \delta = 1, 2, ..., h_0$

In most cases, it suffices to consider 6 = 1 . Since

 $n_0(A_3) + 1 - a_4 \in (0, n_{h_0}(A_3) + a_3) = (h_0A_3^2 - a_3) \cap N_0$

(15.4) fails already for 8 = 1 if N does not belong to the exceptions in (14.3), (14.8) or (14.13). These cases have the

common exception n_0 of (14.2), and $N = n_0$ does in fact lead to a general solution of (15.2):

$$(15.5) \quad a_4 = \hat{a}_4 = n_{h_0}(A_3) - a_3 + r + 2 = n_{h_0}(A_3) - n_{h_0-1}(A_3) \Rightarrow n_{h_0}(A_4) = n_{h_0}(A_3) .$$

This is clear since we cannot use $\delta \ge 2$ in (15.4):

$$2\hat{a}_4 > n_{h_0}(A_3) + 1 \Leftrightarrow n_{h_0}(A_3) > 2a_3 - 2r - 3$$
,

which always holds by (2.8). - Note that $\hat{a}_4 = a_3$ if A_3 is pleasant.

If $a_4 \neq \hat{a}_4$, a necessary condition for (15.2) is that N equals one of the exceptions in (14.8) or (14.13), with t > 1 (since t = 1 corresponds to n_0).

We start with (14.13), hence r = 1. Then $n_{h_0}(A_3)$ is given by (2.28), and we find that we must choose

(15.6)
$$a_4 = a_3 + \tau(a_3 - 1), \tau = 1, 2, ..., \left[\frac{a_2 - 2}{f}\right] - 1$$

(while $\tau = [(a_2 - 2)/f]$ corresponds to \hat{a}_4). We shall see that this is also sufficient for (15.2) to hold.

We consider a representation

$$(15.7) n_{h_0}(A_3) + 1 = x_4 a_4 + x_3 a_3 + x_2 a_2 + x_1,$$

and must show that $\sum x_i > h_0$. This is trivial if $x_4 = 0$, so we can assume $x_4 > 0$, and observe that

$$n_{h_0}(A_3) + 1 \equiv 0$$
, $a_4 \equiv a_3 \equiv 1 \pmod{a_3 - 1 = fa_2}$.

With $x_2 = \kappa f + x_2'$, $0 \le x_2' < f$, (15.7) then gives

common exception n_0 of (14.2), and $N=n_0$ does in fact lead to a general solution of (15.2):

(15.5) $\mathbf{a}_4 = \mathbf{a}_4 = n_{h_0}(\Lambda_3) - \mathbf{a}_3 + \mathbf{r} + 2 = n_{h_0}(\Lambda_3) - n_{h_0-1}(\Lambda_3) - n_{h_0}(\Lambda_4) = n_{h_0}(\Lambda_3)$

This is clear since we cannot use 6 2 2 in (15.4):

 $2\alpha_4 > n_{h_0}(\Lambda_3) + 1 \approx n_{h_0}(\Lambda_3) > 2\alpha_3 - 2\tau - 3$,

which always holds by (2.8). - Note that $a_4 = a_5$ if A_5 is pleasant.

If $a_4 \neq \hat{a}_4$, a necessary condition for (15.2) is that N equals one of the exceptions in (14.8) or (14.15), with t > 1 (since t = 1 corresponds to n_0).

We start with (14.13), hence r=1. Then $n_{\rm h}(\Lambda_3)$ is given by (2.28), and we find that we must choose

(15.6) $a_4 = a_5 + \tau(a_5 - 1), \tau = 1, 2, \dots, \left[\frac{a_2 - 2}{t}\right] = 1$

(while $.\tau = \lfloor (a_2 - 2)/f \rfloor$ corresponds to a_4). We shall see that this is also sufficient for (15.2) to hold.

We consider a representation

(15.7) $n_{h_0}(A_3) + 1 = x_4 a_4 + x_5 a_5 + x_2 a_2 + x_4$

and must show that $\sum x_1 > h_0$. This is trivial if $x_4 \neq 0$, so we can assume $x_4 > 0$, and observe that

 $n_{h_0}(A_3) + 1 \equiv 0$, $a_4 \equiv a_5 \equiv 1$ (mod $a_5 - 1 = fa_2$)

With $x_2 = \kappa f + x_2^2$, $0 \le x_2^2 < f$, (15.7) then gives

 $x_4 + x_3 + x_1 \equiv (f - x_2') a_2 \text{ , hence } x_4 + x_3 + x_1 \geq (f - x_2') a_2$ $x_4 + x_3 + x_2 + x_1 \geq x_4 + x_3 + x_2' + x_1 \geq (f - x_2') a_2 + x_2' \geq a_2 + f - 1 = h_0 + 1 \text{ ,}$ as required. - In particular, we get the known case (15.3) from (15.5-6) with f = 1 .

We next consider (14.8), hence s=q, $a_3=q(a_2-1)$. By (2.29), we now have two possibilities for $n_{h_0}(A_3)$:

$$n_{h_0}(A_3) = \left(\left[\frac{a_2-1}{s}\right]+2\right)a_3-r-\left\{\begin{array}{c} 2 \text{, if } s \nmid (a_2-1) \\ 3 \text{, if } s \mid (a_2-1) \end{array}\right.$$

These two cases must be considered separately.

If $s \nmid (a_2 - 1)$, we find that we must choose

(15.8)
$$a_4 = (\tau + 1)a_3, \quad \tau = 1, 2, \dots, \left[\frac{a_2 - 1}{s}\right] - 1$$

(while $\tau = [(a_2 - 1)/s]$ corresponds to \hat{a}_4). Again, this is also sufficient for (15.2) to hold:

We consider a representation (15.7). Since $a_3 \mid a_4$, we get

$$x_2 a_2 + x_1 \equiv n_{h_0} (A_3) + 1 \equiv -r - 1 = -a_2 + f \pmod{a_3} = q(a_2 - 1)$$
,

from which we draw two conclusions:

1)
$$x_2 a_2 + x_1 \ge a_3 - r - 1$$

2)
$$x_2 a_2 + x_1 \equiv x_2 + x_1 \equiv f - 1 \pmod{a_2 - 1}$$
.

Assuming $\sum x_i \le h_0$ in (15.7), hence $x_4 > 0$, we get $x_2 + x_1 < h_0 = (f-1) + (a_2-1)$, so $x_2 + x_1 = f-1$, and

 $x_4 + x_5 + x_1 \equiv (f - x_2^*) a_2$, hence $x_4 + x_5 + x_1 \ge (f - x_2^*) a_2$

 $x_4 + x_3 + x_2 + x_1 \ge x_4 + x_3 + x_2^2 + x_1 \ge (f - x_2^2)a_2 + x_2^2 \ge a_2 + f - 1 = h_0 + 1$,

as required. - In particular, we get the known case (15.5) from (15.5-6) with f=1.

We next consider (14.8), hence s=q , $a_3=q(a_2-1)$. By (2.29), we now have two possibilities for $n_{h_0}(A_3)$:

$$n_{\text{b_0}}(\Lambda_3) = (\left[\frac{a_2-1}{-s}\right] + 2)a_3 - c - \left\{\begin{array}{ccc} 2 & \text{if } s \nmid (a_2-1) \\ 3 & \text{if } s \mid (a_3-1) \end{array}\right]$$

These two cases must be considered separately.

(15.8)
$$a_1 = (\tau + 1)a_3$$
, $\tau = 1, 2, \dots, \left\lceil \frac{a_2 - 1}{s} \right\rceil - 1$

(while $t = l(a_2 - 1)/s$) corresponds to \hat{a}_3). Again, this is also sufficient for (15.2) to hold:

We consider a representation (15.7). Since at | ad , we get

 $x_2 a_2 + x_1 = n_{10}(A_3) + 1 = -x - 1 = -a_2 + f \pmod{a_3} = q(a_2 - 1))$,

from which we draw two conclusions:

1) x2a2+x1 2.a3-x-1

2) $x_2a_2 + x_1 \equiv x_2 + x_1 \equiv f - 1 \pmod{a_2 - 1}$

Assuming $\sum x_1 \le h_0$ in (15.7), hence $x_4 > 0$, we get $x_2 + x_1 < h_0 = (f-1) + (a_2-1)$, so $x_2 + x_1 = f-1$, and

$$x_2 a_2 + x_1 \le (f - 1) a_2 = a_3 - r - a_2$$
,

contradicting the first conclusion.

If $s \mid (a_2 - 1)$, hence $m = (a_2 - 1)/s$ an integer, we find that we must choose

$$a_4 = (\tau + 1)a_3 - 1$$
, $\tau = 1, 2, \dots, \frac{a_2 - 1}{s} - 1 = m - 1$.

Now (15.4) holds for $\delta = 1$, and we examine $\delta = 2$:

$$n_{h_0}(A_3) + 1 - 2a_4 = (m - 2\tau)a_3 - r = (m - 2\tau - 1)a_3 + fa_2$$
.

If $\tau \geq [\frac{1}{2}(m+1)]$, this expression is negative, and an examination of (15.4) for $\delta \geq 2$ is unnecessary, so (15.2) holds. If $\tau < [\frac{1}{2}(m+1)]$, however, the right hand side belongs to $(h_0 - 2)A_3'$, and (15.4) fails for $\delta = 2$. Thus (15.2) is satisfied only if

(15.9)
$$a_4 = (\tau + 1)a_3 - 1$$
, $\tau = \left[\frac{1}{2}(m + 1)\right], \dots, m - 1$; $m = \frac{a_2 - 1}{s}$.

Summing up, we have the following

THEOREM 15.1. For non-pleasant A_3 , the equality (15.2) holds if and only if we have one of the cases:

(15.5) for arbitrary
$$A_3$$
,

(15.6)
$$for r = 1$$
,

$$(15.8-9)$$
 for $s = q$.

Based on computations by Mossige, this result was conjectured long before a proof was found. The cases r=1 or s=q are also proved in Krätzig-Berle [4, Kap.4], the "if" part along the lines above, the "only if" part by explicit representations for $n_{h_0}(A_3) + 1$ from $n_0(A_4)$ in the remaining cases.

 $x_2a_2 + x_1 \le (f - 1)a_2 = a_3 - r - a_2$

contradicting the first conclusion.

If $s \mid (a_2-1)$, hence $m = (a_2-1)/s$ an integer, we find that e must choose

 $a_2 = (\tau + 1)a_3 - 1$, $\tau = 1, 2, \dots, \frac{a_2 - 1}{s} - 1 = m - 1$

Now (15.4) holds for $\delta = 1$, and we examine $\delta = 2$:

 $n_{h_0}(A_3) + 1 - 2a_4 = (m - 2\tau)a_3 - \tau = (m - 2\tau - 1)a_3 + \epsilon a_2$

If $\tau \ge [\frac{1}{2}(m+1)]$, this expression is negative, and an examination

of (15.4) for 6 2 2 is unnecessary, so (15.2) holds. It

r < [1(m+1)] , however, the right hand side belongs to $(h_0 - 2)A_3^2$

and (15.4) fails for $\delta = 2$. Thus (15.2) is satisfied only if

(15.9) $a_4 = (\tau + 1)a_3 - 1$, $\tau = [1(m+1)], ..., m-1; m = \frac{a_2+1}{2}$

Summing up, we have the following

THEOREM 15.1. For non-pleasant A., the equality (15.2) holds if and only if we have one of the cases:

(15.5) for arbitrary Ag

(15.8-9) for s = q .

Based on computations by Mossige, this result was conjectured long before a proof was found. The cases r = 1 or s = q are also proved in Krätzig-Berle [4, Kap.4], the "if" part along the lines above, the "only if" part by explicit representations for nh (A,) + 1 from hoA, in the remaining cases.

16. The cases with $n_h(A_3 \cup \{a\}) = n_h(A_3)$, $a < a_3$.

In analogy with (3.3), it is quite natural to ask for cases when

$$(16.1) \quad n_h(A_k^*) = n_h(A_{k-1} \cup \{a\}) = n_h(A_{k-1}) , \quad 1 < a < a_{k-1} , \quad a \notin A_{k-1} ,$$

assuming admissible bases.

We need a particular result for the similar problem regarding regular h-ranges:

(16.2)
$$1 < a < a_2 \Rightarrow g_h(A_k^*) > g_h(A_{k-1}).$$

The proof is simple: It follows from Hofmeister [1, Satz 1] that the constant term of the regular representation for $g_h(A_\kappa)$ equals a_2-2 for all admissible A_κ . We conclude that the constant term a_2-1 of $g_h(A_{k-1})+1$ has a regular representation in at most a_2-2 addends 1 and $a \le a_2-1$.

In particular, $g_h(A_3^*) > g_h(A_2)$, and hence also $n_h(A_3^*) > n_h(A_2)$. The first possibility for (16.1) thus occurs when k=4:

$$(16.3) \quad n_h(A_4^*) = n_h(A_3 \cup \{a\}) = n_h(A_3) , \quad 1 < a < a_3 , \quad a \neq a_2 .$$

As in the preceding section, a study of this equality depends entirely on the properties of \mbox{A}_{3} .

If $h=h_0^*$ is the smallest h for which A_4^* is admissible, we clearly have $h_0^* \le h_0$ (where again $h_0=a_2+f-2$ refers to A_3). To be "fair" to A_3 , we restrict the examination of (16.3) to $h \ge h_0$.

16. The cases with $n_h(\Lambda_3 U(a)) = n_h(\Lambda_3)$, a < a -

In analogy with (3.3), it is quite natural to ask for cases when

(16.1) $n_h(A_k^*) = n_h(A_{k-1} \cup \{a\}) = n_h(A_{k-1})$. Is a $A_{k-1} \cup A_{k-1} \cup A_$

assuming admissible bases.

We need a particular result for the similar problem regarding regular h-ranges:

(16.2) 1 < a < a 2 = 8h (AL) > 8h (AL-1)

The proof is simple? It follows from Holmeister [1, Satz 1] that the constant term of the regular representation for $\mathcal{E}_h(A_k)$ equals a_2-2 for all admissible A_k . We conclude that the constant term a_2-1 of $\mathcal{E}_h(A_{k-1})+1$ has a regular representation in at most a_2-2 addends 1 and a_3-2-1 .

In particular, $g_h(A_3^2) > g_h(A_2)$, and hence also $n_h(A_3^2) > n_h(A_2)$ The first possibility for (16.1) thus occurs when k=4:

As in the preceding section, a study of this equality depends entirely on the properties of A₅.

If $h=h_0^*$ is the smallest h for which A_1^* is admissible, we clearly have $h_0^* \le h_0$ (where again $h_0 = a_2 + f - 2$ refers to A_3). To be "fair" to A_3 , we restrict the examination of (16.3) to h > h

Before doing this, we just mention the analogous problem for regular h-ranges. By (16.2), we must then assume a2 < a < a3, and it is not difficult to prove that for h & h0:

(16.4)
$$g_h(A_4^*) = g_h(A_3) \iff a = fa_2 + \rho, 0 \le \rho < r.$$

(My original proof is reproduced in Krätzig-Berle [4, p.27].)

Similar arguments show that (16.3) is impossible with pleasant A_3 . With $n_h(A_4^*) \ge g_h(A_4^*)$ and $n_h(A_3) = g_h(A_3)$, equality in (16.3) could only occur under the conditions of (16.4). But by (2.8-9), we then have

$$n_h(A_3) + 1 = (h - h_0 + 2)a_3 - r - 1 = (h - h_0)a_3 + 1 \cdot a + fa_2 + r - \rho - 1$$
,

with a coefficient sum \leq h $\,$ except in the one case $\,$ r = a_2 - 1 , ρ = 0 , hence $\,$ f \geq 2 . But then

$$n_h(A_3) + 1 = (h - h_0)a_3 + 2a + a_2 - 2$$
, $\Sigma \le h$.

In what follows, we may thus assume non-pleasant A_3 in (16.3).

Since A_3 and A_4^* have a <u>common largest element</u> a_3 , it is possible to use Meures' result (2.16), which in combination with (2.13) shows that for $h \ge h_0 - 1$:

$$n_h(A_k) \le ha_k - g(\overline{A}_k) - 1$$
,

with equality if $h \ge h_1$ ("stabilization", cf. section 3). For non-pleasant A_3 , we know that $h_1 = h_0$. For A_4^* , we put $h_1 = h_1^*$. With

$$\overline{A}_3 = \{a_3 - a_2, a_3 - 1, a_3\}, \overline{A}_4^* = \overline{A}_3 \cup \{a_3 - a\},$$

we thus get, for $h \ge h_0$:

$$n_h(A_3) = ha_3 - g(\overline{A}_3) - 1$$
, $n_h(A_4^*) \le ha_3 - g(\overline{A}_4^*) - 1$.

 $g_h(A_1^*) = g_h(A_2) \iff a = fa_2 + p , 0 \le p < \tau$.

(My original proof is reproduced in Krätzig-Berle (4, p.271.)

Similar arguments show that (16.3) is impossible with pleasant A₃: With $n_h(A_4^*) \ge g_h(A_4^*)$ and $n_h(A_3) = g_h(A_3^*)$, equality in (16.3) could only occur under the conditions of (16.4). But by (2.8-9), we then have

 $n_h(A_3) + 1 = (h - h_0 + 2)a_3 - r - 1 = (h - h_0)a_3 + 1 \cdot a + fa_2 + r - \rho - 1$,

with a coefficient sum \leq h except in the one case $T=a_2-1$, p=0 , hence $f\geq 2$. But then

 $n_h(A_3) + 1 = (h - h_0)a_3 + 2a + a_2 - 2$, $\sum h$

In what follows, we may thus assume non-pleasant A, in (16.3)

Since A_3 and A_4^* have a common largest element a_3 , it is possible to use Meures' result (2.16), which in combination with (2.15) shows that for $h \ge h_0 - 1$:

$n_{\rm h}(\lambda_{\rm K}) \le ha_{\rm k} - g(\Lambda_{\rm K}) - 1$,

with equality if $h \ge h_1$ ("stabilization", cf. section 3). For non-pleasant A_5 , we know that $h_1 = h_0$. For A_4^* , we put $h_4 = h_1^*$. With

 $\overline{A}_3 = \{a_3 - a_2, a_3 - 1, a_3\}, \overline{A}_1^* = \overline{A}_3 \cup \{a_3 - a\}$

we thus get, for h≥ho:

 $n_b(A_3) = ha_3 - g(\overline{A}_3) - 1$, $n_b(A_4^*) \le ha_3 - g(\overline{A}_4^*) - 1$.

Since trivially $n_h(A_4^*) \ge n_h(A_3)$, this shows that

$$(16.5) g(\overline{A}_4^*) = g(\overline{A}_3) \Rightarrow n_h(A_4^*) = n_h(A_3) for h \ge h_0$$

$$(16.6) h \ge h_1^* : n_h(A_4^*) = n_h(A_3) \Rightarrow g(\overline{A}_4^*) = g(\overline{A}_3).$$

We obviously have $g(\overline{A}_4^*) \le g(\overline{A}_3)$. With strict inequality, $g(\overline{A}_3)$ has a representation by \overline{A}_4^* :

$$g(\overline{A}_3) = x_1(a_3 - a) + x_2(a_3 - a_2) + x_3(a_3 - 1) + x_4a_3$$
.

It follows that

$$n_{h_0}(A_3) + 1 = h_0 a_3 - g(\overline{A}_3) = (h_0 - \sum x_i) a_3 + x_1 a + x_2 a_2 + x_3$$

has a representation by A_4^* with coefficient sum $h_0 - x_4 \le h_0$, provided that $\sum x_i \le h_0$. We thus have the following partial converse of (16.5):

$$(16.7) g(\overline{A}_3) \in h_0 \overline{A}_4^* \Rightarrow n_h(A_4^*) > n_h(A_3) for h \ge h_0.$$

We only proved this for $h=h_0$ above, but the general result with $h \ge h_0$ then follows immediately from (2.13-14).

There is one trivial case of equality in (16.3):

(16.8)
$$f = 1$$
, $a_2 = h_0 + 1$, $a_3 = h_0 + r + 1$, $a = a_2 - tr \ge 2$

(16.9)
$$\Rightarrow n_h(A_4^*) = n_h(A_3) \text{ for } h \ge h_0.$$

This follows from (16.5), since \overline{A}_3 and \overline{A}_4^{\star} are "equivalent" as Frobenius bases:

$$\overline{A}_3 = \{r, a_3 - 1, a_3\}, \overline{A}_4^* = \{r, (t+1)r, a_3 - 1, a_3\}.$$

Since trivially $n_h(A_A^*) \ge n_h(A_S)$, this shows that

(16.5) $g(\overline{A}_1^2) = g(\overline{A}_3) + n_h(A_1^2) = n_h(A_3)$ for $h \ge h_0$

(16.6) $h \ge h_1^* : n_h(A_4^*) = n_h(A_3) = g(\overline{A_4}) = g(\overline{A_3})$

We obviously have $g(\overline{A}_4^*) \le g(\overline{A}_3)$. With strict inequality, $g(\overline{A}_3)$ has a representation by \overline{A}_4^* :

 $g(\overline{A}_3) = x_1(a_3 - a) + x_2(a_3 - a_2) + x_3(a_3 - 1) + x_4 a_3$

It follows that

 $n_{h_0}(A_3) + t = h_0 a_3 - g(\overline{A}_3) = (h_0 - \overline{\lambda} x_1) a_3 + x_1 a_1 + x_2 a_2 + x_3$

has a representation by A_4^* with coefficient sum $h_0 - x_4 \le h_0$, provided that $\sum x_1 \le h_0$. We thus have the following partial converse of (16.5):

(16.7) $g(\overline{A}_3) \in h_0 \overline{A}_4^* = n_h(A_4^*) > n_h(A_3)$ for $h \geq h_0$

We only proved this for $h=h_0$ above, but the general result with $h \ge h_0$ then follows immediately from (2.15-14).

There is one trivial case of equality in (16.3):

(16.8) f=1, a2=h0+1, a3=h0+r+1, a=a2-tr≥2

(16.9) \Rightarrow $n_h(A_4^*) = n_h(A_3)$ for $h \ge h_0$

This follows from (16.5), since $\overline{\Lambda}_3$ and $\overline{\Lambda}_4^*$ are "equivalent" as Frobenius bases:

 $\overline{A}_{3} = (r, a_{3} - 1, a_{3}), \overline{A}_{4}^{*} = (r, (t + 1)r, a_{3} - 1, a_{3})$

The second element of \overline{A}_4^{\star} is a multiple of the first one.

We assume that A_3 is non-pleasant. If it is also non-dependent, it follows from Theorem 10.1 that

$$n_{h_0}(A_4^*) \ge n_{h_0}(A_3) \ge (h_0 + 1)a_2 - a_3$$
.

Let $1 < a < a_2$. We then get $h_1^* \le h_0$ by Theorem 3.1, and can combine (16.5-6) to an equivalence for non-dependent A_3 . And for Frobenius-dependent A_3 , Krätzig-Berle [4, p.23] shows very simply that we always have $n_h(A_4^*) > n_h(A_3)$ except in the already settled cases (16.8), hence

(16.10)
$$1 < a < a_2 : g(\overline{A}_4^*) = g(\overline{A}_3) \iff n_h(A_4^*) = n_h(A_3)$$
.

Based on extensive computations by Mossige, I conjectured the following results:

THEOREM 16.1. Let
$$a_2 < a < a_3$$
. Then
$$n_h(A_4^*) > n_h(A_3) \quad \underline{\text{for}} \quad h \geq h_0 .$$

THEOREM 16.2. Let $1 < a < a_2$. In addition to (16.8), there is one more case of equality in (16.9):

f = 1,
$$a_2 = h_0 + 1$$
, $a_3 = h_0 + r + 1$, $a = tr + 1$
 $h_0 = \tau r + \rho$, $0 \le \rho < r - 1$, $\tau \ge \rho$
 $r \equiv -1 \pmod{\rho + 1}$, $t = 1, 2, \dots, \left[\frac{\tau + 1}{\rho + 1}\right]$.

Both theorems were proved in the Master's thesis [2] of my student Kirfel. He used the methods of Rödseth [5] for determining

The second element of \$\tilde{A}_1^*\$ is a multiple of the first one.

We assume that \$A_2\$ is non-pleasant. If it is also nondependent, it follows from Theorem 18.1 that

EB = 2 E(1+0) & (ho +1) = 2 = 83

Let $1 < a < a_2$. We then get $h_1^* \le h_0$ by Theorem 3.1, and can combine (16.5-6) to an equivalence for non-dependent A_5 . And for Probenius-dependent A_5 , Krätzig-Berle (4, p-23) shows very simply that we always have $n_h(A_4^*) > n_h(A_5^*)$ except in the already settled cases (16.8), hence

(16.10) $1 < a < a_2 : g(\overline{\Lambda}_4^*) = g(\overline{\Lambda}_5) \iff n_h(\Lambda_4^*) = n_h(\Lambda_5)$

Based on extensive computations by Mossige, I conjectured the following results:

THEOREM 16.1. Let ay < a < ay . Then

nn(An) > nn(An)

THEOREM 16.2. Let 1 < a < a . In addition to (16.8), there is one more case of equality in (16.9):

f = 1, a2 = h0 + 1, a3 = h0 + r F1, a = tr + 1

0 5 r , f - x > 0 2 0 , q + xr = nd

 $\tau = -1 \pmod{p+1}$, $\tau = 1, 2, \dots, \frac{\tau+1}{p+1}$

Both theorems were proved in the Master's thesis [2] of my student Kirfel. He used the methods of Rödseth [5] for determining

the Frobenius number $g(\overline{A}_3)$. A shortened version [3] is submitted for publication.

Another student of mine, Krätzig-Berle, gave an independent and very elegant proof of Theorem 16.1 in her Diplomarbeit [4, Satz 3.1]. Using the inequalities of Theorems 10.2-5, she could determine a h_0 -representation by A_4^* of $n_{h_0}(A_3) + 1$.

We note that the bases A_3 of Theorem 16.2 satisfy the conditions (8.1-2), and so $n_h(A_3)$ can be determined explicitly by (8.3). It is fairly straightforward (cf. [4, Satz 2.3]) to show that this h-range is not increased when extending the basis with a = tr + 1. The hard problem is of course to show that all other cases (except (16.8)) lead to an increase of the h-range.

the Probenius number $g(\overline{A}_3)$. A shortened version [3] is submitted for publication.

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could determine a ho-representation by Ai of nho(Az) +1.

We note that the bases As of Theorem 46.2 satisfy the conditions (8.1-2), and so nh(Az) can be determined explicitly by (8.3). It is fairly straightforward (cf. [4, Satz 2.3]) to show that this h-range is not increased when extending the basis with a = tr +1. The hard problem is of course to show that all other cases (except (16.8)) lead to an increase of the h-range.

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