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SUB-BASES OF PLEASANT h-BASES

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We assume knowledge of the "postage stamp problem", see for instance [4]. A comprehensive treatment of this problem is contained in the author's research monograph [5] (freely available on request).

A "stamp" basis (an h-basis)

$$A_k = \{1, a_2, \dots, a_k\}, 1 = a_1 < a_2 < \dots < a_k\},$$

is <u>pleasant</u> if and only if the regular representation $n = \Sigma_1^k e_i a_i$ has a minimal coefficient sum among all possible representations $n = \Sigma_1^k \mathbf{x}_i a_i$, for all natural numbers n. Then the h-range $n_h(A_k)$ equals the <u>regular</u> h-range $g_h(A_k)$, which is easily determined.

Let $A_i = \{1, a_2, \ldots, a_i\}, 2 \le i \le k$, be a "partial basis" of A_k . Then A_2 is always pleasant, and Djawadi [1] gave the following criterion for pleasantness in general: Let $\langle x \rangle$ denote the smallest integer $\ge x$, and put

(1)
$$a_i = \gamma_i a_{i-1} - \underbrace{\sum_{j=1}^{i-2} \beta_j^{(i)} a_j}_{\text{regular by } A_{i-2}}, \quad \gamma_i = \left\langle \frac{a_i}{a_{i-1}} \right\rangle$$

Let further A_{i-1} be pleasant. Then A_i is pleasant if and only if

(2)
$$\gamma_{i} > \sum_{j=1}^{i-2} \beta_{j}^{(i)}$$

Djawadi's proof has been simplified by the author [5, Ch. X].

If the condition (2) is satisfied for <u>all</u> i = 3, 4, ..., k, then all partial bases A_i are pleasant, and we call A_k <u>completely</u> pleasant.

Zöllner [6] showed that

(3)
$$k \ge 4$$
, A_k pleasant $\Rightarrow \{1, a_2, a_i\}$ pleasant, $3 \le i \le k$.

The condition was weakened to " A_k weakly pleasant" by Kirfel [3]. In particular, a pleasant A_k always has a pleasant partial basis A_3 , and a pleasant A_4 is thus completely pleasant. For $k \ge 5$, there are pleasant A_k which are <u>not</u> completely pleasant. For k = 5, <u>all</u> such bases were determined by Djawadi [2]:

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is pleasant if and only if the regular representation $a = L_{p,i}^{2} \epsilon_{i}^{2}$, has a minimal coefficient sum among all possible representations $a = L_{x_{i}}^{2} \epsilon_{i}^{2}$. for all natural numbers a. Then the h-range $a_{h}(A_{h})$, equals the regular h-range $g_{h}(A_{h})$, which is easily determined.

of $A_{\rm K}$. Then $A_{\rm S}$ is always pleasant, and Djawadi [1] gave the following criterion for pleasantness in general: let (x) denote the smallest integer $\geq x$, and put

$$\left(\frac{1}{10}\right)^{-1} = 1^{-1} +$$

Let further A_{1-1} be pleasant. Then A_1 is pleasant if and only

Djawadi's proof has been simplified by the author [5, Ch. X]. If the condition (2) is satisfied for <u>ail</u> i = 5, 4, ..., k. then all partial bases A_i are pleasant, and we call A_i <u>completely</u> bleasant.

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(4)
$$A_5 = \{1, 2, b, b+1, 2b\}, b \ge 4$$

(where A_4 is non-pleasant for $b \ge 4$).

For k = 6, the similar bases were characterized by Zöllner [6]. On the average, probably "most" pleasant bases are completely pleasant.

Even if the complete set of conditions (2), for i = 4, 5, ..., k, is not always necessary for pleasantness of A_k , there are <u>some</u> cases of necessity. Djawadi writes (1) as

(5)
$$a_i + \sum_{j=1}^{i-2} \beta_j^{(i)} a_j = \gamma_i a_{i-1}$$
,

where the left hand side is a <u>regular</u> representation by A_i . If then (2) fails, this representation has a <u>larger</u> coefficient sum than the non-regular representation $\gamma_i^a_{i-1}$, and A_i is then not pleasant by definition. In particular, the condition (2) for i = k is thus always necessary for pleasantness of A_k (whether A_{k-1} is pleasant or not).

We have observed the following trivial but perhaps useful generalization: If i < k, and $\gamma_i a_{i-1} < a_{i+1}$, the left hand side of (5) is also a regular representation by the full basis A_k . Hence, if

(6)
$$\left\langle \frac{a_{i}}{a_{i-1}} \right\rangle a_{i-1} < a_{i+1}$$
 (i < k)

the condition (2) is necessary for pleasantness of A_k .

If k > 3, and we remove the basis elements a_3, a_4, \dots, a_{k-1} , it follows from (3) with i = k that the "<u>sub-basis</u>" {1, a_2, a_k } is pleasant if A_k is pleasant (or only weakly pleasant by [3]). We can prove the following generalization:

THEOREM. If $k \ge 5$, $3 \le \kappa \le k - 2$, and the partial bases A_i , i = κ , κ + 1, ..., k, are all pleasant, then

 $A_{k}^{(\kappa)} = \{1, a_{2}, \dots, a_{\kappa}, a_{k}\}$

is also pleasant. If in particular A_k is completely pleasant, so is $A_k^{(\kappa)}$ for all κ .

Before proving this, we make some comments:

(i) We must remove a "block" $a_{\kappa+1}$, ..., a_{k-1} of elements in A_k up to a_{k-1} . The simplest counter-example is given by the completely pleasant basis $A_5 = \{1, 2, 3, 5, 7\}$. Removing a_3 , we get the non-pleasant basis $\{1, 2, 5, 7\}$.

- = (1, 2, b, b+1, 2b), b ≥ 4

where A. is non-pleasant for b 2 4).

For k = 6, the similar bases were characterized by [31] ner [0]. On the Average, probably "most" pleasant bases are completely

Even if the complete set of conditions (2), for 1 = 4, 5, ..., k, is not always necessary for pleasantness of A_k , here are some cases of necessity. Djawadi writes (1) as

$$-1 - 1^{n} 1^{n} = 1^{n} \left[1^{n} \left[\frac{1}{2} \right]^{n} + 1^{n} \right]$$
(2)

where the left hand side is a regular representation by A_1 . If then (2) fails, this representation has a larger coefficient sum that the non-regular representation '1991, and A₁ is then not pleasent by definition. In particular, the condition (2) for i = k is thus always necessary for pleasantness of A₁ (whether A_{k-1} is pleasant or not).

We have observed the following trivial but perhaps user of some side constituation: If i < k, and $\gamma_i^{a_{i-1}} < c_{i+1}$, the left hand side of (5) is also a regular representation by the full basis A_{i-1} . Hence, if

$$(4 > 2) \quad 1 + 1^{\alpha} > 1 - 1^{\alpha} \left\langle \frac{1^{\alpha}}{1 - 1^{\alpha}} \right\rangle$$

the condition (2) is necessary for pleasantness of A, .

If k > 3, and we remove the basis elements a_5 , a_4 , \cdots , b_{k-1} , it follows from (3) with i = k that the "<u>sub-basis</u>" (1, a_2 , a_3) is pleasant if A_k is pleasant (ar only weakly pleasant by [31). We can prove the following generalization:

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(i) We must remove a "block" a_{k+1} ..., b_{k-1} of elements in a_{k+1} up to a_{k-1} . The simplest counter-example is given by the completely pleasant basis $A_{5} = (1, 2, 3, 5, 7)$. Removing a_{5} . We (ii) The condition A_i pleasant for <u>all</u> $i = \kappa, \kappa + 1, \ldots, k$ is not always necessary. For instance, the Djawadi basis (4) leads to $A_5^{(3)} = \{1, 2, b, 2b\}$, which is pleasant by (2).

(iii) As an example where the Theorem <u>fails</u> when A_i is not pleasant for all $i = \kappa, \kappa + 1, \ldots, k$, consider the following extension of (4):

$$A_{c} = \{1, 2, b, b + 1, 2b, a_{6}\}, b \ge 4,$$

which is pleasant if $a_6 > 2b$ is chosen such that (2) holds for i = 6. However, $A_6^{(4)} = \{1, 2, b, b + 1, a_6\}$ is not of the form (4), and is consequently not pleasant since the partial basis A_4 is not.

To prove the Theorem, it will clearly suffice to use repeated removal of the <u>next largest</u> element, hence to show that

(7)
$$A_k^{(k-2)} = \{1, a_2, \dots, a_{k-2}, a_k\}$$

is pleasant. For this purpose, we substitute a_{k-1} from (1) with i = k - 1 into (1) with i = k, and get a_k expressed by A_{k-2} as

$$a_{k} = (\gamma_{k}\gamma_{k-1} - \beta_{k-2}^{(k)})a_{k-2} - \sum_{j=1}^{k-3} (\gamma_{k}\beta_{j}^{(k-1)} + \beta_{j}^{(k)})a_{j}$$

$$= \widetilde{\gamma}a_{k-2} - \sum_{j=1}^{k-3} \widetilde{\beta}_{j}a_{j} \quad (say) \quad .$$

Using (2) for i = k - 1 and i = k, this gives

$$\begin{split} \widetilde{\gamma} & - \sum_{\substack{j=1 \\ k=2 \\ j=1}}^{k-3} \widetilde{\beta}_{j} &= \gamma_{k} (\gamma_{k-1} - \sum_{\substack{j=1 \\ j=1 \\ k=2 \\ k=2 \\ j=1}}^{k-3} \beta_{j}^{(k-1)}) & - \sum_{\substack{j=1 \\ j=1 \\ k=2 \\ j=1}}^{k-2} \beta_{j}^{(k)} \\ & j = 1 \end{split}$$

in analogy with (2). However, we do not know if (8) corresponds to the form (1) for the basis (7), where we now need

(9)
$$a_k = \gamma a_{k-2} - \sum_{j=1}^{k-3} \beta_j a_j, \quad \gamma = \left\langle \frac{a_k}{a_{k-2}} \right\rangle.$$

regular by A_{k-3}

3.

(ii) The condition A, plansant for all i = x, x + 1, ..., kis not always necessary. For instance, the Biawadi basis (4) leads to $A_{(5)}^{(5)} = (1, 2, b, .2b)$, which is pleasant by (2).

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which is pleasant if $a_0 > 2b$ is chosen such that (2) holds for i = b. However, $\delta_0^{(4)} = \{1, 2, b, b + 1, a_0\}$ is not of the form (4), and is consequently not pleasant since the partial basis A_4 is not.

To prove the Theorem, it will clearly suffice to use repeated removal of the next largest element, hence to show that

$$(2) \qquad \qquad A_{1}^{(k-2)} = (1, 2_{2} = \cdots, 4_{k-2}, a_{k})$$

is pleasant. For this purpose, we substitute a_{k-1} from (1) with i = k - 1 into (1) with i = k, and get a_k expressed by A_{k-2} , as

in analogy with (2). Mowever, we do not know if (8) corresponds to the form (1) for the basis (7), where we now need

(9)
$$a_{x} = ra_{x-2} = \int_{-\frac{1}{2}}^{\frac{1}{2}} a_{y}a_{y} + \frac{r}{2} \cdot \frac{a_{x-2}}{a_{x-2}} + \frac{r}{2} \cdot \frac{a_{x-2}}{a_{x-2}} + \frac{r}{2} \cdot \frac{a_{x-2}}{a_{x-2}} + \frac{r}{2} \cdot \frac{r}{a_{x-2}} + \frac{r}{2} \cdot \frac{r}{a_$$

regular by Ag-5

Equating the two expressions for a_k, we get

$$\widetilde{\gamma}^{a}_{k-2} + \sum_{j=1}^{k-3} \beta_{j}^{a}_{j} = \gamma^{a}_{k-2} + \sum_{j=1}^{k-3} \widetilde{\beta}_{j}^{a}_{j}_{j} .$$

The left hand side is a regular representation by the pleasant basis A_{k-2} , and thus has a minimal coefficient sum:

$$\widetilde{\gamma} + \sum_{\substack{j=1 \\ j=1}}^{k-3} \beta_j \leq \gamma + \sum_{\substack{j=1 \\ j=1}}^{k-3} \widetilde{\beta}_j$$

$$- \sum_{\substack{j=1 \\ j=1}}^{k-3} \beta_j \geq \widetilde{\gamma} - \sum_{\substack{j=1 \\ j=1}}^{k-3} \widetilde{\beta}_j > 0.$$

This shows that (2) is satisfied for the form (9). Since A_{k-2} is pleasant, so is also the basis (7), and the Theorem is proved.

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