STATISTICAL REPORT

Integral conditions for Skorohod stochastic differential equations[1]

by

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SUMMARY

We prove the existence and uniqueness for a quasilinear Skorohod stochastic differential equation with an integral type boundary condition. The initial value may depend on the values of the process at any instant later than a fixed time ϵ. The result is a direct extension of a result by Buckdahn and Nualart on Skorohod equations with boundary conditions.

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1 Introduction

The techniques of the anticipating calculus in [NZ86], [NP88] and others have made it possible to study anticipating stochastic differential equations. The direct extension of the Itô-stochastic differential equations are the Skorohod equations using the nonadapted integral originating from [Sko75]. These have been closely studied in the papers by Buckdahn, [Buc91] and [Buc92]. In particular, quasilinear SDE's with the Skorohod integral are studied in [Buc88]. Allowing possibly anticipating coefficients and initial values opens for a lot of new problems previously only considered for deterministic differential equations. For instance, for the quasilinear equation with the Skorohod integral

$$X_t = X_0 + \int_0^t b_s(X_s)\,ds + \int_0^t \sigma_s X_s\,\delta W_s$$

the boundary condition

$$X_0 = \psi(X_1)$$

has been studied in [BN93] and existence and uniqueness of a solution have been proved also for nondeterministic b and ψ. It is clear that this type of boundary condition requires us to consider nonadapted solutions even in the case where b is adapted and ψ deterministic. Boundary conditions for Stratonovich equations are studied in [NP91], [OP89] and [Don91] and conditions for Markov solutions are derived. In [BN93] it is shown that if, in the Skorohod case, both b and ψ are linear and deterministic the solution can be given an explicit form from which it is clear that the solution is Markov. But at the same time they give an example where one may also have Markov solutions when b is nonlinear, at least in the case where b is random. This indicates that it may be hard to find a natural necessary and sufficient condition for Markovian solutions in the Skorohod case.

In the present note we would like to generalize the existence and uniqueness result for the above equation to existence and uniqueness for the same equation but with the more general condition that

$$X_0(\omega) = \int_E g\left[s, \omega, X_s(\omega)\right]\,dl(s)$$

where l is a signed Borel measure, E a Borel subset of $[\epsilon, 1]$ (for some $\epsilon > 0$) and g a given measurable function.

One motivation for this more general condition is that since it is not clear whether X has continuous paths in general (see, however, theorem 4.1 of [BN93]) it may be more natural with a condition that takes into consideration a larger part of the trajectory rather than just the value of X_1. And the condition $X_0 = \psi(X_1)$ can of course still be recovered by letting l be concentrated at $s = 1$. We will prove existence and uniqueness for an equation with this kind of boundary condition. It will be necessary with the stronger nondegeneracy condition $\int_0^\epsilon \sigma_s^2\,ds > 0$ which can be relaxed to $\int_0^1 \sigma_s^2\,ds > 0$ when l is concentrated at $s = 1$.

2 Solutions of equations with integral conditions

We will now consider existence and uniqueness of the equation

$$X_t(\omega) = X_0(\omega) + \int_0^t b_s(\omega, X_s(\omega))\, ds + \int_0^t \sigma_s X_s(\omega)\, \delta W_s \tag{2.1}$$

with the integral condition

$$X_0(\omega) = \int_E g\,[s, \omega, X_s(\omega)]\; dl(s). \tag{2.2}$$

Our result is an extension of theorem 3.3 in [BN93] and we will follow their main lines of proof. But the first parts of the proof will need several modifications, mainly in establishing integrability for certain terms. The reader should consult [BN93] for details when necessary.

We will in the following use the notation $I = [0,1]$ and E a Borel subset of $[\epsilon, 1]$ for some $\epsilon > 0$. Our basic probability space will be the classical Wiener space. See [NZ86], [NP88] and [Nua94] for the elements of stochastic calculus necessary to formulate and solve our problems.

Let us first list the assumptions.

1. $$\sigma \text{ deterministic, } \sigma \in L^2(I) \quad \text{and} \quad \int_0^\epsilon \sigma_\tau^2\, dt > 0$$

2. b is a measurable function on $I \times \Omega \times \mathbb{R}$ such that, for a.a. ω, the following holds:
$$|b_t(\omega, x) - b_t(\omega, y)| \leq \gamma_t\, |x - y| \text{ for all } t \in I, \quad x, y \in \mathbb{R}$$
$$|b_t(\omega, 0)| \leq \Gamma \text{ for all } t \in I$$
where γ is a deterministic function with $\int_0^1 \gamma_s\, ds \leq \Gamma$.

3. g is a measurable function on $I \times \Omega \times \mathbb{R}$ such that, for a.a. ω,
$$|g(t, \omega, x) - g(t, \omega, y)| \leq k_t\, |x - y| \text{ for all } t \in E, \quad x, y \in \mathbb{R} \tag{2.3}$$
$$|g(t, \omega, 0)| \leq c_t \text{ for all } t \in E$$
where k and c are deterministic functions with $\int_E k_s\, d|l|(s) \leq K$, $\int_E c_s\, d|l|(s) \leq C$ and $|l|$ denotes the total variation of l.

Note that the conditions for b are as in [BN93] but for σ we need to know not only that $|\sigma|_2^2 \stackrel{\text{def}}{=} \int_0^1 \sigma_s^2\, ds > 0$ but in fact that $\int_0^\epsilon \sigma_s^2\, ds > 0$.

In [Buc88] it is shown that in the case where $X_0 \in L^\infty(\Omega)$ the solution X_t to the equation 2.1 can be written in the form

$$X_t = Z_t(A_t, X_0(A_t))L_t. \tag{2.4}$$

Here, $Z_t(\omega, x)$ is the solution (for fixed ω) to the equation

$$Z_t(\omega, x) = x + \int_0^t L_s^{-1}(T_s\omega) b_s[T_s\omega, L_s(T_s\omega) Z_s(\omega, x)]\, ds$$

and $L_t = \exp\left(\int_0^t \sigma_\tau \, \delta W_\tau - \frac{1}{2}\int_0^t \sigma_\tau^2 \, d\tau\right)$. T_t and A_t are defined as the following absolutely continuous transformations from Ω to Ω:

$$\begin{aligned} T_t\omega &= \omega + \int_0^{t\wedge\cdot} \sigma_\tau \, d\tau \\ A_t\omega &= \omega - \int_0^{t\wedge\cdot} \sigma_\tau \, d\tau. \end{aligned}$$

It is then the case that $X \in L^2(I \times \Omega)$ and $\mathbf{1}_{[0,t]}\sigma X \in \text{Dom } \delta$ for all t. But as long as g is not bounded, we cannot assume X_0 to be bounded and thus we have to consider solutions $X \in \cap_{p \geq 1} L^p_{\text{loc}}(I \times \Omega)$ for which $\sigma X \in (\text{Dom } \delta)_{\text{loc}}$ (see definition 2.2 later).

We have the following estimates for Z and g [BN93]:

$$|Z_t(x) - Z_t(y)| \leq e^{\Gamma_t} |x - y| \tag{2.5}$$

$$|Z_t(\omega, x)| \leq e^{\Gamma_t} \left(|x| + \Gamma \int_0^t L_s^{-1}(T_s\omega) \, ds\right) \tag{2.6}$$

where $\Gamma_t = \int_0^t \gamma_s \, ds$, and

$$|g(t, \omega, x)| \leq k_t|x| + c_t. \tag{2.7}$$

We are now ready to prove the existence and uniqueness of an appropriate initial condition X_0.

Proposition 2.1 *Let b, σ and g satisfy the conditions 1-3 on page 2. Then there exists a unique random variable X_0 which solves the equation*

$$X_0(\omega) = \int_E g\left[s, \omega, Z_s(A_s\omega, X_0(A_s\omega))L_s(\omega)\right] dl(s) \tag{2.8}$$

and for which

$$\sup_{k \geq 0} \sup_{s_1 \cdots s_k \in I} |X_0(A_{s_1} \cdots A_{s_k})| < \infty \text{ a.s..}$$

Proof.

Let Y_1, Y_2 be stochastic variables and write

$$W_i(\omega) = \int_E g\left[s, \omega, Z_s(A_s\omega, Y_i(A_s\omega))L_s(\omega)\right] d|l|(s), \qquad i = 1, 2$$

if the integrals exist. Then, by using 2.3 and 2.5 we have

$$|W_1(\omega) - W_2(\omega)| \leq e^\Gamma \int_E k_s L_s \, |Y_1(A_s\omega) - Y_2(A_s\omega)| \, d|l|(s) \tag{2.9}$$

and by 2.6 and 2.7 we have

$$|W_i(\omega)| \leq e^\Gamma \int_E k_s L_s(\omega) \, |Y_i(A_s\omega)| \, d|l|(s) + M(\omega), \quad i = 1, 2 \tag{2.10}$$

where

$$M(\omega) = \Gamma e^\Gamma \int_E k_s L_s(\omega) \int_0^s L_u^{-1}(\omega) \, du \, d|l|(s) + C.$$

Observe that

$$\sup_{k \geq 0} \sup_{s_1 \cdots s_k \in I} M(A_{s_1} \cdots A_{s_k}\omega) \leq M(\omega) \quad \text{a.e..}$$

We now define the sequence $X_0^{(n)}$ recursively by

$$\begin{aligned} X_0^{(0)}(\omega) &= 0 \\ X_0^{(n+1)}(\omega) &= \int_E g\left[s,\omega,Z_s(A_s\omega, X_0^{(n)}(A_s\omega))L_s(\omega)\right]\, dl(s). \end{aligned} \quad (2.11)$$

We must first show that $X_0^{(n)}(\omega)$ is well defined, i.e. that the integral on the right hand side exists as a random variable. Clearly, the integrand is, for a.a. ω, a measurable function of s. Assume, by induction, that

$$\sup_{k\geq 0}\sup_{s_1\cdots s_k\in I}\left|X_0^{(n)}(A_{s_1}\cdots A_{s_k}\omega)\right| < \infty.$$

Then, from 2.10, we see that

$$\left|X_0^{(n+1)}(\omega)\right| \leq e^{\Gamma}\int_E k_s L_s(\omega)\left|X_0^{(n)}(A_s\omega)\right|\, d|l|(s) + M(\omega)$$

and from this it is clear that

$$\begin{aligned} \sup_{k\geq 0}\sup_{s_1\cdots s_k\in I}\left|X_0^{(n+1)}(A_{s_1}\cdots A_{s_k}\omega)\right| &\leq e^{\Gamma}\int_E k_s L_s(\omega)\, d|l|(s) \\ &\quad \times\sup_{k\geq 0}\sup_{s_1\cdots s_k\in I}\left|X_0^{(n)}(A_{s_1}\cdots A_{s_k}\omega)\right| + M(\omega), (2.12) \end{aligned}$$

hence $X_0^{(n)}(\omega)$ is well defined for all n and a.a. ω.
The next step is then to establish the existence of the stochastic variable X_0 as the limit of $X_0^{(n)}$. From iterating 2.11 and using 2.9 we can see that

$$\begin{aligned} \left|X_0^{(n+1)} - X_0^{(n)}\right| &\leq e^{\Gamma}\int_E k_{s_1}L_{s_1}\left|X_0^{(n)}(A_{s_1}) - X_0^{(n-1)}(A_{s_1})\right|\, d|l|(s_1) \\ &\vdots \\ &\leq e^{n\Gamma}\int_E\cdots\int_E\prod_{i=1}^{n}\left\{k_{s_i}L_{s_i}(A_{s_1}\cdots A_{s_{i-1}})\right\} \\ &\quad \times\left|X_0^{(1)}(A_{s_1}\cdots A_{s_n}) - X_0^{(0)}(A_{s_1}\cdots A_{s_n})\right|\, d|l|(s_n)\cdots d|l|(s_1) \\ &\leq \int_E\cdots\int_E\prod_{i=1}^{n}\{k_{s_i}L_{s_i}\}\exp\left[-\sum_{1\leq i<j\leq n}\int_0^{s_i\wedge s_j}\sigma_\tau^2\, d\tau\right] d|l|(s_n)\cdots d|l|(s_1) \\ &\quad \times e^{n\Gamma}\sup_{k\geq 0}\sup_{s_1\cdots s_k\in I}\left|X_0^{(1)}(A_{s_1}\cdots A_{s_k})\right| \end{aligned}$$

Since $\int_0^{s_i\wedge s_j}\sigma_\tau^2\, d\tau \geq \int_0^{\epsilon}\sigma_\tau^2\, d\tau$ it follows that

$$\begin{aligned} \left|X_0^{(n+1)} - X_0^{(n)}\right| &\leq e^{n\Gamma}\exp\left[-\frac{n(n-1)}{2}\int_0^{\epsilon}\sigma_\tau^2\, d\tau\right] \\ &\quad \times\left\{\int_E k_s L_s\, d|l|(s)\right\}^n \sup_{k\geq 0}\sup_{s_1\cdots s_k\in I}\left|X_0^{(1)}(A_{s_1}\cdots A_{s_k})\right|. \end{aligned}$$

Now, according to 2.12, $\sup_{k\geq 0}\sup_{s_1\cdots s_k\in I}\left|X_0^{(1)}(A_{s_1}\cdots A_{s_k})\right| < \infty$ and it follows that, a.s.,

$$\sup_{k\geq 0}\sup_{s_1\cdots s_k\in I}\left|X_0^{(n)}(A_{s_1}\cdots A_{s_k}) - X_0(A_{s_1}\cdots A_{s_k})\right| \to 0$$

as $n \to \infty$ for some random variable X_0 with $\sup_{k\geq 0} \sup_{s_1 \cdots s_k \in I} |X_0(A_{s_1} \cdots A_{s_k})| < \infty$.

Then it should be verified that X_0 really satisfies the equation 2.8. But, when letting $n \to \infty$ in

$$X_0^{(n+1)}(\omega) = \int_E g\left[s, \omega, Z_s(A_s\omega, X_0^{(n)}(A_s\omega))L_s(\omega)\right] dl(s),$$

we see, by using the estimate 2.9, that the right hand side converges to $\int_E g\left[s, \omega, Z_s(A_s\omega, X_0(A_s\omega))L_s(\omega)\right] dl(s)$.

It remains to prove the uniqueness of the solution X_0. Let Y_0 be another solution with $\sup_{k\geq 0} \sup_{s_1 \cdots s_k \in I} |Y_0(A_{s_1} \cdots A_{s_k})| < \infty$. By again using 2.9 repeatedly we have

$$\begin{aligned}
&\sup_{k\geq 0} \sup_{s_1 \cdots s_k \in I} |X_0(A_{s_1} \cdots A_{s_k}) - Y_0(A_{s_1} \cdots A_{s_k})| \\
&\quad \leq e^{n\Gamma} \exp\left[-\frac{n(n-1)}{2} \int_0^\epsilon \sigma_\tau^2 \, d\tau\right] \left\{\int_E k_s L_s \, d|l|(s)\right\}^n \\
&\quad \times \sup_{k\geq 0} \sup_{s_1 \cdots s_k \in I} |X_0(A_{s_1} \cdots A_{s_k}) - Y_0(A_{s_1} \cdots A_{s_k})|
\end{aligned}$$

and the right hand side approaches zero as $n \to \infty$.

□

Let us now proceed to the main result. Having proved the existence of X_0 one would of course like to prove that X_t given by 2.4 is a solution to the equation which satisfies the given integral condition. However, since X_0 may be unbounded it is necessary to introduce the localized domain $(\text{Dom } \delta)_{\text{loc}}$ for the Skorohod integral. Consider the following conditions for a sequence of random variables F_n:

$$\begin{aligned}
&F_n \in \mathbb{D}^{1,2} \quad \text{for all } n \\
&\{F_n = 1\} \uparrow \Omega \quad \text{a.s.} \\
&|F_n| \leq 1 \quad \text{a.s.} \quad \text{for all } n
\end{aligned} \tag{2.13}$$

Definition 2.2 *Assume that the measurable process u verifies $\int_0^1 |u_t|^p \, dt < \infty$ a.s. for some $p \geq 2$.*

- *If there exists a sequence F_n satisfying the conditions 2.13 with $\int_0^1 |D_t F_n|^2 \, dt \in L^\infty(\Omega)$ and $F_n u \in L^p(I \times \Omega)$ for all n, we say that $u \in L^p_{\text{loc}}(I \times \Omega)$.*
- *If there exists a sequence F_n satisfying the conditions 2.13 with $E\left[\int_0^1 |u_t D_t F_n| \, dt\right]^2 < \infty$ and $\mathbf{1}_{[0,t]} F_n u \in \text{Dom } \delta$ for all n and $t \in I$, we say that $u \in (\text{Dom } \delta)_{\text{loc}}$.*

Theorem 2.3 *Let b, σ and g satisfy the assumptions 1-3 on page 2. The process X_t defined by 2.4 is a solution to equation 2.1 with the integral condition 2.2 for which $X \in \bigcap_{p\geq 2} L^p_{\text{loc}}(I \times \Omega)$ and $\sigma X \in (\text{Dom } \delta)_{\text{loc}}$, it is unique among the elements of $L^2_{\text{loc}}(I \times \Omega)$ with $\sigma X \in (\text{Dom } \delta)_{\text{loc}}$ provided that $\int_0^1 \gamma_s^2 \, ds < \infty$.*

Proof. We can find an upper bound for X_0 by

$$\begin{aligned}
|X_0| &\le \sum_{k=0}^{\infty} \left| X_0^{(k+1)} - X_0^{(k)} \right| \\
&\le \sum_{k=0}^{\infty} e^{k\Gamma} \exp\left\{ -\frac{k(k-1)}{2} \int_0^{\epsilon} \sigma_s^2 \, ds \right\} \left\{ \int_E k_s L_s \, d|l|(s) \right\}^k \\
&\quad \times \sup_{k \ge 0} \sup_{s_1 \cdots s_k \in I} \left| X_0^{(1)}(A_{s_1} \cdots A_{s_k}) \right| \\
&\le \sum_{k=0}^{\infty} e^{k\Gamma} \exp\left\{ -\frac{k(k-1)}{2} \int_0^{\epsilon} \sigma_s^2 \, ds \right\} \left\{ \int_E k_s L_s \, d|l|(s) \right\}^k \times M.
\end{aligned}$$

Let us denote the last sum on the right by α_1. We see that

$$|X_t| \le L_t e^{\Gamma} \left(\alpha_1 + \Gamma \int_0^1 L_s^{-1} \, ds \right).$$

Define now the localizing sequence F_n as $F_n = f(\frac{1}{n} \int_E k_s L_s \, d|l|(s))$ where $f \in C_0^{\infty}(\mathbb{R})$ is bounded by 1, $f(x) = 0$ when $|x| \ge 2$ and $f(x) = 1$ when $|x| \le 1$. It is now clear that F_n is a localizing sequence for $X \in \cap_{p \ge 2} L^p_{\text{loc}}(I \times \Omega)$ if only we can show that $\int_E k_s L_s \, d|l|(s) \in \mathbb{D}^{1,2}$ and that $D_t \int_E k_s L_s \, d|l|(s) = \sigma_t \int_E \mathbf{1}(t \le s) k_s L_s \, d|l|(s)$ since the last equality implies that $\int_0^1 |D_t F_n|^2 \, dt \in L^{\infty}(\Omega)$. To this end we approximate the integral $\int_E k_s L_s \, d|l|(s)$ by integrals of step functions and then use lemma 1.2.3 of [Nua94]:

Lemma 2.4 *Assume that the sequence $\{G_n\} \subset \mathbb{D}^{1,2}$ converges to G in $L^2(\Omega)$ and that*

$$\sup_n E \int_0^1 |D_t G_n|^2 \, dt < \infty.$$

Then $G \in \mathbb{D}^{1,2}$ and $DG_n \to DG$ weakly in $L^2(I \times \Omega)$.

Define

$$L_s^{(n)} = \sum_{i=0}^{n} \mathbf{1}_{B_i}(s) L_{i/n}$$

where $B_0 = \{0\}, B_i = ((i-1)/n, i/n]$, $i = 1, \ldots, n$. By the continuity of L we have the convergence

$$k_s L_s^{(n)}(\omega) \to k_s L_s(\omega)$$

for all s, a.s. and, by the dominated convergence theorem, this has the consequence that

$$\int_E k_s L_s^{(n)} \, d|l|(s) \to \int_E k_s L_s \, d|l|(s) \quad \text{in} \quad L^2(\Omega).$$

Now, to apply lemma 2.4 we see that $\int_E k_s L_s^{(n)} \, d|l|(s) \in \mathbb{D}^{1,2}$,

$$\begin{aligned}
D_t \left(\int_E k_s L_s^{(n)} \, d|l|(s) \right) &= D_t \left(\sum_{i=0}^{n} L_{i/n} \int_{B_i} k_s \, d|l|(s) \right) \\
&= \sigma_t \sum_{i=0}^{n} \mathbf{1}\{t \le \frac{i}{n}\} L_{i/n} \int_{B_i} k_s \, d|l|(s) \\
&= \sigma_t \int_E k_s \left[\sum_{i=0}^{n} \mathbf{1}\{t \le \frac{i}{n}\} \mathbf{1}_{B_i}(s) L_{i/n} \right] d|l|(s)
\end{aligned}$$

which converges in $L^2(I \times \Omega)$ (strongly) to $\sigma_t \int_E \mathbf{1}(t \leq s) k_s L_s \, d|l|(s)$ and furthermore that

$$E \int_0^1 \left(D_t \int_E k_s L_s^{(n)} \, d|l|(s) \right)^2 dt \leq |\sigma|_2^2 \left(\int_E k_s \, d|l|(s) \right)^2 E \left(\sup_s L_s^2 \right) < \infty.$$

Hence we have proved that $\int_E k_s L_s \, d|l|(s) \in \mathbb{D}^{1,2}$.

The rest of the proof is identical to the proof of theorem 3.3 in [BN93] and will not be included. □

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