Dowker's Theorem by Simplicial Sets & a Category of 0-Interleavings

University of Bergen

Lars Moberg Salbu

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Abstract

In this thesis we look at an alternative proof of Dowker's theorem [4] using simplical sets. We prove the strongest version of the theorem [3], which can be applied to persistence homology in the sense that every nested sequence of relations gives two filtered simplicial complexes with the same persistence homology.

We also compare the category of filtered simplicial complexes with the category of dissimilarities, and see how this leads to a nice category of 0-interleaved filtered simplicial complexes.

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- Lars M. Salbu

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0.1 Introduction

Dowker's theorem was first stated and proved by C. H. Dowker in his original paper [4] from 1952. Starting with two sets X and Y, and a subset of their product $R \subseteq X \times Y$, one can create two different simplicial complexes $N(R)$ and $N(R^T)$ with vertex sets X and Y respectively. The original result was that for a pair $R_2 \subseteq R_1 \subseteq X \times Y$, the relative homology groups $H_*(N(R_1), N(R_2))$ are isomorphic to $H_*(N(R_1^T), N(R_2^T))$, and similarly for cohomology. It turns out that every simplicial complex can be written as $N(R)$ for some R, so Dowker's theorem gives a new perspective for looking at the topological properties of any simplicial complex.

Dowker's result was improved upon by A. Björner in 1995 ([2] Theorem 10.9). He used the nerve theorem to show that not only are the (co)homology groups isomorphic, but the geometric realizations $|N(R)|$ and $|N(R^T)|$ are in fact homotopy equivalent.

With the rise of topological data analysis, Dowker's theorem has become more relevant. It is a theorem about constructing topological spaces from some initial sets and comparing the topology, which is a big part of topological data analysis. A nice example is when you have a distance function $d : X \times X \to \hat{\mathbb{R}}_+ := [0, \infty]$, then you can look at the subsets $R_t \subseteq X \times X$ of pairs with distance less than t. These subsets are nested $R_0 \subseteq R_{t_1} \subseteq \cdots \subseteq R_\infty$ and the nested sequence $N(R_0) \subseteq N(R_{t_1}) \subseteq \cdots \subseteq N(R_\infty)$ turns out to be the Čech-complex of the distance. This motivates the question of if the homotopy equivalence between $|N(R)|$ and $|N(R^T)|$ acts nicely with the maps $|i| : |N(R)| \to |N(R')|$ and $|i^T| : |N(R^T)| \to |N(R'^T)|$ we get from the inclusion $R \subseteq R'$. The question was answered by Chowdhury and Mémoli [3] in 2018 when they showed that the homotopy equivalences commutes up to homotopy with the maps induced by the inclusions. The original 1952 proof consisted of clever arguments around subdivisions and contiguous maps of simplicial complexes, and in [3] they improved the result in very much the same spirit.

In this thesis we will give an alternative proof of this strong form of Dowker's theorem. Our proof uses a different approach using the slightly more modern theory of simplicial sets. One advantage with this proof is that it mostly uses general results from simplicial sets that are well known, with just a small part specialized towards the exact problem. In addition, we also get another classical result regarding contiguous maps (1.1.8) along the way. Dowker's theorem ultimately is about simplicial complexes, so we do need quite some machinery to go back and forth between simplicial complexes and simplicial sets.

The main asset of this strong form of Dowker's theorem, is that it can be applied to persistence homology, which is the main tool in topological data analysis. In persistence homology one constructs a nested family of spaces from some initial data, then each inclusion induces a homomorphism on the homology groups. A homology class is said to be born if it is not in the image of such a homomorphism, and it dies when it merges with an older class. Classes that are long-lived correspond to topological features in the data, while the shorter-lived ones might correspond to noise. In the end we construct persistence diagrams, telling us all we want to know about the topology of the sequence, by plotting when a class is born and dies (more details in [7]).

In our case the data are the sets X and Y from which we look at a nested sequence of subsets of their product. This leads to two different nested sequences (filtration) of simplicial complexes, which by Dowker's theorem will have the same persistence diagrams and thus the same topological features.

A popular kind of question in persistence homology concerns how changing the filtered simplicial complexes will change the corresponding persistence diagrams. We have the notion of ε-interleavings as some measure for how similar two filtered simplicial complexes are. The infimum of $\varepsilon \geq 0$, making two complexes ε-interleaved is called the interleaving distance between them. On the side of persistence diagrams we have the notion of ε-matching, where the infimum of $\varepsilon \geq 0$ is called the bottleneck distance. One can show that two complexes are ε-interleaved if and only if their corresponding persistence diagrams are ε-matched, and in particular that the interleaving distance agrees with the bottleneck distance [1].

In this thesis we will look at the special case when $\varepsilon = 0$. We will find a category of filtered simplicial complexes where isomorphisms are exactly the 0-interleavings, and show that it is equivalent to other categories with interesting properties. We arrive at this category by exploring the connection between filtered simplicial complexes and general functions $\Lambda : V \times W \to \hat{\mathbb{R}}_+$.

We begin in section 1.1 by looking at Dowker's original proof of the theorem named after him. The proof uses simplicial complexes and barycentric subdivision, so those concepts are also introduced in this section.

In 1.2 we will define simplicial sets, which are the tools we will use in our alternative proof. We will in particular look at finite simplicial sets, as they are needed when defining the geometric realization.

In section 1.3 we define the geometric realization of a simplicial set. We use the definition Drinfeld gave in [5], where we first give the set of the realization as a colimit, and then define a metric inducing a topology. This definition is a bit different from the usual definition used in for example [10], however the equivalence of these definitions is given in [6] and is not in the scope of this thesis. We will show that the geometric realization is a functor, and that it preserves products. The realization uses the notion of colimits, and several results surrounding it, which we include in the appendix A.1. We will also in A.2 calculate the geometric realization for standard n-simplices.

In 1.4 we take the nerve of small categories to get simplicial sets, and show some of its properties. The classifying space is the geometric realization of the nerve, and we show that that the classifying space of a category is homeomorphic to the classifying space of the dual category. We will also look at special kinds of functors that gives rise to homotopies on classifying spaces.

As Dowker's theorem is about simplicial complexes, we look in 1.5 at how to get simplicial sets starting with simplicial complexes in a way that acts nicely on the geometric realization. One of the proofs in this section is moved to the appendix A.3.

Finally, in 1.6 we prove Dowker's theorem using the tools we have introduced in the sections before.

In the second part we begin in 2.1 by defining filtered simplicial complexes and dissimilarities. We define maps between them F, $N_<$ and $N_\leq$, and look at some properties of these maps. We will show that the maps in some sense give an upper and lower bound on 0-interleaved complexes for any filtered simplicial complex.

The concepts we introduce in 2.1 will in 2.2 be made categorical. We will define a category of 0-interleaved filtered simplicial complexes, and show that it is equivalent to both a reflective and coreflective subcategory of the category of filtered simplicial complexes. We will use some results about localizations, which we include in A.4.

Part 1

Dowker's Theorem

In this first part we will state Dowker's theorem and prove it in two different ways. We start by looking at the original [4] 1952 proof using barycentric subdivisions and contiguous maps of simplicial complexes, before delving into the theory of simplicial sets and their geometric realization. This theory builds the framework for our alternative proof of the theorem. In the second proof we will show a stronger theorem which was stated and proved in [3], which is applicable in topological data analysis.

1.1 Dowker's Theorem by Simplicial Complexes

We begin by looking at the work of C.H. Dowker [4], but only a simplified case with a single relation R and not pairs (R_1, R_2). Like Dowker, we will in this first section just look at homology, but you can also follow the same arguments for homotopy [3].

We start with some basic definitions about simplicial complexes. Here and in the entire thesis we write $P(S)$ for the **power set** of S, namely the set of all finite, non-empty subsets of a set S.

Definition 1.1.1. *An* ***(abstract) simplicial complex*** (K, V)*, or just* K*, is a set* V *and a subset* $K \subseteq P(V)$ *such that if* $\tau \in K$ *and* $\sigma \subseteq \tau$ *then* $\sigma \in K$*.*

Given a simplicial complex (K, V), then V is called the **vertex set** of K, an element $v \in V$ is called a **vertex**, and an element $\sigma \in K$ is called a **simplex**. Simplices are written with square brackets $\sigma = [v_1, v_2, \dots, v_r] \in K$ where $v_i \in V$.

Definition 1.1.2. *Given two simplicial complexes* (K, V) *and* (K', V') *then a* ***simplicial map*** $F : K \to K'$ *is a function* $F : V \to V'$ *on the vertex sets such that if* $\sigma = [v_1, v_2, \dots, v_r]$ *is a simplex in* K *then* $F(\sigma) := [F(v_1), F(v_2), \dots, F(v_r)]$ *is a simplex in* K'*.*

Simplicial maps are defined on vertices, so we have that $\sigma \subseteq \sigma'$ implies $F(\sigma) \subseteq F(\sigma')$. For two simplicial maps $F : (K, V) \to (K', V')$ and $F' : (K', V') \to (K'', V'')$, the composition $F' \circ F$ is also a simplicial map. We denote the **category of simplicial complexes** by `Cpx`, where morphisms are simplicial maps.

We will now define a relation between sets, and construct simplicial complexes from this relation. Dowker's theorem is about how these complexes relate to each other.

Definition 1.1.3. *A **relation** R between two sets X and Y is a subset $R \subseteq X \times Y$.*

A subset $R \subseteq X \times X$ is called a **binary relation** of X. Given a relation $R \subseteq X \times Y$, then its **transpose relation** $R^T \subseteq Y \times X$ is given by

$$R^T = \{(y, x) \in Y \times X | (x, y) \in R\}. \tag{1.1}$$

Definition 1.1.4. *From a relation $R \subseteq X \times Y$ we define the simplicial complex $(N(R), X)$ called the **Dowker complex** of R:*

$$N(R) = \{\sigma \in P(X) \,|\, \exists\, y \in Y \text{ such that } \sigma \times \{y\} \subseteq R\}. \tag{1.2}$$

We first note that the Dowker complex is indeed a simplicial complex. If there is a $y \in Y$ with $\tau \times \{y\} \subseteq R$ and if $\sigma \subseteq \tau$ then clearly $\sigma \times \{y\} \subseteq R$, and so σ is also in $N(R)$.

If we have two relations $R \subseteq R' \subseteq X \times Y$, and if σ is in $N(R)$. Then there exist a $y \in Y$ such that $\sigma \times \{y\} \subseteq R \subseteq R'$, and so σ is in $N(R')$. So the identity map on vertex sets, defines a simplicial map $i : N(R) \to N(R')$ which we call the **natural inclusion** of Dowker complexes.

When we talk about the **Dowker complexes** of a relation R, we mean both the Dowker complex of R and the one of R^T.

The construction of a Dowker complex is completely general. If (K, V) is a simplicial complex, let $R \subseteq V \times K$ be the relation defined by $R = \{(v, \sigma) \,|\, v \in \sigma\}$. The Dowker complex of this relation is then $N(R) = \{\sigma \in P(V) \,|\, \sigma \subseteq \sigma'$ for some $\sigma' \in K\} = K$. Thus every simplicial complex is the Dowker complex of some relation.

Starting with a simplicial complex K, we can construct a new simplicial complex with K as its vertex set.

Definition 1.1.5. *The **barycentric subdivision** of a simplicial complex (K, V) is the simplicial complex $(Sd\,K, K)$ where the simplices in $Sd\,K$ are the finite sets of simplices in K which can be ordered by inclusion.*

$$Sd\,K = \{[\sigma_1 \subseteq \sigma_2 \subseteq \cdots \subseteq \sigma_n] \,|\, \sigma_i \in K,\, n \geq 1\}$$

If we take away some of the σ_i's, then the ones that are left are still ordered by inclusion, so $\operatorname{Sd} K$ is indeed a simplicial complex. We can also continue subdividing in a similar fashion getting simplicial complexes $(\operatorname{Sd}^{(2)} K, \operatorname{Sd} K)$, $(\operatorname{Sd}^{(3)} K, \operatorname{Sd}^{(2)} K)$, etc. For the barycentric subdivisions of a Dowker complex we write $\operatorname{Sd}^{(j)}(N(R)) = N^{(j)}(R)$ for $j \geq 1$.

Given a simplicial map $F : K \to L$, we get an induced map $\operatorname{Sd} F : \operatorname{Sd} K \to \operatorname{Sd} L$ given by $\operatorname{Sd} F([\sigma_1, \sigma_2, \ldots, \sigma_n]) = [F(\sigma_1), F(\sigma_2), \ldots, F(\sigma_n)]$. Since F is a simplicial map and $\sigma_i \in K$ for all i, then every $F(\sigma_i)$ is a simplex in L. If we have an inclusion $\sigma_i \subseteq \sigma_j$ then $F(\sigma_i) \subseteq F(\sigma_j)$, and so $\operatorname{Sd} F$ is a simplicial map.

Definition 1.1.6. *Given a simplicial complex $(K, V_<)$, where $V_<$ is a totally ordered set, we define the* ***least vertex map*** *$\phi : \operatorname{Sd} K \to K$ by sending vertices in $\operatorname{Sd} K$ (i.e. simplices in K) to their least vertex in $V_<$.*

Note that for $\sigma_i \subseteq \sigma_j$ we have $\phi(\sigma_i) \geq \phi(\sigma_j)$, so ϕ is order reversing on the vertices.

To show that ϕ is a simplicial map, take a simplex $\operatorname{Sd} \sigma = [\sigma_1, \ldots, \sigma_r] \in \operatorname{Sd} K$, with $\sigma_1 \subseteq \sigma_2 \subseteq \cdots \subseteq \sigma_r$ all simplices in K. For all $i = 1, \ldots, r$ we have that $\phi(\sigma_i) \in \sigma_i \subseteq \sigma_r$. So $\phi(\operatorname{Sd} \sigma) = [\phi(\sigma_i), \ldots, \phi(\sigma_r)] \subseteq \sigma_r \in K$, and thus $\phi(\operatorname{Sd} \sigma) \in K$ as a subset of a simplex.

In the definition of the least vertex map, we needed to introduce an ordering on the vertex set. We are interested in complexes with no natural order, so next we want to show that the specific ordering of $V_<$ turns out to be unimportant. To do this we introduce the notion of contiguous maps.

Definition 1.1.7. *We say that two simplicial maps $F, G : K \to L$ are* ***contiguous*** *if for each simplex $\sigma = [v_1, \ldots, v_r] \in K$ there exists a simplex $\gamma \in L$ such that $F(v_i) \in \gamma$ and $G(v_i) \in \gamma$ for all $i = 1, 2, \ldots, r$. Equivalently, they are contiguous if $F(\sigma) \cup G(\sigma)$ is a simplex in L for all $\sigma \in K$.*

If $\phi : \operatorname{Sd} K \to K$ is the least vertex map, then $\phi(\sigma_i) \in \sigma_i \subseteq \sigma_r$ independent of ordering, so the ϕ corresponding to different orderings of V are all contiguous, as the images all are contained in the biggest simplex. The reason this is interesting is that contiguous maps induce homotopic maps on geometric realization. Exactly what we mean by geometric realization of a simplicial complex we will show in section 1.5, for now we will just state some results. In both the following lemmas we will use that if $f = g$ are homotopic maps, then $f_* = g_*$ on homology groups ([14] 1.10).

Lemma 1.1.8. *If $F, G : K \to L$ are contiguous simplicial maps then, $|F|$ and $|G|$ are homotopic. In particular they induce the same maps on homology.*

Proof. We prove this in the discussion after 1.5.11, using simplicial sets. For a classical proof, see [13] Ch. 3.5, Lemma 2. □

Lemma 1.1.9. *If $\phi : \operatorname{Sd} K \to K$ is the least vertex map as in 1.1.6, then $|\phi|$ is a homotopy equivalence. In particular it induces an isomorphism on homology groups.*

Proof. [3], Proposition 22. □

What follows are some technical results about the relationship between barycentric subdivisions, least vertex maps and Dowker complexes, all discussed in Dowker's original paper [4].

Lemma 1.1.10. *Let $(K, V_<)$ be a simplicial complex with ordered vertex set, and let $\phi : Sd\,K \to K$ be the least vertex map. Then $(Sd\,\phi)_* : H_*(Sd^{(2)}K) \to H_*(Sd\,K)$ is an isomorphism.*

Proof. Let $\phi_K : \mathrm{Sd}\,K \to K$ denote the least vertex map with respect to the ordering on $V_<$. Let $\phi_{\mathrm{Sd}\,K} : \mathrm{Sd}^{(2)}K \to \mathrm{Sd}\,K$ be the least vertex map with respect to some ordering $<$ of K that refines the order given by the opposite of inclusions, i.e such that $\sigma \subseteq \tau$ implies $\tau \leq \sigma$. By 1.1.9 we know that both these maps induces isomorphisms on homology, so it is enough to show that the two compositions $\phi_K \circ \phi_{\mathrm{Sd}\,K}$ and $\phi_K \circ \mathrm{Sd}\,\phi_K$ are the same.

If $\sigma^{(1)} = [\sigma_0 \subseteq \cdots \subseteq \sigma_n]$ is a simplex in $\mathrm{Sd}\,K$, then $\phi_K(\sigma^{(1)}) = [\min_V(\sigma_0), \ldots, \min_V(\sigma_n)]$. So let $\sigma^{(2)} = [\sigma_0^{(1)} \subseteq \cdots \subseteq \sigma_k^{(1)}]$ be any simplex in $\mathrm{Sd}^{(2)}K$ where we write $\sigma_i^{(1)} = [\sigma_{i0} \subseteq \cdots \subseteq \sigma_{in_i}]$.

We first look at the composition with the map we are interested in. We have $\phi_K \circ \mathrm{Sd}\,\phi_K(\sigma^{(2)}) = \phi_K[\phi_K(\sigma_0^{(1)}), \ldots, \phi_K(\sigma_k^{(1)})] = [\min_V \phi_K(\sigma_0^{(1)}), \ldots, \min_V \phi_K(\sigma_k^{(1)})]$. Now we know that $\phi_K(\sigma_i^{(1)}) = [\min_V \sigma_{i0}, \ldots, \min_V \sigma_{in_i}]$, and that $\sigma_i \subseteq \sigma_j$ implies that $\min_V \sigma_j \leq \min_V \sigma_i$, and therefore we get $\min_V \phi_K(\sigma_i^{(1)}) = \min_V \sigma_{in_i}$. We conclude that $\phi_K \circ \mathrm{Sd}\phi_K(\sigma^{(2)}) = [\min_V \sigma_{0n_0}, \ldots, \min_V \sigma_{kn_k}]$.

The other way we have $\phi_K \circ \phi_{\mathrm{Sd}\,K}(\sigma^{(2)}) = \phi_K[\min_K \sigma_0^{(1)}, \ldots, \min_K \sigma_k^{(1)}]$, and by the definition of the ordering on K we have $\sigma_i \subseteq \sigma_j$ implies $\sigma_j \leq \sigma_i$. So $\min_K \sigma_i^{(1)} = \sigma_{in_i}$, and thus the composition is $\phi_K \circ \phi_{\mathrm{Sd}\,K}(\sigma^{(2)}) = \phi_K[\sigma_{0n_0}, \ldots, \sigma_{kn_k}] = [\min_V \sigma_{0n_0}, \ldots, \min_V \sigma_{kn_k}]$ which is the same as we got for $\phi_K \circ \mathrm{Sd}\,\phi_K$.

In conclusion we have that since $(\phi_K)_*$ and $(\phi_{\mathrm{Sd}\,K})_*$ both are isomorphisms, and since $(\phi_K)_* \circ (\phi_{K^{(1)}})_* = (\phi_K)_* \circ (\mathrm{Sd}\,\phi_K)_*$ we get that $(\phi_{\mathrm{Sd}\,K})_* = (\mathrm{Sd}\,\phi_K)_*$ and thus $(\phi_K^{(1)})_*$ is also an isomorphism. □

Definition 1.1.11. *Let $N(R)$ and $N(R^T)$ be the Dowker complexes of a relation $R \subseteq X \times Y$ and its transpose $R^T \subseteq Y \times X$. Define the maps*

(a) $\Phi : N^{(1)}(R) \to N(R)$ to be the least vertex map for some ordering on X.

(b) $\Psi : N^{(1)}(R) \to N(R^T)$ by sending vertices $\sigma \in N(R)$ to $\Psi(\sigma) = y \in Y$ such that $(s, y) \in R$ for all $s \in \sigma$. (The existence of y is guaranteed by the definition 1.1.4)

Recall we defined $\mathrm{Sd}^{(j)}(N(R)) = N^{(j)}(R)$. We get similar maps Φ^T and Ψ^T by interchanging $R \longleftrightarrow R^T$ and $X \longleftrightarrow Y$ in the definition above.

To show that Ψ is a simplicial map, let $\sigma^{(1)} = [\sigma_1 \subseteq \cdots \subseteq \sigma_r] \in N^{(1)}(R)$ be a simplex. Let $x_1 \in \sigma_1$ be a vertex, then $x_1 \in \sigma_i$ for all $i = 1, 2, \ldots, r$. By the definition of Ψ we then get that $(x_1, \Psi(\sigma_i)) \in R$ for all the i's. We know from 1.1.4 that $\tau \in N(R^T)$ is a simplex if and only if there exist an $x \in X$ such that $(x, t) \in R$ for all $t \in \tau$, so $\Psi(\sigma^{(1)}) = [\Psi(\sigma_1), \ldots, \Psi(\sigma_r)] \in N(R^T)$ is a simplex by using $x = x_1$.

Note that the definition of Ψ is dependent on choice, but also here we get that different choices will give contiguous maps. If Ψ_1 and Ψ_2 are two such maps, then since we picked $x_1 \in \sigma_1$ independently of Ψ we still have that $(x_1, \Psi_1(\sigma_i)) \in R$ and $(x_1, \Psi_2(\sigma_i)) \in R$ for all $i = 1, 2, \ldots, r$. Again by the definition 1.1.4 of $N(R^T)$ this implies that $[\Psi_1(\sigma_1), \ldots, \Psi_1(\sigma_r), \Psi_2(\sigma_1), \ldots, \Psi_2(\sigma_r)]$ is a simplex in $N(R^T)$ which contains all the images of the vertices of $\sigma^{(1)}$ under Ψ_1 and Ψ_2.

Lemma 1.1.12. *(Lemma 5 and 6 in [4], Claim 1 p.16 in [3])*

(i) $\Phi^T \circ Sd\,\Psi$ *and* $\Psi \circ Sd\,\Phi : N^{(2)}(R) \to N(R^T)$ *are contiguous.*

(ii) $\Phi \circ Sd\,\Phi$ *and* $\Psi^T \circ Sd\,\Psi : N^{(2)}(R) \to N(R)$ *are contiguous.*

Proof. Let $\sigma^{(2)} = [\sigma_1^{(1)}, \ldots, \sigma_r^{(1)}] \in N^{(2)}(R)$ be a such that $\sigma_1^{(1)} \subseteq \cdots \subseteq \sigma_r^{(1)}$.

(i): To start off we look at $\mathrm{Sd}\,\Psi(\sigma^{(2)}) = [\Psi(\sigma_1^{(1)}), \ldots, \Psi(\sigma_r^{(1)})]$. Since Ψ is defined on vertices, we get that $\sigma_i \subseteq \sigma_r$ implies $\Psi(\sigma_i) \subseteq \Psi(\sigma_r)$, and so $\mathrm{Sd}\,\Psi(\sigma^{(2)})$ is a simplex in $N(R^T)$. Now Φ^T picks out a vertex (the least) for each of the simplices $\Psi(\sigma_i^{(1)})$, but since they all are contained in $\Psi(\sigma_r^{(1)})$, each vertex we pick is also in $\Psi(\sigma_r^{(1)})$. So we get that $\Phi^T(\mathrm{Sd}\,\Psi(\sigma^{(2)})) \subseteq \Psi(\sigma_r^{(1)})$.

Next we have $\mathrm{Sd}\,\Phi(\sigma^{(2)}) = [\Phi(\sigma_1^{(1)}), \ldots, \Phi(\sigma_r^{(1)})]$, where Φ picks out a (least) vertex. Since $\sigma_i^{(1)} \subseteq \sigma_r^{(1)}$ for all $i = 1, \ldots, r$, we get as above that $\mathrm{Sd}\,\Phi(\sigma^{(2)}) \subseteq \sigma_r^{(1)}$. Now since Ψ is defined on vertices we also have $\Psi(\mathrm{Sd}\,\Phi(\sigma^{(2)})) \subseteq \Psi(\sigma_r^{(1)})$. We conclude that the images of $\sigma^{(2)}$ under $\Phi^T \circ \mathrm{Sd}\,\Psi$ and $\Psi \circ \mathrm{Sd}\,\Phi$ are both contained in the simplex $\Psi(\sigma_r^{(1)})$, and the maps are therefore contiguous.

(ii): Let $\sigma^{(2)}$ be as above, such that $\sigma_1^{(1)} \subseteq \sigma_i^{(1)}$ for all $i = 1, \ldots, r$. We first look at $\mathrm{Sd}\,\Phi(\sigma^{(2)}) = [\Phi(\sigma_1^{(1)}), \ldots, \Phi(\sigma_r^{(1)})]$. The function Φ picks out the least vertex which we call $\sigma_{i1} = \Phi(\sigma_i^{(1)}) \in \sigma_i^{(1)}$. We have $[\sigma_{11} \leq \cdots \leq \sigma_{1n_1}] = \sigma_1^{(1)} \subseteq \sigma_i^{(1)} = [\sigma_{i1} \leq \cdots \leq \sigma_{in_i}]$, and so $\sigma_{i1} \subseteq \sigma_{11}$ for all $i = 1, \ldots, r$.

Now let $[y_1] = \Psi \circ \mathrm{Sd}\,\Phi[\sigma_1^{(1)}] = \Psi[\Phi(\sigma_1^{(1)})] = \Psi[\sigma_{11}]$. Then $y_1 \in Y$ is such that $\sigma_{11} \times \{y_1\} \in R$, and in particular $\sigma_{i1} \times \{y_1\} \in R$ for all $i =$

$1, \dots, r$. Now since Φ just picks out some vertex we have that $\Phi \circ \operatorname{Sd}\Phi(\sigma^{(2)}) = \Phi[\sigma_{11}, \dots, \sigma_{r1}] = \sigma_{j1}$ for some $1 \leq j \leq r$, and thus $\Phi \circ \operatorname{Sd}\Phi(\sigma^{(2)}) \times \{y_1\} = \sigma_{j1} \times \{y_1\} \subseteq R$.

Next we look at $\Psi^T \circ \operatorname{Sd}\Psi(\sigma^{(2)}) = \Psi^T[\Psi(\sigma_1^{(1)}), \dots, \Psi(\sigma_r^{(1)})]$. First let $[x_i] = \Psi^T[\Psi(\sigma_i^{(1)})]$, then x_i is such that $(x_i, t) \in R$ for all $t \in \Psi(\sigma_i^{(1)})$. We have $\sigma_{11} \in \sigma_1^{(1)} \subseteq \sigma_i^{(1)}$, and since Ψ is defined on vertices we also have that $\sigma_{11} \in \sigma_i^{(1)}$ implies $[y_1] = \Psi[\sigma_{11}] \subseteq \Psi(\sigma_i^{(1)})$. Since now $y_1 \in \Psi(\sigma_i^{(1)})$ is a vertex for all $i = 1, \dots, r$, we get $(x_i, y_1) \in R$, and thus $\Psi^T \circ \operatorname{Sd}\Psi(\sigma^{(2)}) \times \{y_1\} = \{x_1, \dots, x_r\} \times \{y_1\} \subseteq R$.

We conclude that the images of $\sigma^{(2)}$ under $\Phi \circ \operatorname{Sd}\Phi$ and $\Psi^T \circ \operatorname{Sd}\Psi$ are both contained in $\Phi \circ \operatorname{Sd}\Phi(\sigma^{(2)}) \cup \Psi^T \circ \operatorname{Sd}\Psi(\sigma^{(2)})$ which we have just shown is a simplex in $N(R)$ using definition 1.1.4. Therefore the maps are contiguous. □

Note that since the maps Ψ^T and Φ^T are just similar maps but defined for the relation R^T and not R, we get that 1.1.12 also is true by exchanging $\Psi \longleftrightarrow \Psi^T$, $\Phi \longleftrightarrow \Phi^T$ and $R \longleftrightarrow R^T$.

We finally arrive at Dowker's Theorem.

Theorem 1.1.13. ***(Dowker's Theorem)*** *Let R be a relation, and $N(R)$, $N(R^T)$ the corresponding Dowker complexes. Then the homology groups $H_p(N(R))$ and $H_p(N(R^T))$ are isomorphic for all $p \in \mathbb{Z}$.*

Proof. From 1.1.12(ii) together with 1.1.8 we have that $(\Psi^T)_*(\operatorname{Sd}\Psi)_* = (\Phi)_*(\operatorname{Sd}\Phi)_*$. Now using the fact that $(\Phi)_*$ and $(\operatorname{Sd}\Phi)_*$ both are isomorphisms (by 1.1.9 and 1.1.10), we can take the inverse on both sides to get

$$(\Psi^T)_* \circ (\operatorname{Sd}\Psi)_* \circ (\operatorname{Sd}\Phi)_*^{-1} \circ (\Phi)_*^{-1} = \operatorname{Id}_{H_*(N(R))}. \tag{1.3}$$

The contiguity in 1.1.12(i) gives us $(\Phi^T)_*(\operatorname{Sd}\Psi)_* = (\Psi)_*(\operatorname{Sd}\Phi)_*$, so taking inverses we get $(\operatorname{Sd}\Psi)_*(\operatorname{Sd}\Phi)_*^{-1} = (\Phi^T)_*^{-1}(\Psi)_*$. By substituting the middle in (1.3) we get $(\Psi^T)_*(\Phi^T)_*^{-1} \circ (\Psi)_*(\Phi)_*^{-1} = \operatorname{Id}_{H_*(N(R))}$, and similarly by interchanging everything with its corresponding transpose we also get $(\Psi)_*(\Phi)_*^{-1} \circ (\Psi^T)_*(\Phi^T)_*^{-1} = \operatorname{Id}_{H_*(N(R^T))}$. Thus $(\Psi)_*(\Phi)_*^{-1} : H_*(N(R)) \to H_*(N(R^T))$ is an isomorphism with inverse $(\Psi^T)_*(\Phi^T)_*^{-1}$. □

This proof uses the contiguity property for all it is worth, and by cleverly combining it with the barycentric subdivision we get our result. We will next give an alternative proof using simplical sets, but for that we need some more tools.

1.2 Simplicial Sets

We now introduce the notion of simplicial sets which is the main tool we use in the alternative proof of Dowker's theorem. First we look at two new categories, which we will need in the definition.

Definition 1.2.1. *Define* $[n]$ *as the category with objects* $Ob[n] = \{0, \dots, n\}$ *and morphisms* $i \to j \in Mor[n]$ *if and only if* $0 \leq i \leq j \leq n$. *We write* $(i \leq j) \in Mor[n]$ *and composition is given by* $(j \leq k) \circ (i \leq j) = (i \leq k)$.

Definition 1.2.2. *The* ***simplex category*** Δ *is the category with objects* $Ob\,\Delta = \{[n] \,|\, n \geq 0\}$ *and where the morphisms are functors* $Hom_\Delta([m],[n]) = \{functors\ [m] \to [n]\}$.

Note that the functors $[m] \to [n]$ are exactly the order-preserving functions. For if $f : [n] \to [m]$ is a function such that $i \leq j$ implies $f(i) \leq f(j)$, then $i \leq j \leq k$ implies $f(i) \leq f(j) \leq f(k)$ so compositions are preserved, also $f(i) = f(i)$, so f preserves identities and it is a functor. Conversely if $F : [n] \to [m]$ is a functor and $l : i \to j$ is the morphism $i \leq j$ in $[n]$, then $F(l) : F(i) \to F(j)$ is a morphism in $[m]$, i.e. $F(i) \leq F(j)$, and F is order-preserving.

Definition 1.2.3. *A* ***simplicial set*** *is a functor* $X : \Delta^{op} \to \mathtt{Sets}$. *It gives a set* $X_n = X([n])$ *for each* $n \geq 0$ *and functions* $X_n \xrightarrow{X(\alpha)} X_m$ *for each order-preserving map* $[m] \xrightarrow{\alpha} [n]$.

An element $x \in X_n$ is called an n-simplex. A **morphism of simplicial sets** is a natural transformation $\eta : X \to Y$, i.e. a collection of functions $\{\eta_n : X_n \to Y_n \,|\, n \geq 0\}$ such that for all order-preserving maps $\alpha : [m] \to [n]$ we have $\eta_m \circ X(\alpha) = Y(\alpha) \circ \eta_n$, as in the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{X(\alpha)} & X_m \\ \downarrow{\scriptstyle \eta_n} & & \downarrow{\scriptstyle \eta_m} \\ Y_n & \xrightarrow{Y(\alpha)} & Y_m. \end{array} \tag{1.4}$$

We say η is **surjective** (or **injective**) if all functions η_n are surjective (or injective). We say X is a **simplicial subset** of Y, written $X \subseteq Y$, if X_n is a subset of Y_n for all $n \geq 0$. The set X_n is called the **set of degree** n, and an element $x \in X_n$ is called an n**-simplex**.

We denote the category of simplicial sets by `sSet`. One can show that the product and coproduct (defined in A.1) in this category is defined in each degree, $(X \times Y)_n = X_n \times Y_n$ and $(X \amalg Y)_n = X_n \amalg Y_n$. The maps induced by $\alpha : [m] \to [n]$ are $(X \times Y)(\alpha) = (X(\alpha), Y(\alpha)) : X_n \times Y_n \to X_m \times Y_m$ for products, and for coproducts we get the map $(X \amalg Y)(\alpha)$ mapping $x \in X_n \subseteq X_n \amalg Y_n$ to $X(\alpha)(x) \in X_m \subseteq X_m \amalg Y_m$, and similarly for $y \in Y_n$.

An important example of simplicial sets are the standard simplices.

Definition 1.2.4. *The* ***standard n-simplex*** Δ^n *is the simplicial set given by* $\Delta^n := Hom_\Delta(-, [n])$.

Given a simplicial set, we now want to extend it to a functor from a more general category. This will be important later when we define the geometric realization.

Lemma 1.2.5. *A functor $X : \Delta^{op} \to$ `Sets` can be extended to a functor $X' : \Delta_{big}^{op} \to$ `Sets`, where Δ_{big} is the category of finite non-empty totally ordered sets and order-preserving functions. This extension is unique up to unique isomorphism.*

Proof. First we note that we have the inclusion $\Delta \subseteq \Delta_{big}$. Also every element in $T \in \Delta_{big}$ is isomorphic to a unique element $[n]$ in Δ by renaming the elements, we call the isomorphism $\nu_T : T \to [n]$. For example $\{a < b < c\} \in \Delta_{big}$ is isomorphic to $[2] = \{0 < 1 < 2\}$.

To show existence of an extension, let T be an object in Δ_{big} isomorphic to $[n]$, and define $X'(T) := X_n$. If $h : T \to S$ is a morphism in Δ_{big}, then this gives a unique morphism $\alpha_h = \nu_S \circ h \circ \nu_T^{-1} : [n] \to [m]$. We define $X'(h) := X(\alpha_h) : X(S) \to X(T)$. Note that $X'(\nu_{[n]}) = X(\mathrm{Id}_{[n]}) = \mathrm{Id}_{X_n}$, so X' is a well-defined extension which we call the **natural extension** and write $X' = X$.

Let $Y : \Delta_{big}^{op} \to$ `Sets` be a functor such that $Y([n]) = X_n$ for all n and $Y(\alpha) = X(\alpha)$ for all $\alpha \in \Delta$. Then $Y(\nu_T) : Y([n]) \to Y(T)$ is an isomorphism, and every functor $h : T \to S$ can be written as $h = \nu_S^{-1} \circ \alpha_h \circ \nu_T$. Now we calculate $Y(h) = Y(\nu_T) \circ Y(\alpha_h) \circ Y(\nu_S^{-1}) = Y(\nu_T) \circ X(\alpha_h) \circ Y(\nu_S)^{-1}$, and so the collection $\{Y(\nu_T)\}$ defines a unique natural isomorphism between Y and the natural extension X. □

We have two families of morphisms in Δ that are particularly important in relation the simplicial sets, namely the face and degeneracy maps. One can in fact define simplicial sets by the properties of these maps [8].

Definition 1.2.6. *Let $\sigma^i : [n+1] \to [n]$ be the map*

$$\sigma^i(j) = \begin{cases} j & \text{for } j \leq i \\ j-1 & \text{for } j > i, \end{cases} \tag{1.5}$$

and let $\delta^i : [n] \to [n+1]$ be the map

$$\delta^i(j) = \begin{cases} j & \text{for } j < i \\ j+1 & \text{for } j \geq i. \end{cases} \tag{1.6}$$

*Now if X is a simplicial set, then we call $s_i := X(\sigma^i)$ the **degeneracy maps** and $d_i := X(\delta^i)$ the **face maps** of X.*

A composition of degeneracy maps is called a **degeneracy**, also if $x = Sz$ where S is a degeneracy then we say that x is a **degeneracy of** z. It is easy to see that $\sigma^i \circ \delta^i = \mathrm{Id}_{[n]}$, and since simplicial sets are contravariant functors we get that $d_i s_i$ is also the identity.

Definition 1.2.7. *An n-simplex $x \in X_n$ is **degenerate** if it can be written as $s_i\overline{x}$ for some $\overline{x} \in X_{n+1}$ and some $i \in [n]$. It is **non-degenerate** if it is not degenerate.*

Lemma 1.2.8. *Every degenerate simplex is a degeneracy of a unique non-degenerate simplex.*

Proof. [8], Prop. 4.8 □

Next we introduce the subclass of finite simplicial sets. The geometric realization of any simplicial set will be defined by the geometric realization of its finite simplicial subsets.

Definition 1.2.9. *A simplicial set is **finite** if it has finitely many non-degenerate simplices.*

To better understand this definition we look at an important example.

Proposition 1.2.10. *The standard n-simplex Δ^n is finite.*

Proof. Let $x \in \Delta^n_m = \mathrm{Func}([m],[n])$ for $m > n$, so in particular x is not injective. Since x also is order-preserving we know that there is an object i in $[m]$ such that $x(i) = x(i+1)$. Let σ^i be as in (1.5) and δ^i as in (1.6), then we calculate the composition

$$\delta^i \circ \sigma^i(j) = \begin{cases} j & \text{for } j \neq i \\ i+1 & \text{for } j = i. \end{cases}$$

Now $x \circ \delta^i \circ \sigma^i(j) = x(j)$, since $x(i) = x(i+1)$. Recall $s_i = \Delta^n(\sigma^i) = - \circ \sigma^i$, and so $x = x \circ \delta^i \circ \sigma^i = s_i(x \circ \delta^i)$, and x is degenerate.

We conclude that $x \in \Delta^n_m$ can only be non-degenerate if $m \leq n$, but since Δ^n_m has only finitely many elements and n is finite, there is at most finitely many non-degenerate simplices. □

For the rest of the section we will show that several different operations preserve finiteness.

Lemma 1.2.11. *If X and Y are finite, then the coproduct $X \amalg Y$ is also finite.*

Proof. The coproduct of sets is the disjoint union, so if u is an element in $(X \amalg Y)_n = X_n \amalg Y_n$, then u is in X_n or in Y_n. Assume $u \in X_n$ degenerate, i.e. $u = X(\sigma^i)(\overline{u})$ for some $\overline{u} \in X_{n+1}$. By the definition of $(X \amalg Y)(\sigma^i)$ this is true if and only if $(X \amalg Y)(\sigma^i)(\overline{u}) = u$ for the same $\overline{u}$ in $X_{n+1} \amalg Y_{n+1}$. So a simplex $u \in X \amalg Y$ is non-degenerate if and only if it is non-degenerate in X or in Y. Since X and Y both have finitely many non-degenerate simplices, so does $X \amalg Y$. □

Lemma 1.2.12. *Simplicial subsets of finite simplicial sets are finite.*

Proof. Let $Y \subseteq X$ be a simplicial subset, where X is finite. Let $y \in Y_n$ be non-degenerate and assume by contradiction that it is degenerate in X_n, namely $y = s_i\overline{x}$ for some $\overline{x} \in X_{n-1}$. If $\delta^i : [n-1] \to [n]$ is the map (1.6) so that $d_i s_i$ is the identity, then $d_i y \in Y_{n-1}$, and $d_i y = d_i s_i \overline{x} = \overline{x}$. This is a contradiction on the fact that y is non-degenerate in Y_n. We conclude that y is non-degenerate in X_n, and there are finitely many of these. □

Lemma 1.2.13. *If Y is finite and $f : Y \to X$ is a surjective morphism, then X is finite.*

Proof. Let $S = \{x_i \in X_{n_i} \text{ non-degenerate}\}$ be the set of non-degenerate simplices in X. Because f is surjective, the preimage $f^{-1}S$ has more than or the same number of elements as S. Let y be a degenerate simplex in Y, i.e. $y = s_i\overline{y} = Y(\sigma^i)\overline{y}$, and let $x = f(y)$. Since f is a morphism, and thus a natural transformation, we have the commuting diagram

$$\begin{array}{ccc} Y_n & \xrightarrow{Y(\sigma^i)} & Y_{n+1} \\ \big\downarrow f & & \big\downarrow f \\ X_n & \xrightarrow{X(\sigma^i)} & X_{n+1}, \end{array}$$

In particular we get $x = f \circ Y(\sigma^i)\overline{y} = X(\sigma^i) \circ f(\overline{y}) = s_i(f(\overline{y}))$, and thus x is degenerate. So we have that $f(y)$ is degenerate whenever y is. The contrapositive statement is that if x is non-degenerate, then $y \in f^{-1}(x)$ is also non-degenerate. In particular we have that $f^{-1}S$ is a subset of non-degenerate simplices of Y which is finite, therefore S is also finite. $\square$

We use some of these properties to define an equivalent definition of finiteness, which we will use to show that products of finite simplicial sets are finite.

Lemma 1.2.14. *A simplicial set X is finite if and only if there exist a finite indexing set A, and a surjective map*

$$F : \coprod_{\alpha \in A} \Delta^{n_\alpha} \to X. \tag{1.7}$$

Proof.

(⇒): Let X be finite, and let T be the set of all non-degenerate simplices of X. We can now name the elements by some finite indexing set $T = \{x_\alpha\}_{\alpha \in A}$. Let n_α be such that $x_\alpha \in X_{n_\alpha}$. Now let F be the map sending $\beta \in \Delta^{n_\alpha}_m$ to $X(\beta)x_\alpha \in X_m$. If $x = x_\alpha \in X_n$ is non-degenerate, then $x = X(Id_{[n_\alpha]})x_\alpha$ and it is in the image of F. If x is degenerate, then by 1.2.8 there is a non-degenerate simplex $x_\alpha \in X_{n_\alpha}$ such that $x = s_{i_1} \circ \cdots \circ s_{i_k} x_\alpha = X(\sigma^{i_k} \circ \cdots \circ \sigma^{i_1})x_\alpha$. Thus F is surjective.

(⇐): Let A be a finite index set such that (1.7) is surjective. By 1.2.11 this is a surjective morphism from a finite simplicial set, and so by 1.2.13 the simplicial set X is finite. $\square$

Lemma 1.2.15. *The product of two standard simplices $\Delta^n \times \Delta^m$ is finite.*

Proof. Let K be the finite set of all injective functors $\phi : [n+m] \to [n] \times [m]$, and define the map

$$H : \coprod_{\phi \in K} \Delta^{n+m} \to \Delta^n \times \Delta^m$$

by sending each β in the set Δ_k^{n+m} corresponding to ϕ to the composition $\phi \circ \beta$. By 1.2.13 setting $n_\phi = n + m$ for all $\phi \in K$, it is enough to show that H is surjective.

For degree k, we have $(\Delta^n \times \Delta^m)_k = \text{Func}([k],[n]) \times \text{Func}([k],[m]) = \text{Func}([k],[n] \times [m])$. Any functor $f : [k] \to [n] \times [m]$ gives a sequence $f(0) \leq f(1) \leq \cdots \leq f(k)$ of $k+1$ elements in $[n] \times [m]$, where $(r,s) \leq (r',s')$ if and only if $r \leq r'$ and $s \leq s'$.

Starting with $(0,0) \in [n] \times [m]$ we can construct an ordered sequence (not unique) that contains every $f(i)$ in order, ending up in (n,m). We do this inductively by adding one to one of the coordinates that are still less than the next $f(i)$ we want to hit. This sequence will have $n+m+1$ elements as we would have to add n times in one direction and m times in the other, starting with $(0,0)$. This sequence thus corresponds to a functor $\phi : [n+m] \to [n] \times [m]$, which is injective as we always add one to a coordinate in each term. Since it contains every $f(i)$ in order, we can find a functor $f' : [k] \to [n+m]$ such that $f = \phi \circ f'$. Thus f is hit by H. □

To clarify what we just did, let's look at an example. Let $f : [2] \to [3] \times [2]$ be the functor defined by $f(0) = (0,1)$, $f(1) = (1,2)$ and $f(2) = (2,2)$. We then have a non-unique sequence

$$(0,0) \leq (0,1) = f(0) \leq (1,1) \leq (1,2) = f(1) \leq (2,2) = f(2) \leq (3,2),$$

going from $(0,0)$ to $(3,2)$ containing every $f(i)$ in order. This corresponds to the injective functor $\phi : [5] \to [3] \times [2]$ defined by $\phi(0) = (0,0)$, $\phi(1) = (0,1)$, $\phi(2) = (1,1)$, and so on. Now $f(0) = \phi(1)$, $f(1) = \phi(3)$ and $f(2) = \phi(4)$, and we have the map $f' : [2] \to [5]$ given by $f'(0) = 1$, $f'(1) = 3$ and $f'(2) = 4$, such that $f = \phi \circ f'$.

Lemma 1.2.16. *If X and Y is finite then $X \times Y$ is finite.*

Proof. Let $R, S, T \in \texttt{Sets}$ and note that the set

$$(R \amalg S) \times T = \{(x,t) \,|\, x \in R \text{ or } x \in S, \text{ and } t \in T\}$$

and the set

$$(R \times T) \amalg (S \times T) = \{(x,t) \,|\, (x,t) \in R \times T \text{ or } (x,t) \in S \times T\}$$

are isomorphic, by what we call the **distributive bijection**. This can be extended to finite products and disjoint unions. Let X and Y be finite simplicial sets, and let A and B be finite sets with surjective maps $\coprod_{\alpha \in A} \Delta^{n_\alpha} \to X$ and $\coprod_{\beta \in B} \Delta^{n_\beta} \to Y$. We combine the maps to get a surjective map

$$\Big(\coprod_{\alpha \in A} \Delta^{n_\alpha}\Big) \times \Big(\coprod_{\beta \in B} \Delta^{n_\beta}\Big) \to X \times Y \qquad (1.8)$$

Looking closer at the right side, and looking in each degree k we have

$$\Big((\coprod_{\alpha\in A} \Delta^{n_\alpha}) \times (\coprod_{\beta\in B} \Delta^{n_\beta}) \Big)_k = (\coprod_{\alpha\in A} \Delta^{n_\alpha}_k) \times (\coprod_{\beta\in B} \Delta^{n_\beta}_k)$$

by the definition of products and coproducts of simplicial sets. Using the distributive bijection first for the left disjoint union then for the right, we get

$$(\coprod_{\alpha\in A} \Delta^{n_\alpha}_k) \times (\coprod_{\beta\in B} \Delta^{n_\beta}_k) = \coprod_{\alpha\in A} (\Delta^{n_\alpha}_k \times (\coprod_{\beta\in B} \Delta^{n_\beta}_k)) = \coprod_{\alpha\in A} \coprod_{\beta\in B} (\Delta^{n_\alpha}_k \times \Delta^{n_\beta}_k)$$

Now this is finite by 1.2.10 and 1.2.11, so (1.8) is a surjective map from a finite simplicial set, and $X \times Y$ is finite by 1.2.13. □

1.3 Geometric Realization

In this section we will define the geometric realization of a simplicial set. The definition we use is from [5], and it uses results from category theory concerning colimits and filtered categories. These results can be found in the appendix A.1.

We start off by defining a small and filtered category, from which we can take limits and colimits into sets by A.1.10.

Definition 1.3.1. *Let $I = [0, 1]$ be the unit interval, define $I_\subseteq$ as the category with finite subsets $F \subseteq I$ as objects and inclusions as morphisms.*

The category $I_\subseteq$ is small since $\operatorname{Ob} I_\subseteq$ is a subset of the powerset $P(I)$. Also there is at most one morphism between any two objects, so $\operatorname{Mor}(I_\subseteq)$ is a subset of the set $\operatorname{Ob}(I_\subseteq) \times \operatorname{Ob}(I_\subseteq)$. The category $I_\subseteq$ is also filtered. Part (b) in A.1.5 follows trivially from the fact that morphisms between objects in $I_\subseteq$ are unique. In the case of (a), for all finite $F, G \subseteq I$ the union $F \cup G$ is finite with $F \subseteq F \cup G$ and $G \subseteq F \cup G$.

Definition 1.3.2. *Define the functor*

$$\pi_0(I - (-)) : I_\subseteq \to \Delta^{op}_{big},$$

as follows. On objects F, let $\pi_0(I - F)$ be the set of connected components $\{F_0, \dots, F_n\}$ of $I - F$ with the total ordering $F_i \le F_j \iff x_i \le x_j$ for some $x_i \in F_i$ and some $x_j \in F_j$. On morphisms $\kappa : F \subseteq G$, let $\pi_0(I - \kappa) : \pi_0(I - G) \to \pi_0(I - F)$ be the surjective order-preserving function induced by the inclusion $I - G \hookrightarrow I - F$, i.e $\pi_0(I - \kappa)(G_j) = F_i$ whenever $G_j \subseteq F_i$ as subsets of I.

Note that since $\pi_0(I - \kappa)$ is surjective there is an order-preserving map $\alpha : \pi_0(I - F) \to \pi_0(I - G)$ such that $\pi_0(I - \kappa) \circ \alpha = \operatorname{Id}_{\pi_0(I-F)}$. So if X is a simplicial set extended to Δ_{big} as in 1.2.5, then $X(\alpha) \circ X(\pi_0(I - F)) = \operatorname{Id}_{X(\pi_0(I-F))}$. In particular we get that $X(\pi_0(I - F))$ is injective which is one of the conditions needed in A.1.6.

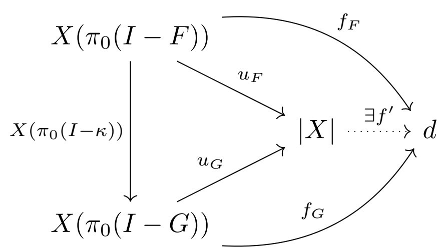
For every simplicial set we get a topological space which we call the geometric realization. We will first just look at the underlying set and later add the topology.

Definition 1.3.3. *Given a simplicial set X, then the* ***underlying set of the geometric realization*** *of X is*

$$|X| = \varinjlim_F X(\pi_0(I - F)). \tag{1.9}$$

Here $X : \Delta^{op} \to \texttt{Sets}$ is extended to Δ^{op}_{big} as in 1.2.5, $\pi_0(I - (-))$ is as in 1.3.2, and $|X|$ is the colimit of the functor $X \circ \pi_0(I - (-)) : I_{\subseteq} \to \texttt{Sets}$ which exists by A.1.10.

Specifically, $|X|$ is a set such that for all finite subsets $F \subseteq I$ there are functions $u_F : X(\pi_0(I - F)) \to |X|$ satisfying the cocone property $u_F = u_G \circ X(\pi_0(I - \kappa))$ for all morphisms $\kappa : F \subseteq G$. This cocone is universal in the sense that if d is a set with functions $f_F : X(\pi_0(I - F)) \to d$ such that $f_F = f_G \circ X(\pi_0(I - \kappa))$ there exists a unique function $f' : |X| \to d$ making the following diagram commute:



(1.10)

Our next goal is to give the geometric realization a topology. We will first define the topology for finite simplicial sets, and later extend this topology to the general case by looking at the finite simplicial subsets. The topology will come from a metric defined from the standard measure on the interval I, so we begin there.

Definition 1.3.4. *For any finite subset $F \subseteq I$ we define the measure μ_F on $\pi_0(I-F)$ induced by the standard length on I. For each element $F_i \in \pi_0(I-F)$ we have that $F_i = (x_i, x_{i+1})$ is some connected component of $I - F$ and so*

$$\mu_F(F_i) = x_{i+1} - x_i.$$

To get a metric from this we first recall from 1.3.2 that $\pi_0(I - F) \in \Delta_{big}$ is a finite non-empty totally ordered set. Any subset $A \subseteq \pi_0(I - F)$ with the induced order will also be in Δ_{big}, and the inclusion map $\alpha : A \hookrightarrow \pi_0(I - F)$ will be order-preserving. Thus for every simplicial set $X : \Delta^{op}_{big} \to \texttt{Sets}$ we get an induced map $X(\alpha) : X(\pi_0(I - F)) \to X(A)$.

Definition 1.3.5. *Let F be an object in $I_{\subseteq}$ and let X be a simplicial set. We define the* (X, F)***-metric*** *on the set $X(\pi_0(I - F))$, where for each $u, v \in X(\pi_0(I - F))$ we have the distance*

$$d_{X,F}(u, v) = \min\{\mu_F(\pi_0(I - F) - A) \,|\, \alpha : A \hookrightarrow \pi_0(I - F),\ X(\alpha)(u) = X(\alpha)(v)\} \tag{1.11}$$

We need to show that this does indeed define a metric. The definition is clearly symmetric so $d_{X,F}(u,v) = d_{X,F}(v,u)$. Since the length of every component of $I - F$ is positive we firstly have that $d_{X,F}(u,v) \geq 0$, and secondly that $\mu(\pi_0(I-F) - A) = 0$ if and only if $A = \pi_0(I-F)$, where α is the identity. Thus $d_{X,F}(u,v) = 0$ if and only if $u = X(\mathrm{Id})(u) = X(\mathrm{Id})(v) = v$.

Finally, to show the triangle inequality for $u, v, w \in X(\pi_0(I-F))$ let $A_1, A_2 \subseteq \pi_0(I-F)$ be the subsets minimizing the distance, such that $X(\alpha_1)(u) = X(\alpha_1)(v)$ and $X(\alpha_2)(v) = X(\alpha_2)(w)$, where α_i are the inclusion maps. Define $B = A_1 \cap A_2$. The order-preserving inclusion map $\beta : B \hookrightarrow \pi_0(I-F)$ can be written as the composition of the inclusions $\gamma_i : B \hookrightarrow A_i$ and $\alpha_i : A_i \hookrightarrow \pi_0(I-F)$ for both $i = 1, 2$. Now $X(\beta) = X(\gamma_i) \circ X(\alpha_i)$, so since $X(\alpha_1)(u) = X(\alpha_1)(v)$ and $X(\alpha_2)(v) = X(\alpha_2)(w)$ we get $X(\beta)(u) = X(\beta)(v) = X(\beta)(w)$. In particular

$$d_{X,F}(u,w) \leq \mu_F(\pi_0(I-F) - B). \tag{1.12}$$

By letting A^C be the complement $\pi_0(I-F) - A$ and using the facts that $A_1^C \cup A_2^C = (A_1 \cap A_2)^C$ and $\mu_F(A) \geq 0$ for all $A \subseteq \pi_0(I-F)$, we conclude

$$\begin{aligned} d_{X,F}(u,v) + d_{X,F}(v,w) &= \mu_F(A_1^C) + \mu_F(A_2^C) = \mu_F(A_1^C \cap A_2^C) + \mu_F(A_1^C \cup A_2^C) \\ &= \mu_F(A_1^C \cap A_2^C) + \mu_F(B^C) \geq \mu_F(B^C) \geq d_{X,F}(u,w). \end{aligned}$$

Where the last inequality comes from 1.12. So the triangle inequality holds, and $d_{X,F}$ defines a metric on $X(\pi_0(I-F))$.

Note that if $X = \Delta^n$ is the standard n-simplex, then $\Delta^n(\alpha)(u) = u \circ \alpha$, and the distance $d_{\Delta^n,F}(u,v)$ tells us the size of the subset of $\pi_0(I-F)$ where u and v disagree.

Next want to extend the (X,F)-metrics to a metric on $|X|$, but to do that we need to show that the distances behave nicely with the maps induced by the inclusions $\kappa : F \subseteq G$.

Lemma 1.3.6. *Let $\kappa : F \subseteq G$ be any morphism of objects in $I_{\subseteq}$. Let X be any simplicial set, and let $u, v \in X(\pi_0(I-F))$ be any elements. Then*

$$d_{X,F}(u,v) = d_{X,G}(X(\pi_0(I-\kappa))(u), X(\pi_0(I-\kappa))(v)). \tag{1.13}$$

Proof. For simplicity we write $X(\pi_0(I-F))(u) = u'$ for all $u \in X(\pi_0(I-F))$. Define $T^F_{uv} = \{A \subseteq \pi_0(I-F) \,|\, \alpha : A \hookrightarrow \pi_0(I-F),\, X(\alpha)(u) = X(\alpha)(v)\}$, so that the distance $d_{X,F}(u,v)$ is given by $\min\{\mu_F(\pi_0(I-F) - A) \,|\, A \in T^F_{uv}\}$.

We write $\pi_0(I-F) = \{F_1 \leq \cdots \leq F_n\}$, and since the map $\pi_0(I-\kappa)$ is surjective and order-preserving we can also write $\pi_0(I-G) = \{G_{1_1} \leq G_{1_2} \leq \cdots \leq G_{1_{s_1}} \leq G_{2_1} \leq \cdots \leq G_{n_{s_n}}\}$ such that $\pi_0(I-\kappa)(G_{i_j}) = F_i$, or in other words such that G_{i_j} is a subset of F_i as subsets of the interval.

$\underline{(\leq)}$: Let $B \in T^G_{u'v'}$ with inclusion $\beta : B \subseteq \pi_0(I-G)$, so we have $X(\beta)(u') = X(\beta)(v')$. Define the subset $\overline{B} = \{F_i \,|\, G_{i_j} \in B \text{ for some } j\} \subseteq \pi_0(I-F)$, which consists of all components of $I - F$ that includes an element of B. In particular, as subsets of I we have $B \subseteq \overline{B}$, and so $\mu_F(\pi_0(I-F)-\overline{B}) \leq \mu_G(\pi_0(I-G)-B)$. Therefore it is enough to show that $\overline{B} \in T^F_{uv}$.

Look at the order-preserving map $\phi : \overline{B} \to B$ given by $\phi(F_i) = \min\{G_{i_j} \in B\}$. Let $F_i \in \overline{B}$ and let $\overline{\beta} : \overline{B} \hookrightarrow \pi_0(I-F)$ be the map induced by the inclusion. Now $\pi_0(I-\kappa) \circ \beta \circ \phi(F_i) = \pi_0(I-\kappa)(G_{i_j})$ for some i_j, and since $G_{i_j} \subseteq F_i$ we get $\pi_0(I-\kappa)(G_{i_j}) = F_i = \overline{\beta}(F_i)$, and thus

$$\overline{\beta}(F_i) = \pi_0(I-\kappa) \circ \beta \circ \phi(F_i).$$

From this we get $X(\overline{\beta})(u) = X(\phi) \circ X(\beta) \circ X(\pi_0(I-\kappa))(u) = X(\phi) \circ X(\beta)(u')$, and similarly for v. Using the fact that B is in $T^G_{u'v'}$, we have $X(\beta)(u') = X(\beta)(v')$, and thus we get the equality $X(\overline{\beta})(u) = X(\overline{\beta})(v)$.

We have thus shown that $\overline{B}$ is in T^F_{uv} and $\mu_F(\pi_0(I-F)-\overline{B}) \leq \mu_F(\pi_0(I-G)-B)$. Since B was arbitrarily chosen, we conclude that $d_{X,F}(u,v) \leq d_{X,G}(u',v')$.

$\underline{(\geq)}$: Let $A \in T^F_{uv}$ and define $\widetilde{A} = \{G_{i_j} \in \pi_0(I-G) \,|\, F_i \in A\}$ consisting of all components of $I-G$ which is included in some element of A. As subsets of I, $\widetilde{A}$ is just A with some finite points in G taken away, so we get that $\mu_F(\pi_0(I-F)-A)$ and $\mu_G(\pi_0(I-G)-\widetilde{A})$ are the same.

Let $\psi : \widetilde{A} \to A$ be the order-preserving map $\psi(G_{i_j}) = F_i$, and let $\alpha : A \hookrightarrow \pi_0(I-F)$ and $\widetilde{\alpha} : \widetilde{A} \hookrightarrow \pi_0(I-G)$ be the maps induced by the inclusions. Now $\alpha \circ \psi(G_{i_j}) = F_i$ and $\pi_0(I-\kappa) \circ \widetilde{\alpha}(G_{i_j}) = F_i$, so we get a commutative diagram, which after taking $X(-)$ is

$$\begin{array}{ccc} X(\pi_0(I-F)) & \xrightarrow{X(\alpha)} & X(A) \\ \downarrow{\scriptstyle X(\pi_0(I-\kappa))} & & \downarrow{\scriptstyle X(\psi)} \\ X(\pi_0(I-G)) & \xrightarrow{X(\widetilde{\alpha})} & X(\widetilde{A}). \end{array}$$

Now $X(\widetilde{\alpha})(u') = X(\widetilde{\alpha}) \circ X(\pi_0(I-F))(u)$, which by the diagram is $X(\psi) \circ X(\alpha)(u)$. Similarly we get $X(\widetilde{\alpha})(v') = X(\psi) \circ X(\alpha)(v)$. Since $A \in T^F_{uv}$ we have $X(\alpha)(u) = X(\alpha)(v)$, and so $X(\widetilde{\alpha})(u') = X(\widetilde{\alpha})(v')$. In conclusion we have $\widetilde{A} \in T^G_{uv}$ with $\mu_F(\pi_0(I-F)-A) = \mu_G(\pi_0(I-G)-\widetilde{A})$, since A was arbitrary we have $d_{X,F}(u,v) \geq d_{X,G}(u',v')$. $\square$

We can finally define a metric on the underlying set of the geometric realization.

Definition 1.3.7. *Let X be a simplicial set, and $(|X|, \{u_f\})$ a colimit diagram of $X(\pi_0(I-(-)))$. The **Drinfeld-metric** d_X on $|X|$ is the metric*

$$d_X(x,y) = d_{X,F}(u_F^{-1}(x), u_F^{-1}(y)). \tag{1.14}$$

Recall that $I_{\subseteq}$ is small and filtered, and $X(\pi_0(I-\kappa))$ injective for all κ. We have by A.1.8 that for all $x, y \in |X|$ there is an F such that both x and y are in the image of u_F, and by A.1.6 the map u_F is injective so the preimages are uniquely defined. Finally, by 1.3.6 and the cocone property of $|X|$ we see that the definition is independent of the choice of F, so the metric is well-defined.

The properties of this metric and what it says about the simplicial set might be interesting in itself. However, to get a realization equivalent to what is commonly used (as shown by [6]), we need an extra step. We give the realization of finite simplicial complexes the metric topology, and define the topology in the general case by looking at the finite simplicial subsets.

Definition 1.3.8. *Let X be a finite simplicial set. The **geometric realization** of X is the topological space with underlying set $|X|$ and the topology given by the Drinfeld-metric.*

We will first check that this definition is functorial. Let `fsSets` be the full subcategory of finite simplicial sets.

Lemma 1.3.9. *$|-| :$ `fsSet` $\to$ `Top` defines a functor. It acts the same as the composition of $\pi_0(I-(-))$ defined in 1.3.2 with the colimit-functor defined in A.1.14 but with added topology.*

Proof. From A.1.14 we know it is a functor from `fsSet` to `Sets`. We just need to show the induced maps are continuous. In particular if $\eta : X \to Y$ is a morphism of finite simplicial sets, and $x, y \in |X|$, then it is enough to show $d_Y(|\eta|(x), |\eta|(y)) \leq d_X(x, y)$.

Let $u_F : X(\pi_0(I-F)) \to |X|$ be the maps associated to $|X|$ as a colimit, and similarly let v_F be associated to $|Y|$. Let F be such that $x, y \in \operatorname{Im} u_F$, and write $x' = u_F^{-1}(x)$ and $y' = u_F^{-1}(y)$. These exist and are unique by A.1.8 and A.1.6. Let $\alpha : A \subseteq \pi_0(I-F)$ be such that $X(\alpha)(x') = X(\alpha)(y')$. Since η is a morphism, and thus a natural transformation, we have

$$\begin{array}{ccc} X(\pi_0(I-F)) & \xrightarrow{X(\alpha)} & X(A) \\ \downarrow{\scriptstyle \eta_F} & & \downarrow{\scriptstyle \eta_A} \\ Y(\pi_0(I-F)) & \xrightarrow{Y(\alpha)} & Y(A), \end{array}$$

where we write $\eta_F := \eta_{\pi_0(I-F)}$. In particular we have $Y(\alpha)(\eta_F(x')) = Y(\alpha)(\eta_F(y'))$, and since A was arbitrary, the distance $d_{Y,F}(\eta_F(x'), \eta_F(y'))$ is less than or equal to $d_{X,F}(x', y') = d_X(x, y)$.

By the definition of maps induced on colimits (diagram (A.6)), we have that $|\eta| \circ u_F = v_F \circ \eta_F$, and since u_F and v_F are injective we have

$$v_F^{-1} \circ |\eta|(x) = \eta_F \circ u_F^{-1}(x) \tag{1.15}$$

whenever x is in the image of u_F. Straight from the definition of the Drinfeld-metric we have $d_Y(|\eta|(x), |\eta|(y)) = d_{Y,F}(v_F^{-1} \circ |\eta|(x), v_F^{-1} \circ |\eta|(y))$. So from (1.15) and the fact that $x, y \in \operatorname{Im} u_F$ by construction, we have that this distance equals $d_{Y,F}(\eta_F \circ u_F^{-1}(x), \eta_F \circ u_F^{-1}(y))$. Now using the fact that $x' = u_F^{-1}(x)$ and $y' = u_F^{-1}(y)$, we get our result that $d_Y(|\eta|(x), |\eta|(y)) = d_{Y,F}(\eta_F(x'), \eta_F(y')) \leq d_X(x, y)$. Thus $|\eta|$ is continuous and $|-|$ defines a functor. □

The definition of the colimit functor chooses a colimit diagram to represent *the* colimit, so we have a similar choice for the geometric realization. From A.1.15 we have that for any two geometric realizations $|-|$ and $\|-\|$, and for any morphism of simplicial sets $\eta : X \to Y$, we have isomorphisms h_X and h_Y, and a commuting diagram

$$\begin{array}{ccc} |X| & \xrightarrow{|\eta|} & |Y| \\ \downarrow{\scriptstyle h_X} & & \downarrow{\scriptstyle h_Y} \\ \|X\| & \xrightarrow{\|\eta\|} & \|Y\|. \end{array} \tag{1.16}$$

The isomorphisms and their inverses are given by the universal property, which by 1.3.9 are continuous, so they are homeomorphisms.

Before extending our definition to general simplicial sets, we will show that products are conserved in the geometric realization for finite ones.

Lemma 1.3.10. *Let X and Y be finite simplicial sets. The natural bijection $|X \times Y| \to |X| \times |Y|$ from A.1.12 is a homeomorphism.*

Proof. We first note that if X is finite, then by 1.2.13 we have a continuous surjective map $\amalg|\Delta^{n_j}| \to |X|$ from a finite disjoint union of compact spaces, so $|X|$ is compact as the continuous image of a compact space ([12] 26.5). In particular $X \times Y$ is finite by 1.2.16, and so $|X \times Y|$ is compact. The space $|X|$ is Hausdorff, as it gets its topology from a metric, and so the product $|X| \times |Y|$ is also Hausdorff ([12] 19.4). The bijection $|X \times Y| \to |X| \times |Y|$ is given by the universal property as in (A.5), giving the diagram

$$\begin{array}{ccccc} X(\pi_0(I-F)) & \xleftarrow{\pi_{X,F}} & (X \times Y)(\pi_0(I-F)) & \xrightarrow{u_F^{X \times Y}} & |X \times Y| \\ \downarrow{\scriptstyle u_F^X} & & \downarrow & & \downarrow \\ |X| & \xleftarrow[\pi_X]{} & |X| \times |Y| & = & |X| \times |Y|. \end{array} \tag{1.17}$$

The bijection is given by the universal property induced from the projection maps, and by the functor properties of geometric realization (1.3.9) this is a continuous map. We thus have a continuous bijection from a compact space to a Hausdorff space, so it is a homeomorphism ([12] 26.6). □

Finally in this section we will extend the definition of geometric realization to all simplicial sets by the geometric realization of their finite simplicial subsets.

Starting with a simplicial set X we can look at the finite simplicial subsets $S \subseteq X$. These form a category $\mathrm{Fin}\, X_{\subseteq}$ where the morphisms are inclusions. For two finite nested subsets $T \subseteq S \subseteq X$, the inclusions defines continuous maps between the geometric realizations $|T| \to |S|$. So the geometric realization defines a functor $\mathrm{Fin}\, X_{\subseteq} \to \mathtt{Top}$, which we will also call $| - |$, sending finite subsets to their realization and inclusions to the continuous maps between them.

Definition 1.3.11. *Let X be any simplicial set. The* ***geometric realization*** *$|X|$ of X is given by*

$$|X| = \varinjlim_{S \in Fin\, X_{\subseteq}} |S|$$

This is a colimit in the category $\mathtt{Top}$, so it exists by A.1.11. Since taking colimits is a functor by A.1.14, we have that $| - | : \mathtt{sSet} \to \mathtt{Top}$ is a functor as a composition of functors.

When looking at products we do get a small problem. If taking the products in the category of topological spaces, the geometric realization will not in general commute as it is not a Cartesian closed category. The fix is to look at a nice subcategory of $\mathtt{Top}$, namely the category $\mathtt{CGHaus}$ of compactly generated Hausdorff spaces. The main property we need is for colimits to be distributive on products, $\varinjlim_{\alpha} (X_\alpha \times Y) \cong (\varinjlim_{\alpha} X_\alpha) \times Y$. Assume this is the case. Now if S and T ranges over the finite simplicial subsets of X and Y respectively, then $|X| \times |Y| = \left(\varinjlim_S |S|\right) \times \left(\varinjlim_T |T|\right) \cong \varinjlim_S \left(|S| \times \varinjlim_T |T|\right) \cong \varinjlim_S \varinjlim_T (|S| \times |T|)$. Now we can combine the colimits and use the homeomorphism for finite subsets to conclude that $|X| \times |Y| \cong \varinjlim_{S \times T} |S \times T| \cong \varinjlim_{R \subseteq X \times Y} |R| = |X \times Y|$.

Grayson ([10]. 2.7) goes into details around this, both showing $|X| \in \mathtt{CGHaus}$ for all simplicial sets X([10]. 2.7.13), and that $|X \times Y| \cong |X| \times |Y|$ when taking the product in $\mathtt{CGHaus}$ ([10]. 2.7.18). He uses a different definition of geometric realization, but the two are shown to be equivalent by Dundas ([6] p.99).

1.4 Nerves and Classifying Spaces

We will in this section define simplicial sets from small categories, and look at how this construction acts with the geometric realization. This construction will be an important link in going from simplicial complexes to simplicial sets (Section 1.5), and we will directly use it in our proof of Dowker's theorem (1.6.4).

Definition 1.4.1. *Let $\mathcal{C}$ be a small category, and define the* ***nerve of the category*** *$\mathcal{C}$ to be the simplicial set $N_s\mathcal{C}$ where $N_s\mathcal{C}_n = Func([n], \mathcal{C})$, and where given a functor $\alpha : [m] \to [n]$ we get the function $N_s\mathcal{C}(\alpha) : N_s\mathcal{C}_n \to N_s\mathcal{C}_m$ sending F to $F \circ \alpha$.*

As a special case we have that the standard n-simplex is the nerve of $[n]$, $\Delta^n = N_s[n]$, where $[n]$ is defined in 1.2.1.

We begin by showing that the nerve defines a functor that preserves products.

Lemma 1.4.2. *N_s- : **Cat** $\to$ **sSet** defines a functor, where if $H : \mathcal{C} \to \mathcal{D}$ is a functor then $(N_sH)_n : N_s\mathcal{C}_n \to N_s\mathcal{D}_n$ sends functors $F : [n] \to \mathcal{C}$ to $H \circ F : [n] \to \mathcal{D}$.*

Proof. The map $H \circ F$ is a composition of functors and thus a functor itself, so we just need to show that N_sH defines a natural transformation. Let $\alpha : [m] \to [n]$ be any order-preserving map, and look at the diagram

$$\begin{array}{ccc} N_s\mathcal{C}_n & \xrightarrow{N_s\mathcal{C}(\alpha)} & N_s\mathcal{C}_m \\ \downarrow{\scriptstyle (N_sH)_n} & & \downarrow{\scriptstyle (N_sH)_m} \\ N_s\mathcal{D}_n & \xrightarrow{N_s\mathcal{D}(\alpha)} & N_s\mathcal{D}_m. \end{array}$$

Let $F \in N_s\mathcal{C}_n$. In one direction of the diagram we have $(N_sH)_m \circ N_s\mathcal{C}(\alpha)(F) = (N_sH)_m(F \circ \alpha) = H \circ F \circ \alpha$, and the other we get $N_s\mathcal{D}(\alpha) \circ (N_sH)_n(F) = N_s\mathcal{D}(\alpha)(H \circ F) = H \circ F \circ \alpha$. □

Lemma 1.4.3. *$N_s(\mathcal{C}_1 \times \mathcal{C}_2)$ is isomorphic to $N_s\mathcal{C}_1 \times N_s\mathcal{C}_2$*

Proof. For any degree n we have $N_s(\mathcal{C}_1 \times \mathcal{C}_2)_n = \mathrm{Func}([n], \mathcal{C}_1 \times \mathcal{C}_2)$. Let $f : [n] \to \mathcal{C}_1 \times \mathcal{C}_2$ be any such functor, then f is uniquely determined by its composition with the projection maps $f = (\pi_1(f), \pi_2(f))$. Conversely, any two functors $g_i : [n] \to \mathcal{C}_i$, for $i = 1, 2$, uniquely determines a functor $g = (g_1, g_2) : [n] \to \mathcal{C}_1 \times \mathcal{C}_2$ by the universal property. Thus we have a bijection sending f in $\mathrm{Func}([n], \mathcal{C}_1 \times \mathcal{C}_2)$ to $(\pi_1(f), \pi_2(f))$ in $\mathrm{Func}([n], \mathcal{C}_1) \times \mathrm{Func}([n], \mathcal{C}_2)$. We see that this also agrees with maps $\alpha : [m] \to [n]$ since $\alpha^*(h) = h \circ \alpha = (\pi_1(h) \circ \alpha, \pi_2(h) \circ \alpha) = (\alpha^*, \alpha^*) \circ (\pi_1(h), \pi_2(h))$. □

We now combine the notion of the nerve, with the geometric realization from last section.

Definition 1.4.4. *The geometric realization of the nerve $|N_s\mathcal{C}|$ for some small category $\mathcal{C}$ is called the **classifying space** of the category. As a set this is*

$$|N_s\mathcal{C}| = \varinjlim_F \mathit{Func}(\pi_0(I - F), \mathcal{C}). \tag{1.18}$$

Corollary 1.4.5. *$|N_s - |$: **Cat** $\to$ **Top** defines a functor.*

Proof. This is the composition of the two functors $|-|$ and N_s- and is thus a functor itself. □

Corollary 1.4.6. *$|N_s(\mathcal{C} \times \mathcal{D})|$ is homeomorphic to $|N_s\mathcal{C}| \times |N_s\mathcal{D}|$, where the product is taken in **CGHaus**.*

Proof. This follows from 1.3.10 with our discussion below 1.3.11, and 1.4.3. □

The next thing we want to show is that the classifying space of a category is homeomorphic to the classifying space of its opposite category. First we introduce a functor from the category $I_\subseteq$ to itself, which we will use to connect the two spaces.

Definition 1.4.7. *Let $\gamma : I_\subseteq \to I_\subseteq$ be the functor sending $F = \{x_0 < \cdots < x_n\}$ to $\gamma(F) = \{1 - x_n < \cdots < 1 - x_0\}$.*

Clearly $F \subseteq G$ implies $\gamma(F) \subseteq \gamma(G)$, so γ is a functor. Also since $1-(1-x_i) = x_i$ we have

$$\gamma^2(F) = F. \tag{1.19}$$

In particular $F \subseteq G$ if and only if $\gamma(F) \subseteq \gamma(G)$, so we have a 1-1 correspondence between $\kappa : F \subseteq G$ and $\gamma(\kappa) : \gamma(F) \subseteq \gamma(G)$.

Proposition 1.4.8. *Let $H : \mathcal{C} \to \mathcal{D}$ be a functor between small categories. There are homeomorphisms $g_\mathcal{C}$ and $g_\mathcal{D}$ such that the following diagram commutes:*

$$\begin{array}{ccc} |N_s\mathcal{C}| & \xrightarrow{g_\mathcal{C}} & |N_s\mathcal{C}^{op}| \\ \downarrow{\scriptstyle |N_sH|} & & \downarrow{\scriptstyle |N_sH^{op}|} \\ |N_s\mathcal{D}| & \xrightarrow{g_\mathcal{D}} & |N_s\mathcal{D}^{op}| \end{array} \tag{1.20}$$

Proof. We will look at four different colimits:

$$\begin{aligned} |N_s\mathcal{C}| &= \varinjlim_F \mathrm{Func}(\pi_0(I-F), \mathcal{C}) \\ \|N_s\mathcal{C}\| &= \varinjlim_F \mathrm{Func}(\pi_0(I-F), \gamma(\mathcal{C})) \\ \langle N_s\mathcal{C}^{op}\rangle &= \varinjlim_F \mathrm{Func}(\pi_0(I-F)^{op}, \mathcal{C}) \\ |N_s\mathcal{C}^{op}| &= \varinjlim_F \mathrm{Func}(\pi_0(I-F), \mathcal{C}^{op}) \end{aligned}$$

We will find bijections between these by finding bijections of the sets before taking the colimit.

Since the opposite of a functor acts the same as its dual counterpart on objects and morphisms, the functor $(-)^{op} : \mathrm{Func}(\pi_0(I-F)^{op}, \mathcal{C}) \to \mathrm{Func}(\pi_0(I-F), \mathcal{C}^{op})$ defines a bijection. The collection of these bijections for all F will define a natural isomorphism from $\mathrm{Func}(\pi_0(I-(-))^{op}, \mathcal{C})$ to $\mathrm{Func}(\pi_0(I-(-)), \mathcal{C}^{op})$, as in the following diagram

$$\begin{array}{ccc} \mathrm{Func}(\pi_0(I-F)^{op}, \mathcal{C}) & \xrightarrow{(-)^{op}} & \mathrm{Func}(\pi_0(I-F), \mathcal{C}^{op}) \\ \downarrow{\scriptstyle (\pi_0(I-\kappa)^{op})^*} & & \downarrow{\scriptstyle \pi_0(I-\kappa)^*} \\ \mathrm{Func}(\pi_0(I-G)^{op}, \mathcal{C}) & \xrightarrow{(-)^{op}} & \mathrm{Func}(\pi_0(I-G), \mathcal{C}^{op}). \end{array} \tag{1.21}$$

From (1.19) we have the commuting diagram

$$\begin{array}{ccc} \mathrm{Func}(\pi_0(I-F), \mathcal{C}) & = & \mathrm{Func}(\pi_0(I-\gamma(\gamma(F))), \mathcal{C}) \\ \downarrow{\scriptstyle \pi_0(I-\kappa)^*} & & \downarrow{\scriptstyle \pi_0(I-\gamma(\gamma(\kappa)))^*} \\ \mathrm{Func}(\pi_0(I-G), \mathcal{C}) & = & \mathrm{Func}(\pi_0(I-\gamma(\gamma(G))), \mathcal{C}). \end{array} \tag{1.22}$$

Now (1.21) in some way connects $\langle N_s\mathcal{C}^{op}\rangle$ with $|N_s\mathcal{C}^{op}|$, and (1.22) connects $|N_s\mathcal{C}|$ with $\|N_s\mathcal{C}\|$, so next we want to find a connection between $\|N_s\mathcal{C}\|$ and $\langle N_s\mathcal{C}^{op}\rangle$. Specifically we want to find a collection of isomorphisms $\{\eta_F : \pi_0(I - \gamma(F)) \to \pi_0(I-F)^{op}\}$ which is natural in the sense that for all $\kappa : F \subseteq G$ we have

$$\begin{array}{ccc} \pi_0(I-\gamma(G)) & \xrightarrow{\eta_G} & \pi_0(I-G)^{op} \\ \downarrow{\scriptstyle \pi_0(I-\gamma(\kappa))} & & \downarrow{\scriptstyle \pi_0(I-\kappa)^{op}} \\ \pi_0(I-\gamma(F)) & \xrightarrow{\eta_F} & \pi_0(I-F)^{op}. \end{array} \tag{1.23}$$

Taking $\mathrm{Func}(-,\mathcal{C})$ on this diagram we will get a natural isomorphism between $\mathrm{Func}(\pi_0(I-(-))^{op},\mathcal{C})$ and $\mathrm{Func}(\pi_0(I-\gamma(-)),\mathcal{C})$.

Write $\pi_0(I-F) = \{F_0 < \cdots < F_n\}$, where $F_i \subseteq I - F$ are the connected components. Then $\pi_0(I-\gamma(F)) = \{1 - F_n < \cdots < 1 - F_0\}$ where $1 - F_i = \{1 - x \,|\, x \in F_i\}$. We also have $\pi_0(I-F)^{op} = \{F_n < \cdots < F_0\}$. We have a clear isomorphisms (in Δ_{big}) by the map $\eta_F : \pi_0(I-\gamma(F)) \to \pi_0(I-F)^{op}$ sending $1 - F_i$ to $\eta_F(1-F_i) = F_i$. We have $G_j \subseteq F_i$ if and only if $1 - G_j \subseteq 1 - F_i$, so $\pi_0(I-\gamma(\kappa))(1-G_j) = 1 - F_i$ if and only if $\pi_0(I-\kappa)(G_j) = F_i$. Thus (1.23) commutes.

By combining the diagrams (1.21)-(1.23), and writing in the colimit cocones, we get a commuting diagram

$$\begin{array}{ccccccc} & & \mathrm{Func}(\pi_0(I-F),\mathcal{C}) & \xrightarrow{K_F} & \mathrm{Func}(\pi_0(I-\gamma(F)),\mathcal{C}^{op}) & & \\ & \swarrow{\scriptstyle u_F} & & & & \searrow{\scriptstyle v_{\gamma(F)}} & \\ |N_s\mathcal{C}| & & \downarrow{\scriptstyle (\pi_0(I-\kappa)^{op})^*} & & \downarrow{\scriptstyle \pi_0(I-\gamma(\kappa))^*} & & |N_s\mathcal{C}^{op}| \\ & \nwarrow{\scriptstyle u_G} & & & & \nearrow{\scriptstyle v_{\gamma(G)}} & \\ & & \mathrm{Func}(\pi_0(I-G),\mathcal{C}) & \xrightarrow{K_G} & \mathrm{Func}(\pi_0(I-\gamma(G)),\mathcal{C}^{op}) & & \end{array}$$

for every $\kappa : F \subseteq G$, where $K_F = (-)^{op} \circ (\eta^{-1}_{\gamma(F)})^*$ are all bijections. From here we see that $(|N_s\mathcal{C}^{op}|, \{v_{\gamma(F)} \circ K_F\})$ is a cocone of $\mathrm{Func}(\pi_0(I-(-)),\mathcal{C})$, and using (1.19) we also have that $(|N_s\mathcal{C}|, \{u_{\gamma(F)} \circ K^{-1}_{\gamma(F)}\})$ is a cocone of $\mathrm{Func}(\pi_0(I-(-)),\mathcal{C}^{op})$. By the universal property of colimits we get unique induced maps $g_{\mathcal{C}} : |N_s\mathcal{C}| \to |N_s\mathcal{C}^{op}|$ and $g_{\mathcal{C}^{op}} : |N_s\mathcal{C}^{op}| \to |N_s\mathcal{C}|$ such that $g_{\mathcal{C}} \circ u_F = v_{\gamma(F)} \circ K_F$ and $g_{\mathcal{C}^{op}} \circ v_F = u_{\gamma(F)} \circ K^{-1}_{\gamma(F)}$. We now get

$$g_{\mathcal{C}^{op}} \circ g_{\mathcal{C}} \circ u_F = g_{\mathcal{C}^{op}} \circ v_{\gamma(F)} \circ K_F = u_F \circ K_F^{-1} \circ K_F = u_F$$

for all finite $F \subseteq I$, where we have used (1.19) in the second equality. By A.1.7 every $x \in |N_s\mathcal{C}|$ is in the image of some u_F, so $g_{\mathcal{C}^{op}} \circ g_{\mathcal{C}} = \mathrm{Id}_{|N_s\mathcal{C}|}$, and similarly $g_{\mathcal{C}} \circ g_{\mathcal{C}^{op}} = \mathrm{Id}_{|N_s\mathcal{C}^{op}|}$. So the unique map $g_{\mathcal{C}} : |N_s\mathcal{C}| \to |N_s\mathcal{C}^{op}|$ is a bijection, and $|N_s\mathcal{C}^{op}|$ and $|N_s\mathcal{C}|$ are colimits of the same functor for every $\mathcal{C}$, and so by A.1.15 the diagram (1.20) commutes.

To show $g_\mathcal{C}$ is a homeomorphism we need to show that K_F is a homeomorphism in the $(N_s\mathcal{C}, F)$-metric. If $H : \pi_0(I - F) \to \mathcal{C}$ is a functor, and $1 - F_i \in \pi_0(I - \gamma(F))$, then $K_F(H)(1 - F_i) = (H \circ \eta^{-1}_{\gamma(F)})^{op}(1 - F_i) = H(F_i)$. Calculating the distance between the images of two functors, we get

$$\begin{aligned} d_{N_s\mathcal{C}^{op},\gamma(F)}(K_F(H), K_F(H')) &= \min\{\mu_{\gamma(F)}(\pi_0(I - \gamma(F)) - A) \,|\, \alpha^*(K_F(H)) = \alpha^*(K_F(H'))\} \\ &= \mu_{\gamma(F)}\{1 - F_i \in \pi_0(I - \gamma(F)) \,|\, K_F(H)(1 - F_i) \neq K_F(H')(1 - F_i)\} \\ &= \mu_F\{F_i \in \pi_0(I - F) \,|\, H(F_i) \neq H'(F_i) \\ &= d_{N_s\mathcal{C},F}(H, H'). \end{aligned}$$

This tells us one way that K_F is continuous, and the other way that K_F^{-1} is continuous. Therefore the map $g_\mathcal{C}$ induced on the colimits is a homeomorphism. □

In the rest of this section we will show that if we have a natural transformation $H_0 \to H_1$ between functors, then their nerves $|N_sH_0|$ and $|N_sH_1|$ are homotopic.

Lemma 1.4.9. *There is a 1-1 correspondence between functors $H : [1] \times \mathcal{C} \to \mathcal{D}$ and natural transformations $H_0 \to H_1$ where $H_0, H_1 : \mathcal{C} \to \mathcal{D}$ are functors.*

Proof.
<u>$(\to)$</u>: Starting with a functor $H : [1] \times \mathcal{C} \to \mathcal{D}$, define $H_i : \mathcal{C} \to \mathcal{D}$ for $i = 0, 1$ such that

$$H_i(c) = H(i, c) \text{ for } c \in \mathrm{Ob}\,\mathcal{C}, \quad H_i(f) = H(\mathrm{Id}_i, f) \text{ for } f \in \mathrm{Mor}\,\mathcal{C}.$$

To show that H_0 and H_1 are functors, let $f : c \to c'$ and $f' : c' \to c''$ be morphisms in $\mathcal{C}$. Then on compositions the map is $H_i(f' \circ f) = H(\mathrm{Id}_i \circ \mathrm{Id}_i, f' \circ f) = H((\mathrm{Id}_i, f') \circ (\mathrm{Id}_i, f)) = H(\mathrm{Id}_i, f') \circ H(\mathrm{Id}_i, f) = H_i(f') \circ H_i(f)$, and on the identity we get $H_i(\mathrm{Id}_c) = H(\mathrm{Id}_i, \mathrm{Id}_c) = H(\mathrm{Id}_{(i,c)}) = \mathrm{Id}_{H(i,c)} = \mathrm{Id}_{H_i(c)}$.

Next let $\tau_c : H_0(c) \to H_1(c)$ be the morphism $H(\leq, \mathrm{Id}_c) : H(0, c) \to H(1, c)$ in $\mathcal{D}$, where $\leq : 0 \to 1$ is the only morphism, and let $f : c \to c'$ be a morphism in $\mathcal{C}$. Then $(\leq, \mathrm{Id}_{c'}) \circ (\mathrm{Id}_0, f) = (\leq, f) = (\mathrm{Id}_1, f) \circ (\leq, \mathrm{Id}_c)$, so by applying the functor H we get $\tau_{c'} \circ H_0(f) = H_1(f) \circ \tau_c$. Thus $\{\tau_c\}$ describes a natural transformation $\tau : H_0 \to H_1$.

$$\begin{array}{ccc} (0,c) & \xrightarrow{(\mathrm{Id}_0,f)} & (0,c') \\ {\scriptstyle(\leq,\mathrm{Id}_c)}\big\downarrow & & \big\downarrow{\scriptstyle(\leq,\mathrm{Id}_{c'})} \\ (1,c) & \xrightarrow{(\mathrm{Id}_1,f)} & (1,c') \end{array} \quad \xrightarrow{H(-)} \quad \begin{array}{ccc} H_0(c) & \xrightarrow{H_0(f)} & H_0(c') \\ {\scriptstyle\tau_c}\big\downarrow & & \big\downarrow{\scriptstyle\tau_{c'}} \\ H_1(c) & \xrightarrow{H_1(f)} & H_1(c') \end{array}$$

($\leftarrow$): Conversely let $H_0, H_1 : \mathcal{C} \to \mathcal{D}$ be functors and $H_0 \xrightarrow{\tau} H_1$ a natural transformation. For any object $(i,c) \in [1] \times \mathcal{C}$ define $H(i,c) = H_i(c)$, and for any morphism $(\leq, f) : (i,c) \to (j,c')$ we define H by

$$H(\leq, f) = \begin{cases} H_i(f) & \text{if } i = j \\ \tau_{c'} \circ H_0(f) = H_1(f) \circ \tau_c & \text{if } i < j \end{cases}$$

On identities we have $H(\mathrm{Id}_{(i,c)}) = H_i(\mathrm{Id}_c) = \mathrm{Id}_{H_i(c)} = \mathrm{Id}_{H(i,c)}$. Next let $(\leq, f) : (i,c) \to (j,c')$ and $(\leq, f') : (j,c') \to (k,c'')$ be morphisms in $[1] \times \mathcal{C}$, so $0 \leq i \leq j \leq k \leq 1$ and $c, c', c'' \in \mathrm{Ob}\,\mathcal{C}$. In general we have $H((\leq, f') \circ (\leq, f)) = H(\leq, f' \circ f)$, and if $i = k$ we just get that this is equal to $H_i(f' \circ f)$ and we use the fact that H_i is a functor to show the rest. If $i = j < k$ then $H(\leq, f' \circ f) = H_1(f' \circ f) \circ \tau_c = H_1(f') \circ H_1(f) \circ \tau_c = H_1(f') \circ \tau_{c'} \circ H_0(f) = H(\leq, f') \circ H(\leq, f)$. Similarly we can show the same for $i < j = k$, and so H is indeed a functor.

Note in particular that $H(\leq, \mathrm{Id}_c) = H_1(\mathrm{Id}_c) \circ \tau_c = \tau_c$. Using the operation we looked at first ($\rightarrow$) on the obtained H, we again end up with the two functors H_0, H_1 and the natural transformation τ. Similarly using both operations on any functor $H : [1] \times \mathcal{C} \to \mathcal{D}$, we will end up with the same functor H. Thus we have described a 1-1 relation. □

Finally we will show that functors $H : [1] \times \mathcal{C} \to \mathcal{D}$ give rise to some homotopies. By the discussion below 1.3.11 about products of general simplicial sets, we think of these homotopies in `CGHaus` if $|N_s\mathcal{C}|$ is not finite. If it is finite, we can by 1.3.9 think of them as homotopies in `Top` as usual.

Lemma 1.4.10. *A functor $H : [1] \times \mathcal{C} \to \mathcal{D}$ gives a homotopy between $|N_s H_1|$ and $|N_s H_0|$ where $H_i : \mathcal{C} \to \mathcal{D}$ are given by $H_i(c) = H(i,c)$ for $i = 0, 1$.*

Proof. From 1.4.9 we know H_0 and H_1 are functors, so using the functor property 1.4.5 of $|N_s -|$ we have that $|N_s H_0|$ and $|N_s H_1|$ are continuous maps from $|N_s\mathcal{C}|$ to $|N_s\mathcal{D}|$. As before we will write $N_s[1] = \Delta^1$.

Let $c \in N_s\mathcal{C}(\pi_0(I - F))$, and let $Q \in \Delta^1(\pi_0(I - \emptyset))$ be one of the two elements, 0 or 1, which we talked about below A.2.1 and below A.2.4. We can think of Q as an element of $\Delta^1(\pi_0(I - F))$ by taking the map induced by the inclusion $\emptyset \subseteq F$. Now $(Q, c) \in (\Delta^1 \times N_s\mathcal{C})(\pi_0(I - F))$, and so $u_F^{\Delta^1 \times N_s\mathcal{C}}(Q, c) \in |\Delta^1 \times N_s\mathcal{C}|$. Looking at the homeomorphism $|\Delta^1 \times N_s\mathcal{C}| \to |\Delta^1| \times |N_s\mathcal{C}|$ from 1.4.5, which is defined as in the diagram (1.17) with $X = \Delta^1$ and $Y = N_s\mathcal{C}$, we get that

$$u_F^{\Delta^1 \times N_s\mathcal{C}}(Q, c) = (u_F^{\Delta^1}(Q), u_F^{N_s\mathcal{C}}(c)).$$

We discussed below A.2.4 that $u_F^{\Delta^1}(0) = y_{[0]} = 1$ and $u_F^{\Delta^1}(1) = y_{[1]} = 0$ in $|\Delta^1|_{\mathbb{R}} = I$. Combining 1.4.5 and A.2.4 we get a homeomorphism $\Phi : |\Delta^1 \times N_s\mathcal{C}| \to |\Delta^1|_{\mathbb{R}} \times |N_s\mathcal{C}| = I \times |N_s\mathcal{C}|$, which sends $u_F(0,c)$ to $(1, u_F^{N_s\mathcal{C}}(c))$, and similarly $u_F(1,c)$ to $(0, u_F^{N_s\mathcal{C}}(c))$.

Define the map $H' : I \times |N_s\mathcal{C}| \to |N_s\mathcal{D}|$ by $H' = |N_sH| \circ \Phi^{-1}$. This is continuous and it sends $(1, u_F^{N_s\mathcal{C}}(c))$ to $|N_sH|(u_F^{\Delta^1 \times N_s\mathcal{C}}(0,c)) = u_F^{N_s\mathcal{D}}(H(0,c))$. By the definition of H_0 this is the same as $u_F^{N_s\mathcal{D}}(H_0(c)) = |N_sH_0|(u_F^{N_s\mathcal{C}}(c))$. By A.1.7, every element in $|N_s\mathcal{C}|$ is in the image of some $u_F^{N_s\mathcal{C}}$, so we conclude that $H'(1,x) = |N_sH_0|(x)$ for all $x \in |N_s\mathcal{C}|$. Similarly we can show that $H'(0,x) = |N_sH_1|(x)$, and thus H' is a homotopy between $|N_sH_1|$ and $|N_sH_0|$. □

1.5 Simplicial Sets from Simplicial Complexes

We will now look at ways of turning simplicial complexes into simplicial sets. This is an important step in proving Dowker's theorem, as it is a theorem about simplicial complexes, using simplicial sets. We will define geometric realization of simplicial complexes, and compare them with their counterpart in simplicial sets. We will also show a connection between the barycentric subdivision and the nerve of a simplicial complex.

The first way we will construct a simplicial set from a simplicial complex requires an ordering on the vertices. The ordering will not matter in the end, as they will all give the same geometric realization. We begin by defining the category of ordered simplicial complexes.

Definition 1.5.1. *The **category of ordered simplicial complexes**, denoted **ordCpx**, is the category whose objects are simplicial complexes $(K, V_<)$ where $V_<$ has a total order, and whose morphisms $f : (K, V_<) \to (L, W_<)$ are (non-strictly) order-preserving simplicial maps.*

If we are given a simplicial map $f : (K, V) \to (L, W_<)$ where $W_<$ has a total order, we can define a partial order on V by saying $v < v'$ whenever $f(v) < f(v')$. Any refinement of this partial order into a total order will make f order-preserving.

Next we will see that the barycentric subdivision of an ordered simplicial complex can be given a total order which makes the least vertex map order preserving.

Definition 1.5.2. *Let $(K, V_<)$ be an ordered simplicial complex, and let $\phi : Sd\,K \to K$ the least vertex map. The **lexicographic order** of the set K with respect to $V_<$ is the total order given by*

$$\sigma < \tau \iff \min(\sigma \cup \tau - \tau) < \min(\sigma \cup \tau - \sigma),$$

where $\min \emptyset = \infty$*.*

This order will look for the smallest vertex that is not in both simplices, and define the simplex that contains it to be the smallest.

We first note that if $\sigma \subsetneq \tau$ then $\min(\sigma \cup \tau - \tau) = \min \emptyset = \infty$, and $\tau < \sigma$. Thus the lexicographic order is a refinement of the partial order given by the opposite of inclusions. Secondly, if σ and τ is such that $\phi(\tau) < \phi(\sigma)$, then $\min(\sigma \cup \tau) = \phi(\tau)$ and $\phi(\tau)$ is not in σ. So we have that $\phi(\tau) = \min(\sigma \cup \tau - \sigma) < \min(\sigma \cup \tau - \tau)$, and therefore $\tau < \sigma$. By looking at the contrapositive result we get in particular that $\sigma < \tau$ implies $\phi(\sigma) \leq \phi(\tau)$, and we conclude that the lexicographic order makes the least vertex map order-preserving.

Lemma 1.5.3. *Let $f : (K, V) \to (L, W)$ be a simplicial map injective on vertex sets, and give W any total order. The following diagram commutes and all arrows are order-preserving*

$$\begin{array}{ccc} (Sd\,K, K) & \xrightarrow{\phi_K} & (K, V) \\ \Big\downarrow{\scriptstyle Sdf} & & \Big\downarrow{\scriptstyle f} \\ (Sd\,L, L) & \xrightarrow{\phi_L} & (L, W), \end{array} \tag{1.24}$$

where V has any total order making f order-preserving, and K and L have the lexicographic order with respect to V and W respectively.

Proof. We know by construction that the least vertex maps ϕ_K and ϕ_L, and the map f are all order-preserving. What's left to show is that the diagram commutes, and that $\mathrm{Sd}f$ is order-preserving. Before starting with the second point, we recall that for simplices $\sigma' = [\sigma_0 \subseteq \cdots \subseteq \sigma_n]$ in $\mathrm{Sd}\,K$, the map $\mathrm{Sd}f$ is defined by $\mathrm{Sd}f(\sigma') = [f(\sigma_0) \subseteq \cdots \subseteq f(\sigma_n)]$. So on vertices (simplices of K) we have $\mathrm{Sd}f(\sigma) = f(\sigma)$.

Now let $\sigma < \tau$ in the lexicographic order of K. Since the lexicographic order is a refinement of the opposite of inclusions, we know σ is not included in τ, and we don't need to look at that case.

If $\tau \subsetneq \sigma$, then since f is injective on vertex sets we have $f(\tau) \subsetneq f(\sigma)$, and thus $f(\sigma) < f(\tau)$.

The last case is when $\tau \not\subseteq \sigma$ and $\sigma \not\subseteq \tau$. Now if $s = \min(\sigma \cup \tau - \tau)$, then in particular s is in σ but not in τ. Using the fact that f is injective on vertices, we have that $f(s) \in f(\sigma) \cup f(\tau) - f(\tau)$. Next let $t = \min(f(\sigma) \cup f(\tau) - f(\sigma))$, so t is in $f(\tau)$ but not in $f(\sigma)$. Again by injectivity we have $f^{-1}(t) \in \tau$ and $f^{-1}(t) \notin \sigma$, and thus $f^{-1}(t) \in \sigma \cup \tau - \sigma$. By assumption $\sigma < \tau$ we have $s < \min(\sigma \cap \tau - \sigma) \leq f^{-1}(t)$. Finally, since f is order-preserving we get $\min(f(\sigma) \cup f(\tau) - f(\tau)) \leq f(s) < t$, and so $f(\sigma) < f(\tau)$. Thus $\mathrm{Sd}f$ is order-preserving.

To show that the diagram commute, we look at where a vertex σ in $\mathrm{Sd}K$ is sent. Calculating directly we get $f \circ \phi_K(\sigma) = f(\min(\sigma))$, and $\phi_L \circ \mathrm{Sd}f(\sigma) = \min(f(\sigma))$. Since f is defined on vertices and is order-preserving, both of these are the same, and the diagram commutes. □

We see from this that the category of ordered simplicial sets are quite general in the sense that we only need to choose one ordering to get all the machinery of barycentric subdivisions directly. We can now define a functor from ordered simplicial complexes to simplicial sets.

Definition 1.5.4. *Define the functor $T : \texttt{ordCpx} \to \texttt{sSet}$:*

For objects $(K, V_<)$, let $T(K)_n = \{\alpha : [n] \to V_< \mid Im\,\alpha \in K,\ \alpha$ order-preserving$\}$ where functors $\beta : [m] \to [n]$ gives functions $T(K)(\beta) : T(K)_n \to T(K)_m$ by $T(K)(\beta)(\alpha) = \alpha \circ \beta$.

For morphisms $f : K \to L$ let $T(f)_n : T(K)_n \to T(L)_n$ the map $T(f)_n(\alpha) = f \circ \alpha$.

Before we continue, we need to check that everything in this definition is well-defined, and that it indeed defines a functor. If $\beta : [m] \to [n]$ is a functor and $\alpha \in T(K)_n$, then $T(K)(\beta)(\alpha) = \alpha \circ \beta : [m] \to V_<$ is a composition of order-preserving functions, and thus order-preserving itself. Also we have $\mathrm{Im}(\alpha \circ \beta) \subseteq \mathrm{Im}\,\alpha \in K$, so $\mathrm{Im}(\alpha \circ \beta)$ is a simplex. Thus $T(K)(\beta)(\alpha) \in T(K)_m$ and $T(K)(\beta)$ is well-defined.

Similarly if $f : (K, V_<) \to (L, W_<)$ is an order-preserving simplicial map then $T(f)_n(\alpha) = f \circ \alpha : [n] \to W_<$ is a composition of order-preserving maps, and $\mathrm{Im}(f \circ \alpha) = f(\mathrm{Im}\,\alpha)$ is an image of a simplex, and thus a simplex itself. So $T(f)_n$ is well defined, and since $T(f)_m \circ T(K)(\beta)(\alpha) = f \circ \alpha \circ \beta = T(L)(\beta) \circ T(f)_n(\alpha)$, the collection of these defines a morphism of simplicial sets.

To show the functoriality we simply calculate $T(\mathrm{Id}_K)_n(\alpha) = \mathrm{Id}_K \circ \alpha = \alpha$ and $T(f \circ g)_n = f \circ g \circ \alpha = T(f)_n \circ T(g)_n(\alpha)$.

The second way of getting a simplicial set from a simplicial complex, we first make a category and then take its nerve. This method does not use any ordering.

Definition 1.5.5. *Given a simplicial complex K, then the* ***inclusion category*** *$K_\subseteq$ is the small category with $Ob(K_\subseteq) = \{\sigma \in K$ simplex$\}$ and morphisms $\sigma \to \sigma' \iff \sigma \subseteq \sigma'$. Compositions of morphisms are $(\sigma' \subseteq \sigma'') \circ (\sigma \subseteq \sigma') = (\sigma \subseteq \sigma'')$, and the identities are $Id_\sigma = (\sigma \subseteq \sigma)$.*

Note that if $f : K \to K'$ is a simplicial map, then we get the functor $f_\subseteq : K_\subseteq \to K'_\subseteq$ defined on morphisms by sending $\sigma \subseteq \sigma'$ in K to $f(\sigma) \subseteq f(\sigma')$ in K'. For compositions of simplicial maps we have $(f \circ g)_\subseteq(\sigma \subseteq \sigma') = (f \circ g(\sigma) \subseteq f \circ g(\sigma')) = f_\subseteq \circ g_\subseteq(\sigma \subseteq \sigma')$, so $(-)_\subseteq : \texttt{Cpx} \to \texttt{Cat}$ defines a functor. Composing this with the nerve functor we get a functor $N_s(-)_\subseteq : \texttt{Cpx} \to \texttt{sSet}$.

Conversely starting with a functor $H : K_\subseteq \to K'_\subseteq$ between inclusion categories, we do not in general get a simplicial map, as the image of simplices consisting of only one vertex can in general consist of multiple vertices.

The two ways of constructing simplicial sets from simplicial complexes turns out to be linked together by the barycentric subdivision in a very natural way.

Lemma 1.5.6. *Let $f : (K, V_<) \to (L, W_<)$ be a morphism of ordered simplicial* *complexes. There are isomorphisms Γ^K and Γ^L, such that the diagram*

$$\begin{array}{ccc} T(SdK) & \xrightarrow{\Gamma^K} & N_s K_{\subseteq}^{op} \\ \big\downarrow {\scriptstyle T(Sdf)} & & \big\downarrow {\scriptstyle N_s f_{\subseteq}^{op}} \\ T(SdL) & \xrightarrow{\Gamma^L} & N_s L_{\subseteq}^{op} \end{array}$$

commutes, where the barycentric subdivisions have the lexicographic order.

Proof. Recall that $T(\mathrm{Sd}K)_m = \{\gamma : [m] \to K_< \,|\, \gamma \text{ order-preserving, } \mathrm{Im}\gamma \in \mathrm{Sd}K\}$. Let $\gamma \in T(\mathrm{Sd}K)_m$ be such a map. If $i \leq j$ is a morphism in $[m]$, then $\mathrm{Im}\gamma \in \mathrm{Sd}K$ tells us that either $\gamma(i) = \gamma(j)$, or one is included in the other. Furthermore since γ is order-preserving, then $\gamma(i) \leq \gamma(j)$ in the lexicographic order, and so $\gamma(i)$ is not strictly included in $\gamma(j)$ by the properties of this order. We conclude that $\gamma(j) \subseteq \gamma(i)$, and that $\gamma : [m] \to K_{\subseteq}^{op}$ defines a functor.

Conversely if $H \in (N_s K_{\subseteq}^{op})_m = \mathrm{Func}([m], K_{\subseteq}^{op})$ and $i \leq j$ in $[m]$, then $H(j) \subseteq H(i)$ and so $\mathrm{Im}H \in \mathrm{Sd}K$. Since the lexicographic order is a refinement of the partial order defined by the opposite of inclusions, we also get $H(i) \leq H(j)$, and H as a map from $[m]$ to $K_<$ is an element in $T(\mathrm{Sd}K)_m$. Thus the function Γ_m^K sends $\gamma : [m] \to K_<$ to the functor $[m] \to K_{\subseteq}^{op}$ sending objects $i \in \mathrm{Ob}[m]$ to $\gamma(i)$. We identify these maps and write $T(\mathrm{Sd}K) = N_s K_{\subseteq}^{op}$.

Finally to check that this all works on maps, let $\alpha : [m] \to [n]$ be a functor. Directly from 1.4.1 and 1.5.4, we have $N_s K_{\subseteq}^{op}(\alpha) = - \circ \alpha = T(\mathrm{Sd})(\alpha)$. Similarly if $f : K \to L$ a simplicial map, then by 1.4.2 and 1.5.4 we have $T(\mathrm{Sd}f)_m = \mathrm{Sd}f \circ -$ and $(N_s f_{\subseteq}^{op})_m = f_{\subseteq} \circ -$. Both $\mathrm{Sd}f$ and $f_{\subseteq}$ send simplices σ to $f\sigma$), and everything is fine. □

We also have a short nice result when K is a **finite simplicial complex**, namely when it has finite vertex and simplex sets.

Lemma 1.5.7. *Let $(K, V_<)$ be a finite ordered simplicial complex, then $T(K)$ and $N_s K_{\subseteq}$ are both finite simplicial sets.*

Proof. If $V_<$ has m elements, then $V_<$ is isomorphic to $[m]$ in Δ_{big}. With this identification $T(K)_n = \{\alpha : [n] \to V_< \,|\, \mathrm{Im}\alpha \in K,\ \alpha \text{ order-preserving}\}$ is a subset of Δ_n^m for all n, and so $T(K)$ is a simplicial subset of Δ^m which is finite by 1.2.10. Thus $T(K)$ is finite by 1.2.12.

If K has r elements, then we can refine the order of $K_{\subseteq}$ into a total order $K_<$, for example by the opposite of the lexicographic order. As above we identify $K_<$ with $[r]$. A map $\beta \in (N_s K_{\subseteq})_n$ is an order-preserving map from $[n]$ to $K_{\subseteq}$, which gives us an order-preserving map from $[n]$ to $K_< = [r]$. This is true for every n so we have $N_s K_{\subseteq} \subseteq \Delta^r$ and $N_s K_{\subseteq}$ is finite. □

At last we define the classical notion of geometric realization of a simplicial complex, before comparing it with the realization of the simplicial sets we have constructed.

Definition 1.5.8. *Let (K, V) be a simplicial complex. The set of the* ***geometric realization*** *of K is*

$$|K| = \{\alpha : V \to I \,|\, \{v \in V \,|\, \alpha(v) \neq 0\} \in K, \sum_{v \in V} \alpha(v) = 1\}.$$

For every simplex $\sigma \in K$ we give the subset $|\sigma| = \{\alpha \in |K| \,|\, \{v \in V \,|\, \alpha(v) \neq 0\} \subseteq \sigma\} \in |K|$ the topology from the metric $d(\alpha, \beta) = \sqrt{\sum_{v \in \sigma}(\alpha(v) - \beta(v))^2}$. We give $|K|$ the ***coherent topology*** *defined by being the finest topology so that the inclusions $|\sigma| \hookrightarrow |K|$ are continuous.*

For a simplicial map $f : K \to L$, we get a continuous map $|f| : |K| \to |L|$ by $|f|(\alpha)(w) = \sum_{f(v)=w} \alpha(v)$ for every vertex w in L. Geometric realization of simplicial complexes are discussed in more detail in [13] 3.1.

Proposition 1.5.9. *If $f : (K, V_<) \to (L, W_<)$ is a simplicial map injective on vertex sets between ordered simplicial complexes, then we have a homeomorphism between $|K|$ and $|T(K)|$, and between $|L|$ and $|T(L)|$, making the following diagram commute*

$$\begin{array}{ccc} |K| & \longrightarrow & |T(K)| \\ \downarrow{\scriptstyle |f|} & & \downarrow{\scriptstyle |T(f)|} \\ |L| & \longrightarrow & |T(L)|\,. \end{array} \tag{1.25}$$

Proof. The proof of this is a bit convoluted and takes a lot of space, so it is moved to A.3. □

Theorem 1.5.10. *Let K and L be simplicial complexes, and let $f : K \to L$ be a simplicial map which is injective on vertex sets. There exist homotopy equivalences $h_K : |K| \to |N_s K_\subseteq|$ and $h_L : |L| \to |N_s L_\subseteq|$ such that the following diagram commutes:*

$$\begin{array}{ccc} |K| & \xrightarrow{h_K} & |N_s K_\subseteq| \\ \downarrow{\scriptstyle |f|} & & \downarrow{\scriptstyle |N_s f_\subseteq|} \\ |L| & \xrightarrow{h_L} & |N_s L_\subseteq| \end{array} \tag{1.26}$$

Proof. From 1.1.9, the least vertex map is a homotopy equivalence between $|K|$ and $|\mathrm{Sd}K|$, this is has a corresponding commuting diagram by 1.5.3. Next, 1.5.9 gives a homeomorphism between $|\mathrm{Sd}K|$ and $|T(\mathrm{Sd}K)|$. From 1.5.6 and using that $| - |$ is a functor, we get a commuting homeomorphism from $|T(\mathrm{Sd}K)|$ to $|N_s K_\subseteq^{op}|$, and finally by 1.4.8 there is a homeomorphism from $|N_s K_\subseteq^{op}|$ to $|N_s K_\subseteq|$. Every step is a homeomorphism or a homotopy equivalence, and they all come with a corresponding commuting diagram. □

Finally we will prove the classical result about contiguous maps which we stated in 1.1.8. Recall from 1.1.7 that $f, g : K \to K'$ are contiguous if $f(\sigma) \cup g(\sigma)$

is a simplex in K' for all simplices $\sigma \in K$. Now let $(f \cup g)_\subseteq : K_\subseteq \to K'_\subseteq$ be the map sending σ to $f(\sigma) \cup g(\sigma)$. If $\sigma \subseteq \sigma'$ then, since simplicial maps are defined on vertices, we get $f(\sigma) \subseteq f(\sigma')$ and $g(\sigma) \subseteq g(\sigma')$. In particular we have $f(\sigma) \cup g(\sigma) \subseteq f(\sigma') \cup g(\sigma')$, and so $(f \cup g)_\subseteq$ defines a functor. We will use this fact in the following proposition.

Proposition 1.5.11. *Let $f, g : K \to K'$ be contiguous simplicial maps between simplicial complexes. Then $|N_s f_\subseteq|$ and $|N_s g_\subseteq|$ are homotopic.*

Proof. We will show that they both are homotopic to $|N_s(f \cup g)_\subseteq|$, and by symmetry it is enough to show that one of them is.

Let $F : [1] \times K_\subseteq \to K'_\subseteq$ be the map sending $(0, \sigma)$ to $f_\subseteq(\sigma)$ and $(1, \sigma)$ to $(f \cup g)_\subseteq(\sigma)$. To see that this is well-defined we need to show that it sends morphisms to morphsms. A morphism in $[1] \times K$ is of the form $(\leq, \subseteq) : (i, \sigma) \to (j, \sigma')$ where $i \leq j$ and $\sigma \subseteq \sigma'$. If $i = j = 0$, then $(\leq, \subseteq)$ is sent to the inclusion $f(\sigma) \subseteq f(\sigma')$, which is a well-defined morphisms by the functoriality of $f_\subseteq$. Similarly if $i = j = 1$ then the morphism $(\leq, \subseteq)$ is sent to the morphism $f(\sigma) \cup g(\sigma) \subseteq f(\sigma') \cup g(\sigma')$. Finally if $i = 0$ and $j = 1$ then the morphism is sent to the inclusion $f(\sigma) \subseteq f(\sigma') \cup g(\sigma')$ which is well-defined since $f(\sigma) \subseteq f(\sigma')$.

The compositions of morphisms are point-wise, so F is a functor. Finally using 1.4.10 we get a homotopy between $|N_s f_\subseteq|$ and $|N_s(f \cup g)_\subseteq|$. ☐

To complete the proof of 1.1.8, let $f, g : K \to L$ be contiguous simplicial maps. Then by 1.5.11, we have a homotopy $|N_s f_\subseteq| \simeq |N_s g_\subseteq|$, and in particular $|N_s f_\subseteq| \circ h_K \simeq |N_s g_\subseteq| \circ h_K$, where h_K is as in 1.5.10. Using that the diagram (1.26) commutes up to homotopy, we have $h_L \circ |f| \simeq |N_s f_\subseteq|$, and similarly for g, so $h_L \circ |f| \simeq h_L \circ |g|$. Finally using that h_L is a homotopy equivalence we get our result $|f| \simeq |g|$.

1.6 Dowker's Theorem by Simplicial Sets

We now have enough general results to prove Dowker's Theorem using simplicial sets. The final ingredient we need is more specialized towards this one particular problem, namely a map between the Dowker complexes of a relation.

Definition 1.6.1. *Let $R \subseteq X \times Y$ be a relation with Dowker complexes (NR, X) and (NR^T, Y). Define the **B-functor of** R as the functor $B : (NR)_\subseteq \to (NR^T)^{op}_\subseteq$ sending a simplex σ to*

$$B(\sigma) = \{y \in Y \mid \sigma \times \{y\} \subseteq R\}$$

Recall that the definition of the Dowker complex (1.2) is $N(R) = \{\sigma \in P(X) \mid \exists\, y \in Y$ such that $\sigma \times \{y\} \subseteq R\}$. We see that $B(\sigma)$ is non-empty, and it is also a simplex in NR^T since $B(\sigma) \times \{s\}$ is in R^T for any s in σ.

To prove that it is a functor we also need to show that inclusions are reversed. If $\sigma' \subseteq \sigma$ and $y \in B(\sigma)$, then $\sigma' \times \{y\} \subseteq \sigma \times \{y\} \subseteq R$. So y is in $B(\sigma')$ and $B(\sigma) \subseteq B(\sigma')$, and B is a well-defined functor.

Similarly we define the C**-functor of** R, $C : (NR^T)^{op}_{\subseteq} \to (NR^{TT})^{op\,op}_{\subseteq} = NR_{\subseteq}$, by sending simplices τ in NR^T to $C(\tau) = \{x \in X \mid \{x\} \times \tau \subseteq R\}$.

Lemma 1.6.2. *Let $R \subseteq X \times Y$ be a relation, with B- and C- functors B and C. Then $|N_s B|$ is a homotopy equivalence, with homotopy inverse $|N_s C|$.*

Proof. Let $s \in \sigma$ be a vertex. Now if y is in $B(\sigma)$, then particularly (s, y) is in R. Thus $\{s\} \times B(\sigma) \subseteq R$, and s is a vertex in $CB(\sigma)$. Since s was arbitrary we have an inclusion $\sigma \subseteq CB(\sigma)$, which is a morphism in $NR_{\subseteq}$. If $\sigma' \subseteq \sigma$ is an inclusion then we also have an inclusion $CB(\sigma') \subseteq CB(\sigma)$ since $CB : NR_{\subseteq} \to NR_{\subseteq}$ is a functor as a composition of functors.

So $\sigma \subseteq CB(\sigma)$ defines a natural transformation from $\mathrm{Id}_{NR_{\subseteq}}$ to CD:

$$\begin{array}{ccc} \mathrm{Id}(\sigma) & \hookrightarrow & CD(\sigma) \\ \downarrow & & \downarrow \\ \mathrm{Id}(\sigma') & \hookrightarrow & CD(\sigma'). \end{array}$$

Using our results from 1.4.9 and 1.4.10 we get a homotopy between $|N_s \mathrm{Id}_{NR_{\subseteq}}|$ and $|N_s(CD)|$. By 1.4.5 the classifying space $|N_s - |$ defines a functor, so we get a homotopy between $\mathrm{Id}_{|N_s(NR)_{\subseteq}|}$ and $|N_s C| \circ |N_s B|$. Similarly we get a homotopy between $\mathrm{Id}_{|N_s(NR^T)^{op}_{\subseteq}|}$ and $|N_s B| \circ |N_s C|$. ☐

Proposition 1.6.3. *Let $R \subseteq R' \subseteq X \times Y$ be two relations, and let $i : NR \to NR'$ and $i^T : NR^T \to NR'^T$ be the natural inclusions of Dowker complexes from under 1.1.4. Let B_R and $B_{R'}$ be the B-functors of R and R' respectively. The following diagram commutes up to homotopy:*

$$\begin{array}{ccc} |N_s(NR)_{\subseteq}| & \xrightarrow{|N_s B_R|} & |N_s(NR^T)^{op}_{\subseteq}| \\ \Big\downarrow {\scriptstyle |N_s i_{\subseteq}|} & & \Big\downarrow {\scriptstyle |N_s (i^T)^{op}_{\subseteq}|} \\ |N_s(NR')_{\subseteq}| & \xrightarrow{|N_s B_{R'}|} & |N_s(NR'^T)^{op}_{\subseteq}| \end{array} \tag{1.27}$$

Proof. We will look at the diagram without the $|N_s - |$. The natural inclusion i, and therefore the functor $i_{\subseteq}$, will send a simplex $\sigma \in NR$ to the simplex σ in NR'. Similarly, since the opposite of a functor sends objects and morphisms to the same as the original functor, we have $(i^T)^{op}_{\subseteq}(\tau) = \tau$ for all simplices $\tau \in NR^T$.

If y is in $B_R(\sigma)$, then $\sigma \times \{y\} \subseteq R \subseteq R'$. Thus y is also in $B_{R'}(\sigma)$ and we have the inclusion $B_R(\sigma) \subseteq B_{R'}(\sigma)$. In particular we have that $B_{R'} \circ i_{\subseteq}(\sigma) = B_{R'}(\sigma)$ and $(i^T)^{op}_{\subseteq} \circ B_R(\sigma) = B_R(\sigma)$ both are contained in the union $B_{R'}(\sigma) \cup B_R(\sigma) = B_{R'}(\sigma)$. So $B_{R'} \circ i_{\subseteq}$ and $(i^T)^{op}_{\subseteq} \circ B_R$ are contiguouos maps, and using 1.5.11 we have that the compositions $|NB_{R'}| \circ |N_s i_{\subseteq}|$ and $|N_s(i^T)^{op}_{\subseteq}| \circ |N_s B_R|$ are homotopic. ☐

We are now finally ready to prove Dowker's theorem using simplicial sets. The theorem we prove here is exactly the same as Theorem 3 in [3].

Theorem 1.6.4. ***(Dowker's Theorem)*** *Let $R \subseteq R' \subseteq X \times Y$ be relations, and let $i : NR \to NR'$ and $i^T : NR^T \to NR'^T$ be the natural inclusion between corresponding Dowker complexes. Then there exist homotopy equivalences $\Gamma_R : NR \to NR^T$ and $\Gamma_{R'} : NR' \to NR'^T$ such that the following diagram commutes up to homotopy:*

$$\begin{array}{ccc} |NR| & \xrightarrow{\Gamma_R} & |NR^T| \\ \downarrow{\scriptstyle |i|} & & \downarrow{\scriptstyle |i^T|} \\ |NR'| & \xrightarrow{\Gamma_{R'}} & |NR'^T| \end{array} \tag{1.28}$$

Proof. The inclusion maps i and i^T act like the identity on vertices, so in particular they are injective on vertex sets. From 1.5.10 setting $K = NR$, $L = NR'$ and $f = i$ we get the commuting diagram

$$\begin{array}{ccc} |NR| & \longrightarrow & |N_s(NR)_\subseteq| \\ \downarrow{\scriptstyle |i|} & & \downarrow{\scriptstyle |N_s i_\subseteq|} \\ |NR'| & \longrightarrow & |N_s(NR')_\subseteq|. \end{array}$$

From 1.6.3 we have the diagram

$$\begin{array}{ccc} |N_s(NR)_\subseteq| & \longrightarrow & |N_s(NR^T)^{op}_\subseteq| \\ \downarrow{\scriptstyle |N_s i_\subseteq|} & & \downarrow{\scriptstyle |N_s(i^T)^{op}_\subseteq|} \\ |N_s(NR')_\subseteq| & \longrightarrow & |N_s(NR'^T)^{op}_\subseteq|, \end{array}$$

commuting up to isomorphism. We can now use 1.4.8, taking $\mathcal{C} = (NR^T)_\subseteq$, $\mathcal{D} = (NR'^T)_\subseteq$, $H = (i^T)_\subseteq$, and mirroring the diagram, giving us

$$\begin{array}{ccc} |N_s(NR^T)^{op}_\subseteq| & \longrightarrow & |N_s(NR^T)_\subseteq| \\ \downarrow{\scriptstyle |N_s(i^T)^{op}_\subseteq|} & & \downarrow{\scriptstyle |N_s(i^T)_\subseteq|} \\ |N_s(NR'^T)^{op}_\subseteq| & \longrightarrow & |N_s(NR'^T)_\subseteq|. \end{array}$$

Finally we can again use 1.5.10 as above, changing the relations with its transpose, and mirroring the diagram:

$$\begin{array}{ccc} |N_s(NR^T)_\subseteq| & \longrightarrow & |NR^T| \\ \downarrow{\scriptstyle |N_s i^T_\subseteq|} & & \downarrow{\scriptstyle |i^T|} \\ |N_s(NR'^T)_\subseteq|. & \longrightarrow & |NR'^T|. \end{array}$$

Combining all diagrams, we get our result since every single diagram commutes or commutes up to homotopy, and the horizontal maps are all homeomorphisms or homotopy equivalences. ☐

Now this is a result that can be used in topological data analysis. Starting with a sequence of relations

$$R_0 \subseteq R_1 \subseteq \ldots R_n \subseteq X \times Y$$

this gives us two nested sequences of simplicial complexes

$$NR_0 \subseteq NR_1 \subseteq \cdots \subseteq NR_n \quad \text{and} \quad NR_0^T \subseteq NR_1^T \subseteq \cdots \subseteq NR_n^T.$$

Now taking homology of this sequence, we get isomorphisms on homology groups $H_*(NR_i) \to H_*(NR_i^T)$ and a commutative diagram:

$$\begin{array}{ccccccc} H_*(NR_0) & \xrightarrow{i_*} & H_*(NR_1) & \xrightarrow{i_*} & \ldots & \xrightarrow{i_*} & H_*(NR_n) \\ \downarrow & & \downarrow & & & & \downarrow \\ H_*(NR_0^T) & \xrightarrow{i_*^T} & H_*(NR_1) & \xrightarrow{i_*^T} & \ldots & \xrightarrow{i_*^T} & H_*(NR_n) \end{array}$$

This follows from Dowker's theorem 1.6.4, and the fact that homotopic maps induce the same map on homology ([14] 1.10.). The **persistent homology** of the nested sequence is given by the rank of of the image of the maps i_* ([7] VI.1), so by commutativity we get that the two different sequences have the same persistent homology.

Note that in the proof of 1.5.10, we referenced the fact 1.1.9 that the least vertex map induces a homotopy on geometric realizations. This is a result about simplicial complexes. It would be nice to instead show that the map $|T(\phi)| : |T(\mathrm{Sd}K)| \to |T(K)|$ is a homotopy equivalence, completing the proof only using simplicial sets. We already know, from the functor property of T and the diagram in 1.24, that this would have a nice commuting diagram, and the proof of 1.5.10 would follow from the sequence of homotopy equivalences $|K| \overset{1.5.9}{\simeq} |T(K)| \simeq |T(\mathrm{Sd}K)| \overset{1.5.6}{\simeq} |N_s K_{\subseteq}^{op}| \overset{1.4.8}{\simeq} |N_s K_{\subseteq}|$.

We could forget the first and last step in the proof of 1.6.4, to get an analogous theorem but for simplicial sets only. Starting with a relation $R \subseteq X \times Y$, we can define two simplicial set $N_s(NR)_{\subseteq}$ and $N_s(NR^T)_{\subseteq}$, and we have shown that their geometric realizations are homotopy equivalent. It might be interesting to explore these kinds of simplicial sets. In [3], they use Dowker's theorem to prove the nerve theorem, so maybe something analogous to that might be done for simplicial sets.

Part 2

0-Interleavings

Starting with a nested sequence of topological spaces and taking the homology we get some unique persistence diagrams telling us about the topological features of the sequence [7]. These diagrams are realized as a multiset in $\hat{\mathbb{R}}^2_+$, or equivalently as a multiset of intervals, called a barcode diagram. Stability theory in topological data analysis in particular looks at questions about how changing the spaces affects the diagrams. When the topological spaces are simplicial complexes we have the notion of two such sequences being ε-interleaved, as a way of saying how similar they are after some bijection on vertex sets.

In this second part we will look at the simplest case where $\varepsilon = 0$. We start by comparing nested sequences of simplicial complexes with functions from a product of sets to the extended line of non-negative numbers. From there we will arrive at a category that identifies the complexes that are 0-interleaved.

2.1 The Maps F, $N_<$ and $N_\leq$

We begin by defining the concepts of filtered simplicial complexes and dissimilarities. We will construct maps between them, and look at the properties of these maps.

Definition 2.1.1. *A* ***filtered simplicial complex*** (K, V)*, or just* K*, is a family of simplicial complexes* $\{(K_t, V)\}_{t\in\mathbb{R}_+}$ *such that* $K_t \subseteq K_{t'}$ *is a subcomplex whenever* $t \leq t'$

As with simplicial complexes say that K has **vertex set** V.

Definition 2.1.2. *Let* V *and* W *be arbitrary sets. A map* $\Lambda : V \times W \to \hat{\mathbb{R}}_+$ *is called a* ***(Dowker) dissimilarity****.*

As an example we have that any distance functions $d : X \times X \to \hat{\mathbb{R}}_+$ is a dissimilarity.

Definition 2.1.3. *Given a filtered simplicial complex K, we define the* ***associated dissimilarity*** $FK : V \times P(V) \to \hat{\mathbb{R}}_+$ *by*

$$FK(v,\sigma) = \begin{cases} \infty & \text{if } v \notin \sigma \\ \infty & \text{if } \sigma \notin \bigcup_{t \in \mathbb{R}_+} K_t \\ \inf\{t | \sigma \in K_t\} & \text{if } v \in \sigma \in \bigcup_{t \in \mathbb{R}_+} K_t. \end{cases} \tag{2.1}$$

Definition 2.1.4. *Given a dissimilarity $\Lambda : V \times W \to \hat{\mathbb{R}}_+$ define the* ***open Dowker nerve*** *of Λ to be the filtered simplicial complex $N_< \Lambda = \{(N_< \Lambda_t, V)\}_{t \in \mathbb{R}_+}$, where*

$$N_< \Lambda_t = \{\sigma \in P(V) \,|\, \exists w \in W \text{ with } \Lambda(s,w) < t \text{ for all } s \in \sigma\}. \tag{2.2}$$

For every t, this is a Dowker complex of some relation $R_t = \{(v,w) \,|\, \Lambda(v,w) < t\} \subseteq V \times W$. Now, if $\sigma \subseteq \tau$ and $\tau \in N_< \Lambda_t$, then $\sigma \in N_< \Lambda_t$ by the same $w \in W$. Also if $\sigma \in N_< \Lambda_t$ and $t' \geq t$, then $\Lambda(s,w) < t \leq t'$ for all $s \in \sigma$, so $\sigma \in N_< \Lambda_{t'}$. Thus $N_< \Lambda$ is indeed a filtered simplicial complex. The same arguments hold if we change the $<$'s with $\leq$'s.

Definition 2.1.5. *For Λ as above, define the* ***closed Dowker nerve***

$$N_\leq \Lambda_t = \{\sigma \in P(V) \,|\, \exists w \in W \text{ with } \Lambda(s,w) \leq t \text{ for all } s \in \sigma\}. \tag{2.3}$$

If $d : X \times X \to \hat{\mathbb{R}}_+$ is a distance function, then $N_< d$ or $N_\leq d$ will be the Čech complex, depending on if you define it by the open or closed balls.

We will compare the open and closed Dowker nerves, and see that a filtered simplicial complex is contained between the open and closed Dowker nerve of its associated dissimilarity.

Lemma 2.1.6. *Let $\Lambda : V \times W \to \hat{\mathbb{R}}_+$ be a dissimilarity. For all $\varepsilon > 0$ and for all $t \in \mathbb{R}_+$ we have $N_< \Lambda_t \subseteq N_\leq \Lambda_t \subseteq N_< \Lambda_{t+\varepsilon}$.*

Proof. We first note that both $N_< \Lambda$ and $N_\leq \Lambda$ have vertex set V, so inclusions can happen. For the first inclusion, let $\sigma \in N_< \Lambda$ and let $w \in W$ such that $\Lambda(s,w) < t$ for all s in σ. Then $\Lambda(s,w) \leq t$ for all $s \in \sigma$, and so σ is also in $N_\leq \Lambda$. Thus $N_< \Lambda \subseteq N_\leq \Lambda$.

Similarly, let $\sigma \in N_\leq \Lambda$ and $w \in W$ be such that $\Lambda(s,w) \leq t$ for all $s \in \sigma$. Now for all $\varepsilon > 0$ we clearly have $t < t + \varepsilon$, and in particular $\Lambda(s,w) < t + \varepsilon$ for all s in σ. We conclude that $N_\leq \Lambda_t \subseteq N_< \Lambda_{t+\varepsilon}$ for all $\varepsilon > 0$. □

Lemma 2.1.7. *Let $K = \{(K_t, V)\}_{t \in \mathbb{R}_+}$ be a filtered simplicial complex. Then*

$$N_< FK_t \subseteq K_t \subseteq N_\leq FK_t$$

Proof. First note that all filtered simplicial complexes have vertex set V.

Assume $\sigma \in N_< FK_t$. This is true if and only if there exists a $\tau \in P(V)$ such that $FK(s,\tau) < t$ for all $s \in \sigma$. By the definition of FK this implies that $\sigma \subseteq \tau$ and that $\inf\{t' | \tau \in K_{t'}\} < t$. Since the greatest lower bound is less than t, there must exist an element $s \in \{t' | \tau \in K_{t'}\}$ such that $s \leq t$, otherwise t would be a greater lower bound. Now because K is a filtered simplicial complex, then $\tau \in K_s$ implies $\tau \in K_t$ for $s \leq t$, and $\sigma \subseteq \tau \in K_t$ implies $\sigma \in K_t$.

For the second inclusion, assume σ in K_t. Then by the definition of FK, for all $s \in \sigma$, we have $FK(s,\sigma) = \inf\{t' | \sigma \in K_{t'}\} \leq t$. In particular we get $\sigma \in N_\leq FK_t$. □

Combining 2.1.6 with 2.1.7 by setting $\Lambda = FK$ we get for all $t \geq 0$ and all $\varepsilon > 0$ the series of inclusions

$$N_< FK_t \subseteq K_t \subseteq N_\leq FK_t \subseteq N_< FK_{t+\varepsilon} \subseteq K_{t+\varepsilon}.$$

By splitting this in two, we get two results similar to that of 2.1.6:

$$N_< FK_t \subseteq K_t \subseteq N_< FK_{t+\varepsilon}, \tag{2.4}$$

$$K_t \subseteq N_\leq FK_t \subseteq K_{t+\varepsilon}. \tag{2.5}$$

The results in 2.1.6, (2.4) and (2.5), are all on the form $K_t \subseteq K'_t \subseteq K_{t+\varepsilon}$ for some filtered simplicial complexes K and K'. We could use this as a definition of some kind of similarity between two filtered simplicial complexes, but this might be a bit restrictive. We would like the similarity to be symmetric, so we are not interested in whether or not we have an inclusion $K_t \subseteq K'_t$ for all t.

Definition 2.1.8. *Two filtered simplicial complexes (K,V) and (K',V), with the same vertex set V, are **strictly 0-interleaved** if $K_t \subseteq K'_{t+\varepsilon}$ and $K'_t \subseteq K_{t+\varepsilon}$ for all $t \in \mathbb{R}_+$ and for all $\varepsilon > 0$.*

The word "strictly" is used because we demand the vertex sets to be the same. We will later generalize the definition to include a wider range of complexes, but for now we continue to show that the results above are just special cases of being strictly 0-interleaved.

Lemma 2.1.9. *If (K,V) and (K',V) are two filtered simplicial complexes such that $K_t \subseteq K'_t \subseteq K_{t+\varepsilon}$ for all $t \in \mathbb{R}_+$ and for all $\varepsilon > 0$. Then K and K' are strictly 0-interleaved.*

Proof. We need to show that $K_t \subseteq K'_{t+\varepsilon}$ and $K'_t \subseteq K_{t+\varepsilon}$ for all $t \in \mathbb{R}_+$, $\varepsilon > 0$. The second part, $K'_t \subseteq K_{t+\varepsilon}$, follows trivially from the assumption. From the assumption we also have $K_t \subseteq K'_t$ which again is included in $K'_{t+\varepsilon}$ by the fact that K' is a filtered simplicial complex. □

The following theorem shows that two filtered simplicial complexes are strictly 0-interleaved if and only if they have the same associated Dowker dissimilarities.

Theorem 2.1.10. *Let $K = \{(K_t, V)\}_{t\in\mathbb{R}_+}$ and $K' = \{(K'_t, V)\}_{t\in\mathbb{R}_+}$ be two filtered simplicial complexes, then the following are equivalent:*

(i) K *and* K' *are strictly 0-interleaved.*

(ii) $K_t \subseteq K'_{t+\varepsilon}$ *and* $K'_t \subseteq K_{t+\varepsilon}$ *for all* $t \in \mathbb{R}_+$ *and for all* $\varepsilon > 0$.

(iii) $FK = FK' : V \times P(V) \to \hat{\mathbb{R}}_+$

Proof.

(i) $\iff$ *(ii)*: Definition.

(ii) $\Rightarrow$ *(iii)*: Let $v \in V$ and $\sigma \in P(V)$ be any elements. We want to show that $FK(v, \sigma) = FK'(v, \sigma)$.

- $v \notin \sigma$:
  By definition $FK(v, \sigma) = FK'(v, \sigma) = \infty$.

- $v \in \sigma$, $\sigma \notin \bigcup K_t$:
  If $\sigma \in \bigcup K_t$ then there is a $t \in \mathbb{R}_+$ such that $\sigma \in K_t$. Since K is filtered we also get $\sigma \in K'_{t+\varepsilon}$ for all $\varepsilon > 0$. Now since $K'_t \subseteq K_{t+\varepsilon}$ by assumption, $\sigma \in K'_t$, and thus $\sigma \in \bigcup K'_t$. Symmetrically by changing K and K' we get that $\sigma \in \bigcup K'_t$ implies $\sigma \in \bigcup K_t$. We conclude that $\sigma \notin \bigcup K_t \iff \sigma \notin \bigcup K'_t$, and that $FK(v, \sigma) = FK'(v, \sigma) = \infty$.

- $v \in \sigma$, $\sigma \in K_t$ for some $t \in \mathbb{R}_+$:
  Let $S = \{t|\sigma \in K_t\}$ and $S' = \{t|\sigma \in K'_t\}$, and assume by contradiction that $FK(v, \sigma) > FK'(v, \sigma)$, i.e. we assume $\inf S > \inf S'$. Then $\inf S = \inf S' + \varepsilon$ for some $\varepsilon > 0$, and there exists a $0 < \delta < \varepsilon$ such that $\delta < \inf S$. Since $\delta > 0$ we have that $K'_t \subseteq K_{t+\delta}$ for all $t \in \mathbb{R}_+$.

  Now if $t \in S'$ then $\sigma \in K'_t \subseteq K_{t+\delta}$, so we get that $t + \delta \in S$ for all $t \in S'$. Since $\inf S \leq s$ for all $s \in S$ we have $\inf S \leq t + \delta$ for all $t \in S'$. By subtracting δ on both sides we get that $\inf S - \delta \leq t$ for all $t \in S'$, so it is a lower bound. Now since $\inf S'$ is the greatest lower bound we know $\inf S - \delta \leq \inf S'$. But now we have $\inf S \leq \inf S' + \delta < \inf S' + \varepsilon = \inf S$ which is a contradiction. Thus $FK(v, \sigma) \leq FK'(v, \sigma)$.

  By the symmetry of K and K' we similarly get that $FK(v, \sigma) \geq FK'(v, \sigma)$, and we conclude that $FK(v, \sigma) = FK'(v, \sigma)$.

(iii) $\Rightarrow$ *(ii)*: We will show the contrapositive. Assume that there exists $t \in \mathbb{R}_+$ and $\varepsilon > 0$ such that $K_t \nsubseteq K'_{t+\varepsilon}$. We want to show that $FK \neq FK'$. Let $\sigma \in K_t$ be such that $\sigma \notin K'_{t+\varepsilon}$, and let $v \in \sigma$ be any vertex. Then $FK(v, \sigma) = \inf\{t|\sigma \in K_t\} \leq t$.

If $\sigma \notin \bigcup K'_t$, then $FK'(v, \sigma) = \infty$ and we are done.If this is not the case then, since K' is a filtered simplicial complex, we have that if $\tau \in K'_t$ then $\tau \in K'_s$ for all $s \geq t$, equivalently if $\tau \notin K'_t$ then $\tau \notin K'_s$ for all $s \leq t$. Since $\sigma \notin K'_{t+\varepsilon}$ we get that $t + \varepsilon \leq t'$ for all $t' \in \{t|\sigma \in K'_t\}$, so $t + \varepsilon$ is a lower bound for $\{t|\sigma \in K'_t\}$. We conclude that $FK(v, \sigma) \leq t < t + \varepsilon \leq \inf\{t|\sigma \in K'_t\} = FK'(v, \sigma)$, and in particular we have $FK(v, \sigma) \neq FK'(v, \sigma)$.

By symmetry we get the same for $K'_t \nsubseteq K_{t+\varepsilon}$. □

From this theorem and the preceding lemmas we get some nice results about our maps F, $N_<$ and $N_\leq$:

Corollary 2.1.11. *For any filtered simplicial complex K and any Dowker dissimilarity Λ, the following holds.*

(i) $FN_<\Lambda = FN_\leq\Lambda$

(ii) $FK = FN_<FK = FN_\leq FK$

Proof.

$\underline{(i)}$: From 2.1.6 we have $N_<\Lambda_t \subseteq N_\leq\Lambda_t \subseteq N_<\Lambda_{t+\varepsilon}$ for all $t \geq 0$ and all $\varepsilon > 0$. Next 2.1.9 says that $N_<\Lambda$ and $N_\leq\Lambda$ are strictly 0-interleaved, and finally 2.1.10 tells us that this is equivalent to $FN_<\Lambda$ and $FN_\leq\Lambda$ being equal.

$\underline{(ii)}$: The second equality follows directly from (i), by setting $\Lambda = FK$. The proof of the first equality is almost identical to the proof above. From (2.4) we have $N_<FK_t \subseteq K_t \subseteq N_<FK_{t+\varepsilon}$ for all $t \geq 0$ and all $\varepsilon > 0$, so by 2.1.9 K and $N_<FK$ are strictly 0-interleaved, and from 2.1.10 we get that $FK = FN_<FK$. ☐

We note that the closed and open dowker nerve in some way give an upper and lower bound of strictly 0-interleaved complexes. If K and K' are strictly 0-interleaved, then $FK = FK'$ and by (2.4) we have $N_<FK_t \subseteq K'_t \subseteq N_\leq FK_t$.

2.2 Category of 0-interleavings

In this section we will begin by making filtered simplicial complexes and dissimilarities into categories, with a structure such that the maps from section 2.1 are functors. We will use the functors to compare the two categories, and create a new interesting category of 0-interleaved simplicial complexes. We will show this category is equivalent to some of the categories constructed by the theory from A.4.

First, we define the two basic categories we will build everything from.

Definition 2.2.1. *The* ***category of filtered simplicial complexes*** **`fsCx`** *is the category where the objects are filtered simplicial complexes, and morphisms $\phi : (K, V) \to (K'V')$ are functions $\phi : V \to V'$ such that $\sigma \in K_t \implies \phi(\sigma) \in K'_t$. Compositions are composition of functions.*

If (K'', V'') is a third filtered simplicial complex, and $\phi' : K' \to K''$ a morphism. Then $\sigma \in K_t \implies \phi(\sigma) \in K'_t \implies \phi'(\phi(\sigma)) \in K''_t$, so the composition $\phi' \circ \phi$ is also a morphism. Associativity and identity of morphisms is induced by the associativity and identity of $\phi : V \to V'$ as a functions. Thus `fsCx` is indeed a category.

Proposition 2.2.2. *Let (K, V) and (K', V') be filtered simplicial complexes. Then $\phi : K \to K'$ is an isomorphism if and only if $\phi : V \to V'$ is bijective and $\sigma \in K_t \iff \phi(\sigma) \in K'_t$.*

Proof.

(⇒): Assume ϕ is an isomorphism. Then there exists a map $\psi : V' \to V$ such that $\tau \in K'_t$ implies $\psi(\tau) \in K_t$, and such that $\psi \circ \phi = \mathrm{Id}_V$ and $\phi \circ \psi = \mathrm{Id}_{V'}$. The last part proves exactly bijection of ϕ as a function on the vertices. For the second part let $\phi(\sigma) \in K'_t$, then $\sigma = \mathrm{Id}_V(\sigma) = \psi(\phi(\sigma)) \in K'_t$.

(⇐): Assume $\phi : V \to V'$ is a bijection such that $\sigma \in K_t \iff \phi(\sigma) \in K'_t$. This is a morphism of filtered simplicial complexes by definition, but we also need to show that ϕ^{-1} is a morphism. Let $\tau \in K'_t \subseteq P(V')$, since ϕ is a bijection there exists a subset $\sigma \subseteq V$ such that $\phi(\sigma) = \tau$. By assumption $\phi(\sigma) \in K'_t$ implies that $\sigma \in K_t$. So $\phi^{-1}(\tau) = \sigma \in K_t$ and ϕ^{-1} is a morphism. □

Definition 2.2.3. *The* ***category of dissimilarities*** *is the category* `Diss` *where the objects are dissimilarities, and where if $\Lambda : V \times W \to \hat{\mathbb{R}}_+$ and $\Lambda' : V' \times W' \to \hat{\mathbb{R}}_+$ are objects then morphisms $(f, g) : \Lambda \to \Lambda'$ are pairs of functions $f : V \to V'$ and $g : W \to W'$ such that $\Lambda'(f(v), g(w)) \leq \Lambda(v, w)$ for all $v \in V$ and all $w \in W$. Compositions are pairwise.*

To check this is a category, let $\Lambda'' : V'' \times W'' \to \hat{\mathbb{R}}_+$ be a third dissimilarity and $(f', g') : \Lambda' \to \Lambda''$ a morphism, then $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$. Now $\Lambda''(f'(f(v)), g'(g(w))) \leq \Lambda'(f(v), g(w)) \leq \Lambda(v, w)$, so the composition is also a morphism. We clearly have the identity morphisms $\mathrm{Id}_\Lambda = (\mathrm{Id}_V, \mathrm{Id}_W) : \Lambda \to \Lambda$, and associativity again follows from associativity of the underlying functions.

Proposition 2.2.4. *Let $\Lambda : V \times W \to \hat{\mathbb{R}}_+$ and $\Lambda' : V' \times W' \to \hat{\mathbb{R}}_+$ be dissimilarities. Then $(f, g) : \Lambda \to \Lambda'$ is an isomorphism if and only if $f : V \to V'$ and $g : W \to W'$ both are bijective and $\Lambda'(fv, gw) = \Lambda(v, w)$ for all $v \in V$ and $w \in W$.*

Proof.

(⇒): Assume (f, g) is an isomorphism. Then there exist a morphism $(f'g') : \Lambda' \to \Lambda$ such that $(f' \circ f, g' \circ g) = (\mathrm{Id}_V, \mathrm{Id}_W)$ and $(f \circ f', g \circ g') = (\mathrm{Id}_{V'}, \mathrm{Id}_{W'})$. The identities implies that f and g are bijective. Since (f, g) is a morphism we know $\Lambda'(fv, gw) \leq \Lambda(v, w)$, and since (f', g') is a morphism we know $\Lambda(f'v', g'w') \leq \Lambda'(v', w')$ for all $v' \in V'$, $w' \in W'$. In particular $\Lambda(f'(f(v)), g'(g(w))) = \Lambda(v, w) \leq \Lambda'(f(v), g(w))$, so $\Lambda(v, w) = \Lambda'(f(v), g(w))$.

(⇐): Let f and g both be bijective such that $\Lambda'(fv, gw) = \Lambda(v, w)$ for all $v \in V$ and $w \in W$. This is a morphism of dissimilarities, since $\Lambda(v, w) \leq \Lambda(v, w)$. We need to show that (f^{-1}, g^{-1}) also is a morphism. Since f and g are bijective, then all $v' \in V'$ and all $w' \in W'$ is the image of some element v and w under f and g respectively, so $\Lambda(f^{-1}v', g^{-1}w') = \Lambda(f^{-1} \circ f(v), g^{-1} \circ g(w)) = \Lambda(v, w) = \Lambda'(fv, gw) = \Lambda'(v', w')$, and in particular $\Lambda(f^{-1}v', g^{-1}w') \leq \Lambda'(v', w')$ for all v' and w'. □

The next step is to show that the maps F, $N_<$ and $N_\leq$ are in some way functors between the two categories.

Proposition 2.2.5. *Let $F : \texttt{fsCx} \to \texttt{Diss}$ be the map $K \mapsto FK$ on objects and $\phi \mapsto (\phi, \phi)$ on morphisms. Then F is a functor.*

Proof. Recall from (2.1) that if we start with a filtered simplicial complex (K, V) then $FK : V \times P(V) \to \hat{\mathbb{R}}_+$ is the Dowker dissimilarity given by

$$FK(v, \sigma) = \begin{cases} \infty & \text{if } v \notin \sigma \\ \infty & \text{if } \sigma \notin \bigcup_{t \in \mathbb{R}_+} K_t \\ \inf\{t | \sigma \in K_t\} & \text{otherwise} \end{cases}$$

We need to show that given a morphism $\phi : K \to K'$ in $\texttt{fsCx}$ then $(\phi, \phi) : FK \to FK'$ is also a morphism. Specifically we need to show that if $\phi : V \to V'$ is such that $\phi(K_t) \subseteq K'_t$ then $FK'(\phi(v), \phi(\sigma)) \leq FK(v, \sigma)$ for all $v \in V$ and all $\sigma \in P(V)$.

First we look at the cases when $FK'(\phi(v), \phi(\sigma)) = \infty$. One way this can happen is if $\phi(v) \notin \phi(\sigma)$, but if this is the case then $v \notin \sigma$ by the fact that ϕ acts on vertices, and thus $FK(v, \sigma) = \infty$. The other way this can happen is if $\phi(\sigma) \notin K'_t$ for all $t \in \mathbb{R}_+$. Then by the assumption that ϕ is a morphism in $\texttt{fsCx}$, we have $\sigma \notin K_t$ for all $t \in \mathbb{R}_+$. So for both cases we have $FK(v, \sigma) = \infty$.

Next we look at when $FK'(\phi(v), \phi(\sigma))$ is finite. If $FK(v, \sigma)$ is infinite we are done. If it is finite then $v \in \sigma$ and there is a $t \in \mathbb{R}_+$ such that $\sigma \in K_t$. Let $s \in \{t \,|\, \sigma \in K_t\}$. Again since ϕ is a morphism we have $\phi(\sigma) \in K'_s$, and thus $\{t \,|\, \sigma \in K_t\} \subseteq \{t \,|\, \phi(\sigma) \in K'_t\}$. The infimum of a subset is greater than or equal to the infimum of its superset, so we have $FK'(\phi(v), \phi(\sigma)) = \inf\{t \,|\, \phi(\sigma) \in K'_t\} \leq \inf\{t \,|\, \sigma \in K_t\} = FK(v, \sigma)$

Hence we have $FK'(\phi(v), \phi(\sigma)) \leq FK(v, \sigma)$ for all $v \in V$ and all $\sigma \in P(V)$, and so F sends morphisms to morphisms. Clearly $F(\mathrm{Id}_K) = F(\mathrm{Id}_V) = (\mathrm{Id}_V, \mathrm{Id}_V) = \mathrm{Id}_{FK}$ and $F(\psi \circ \phi) = (\psi \circ \phi, \psi \circ \phi) = (\psi, \psi) \circ (\phi, \phi) = F(\psi) \circ F(\phi)$, so F is a functor. $\square$

Proposition 2.2.6. *Let $N_* : \texttt{Diss} \to \texttt{fsCx}$ be the map $\Lambda \mapsto N_*\Lambda$ on objects and $(f, g) \mapsto f$ on morphisms. Then N_* is a functor for both $*$ being $\leq$ and $<$.*

Proof. As in the previous proposition we need to show that N_* sends morphisms to morphisms. Let $(f, g) : \Lambda \to \Lambda'$ be a morphism, where $\Lambda : V \times W \to \hat{\mathbb{R}}_+$ and $\Lambda' : V' \times W' \to \hat{\mathbb{R}}_+$. Look at the case when $*$ is strictly less than, $<$. The filtered simplicial complex $N_<\Lambda$ is, in (2.2), defined by

$$N_<\Lambda_t = \{\sigma \in P(V) \,|\, \exists w \in W \text{ s.th. } \Lambda(s, w) < t \ \forall s \in \sigma.$$

Now let $\sigma \in N_<\Lambda_t$, and let $w \in W$ be such that $\Lambda(s, w) < t$ for all $s \in \sigma$. Then by the fact that (f, g) is a morphism, we get in particular that there is an $g(w) \in W'$ such that $\Lambda'(f(s), g(w)) \leq \Lambda(s, w) < t$. for all $f(s) \in f(\sigma)$. So $f(\sigma) \in N_<\Lambda'_t$, and f is a morphism from $N_<\Lambda_t$ to $N_<\Lambda'_t$.

By changing all $<$ with $\leq$ the proof is still valid, so N_* sends morphisms to morphisms. N_* also preserve identities and composition, since $N_*(\mathrm{Id}_\Lambda) = N_*(\mathrm{Id}_V, \mathrm{Id}_W) = \mathrm{Id}_V = \mathrm{Id}_{N_*\Lambda}$ and $N_*((f,g) \circ (f',g')) = N_*(f \circ f', g \circ g') = f \circ f' = N_*(f,g) \circ N_*(f',g')$. Thus N_* is a functor for both $<$ and $\leq$. $\square$

We sum up what we did in the two previous proofs in a corollary.

Corollary 2.2.7.

*(i): If $(f,g) : \Lambda \to \Lambda'$ is a morphism in **Diss**, then $f : N_*\Lambda \to N_*\Lambda'$ is a morphism in **fsCx**.*

*(ii): If $\phi : K \to K'$ is a morphism in **fsCx**, then $(\phi,\phi) : FK \to FK'$ is a morphism in **Diss**.* $\square$

Next we want to use these functors within the framework of localizations (Section A.4) to create some interesting categories. In particular, we want to look at localizations of sets of morphisms in the category of filtered simplicial complexes, but the theory we developed in A.4 only works for small categories, which `fsCx` is not. We do however have some ways around this problem. One option would be to use a more sophisticated localization, but that is beyond the reach of this thesis. Instead, we can look at filtered simplicial complexes whose vertex set is a subset of some fixed universe, $V \subseteq U$, then since $K_t \subseteq P(V) \subseteq P(U)$, every family $\{K_t\}_{t \in \mathbb{R}_+}$ can be viewed as an element of the set $\mathbb{R}_+ \times P(U)$.

There might be other interesting subcategories of `fsCx` to look at, so the only restrictions we will look at is small subcategories which is nice in the following way.

Definition 2.2.8. *A subcategory $\mathcal{F}$ of **fsCx** is a **nice subcategory** if it has the property that if $K \in \mathcal{F}$ then $N_< FK \in \mathcal{F}$ and $N_\leq FK \in \mathcal{F}$.*

Note that `fsCx` is nice, but not small. Since the vertex set of $N_< FK$ and $N_\leq FK$ are the same as the vertex set of K, every subcategory created by some restriction on the vertex sets will be nice. In particular, both the subcategory of filtered simplicial complexes with finite vertex sets, and the subcategory where the vertex sets are subsets of some fixed set are nice subcategories, but only the second one is small.

Definition 2.2.9. *Define the full subcategories $\mathcal{SF}_<$ and $\mathcal{SF}_\leq$ of the some nice subcategory $\mathcal{F} \subseteq$ **fsCx**, with objects $Ob\,\mathcal{SF}_< = \{N_< FK \mid K \in Ob\,\mathcal{F}\}$, and $Ob\,\mathcal{SF}_\leq = \{N_\leq FK \mid K \in Ob\,\mathcal{F}\}$.*

Proposition 2.2.10. *$\mathcal{SF}_\leq$ is a reflective subcategory of $\mathcal{F}$ with reflective functor $N_\leq F$, and $\mathcal{SF}_<$ is a coreflective subcategory with coreflective functor $N_< F$.*

Proof. For the reflective case, we need to show that we have a natural (A.22) bijection $\mathrm{Hom}_{\mathcal{SF}_\leq}(N_\leq FK, N_\leq FK') \xrightarrow{\phi} \mathrm{Hom}_{\mathcal{F}}(K, N_\leq FK')$ for all objects K and K' in $\mathcal{F}$.

Given a morphism $k : K \to N_{\leq}FK'$, then applying the functors F and $N_{\leq}$ gives us a new morphism $N_{\leq}F(k) : N_{\leq}FK \to N_{\leq}FN_{\leq}FK' = N_{\leq}FK'$ where the last equality follows from 2.1.11(ii). From the definition of the functors, we have $N_{\leq}F(k) = k$ as functions on the vertex set.

Conversely, given a morphism $k : N_{\leq}FK \to N_{\leq}FK'$, we want to show that the function k on vertex sets also is a morphism from K to $N_{\leq}FK'$. So let $\sigma \in K_t$. By 2.1.7 we have $\sigma \in N_{\leq}FK_t$, and by the definition of morphisms in `fsCx` this again implies $k(\sigma) \in N_{\leq}FK'_t$. Thus $\sigma \in K_t$ implies $k(\sigma) \in N_{\leq}FK'_t$ and $k : K \to N_{\leq}FK'$ is a morphism.

We now have a bijection given by $\phi(k) = k$, where the naturality properties (A.22) follows trivially from the fact that both the left and right adjoint and the bijection is the identity on the vertex maps. So the inclusion $i : \mathcal{SF}_{\leq} \hookrightarrow \mathcal{F}$ has $N_{\leq}F$ as left adjoint.

To show that $\mathcal{SF}_{<}$ is a coreflective subcategory we want to find a natural bijection $\mathrm{Hom}_{\mathcal{F}}(N_{<}FK, K') \to \mathrm{Hom}_{\mathcal{SF}_{<}}(N_{<}FK, N_{<}FK')$.

Starting with a morphism $h : N_{<}FK \to K'$, then applying the functor $N_{<}F$ we get a new morphism $N_{<}F(h) = h : N_{<}FN_{<}FK = N_{<}FK \to N_{<}FK'$, again using 2.1.11$(ii)$.

The other way, starting with a morphism $h : N_{<}FK \to N_{<}FK'$, then $\sigma \in N_{<}FK_t$ implies $h(\sigma) \in N_{<}FK'_t$. From 2.1.7 we get that $N_{<}FK'_t \subseteq K'_t$, so $h(\sigma) \in K'_t$, and $h : N_{<}FK \to K'$ is a morphism.

We see again that the natural bijection is just the identity on the vertex set, and naturality follows trivially. □

Corollary 2.2.11. *If $\mathcal{F}$ is a nice and small subcategory of* `fsCx`*, then the closed subcategory $\mathcal{SF}_{\leq}$ of $\mathcal{F}$ is equivalent to the localization $\mathcal{F}[\Sigma^{-1}]$ with respect to the set of morphisms $\Sigma = \{\phi \in Mor\,\mathcal{F} \mid N_{\leq}F(\phi) \text{ is an iso}\}$.*

Proof. This is a direct consequence of the previous proposition and A.4.13. □

Definition 2.2.12. *Let $\mathcal{F}$ be any subcategory of* `fsCx`*. Define the category $\mathcal{CF}$ with objects $Ob\,\mathcal{CF} = Ob\,\mathcal{F}$, and where morphisms $\phi : K \to K'$ are morphisms of the associated dissimilarities $(\phi, \phi) : FK \to FK'$.*

Note that a morphism $\phi : K \to K'$ in $\mathcal{CF}$ is a function $\phi : V \to V'$ on the vertex sets, such that $FK'(\phi(v), \phi(\sigma)) \leq FK(v, \sigma)$ for all $v \in V$ and all $\sigma \in P(V)$.

Also note that if $\mathcal{F}' \subseteq \mathcal{F}$ is a subcategory, then $\mathcal{CF}'$ is a subcategory of $\mathcal{CF}$. In particular we have $\mathcal{CF} \subseteq \mathcal{C}($`fsCx`$)$ for all $\mathcal{F}$.

Definition 2.2.13. *Let $\mathcal{F}$ be any subcategory of* `fsCx`*. Define the category $\mathcal{WF}_{<}$ of $\mathcal{F}$, with objects $Ob\,\mathcal{WF}_{<} = Ob\,\mathcal{F}$ and the morphisms $K \to K'$ are morphisms $N_{<}FK \to N_{<}FK'$ as filtered simplicial complexes. Similarly define the $\mathcal{WF}_{\leq}$ with the same objects, but where morphisms $K \to K'$ are morphisms of filtered complexes $N_{\leq}FK \to N_{\leq}FK'$.*

A morphism $\phi : K \to K'$ in $\mathcal{WF}_{<}$, is a function on vertex sets $\phi : V \to V'$, such that $\phi(\sigma) \in N_{<}FK'_t$ whenever $\phi \in N_{<}FK_t$.

Proposition 2.2.14. *Let $\mathcal{F}$ be any subcategory of* **fsCx**. *The categories $\mathcal{CF}$, $\mathcal{WF}_<$ and $\mathcal{WF}_\leq$ are isomorphic. Furthermore this isomorphism is the identity on objects and on morphisms as functions on vertex sets, so $\phi : K \to K$ is an isomorphism in $\mathcal{CF}$ if and only if $\phi : K \to K'$ is an isomorphism in $\mathcal{WF}_<$, and similarly for $\mathcal{WF}_\leq$.*

Proof. Look at the map $\mathcal{CF} \to \mathcal{WF}_*$, for $*$ either $<$ or $\leq$, being the identity on objects and the identity on morphisms as vertex sets. This will send morphism to morphism by 2.2.7(i), and it preserves composition, identities and associativity in the same way as the functor in 2.2.6.

The other way, let $\mathcal{WF}_* \to \mathcal{CF}$ send objects to themselves and morphisms to the morphism acting the same on vertex sets. A morphism $\phi : K \to K'$ in $\mathcal{WF}_*$, is a morphism of filtered simplicial complexes $\phi : N_*FK \to N_*FK'$. Using 2.2.7(ii) we get a morphism of dissimilarities $(\phi, \phi) : FN_*FK \to FN_*FK'$, and by 2.1.11($ii$) this is a morphism $(\phi, \phi) : FK \to FK'$, so $\phi : K \to K'$ is a morphism in $\mathcal{CF}$. This map preserves composition, identities and associativity by 2.2.5.

Clearly the compositions of these functors are the identity functors, so $\mathcal{CF}$ and $\mathcal{WF}_*$ are isomorphic categories, both when $*$ is $<$ and when it is $\leq$. □

By 2.2.4, two filtered complexes K and K' are isomorphic in $\mathcal{CF}$ if and only if there exist a bijection $\phi : V \to V'$ between their vertex sets, such that $FK(v, \sigma) = FK'(\phi(v), \phi(\sigma))$. This is close to the properties in 2.1.10, but where one side is precomposed with ϕ. We will now extend the definition of strictly 0-interleaved complexes to an equally (or more) interesting class which includes complexes with different vertex sets.

Definition 2.2.15. *Let (K, V) and (K', V') be filtered simplicial complexes and let $\phi : V \to V'$ be any function. Then the* ***image*** *of K by ϕ, denoted by $\phi(K)$, is the filtered simplicial complex with vertex set V' and simplices*

$$\phi(K)_t := \phi(K_t) = \{\tau \in K'_t \,|\, \tau = \phi(\sigma) \text{ for some } \sigma \in K_t\} \tag{2.6}$$

To show that $\phi(K)$ is a filtered simplicial complex, we first need to show that $\phi(K)_t$ is a simplicial complex for every t. Let $\tau = \phi(\sigma)$ for some $\sigma \in K_t$ and let τ' be any subset of τ. Now since ϕ is a defined on vertices we have $\phi^{-1}(\tau') \cap \sigma \subseteq \phi^{-1}(\tau) \cap \sigma = \sigma \in K_t$, and so $\phi^{-1}(\tau') \cap \sigma$ is a simplex in K_t. Now $\tau' = \phi(\phi^{-1}(\tau') \cap \sigma)$, so τ' is in $\phi(K)_t$, and $\phi(K)_t$ is a simplicial complex.

We also need $\phi(K_t) \subseteq \phi(K_{t'})$ whenever $t \leq t'$. If $\tau \in \phi(K_t)$, then there is a $\sigma \in K_t$ with $\phi(\sigma) = \tau$, but K is filtered so $\sigma \in K_{t'}$ and thus $\tau \in \phi(K_{t'})$.

Lemma 2.2.16. *Let (K, V) and (K', V') be filtered simplicial complexes and $\phi : V \to V'$ a bijection. Then $FK \circ \phi^{-1} = F(\phi(K))$*

Proof. Let $v' \in V'$ and $\tau \in P(V')$ be any elements.

We first start with the infinite case. Recall from the definition of F in (2.1) that $F(\phi(K))(v', \tau) = \infty$ if and only if $v' \notin \tau$ or $\tau \notin \phi(K)_t$ for every

$t \in \mathbb{R}_+$. Since ϕ is a bijection by assumption, we have that $v' \notin \tau$ if and only if $\phi^{-1}(v') \notin \phi^{-1}(\tau)$. By (2.6), τ is not in $\phi(K)_t$ if and only if it is not the image of some σ in K_t, or since we have a bijection $\phi^{-1}(\tau) \neq \sigma$ for any $\sigma \in K_t$, which is the case if and only if ϕ^{-1} is not a simplex in $\notin K_t$. So we have $F\phi(K)(v', \tau) = \infty$ if and only if $\phi^{-1}(v') \notin \phi^{-1}(\tau)$ or $\phi^{-1}(\tau) \notin K_t$ for all $t \in \mathbb{R}_+$, which is exactly when $FK(\phi^{-1}(v'), \phi^{-1}(\tau)) = \infty$.

We have just shown that $\tau \notin \phi(K_t)$ if and only if $\phi^{-1}(\tau) \notin K_t$, so by contraposition $\tau \in \phi(K_t)$ if and only if $\phi^{-1}(\tau) \in K_t$. Therefore in the finite case we have $\{t \,|\, \tau \in \phi(K_t)\} = \{t \,|\, \phi^{-1}(\tau) \in K_t\}$, and so $F\phi(K)(v', \tau) = \inf\{t \,|\, \tau \in \phi(K_t)\} = \inf\{t \,|\, \phi^{-1}(\tau) \in K_t\} = FK(\phi^{-1}(v'), \phi^{-1}(\tau))$. □

Corollary 2.2.17. *Let (K, V) and (K', V') be filtered simplicial complexes. Then the following are equivalent:*

*(i): K and K are isomorphic in $\mathcal{CF}$ for every subcategory $\mathcal{F} \subseteq$ **fsCx** containing both K and K'.*

*(ii): K and K are isomorphic in $\mathcal{CF}$ for some subcategory $\mathcal{F} \subseteq$ **fsCx** containing both K and K'.*

*(iii): K and K' are isomorphic in $\mathcal{C}($**fsCx**$)$.*

(iv): There exists a bijection $\phi : V' \to V$ such that $K_t \subseteq \phi(K'_{t+\varepsilon})$ and $\phi(K'_t) \subseteq K_{t+\varepsilon}$ for all $t \in \hat{\mathbb{R}}_+$ and all $\varepsilon > 0$.

(v): K and $\phi(K')$ are strictly 0-interleaved for some bijection $\phi : V' \to V$.

Proof.

(*iv*) $\iff$ (*v*): The statement (*iv*) with the fact that $\phi(K')_t := \phi(K'_t)$ is exactly the definition of K and $\phi(K')$ being strictly 0-interleaved.

(*v*) $\iff$ (*i*): Assume there exists a bijection $\phi : V' \to V$ on vertex sets such that K and $\phi(K')$ are strictly 0-interleaved. This is by definition equivalent to the statement (*iii*) in 2.1.10, namely that $FK = F\phi(K')$. By using 2.2.16 we have that this is equivalent to $FK = FK' \circ \phi^{-1}$. Isomorphisms $(K, V) \to (K', V')$ in $\mathcal{CF}$ are bijections $\phi' : V \to V'$ such that $FK = FK' \circ \phi'$. So $\phi^{-1} : K \to K'$ is an isomorphism in $\mathcal{CF}$.

(*i*) $\implies$ (*iii*) $\implies$ (*ii*): Trivially true.

(*ii*) $\implies$ (*i*): Let (K, V) and (K', V') be objects in some subcategory $\mathcal{F}$ of `fsCx`, so that they are isomorphic in $\mathcal{CF}$. Let $\mathcal{F}'$ be another subcategory containing K and K'. The isomorphism in $\mathcal{CF}$ is a bijection $\phi : V \to V'$, such that $FK = FK' \circ \phi$, but this definition is independent of $\mathcal{F}$, so this is also an isomorphism in $\mathcal{CF}'$. □

Definition 2.2.18. *Define* **0-int** $:= \mathcal{C}($**fsCx**$)$ *as the* ***category of 0-interleaved filtered simplicial complexes****. We say K and K' are* ***0-interleaved*** *if they are isomorphic in* **0-int** *(and thus satisfy every property in 2.2.17), and the isomorphism is called a* ***0-interleaving****.*

In particular, by 2.2.17, a 0-interleaving $\phi : K \to K'$ is a bijection on vertex sets $\phi : V \to V'$ such that $\phi(K_t) \subseteq K'_{t+\varepsilon}$ and $K'_t \subseteq \phi(K_{t+\varepsilon})$ for all $t \in \mathbb{R}_+$ and all $\varepsilon > 0$.

The category $\mathcal{CF}$ is a subcategory of `0-int` for every subcategory $\mathcal{F} \subseteq$ `fsCx`. So two filtered simplicial complexes $K, K' \in \mathcal{F}$ are 0-interleaved if and only if they are isomorphic in $\mathcal{CF}$. We call $\mathcal{CF}$ the **category of 0-interleaved complexes in $\mathcal{F}$**.

From 2.2.14 we have another way of looking at the category $\mathcal{CF}$. We will now use $\mathcal{WF}_*$ to show that the category of 0-interleaved complexes in $\mathcal{F}$ is equivalent to both the reflective and coreflective subcategories from 2.1.6.

Lemma 2.2.19. *If $\mathcal{F}$ is a nice subcategory of* `fsCx`*, then the categories $\mathcal{CF}$ $\mathcal{SF}_<$, $\mathcal{WF}_<$, $\mathcal{SF}_\leq$, and $\mathcal{WF}_\leq$ are all equivalent.*

Proof. We already known from 2.2.14 that $\mathcal{WF}_\leq$, $\mathcal{WF}_<$ and $\mathcal{CF}$ are isomorphic, so we just need to show that $\mathcal{SF}_*$ and $\mathcal{WF}_*$ are equivalent whenever $*$ is $<$ and $\leq$.

Remember that morphisms $\phi : (K, V) \to (K', V')$ in $\mathcal{WF}_*$ are morphisms of filtered simplicial complexes $\phi : N_*FK \to N_*FK'$, which again are some function $\phi : V \to V'$ on vertex sets. We have an injective functor $\mathcal{SF}_* \hookrightarrow \mathcal{WF}_*$ sending objects N_*FK to N_*FK, and morphisms $\phi : N_*FK \to N_*FK'$ to the morphism of filtered simplicial complexes $\phi : N_*FN_*FK \to N_*FN_*FK'$, which by 2.1.11 is just the original morphism $\phi : N_*FK \to N_*FK$. So $\mathcal{SF}_*$ are in some way included in $\mathcal{WF}_*$.

The other way we have a functor $\mathcal{WF}_* \to \mathcal{SF}_*$ sending objects $K \mapsto N_*FK$, and morphisms $\phi : N_*FK \to N_*FK'$ in $\mathcal{SF}_*$ to the morphisms of filtered simplicial complexes $\phi : N_*FN_*FK \to N_*FN_*FK'$, which again is the original morphism by 2.1.11.

The composition $\mathcal{SF}_* \hookrightarrow \mathcal{WF}_* \to \mathcal{SF}_*$ is now the identity, using $N_*FK = N_*FN_*FK$. The other composition $\mathcal{WF}_* \to \mathcal{SF}_* \hookrightarrow \mathcal{WF}_*$, lets call it $D_* : \mathcal{WF}_* \to \mathcal{WF}_*$, maps morphisms $\phi : K \to K'$ in $\mathcal{WF}_*$ to morphisms $\phi : N_*FK \to N_*FK'$. For every object (K, V) in $\mathcal{WF}_*$, the identity map on vertices induce an isomorphism of filtered simplicial complexes $\mathrm{Id}_V : N_*FK \to N_*FK' = N_*FN_*FK$, which corresponds to an isomorphisms $\mathrm{Id}_V : K \to N_*FK$ in $\mathcal{WF}_*$. Since $D_*(\phi) = \phi$ on vertex sets, we have the commuting diagram

$$\begin{array}{ccc} K & \xrightarrow{\mathrm{Id}_V} & (D_*)(K) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle (D_*)(\phi)} \\ K' & \xrightarrow{\mathrm{Id}_{V'}} & (D_*)(K'). \end{array}$$

This defines a natural isomorphism between D_* and $\mathrm{Id}_{\mathcal{WF}_*}$. So the composition are the identity functor one way, and naturally isomorphic to the identity the other way. Hence we have an equivalence of categories $\mathcal{WF}_* \to \mathcal{SF}_*$. $\square$

In particular, when $\mathcal{F} = \texttt{fsCx}$, we have that the category of 0-interleaved complexes `0-int` is equivalent to both a reflective and a coreflective subcategory of `fsCx`.

Finally we will see that the 0-interleaved complexes with objects in $\mathcal{F}$ are equivalent to the category $\mathcal{F}$ localized at the set of 0-interleavings.

Lemma 2.2.20. *Let $\mathcal{F}$ be any nice subcategory of $\texttt{fsCx}$, and let $C_* : \mathcal{F} \to \mathcal{WF}_*$ be the functor which is the identity on objects and sending morphisms $\phi : K \to K'$ to the morphisms $\phi : K \to K'$ in $\mathcal{WF}_*$ given by $N_*F(\phi) = \phi : N_*FK \to N_*FK'$. Then the set $\Sigma_* := \{\phi \in Mor\,\mathcal{F} \,|\, C_*(\phi) \text{ is an iso}\}$ is the same as the set $\Sigma = \{\phi \in Mor\,\mathcal{F} \,|\, N_{\leq}F(\phi) \text{ is an iso}\}$ used in 2.2.11, i.e. $\Sigma_< = \Sigma = \Sigma_{\leq}$.*

Proof. The functorial properties of C_* comes from the fact that $N_*F : \mathcal{F} \to \mathcal{F}$ is a functor (2.2.5 and 2.2.6).

Let $\phi \in \mathrm{Mor}\,\mathcal{F}$. Then by 2.2.14, the map $C_<(\phi)$ is an isomorphism in $\mathcal{WF}_<$ if and only if $C_<(\phi)$ is an isomorphism in $\mathcal{CF}$ if and only if $C_<(\phi)$ is an isomorphism in $\mathcal{WF}_{\leq}$, and similarly for $C_{\leq}(\phi)$. Thus we have $\Sigma_< = \Sigma_{\leq}$.

Morphisms $\phi : K \to K'$ in $\mathcal{WF}_{\leq}$, are morphisms $\phi : N_{\leq}FK \to N_{\leq}FK'$ in $\mathcal{F}$. So the image $C_{\leq}(\phi) = N_{\leq}F(\phi) : K \to K'$ is an isomorphism if and only if $\phi = N_{\leq}F(\phi) : N_{\leq}FK \to N_{\leq}FK'$ is an isomorphism of filtered simplicial complexes. Thus we have the equality $\Sigma_{\leq} = \Sigma$. □

Corollary 2.2.21. *Let $\mathcal{F}$ be a small and nice subcategory of filtered simplicial complexes. The subcategory of 0-interleaved complexes $\mathcal{CF} \subseteq \texttt{0-int}$ with objects in $\mathcal{F}$ is equivalent to the localization $\mathcal{F}[\Sigma^{-1}]$ at the set $\Sigma = \{\phi : (K, V) \to (K', V') \,|\, \phi : V \to V' \text{ is a 0-interleaving}\}$.*

Proof. We know from 2.2.19 that the category $\mathcal{CF}$ is equivalent to the $\mathcal{SF}_{\leq}$ which by 2.2.11 is equivalent to the localization $\mathcal{F}[\Sigma^{-1}]$ at the set of morphisms $\Sigma = \{\phi \in \mathrm{Mor}\mathcal{F} \,|\, N_{\leq}F(\phi) \text{ is an iso. in } \mathcal{F}\}$. By 2.2.20, we get $\Sigma = \{\phi \in \mathrm{Hom}_{\mathcal{F}}(K, K') \,|\, N_{\leq}F(\phi) = \phi : K \to K' \text{ is an isomorphism in } \mathcal{WF}_*\}$, but from 2.2.14 we know that $\phi : K \to K'$ is an isomorphism in $\mathcal{WF}_*$ if and only if it is an isomorphism in $\mathcal{CF}$ which is true if and only if it is a 0-interleaving. □

The localization of a category $\mathcal{C}$ at Σ has the same objects as $\mathcal{C}$, but it adds extra inverse morphisms to every element in Σ making them isomorphisms. So the category of 0-interleavings is the same as the category of filtered simplicial complexes, but with extra inverse morphisms for the 0-interleavings.

Starting with a filtered simplicial complex, we get a unique persistence diagram by the homology functor [7]. This functor sends 0-interleaved simplicial complexes to persistence diagrams with bottleneck distance 0 [1]. We say the diagrams are 0-matched. It would be interesting to see if we could find some analogous category of 0-matched persistence diagrams, as a localization maybe in the form of some diagram

$$\begin{array}{ccc} \texttt{fsCx} & \xrightarrow{H(-)} & \texttt{PersDiag} \\ \downarrow{\scriptstyle P_\Sigma} & & \downarrow{\scriptstyle P_{\Sigma'}} \\ \texttt{0-int} & \xrightarrow{H(-)} & \texttt{0-match}. \end{array}$$

One way of representing persistence diagrams is as a multiset of intervals, called a barcode diagram. One obvious guess is that the corresponding localization `0-match` would identify all intervals with the same endpoints, be it open, closed or half-open intervals.

In general, two filtered simplicial complexes (K, V) and (K', V') are called ε**-interleaved** if there is a bijection on vertex sets $\phi : V \to V'$ such that $\phi(K_t) \subseteq K'_{t+\delta}$ and $K'_t \subseteq \phi(K_{t+\delta}$ for all $t \in \mathbb{R}_+$ and all $\delta > \varepsilon$. One might want to extend the category of 0-interleavings to a general ε. This will probably not work without problems, since being ε-interleaved is not an equivalence class, as it is not transitive. All we know is that if K and K' are interleaved by ε, and if K' and K'' by ε', then K and K'' are $(\varepsilon + \varepsilon')$-interleaved. Trying to do the same for a general ε, in the way we have done it for 0, we would end up with a composition of isomorphisms that is not an isomorphism, so something different would be needed.

Appendix A

Appendix

A.1 Colimit Diagrams

In this section we will go through the basic definitions and results concerning limits and colimits. This is used in section 1.3 and beyond, as we use it to define geometric realizations. Much of this section is picked from [11].

We start by defining the diagonal functor and a universal morphism, which we will combine to define colimits. ([11], III,3).

Definition A.1.1. *The **diagonal functor*** $\Delta : \mathcal{C} \to Func(J,\mathcal{C})$ *of a small category* J *and a category* $\mathcal{C}$ *is the functor sending objects* $c \in \mathcal{C}$ *to the constant functor* $\Delta c(j) = c$ *and* $\Delta c(f) = Id_c$ *for all* $j \in Ob\,J$ *and* $f \in Mor\,J$*, and morphisms* $f : c \to d$ *to the natural transformation* $\Delta c \xrightarrow{\Delta f} \Delta d$ *given by:*

$$\begin{array}{ccc} \Delta c(i) & \xrightarrow{f} & \Delta d(i) \\ {\scriptstyle \Delta c(g)=Id_c}\big\downarrow & & \big\downarrow{\scriptstyle \Delta d(g)=Id_d} \\ \Delta c(j) & \xrightarrow{f} & \Delta d(j) \end{array}$$

where $g : i \to j$ *is any morphism in* J*.*

Definition A.1.2. *Let* $S : \mathcal{D} \to \mathcal{C}$ *be a functor and* c *and object in* $\mathcal{C}$*. A **universal morphism** from* c *to* S *is a pair* (r, u) *where* r *is an object in* $\mathcal{D}$ *and* $u : c \to Sr$ *is a morphism in* $\mathcal{C}$ *such that for any other such pair* $(d \in Ob\,\mathcal{D}, f : c \to Sd)$ *there is a unique morphism* $f' : r \to d$ *where* $Sf' \circ u = f$*.*

Lemma A.1.3. *If* (r, u) *and* (r', u') *are two universal morphisms from* c *to* S*, then* r *and* r' *are isomorphic in* $\mathcal{D}$*, and the isomorphism is the unique map given by the property of universal morphisms.*

Proof. By applying the definition both ways we get two unique morphisms $f : r \to r'$ and $f' : r' \to r$ such that $Sf \circ u = u'$ and $Sf' \circ u' = u$. In particular we have

$$u' = Sf \circ Sf' \circ u' = S(f \circ f') \circ u'. \tag{A.1}$$

Now $(r', S(f \circ f') \circ u')$ is a pair such that $r' \in \operatorname{Ob} \mathcal{D}$ and $S(f \circ f') \circ u' : c \to Sr'$, so there is a unique morphism $g : r' \to r'$ with $S(g) \circ u' = S(f \circ f') \circ u'$. From (A.1) we see that this is true both for $g = \mathrm{Id}_{r'}$ and for $g = f \circ f'$ which both are morphisms in $\mathcal{D}$. By the uniqueness of g we have $\mathrm{Id}_{r'} = f \circ f'$, and symmetrically we can show that $\mathrm{Id}_r = f' \circ f$. □

Combining the definitions above we get the definition of the colimit of a functor.

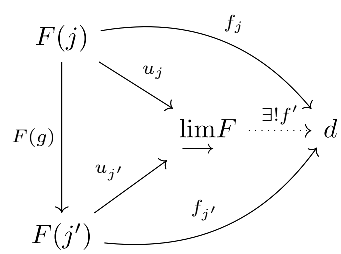
Definition A.1.4. *Let J be a small category, $\mathcal{C}$ a category, $F \in Func(J, \mathcal{C})$ and let $\Delta : \mathcal{C} \to Func(J, \mathcal{C})$ be the diagonal functor of J and $\mathcal{C}$. Then a universal morphism (r, u) from F to Δ is called a* ***colimit diagram*** *for F.*

The object $r \in \operatorname{Ob} \mathcal{C}$ is called the **colimit** of F and is denoted $r = \varinjlim F$ or $r = \varinjlim_j F(j)$. The morphism $u : F \to \Delta(\varinjlim F)$ is a natural transformation, i.e. a collection of morphisms $\{u_j : F(j) \to \Delta(\varinjlim F)(j) = \varinjlim F\}_{j \in J}$, called the **maps associated with** $\varinjlim F$. They have the following two properties:

(a) For all morphisms $g : j \to j'$, we have $u_j = u_{j'} \circ F(g)$. We say $(\varinjlim F, \{u_j\})$ is a **cocone** of F.

(b) For any cocone of F, i.e. every pair $(d \in \operatorname{Ob} \mathcal{C}, \{f_j : F(j) \to d\}_{j \in J})$ where $f_j = f_{j'} \circ F(g)$ for any morphism $g : j \to j'$, we have a unique morphism $f' : \varinjlim F \to d$ in $\mathcal{C}$ such that $f' \circ u_j = f_j$ for all $j \in J$. We say $(\varinjlim F, \{u_j\}_{j \in J})$ is **universal** over every cocone, and the morphism f' is **given by the universal property**.

So we get commuting diagrams for all g, and all cocones $(d, \{f_j\})$:



(A.2)

Note that the colimit is unique up to isomorphism. By A.1.3 we have that if (r, u) and (r', u') both are colimit diagrams for F then r and r' are isomorphic in $\mathcal{C}$, and the isomorphism between them is given by the universal property.

Putting some extra restraints on on the category J, we get some additional nice properties of the colimits. One condition we will have in our work with geometric realization of simplicial set, colimits when J is filtered ([11] XI.1):

Definition A.1.5. *A non-empty category J is* ***filtered*** *if the following are satisfied:*

(a) For any two objects i and j in J there is a third object k with morphisms $i \to k$ and $j \to k$.

(b) For any two arrows $u, v : i \to j$ there is an object k in J and a morphism $w : j \to k$ such that $wu = wv$.

Another property we will look at is when the image of every morphism under the functor of which we take our colimit is injective. With these two conditions we get some nice properties of the maps associated with the colimit, whenever the functor goes into the category of sets.

Lemma A.1.6. *Let J be a small filtered category and let $F : J \to$ **`Sets`** be a functor such that $F(\alpha)$ is injective for all $\alpha : j \to j'$. Then all the maps $u_j : F(j) \to \varinjlim F$ associated with $\varinjlim F$ are injective.*

Proof. We look at the quotient $F_\sim := \coprod_{j \in J} F(j)/\sim$ where if $x \in F(j)$ and $y \in F(j')$ then $x \sim y$ if and only if for some object k in J there are morphisms $u : j \to k$ and $u' : j' \to k$ such that $F(u)x = F(u')y$.

Let $v_j : F(j) \to F_\sim$ be the map sending $x \in F(j)$ to its class $[x]$. For any $x \in F(j)$ and $\alpha : j \to j'$, since F is a functor, we get the equality $F(\alpha)x = F(\mathrm{Id}_{j'})F(\alpha)x$ where $\mathrm{Id}_{j'} : j' \to j'$ is the identity. By the definition of the equivalence relation we get that $x \sim F(\alpha)x$ and so $v_j(x) = [x] = [F(\alpha)x] = v_{j'} \circ F(\alpha)(x)$. Since x and α are arbitrary we have that $(F_\sim, \{v_j\})$ is a cocone of F, and there is a unique function $f : \varinjlim F \to F_\sim$ such that $v_j = f \circ u_j$ for all $j \in J$. Now to show that u_j in injective it is enough to show that v_j is injective.

Let $x, y \in F(j)$ and assume $v_j(x) = v_j(y)$. By the definition of the equivalence this is true if and only if for some $k \in \mathrm{Ob}\, J$ there are $u, v : j \to k$ such that $F(u)x = F(v)y$. Now using property (b) of filtered categories there is a morphism w from k to some other k' in J such that $wu = wv$. Composing with $F(w)$ we get $F(wu)x = F(wv)y$, and since all $F(\alpha)$ are injective we have $x = y$. Thus the v_j are all injective and so are the u_j. □

Lemma A.1.7. *Let J be small and filtered, and let $F : J \to$ **`Sets`** be a functor. Then every element $s \in \varinjlim F$ is in the image of some map u_j associated with the colimit.*

Proof. We will show that $(F_\sim, \{v_j\})$ we defined in the previous proof is a colimit diagram, and thus the map $f : \varinjlim F \to F_\sim$ is an isomorphism by A.1.3. Let $(d, \{f_j\})$ be a cocone of F. Define the map $g : F_\sim \to d$ by $f_j = g \circ v_j$, so if $x \in F(j)$ then $g([x]) = f_j(x)$. Now if g is well-defined map on the entire $F_\sim$ then it is clearly the unique map where $f_j = g \circ v_j$ for all $j \in J$.

To show that it is well-defined, let $x \sim y$ where $x \in F(j)$ and $y \in F(j')$. We want to show that $f_j(x) = f_{j'}(y)$. The equivalence gives an object k in J and morphisms $u : j \to k$ and $u' : j' \to k$ such that $F(u)x = F(u')y$. Now since $(d, \{f_j\})$ is a cocone of F we have that $f_j = f_k \circ F(u)$ and $f_{j'} = f_k \circ F(u')$, so in particular

$$f_j(x) = f_k \circ F(u)x = f_k \circ F(u')y = f_{j'}(y).$$

Finally we need to show that the function g is defined on the entire $F_\sim$,, but clearly every element $[x] \in F_\sim$ is represented by some $x \in F(j)$ for some j and thus $g([x]) = g \circ v_j(x) = f_j(x)$.

So $(F_\sim, \{v_j\})$ is indeed a colimit diagram, and $f : \varinjlim F \to F_\sim$ is an isomorphism. If $s \in \varinjlim F$, then $f(s) \in F_\sim$ is some equivalence class, say $f(s) = [x]$ where $x \in F(j)$. By A.1.3 we have that f^{-1} is the unique map such that $u_j = f^{-1} \circ v_j$, and so

$$u_j(x) = f^{-1} \circ v_j(x) = f^{-1}([x]) = f^{-1}(f(s)) = s$$

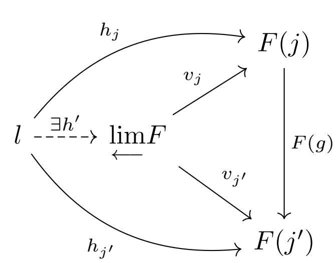
Thus s is in the image of u_j. Since s was arbitrary, we get our result. □

We immediately get the following corollary:

Corollary A.1.8. *Let J be a small filtered category and let $F : J \to$ **Sets** be a functor. Then every pair or elements $s, r \in \varinjlim F$ is in the image of some u_j.*

Proof. Let $s, r \in \varinjlim F$ be any two elements. Using the A.1.7, we have $s = u_j(x)$ and $r = u_{j'}(y)$ for some $x \in F(j)$ and some $y \in F(j')$. Since J is filtered we have $k \in J$ with morphisms $u : j \to k$ and $u' : j' \to k$, and by the cocone property we have $u_j = u_k \circ F(u)$ and $u_{j'} = u_k \circ F(u')$. So $s = u_j(x) = u_k(F(u)x)$ and $r = u_{j'}(y) = u_k(F(u')y)$, thus $s, r \in \operatorname{Im} u_k$. □

The dual notion of a colimit diagram, is unsurprisingly called a limit diagram. A **limit diagram** for $F : J \to \mathcal{C}$, where J is small, is an object $\varprojlim F \in \operatorname{Ob} \mathcal{C}$ together with a collection of morphisms $\{v_j : \varprojlim F \to F(j)\}_{j \in \operatorname{Ob} J}$ which is a **cone** over F, i.e. for all morphisms $g : j \to j'$ in J we have $F(g) \circ v_j = v_{j'}$, and which is **universal** in the sense for any other cone $(l, \{h_j : l \to F(j)\}_{j \in \operatorname{Ob} J})$ we get a unique morphism $h' : l \to \varprojlim F$ in $\mathcal{C}$ where $h_j = v_j \circ h'$. Thus we have the following commuting diagram for all g:



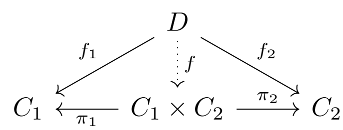
(A.3)

Limits can be defined from universal arrows ([11] III), so limit diagrams are also unique up to unique isomorphism by A.1.3.

A special case of a limit diagram is the product. If `Dis(n)` is the discrete category with n objects $\{1, \ldots, n\}$ and n morphisms $\{\mathrm{Id}_1, \ldots, \mathrm{Id}_n\}$, then a functor $F : \texttt{Dis(n)} \to \mathcal{C}$ is uniquely determined by $F(i) = C_i$. We will look at $n = 2$.

Definition A.1.9. *Let $\mathcal{C}$ be a category with objects C_1 and C_2. A **product** of C_1 and C_2 is a limit diagram for the functor $H : \texttt{Dis}(2) \to \mathcal{C}$ sending $H(i)$ to C_i for $i = 1, 2$. We write $\varprojlim H = C_1 \times C_2$.*

So a product is an object $C_1 \times C_2$ in $\mathcal{C}$, unique up to isomorphism, together with two morphisms $\pi_i : C_1 \times C_2 \to C_i$ for $i = 1, 2$ such that for any other object D in $\mathcal{C}$ with morphisms $f_i : D \to C_i$ there is a unique morphism $f : D \to C_1 \times C_2$ making the following diagram commute:



(A.4)

We can in a similar fashion define products of n elements by looking at functors from `Dis(n)`. We can also define **coproducts** written $\amalg_i C_i$ by looking at colimits of such functors. All of this is described in more details in [11] III.

As an example let S_1 and S_2 be any two sets. We have the Cartesian product $S_1 \times S_2 = \{(s_1, s_2) \,|\, s_1 \in S_1,\, s_2 \in s_2\}$ together with the usual projection maps of S_1 and S_2 sending (s_1, s_2) to s_1 and s_2 respectively. These give a product in the category `Sets`, for if R is any set with functions $f_i : R \to S_i$ for $i = 1, 2$, then the function $(f_1, f_2) : R \to S_1 \times S_2$ sending $r \in R$ to $(f_1(r), f_2(r))$ is the unique function that makes everything commute. The coproduct $S_1 \amalg S_2$ of sets is the disjoint union.

What follows are some results concerning limits and colimits, which we use throughout the thesis.

Theorem A.1.10. *The limit and colimit exists for any functor $F : J \to \texttt{Sets}$ where J is any small category.*

Proof. [11], V.1. Thm. 1 and Ex. 8. □

Theorem A.1.11. *The colimit exists for any functor $F : J \to \texttt{Top}$ where J is any small category. The set of the colimit is the colimit of the functor $F : J \to \texttt{Sets}$, and the topology is the finest topology making the maps associated with this colimit all continuous.*

Proof. Let $(\varinjlim F, \{u_F\})$ be a colimit diagram of the functor $F : J \to \texttt{Sets}$, and let τ be the finest topology on $\varinjlim F$ such that $u_F : F(j) \to \varinjlim F$ are continuous for all objects j in J. Let $(d, \{f_j\})$ be any cocone of the functor $F : J \to \texttt{Top}$. As sets, the colimit property of $\varinjlim F$ gives a commuting diagram as the one in (A.2). We have a function $f' : \varinjlim F \to d$, such that $f' \circ u_j = f_j$.

Assume by contradiction that f' is not continuous, so there is an open set $U \subseteq d$ such that $V = f'^{-1}(U)$ is not open. The collection $\{f_j\}$ is a collection of morphisms in $\texttt{Top}$, so f_j are continuous for all j. In particular $f_j^{-1}(U) = u_j^{-1}(f'^{-1}(U)) = u_j^{-1}(V)$ are open for all j, The topology on $\varinjlim F$ was the finest topology such that u_j are continuous, but we can now add V making it even finer. Thus f' is continuous, and $\varinjlim F$ with the finest topology is a colimit of $F : J \to \texttt{Top}$.

□

Theorem A.1.12. *Let J be a filtered small category, P a finite category and $F : P \times J \to \textit{Sets}$ a functor, where the product is in the category of small categories. Then we have a natural bijection*

$$\varinjlim_j \varprojlim_p F(p,j) \to \varprojlim_p \varinjlim_j F(p,j)$$

Proof. [11], IX.2 Thm. 1. The naturality is given by the diagram:

$$\begin{array}{ccccc} F(p,j) & \xleftarrow{v_{p,j}} & \varprojlim_p F(p,j) & \xrightarrow{u_j} & \varinjlim_j \varprojlim_p F(p,j) \\ \downarrow{\scriptstyle u_{p,j}} & & \downarrow & & \downarrow \\ \varinjlim_j F(p,j) & \xleftarrow{v_p} & \varprojlim_p \varinjlim_j F(p,j) & = & \varprojlim_p \varinjlim_j F(p,j) \end{array} \tag{A.5}$$

□

Theorem A.1.13. *Let $\mathcal{C}$ be a category where colimits exists for all functors $F : J \to \mathcal{C}$, whenever J is small. Let $H : J \times J' \to \mathcal{C}$ be a functor, where J and J' are small, then there is an isomorphism*

$$\varinjlim_j \varinjlim_{j'} F(j,j') \to \varinjlim_{j'} \varinjlim_j F(j,j')$$

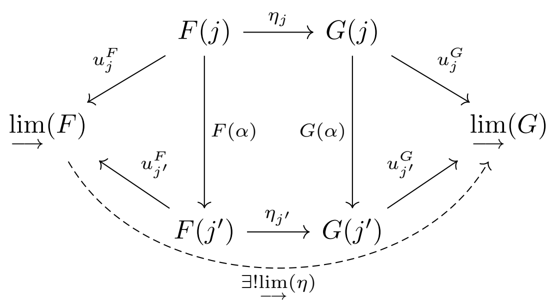
Proof. [11], IX.2 (2). □

Finally we will show that taking the colimit defines a functor.

Lemma A.1.14. *If $\mathcal{C}$ is such that colimits exist for all functors $F : J \to \mathcal{C}$ where J is small, then $\varinjlim : Func(J, \mathcal{C}) \to \mathcal{C}$ defines a functor.*

Proof. ([11] ex V.2.3) We first look at objects. Let $F : J \to \mathcal{C}$ be a functor, then we define $\varinjlim(F) = \varinjlim_j F(j)$, which exists by assumption. Note that this definition makes a choice, choosing one colimit diagram to represent *the* colimit. We will address this choice in A.1.15.

Now looking at morphisms, let $\eta : F \to G$ be a natural transformation of functors $F, G : J \to \mathcal{C}$, then for all morphisms $\alpha : j \to j'$ in J we have the following diagram:



(A.6)

Clearly if $F = G$ and $\eta = \mathrm{Id}_F$, then the unique morphism making the diagram commute will be the identity, and so $\varinjlim(\mathrm{Id}_F) = \mathrm{Id}_{\varinjlim(F)}$. If we have a composition of natural transformations $F \xrightarrow{\eta} G \xrightarrow{\delta} H$ then the composition $\varinjlim(\delta) \circ \varinjlim(\eta)$ will make the diagram commute, thus $\varinjlim : \mathrm{Func}(J, \mathcal{C}) \to \mathcal{C}$ is a functor. $\square$

As we noted in the definition of the functor in the previous lemma, we just choose an arbitrary colimit diagram and say that is *the* colimit. The following lemma shows that we have a natural isomorphism between any two such choices, whenever J is filtered and $\mathcal{C} = \texttt{Sets}$

Lemma A.1.15. *Let J be small and filtered, and let $\varinjlim, \overline{\varinjlim} : Func(J, \texttt{Sets}) \to \texttt{Sets}$ be two choices of colimit functors as defined in A.1.14. Let $\eta = \{\eta_j : F(j) \to G(j)\}_{j \in J}$ be a natural transformation between functors F and G from J to $\mathcal{C}$. We then have a commuting diagram*

$$\begin{array}{ccc} \varinjlim F & \xrightarrow{\varinjlim(\eta)} & \varinjlim G \\ \downarrow{\scriptstyle h_F} & & \downarrow{\scriptstyle h_G} \\ \overline{\varinjlim} F & \xrightarrow{\overline{\varinjlim}(\eta)} & \overline{\varinjlim} G, \end{array}$$

where the h_F's are the isomorphisms given from the universal property (like in A.1.3).

Proof. We write the different colimit diagrams as $(\varinjlim F, \{u_j\})$, $(\varinjlim G, \{v_j\})$, $(\overline{\varinjlim} F, \{\overline{u_j}\})$ and $(\overline{\varinjlim} G, \{\overline{v_j}\})$. Since h_F and h_G comes from the universal property we have

$$h_F \circ u_j = \overline{u_j} \tag{A.7}$$

$$h_G \circ v_j = \overline{v_j}, \tag{A.8}$$

for all objects j in J. Similarly, $\varinjlim(\eta)$ and $\overline{\varinjlim}(\eta)$ are given by the universal property like in diagram (A.6), so we get

$$\varinjlim(\eta) \circ u_j = v_j \circ \eta_j \tag{A.9}$$

$$\overline{\varinjlim}(\eta) \circ \overline{u_j} = \overline{v_j} \circ \eta_j. \tag{A.10}$$

Now let $k \in \varinjlim F$ be any element. From A.1.7 we know that there is an object j in J and $x \in F(j)$ such that $u_j(x) = k$. Using this with with (A.9) we get $h_G \circ \varinjlim(\eta)(k) = h_G \circ v_j \circ \eta_j(x)$, which by (A.8) is $\overline{v_j} \circ \eta_j(x)$. Now by (A.10) this again equals $\overline{\varinjlim}(\eta) \circ \overline{u_j}(x)$. Finally, using (A.7) and $u_j(x) = k$, we get that this is indeed $\overline{\varinjlim}(\eta) \circ h_F(k)$. Since k was arbitrarily chosen, we get our desired commutative diagram. □

A.2 Geometric Realization of Standard n-Simplex

In this section we will calculate the geometric realization of a standard n-simplex using the definition we introduced in Section 1.3. In Drinfeld's paper ([5], Example) there were some of the same arguments, but in much less detail. Grayson also had similar ideas ([10], 2.4), but uses a different definition of geometric realization.

We start with an intermediate step, looking only at the underlying set of the geometric realization.

Proposition A.2.1. *Define $|\Delta^n|_T := \{K : I \to [n]$ piecewise constant, non-decreasing functions$\}/\sim$, where $K \sim K'$ are equivalent if and only if $K(t) \neq K'(t)$ only for a finite number of $t \in I$. Then $|\Delta^n|_T$ is a colimit of $\Delta^n(\pi_0(I - (-)))$, and thus isomorphic to $|\Delta^n|$ as sets.*

Proof. Fixing F, let $\pi_F : I \to \pi_0(I - F)$ be the map sending $t \in I - F$ to its component in $\pi_0(I - F)$ and $t \in F$ to one of its two neighboring components. Let $u_F : \mathrm{Func}(\pi_0(I - F), [n]) \to |\Delta^n|_T$ be the map sending H to $[H \circ \pi_F]$. The choices for π_F are only for $t \in F$ a finite number of points, also both H and π_F are non-decreasing, so u_F is well-defined.

We will show that $(|\Delta^n|_T, \{u_F\})$ is a colimit diagram for $\mathrm{Func}(\pi_0(I-(-)), [n])$. The first thing we need to show is the cocone property, namely that $u_F =$

$u_G \circ \pi_0(I-\kappa)^*$ for all morphisms $\kappa : F \hookrightarrow G$ in $I_\subseteq$.

$$\begin{array}{ccc} \operatorname{Func}(\pi_0(I-F),[n]) & & \\ \downarrow{\scriptstyle \pi_0(I-\kappa)^*} & \searrow^{u_F} & \\ & & |\Delta^n|_T \\ & \nearrow_{u_G} & \\ \operatorname{Func}(\pi_0(I-G),[n]) & & \end{array} \tag{A.11}$$

Let $\kappa : F \hookrightarrow G$ be a morphism in $I_\subseteq$, and let $H : \pi_0(I-F) \to [n]$ be any functor. Now $u_F(H) = [H \circ \pi_F]$ and $u_G \circ \pi_0(I-\kappa)^*(H) = [H \circ \pi_0(I-\kappa) \circ \pi_G]$. Let $t \in I$ be such that t is not in G, and thus also not in F since $F \subseteq G$. By definition π_F sends t to the component of $I-F$ containing t. The function $\pi_0(I-\kappa)$ is induced by the inclusion $I-G \hookrightarrow I-F$ so it sends each component G_i of $I-G$ to the component in $I-F$ containing G_i. Thus $\pi_F(t) = \pi_0(I-\kappa)\circ\pi_G(t)$ for all $t \in I-G$, and so the functions can therefore only disagree on a finite number of t's. In particular we get $[H \circ \pi_F] = [H \circ \pi_0(I-\kappa) \circ \pi_G]$ which is what we wanted.

To show the universal property, let $(d, \{f_F : \operatorname{Func}(\pi_0(I-F),[n]) \to d\})$ be another cocone. Let $\kappa : F \subseteq G$, and let $H : \pi_0(I-F) \to [n]$ be any functor and define the map $f : |\Delta^n|_T \to d$ by $f \circ u_F(H) = f_F(H)$. We need to show that this is indeed a well-defined function on all of $|\Delta^n|_T$, so we need that every element in $|\Delta^n|_T$ is in the image of some u_F and that $f_F(H) = f_F(H')$ whenever $u_F(H) = u_F(H')$.

For the first point, let $[K] \in |\Delta^n|_T$ be represented by $K : I \to [n]$. Let $F_r = \operatorname{int} K^{-1}(r)$ be the interior of the preimage, which are each connected since K is non-decreasing. Then the subset $F = I - \bigcup_r F_r = \bigcup_r \subseteq I$ is finite, and $\pi_0(I-F)$ consists of the non-empty F_r where $F_i \leq F_j$ whenever $i \leq j$. Let $H : \pi_0(I-F) \to [n]$ be the functor defined by $H(F_r) = r$, which is non-decreasing by definition. If $t \in F_r$ then $H \circ \pi_F(t) = H(F_r) = r$ and $K(t) = r$ since $t \in F_r \subseteq K^{-1}(r)$. This is true for all $t \notin F$ and F is finite, so $H \circ \pi_F \sim K$, and thus $u_F(H) = [H \circ \pi_F] = [K]$.

To show that $u_F(H) = u_F(H')$ implies $f_F(H) = f_F(H')$ we will show the stronger statement that it in fact implies $H = H'$. We already know from A.1.6 that this has to be the case for it to be a colimit diagram. We will show the contrapositive statement, so let $H, H' : \pi_0(I-F) \to [n]$ be two functors such that $H \neq H'$, i.e. there is a component $F_i \in \pi_0(I-F)$ where $H(F_i) \neq H'(F_i)$. Now let $t \in F_i$ be any point in the component, then $H \circ \pi_F(t) = H(F_i) \neq H'(F_i) = H \circ \pi_F(t)$. All components in $\pi_0(I-F)$ are open non-empty subsets of I, hence they have infinite elements. So $H \circ \pi_F \neq H' \circ \pi_F$ for an infinite number of points, and so $u_F(H) = [H \circ \pi_F] \neq [H' \circ \pi_F] = u_F(H')$. $\square$

If we look at $|\Delta^1|_T$ with $F = \emptyset$. Then $\Delta^1(I - \emptyset) \cong \Delta^1([0]) = \mathrm{Func}([0],[1])$ which consists of two elements, the inclusion into zero and into one, called 0 and 1 respectively. The element 0 is sent to the class $[0]$ containing the constant zero-map $0 : I \to [1]$, and similarly 1 is sent to the class $[1]$ containing the constant map sending everything to 1.

We continue looking at the geometric realization of the standard n-simplices, now looking at topology as well. We know from 1.2.10 that standard n-simplices are finite, so we can use the topology from 1.3.7.

Lemma A.2.2. *The geometric realization of the standard n-simplex $|\Delta^n|$ is homeomorphic to the subset $|\Delta^n|_{\mathbb{R}} := \{(x_1, \ldots, x_n) \in I^n \mid 0 \leq x_1 \leq \cdots \leq x_n \leq 1\} \subseteq \mathbb{R}$ with the standard subspace topology.*

Proof. We have from A.2.1 that $|\Delta^n|_T$ is a colimit, so we can give it the Drinfeld-metric making it homeomorphic to the geometric realization $|\Delta^n|$. We will construct a bijection between equivalence classes $[K]$ and families $x = (0 = x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} = 1)$.

Starting with such a family x we construct any non-decreasing function $K_x : I \to [n]$ such that K_x changes value at each x_i and $K_x(t) = i$ whenever $x_i < t < x_{i+1}$. Each such function K_x will represent the same unique equivalence class $[K_x]$ since the choices are made only for $t = x_i$ which there are finitely many of.

Conversely let $[K]$ be represented by $K : I \to [n]$. Since K is non-decreasing, the preimage $K^{-1}([0, i-1])$ will be some open or closed segment $[0, y_i)$. We write the standard measure of this segment as

$$y_i = \mu(K^{-1}([0, i-1]). \tag{A.12}$$

We have $y_i \leq y_j$ whenever $i \leq j$, and the measure of any subset is between 0 and 1. So we get an element $y_K = \{y_1 \leq \cdots \leq y_n\}$ in $|\Delta^n|_{\mathbb{R}}$

To see that this is a bijection, let $y_i < t < y_{i+1}$. Then t is in $[0, y_{i+1}) = K^{-1}([0,i])$ but it is not in $K^{-1}([0, i-1])$, so $K(t) = i = K_{y_K}(t)$, so we get the same class $[K_{y_K}] = [K]$.

The other way, let $x = \{0 \leq x_1 \leq \cdots \leq x_n \leq 1\}$. Now let K_x be a function such that $K_x(t) = i$ for $t \in (x_i, x_{i+1})$, then $y_i = \mu(K^{-1}([0, i-1]) = \mu(K^{-1}[0, x_i)) = x_i$. Thus $y_{K_x} = x$, and we have a bijection.

Next, we need to show that the bijection is a homeomorphism, but first we simplify the distances we use both for $|\Delta^n|_T$ and $|\Delta^n|_{\mathbb{R}}$. From [12] 20.3 we have that the usual Euclidean metric is equivalent to the l^∞ metric. So when $x = (x_1 \leq \cdots \leq x_n)$ and $y = (y_1 \leq \cdots \leq y_n)$ are elements in $|\Delta^n|_{\mathbb{R}}$, we can use the distance

$$d_l(x, y) = \max |x_i - y_i|. \tag{A.13}$$

As we noted under 1.3.5, the distance $d_{\Delta^n,F}$ measures the size of the subset of $\pi_0(I-F)$ where two functions differ. If $K : I \to [n]$ is piecewise constant and non-decreasing, then from A.2.1 we have an F and a function $H : \pi_0(I-F) \to [n]$ such that $H(F_i) = K(t)$ for all $t \in F_i$ and for all i. If K' and H' is another such pair of functions, then

$$d_T([K],[K']) = d_{\Delta^n,F}(H,H') = \mu_F(\{F_i \mid H(F_i) \neq H'(F_i)\}) = \mu(\{t \in I \mid K(t) \neq K'(t)\}),$$

where in the last equality we have used the fact that finitely many points have measure zero. Now using the bijection we can simplify this further. Except for finitely many points we know that $K(t) = K_{y_K}(t)$, so we see that $K(t) = K'(t)$ if and only if t is in $(y_i, y_{i+1}) \cap (y'_i, y'_{i+1})$ for some i. We finally get the distance

$$d_T([K],[K']) = \mu\left(I - \amalg_{i=0}^{n}(y_i, y_{i+1}) \cap (y'_i, y'_{i+1})\right). \tag{A.14}$$

To show that the bijection is indeed a homeomorphism, we note that $|\Delta^n|_{\mathbb{R}}$ is a closed and bounded subset of $\mathbb{R}$, and so it's compact. We also note that since the topology comes from a metric, both $|\Delta^n|_T$ and $|\Delta^n|_{\mathbb{R}}$ are Hausdorff. In particular the map $|\Delta^n|_{\mathbb{R}} \to |\Delta^n|_T$ sending y to K_y is a bijection from a compact to a Hausdorff space. If this is continuous, then it is a homeomorphism by [12] 26.6.

We will show continuity by induction on n showing that $d_T([K_x],[K_y]) \leq n \cdot d_l(x,y)$.

Start: Let $n = 1$, so that $x = \{x_1\}$ and $y = \{y_1\}$ are just a single point. By symmetry we can assume $x_1 \leq y_1$, and we calculate

$$\begin{aligned} d_T([K_x],[K_y]) &= 1 - \mu\left((0,x_1) \cap (0,y_1)\right) - \mu\left((x_1,1) \cap (y_1,1)\right) \\ &= 1 - (x_1 - 0) - (1 - y_1) = y_1 - x_1 = 1 \cdot d_l(x,y). \end{aligned}$$

Step: Assume true for $n-1$, i.e. if $x, y \in |\Delta^{n-1}|$ then $d_T([K_x],[K_y]) \leq (n-1) \cdot d_l(x,y)$. Let $x = \{x_1, \ldots, x_n\}$ and $y = \{y_1, \ldots, y_n\}$ be elements in $|\Delta^n|_{\mathbb{R}}$, and define $\overline{x} = x - \{x_n\}$ and $\overline{y} = y - \{y_n\}$ both elements in $|\Delta^n|_{\mathbb{R}}$. First we note that $d_l(x,y) \geq d_l(\overline{x},\overline{y})$, where it is bigger only if the right side is $|x_n - y_n|$. Writing out the definitions we have

$$\begin{aligned} d_T([K_{\overline{x}}],[K_{\overline{y}}]) &= 1 - \amalg_{i=0}^{n-2}\mu\left((x_i,x_{i+1}) \cap (y_i,y_{i+1})\right) - \mu\left((x_{n-1},1) \cap (y_{n-1},1)\right) \\ d_T([K_x],[K_y]) &= 1 - \amalg_{i=0}^{n-2}\mu\left((x_i,x_{i+1}) \cap (y_i,y_{i+1})\right) - \mu\left((x_{n-1},x_n) \cap (y_{n-1},y_n)\right) \\ &\quad - \mu\left((x_n,1) \cap (y_n,1)\right). \end{aligned}$$

We now have a several different cases of how the size of x_{n-1}, x_n, y_{n-1} and y_n all relate to each other. For example, if $x_{n-1} \leq x_n \leq y_{n-1} \leq y_n$, then

$$\begin{aligned} d_T([K_x],[K_y]) &= d_T([K_{\overline{x}}],[K_{\overline{y}}]) + \mu(y_{n-1},1) - \mu(\emptyset) - \mu(y_n,1) \\ &\leq (n-1) \cdot d_l(\overline{x},\overline{y}) + (y_n - y_{n_1}) \\ &\leq (n-1) \cdot d_l(\overline{x},\overline{y}) + |y_n - x_n|. \end{aligned}$$

Calculating for each case we will similarly get $d_T([K_x],[K_y]) \leq (n-1)\cdot d_l(\overline{x},\overline{y}) + |y_n - x_n|$. Here the first part of the right hand side is less than or equal to $(n-1)\cdot d_l(x,y)$, and the second part is less than or equal to $d_l(x,y)$. So we conclude that $d_T([K_x],[K_y]) \leq n \cdot d_l(x,y)$, and the bijection is thus a homeomorphism. □

We will just write out some quick results from this.

Corollary A.2.3. $|\Delta^n|$ *is compact Hausdorff for all* $n \geq 0$. □

Corollary A.2.4. *The geometric realization of the 1-simplex is homeomorphic to the closed interval* I. □

Note that if $[0]$ and $[1]$ in $|\Delta^1|_T$ are the classes of the constant maps as we looked at below A.2.1, then $y_{[0]} = \{1\}$ and $y_{[1]} = \{0\}$. We use this in 1.4.10.

A.3 Proof of Proposition 1.5.9

In this section we will prove 1.5.9.

Let $(K, V_<)$ be an ordered simplicial complex. By a slight abuse of notation we define a new functor $T : K_\subseteq \to \mathtt{sSet}$ sending a simplex σ to the simplicial set $T(\sigma)$ as in 1.5.4, where we think of σ as a simplicial complex with vertex set $V_<$. Recall that $\beta \in T(\sigma)_n$ is an order-preserving map $\beta : [n] \to V_<$ such that $\operatorname{Im}\beta \subseteq \sigma$, so if $\sigma \subseteq \tau$ then $\beta \in T(\tau)_n$ and there is a natural inclusion $T(\sigma) \hookrightarrow T(\tau)$ making T a functor. We compose this with the geometric realization to get a functor $|T(-)| : K_\subseteq \to \mathtt{Top}$.

Lemma A.3.1. *If* K *is an ordered finite simplicial complex, then we have a homeomorphism* $|T(K)| \cong \varinjlim_{\sigma \in K_\subseteq} |T(\sigma)|$.

Proof. We first note that if $\beta \in T(\sigma)_n$ for some simplex, then $\operatorname{Im}\beta \subseteq \sigma$ is a simplex in K, and so β is in $T(K)_n$. Thus we have a family of inclusions $i_\sigma = \{i_{\sigma,n}\} : T(\sigma) \to T(K)$ for every simplex σ.

We claim that $(T(K), \{i_\sigma\})$ is a colimit diagram of the functor $T : K_\subseteq \to \mathtt{sSet}$. Morphisms $\iota : \sigma \subseteq \tau$ are sent by T to inclusions $T(\iota) : T(\sigma) \hookrightarrow T(\tau)$, and since every map involved are inclusions we get $i_\sigma = i_\tau \circ T(\iota)$. Thus $(T(K), \{i_\sigma\})$ is a cocone.

As in A.2.1, to show the universal property it is enough to show that every element $\beta \in T(K)_n$ is in the image of some $i_{\sigma,n}$, and that $\beta = \beta'$ whenever $i_{\sigma,n}(\beta) = i_{\sigma,n}(\beta')$. Then if $(Y, \{f_\sigma\})$ is another cocone, the map $f' : T(K) \to Y$ defined by $i_{\sigma,n} \circ f'_n = f_{\sigma,n}$ is well-defined.

For the first point, we note that if $\beta \in T(K)_n$, then $\operatorname{Im}\beta$ is a simpliex by definition, and $\beta \in T(\operatorname{Im}\beta)_n$, so $\beta \in \operatorname{Im}(i_{\operatorname{Im}\beta,n})$. For the second point, we assume by contraposition that $\beta \neq \beta' \in T(\sigma)_n$, then they are also not equal after including them into a bigger set, i.e. $i_{\sigma,n}(\beta) \neq i_{\sigma,n}(\beta')$. So we conclude that $(T(K), \{i_\sigma\})$ is a colimit diagram, and we have an isomorphism $T(K) \cong \varinjlim_\sigma T(\sigma)$.

Looking at the geometric realization, the functor properties gives a homeomorphism $|T(K)| \cong |\varinjlim_\sigma T(\sigma)| = \varinjlim_F \varinjlim_\sigma T(\sigma)(\pi_0(I - F))$. By A.1.11 and A.1.13 using the fact that $K_\subseteq$ are small, we can change order of the colimits with an isomorphism. Isomorphisms of topological spaces are homeomorphisms, so $|T(K)| \cong \varinjlim_\sigma \varinjlim_F T(\sigma)(\pi_0(I - F)) = \varinjlim_\sigma |T(\sigma)|$. □

We will do something very similar for $|K|$. Note that also the geometric realization of a simplicial complex defines a functor $| - | : K_\subseteq \to \mathtt{Top}$, for if $\sigma \subseteq \tau$ and $\alpha \in |\sigma|$, then $\{v \in V \mid \alpha(v) \neq 0\} \subseteq \sigma \subseteq \tau$, and we have the inclusion $|\sigma| \hookrightarrow |\tau|$. Inclusions are clearly continuous with respect to the euclidean metric.

Lemma A.3.2. *If K is an ordered finite simplicial complex, then we have a homeomorphism $|K| \cong \varinjlim_{\sigma \in K_\subseteq} |\sigma|$.*

Proof. This proof is similar to the one for A.3.1. We again look at the inclusions $i_\sigma : |\sigma| \to |K|$, and show that $(|K|, \{i_\sigma\})$ is a colimit diagram. As in A.3.1, all maps involved are inclusions, so it is a cocone.

If $\alpha \in |K|$ then $\sigma_\alpha := \{v \in V \mid \alpha(v) \neq 0\}$ is a simplex and $\alpha \in |\sigma_\alpha|$. So we have $\alpha \in \operatorname{Im} i_{\sigma_\alpha}$.

Finally if $\alpha \neq \alpha' \in |\sigma|$, then they are still not equal after including them into a bigger space $|K|$, so $\alpha = \alpha'$ whenever $i_\sigma(\alpha) = i_\sigma(\alpha)$. We conclude that $(|K|, \{i_\sigma\})$ is a colimit diagram as sets, and we have a bijection $|K| \cong \varinjlim_\sigma |\sigma|$.

From A.1.11 the colimit is given the finest topology making the maps associated with the colimit as sets continuous. The maps associated with $|K|$ as a colimit are all the inclusions $|\sigma| \hookrightarrow |K|$, so the colimit topology agrees with the coherent topology, and the bijection is a homeomorphism $|K| \cong \varinjlim_\sigma |\sigma|$. □

Looking at a general ordered simplicial complex $(K, V_<)$, we can look at the category of finite simplicial subcomplexes $K' \subseteq K$, where morphisms are inclusions, and by the exact same arguments as A.3.1 and A.3.2 we get homeomorphisms

$$|K| \cong \varinjlim_{K' \subseteq K} |K'| \quad \text{and} \quad |T(K)| \cong \varinjlim_{K' \subseteq K} |T(K')|. \tag{A.15}$$

We will now show that the geometric realization of a simplex is homeomorphic to the realization of a standard simplex.

Lemma A.3.3. *Let $(K, V_<)$ be an ordered simplicial complex, and $\sigma = [v_1 < \cdots < v_m] \in K$ a simplex. Then there is a homeomorphism between $|\sigma|$ and $|\Delta^{m-1}|_\mathbb{R}$, where $|\Delta^{m-1}|_\mathbb{R}$ is defined as in A.2.2.*

Proof. Let $\alpha \in |\sigma| = \{\alpha : V \to I \mid \{v \in V \mid \alpha(v) \neq 0\} \subseteq \sigma, \sum_{v \in \sigma} \alpha(v) = 1\}$, and define

$$x_0 = 0, \quad x_i = \alpha(v_1) + \cdots + \alpha(v_i). \tag{A.16}$$

Note in particular that $x_m = \sum_{v\in\sigma} \alpha(v) = 1$. Clearly $0 = x_0 \leq x_1 \leq \cdots \leq x_m = 1$, so $x_\alpha := (x_1, x_2, \ldots, x_{m-1})$ is an element in $|\Delta^{m-1}|_{\mathbb{R}}$.

Conversely, starting with $x = (x_1, x_2, \ldots, x_{m-1}) \in |\Delta^{m-1}|_{\mathbb{R}}$, let $\alpha_x : V_< \to I$ be the map

$$\alpha_x(v) = \begin{cases} x_i - x_{i-1} & \text{for } v = v_i \in \sigma \\ 0 & \text{for } v \neq v_i \in \sigma. \end{cases} \tag{A.17}$$

Then $\{v \in V \mid \alpha_x(v) \neq 0\} \subseteq \sigma$ and $\sum_{v\in V} \alpha_x(v) = \sum_{i=1}^{m} x_i - \sum_{i=0}^{m-1} x_i = x_m = 1$. So α_x is in $|\sigma|$.

We now have a bijection $|\Delta^{m-1}|_{\mathbb{R}} \to |\sigma|$. We know $|\Delta^{m-1}|_{\mathbb{R}}$ is compact, and $|\sigma|$ is a metric space, so it is Hausdorff, therefore it is enough to show that the bijection is continuous ([12], 26.6). Let $x, y \in |\Delta^{m-1}|_{\mathbb{R}}$, and let α_x and α_y be their image in $|\sigma|$, as given by (A.17). Again we use that the Euclidean and square metrics are equivalent ([12], 20.3), and look at the distances $d(\alpha_x, \alpha_y) = \max_{v_i\in\sigma} |\alpha_x(v_i) - \alpha_y(v_i)|$, and $d_l(x, y) = \max_{0<i<m} |x_i - y_i|$. We need to show that the bijection is continuous with respect to these distances, so let $v_j \in \sigma$ be a vertex such that $d(\alpha_x, \alpha_y) = |\alpha_x(v_j) - \alpha_y(v_j)|$. Then by (A.17), we have $d(\alpha_x, \alpha_y) = |(x_j - x_{j-1}) - (y_j - y_{j-1})| = |(x_j - y_j) + (y_{j-1} - x_{j-1})|$. Using the triangle inequality and the fact that $|a| = |-a|$, this less than or equal to $|x_j - y_j| + |x_{j-1} - y_{j-1}| \leq d_l(x, y) + d_l(x, y)$. So $d(\alpha_x, \alpha_y) \leq 2d_l(x, y)$, and the bijection is continuous, and thus a homeomorphism. □

We will now prove 1.5.9. For clarity we restate it.

Proposition A.3.4. *If $f : (K, V_<) \to (L, W_<)$ is a simplicial map injective on vertex sets between ordered simplicial complexes, then we have a homeomorphism between $|K|$ and $|T(K)|$, and between $|L|$ and $|T(L)|$, making the following diagram commute*

$$\begin{array}{ccc} |K| & \longrightarrow & |T(K)| \\ \downarrow{\scriptstyle |f|} & & \downarrow{\scriptstyle |T(f)|} \\ |L| & \longrightarrow & |T(L)|\,. \end{array} \tag{A.18}$$

Proof. We will show that $|K|$ is a colimit of $T(K)(\pi_0(I-(-))$, to get a bijection between $|K|$ and $|T(K)|$. Then we will show that this bijection is a homeomorphism by reducing to the case of simplices and comparing the topologies.

We have $T(K)(\pi_0(I - F)) = \{\beta : \pi_0(I - F) \to V_< \text{ order-preserving} \mid \mathrm{Im}\beta \in K\}$. Define $u_F : T(K)(\pi_0(I - F)) \to |K|$, by $u_F(\beta)(v) = \mu_F(\beta^{-1}(v))$. We want to show that this is well-defined by showing $u_F(\beta) \in |K|$. The elements in $\pi_0(I - F)$ are all open intervals, so they all have measure different from zero. Thus we get $\mu_F(\beta^{-1})(v) \neq 0$ if and only if $\beta^{-1}(v) \neq \emptyset$, and so $\{v \in$

$V \mid u_F(\beta)(v) \neq 0\} = \mathrm{Im}\beta \in K$. Since β is well-defined the preimages of different vertices are disjoint, $\beta^{-1}(v) \cap \beta^{-1}(w) = \emptyset$ for $v \neq w$, and the preimage of the entire vertex set $\beta^{-1}(V)$ is the entire set $\pi_0(I - F)$. From the first point we can pull the sum inside the measure, $\sum_{v\in V} \mu_F(\beta^{-1}(v)) = \mu_F(\cup_{v\in V}\beta^{-1}(v)) = \mu_F(\beta^{-1}(V))$, which from the second point is just the measure of the entire $\pi_0(I - F)$, which again is just one. So $\sum_{v\in v} u_F(\beta)(v) = 1$, and u_F is well-defined.

We now want to show that $(|K|, \{u_F\})$ is a colimit diagram of $T(K)(\pi_0(I - (-))$. We calculate the cocone property directly, $u_F = u_G \circ \pi_0(I - \kappa)$ for $\kappa : G \subseteq F$. We know $G_i \subseteq F_j$ if and only if $\pi_0(I - F)(G_i) = F_j$, so if $\beta \in T(K)(\pi_0(I - F))$ then

$$\begin{aligned} u_G \circ T(K)(\pi_0(I-\kappa))(\beta)(v) &= u_G(\beta \circ \pi_0(I-\kappa))(v) \\ &= \mu_G(\pi_0(I-\kappa)^{-1}(\beta^{-1}(v))) \\ &= \mu_G\{G_i \mid \beta \circ \pi_0(I-\kappa)(G_i) = v\} \\ &= \mu_F\{F_j \mid G_i \subseteq F_j,\, \beta \circ \pi_0(I-F)(G_i) = v\} \\ &= \mu_F\{F_j \mid \beta(F_j) = v\} = u_F(\beta)(v) \end{aligned}$$

For universality let $(d, \{f_F\})$ be a cocone of $T(K)(\pi_0(I - F))$, and define $f : |K| \to d$ by $f \circ u_F = f_F$. As in A.2.1 we will show that this is well-defined by showing that every $\alpha \in |K|$ is in the image of some u_F, and that $\beta = \beta'$ whenever $u_F(\beta) = u_F(\beta')$.

So let $\alpha : V_< \to I$ be such that $\sum_{v\in V_<} \alpha(v) = 1$ and $\sigma_\alpha = \{v \mid \alpha(v) \neq 0\} \in K$. Since simplices are finite we have $\alpha(v) \neq 0$ for finitely many $v \in V_<$, say $\sigma_\alpha = [v_1 < \cdots < v_n]$. Similar to (A.16) we define

$$x_0 = 0, \quad x_i = \alpha(v_i) + \cdots + \alpha(v_i) \tag{A.19}$$

for $i = 1, \ldots, n$, and note that we still have $x_n = 1$. Let $F = \{x_0 < \cdots < x_n\}$ and define $\beta : \pi_0(I - F) \to V_<$ by $\beta(F_i) = v_i$, where $F_i = (x_{i-1}, x_i) \in \pi_0(I - F)$. If v is not in σ_α, then $\alpha(v) = 0$ and $u_F(\beta)(v) = \mu_F(\beta^{-1}(v)) = \mu_F(\emptyset) = 0$. If v_i is in σ_α, then $u_F(\beta)(v_i) = \mu(F_i) = x_i - x_{i-1} = \alpha(v_i)$. So $u_F(\beta) = \alpha$ and in particular $\alpha \in \mathrm{Im}\, u_F$ for some F.

For the second part of the universality we will show the contrapositive statement, namely that $\beta \neq \beta' : \pi_0(I - F) \to V_<$ implies that $u_F(\beta) \neq u_F(\beta')$. If $\beta \neq \beta'$, then there is an $F_i \in \pi_0(I - F)$ such that $w := \beta(F_i) \neq \beta'(F_i)$. We can assume by symmetry that $w > \beta'(F_i)$. By the order-preserving property of β and β', we get the strict inclusion $\{F_j \mid \beta(F_j) < w\} \subsetneq \{F_j \mid \beta'(F_j) < w\}$ using the fact that $\beta'(F_i) < w = \beta(F_i)$. The sum $\sum_{v<w} \mu_F(\beta^{-1}(v))$ is then smaller than the sum $\sum_{v<w} \mu_F(\beta'^{-1}(v))$ and there must exist a vertex $u < w$ such that $\mu_F(\beta^{-1}(u)) < \mu_F(\beta'^{-1}(u))$. We conclude that $u_F(\beta) \neq u_F(\beta')$, and that $(|K|, \{u_F\})$ is a colimit diagram of $T(K)(\pi_0(I-(-)))$. We thus have a bijection $|K| \cong |T(K)|$.

To show that (A.18) commutes, we need to show that the map $|f|$ is the same as the one we get by the universal property between colimits, i.e. we want the following diagram to commute for all F:

$$\begin{array}{ccc} T(K)(\pi_0(I-F) & \xrightarrow{T(f)} & T(L)(\pi_0(I-F)) \\ \downarrow{\scriptstyle u_F^K} & & \downarrow{\scriptstyle u_F^L} \\ |K| & \xrightarrow{\quad |f| \quad} & |L| \end{array}$$

If this is the case we use A.1.15 to show that (A.18) commutes. Let $\beta \in T(K)(\pi_0(I-F))$ and $w \in W_<$. In one way of the diagram we have $u_F^L \circ T(f)(\beta)(w) = u_F^L(f \circ \beta)(w) = \mu_F(\beta^{-1}(f^{-1}(w)))$, and the other way is $|f| \circ u_F^K(\beta)(w) = \sum_{f(v)=w} \mu_F(\beta^{-1}(v))$. These are the same since we can move the sum inside the measure by the fact that $\beta^{-1}(v) \cap \beta^{-1}(v')$ whenever $v \neq v'$.

Finally we need to show that the bijection $|K| \to |T(K)|$ is a homeomorphism. From A.3.1, A.3.2 and (A.15) it is enough to show that the induced maps on simplices $|\sigma| \to |T(\sigma)|$ are homeomorphisms for every simplex $\sigma = [v_1 < \cdots < v_m]$.

Recall that maps $\beta \in T(\sigma)_n$ are order-preserving maps $\beta : [n] \to V_<$ such that $\beta([n]) \subseteq \sigma$. There is a bijection between such maps and order preserving maps $\beta : [n] \to \sigma$, by just removing the vertices that are never hit. Since σ is isomorphic to $[m-1]$ in Δ_{big} by $v_i \mapsto i-1$, we have that $T(\sigma)$ is isomorphic to the standard $(m-1)$-simplex Δ^{m-1}. We now have a bijection $|\sigma| \to |T(\sigma)|$, a homeomorphism $|T(\sigma)| \cong |\Delta^{m-1}| \to |\Delta^{m-1}|_{\mathbb{R}}$ by A.2.2, and a homeomorphism $|\sigma| \to |\Delta^{m-1}|$ by A.3.3. We will show that the following diagram commutes

$$\begin{array}{ccc} |\sigma| & \xrightarrow{\qquad\qquad} & |T(\sigma)| \\ & \searrow \quad \swarrow & \\ & \left|\Delta^{m-1}\right|_{\mathbb{R}} & \end{array} \tag{A.20}$$

Let $\alpha \in |\sigma|$. By (A.16) this is mapped to $x_\alpha = (x_1 < \cdots < x_{m-1}) \in |\Delta^{m-1}|_{\mathbb{R}}$, where $x_i = \alpha(v_1) + \cdots + \alpha(v_i)$.

The other way, let $F = (0 = x_0 \leq x_1 \leq \cdots \leq x_m = 1)$, $F_i = (x_{i-1}, x_i)$ for $i = 1, \ldots, m$, and let $\beta : \pi_0(I-F) \to V_<$ be the map $\beta(F_i) = v_i$, as we showed above $u_F(\beta) = \alpha$. Forgetting the vertices not in σ, and using the isomorphisms $\sigma \cong [m-1]$, then β can be viewed as the map $\beta : \pi_0(I-F) \to [m-1]$ sending F_i to $i-1$. By (A.12) this map is sent to $y = (y_1, \ldots, y_{m-1})$ in $|\Delta^{m-1}|_{\mathbb{R}}$, where

$y_i = \mu(\pi_F^{-1} \circ \beta^{-1}[0, i-1])$. Recall from the start of A.2.1, that $\pi_F : I \to \pi_0(I-F)$ is any map that sends $t \in F_i$ to F_i. Since $\beta(F_j) = j - 1$, we have

$$\begin{aligned} y_i &= \mu(\pi_F^{-1} \circ \beta^{-1}([0, i-1])) \\ &= \mu(\pi_F^{-1}(\{F_j \mid 1 \leq j \leq i\})) \\ &= \mu(\{F_j \mid 1 \leq j \leq i\}) \\ &= \Sigma_{j=1}^{i}(x_j - x_{j-1}) = x_i - x_0 = x_i \end{aligned}$$

So $y = x_\alpha$, and (A.20) commutes. Thus the bijection $|\sigma| \to |T(\sigma)|$ is a homeomorphism, and so is $|K| \to |T(K)|$. □

A.4 Localizations

In this section we will build up all the machinery to define localization of a category. It is mostly based on [11] II,7, II,8 and IV.1, and Chapter 1 in [9]. Intuitively a localization of a category $\mathcal{C}$ at $\Sigma \subseteq \operatorname{Mor}\mathcal{C}$ adds additional morphism to the category such that every map in Σ becomes an isomorphism. We start of by defining a graph, and looking at its properties.

Definition A.4.1. *A **(directed) graph** $G : A \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} O$ is a set of objects O, a set of arrows A and two functions $\partial_0, \partial_1 : A \to O$. Given an arrow $a \in A$ we say that $\partial_0 a$ is the **domain** of a and $\partial_1 a$ is the **codomain** of a, and we write $a : \partial_0 a \to \partial_1 a$.*

Note that different names are used for what we call directed graphs in the literature. For instance in representation theory the word **quiver** is used and perhaps more historically (in [9]) they used **diagram scheme**. Some authors do not allow loops or multiple arrows with the same domain or codomain, but we do not have any such restrictions.

Definition A.4.2. *Let $G : A \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} O$ and $G' : A' \overset{\partial_0'}{\underset{\partial_1'}{\rightrightarrows}} O'$ be two graphs, then a **morphism of graphs** $D : G \to G'$ is a pair of functions $D_O : O \to O'$, $D_A : A \to A'$ such that*

$$D_O \partial_0 a = \partial_0' D_A a \text{ and } D_O \partial_1 a = \partial_1' D_A a \text{ for all } a \in A. \tag{A.21}$$

Letting composition of two morphisms be pairwise composition of their functions, we get the **category of directed graphs**, which we denote by `Grph`.

Definition A.4.3. *Starting with a graph $G : A \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} O$, we can construct the **path category** $Pa(G)$ of G, with $Ob\,Pa(Q) = O$ and $Mor\,Pa(G) = \{c_0 \xrightarrow{f_0} \dots \xrightarrow{f_{n-1}} c_n \mid c_i \in O,\ f_0, \dots, f_{n-1} \in A,\ n \geq 0\}$.*

A morphism here is called a **path** from c_0 to c_n and the integer n is called the **length** of the path. Composition of two paths is defined by joining their common endpoint, e.g. $(b \xrightarrow{g} c) \circ (a \xrightarrow{f} b) = (a \xrightarrow{f} b \xrightarrow{g} c)$, and the identities are the paths of length 0, $\mathrm{Id}_c = (c)$. Any path of length $n > 0$ can be written as a composition of of paths of length 1,

$$\left(c_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} c_n\right) = \left(c_{n-1} \xrightarrow{f_{n-1}} c_n\right) \circ \dots \circ \left(c_1 \xrightarrow{f_1} c_2\right).$$

Let $G : A \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} O$ and $G' : A' \underset{\partial'_1}{\overset{\partial'_0}{\rightrightarrows}} O'$ be two graphs, and $D : G \to G'$ a morphism between them. Look at the map $\mathrm{Pa}(D) : \mathrm{Pa}(G) \to \mathrm{Pa}(G')$ defined by

$$\mathrm{Pa}(D)\left(c_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} c_n\right) = \left(D_O(c_1) \xrightarrow{D_A(f_1)} \dots \xrightarrow{D_A(f_{n-1})} D_O(c_n)\right).$$

This is well defined by the properties in (A.21), we have $\mathrm{Pa}(D)(\mathrm{Id}_c) = \mathrm{Pa}(D)(c) = (D_O(c)) = \mathrm{Id}_{D_O(c)}$, where we think of (c) as the path of length 0. Also

$$\begin{aligned}
\mathrm{Pa}(D)\left((b \xrightarrow{g} c) \circ (a \xrightarrow{f} b)\right) &= \mathrm{Pa}(D)(a \xrightarrow{f} b \xrightarrow{g} c) \\
&= \left(D_O(a) \xrightarrow{D_A(f)} D_O(b) \xrightarrow{D_A(g)} D_O(c)\right) \\
&= \left(D_O(b) \xrightarrow{D_A(g)} D_O(c)\right) \circ \left(D_O(a) \xrightarrow{D_A(f)} D_O(b)\right) \\
&= \mathrm{Pa}(D)\left(b \xrightarrow{g} c\right) \circ \mathrm{Pa}(D)\left(a \xrightarrow{f} b\right).
\end{aligned}$$

Thus $\mathrm{Pa}(D)$ is a functor, and we have a functor $\mathrm{Pa} : \mathtt{Grph} \to \mathtt{Cat}$ called the **path functor**.

Next we want to introduce quotient categories with respect to relations. The localization is a quotient category.

Definition A.4.4. *Let $\mathcal{C}$ be a category. A **relation** on $\mathcal{C}$ is a binary relation $R \subseteq Mor\mathcal{C} \times Mor\mathcal{C}$ such that $R = \coprod_{a,b \in \mathcal{C}} R_{a,b}$ is a disjoint union of binary relations $R_{a,b} \subseteq Hom_{\mathcal{C}}(a,b) \times Hom_{\mathcal{C}}(a,b) \subseteq Mor\mathcal{C} \times Mor\mathcal{C}$.*

The union R is disjoint because if $(a,b) \neq (a',b')$ then $\mathrm{Hom}_{\mathcal{C}}(a,b) \cap \mathrm{Hom}_{\mathcal{C}}(a',b') = \emptyset$.

Definition A.4.5. *A relation R is a **congruence (relation)** on $\mathcal{C}$, if for all $R_{a,b}$ we have the following:*

(i): $(f,f) \in R_{a,b}$ *for all* $f \in Hom_{\mathcal{C}}(a,b)$.

(ii): $(f,f') \in R_{a,b}$ *implies* $(f',f) \in R_{a,b}$.

(iii): $(f,f') \in R_{a,b}$ *and* $(f',f'') \in R_{a,b}$ *implies* $(f,f'') \in R_{a,b}$.

(iv): If $(f,f')\in R_{a,b}$, then $(hfg,hf'g)\in R_{a',b'}$ for all morphisms $g:a'\to a$, $h:b\to b'$.

The first three just say that $R_{a,b}$ is an equivalence relation, and the last one gives an extra condition on the relationship between the different parts of the disjoint union.

Lemma A.4.6. *Let $\mathcal{C}$ be a category, and R a relation on $\mathcal{C}$. Then there is a least congruence on $\mathcal{C}$ containing R.*

Proof. Let $A=\coprod_{a,b\in\mathcal{C}}A_{a,b}=\coprod_{a,b\in\mathcal{C}}\mathrm{Hom}_{\mathcal{C}}(a,b)\times\mathrm{Hom}_{\mathcal{C}}(a,b)\subseteq\mathrm{Mor}\,\mathcal{C}\times\mathrm{Mor}\,\mathcal{C}$. Now $(f,f')\in A_{a,b}$ if and only if f and f' both are in $\mathrm{Hom}_{\mathcal{C}}(a,b)$. This is clearly an equivalence relation, and if $g:a'\to a$ and $h:b\to b'$ are morphisms then $hfg,hf'g:a'\to b'$ is a morphism, i.e. $(hfg,hf'g)\in A_{a,b}$. In addition $R_{a,b}\subseteq A_{a,b}$ by definition, so A is a congruence containing R.

Let R' be the intersection of all congruence relations R'' containing R. Clearly $R\subseteq R'$ and $R'\subseteq R''$ for all congruence relations containing R. So we just need to show that R' is a congruence. All parts of this problem follows trivially from the fact that the R'' are congruence relations, and the fact that $(f,f')\in R'_{a,b}$ if and only if $(f,f')\in R''_{a,b}$ for all R''. For example, for property *(ii)* of R' we have $(f,f')\in R'_{a,b}$ if and only if $(f,f')\in R''_{a,b}$ for all R'' which implies (by property *(ii)* of R'') that $(f',f)\in R''_{a,b}$ for all R'' which is true if and only if $(f',f)\in R'_{a,b}$. □

Definition A.4.7. *Let $\mathcal{C}$ be any category and R a relation on $\mathcal{C}$. The* ***quotient category*** *$\mathcal{C}/R$ of $\mathcal{C}$ by R, is the category with $Ob(\mathcal{C}/R)=Ob\,\mathcal{C}$ and $Hom_{\mathcal{C}/R}(a,b)=Hom_{\mathcal{C}}(a,b)/R'_{a,b}$, where R' is the least congruence containing R.*

Let $\pi_R:\mathcal{C}\to\mathcal{C}/R$ be the **quotient functor** of $\mathcal{C}$ by R that act like identity on objects and sending morphisms to their equivalence class. To show that this indeed is a functor, we need to show that compositions in the quotient are well defined. Let $(f,f')\in R'_{a,b}$ and $(g,g')\in R'_{b,c}$, so that in particular $f=f'$ and $g=g'$ in $\mathcal{C}/R$. By the property *(iv)* of congruence we have $(gf,gf')\in R_{a,c}$ and $(gf',g'f')\in R_{a,c}$. So by transitivity (property *(iii)*) we have $(gf,g'f')\in R_{a,c}$, and $gf=g'f'$ in $\mathcal{C}/R$. So we can define compositions in $\mathcal{C}/R$ by picking two arbitrary representatives.

Proposition A.4.8. *Let $\mathcal{C}$ be a category with relation R. Then the quotient functor $\pi_R:\mathcal{C}\to\mathcal{C}/R$ has the following properties:*

(i): $(f,f')\in R_{a,b}$ implies $\pi_R(f)=\pi_R(f')$.

(ii): Let $\mathcal{D}$ be any category and $H:\mathcal{C}\to\mathcal{D}$ be a functor such that $(f,f')\in R_{a,b}$ implies $Hf=Hf'$ for all $f,f'\in Mor\mathcal{C}$. Then there exist a unique functor $H':\mathcal{C}/R\to\mathcal{D}$ such that $H'\circ\pi_R=H$.

Proof. [11] II,8 Proposition 1. □

We now have the tools to construct the localization, so let $\mathcal{C}$ be a small category and let Σ be a subset of $\operatorname{Mor}\mathcal{C}$. Let $\delta_0, \delta_1 : \operatorname{Mor}\mathcal{C} \to \operatorname{Ob}\mathcal{C}$ be the maps such that for all morphisms $f : a \to b$ we have $\delta_0(f) = a$ and $\delta_1(f) = b$. Define the graph $G_\Sigma : \operatorname{Mor}\mathcal{C} \amalg \Sigma \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} \operatorname{Ob}\mathcal{C}$, where ∂_0 and ∂_1 is given by

$$\partial_0 \circ i_1 = \delta_0, \quad \partial_1 \circ i_1 = \delta_1, \quad \partial_0 \circ i_2 = \delta_1|_\Sigma, \text{ and } \partial_1 \circ i_2 = \delta_0|_\Sigma.$$

Here $i_1 : \operatorname{Mor}\mathcal{C} \to \operatorname{Mor}\mathcal{C} \amalg \Sigma$ and $i_2 : \Sigma \to \operatorname{Mor}\mathcal{C} \amalg \Sigma$ are the natural inclusions into the disjoint union.

Definition A.4.9. *The **localization** of $\mathcal{C}$ at Σ, written $\mathcal{C}[\Sigma^{-1}] := Pa(G_\Sigma)/R$, is the quotient of the category of paths $Pa(G_\Sigma)$ by the **localization relation** R defined by:*

(a) $(i_1 g) \circ (i_1 f) \sim i_1(g \circ f)$ *whenever* $g \circ f$ *is defined in* $\mathcal{C}$.

(b) $i_1(Id_a^{\mathcal{C}}) \sim Id_a^{Pa(G_\Sigma)}$ *for all* $a \in Ob\,\mathcal{C}$.

(c) $(i_2\sigma) \circ (i_1\sigma) \sim Id_{\partial_0\sigma}^{Pa(G_\Sigma)}$ *and* $(i_1\sigma) \circ (i_2\sigma) \sim Id_{\partial_1\sigma}^{Pa(G_\Sigma)}$ *for all* $\sigma \in \Sigma$.

The quotient functor of $\operatorname{Pa}(G_\Sigma)$ by R is called the **localization functor**, and we write it like $P_\Sigma : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$.

Definition A.4.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$ be functors between them. Then F is a **left adjoint** and G is the corresponding **right adjoint** if there exists a natural bijection*

$$\phi : Hom_{\mathcal{B}}(F(a), b) \to Hom_{\mathcal{A}}(a, G(b)) \text{ for all } a \in Ob\,\mathcal{A}, b \in Ob\,\mathcal{B}$$

The **naturality** of the bijection is that for all $\alpha : a' \to a$, $\beta : b \to b'$, $f : F(a) \to b$ and $g : a \to G(b)$ we have the following.

$$\begin{array}{ll} \text{(i): } \phi(\beta \circ f) = G\beta \circ \phi(f) & \text{(iii): } \phi^{-1}(g \circ \alpha) = \phi^{-1}(g) \circ F\alpha \\ \text{(ii): } \phi(f \circ F\alpha) = \phi(f) \circ \alpha & \text{(iv): } \phi^{-1}(G\beta \circ g) = \beta \circ \phi^{-1}(g) \end{array} \tag{A.22}$$

Theorem A.4.11. *Lat $\mathcal{A}$ and $\mathcal{B}$ be small categories. Let $F : \mathcal{A} \to \mathcal{B}$ be left adjoint and $G : \mathcal{B} \to \mathcal{A}$ the corresponding right adjoint. Let $\Sigma = \{\sigma \in Mor\,\mathcal{A} | F(\sigma)$ is an isomorphism$\}$, and $P_\Sigma : \mathcal{A} \to \mathcal{A}[\Sigma^{-1}]$ the localization functor. Then the following are equivalent:*

(i) *G is full and faithful*

(ii) *The functor $H : \mathcal{A}[\Sigma^{-1}] \to \mathcal{B}$ from A.4.8 such that $F = H \circ P_\Sigma$ is an equivalence functor.*

Proof. [9] Ch. 1, Proposition 1.3. □

Definition A.4.12. *Let $\mathcal{A}$ be any category.*
A ***reflective subcategory*** *is a full subcategory $\mathcal{B} \subseteq \mathcal{A}$ such that the inclusion functor $i : \mathcal{B} \hookrightarrow \mathcal{A}$ has a left adjoint $L : \mathcal{A} \to \mathcal{B}$ called the* ***reflection functor****.*
A ***coreflective subcategory*** *is a full subcategory $\mathcal{B} \subseteq \mathcal{A}$ such that the inclusion functor has a right adjoint $R : \mathcal{A} \to \mathcal{B}$ called the* ***coreflective functor****.*

Corollary A.4.13. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a reflective subcategory of a small category with reflection functor $L : \mathcal{A} \to \mathcal{B}$. Then $\mathcal{B}$ is equivalent to the localization $\mathcal{A}[\Sigma^{-1}]$ of $\mathcal{A}$ at $\Sigma = \{\sigma \in Mor\,\mathcal{A}\,|\,L(\sigma) \text{ is an iso}\}$.*

Proof. $\mathcal{B}$ is a full subcategory so the inclusion is fully faithful. Thus by A.4.11 we have $\mathcal{A}[\Sigma^{-1}]$ is equivalent to $\mathcal{B}$. □

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