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Some Computing Aspects of Projective Geometry I.

Basic functions, algorithms and procedures.

By

Audun Holme
Department of Mathematics
Bergen, Norway

UNIVERSITY OF BERGEN

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Introduction.

The aim of this paper is to lay the foundation for certain investigations in projective geometry by means of a computer.

Here I only give some of the basic techniques which will have to play a central role in these investigations, to appear later as [Hm 8, 9]. Thus the present work is of a preliminary nature.

One of the main points at this stage has been to investigate the feasability of some of those basic techniques, implemented on a computer. For this I have been able to use the MACSYMA-system of the Mathlab-group at the Computer Science Laboratory of MIT, Cambridge, Mass. MACSYMA is a very sophisticated system for symbolic manipulation, in principle very well suited for the kind of investigation I am undertaking here and in the articles announced above.

However, there is a difficulty which became serious already at the current preliminary stage: Namely, as the size of the problem grows - for instance the number of transcendentals in the expressions one works with - then the size of the computations in some cases tend to grow exponentially. And since the elegant and flexible facilities available in MACSYMA also tend to require considerable core memory space, one frequently finds that the swell in intermediate calculations severely limits the size of the problems which can be treated. This is a serious obstacle. Moreover, for similar reasons some of the computations would tend to run for an unexpectedly long time, even in relatively simple cases.

It therefore is appearent that the present project can not be carried out exclusively within the environment of symbolic manipulation. Nevertheless, the use of MACSYMA has been indispensable, and tools of this kind will have to be used extensively in the future. In fact, aside from the obvious approach of trying to implement functions

written in a language for symbolic manipulation in a lower level language, one may develop programs by obtaining intermediate results for instance by MACSYMA and use these in programs written in a lower level language. A typical example of this is given by the expressions RE[1], RE[2] and RE[3] which have been found using MACSYMA. RE[3] could simply not have been computed "by hand", while RE[1] is easy. These expressions make it possible to find the Chern- and Segre classes for Grassmanians of lines, of planes and of 3-spaces in $\mathbb{P}^N$ for all N, using conventional programming languages. Unfortunately MACSYMA was not able to compute RE[4]. See sections 2 and 5 for details on this.

Moreover, in my opinion there is no question that computing with MACSYMA competes favorably with traditional approaches such as the one taken by <u>A. Lascoux</u> in [Lx] to compute Chern classes of tensor products and the 2 nd. symmetric and exterior power: Indeed, in section 1 it is shown how this goes through smoothly for tensor products, which is used later. Also the second exterior power is treated as an example, and it is clear how to generalize this to any exterior power. The symmetric powers are dealt with similarly, but this is omitted here. From a computational point of view I believe that the present approach is preferable.

Also, it is instructive to compare <u>R. Donagi's</u> computations in [Do], the appendix, to the material in section 3.

Thus in section 1, I give a procedure*) for computing the Chern polynomial of a tensor product in terms of the Chern polynomials of the factors. Using the procedure, a function carrying out the computation is then written in MACSYMA. In section 2, procedures for the Chern and Segre classes of Grassmanians are similarly given and implemented on MACSYMA. In section 3 an algorithm is given which will

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*) The word "procedure" is used here in a less technical sense than the word "algorithm".

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convert expressions in the Chern classes of the universal quotient bundle of a Grassmanian into expressions in the Schubert symbols. This is done by repeatedly applying Pieri's formula. The algorithm is implemented on MACSYMA, but unfortunately the computations tend to be rather time-consuming.

In section 4 I give some basic formulae and functions for computing the embedding- and duality properties of projective varieties. This material should be viewed in light of section 6, where the continuation of this work is outlined.

Section 5 contains some combinatorical aspects of Grassmanians of lines, which also illustrates how parts of this project can be carried out with conventional computer programs.

I would like to thank Professor Joel Moses of the Computer Science Laboratory at MIT for giving me access to the MACSYMA system, and the entire Mathlab group for their patience and help during my work with MACSYMA. In particular I would like to thank Dr. B. Trager, whith whoom I had many enlightening conversations.

§ 1. Computing the Chern classes of a tensor product.

One of the techniques which we will need in this paper, is a practicable algorithm for computation of the Chern classes of a tensor product $E \otimes F$ of two locally free O_X - Modules on a scheme X, in terms of the Chern classes of E and F.

The need for such a method, as well as the related one for symmetric and exterior powers, arises in many situations. In addition to the questions studied in this paper, they are also needed for the higher order Thorn-Boardman singularities.

This is the motivation for a recent article by A. Laxoux, [Lx] where the theory of Schur-functions is utilized to obtain explicit formulae for these Chern classed.

However, while Laxoux's expressions are nice to have, and of some interest in their own right, they do not lend themselves easily to computation: The Chern classed in question are obtained in terms of certain determinants in the Segre classes of E and F, and even though the passage from Chern polynomials to Segre polynomials is trivial on a computer, the further computations with Laxoux's formulas would still be rather large.

Here we take a different and much simpler approach to this problem: Using only substitutions, expansions and simplifications, as well as the function RESULTANT, we write a function in MACSYMA,

$$\text{TENSOR}\ (arg_1, arg_2, arg_3, arg_4)\ ,$$

which when given the arguments

$$arg_1 = 1 + c_1(E)T + \ldots + c_e(E)T^e\ ,$$

the Chern polynomial of E;

$$\arg_2 = e \ ,$$

the rank of E;

$$\arg_3 = 1 + c_1(F)T + \ldots c_f(E)T^f ,$$

the Chern polynomial of F;

$$\arg_4 = f \ ,$$

the rank of f; will return the Chern polynomial of $E \otimes F$ in terms of the Chern classes

$$c_1(E) , \ldots, c_e(E) \ ; \ c_1(F) , \ldots c_f(E) \ .$$

The indeterminate must be T in $\arg_1$ and $\arg_3$.

We obtain the function as follows: First, write down the <u>reverse Chern polynomials</u> of E, F and $E \otimes F$ with X, Y and T as indeterminates:

$$P(X) = X^e + c_1(E)X^{e-1} + \ldots + c_e(E)$$
$$Q(Y) = Y^f + c_1(F)Y^{f-1} + \ldots + c_f(F)$$
$$R(T) = T^{ef} + c_1(F \otimes F)T^{ef-1} + \ldots + c_{ef}(E \otimes F) \ .$$

Now regard the coefficients of P and Q as transcendentals. Then in some field extension of Q we have

$$P(X) = \prod_i (X - \ell_i)$$

$$Q(Y) = \prod_j (Y - m_j)$$

This being so, it now follows from [Fℓ] or any other standard source on Chern classes that

$$R(T) = \prod_{i,j} (T - (\ell_i + m_j)) \ .$$

For more details, see [Hm 6] .

Hence it is natural to introduce the relation

$$T = X + Y ,$$

and we get

$$Q(T - X) = (-1)^f \prod_j (X - (T - m_j)) .$$

Recall, [VdWa] Vol I section 28, that the resultant of two polynomials

$$f(X) = a_0 X^n + a_1 X^{n-1} + \ldots + a_n$$

$$g(X) = b_0 X^m + b_1 X^{m-1} + \ldots + b_m$$

where a_0 and b_0 are non-zero, is equal to

$$r = a_0{}^m b_0{}^n \prod_{i,j} (x_j - y_i) ;$$

$x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ being all the roots of

$$f(X) = g(X) = 0$$

in some splitting field. Thus letting

$$f(X) = Q(T - X) , \; g(X) = P(X) ,$$

we get that

$$r = (-1)^f \prod_{i,j} (T - m_j - \ell_i) ,$$

so that up to a sign, the resultant of $Q(T - X)$ and $P(X)$ with respect to X is equal to $R(T)$.

The function RESULTANT in MACSYMA may use different algorithms, see [MAC] p. 118. Normally the usual determinant is not computed directly. This may in some cases yield a sign different from what one expects. Rather than to keep track of this, it is better to adjust the sign in the end, which is easy because any Chern polynomial has 1 as its constant term.

Our function TENSOR is written as follows:

```
TENSOR(E,M,F,N):=BLOCK([L,P,Q,R,S,U,V],L:M*N,
        P:X^M*SUBST(1/X,T,E),Q:(T-X)^N*SUBST(1/(T-X),T,F),
        R:RESULTANT(P,Q,X),S:T^L*SUBST(1/T,T,R),S:RATEXPAND(S),
        U:S*COEFF(S,T,0),V:SUM(COEFF(U,T,I)*T^I,I,0,L),V);
```

In order to demonstrate this, we wish to generate Chern polynomials of two Modules, denoted by A and B. We do this by the array-defined function

```
A[J]:=1+SUM(CONCAT(C,I,A)*T^I,I,1,J)$
```

and a similar definition involving B. The result is as follows:

```
(C12) A[1];
(D12)                          C1A T + 1

(C13) A[2];
                                     2
(D13)                         C2A T  + C1A T + 1

(C14) A[3];
                                3        2
(D14)                     C3A T  + C2A T  + C1A T + 1
```

and for B,

```
(C15) B[1];
(D15)                          C1B T + 1

(C16) B[2];
                                    2
(D16)                        C2B T  + C1B T + 1

(C17) B[3];
                               3        2
(D17)                    C3B T  + C2B T  + C1B T + 1
```

If for instance F is of rank 1, then of course we have a well known formula for the Chern classes of $E \otimes F$, namely

$$c_k(E \otimes F) = \sum_{i=0}^{k} \binom{e-k+1}{i} c_{k-i}(E) c_1(F)^i .$$

Moreover, for $e = f = 2$ the polynomial is rather simple, and may be familiar. We get:

```
(C18) TENSOR(A[1],1,B[1],1);
(D18)                         (C1B + C1A) T + 1

(C19) TENSOR(A[1],1,B[2],2);
                                          2   2
(D19)       (C2B + C1A C1B + C1A ) T  + (C1B + 2 C1A) T + 1

(C20) TENSOR(A[1],1,B[3],3);
                           2          3    3                                2   2
(D20) (C3B + C1A C2B + C1A  C1B + C1A ) T  + (C2B + 2 C1A C1B + 3 C1A ) T

                                                          + (C1B + 3 C1A) T + 1

(C21) TENSOR(A[2],2,B[2],2);
         2                                       2          2      2
(D21) (C2B  - 2 C2A C2B + C1A C1B C2B + C1A  C2B + C2A  + C1B  C2A

                      4                                                              2
 + C1A C1B C2A) T  + (2 C1B C2B + 2 C1A C2B + 2 C1B C2A + 2 C1A C2A + C1A C1B

       2        3                              2                      2   2
 + C1A  C1B) T  + (2 C2B + 2 C2A + C1B  + 3 C1A C1B + C1A ) T

 + (2 C1B + 2 C1A) T + 1
```

However, already for $e = f = 3$ the expressions become quite formidable, and completely unsuited for processing "by hand". See the appendix, section 1.

We may use a similar method to obtain functions which return the Chern polynomials of any exterior or symmetric power as well. This will not be needed here, but to illustrate we shall give a function

$$\text{EXTERIOR2}\,(\text{arg}_1, \text{arg}_2)\ ,$$

which when given the arguments

$$\text{arg}_1 = 1 + c_1(E)T + \ldots + c_e(E)T^e\ ,$$

the Chern polynomial of E in which T must be the indeterminate, and

$$\arg_2 = e \ ,$$

the rank of E which must be ≥ 2 ; will return the Chern polynomial of $\Lambda^2 E$. It is clear from what follows how to generalize this to a function

$$\text{EXTERIOR}(\arg_1, \arg_2, \arg_3) \ ,$$

which when given a third argument

$$\arg_3 = r \ ,$$

which must be $\leq e$; will return the Chern polynomial of $\Lambda^r E$.

The function is:

```
EXTERIOR2(E,M):=BLOCK([L,B,P,Q,R,S,U,V],L:M^2,B:M*(M-1)/2,
        P:X^M*SUBST(1/X,T,E),Q:SUBST(T-X,X,P),P:RATEXPAND(P),
        Q:RATEXPAND(Q),R:RESULTANT(P,Q,X),
        S:R/(2^M*SUBST(1/2*T,X,P)),S:RATEXPAND(S),U:FACTOR(S),
        V:SQRT(U),V:PART(V,1),V:T^(M*(M-1)/2)*SUBST(1/T,T,V),
        V:RATSIMP(V),V:SUM(COEFF(V,T,I)*T^I,I,0,M*(M-1)/2),V);
```

To see why this yields the correct result, denote the reverse Chern polynomials of E , $E \otimes E$ and $\Lambda^2 E$ by, respectively $P(T)$, $Q(T)$ and $R(T)$. As above, writing

$$P(T) = \prod_i (T - \ell_i) \ ,$$

we have

$$Q(T) = \prod_{i_1, i_2} (T - (\ell_{i_1} + \ell_{i_2})) \ ;$$

and

$$R(T) = \prod_{i_1 < i_2} (T - (\ell_{i_1} + \ell_{i_2})) \ ;$$

the latter being the standard way in which the Chern classes of exterior powers are determined. Using this, we get

$$Q(T) = \prod_i (T - 2\ell_i)\{\prod_{i_1<i_2} (T - (\ell_{i_1} + \ell_{i_2}))\}^2$$

$$= 2^e P(\tfrac{1}{2}T)R(T)^2 .$$

This should explain everything in the function body above, except possibly the use of the function PART. This is necessary since SQRT applied to $R(T)^2$ will return ABS(R(T)) . The PART-function picks out the expression R(T) .

We obtain the following polynomials for $2 \leq e \leq 4$:

```
(C23) EXTERIOR\2(A[1],1);
Part fell off end.
(C24) EXTERIOR\2(A[2],2);
(D24)                                  C1A T + 1

(C25) EXTERIOR\2(A[3],3);
                                    3        2  2
(D25)                        - C3A T  + C1A  T  + 2 C1A T + 1

(C26) EXTERIOR\2(A[4],4);

You have run out of LIST space.
Do you want more?
Type ALL; NONE; a level-no. or the name of a space.
ALL;
                        2  6        3  3          2  2
(D26)              - C3A   T  + C1A   T  + 3 C1A   T  + 3 C1A T + 1
```

As we see, the improper arguments given to the function on line (C 23) results in an error message. The function call on line (C 26), while returning a simple polynomial as the answer, clearly generates large intermediate expressions.

Unfortunately this is not an infrequent situation, which tends to limit the range of results obtainable by methods such as the ones developed in this paper.

§ 2. Chern- and Segre classes of Grassmanians.

Let $G_k(r, A) = G(r, A) = G$ be the Grassmanian which parametrizes the linear r-subspaces of $\mathbb{P}_k^A = \mathbb{P}^A$. Equivalently, $G(r, A)$ is the scheme which represents the functor of $r + 1$ - quotients of $V = k^{A+1}$, so in particular it carries the universal quotient

$$V_G \longrightarrow Q \longrightarrow 0$$

where Q is locally free of rank $r + 1$.

Here (i.e. in this and the following section), we only summarize the basic formulae needed from the theory of Grassmanians in algebraic geometry, and the closely related theory of Schubert Calculus. For details, the reader is referred to [Hm 7] or the references given there, such as [Lk 1]. Last but not least, the recent paper of R. Donagi [Do] contains among other things an exellent account of some of the material we need here.

The reader not familiar with the material which follows, may consult [Hm 7] for a more extensive survey, with references to the literature. In using these, one should of course beware of the distinction between the projective and the affine notation. Thus the Grassmanian which we denote by $G(r, A)$ would be denoted by $G(r + 1, A + 1)$ in affine notation, used for instance in [Do].

First, we have the basic formula

(2.1) $$\Omega_G^1 = Q^\vee \otimes M ,$$

M being defined by the exact sequence

(2.2) $$0 \longrightarrow M \longrightarrow V_G \longrightarrow Q \longrightarrow 0$$

and where

$$Q^\vee = \underline{\mathrm{Hom}}_G(Q, O_X)$$

is the dual of Q . Thus in particular G is non-singular and

$$\dim(G) = (r+1)(A-r) .$$

Moreover, as the sequence (2.2) is split we have the identification

$$G(r, A) = G(A - r - 1, A)$$

and it thus suffices to consider the cases

$$r \leq \left[\frac{A-1}{2}\right] .$$

Since the case

$$G(0, A) = \mathbb{P}^A$$

is trivial as far as this investigation is concerned, we shall assume that

$$1 \leq r \leq \left[\frac{A-1}{2}\right] .$$

Using the function TENSOR of the previous paragraph, together with the well known relation

$$c_t(Q) = c_{(-t)}(Q^\vee)$$

where as always $c_t(E)$ denotes the Chern polynomial of E , it is now easy to write a function in MACSYMA which computes the Chern polynomial of G(r, A), i.e. the Chern polynomial of

$$T_G = \Omega_G^{1\vee} .$$

First, we introduce the function

CHERNPOLYBUNDLE(arg_1, arg_2, arg_3, arg_4) ,

which returns the Chern polynomial in the indeterminate given by arg_4 , of a bundle (i.e. a locally free Module) denoted by arg_1 of rank = arg_2 on a variety of dimension = arg_3 . We include also the function

SEGREPOLYBUNDLE(arg_1, arg_2, arg_3, arg_4) ,

which when given the same arguments as CHERNPOLYBUNDLE above, will return the Segre polynomial in terms of the Chern classes of the bundle.

```
CHERNPOLYBUNDLE(Q,RANK,DIM,T):=BLOCK([P],
              P:1+SUM(CONCAT(C,I,Q)*T^I,I,1,MIN(RANK,DIM)),P);

SEGREPOLYBUNDLE(Q,RANK,DIM,T):=TAYLOR(
              1/CHERNPOLYBUNDLE(Q,RANK,DIM,-T),T,0,DIM);
```

Furthermore, the function

CHERNPOLYMDUAL(arg_1, arg_2, arg_3)

will return the Chern-polynomial with arg_3 as the indeterminate of the bundle M^V on G(r, A), when given the arguments $arg_1 = r$, $arg_2 = A$. The result is expressed in terms of the Chern classes of Q. The function is:

```
CHERNPOLYMDUAL(R,A,T):=BLOCK([RANK,DIM,P],RANK:R+1,DIM:A-R,
              P:SEGREPOLYBUNDLE(Q,RANK,DIM,T),EXPAND(P));
```

Using the above CHERNPOLYBUNDLE and CHERNPOLYMDUAL, together with TENSOR, we now write a function

POLYS(arg_1, arg_2),

which when given the arguments

$arg_1 = r$, $arg_2 = A$;

will do the following: First, the Chern polynomial of G(r, A) is computed by the function CHERNPOLYG given below and assigned to the variable CHEPOrA. Next, the result is printed out as

"CHEPOrA = c_t(G(r, A))".

Finally the same is done for the Segre polynomial of G(r, A). The function is:

```
POLYSG(R,A):=BLOCK([DIM,CH,D,SE],DIM:(R+1)*(A-R),
      CH:CHERNPOLYG(R,A,T),CONCAT('CHEPO,R,A)::CH,
      PRINT("CHEPO",R,A,"=",CH),D:1/CH,SE:TAYLOR(D,T,0,DIM),
      SE:1+SUM(EXPAND(COEFF(SE,T,I))*T^I,I,1,DIM),
      CONCAT('SEPO,R,A)::SE,PRINT("SEPO",R,A,"=",SE),DIM);
```

The function CHERNPOLYG(arg_1, arg_2, arg_3) which computes the Chernpolynomial of $G(r, A)$ in terms of the Chern classes of Q, with arg_3 as the indeterminate and where

$$arg_1 = r, \; arg_2 = A ,$$

is as follows:

```
CHERNPOLYG(R,A,T):=BLOCK([RQ,DIM,RM,F,G,H],RQ:R+1,
      DIM:(R+1)*(A-R),RM:A-R,F:CHERNPOLYBUNDLE(Q,RQ,DIM,T),
      G:CHERNPOLYMDUAL(R,A,T),H:TENSOR(F,RQ,G,RM),H);
```

Note that $M^\vee$ is of rank $A - r$, so the polynomial $c_t(M^\vee)$ is of degree $A - r$. This observation yields the set of relations (actually: a set of generators for the ideal of relations) among the Chern classes of Q: In fact, the inverse of the polynomial $c_t(Q^\vee)$, where $c_1(Q), \ldots, c_{r-1}(Q)$ are regarded as transcendentals for the moment, contains terms

$$\left\{\rho_i(c_1(Q), \ldots, c_{r+1}(Q))t^i \mid A - r < i \leq (r + 1)(A - r)\right\}$$

and these are all equal to zero, thus giving $r(A - r)$ relations among the Chern classes of Q. It is a classical result that these are all the relations.

This observation enables us to write the function

RELATIONSOFCHERNCLASSES(arg_1, arg_2),

which when given the arguments

$$\arg_1 = r \ , \ \arg_2 = A \ ;$$

will return a print out as follows

$$"\rho_i(c_1(Q), \ \ldots, \ c_{r+1}(Q)) = 0"$$

for $i = A - r + 1, \ \ldots, \ (r + 1)(A - r)$. We are not using this in the sequel, otherwise it would of course be better to let the function return a list of the relations, which could then be further manipulated on by other functions. We have:

```
RELATIONSOFCHERNCLASSESG(R,A):=BLOCK([RANK,RANKM,P,RE],RANK:R+1,
                               RANKM:A-R,DIM:RANK*RANKM,
                               P:SEGREPOLYBUNDLE(Q,RANK,DIM,T),
                               FOR I FROM RANKM+1 THRU DIM DO
                                 (RE:EXPAND(COEFF(P,T,I)),
                                 PRINT(RE," = 0 ")));
```

The following function generates the Chern and the Segre polynomials, as well as the relations of the Chern classes of Q for $G(r, A)$ for all

$$1 \leq r \leq \left[\frac{A - 1}{2}\right] :$$

```
G(A):=BLOCK([E],E:ENTIER((A-1)/2),
  FOR I THRU E DO (POLYSG(I,A),RELATIONSOFCHERNCLASSESG(I,A)));
```

For $A = 3$ we get the following:

```
(C5) G(3);
                         2          2           4  4          3  3          2  2
CHEPO 1 3 = (4 C2Q  - 4 C1Q  C2Q + 3 C1Q ) T  + 6 C1Q  T  + 7 C1Q  T
                                                                    + 4 C1Q T + 1
                         2          2            4  4          3  3
SEPO 1 3 = (- 4 C2Q  + 4 C1Q  C2Q + 14 C1Q ) T  - 14 C1Q  T

                                                          2  2
                                                  + 9 C1Q  T  - 4 C1Q T + 1
   3
C1Q  - 2 C1Q C2Q = 0
   2        2        4
C2Q  - 3 C1Q  C2Q + C1Q  = 0
```

A full list of results for $3 \leq A \leq 6$ is given in the appendix, § 2.

There is a somewhat different method for computing the Chern- and Segre classes of Grassmanians.

This utilizes the exact sequence

$$(2.3) \qquad 0 \longrightarrow \Omega_G^1 \longrightarrow (Q^V)^{A+1} \longrightarrow Q \otimes Q^V \longrightarrow 0$$

obtained from (2.2) by tensoring with Q^V . It yields the formulae

$$(2.4) \qquad c_t(Gr, A)) = (1 + c_1 t + \ldots c_{r+1} t^{r+1})^{A+1} / (c_t(Q \otimes Q^V))$$

$$(2.5) \qquad s_t(G(r, A) = c_t(Q \otimes Q^V) / (1 + c_1 t + \ldots + c_{r+1} t^{r+1})^{A+1}$$

Using the function TENSOR, it is of course clear how to write functions which compute the polynomials according to the formulae above. It is best to do so by an <u>array-defined</u> function,

RE[arg] ,

which when given the argument $r = arg$ will return $c_t(Q \otimes Q^V)$ for Q of rank $r + 1$. Thus once RE is computed for a given value of r , it is stored and may be used again when needed for a different value of A . We delete the details. One then uses the function

```
SEGREPOLYGRASS(R,A):=BLOCK([DIM,S],DIM:(R+1)*(A-R),
                 S:RE[R]/(1+SUM(CONCAT(C,I,Q)*T^I,I,1,R+1))^(A+1),
                 S:TAYLOR(S,T,0,DIM),
                 S:1+SUM(EXPAND(COEFF(S,T,I))*T^I,I,1,DIM),S);
```

We obtain the following results, via a function

GG(arg)

which when given the argument

arg = A

will print out the new Chern and Segre polynomials in an analogous way to the function G(arg) .

```
(C6) GG(3);
NEWCHEPO 1 3 = (6 C2Q^2 - 16 C1Q^2 C2Q + 8 C1Q^4) T^4
               + (8 C1Q^3 - 4 C1Q C2Q) T^3 + 7 C1Q^2 T^2 + 4 C1Q T + 1
NEWSEPO 1 3 = (- 6 C2Q^2 - 16 C1Q^2 C2Q + 25 C1Q^4) T^4
               + (4 C1Q C2Q - 16 C1Q^3) T^3 + 9 C1Q^2 T^2 - 4 C1Q T + 1
(D6)                          DONE
(C7) GG(4);
NEWCHEPO 1 4 = (- 14 C2Q^3 + 88 C1Q^2 C2Q^2 - 72 C1Q^4 C2Q + 16 C1Q^6) T^6
  + (30 C1Q C2Q^2 - 40 C1Q^3 C2Q + 16 C1Q^5) T^5
  + (6 C2Q^2 - 13 C1Q^2 C2Q + 16 C1Q^4) T^4 + 15 C1Q^3 T^3
  + (C2Q + 11 C1Q^2) T^2 + 5 C1Q T + 1
NEWSEPO 1 4 = (25 C2Q^3 - 15 C1Q^2 C2Q^2 - 245 C1Q^4 C2Q + 140 C1Q^6) T^6
  + (15 C1Q C2Q^2 + 110 C1Q^3 C2Q - 91 C1Q^5) T^5
  + (- 5 C2Q^2 - 40 C1Q^2 C2Q + 55 C1Q^4) T^4 + (10 C1Q C2Q - 30 C1Q^3) T^3
  + (14 C1Q^2 - C2Q) T^2 - 5 C1Q T + 1
(D7)                          DONE
```

A full list of results for $3 \leq A \leq 6$ is given in the appendix, § 2.

We see that the Chern and Segre polynomials returned by the two methods are not identical. This is of course due to the relations among the Chern classes of Q .

As we shall see in following paragraphs, it will be essential to find the simplest possible expressions for the Cern and Segre classes, at least when r and A are large.

The above data seem to indicate that the first method yields a simpler result than the second. However, for small values of r the second method is the best. In fact, once RE[R] is given, we can compute the polynomials for any A using less sophisticated systems than MACSYMA, and even obtain explicit if messy formulae in some cases. We return to this in section 5 .

There we will treat the case $r = 1$, i.e. G(1, N)'s in this manner, and see how the computations we are interested in, and where we use the Chern and Segre polynomials, reduce to the evaluation of straight forward combinatorical identities, which present no difficulties from a computational point of view.

Moreover, there should be a good possibility that the case $r = 2$ can be treated analogously, as we shall indicate. But for $r = 3$ there is no hope of obtaining similar combinatorical expressions, even though the method might still be usefull from a numerical point of view. This becomes clear from the expressions for RE[1], RE[2] and RE[3] which are listed below. For $R = 4$ the computation became too large for the system, in that the intermediate expressions filled up all available space.

Thus the first method for computing the Chern and Segre polynomials

would yield new information for the first time for G(4,9).

But at this level the method appears to be impracticable on MACSYMA. I say <u>appear</u>, because I have not tried to have this done as a background job with disk use.

My present opinion is that it would be better to have the functions implemented directly in a lower level language.

```
(C11) RE[1];
                                   2   2
(D11)              (4 C2Q - C1Q ) T  + 1

(C12) RE[2];
              2                         3               3       2     2    6
(D12) (27 C3Q   - 18 C1Q C2Q C3Q + 4 C1Q  C3Q + 4 C2Q  - C1Q  C2Q ) T

                      2         2           4  4                   2   2
           + (9 C2Q  - 6 C1Q  C2Q + C1Q ) T  + (6 C2Q - 2 C1Q ) T  + 1

(C13) RE[3];

You have run out of LIST space.
Do you want more?
Type ALL; NONE; a level-no. or the name of a space.
ALL;
               3                   2           2    2           2          2
(D13) (256 C4Q   - 192 C1Q C3Q C4Q  - 128 C2Q  C4Q  + 144 C1Q  C2Q C4Q

         4    2               2         2     2
 - 27 C1Q  C4Q  + 144 C2Q C3Q  C4Q - 6 C1Q  C3Q  C4Q

             2                    3                      4
 - 80 C1Q C2Q  C3Q C4Q + 18 C1Q  C2Q C3Q C4Q + 16 C2Q  C4Q

        2     3                4                      3         3     3
 - 4 C1Q   C2Q  C4Q - 27 C3Q  + 18 C1Q C2Q C3Q  - 4 C1Q  C3Q

        3     2       2     2     2  12
 - 4 C2Q   C3Q  + C1Q   C2Q   C3Q ) T

                    2          2    2           2
 + (- 192 C2Q C4Q  + 72 C1Q  C4Q  + 216 C3Q  C4Q - 120 C1Q C2Q C3Q C4Q

         3                3          2    2                        3
 + 18 C1Q   C3Q C4Q + 32 C2Q  C4Q - 6 C1Q  C2Q  C4Q - 54 C1Q C3Q

         2     2           2         2        4    2                3
 + 18 C2Q   C3Q  + 42 C1Q  C2Q C3Q  - 9 C1Q  C3Q  - 26 C1Q C2Q  C3Q

        3     2            5      2     4  10
 + 6 C1Q   C2Q  C3Q + 4 C2Q  - C1Q  C2Q ) T
```

```
                   2                            2             2
+ (- 112 C4Q  + 56 C1Q C3Q C4Q + 24 C2Q  C4Q - 32 C1Q  C2Q C4Q

        4                   2           2   2             2
+ 6 C1Q  C4Q + 48 C2Q C3Q  - 25 C1Q  C3Q  - 54 C1Q C2Q  C3Q

         3                 5             4          2    3
+ 38 C1Q  C2Q C3Q - 6 C1Q  C3Q + 17 C2Q  - 12 C1Q  C2Q

        4   2   8                      2              2
+ 2 C1Q  C2Q ) T  + (16 C2Q C4Q - 6 C1Q  C4Q + 26 C3Q

                             3             3          2    2          4
- 30 C1Q C2Q C3Q + 8 C1Q  C3Q + 28 C2Q  - 24 C1Q  C2Q  + 8 C1Q  C2Q

     6   6                              2           2             4  4
 - C1Q ) T  + (8 C4Q - 2 C1Q C3Q + 22 C2Q  - 16 C1Q  C2Q + 3 C1Q ) T

                     2   2
 + (8 C2Q - 3 C1Q ) T  + 1
(C14) RE[4];

You have run out of LIST space.
Do you want more?
Type ALL; NONE; a level-no. or the name of a space.
ALL;

You have run out of LIST space.
Do you want more?
Type ALL; NONE; a level-no. or the name of a space.
ALL;

CORE capacity exceeded (while requesting FIXNUM space)

33037 msec. so far
```

§ 3. A basis for some Schubert Calculus on a computer.

Here we take the term Schubert Calculus to mean the formal computation with "Schubert Cycles" $\Omega(a_1, \ldots, a_q)$ in $A(G(r, A))$, where $q = A - r$.

Different notations are in use, for the ones utilized here we refer to [Hm 7]. Also, there the reader will find references to some of the literature, of which we rely particularly on the fundamental and classical source [HP].

We shall make no attempt here to pursue the Schubert Calculus on a computer for its own sake, even though such a project would certainly be a very interesting one. Rather, we develope the minimum reqired for our present purpose.

Indeed, recall the Plücker-embedding

$$p : G(r, A) \hookrightarrow \mathbb{P}(\Lambda^{r+1} V) = \mathbb{P}^N$$

where

$$N = \binom{A + 1}{r + 1} - 1 .$$

Then we have that

$$\mathcal{O}_{G(r,A)}(1) = p^*(\mathcal{O}_{\mathbb{P}^N}(1)) = \Lambda^{r+1} Q .$$

Hence in particular if D is a very ample divisor giving the embedding p, then

$$[D] = c_1(Q) .$$

In the next sections we shall see how this makes it possible to describe properties of embeddings and duality for Grassmanians, as

well as for a large class of varieties "generated" by Grassmanians in a sense made precise later, in terms of the Chern numbers of the universal quotient bundles Q: That is to say, in terms of the degrees of the elements

$$c_1(Q)^{i_1} \ldots c_{r+1}(Q)^{i_{r+1}} \in A^{\dim}(G(r, A))$$

where $i_1 + \ldots + i_{r+1} = \dim = (r+1)(A-r)$. The degree map is in this case an isomorphism

$$A^{\dim}(G(r, A)) \xrightarrow[\cong]{\deg} \mathbb{Z}$$

so we may refer to the momomials above as the Chern numbers of Q .

Hence we need an algorithm to compute the Chern numbers of Q for any $G(r, A)$. For this we proceed as follows: It is possible to convert any polynomial F in $c_1(Q),\ldots,c_{r+1}(Q)$ with integral coefficients into a linear combination in the elements $\Omega(a_1, \ldots, a_q) \in A(G(r,A))$, $q = A - r$ and $1 \leq a_1 < a_2 < \ldots < a_q \leq A + 1$. This is done by repeated application of Pieri's Formula, which asserts the following:

(3.1) $$\Omega(a_1, \ldots, a_q)c_h(Q) = \sum \Omega(b_1, \ldots, b_q)$$

where the sum is extended over all indicies for which

$$\sum_{j=1}^{q} b_j = \sum_{j=1}^{q} a_j - h$$

and

$$1 \leq b_1 \leq a_1 < b_2 \leq a_2 < \ldots < b_q \leq a_q \leq A + 1 .$$

Since

well as for a large class of varieties "generated" by Grassmanians in a sense made precise later, in terms of the Chern numbers of the universal quotient bundles Q: That is to say, in terms of the degrees of the elements

$$c_1(Q)^{i_1} \cdots c_{r+1}(Q)^{i_{r+1}} \in A^{\dim}(G(r, A))$$

where $i_1 + \dots + i_{r+1} = \dim = (r+1)(A-r)$. The degree map is in this case an isomorphism

$$A^{\dim}(G(r, A)) \xrightarrow[\cong]{\deg} \mathbb{Z}$$

so we may refer to the monomials above as the <u>Chern numbers of Q</u>.

Hence we need an algorithm to compute the Chern numbers of Q for any $G(r, A)$. For this we proceed as follows: It is possible to convert any polynomial F in $c_1(Q), \dots, c_{r+1}(Q)$ with integral coefficients into a linear combination in the elements $\Omega(a_1, \dots, a_d) \in A(G(r,A))_p$, $d = A - r$ and $1 \le a_1 < a_2 < \dots < a_d \le A + 1$. This is done by repeated application of <u>Pieri's Formula</u>, which asserts the following:

$$\Omega(a_1, \dots, a_d)c_h(Q) = \sum \Omega(b_1, \dots, b_d) \tag{3.1}$$

where the sum is extended over all indicies for which

$$\sum_{j=1}^{d} b_j = \sum_{j=1}^{d} a_j - h$$

and

$$1 \le b_1 \le a_1 < b_2 \le a_2 < \dots < b_d \le a_d \le A + 1$$

Since

$$1 = \Omega(r + 2, \ldots, A + 1)$$

in $A(G(r, A))$ we obtain the algorithm in question as follows:

First multiply F by $\Omega(r + 2, \ldots, A + 1)$, then perform the substitution (3.1) in F repeatedly until F is free of $c_h(Q)$, for $h = 1, \ldots, r + 1$.

Since there is only one $\Omega(a_1, \ldots, a_q) \in A^{\dim}(G(r, A))$, namely,

$$a_1 = 1, a_2 = 2, \ldots, a_q = A - r$$

and this element is easily seen to have degree 1, the case when F is homogeneous of (weighed) degree dim will yield an integral multiplum of $\Omega(1, \ldots, A - r)$, and the numerical factor is the sought degree.

In principle this will solve our problem. However, it may be better to proceed in a slightly different way. This is due to the fact that the above procedure, when implemented in MACSYMA, tends to be quite time-consuming. What we can do, is first of all to observe that

$$\deg(c_1(Q)^i) = \deg(G(r, A)) = \frac{\prod_{i=1}^{r} (i!)((r + 1)(A - r))!}{\prod_{i=A-r}^{A} (i!)}$$

for all $i \leq \dim$. Here the degree is with respect to the Plücker embedding, [Hm 7]. Moreover, with the same interpretation of "deg", we have

$$\deg(c_1(Q)^{i_1} c_2(Q)^{i_2} \ldots c_{r+1}(Q)^{i_{r+1}}) =$$

$$\deg(c_2(Q)^{i_2} \ldots c_{r+1}(Q)^{i_{r+1}})$$

and since there are formulae for computing the degrees of $\Omega(a_1, \ldots a_q)$

in general, all we need to do is to reduce monomials of the form

$$c_2(Q)^{i_2} \ldots c_{r+1}(Q)^{i_{r+1}}$$

to a linear combination in the Ω's . We return to this below.

We now give the function which transforms a polynomial in the $c_i(Q)$'s to a linear combination in the Ω's .

First, the function DOMAIN(arg_1, arg_2) generates the list of all lists of indicies $[a_1, \ldots, a_q]$ where $q = arg_1$, $\Sigma a_i = arg_2$ and $1 \leq a_1 < \ldots < a_q$:

```
DOMAIN(M,N):=BLOCK([S,R,L],L:[],
        IF N >= M*(M+1)/2
            THEN (IF EQUAL(M,1) THEN L:[[N]]
                      ELSE FOR I THRU N DO
                              FOR S IN DOMAIN(M-1,N-I) DO
                                  (IF S[M-1] < I
                                      THEN (R:ENDCONS(I,S),
                                            L:CONS(R,L))))),L);
```

The function

$$\text{DOMAING}(arg_1, arg_2, arg_3)$$

will, when given the arguments

$$arg_1 = r, \; arg_2 = A, \; arg_3 = N \;;$$

return the list of all lists of indicies

$$[a_1, \ldots, a_{A-r}] \;,$$

where

$$1 \leq a_1 < \ldots < a_{N-r} \leq A + 1 \;,$$

and

$$\sum a_i = N \;:$$

```
DOMAING(R,A,N):=BLOCK([S,RR,L,M],L:[],M:A-R,
        IF N >= M*(M+1)/2
           THEN (IF EQUAL(M,1) THEN (IF N <= A+1 THEN L:[[N]])
                     ELSE FOR I THRU MIN(N,A+1) DO
                              FOR S IN DOMAIN(M-1,N-I) DO
                                 (IF S[M-1] < I
                                     THEN (RR:ENDCONS(I,S),
                                           L:CONS(RR,L))))),L);
```

The function

$$\text{SUMDOMAIN}(arg_1\ arg_2,\ arg_3,\ arg_4)$$

will, when given the arguments

$$arg_1 = q,\ arg_2 = \sum a_j,\ arg_3 = h$$

$$arg_4 = [a_1,\ \ldots,\ a_q]\ ;$$

return the list of all lists of indicies

$$[b_1,\ \ldots,\ b_q]$$

such that the relations in (3.1) hold:

```
SUMDOMAIN(M,N,H,L):=BLOCK([D,S],D:DOMAIN(M,N-H),
        FOR S IN D DO
           (FOR I THRU M-1 DO
               (IF NOT (S[I] <= L[I] AND L[I] < S[I+1])
                   THEN D:DELETE(S,D)),
            IF S[M] > L[M] THEN D:DELETE(S,D)),D);
```

The function

$$\text{FUNDCLASSG}(arg_1,\ arg_2)$$

will, when given the arguments

$$arg_1 = r,\ arg_2 = A\ ;$$

generate the fundamental class $\Omega(r + 2,\ \ldots,\ A + 1)$ of $G(r,\ A)$.

For this it uses the function FUNDLIST:

```
FUNDLIST(R,A):=BLOCK([L],L:[],
         FOR I FROM R+2 THRU A+1 DO L:ENDCONS(I,L),L);
```

```
FUNDCLASSG(R,A):=APPLY(OMEGA,FUNDLIST(R,A));
```

The function

$$\text{TOTALDOMAIN}(arg_1, arg_2)$$

will, when given the arguments

$$arg_1 = r, arg_2 = A ;$$

return the list of all possible lists of indicies

$$[a_1, \ldots, a_{A-r}]$$

where

$$1 \leq a_1 < a_2 < \ldots < a_{A-r} \leq A + 1 :$$

```
TOTALDOMAIN(R,A):=BLOCK([S,LO,HI],LO:(A-R)*(A-R+1)/2,
      HI:SUM(I,I,R+2,A+1),S:[],
      FOR J FROM LO THRU HI DO S:APPEND(DOMAING(R,A,J),S),
      S);
```

The function

$$\text{OMEGATRANSFORM}(arg_1, arg_2, arg_3)$$

when given the arguments:

$$arg_1 = r, arg_2 = B, arg_3 = F(c_1(Q), \ldots, c_{r+1}(Q)) ,$$

the last one being a polynomial in the Chern-classes of the universal quotient bundle of $G(r, A)$, will apply the substitution (3.1) repeatedly to $F = F \cdot \Omega(r + 2, \ldots, A + 1)$ untill the expression is free of $c_1(Q), \ldots, c_{r+1}(Q)$.

```
OMEGATRANSFORM(R,A,F):=BLOCK([M,TOT,N,U,OLD,NEW],
                 F:F*FUNDCLASSG(R,A),M:A-R,TOT:TOTALDOMAIN(R,A),
                 FOR I THRU R+1 DO
                    (FOR J THRU INF UNLESS
                       FREEOF(CONCAT(C,I,Q),F) DO.
                       FOR S IN TOT DO
                       (N:SUM(S[K],K,1,M),
                        U:SUMDOMAIN(A-R,N,I,S),
                        OLD:APPLY(OMEGA,S)*CONCAT(C,I,Q),
                        NEW:SUM(APPLY(OMEGA,U[K]),K,1,
                                 LENGTH(U)),
                        F:RATSUBST(NEW,OLD,F))),F);
```

It is best to assume that the (weighed) degree of F with respect to the (weighed) degrees of $c_1(Q), \ldots, c_{r+1}(Q)$ is $\leq \dim = (r+1)(A-r)$. However, with the usual intersection-theoretic interpretation of monomials in the $c(Q)$'s , the function returns the correct result in this case as well, namely zero . On the other hand, the term

$$\deg^2 = (c_1(Q)^{\dim})^2$$

which might occur in F (and indeed it will later on) is clearly not intended to mean the element $c_1(Q)$ of $A(G(r, A))$ raised to the 2dim power, but rather the square of the Chern number $c_1(Q)^{\dim}$. Thus the function OMEGATRANSFORM has to be used with caution in such cases.

The function will for instance return the following results:

(C 11) OMEGATRANSFORM(2, 5, c2Q**2) ;

(D 10) OMEGA(2, 3, 6) + OMEGA(1, 4, 6)

or

(C 14) OMEGATRANSFORM(2, 5, c1Q**2*c2Q - c2Q**2 - c1Q*c3Q) ;

(D 14) OMEGA(2, 4, 5) .

The function OMEGATRANSFORM will also convert any polynomial expression in $c_1(Q), \ldots, c_{r+1}(Q)$ into the corresponding one involving the Ω's . Thus for instance we obtain the following:

(C 5) OMEGATRANSFORM(1, 3, CHEPO13) ;

(D 5) 6OMEGA(1,2)T^4 + 12OMEGA(1,3)T^3 + (7OMEGA(2,3) + 7OMEGA(1,4))T^2
+ 4OMEGA(2,4)T + OMEGA(3,4)

Here the Chern polynomial of G(1, 3) has previously been computed and assigned to the variable CHEPO13 , using the function G .

Similarly,

(C 6) OMEGATRANSFORM(1, 3, SEPO13) ;

(D 7) 28OMEGA(1,2)T^4 - 28OMEGA(1,3)T^3 + (9OMEGA(2,3) + 9OMEGA(1,4))T^2
- 4OMEGA(2,4)T + OMEGA(3,4) .

The results of the function calls with $r = 1$, $A = 4, 5$ are given in the appendix, § 3.

What we really need are the degrees of the Chern- and Segre classes, however. To compute these, we may use the following classical formula, given in [HP]:

$$\dim\left(\Omega_{\alpha_0, \ldots, \alpha_r}\right) = \sum_{i=0}^{r} \alpha_i - \tfrac{1}{2}r(r+1)$$

$$\deg\left(\Omega_{\alpha_0, \ldots, \alpha_r}\right) = \frac{\left(\dim \Omega_{\alpha_0, \ldots, \alpha_r}\right)!}{\prod_{i=0}^{r} (\alpha_i !)} \prod_{\lambda>\mu} (\alpha_\lambda - \alpha_\mu) \qquad (3.2)$$

To get the formulae in a reasonable form, we had to introduce the Schubert-symbols $\Omega_{\alpha_0, \ldots, \alpha_r}$, which are related to $\Omega(a_1, \ldots, a_{A-r})$

in the following way (for proof, see [Hm 7] Lemma 3.6):

Lemma 3.3 $\Omega_{\alpha_0, \ldots, \alpha_r} = \Omega(a_1, \ldots, a_{A-r})$, where

$$1 \leq a_1 < a_2 < \ldots < a_{A-r} \leq A + 1$$

are the numbers obtained from

$$\{1, 2, \ldots, A + 1\}$$

by deleting

$$\{A - \alpha_r + 1, \ldots, A - \alpha_0 + 1\} .$$

Thus for instance

$$\Omega_{0,1, \ldots, r} = \Omega(1, \ldots, A - r)$$

and in fact this is the only Schubert cycle of dimension zero. Obviously it has degree 1.

We therefore have an alternative way of computing the degrees of the Chern- and Segre classes: Writing

$$P(t) = 1 + c_1(G)t + \ldots + c_{dim}(G)t^{dim}$$

where

$$G = G(r, A), \; dim = (r + 1)(A - r)$$

and $c_1, \ldots, c_{dim}$ are the Chern classes, we put

$$Q(t) = c_1(Q)^{dim} + c_1(G)c_1(Q)^{dim-1}t + \ldots$$

$$+ c_{dim-1}(G)c_1(Q)t^{dim-1} + c_{dim}(G)t^{dim}$$

Then putting

$$\omega = \Omega(1, \ldots, A - r) ,$$

OMEGATRANSFORM applied to $Q(t)$ will return

$$e_0\omega + e_1\omega t + \ldots + e_{dim}\omega t^{dim}$$

where

$$e_i = \deg(c_i(G)) .$$

Similarly for the Segre Polynomial.

The removal of ω in the output is a matter of stream-lining which I did not carry out. For simplicity I gave the output as

"P(t) = Q(t)"

which is of course incorrect. Q(t) is the "degree" of P(t) .

Since OMEGATRANSFORM is rather time-consuming, an attempt was made to save some time by arranging the computation somewhat differently, through the function PIERI and POLYSGOMEGA. Their definitions, as well as the results for G(1, 3) and G(1, 4) , is given in the appendix, § 3.

Unfortunately the computation became too long already for G(1, 5), so that this approach is clearly not feasible on MACSYMA. However, the principle itself might still work on a computer for higher Grassmanians. What we have demonstrated here, then, is that Schubert Calculus can indeed be implemented on a computer; at least the amount of Schubert Calculus needed for the current project.

Some alternatives to this method will be discussed in Sections 5 and 6.

§ 4. Projective embeddings and duality.

Let X be a non-singular, projective variety over the algebraically closed field k, embedded in projective N-space by the embedding

$$i : X \hookrightarrow \mathbb{P}^N_k .$$

In this setting, we define the m th embedding-obstruction of the embedded variety X as

$$\gamma_m = \deg(X)^2 - \sum_{j=0}^{m-\dim(X)} \binom{m+1}{m-\dim(X)-j} p_j(X, i)$$

provided that $m \leq 2\dim(X)$, while $\gamma_m = 0$ for $m \geq 2\dim(X) + 1$. Here $\deg(X)$ is the degree of X with respect to the embedding i, $\dim(X)$ is the dimension of X and

$$p_j(X, i) = \text{degree}(s_j(X))$$

is the degree, with respect to the embedding i, of the j th Segre class of X. One then has the following

<u>Theorem 4.1</u>. X <u>may be embedded into</u> $\mathbb{P}^m_k$ <u>by a projection from</u> $\mathbb{P}^N_k$ <u>if and only if</u> $\gamma_m = 0$.

For proofs of this and related results, see for instance [Hm 2, 3, 4, 7], [HR], [Jn], [Lk 2-4] and [Rb 1-5] as well as [Kl].

It is sometimes convenient to express γ_m in a slightly different form. In fact, letting D denote the divisor class which corresponds to the embedding i, we have

$$\deg(X) = D^{\dim(X)}$$

$$p_j(X, i) = D^{\dim(X)-j} s_j(X) ,$$

where we identify $A^{\dim(X)}(X)$ and $\mathbb{Z}$ via the degree-map

$$\deg: A^{\dim(X)}(X) \xrightarrow[\cong]{} \mathbb{Z}$$

Thus

$$\gamma_m = \left(D^{\dim(X)}\right)^2 - \sum_{j=0}^{m-\dim(X)} \binom{m+1}{m-\dim(X)-j} D^{\dim(X)-j} s_j(X) .$$

Furthermore, the generic projection from $\mathbb{P}_k^N$ to $\mathbb{P}_k^m$ induces a morphism

$$p_m: \ X \longrightarrow X' \subset \mathbb{P}_k^m ,$$

which has a ramification cycle on X denoted by $\mathrm{Ram}(p_m)$. We now have (see [Rb], [Jn], [HR]) the

<u>Theorem 4.2.</u> <u>The degree of the cycle</u> $\mathrm{Ram}(p_m)$ <u>is given by the expression</u>

$$\mathrm{ram}_m = \sum_{j=0}^{m+1-\dim(X)} \binom{m+1}{j+1} D^{\dim(X)-j} s_j(X)$$

The observation that for $m \leq 2\dim(X)$,

$$\gamma_m - \gamma_{m-1} = \mathrm{ram}_{m-1}$$

yields the

<u>Corollary 4.2.1.</u> <u>Assume that</u> $\gamma_{2\dim(X)} = 0$. <u>Then</u> X <u>can be embedded into</u> $\mathbb{P}^m$ <u>via a projection from</u> $\mathbb{P}^N$ <u>if and only if</u>

$$\mathrm{ram}_m = 0 .$$

K. Johnson in [Jn] conjectured that

$$\mathrm{ram}_{2\dim(X)-1} = 0 \Rightarrow \gamma_{2\dim(X)} = 0 .$$

This is shown by W. Fulton and J. Hansen in F-H as a corollary of their remarkable connectedness-theorem. One of the initial motivations for the computations below was to check this conjecture against various families of examples, to be generated as described in section 6.

The following functions will generate the Chern polynomial and the Segre polynomial of X in terms of the Chern classes $c_1, \ldots, c_{dim}$ of X:

```
CHERNPOLY(DIM,T):=1+SUM(CONCAT(C,J)*T^J,J,1,DIM);

SEGREPOLY(DIM,T):=TAYLOR(1/CHERNPOLY(DIM,T),T,0,DIM);
```

We proceed with the functions which return the m th embedding obstruction and the degree of the ramification cycle $\mathrm{Ram}(p_m)$:

```
GAMMASEGRE(DIM,M):=DEG^2
     -(BINOMIAL(M+1,M-DIM)*D^DIM
     +SUM(BINOMIAL(M+1,M-DIM-J)*D^(DIM-J)*CONCAT(S,J),J,
          1,M-DIM));
```

It is convenient not to substitute D^{dim} for deg in the first term, see for instance the remark on the use of OMEGATRANSFORM in section 3.

Next, we write the same entity in terms of the Chern classes of X:

```
GAMMACHERN(DIM,M):=BLOCK([Q,P],Q:GAMMASEGRE(DIM,M),
            P:SEGREPOLY(DIM,T),
            FOR I THRU DIM DO
                Q:SUBST(COEFF(P,T,I),CONCAT(S,I),Q),EXPAND(Q));
```

Analogously we express the degree of the ramification cycle as follows:

```
RAMSEGRE(DIM,M):=BINOMIAL(M+1,DIM)*D^DIM
        +SUM(BINOMIAL(M+1,J+DIM)*D^(DIM-J)*CONCAT(S,J),J,1,
            M-DIM+1);
```

```
RAMCHERN(DIM,M):=BLOCK([Q,P],Q:RAMSEGRE(DIM,M),
        P:SEGREPOLY(DIM,T),
        FOR I THRU DIM DO Q:SUBST(COEFF(P,T,I),CONCAT(S,I),Q),
        EXPAND(Q));
```

Finally we define the m th defect-obstruction of the embedded variety $X \hookrightarrow \mathbb{P}^N$ as

$$\delta_s = \sum_{i=s}^{\dim} \binom{i+1}{s+1} e_{\dim - i} = \sum_{i=s}^{\dim} \binom{i+1}{s+1} D^i c_{\dim - i}(X) .$$

The reason for this name is as follows: The dual variety $X^\vee$ of X with respect to the embedding i is "normally" a hypersurface, i.e. a subvariety of $\mathbb{P}^N$ of dimension $N - 1$, see for instance [Kl]. But more precisely we have the following result, which is proved in [Hm 7]:

<u>Theorem 4.3.</u> <u>Assume that</u>

$$\delta_0 = \delta_1 = \ldots = \delta_{m-1} = 0 , \delta_m \neq 0 .$$

<u>Then</u>

$$\dim(X^\vee) = N - 1 - m .$$

Following A. Landman [Ln 1, 2] we shall call

$$m = N - 1 - \dim(X^\vee)$$

the <u>(duality) defect</u> of X.

The corresponding function in MACSYMA is:

```
DELTACHERN(N, DIM, M): = SUM(BINOMIAL(I + 1, M + 1)
        *D^I*CONCAT(C, DIM - I), I, M, DIM - 1)
        + BINOMIAL(DIM + 1, M + 1)*D^DIM
```

Our intended use of the above functions is the following:

We will consider a certain class of embedded varieties, for which there is given a procedure for computing the Chern classes $c_1, \ldots, c_{dim}$ as well as the intersection numbers

$$D^{n-(i_1+\ldots+i_n)} c_1^{i_1} \ldots c_n^{i_n}, \quad n = \dim .$$

This could for instance be the varieties generated from a given one by a finite number of general hypersurface sections, or the ones obtained by blowing up certain loci; possibly both. Reasonable choices for the given variety - which we might call the <u>generating variety</u> - would be a Veronese-variety, or products of some simple given varieties embedded by the Segre-embedding, or finally a Grassmanian or more generally a flag-manifold. We return to this in Section 5.

In an environment established in this way, by a generating variety and a generating procedure, one would then compute the numbers

$$\gamma_m, \; ram_m, \; \delta_m .$$

We return to the significance of this in Section 5.

The most straightforward approach, which would also be the most flexible and therefore preferable to other alternatives, would be to compute the Chern polynomial of a generated variety, then the intersection numbers above and substitute them into the results of

RAMSEGRE(DIM, M), RAMCHERN(DIM, M)

DELTACHERN(N, DIM, M).

In order to explore the feasability of this, and also to establish the base for using Grassmanians as generating varieties, we carried out some of the above computations for Grassmanians up to G(2, 6) . Of course both embedding and duality properties of G(1, A) is well known from geometric theory, see [Hm 7], so no new results were expected from the computations in these cases. In fact, it is known that G(1, A) , which is of dimension

$$n = 2(A - 1) ,$$

can be embedded into $\mathbb{P}^{2n-3}$ via a projection from the Plücker embedding, and this is the best possible. Moreover, if A is even and ≥ 4 , then

$$\dim(X^{\vee}) = N - 3$$

where

$$N = \binom{A+1}{2} - 1$$

is the embedding-dimension of the Plücker embedding. If A is odd, then

$$\dim(X^{\vee}) = N - 1 .$$

These results are due to A. Landman, [Lm 1, 2].

However, from what follows we do make the observation that both G(2, 5) and G(2, 6) can not be projected into a space of lower dimension than 2n + 1 , where n is the dimension of G(2, 5) resp. G(2, 6), starting with the Plücker embedding. But this can probably be shown easily by direct geometric and elementary methods, and is

not the main point of what follows.

The main point was to see if the computational method referred to above is feasible. Unfortunately the outcome is that at the present time, at least, this direct approach does not seem to be practicable. In sections 5 and 6 we establish an alternative procedure which is less flexible, but which is better suited for large computations.

The function

$$\mathrm{GRASS}(arg_1, arg_2)$$

will, when given the arguments

$$arg_1 = r, arg_2 = A ;$$

print the Chern polynomial of $G(r, A)$ in terms of the Chern classes of Q, and finally evaluate

$$\gamma_{2\dim(X)}, ram_{2\dim(X)-1}, \ldots$$

stopping with the first non-zero one. I also worked with a variation of GRASS which always computes at least one ram_i, in order to check Johnson's conjecture. The functions are as follows:

```
GRASS(R,A):=BLOCK([DIM,M,GS,DE,N,TEST],DIM:(R+1)*(A-R),
      CHEPO:CHERNPOLYGRASS(R,A,T),
      PRINT("THE CHERNPOLYNOMIAL OF GRASS(",R,",",A,") IS: ",
            CHEPO),
      PRINT("THE RELATIONS OF THE CHERNCLASSES OF Q ARE:"),
      RELATIONSOFCHERNCLASSESG(R,A),M:2*DIM,
      GS:GAMMASEGRE(DIM,M),PRINT(ARRAYAPPLY(GAMMA,[M]),"=",GS),
      GC:GAMMACHERN(DIM,M),GC:GRASSEVAL(GC),
      GC:OMEGATRANSFORM(R,A,GC-DEG^2),
      DE:PROD(I!,I,1,R)*DIM!/PROD(I!,I,A-R,A),


         N:NUMFACTOR(GC)+DE^2,PRINT("= ",N),TEST:0,
         FOR I THRU DIM WHILE EQUAL(TEST,0) DO
             (M:M-1,GS:RAMSEGRE(DIM,M),
              PRINT(ARRAYAPPLY(RAM,[M]),"=",GS),GC:RAMCHERN(DIM,M),
              GC:GRASSEVAL(GC),GC:OMEGATRANSFORM(R,A,GC),
              N:NUMFACTOR(GC),PRINT("=",N),TEST:N),
         PRINT("DEG =",DE));
```

not the main point of what follows.

The main point was to see if the computational method referred to above is feasible. Unfortunately the outcome is that at the present time, at least, this direct approach does not seem to be practicable. In sections 5 and 6 we establish an alternative procedure which is less flexible, but which is better suited for large computations.

The function

$$GRASS(arg_1, arg_2)$$

will, when given the arguments

$$arg_1 = r, \; arg_2 = A \; ;$$

print the Chern polynomial of $Q(r, A)$ in terms of the Chern classes of Q, and finally evaluate

$$\gamma_{2dim(X)}, \; ram_{2dim(X)-1}, \; \ldots$$

stopping with the first non-zero one. I also worked with a variation of GRASS which always computes at least one ram_i, in order to check Johnson's conjecture. The functions are as follows:

```
GRASS(R,A):=BLOCK([DIM,M,GS,DE,N,TEST],DIM:(R+1)*(A-R),
    CHEPO:CHERNPOLYGRASS(R,A,T),
    PRINT("THE CHERNPOLYNOMIAL OF GRASS(",R,",",A,") IS: ",
      CHEPO),
    PRINT("THE RELATIONS OF THE CHERNCLASSES OF Q ARE:"),
    RELATIONSOFCHERNCLASSESQ(R,A),M:2*DIM,
    GS:GAMMASECHE(DIM,M),PRINT(ARRAYAPPLY(GAMMA,[M]),"=",GS),
    GC:GAMMACHERN(DIM,M),GC:GRASSEVAL(GC),
    GC:OMEGATRANSFORM(R,A,GC,DEG^2),
    DE:PROD(I!,I,1,R)*DIM!/PROD(I!,I,A-R,A),



    N:NUMFACTOR(GC)+DE^2,PRINT("= ",N),TEST:0,
    FOR I THRU DIM WHILE EQUAL(TEST,0) DO
      (M:M-1,GS:RAMSECHE(DIM,M),
       PRINT(ARRAYAPPLY(RAM,[M]),"=",GS),GC:RAMCHERN(DIM,M),
       GC:GRASSEVAL(GC),GC:OMEGATRANSFORM(R,A,GC),
       N:NUMFACTOR(GC),PRINT("=",N),TEST:N),
    PRINT("DEG =",DE));
```

The function GRASSEVAL substitutes the Chern-classes for the Grassmanians into the general expressions:

```
GRASSEVAL(P):=BLOCK([H],
          FOR I THRU DIM DO
              H:RATSUBST(COEFF(CHEPO,T,I),CONCAT(C,I),P),
          H:RATSUBST(C1Q,D,H),EXPAND(H),
          FOR I THRU DIM DO
              H:RATSUBST(COEFF(CHEPO,T,I),CONCAT(C,I),H),
          EXPAND(H));
```

Output from this function is listed in the appendix, § 4. The assertions made above about G(2, 5) and G(2, 6) follow from that.

If one were only interested in the data actually printed out by the function, this would not be the most efficient way to proceed: Indeed, one could then find the degrees of the Segre classes as in section 3, and substitute the result directly into the expressions for γ_m and ram_m . But the function GRASS was also used to give γ_m and ram_m expressed in terms of $c_1(Q), \ldots, c_{r+1}(Q)$, by inserting a PRINT statement after the statement

GC : GRASSEVAL(FC)

This was done in order to look for a possible pattern, which conceivably could lead to simpler formulae for γ_m and ram_m in terms of the Chern classes of Q . But as no pattern seemed to emerge, it was abandoned. Another approach, which also turned out to be impracticable on MACSYMA, was to determine whether or not γ_m was zero by checking if the expression returned for γ_m in terms of $c_1(Q), \ldots, c_{r+1}(Q)$ was contained in the ideal in $\mathbf{Z}[c_1, \ldots, c_{r+1}]$ generated by the relations of the Chern classes. For instance, we obtained the following results from the last version of GRASS:

```
(C9) grass(1,3);

THE CHERNPOLYNOMIAL OF GRASS( 1 , 3 ) IS:

       2         2            4      4           3   3           2  2
(4 C2Q   - 4 C1Q   C2Q + 3 C1Q  ) T    + 6 C1Q   T   + 7 C1Q   T   + 4 C1Q T + 1

 THE RELATIONS OF THE CHERNCLASSES OF Q ARE:
    3
 C1Q   - 2 C1Q C2Q  = 0
    2       2        4
 C2Q   - 3 C1Q  C2Q + C1Q   = 0
                             2           3           2        4
 GAMMA = - S4 - 9 D S3 - 36 D   S2 - 84 D   S1 + DEG   - 126 D
      8
      2       2        2          4
 = DEG  + 4 C2Q  - 4 C1Q  C2Q - 2 C1Q

 = 0
                            2          3          4
 RAM  = S4 + 8 D S3 + 28 D   S2 + 56 D   S1 + 70 D
    7
          2          2
 = 4 C1Q   C2Q - 4 C2Q
 = 0
                    2          3           4
 RAM  = D S3 + 7 D   S2 + 21 D   S1 + 35 D
    6
 = 0
 = 0
         2          3          4
 RAM  = D  S2 + 6 D   S1 + 15 D
    5
 = 0
 = 0
         3          4
 RAM  = D   S1 + 5 D
    4
      4
 = C1Q
 = 2
 DEG = 2
```

This certainly looks promising: Indeed, both ram_5 and ram_6 vanish identically here. However, it should be pointed out that had the function SEGREPOLYGRASS been used as in GG, then the resulting Chern polynomial would have been more complicated even in this simple case, so that only ram_5 would have vanished identically. We then would have gotten

$$RAM_6 = 4C1Q^2C2Q - 2C1Q^4$$

Moreover, we get

$$\gamma_7 = 4c_1(Q)^2c_2(Q) - 4c_2(Q)^2$$

and since

$$c_1(c_1^3 - 2c_1c_2) - (c_2^2 - c_1^2c_2 - 2c_1^2c_2 + c_1^4)$$
$$= c_1^2c_2 - c_2^2$$

we also get $\gamma_7 = 0$ by the indicated method.

For G(1, 4) the expressions are more complicated, and in fact none of them vanish identically. But the expressions are still managable enough, so that the method indicated would work. But for G(1, 5), G(2, 5), G(1, 6) and G(2, 6) the expressions become very large, and so do the relations of the Chern classes. Moreover, already the computation of the Chern polynomial beyond G(2, 6) with the method used is a large affair, and even if such computations could be carried out with disk use and time, it is not clear that the result would be of much use in the given environment.

§ 5. Grassmanians of lines and combinatorical identities.

In this section we show how the numerical study of section 4 may be continued along somewhat different lines from those suggested by the material of section 3, at least in certain special cases.

Indeed, we now take up the case $G = G(1, N)$, Grassmanians of lines in $\mathbb{P}^N$.

We use the expression

$$c_t(Q \otimes Q^\vee) = (4c_2Q - c_1Q^2)t^2 + 1$$

which we computed in section 2 in order to find the Chern and Segre classes of G , by means of (2.4) and (2.5). Actually, using the available information on $c_t(Q \otimes Q^\vee)$, we may use this method up to Grassmanians of 3-spaces in some $\mathbb{P}^N$. Of course the expressions then become quite unmanagable, at least "by hand".
Moreover, since Q is of rank 2 , and $c_1(Q)$ is the pull back of a hyperplane class via the Plücker embedding, in order to find the degrees of the Chern and Segre classes of G , all we need are the degrees of $c_2(Q)^j$ for $j \leq N - 1$. By Proposition 3.6 of [Hm 7] we now have

$$c_2(Q)^j = \Omega(1, 2, \ldots, j, j + 3, \ldots, N + 1) ,$$

and since

$$\{1, 2, \ldots, j, j + 3, \ldots, N + 1\} =$$
$$\{1, 2, \ldots, N + 1\} - \{N - a_1 + 1, N - a_2 + 1\}$$

gives

$$N - a_1 + 1 = j + 1, N - a_0 + 1 = j + 2$$

so that

$$a_1 = N - j,\ a_0 = N - j - 1$$

we have that

$$c_2(Q)^j = \Omega_{N-j-1,N-j}$$

by Lemma 3.3. Thus (3.2) gives

$$\deg(c_2(Q)^j) = \frac{(2(N-1-j))!}{(N-1-j)!(N-j)!} \tag{5.1}$$

where the degree is taken with respect to the Plücker-embedding.

This method may also be used to compute the degrees of the monomials in the Chern classes of Q for higher Grassmanians, i.e. for $G(r, N)$'s with $r > 1$. But of course the size of the computations rapidly become quite large, and simple closed form expressions like (5.1) can not be expected.

If the intention with these computations were merely to compute the embedding- and duality numbers of section 4 for Grassmanians, it might not be worth the effort: Indeed, the embedding dimension of $G(1, N)$ via a projection from the Plücker embedding is of course well known and the result easy to prove, [Hm 7] section 3. Further, A. Landman proved in [Lm 1, 2] that the dual variety of $G(1, N)$ with respect to the Plücker embedding is of codimension 3 provided N is even and ≥ 4 , and of codimension 1 otherwise.

However, the main point with these computations is that it becomes possible to use $G(1, N)$'s - and with the extention indicated above $G(r, N)$'s with $r \leq 3$ - as generating varieties in the sense of section 6.

Moreover, it turns out as we shall see below, that for $G(1, N)$'s the above known information on embeddings and duality yields certain

combinatorical identities, which appear to have no easy direct proofs. Thus the geometry is in this case more likely to yield combinatorical information than the other way around. This is not a new situation in Schubert Calculus.

We start with γ_m and ram_m for $G(1, N)$, and get the following (see [Hm 7] section 4; $c_i = c_i(Q)$):

$$s_\ell(G(1,N)) =$$

$$\sum_{j=0}^{[\frac{\ell}{2}]} (-1)^{\ell-j}\left\{\binom{N+\ell-j}{\ell-j}\binom{\ell-j}{j} - 4\binom{N+\ell-1-j}{\ell-1-j}\binom{\ell-1-j}{j-1} - \binom{N-\ell-2-j}{\ell-2-j}\binom{\ell-2-j}{j}\right\} c_1^{\ell-2j} c_2^j$$

Hence

$$\deg(s_\ell(G(1,N))) =$$

$$\sum_{j=0}^{[\frac{\ell}{2}]} (-1)^{\ell-j}\left\{\binom{N+\ell-j}{\ell-j}\binom{\ell-j}{j} - 4\binom{N+\ell-1-j}{\ell-1-j}\binom{\ell-1-j}{j-1} - \binom{N-\ell-2-j}{\ell-2-j}\binom{\ell-2-j}{j}\right\} \frac{(2(N-1-j))!}{(N-1-j)!(N-j)!}$$

Letting $n = \dim(G(1, N)) = 2N - 2$, the fact that $G(1, N)$ may be embedded into $\mathbb{P}^{2n-3}$ via a projection from the Plücker embedding is equivalent to

$$\gamma_{2n-3} = 0 ,$$

which after straightforward computations yields the combinatorical identity (for $N \geq 3$):

$$\sum_{j=0}^{N-3}\sum_{\ell=2j}^{2N-5}(-1)^{\ell-j}\binom{4N-6}{2N-5-\ell}\left\{\binom{N+\ell-j}{\ell-j}\binom{\ell-j}{j}\right.$$

$$\left.-4\binom{N+\ell-1-j}{\ell-1-j}\binom{\ell-1-j}{j-1}-\binom{N+\ell-2-j}{\ell-2-j}\binom{\ell-2-j}{j}\right\}\frac{(2(N-1-j))!}{(N-1-j)!(N-j)!}$$

$$=\left\{\frac{(2(N-1))!}{(N-1)!N!}\right\}^2 .$$

Similarly one obtains combinatorical identities from

$ram_{2n-1} = ram_{2n-2} = ram_{2n-3} = 0$.

We next turn to the duality-deficiency. First we obtain the following expressions for the Chern classes:

$$c_r(G(1,N)) =$$

$$\sum_{\substack{\ell+2s=r\\0\le\ell\le r}}(-1)^s(4c_2-c_1^{\,2})^s\sum_{j=0}^{[\frac{\ell}{2}]}\binom{N+1}{\ell-j}\binom{\ell-j}{j}c_1^{\,\ell-2j}c_2^{\,j} =$$

$$\sum_{s=0}^{[\frac{r}{2}]}\sum_{j=0}^{[\frac{r}{2}]-s}\sum_{i=0}^{s}(-1)^{s+i}4^{s-i}\binom{s}{i}\binom{N+1}{r-2s-j}\binom{r-2s-j}{j}c_1^{\,r+2(i-s-j)}c_2^{\,s+j-i}$$

Thus

$$\deg(e_r(G(1,N))) =$$

$$\sum_{s=0}^{[\frac{r}{2}]}\sum_{j=0}^{[\frac{r}{2}]-s}\sum_{i=0}^{s}(-1)^{s+i}4^{s-i}\binom{s}{i}\binom{N+1}{r-2s-j}\binom{r-2s-j}{j}\frac{(2(N-1-s-j+i))!}{(N-1-s-j+i)!(N-s-j+i)!}$$

We get

$$\delta_m = \sum_{i=m}^{n}\binom{i+1}{m+1}\deg(c_{n-i}(\Omega^1_{G(1,N)})) =$$

$$\sum_{i=m}^{n}\sum_{j=0}^{[\frac{n-i}{2}]}\sum_{k=0}^{[\frac{n-i}{2}]}\sum_{\ell=0}^{j}(-1)^{n-i+j+\ell}4^{j-\ell}\binom{i+1}{m+1}\binom{j}{\ell}\binom{N+1}{n-i-j-k}\binom{n-i-j-k}{k-j}\frac{(2(N-1-k+\ell))!}{(N-1-k+\ell)!(N-k+\ell)!}$$

after a straightforward computation.

Computation of the first few values of δ_m yields the table on the following page.

" ' ' . . . . . .

As we see, this is in good agreement with Landmans results quoted above. Moreover, there are many clearly appearing patterns in the table. It would be nice to have <u>geometric</u> proofs for these. So far, we can only ecplain the zeroes, via Landman's results qouted above.

m \ N	4	5	6	7	8	9	10	11	12	13	14	15
0	0	3	0	4	0	5	0	6	0	7	0	8
1	0	6	0	12	0	20	0	30	0	42	0	56
2	5	12	14	36	30	80	55	150	91	252	140	392
3	10	24	56	108	180	320	440	750	910	1512	1680	2744
4	12	48	140	324	684	1280	2244	3750	5824	9072	12936	19208
5	10	68	266	804	2022	4460	8998	16748	29484	49336	79100	123032
6	5	66	395	1572	4887	12890	30041	63720	125502	232316	409170	689864
7		42	434	2412	9612	30770	84282	205344	456144	940716	1824540	3360248
8		14	336	2868	15232	60710	198704	562256	1421888	3287396	7061920	14268296
9			168	2556	19224	98564	393184	1310352	3810560	9952432	23825712	53065784
10			42	1620	18954	130520	650952	2598096	8791016	26168352	70291140	173525000
11				660	14124	139040	896104	4372000	17456088	59829392	181727240	500302360
12				132	7524	116600	1015388	6214656	29779860	118936552	412166460	1274183560
13					2574	74360	932646	7405944	43488822	205289832	820084110	2869261000
14					429	34034	678535	7316036	54048189	306773054	1429858495	5712970696
15						10010	377234	5893524	56679012	395068198	2179563630	10048890712
16						1430	151008	3779100	49548096	435557178	2893948320	15585370984
17							38896	1859052	35490832	407295588	3329380640	21249994968
18							4862	660348	20317414	318938088	3294841740	25362343656
19								151164	8952268	205443504	2777402064	26344420056
20								16796	2855320	106119712	1967883960	23632021096
21									587860	42307832	1150726260	18120798712
22									58786	12236532	541118670	11715832744
23										2288132	196868500	6268827928
24										208012	52062432	2703977704
25											8914800	904257880
26											742900	220195560
27												34767720
28												2674440
29												
30												

§ 6. Generating varieties and classes of projective varieties.

A well known conjecture by Horrocks and Mumford, [HM], asserts that if X is a variety embedded in $\mathbb{P}^n$, and if X is non-singular and of codimension 2, then X is a complete intersection provided $n \geq 6$.

A related conjecture by Hartshorne, [Hn] asserts the same conclusion under the assumption that X is nonsingular and of dimension $> \frac{2}{3} n$, provided $n \geq 7$.

These conjectures clearly testify to the current lack of information on examples of non singular projective varieties of high dimension.

If a classification theory of the type carried out by Swinnerton-Dyer [Sw] for varieties of degree 4 could be carried out in general, these and many other questions would of course be settled. But such a general classification-theory is clearly not in sight at this time.

However, the fundamental idea underlying Swinnerton-Dyers classification is to obtain all varieties from a fundamental set of varieties by processes such as blowing up subvarieties and taking hyperplane sections. While such a procedure might not yield a complete classification in general, it still can be used to generate lists of examples, and provide the starting point for a systematical search for counterexamples to conjectures such as the above.

Initially one should search for subvarieties of projective space, of high diemnsion and of relatively low codimension. One would then be able to get some idea of how frequent non-singular varieties of this type are, and of course examine such questions as whether they are complete intersections or not.

The procedure which we propose is therefore to start out with a variety such as for instance a suitable Grassmanian, a Veronese-

variety or a product of projective spaces embedded by the Segre-embedding (see [Hm 7], [Da] and [Vi] for details on the last two cases). We would then modify the situation by blowing up suitable subschemes, such as a finite number of possibly mutiple points or Schubert-varieties in the case of Grassmanians. In all of these cases there are well known algorithms for computation of the Chern- and Segre polynomials of the new variety in terms of that of the old one and data associated with the center of blowing up. Moreover, the new variety comes with a natural projective embedding, given by the projective embedding of the blow-up of the ambient space with the given center induced from the interpretation of a blow-up as a monoidal transformation.

Thus there are natural projective embeddings of the results of each modification, and furthermore algorithms for the computation of the corresponding degrees of the Chern- and Segre classes.

Another possible modification is for instance to take sections with hypersurfaces, in which case the degrees of the new Chern- and Segre classes are expressed in terms of the corresponding data for the original variety and the degree of the hypersurface by particularly simple formulae.

Thus once an initial variety is selected - which we will call the generating variety - a whole class of projective varieties is generated by all possible modifications of the type described above applied repeatedly in any order. Of course the class of admissible modifications may vary, one could for example start with the simplest one which is to take hypersurface sections. Also, it is no essential limitation to restrict the permissible zero dimensional centers for the blowing ups to simple points.

To get a good picture of each such generated class of projective varieties, one should ideally have classified the varieties in them according to projective equivalence. In principle this should be

possible to do on a computer, with the availability of a sufficiently powerfull system for symbolic manipulation like MACSYMA. But given the results from section 3 in particular, I rather doubt that an investigation along these lines is feasible at this time. This is mainly due to the size such computations would necessarily have.

However, there is a coarser equivalence of a homological nature which is well suited for a computer. Under this equivalence varieties for which the degrees of all monomials in the Chern classes are equal, and for which the Chow rings are not "too different", will be identified.

All of the above adresses itself to non-singular varieties. But singular varieties may also be considered. In the singular case we will have to compute the invariants introduced in [Hm 4], and which are actually degrees of different types of Segre classes introduced later in the singular case by Fulton and Mc Phersson, [FM]. Moreover, one also needs the degrees of Fultons singular Chern classes, [F] and [Hm 6]. If one allows singularities, the computation of the invariants of the modified variety in terms of those of the original one becomes more difficult, and further development of the general theory is needed before this can be done. The reason why this would be of interest, is among other things the following: We may obtain more <u>non singular</u> varieties if we allow singular generating varieties. Also, the choice of admissible modifications is greatly expanded, since we may take <u>cones</u> over given subvarieties, or more generally form the <u>join</u> of two given projective subvarieties of some projective space, see [AK]. Furthermore, we can modify by <u>deformations</u> as in [Hm 5] or as in [Hm 1].

The program outlined above will constitute the continuation of this paper, [Hm 8] and [Hm 9].

Appendix Printouts

§ 1. Output from the functions TENSOR and EXTERIOR 2.

```
(C12) A[1];
(D12)                         C1A T + 1

(C13) A[2];
                               2
(D13)                     C2A T  + C1A T + 1

(C14) A[3];
                         3        2
(D14)               C3A T  + C2A T  + C1A T + 1

(C15) B[1];
(D15)                         C1B T + 1

(C16) B[2];
                               2
(D16)                     C2B T  + C1B T + 1

(C17) B[3];
                         3        2
(D17)               C3B T  + C2B T  + C1B T + 1

(C18) TENSOR(A[1],1,B[1],1);
(D18)                   (C1B + C1A) T + 1

(C19) TENSOR(A[1],1,B[2],2);
                                     2   2
(D19)         (C2B + C1A C1B + C1A ) T  + (C1B + 2 C1A) T + 1

(C20) TENSOR(A[1],1,B[3],3);
                                2            3   3                              2   2
(D20) (C3B + C1A C2B + C1A  C1B + C1A ) T  + (C2B + 2 C1A C1B + 3 C1A ) T

                                                        + (C1B + 3 C1A) T + 1

(C21) TENSOR(A[2],2,B[2],2);
          2                                    2           2          2
(D21) (C2B  - 2 C2A C2B + C1A C1B C2B + C1A  C2B + C2A  + C1B  C2A

                       4                                                              2
 + C1A C1B C2A) T  + (2 C1B C2B + 2 C1A C2B + 2 C1B C2A + 2 C1A C2A + C1A C1B

       2         3                           2                       2   2
 + C1A  C1B) T  + (2 C2B + 2 C2A + C1B  + 3 C1A C1B + C1A ) T

 + (2 C1B + 2 C1A) T + 1

(C22) TENSOR(A[3],3,B[3],3);
         3              2                        2                    2                       2
(D22) (C3B  + 3 C3A C3B  + C1A C2B C3B  - 2 C1B C2A C3B  - 3 C1A C2A C3B

       2            2       3    2          2
 + C1A  C1B C3B  + C1A  C3B  + 3 C3A  C3B - 3 C1B C2B C3A C3B

                                                                                2
 - C1A C2B C3A C3B - C1B C2A C3A C3B - 3 C1A C2A C3A C3B - 2 C1A C1B  C3A C3B

         2                           2            2
 - 2 C1A  C1B C3A C3B + C2A C2B  C3B - 2 C2A  C2B C3B + C1A C1B C2A C2B C3B
```

$$+ C1A^{2}\ C2A\ C2B\ C3B + C2A^{3}\ C3B + C1B^{2}\ C2A^{2}\ C3B + C1A\ C1B\ C2A^{2}\ C3B + C3A^{3}$$

$$- 3\ C1B\ C2B\ C3A^{2} - 2\ C1A\ C2B\ C3A^{2} + C1B\ C2A\ C3A^{2} + C1B^{3}\ C3A^{2} + C1A\ C1B^{2}\ C3A^{2}$$

$$+ C2B^{3}\ C3A - 2\ C2A\ C2B^{2}\ C3A + C1A\ C1B\ C2B^{2}\ C3A + C1A^{2}\ C2B^{2}\ C3A + C2A^{2}\ C2B\ C3A$$

$$+ C1B^{2}\ C2A\ C2B\ C3A + C1A\ C1B\ C2A\ C2B\ C3A)\ T^{9}$$

$$+ (3\ C2B\ C3B^{2} - 6\ C2A\ C3B^{2} + 2\ C1A\ C1B\ C3B^{2} + 3\ C1A^{2}\ C3B^{2} - 3\ C2B\ C3A\ C3B$$

$$- 3\ C2A\ C3A\ C3B - 6\ C1B^{2}\ C3A\ C3B - 14\ C1A\ C1B\ C3A\ C3B - 6\ C1A^{2}\ C3A\ C3B$$

$$+ 2\ C1A\ C2B^{2}\ C3B - 3\ C1A\ C2A\ C2B\ C3B + 2\ C1A^{2}\ C1B\ C2B\ C3B + 2\ C1A^{3}\ C2B\ C3B$$

$$+ 2\ C1B\ C2A^{2}\ C3B + 3\ C1A\ C2A^{2}\ C3B + 2\ C1A\ C1B^{2}\ C2A\ C3B + 2\ C1A^{2}\ C1B\ C2A\ C3B$$

$$- 6\ C2B\ C3A^{2} + 3\ C2A\ C3A^{2} + 3\ C1B^{2}\ C3A^{2} + 2\ C1A\ C1B\ C3A^{2} + 3\ C1B\ C2B^{2}\ C3A$$

$$+ 2\ C1A\ C2B^{2}\ C3A - 3\ C1B\ C2A\ C2B\ C3A + 2\ C1A\ C1B^{2}\ C2B\ C3A$$

$$+ 2\ C1A^{2}\ C1B\ C2B\ C3A + 2\ C1B\ C2A^{2}\ C3A + 2\ C1B^{3}\ C2A\ C3A + 2\ C1A\ C1B^{2}\ C2A\ C3A$$

$$+ C2A\ C2B^{3} - 2\ C2A^{2}\ C2B^{2} + C1A\ C1B\ C2A\ C2B^{2} + C1A^{2}\ C2A\ C2B^{2} + C2A^{3}\ C2B$$

$$+ C1B^{2}\ C2A^{2}\ C2B + C1A\ C1B\ C2A^{2}\ C2B)\ T^{8}$$

$$+ (3\ C1B\ C3B^{2} + 3\ C1A\ C3B^{2} - 21\ C1B\ C3A\ C3B - 21\ C1A\ C3A\ C3B + 3\ C2B^{2}\ C3B$$

$$- 6\ C2A\ C2B\ C3B + 6\ C1A\ C1B\ C2B\ C3B + 6\ C1A^{2}\ C2B\ C3B + 3\ C2A^{2}\ C3B$$

$$+ 3\ C1A\ C1B\ C2A\ C3B + 3\ C1A^{2}\ C2A\ C3B + 2\ C1A^{2}\ C1B^{2}\ C3B + 2\ C1A^{3}\ C1B\ C3B$$

$$+ 3\ C1B\ C3A^{2} + 3\ C1A\ C3A^{2} + 3\ C2B^{2}\ C3A - 6\ C2A\ C2B\ C3A + 3\ C1B^{2}\ C2B\ C3A$$

$$+ 3\ C1A\ C1B\ C2B\ C3A + 3\ C2A^{2}\ C3A + 6\ C1B^{2}\ C2A\ C3A + 6\ C1A\ C1B\ C2A\ C3A$$

$$+ 2\ C1A\ C1B^{3}\ C3A + 2\ C1A^{2}\ C1B^{2}\ C3A + C1A\ C2B^{3} + 3\ C1B\ C2A\ C2B^{2}$$

$$+ C1A^{2}\ C1B\ C2B^{2} + C1A^{3}\ C2B^{2} + 3\ C1A\ C2A^{2}\ C2B + 3\ C1A\ C1B^{2}\ C2A\ C2B$$

```
         2                         3     3  2               2    2  7
 + 3 C1A  C1B C2A C2B + C1B C2A  + C1B  C2A  + C1A C1B  C2A  ) T

           2
 + (3 C3B  - 21 C3A C3B + 6 C1B C2B C3B + 8 C1A C2B C3B - 4 C1B C2A C3B

                                  2              2                  3          2
 + 3 C1A C2A C3B + 4 C1A C1B  C3B + 8 C1A  C1B C3B + 2 C1A  C3B + 3 C3A

                                                                                  3
 + 3 C1B C2B C3A - 4 C1A C2B C3A + 8 C1B C2A C3A + 6 C1A C2A C3A + 2 C1B  C3A

              2             2               3          2                 2
 + 8 C1A C1B  C3A + 4 C1A  C1B C3A + C2B  + C2A C2B  + 4 C1A C1B C2B

        2     2       2            2
 + 3 C1A  C2B  + C2A  C2B + 4 C1B  C2A C2B + 7 C1A C1B C2A C2B

        2                2    2              3              3          2     2
 + 4 C1A  C2A C2B + 2 C1A  C1B  C2B + 2 C1A  C1B C2B + C2A  + 3 C1B  C2A

                  2            3             2    2      6
 + 4 C1A C1B C2A  + 2 C1A C1B  C2A + 2 C1A  C1B  C2A) T

                                        2                             2
 + (6 C2B C3B - 3 C2A C3B + 3 C1B  C3B + 10 C1A C1B C3B + 6 C1A  C3B

                                        2                             2
 - 3 C2B C3A + 6 C2A C3A + 6 C1B  C3A + 10 C1A C1B C3A + 3 C1A  C3A

              2              2                                                2
 + 3 C1B C2B  + 5 C1A C2B  + 6 C1B C2A C2B + 6 C1A C2A C2B + 5 C1A C1B  C2B

        2                  3                 2             2       3
 + 8 C1A  C1B C2B + 2 C1A  C2B + 5 C1B C2A  + 3 C1A C2A  + 2 C1B  C2A

              2              2                2     3       3    2   5
 + 8 C1A C1B  C2A + 5 C1A  C1B C2A + C1A  C1B  + C1A  C1B ) T

                                                                 2
 + (6 C1B C3B + 6 C1A C3B + 6 C1B C3A + 6 C1A C3A + 3 C2B  + 3 C2A C2B

        2                                 2             2        2
 + 3 C1B  C2B + 12 C1A C1B C2B + 6 C1A  C2B + 3 C2A  + 6 C1B  C2A

                              2                  3         2    2         3      4
 + 12 C1A C1B C2A + 3 C1A  C2A + 2 C1A C1B  + 5 C1A  C1B  + 2 C1A  C1B) T

                                                                              3
 + (3 C3B + 3 C3A + 6 C1B C2B + 7 C1A C2B + 7 C1B C2A + 6 C1A C2A + C1B

              2         2          3   3                             2
 + 7 C1A C1B  + 7 C1A  C1B + C1A ) T  + (3 C2B + 3 C2A + 3 C1B  + 8 C1A C1B

        2   2
 + 3 C1A ) T  + (3 C1B + 3 C1A) T + 1

(C23) EXTERIOR\2(A[1],1);
Part fell off end.
(C24) EXTERIOR\2(A[2],2);
```

(D24) $C1A\ T + 1$

(C25) EXTERIOR\2(A[3],3);

(D25) $- C3A\ T^{3} + C1A^{2}\ T^{2} + 2\ C1A\ T + 1$

(C26) EXTERIOR\2(A[4],4);

(D26) $- C3A^{2}\ T^{6} + C1A^{3}\ T^{3} + 3\ C1A^{2}\ T^{2} + 3\ C1A\ T + 1$

§ 2. Output from the functions G, GG and RE .

(C5) G(3);

$$\mathrm{CHEPO\ 1\ 3} = (4\,C2Q^{2} - 4\,C1Q^{2}\,C2Q + 3\,C1Q^{4})\,T^{4} + 6\,C1Q^{3}\,T^{3} + 7\,C1Q^{2}\,T^{2} + 4\,C1Q\,T + 1$$

$$\mathrm{SEPO\ 1\ 3} = (-4\,C2Q^{2} + 4\,C1Q^{2}\,C2Q + 14\,C1Q^{4})\,T^{4} - 14\,C1Q^{3}\,T^{3} + 9\,C1Q^{2}\,T^{2} - 4\,C1Q\,T + 1$$

$$C1Q^{3} - 2\,C1Q\,C2Q = 0$$

$$C2Q^{2} - 3\,C1Q^{2}\,C2Q + C1Q^{4} = 0$$

(C6) G(4);

$$\mathrm{CHEPO\ 1\ 4} = (4\,C2Q^{3} + 12\,C1Q^{2}\,C2Q^{2} - 13\,C1Q^{4}\,C2Q + 4\,C1Q^{6})\,T^{6}$$
$$+ (20\,C1Q\,C2Q^{2} - 20\,C1Q^{3}\,C2Q + 10\,C1Q^{5})\,T^{5}$$
$$+ (4\,C2Q^{2} - 7\,C1Q^{2}\,C2Q + 14\,C1Q^{4})\,T^{4} + 15\,C1Q^{3}\,T^{3}$$
$$+ (C2Q + 11\,C1Q^{2})\,T^{2} + 5\,C1Q\,T + 1$$

$$\mathrm{SEPO\ 1\ 4} = (3\,C2Q^{3} + 79\,C1Q^{2}\,C2Q^{2} - 426\,C1Q^{4}\,C2Q + 198\,C1Q^{6})\,T^{6}$$
$$+ (5\,C1Q\,C2Q^{2} + 150\,C1Q^{3}\,C2Q - 105\,C1Q^{5})\,T^{5}$$
$$+ (-3\,C2Q^{2} - 46\,C1Q^{2}\,C2Q + 57\,C1Q^{4})\,T^{4} + (10\,C1Q\,C2Q - 30\,C1Q^{3})\,T^{3}$$
$$+ (14\,C1Q^{2} - C2Q)\,T^{2} - 5\,C1Q\,T + 1$$

$$C2Q^{2} - 3\,C1Q^{2}\,C2Q + C1Q^{4} = 0$$

$$3\,C1Q\,C2Q^{2} - 4\,C1Q^{3}\,C2Q + C1Q^{5} = 0$$

$$-C2Q^{3} + 6\,C1Q^{2}\,C2Q^{2} - 5\,C1Q^{4}\,C2Q + C1Q^{6} = 0$$

(C7) G(5);

$$\mathrm{CHEPO\ 1\ 5} = (9\,C2Q^{4} - 18\,C1Q^{2}\,C2Q^{3} + 42\,C1Q^{4}\,C2Q^{2} - 26\,C1Q^{6}\,C2Q$$
$$+ 5\,C1Q^{8})\,T^{8} + (75\,C1Q^{3}\,C2Q^{2} - 60\,C1Q^{5}\,C2Q + 15\,C1Q^{7})\,T^{7}$$
$$+ (-6\,C2Q^{3} + 78\,C1Q^{2}\,C2Q^{2} - 66\,C1Q^{4}\,C2Q + 25\,C1Q^{6})\,T^{6}$$

$$+ (30\,\mathrm{C1Q}\,\mathrm{C2Q}^{2} - 30\,\mathrm{C1Q}^{3}\,\mathrm{C2Q} + 30\,\mathrm{C1Q}^{5})\,T^{5}$$

$$+ (7\,\mathrm{C2Q}^{2} - 2\,\mathrm{C1Q}^{2}\,\mathrm{C2Q} + 31\,\mathrm{C1Q}^{4})\,T^{4} + (6\,\mathrm{C1Q}\,\mathrm{C2Q} + 26\,\mathrm{C1Q}^{3})\,T^{3}$$

$$+ (2\,\mathrm{C2Q} + 16\,\mathrm{C1Q}^{2})\,T^{2} + 6\,\mathrm{C1Q}\,T + 1$$

$$\mathrm{SEPO}\ 1\ 5 = (- 52\,\mathrm{C2Q}^{4} + 1318\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}^{3} - 828\,\mathrm{C1Q}^{4}\,\mathrm{C2Q}^{2} - 902\,\mathrm{C1Q}^{6}\,\mathrm{C2Q}$$

$$+ 500\,\mathrm{C1Q}^{8})\,T^{8} + (- 252\,\mathrm{C1Q}\,\mathrm{C2Q}^{3} + \mathrm{C1Q}^{3}\,\mathrm{C2Q}^{2} + 904\,\mathrm{C1Q}^{5}\,\mathrm{C2Q}$$

$$- 455\,\mathrm{C1Q}^{7})\,T^{7} + (26\,\mathrm{C2Q}^{3} + 18\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}^{2} - 522\,\mathrm{C1Q}^{4}\,\mathrm{C2Q} + 319\,\mathrm{C1Q}^{6})$$

$$T^{6} + (6\,\mathrm{C1Q}\,\mathrm{C2Q}^{2} + 230\,\mathrm{C1Q}^{3}\,\mathrm{C2Q} - 194\,\mathrm{C1Q}^{5})\,T^{5}$$

$$+ (- 3\,\mathrm{C2Q}^{2} - 78\,\mathrm{C1Q}^{2}\,\mathrm{C2Q} + 105\,\mathrm{C1Q}^{4})\,T^{4} + (18\,\mathrm{C1Q}\,\mathrm{C2Q} - 50\,\mathrm{C1Q}^{3})\,T^{3}$$

$$+ (20\,\mathrm{C1Q}^{2} - 2\,\mathrm{C2Q})\,T^{2} - 6\,\mathrm{C1Q}\,T + 1$$

$$3\,\mathrm{C1Q}\,\mathrm{C2Q}^{2} - 4\,\mathrm{C1Q}^{3}\,\mathrm{C2Q} + \mathrm{C1Q}^{5} = 0$$

$$- \mathrm{C2Q}^{3} + 6\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}^{2} - 5\,\mathrm{C1Q}^{4}\,\mathrm{C2Q} + \mathrm{C1Q}^{6} = 0$$

$$- 4\,\mathrm{C1Q}\,\mathrm{C2Q}^{3} + 10\,\mathrm{C1Q}^{3}\,\mathrm{C2Q}^{2} - 6\,\mathrm{C1Q}^{5}\,\mathrm{C2Q} + \mathrm{C1Q}^{7} = 0$$

$$\mathrm{C2Q}^{4} - 10\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}^{3} + 15\,\mathrm{C1Q}^{4}\,\mathrm{C2Q}^{2} - 7\,\mathrm{C1Q}^{6}\,\mathrm{C2Q} + \mathrm{C1Q}^{8} = 0$$

$$\mathrm{CHEPO}\ 2\ 5 = (8\,\mathrm{C3Q}^{3} - 24\,\mathrm{C1Q}\,\mathrm{C2Q}\,\mathrm{C3Q}^{2} + 4\,\mathrm{C1Q}^{3}\,\mathrm{C3Q}^{2} + 52\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}^{2}\,\mathrm{C3Q}$$

$$- 36\,\mathrm{C1Q}^{4}\,\mathrm{C2Q}\,\mathrm{C3Q} + 7\,\mathrm{C1Q}^{6}\,\mathrm{C3Q} - 8\,\mathrm{C1Q}\,\mathrm{C2Q}^{4} - 20\,\mathrm{C1Q}^{3}\,\mathrm{C2Q}^{3}$$

$$+ 38\,\mathrm{C1Q}^{5}\,\mathrm{C2Q}^{2} - 21\,\mathrm{C1Q}^{7}\,\mathrm{C2Q} + 4\,\mathrm{C1Q}^{9})\,T^{9}$$

$$+ (- 22\,\mathrm{C1Q}^{2}\,\mathrm{C3Q}^{2} + 56\,\mathrm{C1Q}\,\mathrm{C2Q}^{2}\,\mathrm{C3Q} - 12\,\mathrm{C1Q}^{3}\,\mathrm{C2Q}\,\mathrm{C3Q} + 2\,\mathrm{C1Q}^{5}\,\mathrm{C3Q}$$

$$- 4\,\mathrm{C2Q}^{4} - 48\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}^{3} + 85\,\mathrm{C1Q}^{4}\,\mathrm{C2Q}^{2} - 57\,\mathrm{C1Q}^{6}\,\mathrm{C2Q} + 14\,\mathrm{C1Q}^{8})\,T^{8}$$

$$+ (- 30\,\mathrm{C1Q}\,\mathrm{C3Q}^{2} + 24\,\mathrm{C2Q}^{2}\,\mathrm{C3Q} + 36\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}\,\mathrm{C3Q} + 2\,\mathrm{C1Q}^{4}\,\mathrm{C3Q}$$

$$- 24\,\mathrm{C1Q}\,\mathrm{C2Q}^{3} + 70\,\mathrm{C1Q}^{3}\,\mathrm{C2Q}^{2} - 78\,\mathrm{C1Q}^{5}\,\mathrm{C2Q} + 28\,\mathrm{C1Q}^{7})\,T^{7}$$

$$+ (- 15\,\mathrm{C3Q}^{2} + 30\,\mathrm{C1Q}\,\mathrm{C2Q}\,\mathrm{C3Q} + 26\,\mathrm{C1Q}^{3}\,\mathrm{C3Q} + 43\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}^{2}$$

$$- 84\,\mathrm{C1Q}^{4}\,\mathrm{C2Q} + 43\,\mathrm{C1Q}^{6})\,T^{6} + (41\,\mathrm{C1Q}^{2}\,\mathrm{C3Q} + 22\,\mathrm{C1Q}\,\mathrm{C2Q}^{2}$$

$$- 63\ C1Q^{3}\ C2Q + 50\ C1Q^{5}\ )\ T^{5} + (24\ C1Q\ C3Q + 3\ C2Q^{2} - 27\ C1Q^{2}\ C2Q$$

$$+ 45\ C1Q^{4}\ )\ T^{4} + (6\ C3Q - 6\ C1Q\ C2Q + 32\ C1Q^{3}\ )\ T^{3} + 17\ C1Q^{2}\ T^{2}$$

$$+ 6\ C1Q\ T + 1$$

$$SEPO\ 2\ 5 = (- 404\ C3Q^{3} + 1212\ C1Q\ C2Q\ C3Q^{2} - 9364\ C1Q^{3}\ C3Q^{2}$$

$$- 1750\ C1Q^{2}\ C2Q^{2}\ C3Q + 19266\ C1Q^{4}\ C2Q\ C3Q - 9684\ C1Q^{6}\ C3Q$$

$$- 70\ C1Q\ C2Q^{4} + 1082\ C1Q^{3}\ C2Q^{3} - 10860\ C1Q^{5}\ C2Q^{2} + 10572\ C1Q^{7}\ C2Q$$

$$- 2586\ C1Q^{9}\ )\ T^{9} + (2596\ C1Q^{2}\ C3Q^{2} - 8\ C1Q\ C2Q^{2}\ C3Q$$

$$- 5184\ C1Q^{3}\ C2Q\ C3Q + 3878\ C1Q^{5}\ C3Q + 13\ C2Q^{4} - 18\ C1Q^{2}\ C2Q^{3}$$

$$+ 2923\ C1Q^{4}\ C2Q^{2} - 4200\ C1Q^{6}\ C2Q + 1271\ C1Q^{8}\ )\ T^{8}$$

$$+ (- 510\ C1Q\ C3Q^{2} + 12\ C2Q^{2}\ C3Q + 1008\ C1Q^{2}\ C2Q\ C3Q - 1530\ C1Q^{4}\ C3Q$$

$$- 12\ C1Q\ C2Q^{3} - 558\ C1Q^{3}\ C2Q^{2} + 1590\ C1Q^{5}\ C2Q - 632\ C1Q^{7}\ )\ T^{7}$$

$$+ (51\ C3Q^{2} - 102\ C1Q\ C2Q\ C3Q + 586\ C1Q^{3}\ C3Q + 35\ C1Q^{2}\ C2Q^{2}$$

$$- 570\ C1Q^{4}\ C2Q + 322\ C1Q^{6}\ )\ T^{6} + (- 197\ C1Q^{2}\ C3Q + 14\ C1Q\ C2Q^{2}$$

$$+ 183\ C1Q^{3}\ C2Q - 168\ C1Q^{5}\ )\ T^{5} + (48\ C1Q\ C3Q - 3\ C2Q^{2} - 45\ C1Q^{2}\ C2Q$$

$$+ 88\ C1Q^{4}\ )\ T^{4} + (- 6\ C3Q + 6\ C1Q\ C2Q - 44\ C1Q^{3}\ )\ T^{3} + 19\ C1Q^{2}\ T^{2}$$

$$- 6\ C1Q\ T + 1$$

$$2\ C1Q\ C3Q + C2Q^{2} - 3\ C1Q^{2}\ C2Q + C1Q^{4} = 0$$

$$- 2\ C2Q\ C3Q + 3\ C1Q^{2}\ C3Q + 3\ C1Q\ C2Q^{2} - 4\ C1Q^{3}\ C2Q + C1Q^{5} = 0$$

$$C3Q^{2} - 6\ C1Q\ C2Q\ C3Q + 4\ C1Q^{3}\ C3Q - C2Q^{3} + 6\ C1Q^{2}\ C2Q^{2} - 5\ C1Q^{4}\ C2Q^{4}$$

$$+ C1Q^{6} = 0$$

$$3\ C1Q\ C3Q^{2} + 3\ C2Q^{2}\ C3Q - 12\ C1Q^{2}\ C2Q\ C3Q + 5\ C1Q^{4}\ C3Q - 4\ C1Q\ C2Q^{3}$$

$$+ 10\ C1Q^{3}\ C2Q^{2} - 6\ C1Q^{5}\ C2Q + C1Q^{7} = 0$$

$$- 3\ C2Q\ C3Q^{2} + 6\ C1Q^{2}\ C3Q^{2} + 12\ C1Q\ C2Q^{2}\ C3Q - 20\ C1Q^{3}\ C2Q\ C3Q$$

$$+ 6\ C1Q^{5}\ C3Q + C2Q^{4} - 10\ C1Q^{2}\ C2Q^{3} + 15\ C1Q^{4}\ C2Q^{2} - 7\ C1Q^{6}\ C2Q + C1Q^{8}$$

$$= 0$$

$$C3Q^{3} - 12\ C1Q\ C2Q\ C3Q^{2} + 10\ C1Q^{3}\ C3Q^{2} - 4\ C2Q^{3}\ C3Q + 30\ C1Q^{2}\ C2Q^{2}\ C3Q$$

$$- 30\ C1Q^{4}\ C2Q\ C3Q + 7\ C1Q^{6}\ C3Q + 5\ C1Q\ C2Q^{4} - 20\ C1Q^{3}\ C2Q^{3}$$

$$+ 21\ C1Q^{5}\ C2Q^{2} - 8\ C1Q^{7}\ C2Q + C1Q^{9} = 0$$

(C8) G(6);

$$CHEPO\ 1\ 6 = (9\ C2Q^{5} + 36\ C1Q^{2}\ C2Q^{4} - 102\ C1Q^{4}\ C2Q^{3} + 106\ C1Q^{6}\ C2Q^{2}$$

$$- 43\ C1Q^{8}\ C2Q + 6\ C1Q^{10})\ T^{10} + (63\ C1Q\ C2Q^{4} - 168\ C1Q^{3}\ C2Q^{3}$$

$$+ 252\ C1Q^{5}\ C2Q^{2} - 126\ C1Q^{7}\ C2Q + 21\ C1Q^{9})\ T^{9}$$

$$+ (3\ C2Q^{4} - 87\ C1Q^{2}\ C2Q^{3} + 310\ C1Q^{4}\ C2Q^{2} - 194\ C1Q^{6}\ C2Q + 41\ C1Q^{8})\ T^{8}$$

$$+ (- 42\ C1Q\ C2Q^{3} + 259\ C1Q^{3}\ C2Q^{2} - 182\ C1Q^{5}\ C2Q + 56\ C1Q^{7})\ T^{7}$$

$$+ (C2Q^{3} + 127\ C1Q^{2}\ C2Q^{2} - 93\ C1Q^{4}\ C2Q + 62\ C1Q^{6})\ T^{6}$$

$$+ (49\ C1Q\ C2Q^{2} - 14\ C1Q^{3}\ C2Q + 63\ C1Q^{5})\ T^{5}$$

$$+ (9\ C2Q^{2} + 20\ C1Q^{2}\ C2Q + 57\ C1Q^{4})\ T^{4} + (14\ C1Q\ C2Q + 42\ C1Q^{3})\ T^{3}$$

$$+ (3\ C2Q + 22\ C1Q^{2})\ T^{2} + 7\ C1Q\ T + 1$$

$$SEPO\ 1\ 6 = (- 1414\ C1Q^{2}\ C2Q^{4} - 18914\ C1Q^{4}\ C2Q^{3} + 47362\ C1Q^{6}\ C2Q^{2}$$

$$- 33264\ C1Q^{8}\ C2Q + 6916\ C1Q^{10})\ T^{10}$$

$$+ (700\ C1Q\ C2Q^{4} + 2254\ C1Q^{3}\ C2Q^{3} - 14448\ C1Q^{5}\ C2Q^{2} + 14616\ C1Q^{7}\ C2Q$$

$$- 3808\ C1Q^{9})\ T^{9} + (- 78\ C2Q^{4} + 162\ C1Q^{2}\ C2Q^{3} + 4218\ C1Q^{4}\ C2Q^{2}$$

$$- 6618\ C1Q^{6}\ C2Q + 2196\ C1Q^{8})\ T^{8} + (- 154\ C1Q\ C2Q^{3} - 1120\ C1Q^{3}\ C2Q^{2}$$

$$+ 2954\ C1Q^{5}\ C2Q - 1274\ C1Q^{7})\ T^{7} + (26\ C2Q^{3} + 236\ C1Q^{2}\ C2Q^{2}$$

$$- 1228\ C1Q^{4}\ C2Q + 716\ C1Q^{6})\ T^{6} + (- 28\ C1Q\ C2Q^{2} + 448\ C1Q^{3}\ C2Q$$

$$- 378\ C1Q^{5})\ T^{5} + (182\ C1Q^{4} - 133\ C1Q^{2}\ C2Q)\ T^{4}$$

$$+ (28\ C1Q\ C2Q - 77\ C1Q^{3})\ T^{3} + (27\ C1Q^{2} - 3\ C2Q)\ T^{2} - 7\ C1Q\ T + 1$$

$$- C2Q^{3} + 6\ C1Q^{2}\ C2Q^{2} - 5\ C1Q^{4}\ C2Q + C1Q^{6} = 0$$

$$- 4\ C1Q\ C2Q^{3} + 10\ C1Q^{3}\ C2Q^{2} - 6\ C1Q^{5}\ C2Q + C1Q^{7} = 0$$

$$C2Q^{4} - 10\ C1Q^{2}\ C2Q^{3} + 15\ C1Q^{4}\ C2Q^{2} - 7\ C1Q^{6}\ C2Q + C1Q^{8} = 0$$

$$5\ C1Q\ C2Q^{4} - 20\ C1Q^{3}\ C2Q^{3} + 21\ C1Q^{5}\ C2Q^{2} - 8\ C1Q^{7}\ C2Q + C1Q^{9} = 0$$

$$- C2Q^{5} + 15\ C1Q^{2}\ C2Q^{4} - 35\ C1Q^{4}\ C2Q^{3} + 28\ C1Q^{6}\ C2Q^{2} - 9\ C1Q^{8}\ C2Q$$

$$+ C1Q^{10} = 0$$

$$\mathrm{CHEPO\ 2\ 6} = (8\ C3Q^{4} + 8\ C1Q\ C2Q\ C3Q^{3} + 48\ C1Q^{3}\ C3Q^{3} + 16\ C2Q^{3}\ C3Q^{2}$$

$$- 6\ C1Q^{2}\ C2Q^{2}\ C3Q^{2} - 150\ C1Q^{4}\ C2Q\ C3Q^{2} + 47\ C1Q^{6}\ C3Q^{2}$$

$$- 6\ C1Q\ C2Q^{4}\ C3Q - 40\ C1Q^{3}\ C2Q^{3}\ C3Q + 211\ C1Q^{5}\ C2Q^{2}\ C3Q$$

$$- 132\ C1Q^{7}\ C2Q\ C3Q + 23\ C1Q^{9}\ C3Q + 9\ C2Q^{6} - 45\ C1Q^{2}\ C2Q^{5}$$

$$+ 105\ C1Q^{4}\ C2Q^{4} - 170\ C1Q^{6}\ C2Q^{3} + 125\ C1Q^{8}\ C2Q^{2} - 41\ C1Q^{10}\ C2Q$$

$$+ 5\ C1Q^{12})\ T^{12} + (8\ C2Q\ C3Q^{3} + 114\ C1Q^{2}\ C3Q^{3} + 90\ C1Q\ C2Q^{2}\ C3Q^{2}$$

$$- 414\ C1Q^{3}\ C2Q\ C3Q^{2} + 86\ C1Q^{5}\ C3Q^{2} - 9\ C2Q^{4}\ C3Q - 102\ C1Q^{2}\ C2Q^{3}\ C3Q$$

$$+ 575\ C1Q^{4}\ C2Q^{2}\ C3Q - 345\ C1Q^{6}\ C2Q\ C3Q + 62\ C1Q^{8}\ C3Q - 18\ C1Q\ C2Q^{5}$$

$$+ 120\ C1Q^{3}\ C2Q^{4} - 387\ C1Q^{5}\ C2Q^{3} + 364\ C1Q^{7}\ C2Q^{2} - 141\ C1Q^{9}\ C2Q$$

$$+ 20\ C1Q^{11})\ T^{11} + (56\ C1Q\ C3Q^{3} + 63\ C2Q^{2}\ C3Q^{2} - 259\ C1Q^{2}\ C2Q\ C3Q^{2}$$

$$- 77\ C1Q^{4}\ C3Q^{2} - 56\ C1Q\ C2Q^{3}\ C3Q + 539\ C1Q^{3}\ C2Q^{2}\ C3Q$$

$$- 259\ C1Q^{5}\ C2Q\ C3Q + 56\ C1Q^{7}\ C3Q - 15\ C2Q^{5} + 123\ C1Q^{2}\ C2Q^{4}$$

$$- 516\ C1Q^{4}\ C2Q^{3} + 564\ C1Q^{6}\ C2Q^{2} - 262\ C1Q^{8}\ C2Q + 45\ C1Q^{10})\ T^{10}$$

$$+ (- 7\,\mathrm{C3Q}^{3} - 28\,\mathrm{C1Q}\,\mathrm{C2Q}\,\mathrm{C3Q}^{2} - 266\,\mathrm{C1Q}^{3}\,\mathrm{C3Q}^{2} + 22\,\mathrm{C2Q}^{3}\,\mathrm{C3Q}$$

$$+ 230\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}^{2}\,\mathrm{C3Q} + 47\,\mathrm{C1Q}^{4}\,\mathrm{C2Q}\,\mathrm{C3Q} + 29\,\mathrm{C1Q}^{6}\,\mathrm{C3Q} + 51\,\mathrm{C1Q}\,\mathrm{C2Q}^{4}$$

$$- 375\,\mathrm{C1Q}^{3}\,\mathrm{C2Q}^{3} + 537\,\mathrm{C1Q}^{5}\,\mathrm{C2Q}^{2} - 337\,\mathrm{C1Q}^{7}\,\mathrm{C2Q} + 75\,\mathrm{C1Q}^{9}\,)\,T^{9}$$

$$+ (- 15\,\mathrm{C2Q}\,\mathrm{C3Q}^{2} - 247\,\mathrm{C1Q}^{2}\,\mathrm{C3Q}^{2} + 66\,\mathrm{C1Q}\,\mathrm{C2Q}^{2}\,\mathrm{C3Q} + 185\,\mathrm{C1Q}^{3}\,\mathrm{C2Q}\,\mathrm{C3Q}$$

$$+ 64\,\mathrm{C1Q}^{5}\,\mathrm{C3Q} + 22\,\mathrm{C2Q}^{4} - 185\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}^{3} + 395\,\mathrm{C1Q}^{4}\,\mathrm{C2Q}^{2}$$

$$- 362\,\mathrm{C1Q}^{6}\,\mathrm{C2Q} + 106\,\mathrm{C1Q}^{8}\,)\,T^{8} + (- 105\,\mathrm{C1Q}\,\mathrm{C3Q}^{2} - 15\,\mathrm{C2Q}^{2}\,\mathrm{C3Q}$$

$$+ 122\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}\,\mathrm{C3Q} + 143\,\mathrm{C1Q}^{4}\,\mathrm{C3Q} - 44\,\mathrm{C1Q}\,\mathrm{C2Q}^{3} + 232\,\mathrm{C1Q}^{3}\,\mathrm{C2Q}^{2}$$

$$- 322\,\mathrm{C1Q}^{5}\,\mathrm{C2Q} + 127\,\mathrm{C1Q}^{7}\,)\,T^{7} + (- 9\,\mathrm{C3Q}^{2} + 20\,\mathrm{C1Q}\,\mathrm{C2Q}\,\mathrm{C3Q}$$

$$+ 169\,\mathrm{C1Q}^{3}\,\mathrm{C3Q} - 11\,\mathrm{C2Q}^{3} + 115\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}^{2} - 221\,\mathrm{C1Q}^{4}\,\mathrm{C2Q} + 128\,\mathrm{C1Q}^{6}\,)$$

$$T^{6} + (6\,\mathrm{C2Q}\,\mathrm{C3Q} + 110\,\mathrm{C1Q}^{2}\,\mathrm{C3Q} + 33\,\mathrm{C1Q}\,\mathrm{C2Q}^{2} - 100\,\mathrm{C1Q}^{3}\,\mathrm{C2Q}$$

$$+ 109\,\mathrm{C1Q}^{5}\,)\,T^{5} + (42\,\mathrm{C1Q}\,\mathrm{C3Q} + 6\,\mathrm{C2Q}^{2} - 25\,\mathrm{C1Q}^{2}\,\mathrm{C2Q} + 80\,\mathrm{C1Q}^{4}\,)\,T^{4}$$

$$+ (7\,\mathrm{C3Q} + 49\,\mathrm{C1Q}^{3}\,)\,T^{3} + (\mathrm{C2Q} + 23\,\mathrm{C1Q}^{2}\,)\,T^{2} + 7\,\mathrm{C1Q}\,T + 1$$

$$\mathrm{SEPO\ 2\ 6} = (3699\,\mathrm{C3Q}^{4} - 12142\,\mathrm{C1Q}\,\mathrm{C2Q}\,\mathrm{C3Q}^{3} + 3105\,\mathrm{C1Q}^{3}\,\mathrm{C3Q}^{3}$$

$$+ 4758\,\mathrm{C2Q}^{3}\,\mathrm{C3Q}^{2} - 68043\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}^{2}\,\mathrm{C3Q}^{2} + 131775\,\mathrm{C1Q}^{4}\,\mathrm{C2Q}\,\mathrm{C3Q}^{2}$$

$$- 137452\,\mathrm{C1Q}^{6}\,\mathrm{C3Q}^{2} - 8338\,\mathrm{C1Q}\,\mathrm{C2Q}^{4}\,\mathrm{C3Q} + 197449\,\mathrm{C1Q}^{3}\,\mathrm{C2Q}^{3}\,\mathrm{C3Q}$$

$$- 480172\,\mathrm{C1Q}^{5}\,\mathrm{C2Q}^{2}\,\mathrm{C3Q} + 442503\,\mathrm{C1Q}^{7}\,\mathrm{C2Q}\,\mathrm{C3Q} - 102743\,\mathrm{C1Q}^{9}\,\mathrm{C3Q}$$

$$+ 603\,\mathrm{C2Q}^{6} - 738\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}^{5} - 121479\,\mathrm{C1Q}^{4}\,\mathrm{C2Q}^{4} + 344794\,\mathrm{C1Q}^{6}\,\mathrm{C2Q}^{3}$$

$$- 354873\,\mathrm{C1Q}^{8}\,\mathrm{C2Q}^{2} + 142370\,\mathrm{C1Q}^{10}\,\mathrm{C2Q} - 19272\,\mathrm{C1Q}^{12}\,)\,T^{12}$$

$$+ (528\,\mathrm{C2Q}\,\mathrm{C3Q}^{3} - 11341\,\mathrm{C1Q}^{2}\,\mathrm{C3Q}^{3} + 12100\,\mathrm{C1Q}\,\mathrm{C2Q}^{2}\,\mathrm{C3Q}^{2}$$

$$+ 914\,\mathrm{C1Q}^{3}\,\mathrm{C2Q}\,\mathrm{C3Q}^{2} + 26340\,\mathrm{C1Q}^{5}\,\mathrm{C3Q}^{2} + 77\,\mathrm{C2Q}^{4}\,\mathrm{C3Q}$$

$$- 40312\,\mathrm{C1Q}^{2}\,\mathrm{C2Q}^{3}\,\mathrm{C3Q} + 107674\,\mathrm{C1Q}^{4}\,\mathrm{C2Q}^{2}\,\mathrm{C3Q} - 116728\,\mathrm{C1Q}^{6}\,\mathrm{C2Q}\,\mathrm{C3Q}$$

$$+ 28184\ C1Q^{8}\ C3Q + 882\ C1Q\ C2Q^{5} + 29844\ C1Q^{3}\ C2Q^{4} - 103008\ C1Q^{5}\ C2Q^{3}$$

$$+ 117192\ C1Q^{7}\ C2Q^{2} - 49076\ C1Q^{9}\ C2Q + 6644\ C1Q^{11}\ )\ T^{11}$$

$$+ (3696\ C1Q\ C3Q^{3} - 1188\ C2Q^{2}\ C3Q^{2} - 6509\ C1Q^{2}\ C2Q\ C3Q^{2}$$

$$+ 281\ C1Q^{4}\ C3Q^{2} + 5734\ C1Q\ C2Q^{3}\ C3Q - 21074\ C1Q^{3}\ C2Q^{2}\ C3Q$$

$$+ 20790\ C1Q^{5}\ C2Q\ C3Q - 3252\ C1Q^{7}\ C3Q - 125\ C2Q^{5} - 5617\ C1Q^{2}\ C2Q^{4}$$

$$+ 28028\ C1Q^{4}\ C2Q^{3} - 33579\ C1Q^{6}\ C2Q^{2} + 13005\ C1Q^{8}\ C2Q - 1322\ C1Q^{10}\ )$$

$$T^{10} + (- 462\ C3Q^{3} + 1358\ C1Q\ C2Q\ C3Q^{2} - 3206\ C1Q^{3}\ C3Q^{2} - 376\ C2Q^{3}\ C3Q$$

$$+ 4219\ C1Q^{2}\ C2Q^{2}\ C3Q + 339\ C1Q^{4}\ C2Q\ C3Q - 2616\ C1Q^{6}\ C3Q$$

$$+ 641\ C1Q\ C2Q^{4} - 7085\ C1Q^{3}\ C2Q^{3} + 8032\ C1Q^{5}\ C2Q^{2} - 1367\ C1Q^{7}\ C2Q$$

$$- 515\ C1Q^{9}\ )\ T^{9} + (- 66\ C2Q\ C3Q^{2} + 1667\ C1Q^{2}\ C3Q^{2} - 824\ C1Q\ C2Q^{2}\ C3Q$$

$$- 2035\ C1Q^{3}\ C2Q\ C3Q + 2579\ C1Q^{5}\ C3Q - 25\ C2Q^{4} + 1601\ C1Q^{2}\ C2Q^{3}$$

$$- 1538\ C1Q^{4}\ C2Q^{2} - 1219\ C1Q^{6}\ C2Q + 885\ C1Q^{8}\ )\ T^{8}$$

$$+ (- 462\ C1Q\ C3Q^{2} + 90\ C2Q^{2}\ C3Q + 808\ C1Q^{2}\ C2Q\ C3Q - 1481\ C1Q^{4}\ C3Q$$

$$- 268\ C1Q\ C2Q^{3} + 274\ C1Q^{3}\ C2Q^{2} + 1134\ C1Q^{5}\ C2Q - 748\ C1Q^{7}\ )\ T^{7}$$

$$+ (58\ C3Q^{2} - 146\ C1Q\ C2Q\ C3Q + 657\ C1Q^{3}\ C3Q + 22\ C2Q^{3} - 84\ C1Q^{2}\ C2Q^{2}$$

$$- 620\ C1Q^{4}\ C2Q + 506\ C1Q^{6}\ )\ T^{6} + (8\ C2Q\ C3Q - 229\ C1Q^{2}\ C3Q$$

$$+ 30\ C1Q\ C2Q^{2} + 254\ C1Q^{3}\ C2Q - 298\ C1Q^{5}\ )\ T^{5}$$

$$+ (56\ C1Q\ C3Q - 5\ C2Q^{2} - 76\ C1Q^{2}\ C2Q + 155\ C1Q^{4}\ )\ T^{4}$$

$$+ (- 7\ C3Q + 14\ C1Q\ C2Q - 70\ C1Q^{3}\ )\ T^{3} + (26\ C1Q^{2} - C2Q)\ T^{2} - 7\ C1Q\ T$$

$$+ 1$$

$$- 2\ C2Q\ C3Q + 3\ C1Q^{2}\ C3Q + 3\ C1Q\ C2Q^{2} - 4\ C1Q^{3}\ C2Q + C1Q^{5} = 0$$

$$C3Q^{2} - 6\ C1Q\ C2Q\ C3Q + 4\ C1Q^{3}\ C3Q - C2Q^{3} + 6\ C1Q^{2}\ C2Q^{2} - 5\ C1Q^{4}\ C2Q$$

$$+ C1Q^{6} = 0$$

$$3\ C1Q\ C3Q^{2} + 3\ C2Q^{2}\ C3Q - 12\ C1Q^{2}\ C2Q\ C3Q + 5\ C1Q^{4}\ C3Q - 4\ C1Q\ C2Q^{3}$$

$$+ 10\ C1Q^{3}\ C2Q^{2} - 6\ C1Q^{5}\ C2Q + C1Q^{7} = 0$$

$$- 3\ C2Q\ C3Q^{2} + 6\ C1Q^{2}\ C3Q^{2} + 12\ C1Q\ C2Q^{2}\ C3Q - 20\ C1Q^{3}\ C2Q\ C3Q$$

$$+ 6\ C1Q^{5}\ C3Q + C2Q^{4} - 10\ C1Q^{2}\ C2Q^{3} + 15\ C1Q^{4}\ C2Q^{2} - 7\ C1Q^{6}\ C2Q + C1Q^{8}$$

$$= 0$$

$$C3Q^{3} - 12\ C1Q\ C2Q\ C3Q^{2} + 10\ C1Q^{3}\ C3Q^{2} - 4\ C2Q^{3}\ C3Q + 30\ C1Q^{2}\ C2Q^{2}\ C3Q$$

$$- 30\ C1Q^{4}\ C2Q\ C3Q + 7\ C1Q^{6}\ C3Q + 5\ C1Q\ C2Q^{4} - 20\ C1Q^{3}\ C2Q^{3}$$

$$+ 21\ C1Q^{5}\ C2Q^{2} - 8\ C1Q^{7}\ C2Q + C1Q^{9} = 0$$

$$4\ C1Q\ C3Q^{3} + 6\ C2Q^{2}\ C3Q^{2} - 30\ C1Q^{2}\ C2Q\ C3Q^{2} + 15\ C1Q^{4}\ C3Q^{2}$$

$$- 20\ C1Q\ C2Q^{3}\ C3Q + 60\ C1Q^{3}\ C2Q^{2}\ C3Q - 42\ C1Q^{5}\ C2Q\ C3Q + 8\ C1Q^{7}\ C3Q$$

$$- C2Q^{5} + 15\ C1Q^{2}\ C2Q^{4} - 35\ C1Q^{4}\ C2Q^{3} + 28\ C1Q^{6}\ C2Q^{2} - 9\ C1Q^{8}\ C2Q$$

$$+ C1Q^{10} = 0$$

$$- 4\ C2Q\ C3Q^{3} + 10\ C1Q^{2}\ C3Q^{3} + 30\ C1Q\ C2Q^{2}\ C3Q^{2} - 60\ C1Q^{3}\ C2Q\ C3Q^{2}$$

$$+ 21\ C1Q^{5}\ C3Q^{2} + 5\ C2Q^{4}\ C3Q - 60\ C1Q^{2}\ C2Q^{3}\ C3Q + 105\ C1Q^{4}\ C2Q^{2}\ C3Q$$

$$- 56\ C1Q^{6}\ C2Q\ C3Q + 9\ C1Q^{8}\ C3Q - 6\ C1Q\ C2Q^{5} + 35\ C1Q^{3}\ C2Q^{4}$$

$$- 56\ C1Q^{5}\ C2Q^{3} + 36\ C1Q^{7}\ C2Q^{2} - 10\ C1Q^{9}\ C2Q + C1Q^{11} = 0$$

$$C3Q^{4} - 20\ C1Q\ C2Q\ C3Q^{3} + 20\ C1Q^{3}\ C3Q^{3} - 10\ C2Q^{3}\ C3Q^{2}$$

$$+ 90\ C1Q^{2}\ C2Q^{2}\ C3Q^{2} - 105\ C1Q^{4}\ C2Q\ C3Q^{2} + 28\ C1Q^{6}\ C3Q^{2}$$

$$+ 30\ C1Q\ C2Q^{4}\ C3Q - 140\ C1Q^{3}\ C2Q^{3}\ C3Q + 168\ C1Q^{5}\ C2Q^{2}\ C3Q$$

$$- 72\ C1Q^{7}\ C2Q\ C3Q + 10\ C1Q^{9}\ C3Q + C2Q^{6} - 21\ C1Q^{2}\ C2Q^{5} + 70\ C1Q^{4}\ C2Q^{4}$$

$$- 84\ C1Q^{6}\ C2Q^{3} + 45\ C1Q^{8}\ C2Q^{2} - 11\ C1Q^{10}\ C2Q + C1Q^{12}$$

```
(D4)                                  [DSK, USERS]

(C5) LINEL:70;
(D5)                                  70

(C6) GG(3);
                      2          2            4   4
NEWCHEPO 1 3 = (6 C2Q  - 16 C1Q  C2Q + 8 C1Q ) T

                           3                    3           2  2
                   + (8 C1Q  - 4 C1Q C2Q) T  + 7 C1Q  T  + 4 C1Q T + 1
                        2        2               4   4
NEWSEPO 1 3 = (- 6 C2Q  - 16 C1Q  C2Q + 25 C1Q ) T

                                          3 3          2  2
                  + (4 C1Q C2Q - 16 C1Q ) T  + 9 C1Q  T  - 4 C1Q T + 1
(D6)                                 DONE

(C7) GG(4);
                         3            2    2            4               6   6
NEWCHEPO 1 4 = (- 14 C2Q  + 88 C1Q  C2Q  - 72 C1Q  C2Q + 16 C1Q ) T

               2         3              5   5
 + (30 C1Q C2Q  - 40 C1Q  C2Q + 16 C1Q ) T

          2          2            4   4        3 3
 + (6 C2Q  - 13 C1Q  C2Q + 16 C1Q ) T  + 15 C1Q  T

                 2   2
 + (C2Q + 11 C1Q ) T  + 5 C1Q T + 1
                        3           2    2             4                6   6
NEWSEPO 1 4 = (25 C2Q  - 15 C1Q  C2Q  - 245 C1Q  C2Q + 140 C1Q ) T

               2         3               5   5
 + (15 C1Q C2Q  + 110 C1Q  C2Q - 91 C1Q ) T

            2          2              4   4                              3   3
 + (- 5 C2Q  - 40 C1Q  C2Q + 55 C1Q ) T  + (10 C1Q C2Q - 30 C1Q ) T

           2         2
 + (14 C1Q  - C2Q) T  - 5 C1Q T + 1
(D7)                                 DONE

(C8) GG(5);
                        4            2    3            4    2            6
NEWCHEPO 1 5 = (47 C2Q  - 368 C1Q  C2Q  + 504 C1Q  C2Q  - 224 C1Q  C2Q

          8  8                  3              3    2               5
 + 32 C1Q ) T  + (- 84 C1Q C2Q  + 248 C1Q  C2Q  - 160 C1Q  C2Q

          7  7              3            2    2            4                6   6
 + 32 C1Q ) T  + (- 8 C2Q  + 105 C1Q  C2Q  - 96 C1Q  C2Q + 32 C1Q ) T

               2         3              5   5
 + (36 C1Q C2Q  - 38 C1Q  C2Q + 32 C1Q ) T

         2          2             4   4                        3   3
 + (7 C2Q  - 2 C1Q  C2Q + 31 C1Q ) T  + (6 C1Q C2Q + 26 C1Q ) T
```

$$+ (2\ C2Q + 16\ C1Q^{2})\ T^{2} + 6\ C1Q\ T + 1$$

$$NEWSEPO\ 1\ 5 = (- 98\ C2Q^{4} + 560\ C1Q^{2}\ C2Q^{3} + 1134\ C1Q^{4}\ C2Q^{2}$$

$$- 2436\ C1Q^{6}\ C2Q + 825\ C1Q^{8})\ T^{8} + (- 168\ C1Q\ C2Q^{3} - 336\ C1Q^{3}\ C2Q^{2}$$

$$+ 1260\ C1Q^{5}\ C2Q - 540\ C1Q^{7})\ T^{7} + (28\ C2Q^{3} + 63\ C1Q^{2}\ C2Q^{2}$$

$$- 588\ C1Q^{4}\ C2Q + 336\ C1Q^{6})\ T^{6} + (238\ C1Q^{3}\ C2Q - 196\ C1Q^{5})\ T^{5}$$

$$+ (- 3\ C2Q^{2} - 78\ C1Q^{2}\ C2Q + 105\ C1Q^{4})\ T^{4} + (18\ C1Q\ C2Q - 50\ C1Q^{3})\ T^{3}$$

$$+ (20\ C1Q^{2} - 2\ C2Q)\ T^{2} - 6\ C1Q\ T + 1$$

$$NEWCHEPO\ 2\ 5 = (- 142\ C3Q^{3} + 1062\ C1Q\ C2Q\ C3Q^{2} - 1032\ C1Q^{3}\ C3Q^{2}$$

$$- 162\ C2Q^{3}\ C3Q + 198\ C1Q^{2}\ C2Q^{2}\ C3Q - 36\ C1Q^{4}\ C2Q\ C3Q + 88\ C1Q^{6}\ C3Q$$

$$+ 648\ C1Q\ C2Q^{4} - 2058\ C1Q^{3}\ C2Q^{3} + 2090\ C1Q^{5}\ C2Q^{2} - 876\ C1Q^{7}\ C2Q$$

$$+ 128\ C1Q^{9})\ T^{9} + (132\ C2Q\ C3Q^{2} - 393\ C1Q^{2}\ C3Q^{2} + 162\ C1Q\ C2Q^{2}\ C3Q$$

$$- 174\ C1Q^{3}\ C2Q\ C3Q + 140\ C1Q^{5}\ C3Q + 81\ C2Q^{4} - 684\ C1Q^{2}\ C2Q^{3}$$

$$+ 1078\ C1Q^{4}\ C2Q^{2} - 612\ C1Q^{6}\ C2Q + 112\ C1Q^{8})\ T^{8}$$

$$+ (- 102\ C1Q\ C3Q^{2} + 42\ C2Q^{2}\ C3Q - 120\ C1Q^{2}\ C2Q\ C3Q + 144\ C1Q^{4}\ C3Q$$

$$- 162\ C1Q\ C2Q^{3} + 474\ C1Q^{3}\ C2Q^{2} - 396\ C1Q^{5}\ C2Q + 96\ C1Q^{7})\ T^{7}$$

$$+ (- 12\ C3Q^{2} - 42\ C1Q\ C2Q\ C3Q + 116\ C1Q^{3}\ C3Q - 20\ C2Q^{3}$$

$$+ 166\ C1Q^{2}\ C2Q^{2} - 228\ C1Q^{4}\ C2Q + 80\ C1Q^{6})\ T^{6}$$

$$+ (- 6\ C2Q\ C3Q + 72\ C1Q^{2}\ C3Q + 42\ C1Q\ C2Q^{2} - 108\ C1Q^{3}\ C2Q + 64\ C1Q^{5})$$

$$T^{5} + (30\ C1Q\ C3Q + 6\ C2Q^{2} - 36\ C1Q^{2}\ C2Q + 48\ C1Q^{4})\ T^{4}$$

$$+ (6\ C3Q - 6\ C1Q\ C2Q + 32\ C1Q^{3})\ T^{3} + 17\ C1Q^{2}\ T^{2} + 6\ C1Q\ T + 1$$

$$NEWSEPO\ 2\ 5 = (- 218\ C3Q^{3} + 1746\ C1Q\ C2Q\ C3Q^{2} - 3720\ C1Q^{3}\ C3Q^{2}$$

```
          3             2     2                   4
- 150 C2Q  C3Q - 666 C1Q  C2Q  C3Q + 5502 C1Q  C2Q C3Q

           6                    4              3      3             5     2
- 3416 C1Q  C3Q + 420 C1Q C2Q  - 1610 C1Q  C2Q  - 364 C1Q  C2Q

           7                9    9                      2             2     2
+ 2016 C1Q  C2Q - 670 C1Q  ) T  + (- 204 C2Q C3Q  + 1281 C1Q  C3Q

                 2                3                      5                4
- 18 C1Q C2Q  C3Q - 1998 C1Q  C2Q C3Q + 1890 C1Q  C3Q - 45 C2Q

          2     3             6                8    8
+ 580 C1Q  C2Q  - 1176 C1Q  C2Q + 489 C1Q  ) T

                   2            2                 2                    4
+ (- 330 C1Q C3Q  + 30 C2Q  C3Q + 564 C1Q  C2Q C3Q - 954 C1Q  C3Q

              3            3     2             5                7    7
- 150 C1Q C2Q  + 90 C1Q  C2Q  + 630 C1Q  C2Q - 344 C1Q  ) T

         2                                 3              3           2     2
+ (48 C3Q  - 102 C1Q C2Q C3Q + 424 C1Q  C3Q + 20 C2Q  - 70 C1Q  C2Q

          4                  6    6                               2
- 300 C1Q  C2Q + 231 C1Q  ) T  + (6 C2Q C3Q - 156 C1Q  C3Q

               2            3                5    5
+ 30 C1Q C2Q  + 120 C1Q  C2Q - 146 C1Q  ) T

                         2          2                4    4
+ (42 C1Q C3Q - 6 C2Q  - 36 C1Q  C2Q + 85 C1Q  ) T

                                     3    3          2    2
+ (- 6 C3Q + 6 C1Q C2Q - 44 C1Q  ) T  + 19 C1Q  T  - 6 C1Q T + 1
(D8)                                 DONE

(C9) GG(6);
                              5              2     4              4     3
NEWCHEPO 1 6 = (- 135 C2Q  + 1532 C1Q  C2Q  - 2936 C1Q  C2Q

           6     2            8               10    10
+ 2064 C1Q  C2Q  - 608 C1Q  C2Q + 64 C1Q   ) T

                 4             3     3             5     2            7
+ (329 C1Q C2Q  - 1176 C1Q  C2Q  + 1232 C1Q  C2Q  - 480 C1Q  C2Q

        9    9            4            2     3            4     2
+ 64 C1Q  ) T  + (39 C2Q  - 347 C1Q  C2Q  + 656 C1Q  C2Q

          6               8    8                   3            3     2
- 352 C1Q  C2Q + 64 C1Q  ) T  + (- 56 C1Q C2Q  + 315 C1Q  C2Q

          5               7    7           3            2     2            4
- 224 C1Q  C2Q + 64 C1Q  ) T  + (- C2Q  + 139 C1Q  C2Q  - 103 C1Q  C2Q

        6    6                  2           3               5    5
+ 64 C1Q  ) T  + (49 C1Q C2Q  - 14 C1Q  C2Q + 63 C1Q  ) T

         2           2                  4    4                            3    3
+ (9 C2Q  + 20 C1Q  C2Q + 57 C1Q  ) T  + (14 C1Q C2Q + 42 C1Q  ) T
```

$$+ (3\ C2Q^{2} + 22\ C1Q^{2})\ T^{2} + 7\ C1Q\ T + 1$$

$$NEWSEPO\ 1\ 6 = (378\ C2Q^{5} - 4830\ C1Q^{2}\ C2Q^{4} + 22176\ C1Q^{6}\ C2Q^{2}$$

$$- 21021\ C1Q^{8}\ C2Q + 5005\ C1Q^{10})\ T^{10}$$

$$+ (1050\ C1Q\ C2Q^{4} - 840\ C1Q^{3}\ C2Q^{3} - 9240\ C1Q^{5}\ C2Q^{2} + 11616\ C1Q^{7}\ C2Q$$

$$- 3289\ C1Q^{9})\ T^{9} + (- 126\ C2Q^{4} + 504\ C1Q^{2}\ C2Q^{3} + 3360\ C1Q^{4}\ C2Q^{2}$$

$$- 6006\ C1Q^{6}\ C2Q + 2079\ C1Q^{8})\ T^{8} + (- 168\ C1Q\ C2Q^{3} - 1008\ C1Q^{3}\ C2Q^{2}$$

$$+ 2856\ C1Q^{5}\ C2Q - 1254\ C1Q^{7})\ T^{7} + (28\ C2Q^{3} + 224\ C1Q^{2}\ C2Q^{2}$$

$$- 1218\ C1Q^{4}\ C2Q + 714\ C1Q^{6})\ T^{6} + (- 28\ C1Q\ C2Q^{2} + 448\ C1Q^{3}\ C2Q$$

$$- 378\ C1Q^{5})\ T^{5} + (182\ C1Q^{4} - 133\ C1Q^{2}\ C2Q)\ T^{4}$$

$$+ (28\ C1Q\ C2Q - 77\ C1Q^{3})\ T^{3} + (27\ C1Q^{2} - 3\ C2Q)\ T^{2} - 7\ C1Q\ T + 1$$

$$NEWCHEPO\ 2\ 6 = (197\ C3Q^{4} + 6924\ C1Q\ C2Q\ C3Q^{3} - 7764\ C1Q^{3}\ C3Q^{3}$$

$$+ 3078\ C2Q^{3}\ C3Q^{2} - 29871\ C1Q^{2}\ C2Q^{2}\ C3Q^{2} + 37245\ C1Q^{4}\ C2Q\ C3Q^{2}$$

$$- 10471\ C1Q^{6}\ C3Q^{2} + 2106\ C1Q\ C2Q^{4}\ C3Q + 4962\ C1Q^{3}\ C2Q^{3}\ C3Q$$

$$- 12324\ C1Q^{5}\ C2Q^{2}\ C3Q + 5714\ C1Q^{7}\ C2Q\ C3Q - 612\ C1Q^{9}\ C3Q + 972\ C2Q^{6}$$

$$- 11367\ C1Q^{2}\ C2Q^{5} + 26685\ C1Q^{4}\ C2Q^{4} - 26440\ C1Q^{6}\ C2Q^{3}$$

$$+ 13292\ C1Q^{8}\ C2Q^{2} - 3360\ C1Q^{10}\ C2Q + 336\ C1Q^{12})\ T^{12}$$

$$+ (1064\ C2Q\ C3Q^{3} - 3311\ C1Q^{2}\ C3Q^{3} - 6048\ C1Q\ C2Q^{2}\ C3Q^{2}$$

$$+ 15036\ C1Q^{3}\ C2Q\ C3Q^{2} - 6013\ C1Q^{5}\ C3Q^{2} + 567\ C2Q^{4}\ C3Q$$

$$- 630\ C1Q^{2}\ C2Q^{3}\ C3Q - 2954\ C1Q^{4}\ C2Q^{2}\ C3Q + 2002\ C1Q^{6}\ C2Q\ C3Q$$

$$- 140\ C1Q^{8}\ C3Q - 2268\ C1Q\ C2Q^{5} + 9639\ C1Q^{3}\ C2Q^{4} - 13356\ C1Q^{5}\ C2Q^{3}$$

$$+ 8532\ C1Q^{7}\ C2Q^{2} - 2608\ C1Q^{9}\ C2Q + 304\ C1Q^{11})\ T^{11}$$

$$+ (- 994\ C1Q\ C3Q^{3} - 582\ C2Q^{2}\ C3Q^{2} + 4791\ C1Q^{2}\ C2Q\ C3Q^{2}$$

$$- 3063\ C1Q^{4}\ C3Q^{2} - 648\ C1Q\ C2Q^{3}\ C3Q - 12\ C1Q^{3}\ C2Q^{2}\ C3Q$$

$$+ 266\ C1Q^{5}\ C2Q\ C3Q + 156\ C1Q^{7}\ C3Q - 243\ C2Q^{5} + 2781\ C1Q^{2}\ C2Q^{4}$$

$$- 5976\ C1Q^{4}\ C2Q^{3} + 5100\ C1Q^{6}\ C2Q^{2} - 1952\ C1Q^{8}\ C2Q + 272\ C1Q^{10}\ )\ T^{10}$$

$$+ (- 154\ C3Q^{3} + 1050\ C1Q\ C2Q\ C3Q^{2} - 1309\ C1Q^{3}\ C3Q^{2} - 140\ C2Q^{3}\ C3Q$$

$$+ 406\ C1Q^{2}\ C2Q^{2}\ C3Q - 294\ C1Q^{4}\ C2Q\ C3Q + 308\ C1Q^{6}\ C3Q + 567\ C1Q\ C2Q^{4}$$

$$- 2268\ C1Q^{3}\ C2Q^{3} + 2772\ C1Q^{5}\ C2Q^{2} - 1392\ C1Q^{7}\ C2Q + 240\ C1Q^{9}\ )\ T^{9}$$

$$+ (114\ C2Q\ C3Q^{2} - 423\ C1Q^{2}\ C3Q^{2} + 204\ C1Q\ C2Q^{2}\ C3Q - 286\ C1Q^{3}\ C2Q\ C3Q$$

$$+ 348\ C1Q^{5}\ C3Q + 61\ C2Q^{4} - 680\ C1Q^{2}\ C2Q^{3} + 1324\ C1Q^{4}\ C2Q^{2}$$

$$- 928\ C1Q^{6}\ C2Q + 208\ C1Q^{8}\ )\ T^{8} + (- 84\ C1Q\ C3Q^{2} + 42\ C2Q^{2}\ C3Q$$

$$- 126\ C1Q^{2}\ C2Q\ C3Q + 308\ C1Q^{4}\ C3Q - 140\ C1Q\ C2Q^{3} + 532\ C1Q^{3}\ C2Q^{2}$$

$$- 560\ C1Q^{5}\ C2Q + 176\ C1Q^{7}\ )\ T^{7} + (- 6\ C3Q^{2} - 24\ C1Q\ C2Q\ C3Q$$

$$+ 220\ C1Q^{3}\ C3Q - 14\ C2Q^{3} + 172\ C1Q^{2}\ C2Q^{2} - 288\ C1Q^{4}\ C2Q + 144\ C1Q^{6}\ )$$

$$T^{6} + (119\ C1Q^{2}\ C3Q + 42\ C1Q\ C2Q^{2} - 112\ C1Q^{3}\ C2Q + 112\ C1Q^{5}\ )\ T^{5}$$

$$+ (42\ C1Q\ C3Q + 6\ C2Q^{2} - 25\ C1Q^{2}\ C2Q + 80\ C1Q^{4}\ )\ T^{4}$$

$$+ (7\ C3Q + 49\ C1Q^{3}\ )\ T^{3} + (C2Q + 23\ C1Q^{2}\ )\ T^{2} + 7\ C1Q\ T + 1$$

$$NEWSEPO\ 2\ 6 = (966\ C3Q^{4} - 18312\ C1Q\ C2Q\ C3Q^{3} + 39592\ C1Q^{3}\ C3Q^{3}$$

$$- 1484\ C2Q^{3}\ C3Q^{2} + 53396\ C1Q^{2}\ C2Q^{2}\ C3Q^{2} - 156618\ C1Q^{4}\ C2Q\ C3Q^{2}$$

$$+ 85932\ C1Q^{6}\ C3Q^{2} - 5544\ C1Q\ C2Q^{4}\ C3Q - 6552\ C1Q^{3}\ C2Q^{3}\ C3Q$$

$$+ 114828\ C1Q^{5}\ C2Q^{2}\ C3Q - 140448\ C1Q^{7}\ C2Q\ C3Q + 41272\ C1Q^{9}\ C3Q$$

$$- 294\ C2Q^{6} + 10332\ C1Q^{2}\ C2Q^{5} - 35700\ C1Q^{4}\ C2Q^{4} + 14322\ C1Q^{6}\ C2Q^{3}$$

$$+ 33957\ C1Q^{8}\ C2Q^{2} - 28028\ C1Q^{10}\ C2Q + 5551\ C1Q^{12}\ )\ T^{12}$$

$$+ (1848\ C2Q\ C3Q^{3} - 11256\ C1Q^{2}\ C3Q^{3} - 8820\ C1Q\ C2Q^{2}\ C3Q^{2}$$

$$+ 51632\ C1Q^{3}\ C2Q\ C3Q^{2} - 40530\ C1Q^{5}\ C3Q^{2} + 686\ C2Q^{4}\ C3Q$$

$$- 1736\ C1Q^{2}\ C2Q^{3}\ C3Q - 38178\ C1Q^{4}\ C2Q^{2}\ C3Q + 68712\ C1Q^{6}\ C2Q\ C3Q$$

$$- 25179\ C1Q^{8}\ C3Q - 1764\ C1Q\ C2Q^{5} + 12432\ C1Q^{3}\ C2Q^{4} - 9240\ C1Q^{5}\ C2Q^{3}$$

$$- 16830\ C1Q^{7}\ C2Q^{2} + 17842\ C1Q^{9}\ C2Q - 4082\ C1Q^{11}\ )\ T^{11}$$

$$+ (2352\ C1Q\ C3Q^{3} + 756\ C2Q^{2}\ C3Q^{2} - 13776\ C1Q^{2}\ C2Q\ C3Q^{2}$$

$$+ 17262\ C1Q^{4}\ C3Q^{2} + 1064\ C1Q\ C2Q^{3}\ C3Q + 10136\ C1Q^{3}\ C2Q^{2}\ C3Q$$

$$- 30660\ C1Q^{5}\ C2Q\ C3Q + 14616\ C1Q^{7}\ C3Q + 154\ C2Q^{5} - 3472\ C1Q^{2}\ C2Q^{4}$$

$$+ 4998\ C1Q^{4}\ C2Q^{3} + 7518\ C1Q^{6}\ C2Q^{2} - 10857\ C1Q^{8}\ C2Q + 2926\ C1Q^{10}\ )\ T^{10}$$

$$+ (- 273\ C3Q^{3} + 2646\ C1Q\ C2Q\ C3Q^{2} - 6412\ C1Q^{3}\ C3Q^{2} - 196\ C2Q^{3}\ C3Q$$

$$- 1841\ C1Q^{2}\ C2Q^{2}\ C3Q + 12124\ C1Q^{4}\ C2Q\ C3Q - 7980\ C1Q^{6}\ C3Q$$

$$+ 686\ C1Q\ C2Q^{4} - 2240\ C1Q^{3}\ C2Q^{3} - 2898\ C1Q^{5}\ C2Q^{2} + 6252\ C1Q^{7}\ C2Q$$

$$- 2035\ C1Q^{9}\ )\ T^{9} + (- 273\ C2Q\ C3Q^{2} + 1960\ C1Q^{2}\ C3Q^{2} + 126\ C1Q\ C2Q^{2}\ C3Q$$

$$- 4060\ C1Q^{3}\ C2Q\ C3Q + 4032\ C1Q^{5}\ C3Q - 70\ C2Q^{4} + 791\ C1Q^{2}\ C2Q^{3}$$

$$+ 882\ C1Q^{4}\ C2Q^{2} - 3360\ C1Q^{6}\ C2Q + 1365\ C1Q^{8}\ )\ T^{8}$$

$$+ (- 441\ C1Q\ C3Q^{2} + 21\ C2Q^{2}\ C3Q + 1064\ C1Q^{2}\ C2Q\ C3Q - 1841\ C1Q^{4}\ C3Q$$

$$- 196\ C1Q\ C2Q^{3} - 161\ C1Q^{3}\ C2Q^{2} + 1652\ C1Q^{5}\ C2Q - 876\ C1Q^{7}\ )\ T^{7}$$

$$+ (55\ C3Q^{2} - 186\ C1Q\ C2Q\ C3Q + 732\ C1Q^{3}\ C3Q + 25\ C2Q^{3} - 15\ C1Q^{2}\ C2Q^{2}$$

```
         4           6   6
- 721 C1Q  C2Q + 532 C1Q ) T  + (14 C2Q C3Q - 238 C1Q  C3Q
                                                     2
             2               3                   5   5
+ 21 C1Q C2Q  + 266 C1Q  C2Q  - 301 C1Q ) T
                                                       .
                    2             2                4   4
+ (56 C1Q C3Q - 5 C2Q  - 76 C1Q  C2Q + 155 C1Q ) T

                                              3   3               2         2
+ (- 7 C3Q + 14 C1Q C2Q - 70 C1Q ) T  + (26 C1Q  - C2Q) T  - 7 C1Q T

+ 1
(D9)                                 DONE

(C10) CLOSEFILE(HOLME,OUT1);
```

```
(D2)                                  [DSK, USERS]

(C3) GRIND(RE);
RE[R]:=BLOCK([F,G,H,K,L,M,N],F:X^(R+1)+SUM(CONCAT(C,I,Q)*X^(R+1-I),I,1,R+1),
   G:RATSUBST(X-1,X,F),H:F-G,K:RESULTANT(F,H,X),L:K,
   FOR I THRU R+1 DO L:RATSUBST(CONCAT(C,I,Q)*T^I,CONCAT(C,I,Q),L),
   M:L*COEFF(L,T,0),N:SUM(EXPAND(COEFF(M,T,I))*T^I,I,0,(R+1)^2),N)$
(D3)                                      DONE

(C4) RE[1];
                                             2  2
(D4)                              (4 C2Q - C1Q ) T  + 1

(C5) RE[2];

CONCAT FASL DSK MAXOUT being loaded
loading done
                2                        3            3      2    2   6
(D5) (27 C3Q  - 18 C1Q C2Q C3Q + 4 C1Q  C3Q + 4 C2Q  - C1Q  C2Q ) T

                             2          2              4   4                2   2
                      + (9 C2Q  - 6 C1Q  C2Q + C1Q ) T  + (6 C2Q - 2 C1Q ) T  + 1

(C6) RE[3];
                3                          2          2   2             2           2
(D6) (256 C4Q  - 192 C1Q C3Q C4Q  - 128 C2Q  C4Q  + 144 C1Q  C2Q C4Q

          4     2                 2             2     2                    2
 - 27 C1Q  C4Q  + 144 C2Q C3Q  C4Q - 6 C1Q  C3Q  C4Q - 80 C1Q C2Q  C3Q C4Q

         3                           4              2    3             4
 + 18 C1Q  C2Q C3Q C4Q + 16 C2Q  C4Q - 4 C1Q  C2Q  C4Q - 27 C3Q

                     3          3    3          3     2         2     2     2   12
 + 18 C1Q C2Q C3Q  - 4 C1Q  C3Q  - 4 C2Q  C3Q  + C1Q  C2Q  C3Q ) T

                     2          2    2            2
 + (- 192 C2Q C4Q  + 72 C1Q  C4Q  + 216 C3Q  C4Q - 120 C1Q C2Q C3Q C4Q

         3                     3             2    2                     3
 + 18 C1Q  C3Q C4Q + 32 C2Q  C4Q - 6 C1Q  C2Q  C4Q - 54 C1Q C3Q

          2     2          2                2         4     2                3
 + 18 C2Q  C3Q  + 42 C1Q  C2Q C3Q  - 9 C1Q  C3Q  - 26 C1Q C2Q  C3Q

        3     2                5        2     4   10
 + 6 C1Q  C2Q  C3Q + 4 C2Q  - C1Q  C2Q ) T

               2                               2             2                  4
 + (- 112 C4Q  + 56 C1Q C3Q C4Q + 24 C2Q  C4Q - 32 C1Q  C2Q C4Q + 6 C1Q  C4Q

               2          2     2                 2             3                  5
 + 48 C2Q C3Q  - 25 C1Q  C3Q  - 54 C1Q C2Q  C3Q + 38 C1Q  C2Q C3Q - 6 C1Q  C3Q

          4          2     3         4     2   8
 + 17 C2Q  - 12 C1Q  C2Q  + 2 C1Q  C2Q ) T

                             2               2                         3               3
 + (16 C2Q C4Q - 6 C1Q  C4Q + 26 C3Q  - 30 C1Q C2Q C3Q + 8 C1Q  C3Q + 28 C2Q
```

```
          2     2           4           6  6
 - 24 C1Q  C2Q  + 8 C1Q  C2Q - C1Q ) T

                                         2              2                 4  4
 + (8 C4Q - 2 C1Q C3Q + 22 C2Q  - 16 C1Q  C2Q + 3 C1Q ) T

                            2  2
 + (8 C2Q - 3 C1Q ) T  + 1

(C7) RE[4];

You have run out of LIST space.
Do you want more?
Type ALL; NONE; a level-no. or the name of a space.
ALL;

You have run out of LIST space.
Do you want more?
Type ALL; NONE; a level-no. or the name of a space.
ALL;

You have run out of LIST space.
Do you want more?
Type ALL; NONE; a level-no. or the name of a space.
ALL;

You have run out of FIXNUM space.
Do you want more?
Type ALL; NONE; a level-no. or the name of a space.
ALL;

115235 msec.

139662 msec.

170254 msec.

202918 msec.

max allocation exceeded
FIXNUM storage capacity exceeded

203907 msec. so far

(C8) CLOSEFILE(HOLME,OUT2);
```

§ 3. Output from Schubert Calculus.

```
(C5) GRIND(PIERI);
PIERI(R,A,TOT,F):=BLOCK([M,N,U,OLD,NEW],
     F:F*FUNDCLASSG(R,A),M:A-R,
     FOR I THRU R+1 DO
          (FOR J THRU INF UNLESS FREEOF(CONCAT(C,I,Q),F) DO
                FOR S IN TOT DO
                     (N:SUM(S[K],K,1,M),
                      U:SUMDOMAIN(A-R,N,I,S),
                      OLD:APPLY(OMEGA,S)*CONCAT(C,I,Q),
                      NEW:SUM(APPLY(OMEGA,U[K]),K,1,
                               LENGTH(U)),
                      F:RATSUBST(NEW,OLD,F))),F)$
(D5)                           DONE

(C6) GRIND(POLYSGOMEGA);
POLYSGOMEGA(R,A):=BLOCK([DIM,CH,D,SE],DIM:(R+1)*(A-R),
            CH:CHERNPOLYG(R,A,T),TOT:TOTALDOMAIN(R,A),
            PRINT("CHEPO",R,A,"=",CH),H:CH,
            H:PIERI(R,A,TOT,H),PRINT("= ",H),
            H:C1Q^DIM*RATSUBST(T/C1Q,T,CH),
            H:PIERI(R,A,TOT,H),PRINT("= ",H),D:1/CH,
            SE:TAYLOR(D,T,0,DIM),
            SE:1+SUM(EXPAND(COEFF(SE,T,I))*T^I,I,1,DIM),
            PRINT("SEPO",R,A,"=",SE),E:PIERI(R,A,TOT,SE),
            PRINT("= ",E),E:C1Q^DIM*RATSUBST(T/C1Q,T,SE),
            E:PIERI(R,A,TOT,E),PRINT("= ",E),PRINT("DONE"))$
(D6)                           DONE

(C7) LINEL:70;
(D7)                                    70

(C8) POLYSGOMEGA(1,3);

CONCAT FASL DSK MAXOUT being loaded
loading done

HAYAT FASL DSK MACSYM being loaded
loading done
                  2       2                 4   4         3 3       2 2
CHEPO 1 3 = (4 C2Q  - 4 C1Q  C2Q + 3 C1Q ) T  + 6 C1Q  T  + 7 C1Q  T

                                                              + 4 C1Q T + 1
                      4                 3
=  6 OMEGA(1, 2) T  + 12 OMEGA(1, 3) T

                                     2
 + (7 OMEGA(2, 3) + 7 OMEGA(1, 4)) T  + 4 OMEGA(2, 4) T + OMEGA(3, 4)
                   4                   3                   2
=  6 OMEGA(1, 2) T  + 12 OMEGA(1, 2) T  + 14 OMEGA(1, 2) T

                                          + 8 OMEGA(1, 2) T + 2 OMEGA(1, 2)
                      2       2            4   4          3 3
SEPO 1 3 = (- 4 C2Q  + 4 C1Q  C2Q + 14 C1Q ) T  - 14 C1Q  T
```

```
                                                              + 9 C1Q²T²- 4 C1Q T + 1
                          4                         3
- 28 OMEGA(1, 2) T  - 28 OMEGA(1, 3) T

                                              2
 + (9 OMEGA(2, 3) + 9 OMEGA(1, 4)) T  - 4 OMEGA(2, 4) T + OMEGA(3, 4)
                          4                         3                        2
- 28 OMEGA(1, 2) T  - 28 OMEGA(1, 2) T  + 18 OMEGA(1, 2) T

                                                  - 8 OMEGA(1, 2) T + 2 OMEGA(1, 2)
DONE
(D8)                                      DONE

(C9) POLYSGOMEGA(1,4);
                         3                2      2                4                  6    6
CHEPO 1 4 = (4 C2Q  + 12 C1Q  C2Q  - 13 C1Q  C2Q + 4 C1Q ) T

                  2              3                    5    5
 + (20 C1Q C2Q  - 20 C1Q  C2Q + 10 C1Q ) T

              2              2                  4    4              3    3
 + (4 C2Q  - 7 C1Q  C2Q + 14 C1Q ) T  + 15 C1Q  T

                         2    2
 + (C2Q + 11 C1Q ) T  + 5 C1Q T + 1
                                  6                                    5
- 10 OMEGA(1, 2, 3) T  + 30 OMEGA(1, 2, 4) T

                                                        4
 + (35 OMEGA(1, 3, 4) + 25 OMEGA(1, 2, 5)) T

                                                        3
 + (15 OMEGA(2, 3, 4) + 30 OMEGA(1, 3, 5)) T

                                                        2
 + (11 OMEGA(2, 3, 5) + 12 OMEGA(1, 4, 5)) T  + 5 OMEGA(2, 4, 5) T

 + OMEGA(3, 4, 5)

You have run out of LIST space.
Do you want more?
Type ALL; NONE; a level-no. or the name of a space.
ALL;
                                  6                                5                              4
-  10 OMEGA(1, 2, 3) T  + 30 OMEGA(1, 2, 3) T  + 60 OMEGA(1, 2, 3) T

                                  3                                2
 + 75 OMEGA(1, 2, 3) T  + 57 OMEGA(1, 2, 3) T  + 25 OMEGA(1, 2, 3) T

 + 5 OMEGA(1, 2, 3)
                         3                 2      2                  4                    6    6
SEPO 1 4 = (3 C2Q  + 79 C1Q  C2Q  - 426 C1Q  C2Q + 198 C1Q ) T

                2               3                     5    5
 + (5 C1Q C2Q  + 150 C1Q  C2Q - 105 C1Q ) T

              2            2                  4    4                                    3    3
 + (- 3 C2Q  - 46 C1Q  C2Q + 57 C1Q ) T  + (10 C1Q C2Q - 30 C1Q ) T
```

```
 + (14 C1Q 2 - C2Q) T 2 - 5 C1Q T + 1

63503 msec.
                          6                      5
- 220 OMEGA(1, 2, 3) T  - 220 OMEGA(1, 2, 4) T

                                                4
 + (125 OMEGA(1, 3, 4) + 65 OMEGA(1, 2, 5)) T

                                               3
 + (- 30 OMEGA(2, 3, 4) - 50 OMEGA(1, 3, 5)) T

                                              2
 + (14 OMEGA(2, 3, 5) + 13 OMEGA(1, 4, 5)) T  - 5 OMEGA(2, 4, 5) T

 + OMEGA(3, 4, 5)
83168 msec.
                          6                      5
-  220 OMEGA(1, 2, 3) T  - 220 OMEGA(1, 2, 3) T
                          4                      3
 + 190 OMEGA(1, 2, 3) T  - 130 OMEGA(1, 2, 3) T
                         2
 + 68 OMEGA(1, 2, 3) T  - 25 OMEGA(1, 2, 3) T + 5 OMEGA(1, 2, 3)
DONE
(D9)                               DONE

(C10) CLOSEFILE(HOLME,OUT4);
```

§ 4. Output from Grass.

```
                                                        2         2              4    4          3  3           2  2
THE CHERNPOLYNOMIAL OF GRASS( 1 , 3 ) IS:  (4 C2Q  - 4 C1Q  C2Q + 3 C1Q ) T  + 6 C1Q  T  + 7 C1Q  T  + 4 C1Q T + 1
THE RELATIONS OF THE CHERNCLASSES OF Q ARE:
   3
C1Q  - 2 C1Q C2Q  = 0
   2        2          4
C2Q  - 3 C1Q  C2Q + C1Q   = 0
                               2          3          2        4
GAMMA  = - S4 - 9 D S3 - 36 D  S2 - 84 D  S1 + DEG  - 126 D
       8
=  0
                           2          3          4
RAM  = S4 + 8 D S3 + 28 D  S2 + 56 D  S1 + 70 D
     7
= 0
                 2          3          4
RAM  = D S3 + 7 D  S2 + 21 D  S1 + 35 D
     6
= 0
        2         3         4
RAM  = D  S2 + 6 D  S1 + 15 D
     5
= 0
        3         4
RAM  = D  S1 + 5 D
     4
= 2
DEG = 2
(IN25) GRASS(1,4);
                                                      3         2    2          4             6   6
THE CHERNPOLYNOMIAL OF GRASS( 1 , 4 ) IS:  (4 C2Q  + 12 C1Q  C2Q  - 13 C1Q  C2Q + 4 C1Q ) T

                 2          3            5  5         2          2            4   4          3  3                 2   2
  + (20 C1Q C2Q  - 20 C1Q  C2Q + 10 C1Q ) T  + (4 C2Q  - 7 C1Q  C2Q + 14 C1Q ) T  + 15 C1Q  T  + (C2Q + 11 C1Q ) T  + 5 C
THE RELATIONS OF THE CHERNCLASSES OF Q ARE:
   2        2          4
C2Q  - 3 C1Q  C2Q + C1Q   = 0
            2        3          5
3 C1Q C2Q  - 4 C1Q  C2Q + C1Q   = 0
    3         2    2          4          6
- C2Q  + 6 C1Q  C2Q  - 5 C1Q  C2Q + C1Q   = 0
                                    2          3           4            5          2        6
GAMMA   = - S6 - 13 D S5 - 78 D  S4 - 286 D  S3 - 715 D  S2 - 1287 D  S1 + DEG  - 1716 D
       12
=  0
                              2           3           4           5           6
RAM   = S6 + 12 D S5 + 66 D  S4 + 220 D  S3 + 495 D  S2 + 792 D  S1 + 924 D
     11
= 0
                       2           3            4            5           6
RAM   = D S5 + 11 D  S4 + 55 D  S3 + 165 D  S2 + 330 D  S1 + 462 D
     10
= 0
         2          3           4            5           6
RAM  = D  S4 + 10 D  S3 + 45 D  S2 + 120 D  S1 + 210 D
     9

= 0
         3          4           5           6
RAM  = D  S3 + 9 D  S2 + 36 D  S1 + 84 D
     8
= 2
DEG = 5
```

(IN26) GRASS(1,5);

THE CHERNPOLYNOMIAL OF GRASS(1 , 5) IS: $(9 C2Q^{4} - 18 C1Q^{2} C2Q^{3} + 42 C1Q^{4} C2Q^{2} - 26 C1Q^{6} C2Q + 5 C1Q^{8}) T^{8}$

$+ (75 C1Q^{3} C2Q^{2} - 60 C1Q^{5} C2Q + 15 C1Q^{7}) T^{7} + (- 6 C2Q^{3} + 78 C1Q^{2} C2Q^{2} - 66 C1Q^{4} C2Q + 25 C1Q^{6}) T^{6}$

$+ (30 C1Q C2Q^{2} - 30 C1Q^{3} C2Q + 30 C1Q^{5}) T^{5} + (7 C2Q^{2} - 2 C1Q^{2} C2Q + 31 C1Q^{4}) T^{4} + (6 C1Q C2Q + 26 C1Q^{3}) T^{3} + (2 C2Q + 16$

$+ 6 C1Q T + 1$

THE RELATIONS OF THE CHERNCLASSES OF Q ARE:

$3 C1Q C2Q^{2} - 4 C1Q^{3} C2Q + C1Q^{5} = 0$

$- C2Q^{3} + 6 C1Q^{2} C2Q^{2} - 5 C1Q^{4} C2Q + C1Q^{6} = 0$

$- 4 C1Q C2Q^{3} + 10 C1Q^{3} C2Q^{2} - 6 C1Q^{5} C2Q + C1Q^{7} = 0$

$C2Q^{4} - 10 C1Q^{2} C2Q^{3} + 15 C1Q^{4} C2Q^{2} - 7 C1Q^{6} C2Q + C1Q^{8} = 0$

$GAMMA_{16} = - S8 - 17 D S7 - 136 D^{2} S6 - 680 D^{3} S5 - 2380 D^{4} S4 - 6188 D^{5} S3 - 12376 D^{6} S2 - 19448 D^{7} S1 + DEG^{2} - 24310 D^{8}$

$= 0$

$RAM_{15} = S8 + 16 D S7 + 120 D^{2} S6 + 560 D^{3} S5 + 1820 D^{4} S4 + 4368 D^{5} S3 + 8008 D^{6} S2 + 11440 D^{7} S1 + 12870 D^{8}$

$= 0$

$RAM_{14} = D S7 + 15 D^{2} S6 + 105 D^{3} S5 + 455 D^{4} S4 + 1365 D^{5} S3 + 3003 D^{6} S2 + 5005 D^{7} S1 + 6435 D^{8}$

$= 0$

$RAM_{13} = D^{2} S6 + 14 D^{3} S5 + 91 D^{4} S4 + 364 D^{5} S3 + 1001 D^{6} S2 + 2002 D^{7} S1 + 3003 D^{8}$

$= 0$

$RAM_{12} = D^{3} S5 + 13 D^{4} S4 + 78 D^{5} S3 + 286 D^{6} S2 + 715 D^{7} S1 + 1287 D^{8}$

$= 6$

DEG = 14

(IN27) GRASS(2,5);

THE CHERNPOLYNOMIAL OF GRASS(2 , 5) IS: $(8 C3Q^{3} - 24 C1Q C2Q C3Q^{2} + 4 C1Q^{3} C3Q^{2} + 52 C1Q^{2} C2Q^{2} C3Q - 36 C1Q^{4} C2Q C3Q$

$+ 7 C1Q^{6} C3Q - 8 C1Q C2Q^{4} - 20 C1Q^{3} C2Q^{3} + 38 C1Q^{5} C2Q^{2} - 21 C1Q^{7} C2Q + 4 C1Q^{9}) T^{9}$

$+ (- 22 C1Q^{2} C3Q^{2} + 56 C1Q C2Q^{2} C3Q - 12 C1Q^{3} C2Q C3Q + 2 C1Q^{5} C3Q - 4 C2Q^{4} - 48 C1Q^{2} C2Q^{3} + 85 C1Q^{4} C2Q^{2} - 57 C1Q^{6} C2Q$

$+ 14 C1Q^{8}) T^{8} + (- 30 C1Q C3Q^{2} + 24 C2Q^{2} C3Q + 36 C1Q^{2} C2Q C3Q + 2 C1Q^{4} C3Q - 24 C1Q C2Q^{3} + 70 C1Q^{3} C2Q^{2} - 78 C1Q^{5} C2Q +$

$T^{7} + (- 15 C3Q^{2} + 30 C1Q C2Q C3Q + 26 C1Q^{3} C3Q + 43 C1Q^{2} C2Q^{2} - 84 C1Q^{4} C2Q + 43 C1Q^{6}) T^{6}$

$+ (41 C1Q^{2} C3Q + 22 C1Q C2Q^{2} - 63 C1Q^{3} C2Q + 50 C1Q^{5}) T^{5} + (24 C1Q C3Q + 3 C2Q^{2} - 27 C1Q^{2} C2Q + 45 C1Q^{4}) T^{4}$

$+ (6 C3Q - 6 C1Q C2Q + 32 C1Q^{3}) T^{3} + 17 C1Q^{2} T^{2} + 6 C1Q T + 1$

THE RELATIONS OF THE CHERNCLASSES OF Q ARE:

$2 C1Q C3Q + C2Q^{2} - 3 C1Q^{2} C2Q + C1Q^{4} = 0$

$- 2 C2Q C3Q + 3 C1Q^{2} C3Q + 3 C1Q C2Q^{2} - 4 C1Q^{3} C2Q + C1Q^{5} = 0$

```
 C3Q2 - 6 C1Q C2Q C3Q + 4 C1Q3 C3Q - C2Q3 + 6 C1Q2 C2Q2 - 5 C1Q4 C2Q + C1Q   = 0
         2            2             2               4                 3            3     2           5           7
 3 C1Q C3Q   + 3 C2Q   C3Q - 12 C1Q   C2Q C3Q + 5 C1Q   C3Q - 4 C1Q C2Q   + 10 C1Q   C2Q   - 6 C1Q   C2Q + C1Q    = 0
            2           2              2                3              5          4              2     3         4     2             6
 - 3 C2Q C3Q   + 6 C1Q   C3Q   + 12 C1Q C2Q   C3Q - 28 C1Q   C2Q C3Q + 6 C1Q   C3Q + C2Q   - 10 C1Q   C2Q   + 15 C1Q   C2Q   - 7 C1Q   C

                                                                                                                                   8
                                                                                                                            + C1Q
    3                 2           3     2          3               2   2                   4           6            4              3
 C3Q   - 12 C1Q C2Q C3Q   + 10 C1Q   C3Q   - 4 C2Q   C3Q + 30 C1Q   C2Q   C3Q - 30 C1Q   C2Q C3Q + 7 C1Q   C3Q + 5 C1Q C2Q   - 28 C1Q

                                                                                                          5     2          7         9
                                                                                                 + 21 C1Q   C2Q   - 8 C1Q   C2Q + C1Q
                                  2            3           4            5              6               7               8
 GAMMA     = - S9 - 19 D S8 - 171 D  S7 - 969 D  S6 - 3876 D  S5 - 11628 D  S4 - 27132 D  S3 - 58388 D  S2 - 75582 D  S1 + DEG
      18


                                                                                                                            - 92
 . 2
                              2           3            4              5                 6                  7                  8           9
 RAM     = S9 + 18 D S8 + 153 D  S7 + 816 D  S6 + 3060 D  S5 + 8568 D  S4 + 18564 D  S3 + 31824 D  S2 + 43758 D  S1 + 48620 D
    17
 , 4
 DEG = 42

 (IN28) GRASS(2,6);

 CORE capacity exceeded (while requesting LIST space)

 01774 msec. so far
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