Semi-H-type groups and Semi-Damek-Ricci spaces

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1 Introduction

In 1980 A. Kaplan [14] introduced the notion of Heisenberg-type Lie algebras and Heisenberg-type groups as a natural generalization of the Lie algebra of the Heisenberg group and the Heisenberg group itself. He introduced them to study the composition of two positive definite quadratic forms and certain explicit solutions of sub-elliptic operators. Later he discovered an intimate connection between Heisenberg type algebras and certain representations of Clifford algebras. Kaplan continued by endowing the Heisenberg-type groups with a left invariant metric and studied the Riemannian geometry of these spaces such as curvature, geodesics and isometries. The Heisenberg group and the Heisenberg-type groups can in a natural way be considered as sub-Riemannian manifolds and are considered as core examples in the study of sub-Riemannian manifolds.

A. Lichnerowicz [18] showed in 1944 that in dimensions not greater then four, harmonic spaces coincides with rank-one symmetric spaces. He also conjectured that this is true for higher dimensions. Some years after Kaplan introduced the notion of Heisenberg-type groups, E. Damek [8][7], considered the geometry and curvature of semi-direct extensions of a Heisenberg-type group with a one-dimensional simply connected abelian group and together with F.Ricci in [9], proved that these are harmonic but not always symmetric, giving a counterexample to the so called Lichnerowicz conjecture. These spaces are known as Damek-Ricci spaces.

Some years later P. Ciatti in [4] generalized Kaplans Heisenberg-type Lie algebras to allow for non-degenerate quadratic forms, instead of only positive definite. He called these Lie algebra as semi-H-type Lie algebra and showed their existence and classified them using representations of Clifford algebra, that satisfies a compatibility condition. Analogous to how Kaplan introduced the Heisenberg-type groups as Riemannian manifolds, we can consider the simply connected two-step nilpotent Lie group attached to a semi-H-type Lie algebra and equip it with a left invariant metric. These semi-Riemannian manifolds, were called semi-H-type groups and were studied by L. Cordero and P. Parker in [5].

In this thesis we will study the geometry and curvature of semi-H-type groups, in particular we are interested in finding the semi-Riemannian geodesics. Moreover we want to find geodesics when we consider semi-H-type groups as a sub-semi-Riemannian manifolds. We will also generalize the notion of a Damek-Ricci space, by instead considering the semi-direct extension of a semi-H-type group with a one-dimensional simply connected abelian group. We will denote these spaces as semi-Damek-Ricci spaces and study their curvature and geodesics.

The structure of this thesis is the following:

- In Chapter 2 we will introduce the basic notions, definitions, conventions and results that will be used in this thesis. We assume that reader is familiar with linear algebra and the theory of smooth manifolds.

- In Chapter 3 we give the definition of a semi-H-type Lie algebra and group. We study the curvature and the semi-Riemannian geodesics of a semi-H-type group. Moreover we find the sub-semi-Riemannian geodesics of semi-H-type groups, when considering them as sub-semi-Riemannian manifolds.

- In Chapter 4 we generalize the definition of a Damek-Ricci spaces and study their curvature and geodesics.

- In chapter 5 we summarize the main results of this thesis. Moreover we present some open questions about semi-H-type groups and semi-Damek-Ricci spaces and potential topics for further research.

2 Prerequisites

This section will be devoted to the basic definitions, conventions and results that will be used throughout this thesis. We start by going through the linear algebra theory and work our way to semi-Riemannian manifolds and sub-semi-Riemannian manifolds.

2.1 Vector Spaces

We assume that the reader is familiar with the definition of a vector space and basic linear algebra. In this thesis we will deal with finite dimensional vector space over $\mathbb{R}$, unless said otherwise. We will also use Einstein summation convention, when it is convenient.

2.1.1 Scalar product spaces

Definition 2.1. Let V be a n-dimensional vector space over $\mathbb{R}$. A *scalar product* on V is a nondegenerate symmetric bilinear form

$$g : V \times V \xrightarrow{\sim} \mathbb{R}$$

The tuple (V, g) is called a *scalar product space.* Let $x \in V$, define $g(x, x) = \|x\|^2$ and call it the *square of the norm of* x, we say that x is

$$\begin{array}{lll} \textit{spacelike} & \text{if} & \|x\|^2 > 0 \\ \textit{null} & \text{if} & \|x\|^2 = 0 \\ \textit{timelike} & \text{if} & \|x\|^2 < 0. \end{array}$$

We define $|x| = |g(x, x)|^{\frac{1}{2}} = |\|x\|^2|^{\frac{1}{2}}$ and we have that

$$\begin{array}{lll} |x|^2 = \|x\|^2 & \text{if} & x \text{ is spacelike} \\ |x|^2 = 0 & \text{if} & x \text{ is null} \\ |x|^2 = -\|x\|^2 & \text{if} & x \text{ is timelike} \end{array}$$

Notice that $0 \in V$ is a null vector, since $\|0\|^2 = \|x - x\|^2 = 0$. We say that two vectors $x, y \in V$ are *orthogonal* to each other if $g(x, y) = 0$ and they are *orthonormal* if they are orthogonal and $\|x\|^2 = \pm 1$ and $\|y\|^2 = \pm 1$. We define the following three sets

$$\begin{aligned} \text{Space}(V, g) &= \left\{x \in V \middle|\ x \text{ is spacelike}\right\} \\ \text{Null}(V, g) &= \left\{x \in V \middle|\ x \text{ is null}\right\} \\ \text{Time}(V, g) &= \left\{x \in V \middle|\ x \text{ is timelike}\right\} \end{aligned}$$

Definition 2.2. Let $\{e_i\}_{i=1}^n$ be a basis for V. Then $\{e_i\}_{i=1}^n$ is said to be an *orthonormal basis* if

$$\|e_i\|^2 = \pm 1 \quad \text{and} \quad g(e_i, e_j) = 0 \ \text{ for } i \neq j$$

Proposition 2.3. *[22]*
A scalar product space (V, g) *has an orthonormal basis.*

Let $\{e_i\}_{i=1}^n$ be a basis for V such that $x = x^i e_i$ and $y = y^i e_i$. Then $g(x, y) = x^i y^j g(e_i, e_j)$ is coordinate expression for the scalar product of x with y.

Definition 2.4. Let $\{e_i\}_{i=1}^n$ be a basis for the scalar product space (V, g). We define

$$g_{ij} := g(e_i, e_j)$$

and say that g_{ij} are the *components of* g relative to the basis $\{e_i\}_{i=1}^n$.

Proposition 2.5. *[22]*
Let (V, g) be a scalar product space and $\{e_i\}_{i=1}^n$ be an orthonormal basis for V. Then each $x \in V$ has the unique expression

$$x = g(x, e_i)g_{ii}e_i$$

The requirement that the symmetric bilinear form is nondegenerate is of great importance and we would like to know when a symmetric bilinear form is nondegenerate.

Proposition 2.6. *[22]*
A symmetric bilinear form is nondegenerate if and only if its matrix relative to one basis is invertible.

By proposition 2.6 we have that any scalar product has an invertible matrix associated with it and in an orthonormal basis $\{e_i\}_{i=1}^n$ this matrix has the form

$$g = (g_{ij}) = (\delta_{ij}\epsilon_j) = \begin{bmatrix} \pm 1 & 0 & \dots & 0 \\ 0 & \pm 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \pm 1 \end{bmatrix} \quad \text{where } \epsilon_j = g_{jj} = \pm 1. \tag{2.1.1}$$

Definition 2.7. Let $\{e_i\}_{i=1}^n$ be an ordered orthonormal basis for (V, g) such that the n-tuple $(\epsilon_1, \dots, \epsilon_n)$ has r positive signs at the front and s negative signs at the end. Then the tuple $(\sum_{i=1}^r \epsilon_i, \sum_{i=r}^n -\epsilon_i) = (r, s)$ is called the *signature of* g.

In general the scalar product space $(\mathbb{R}^n, g)$, when g has signature (r, s) will be denoted $\mathbb{R}^{r,s}$. The dot product from linear algebra is a scalar product and in this case two orthogonal vectors are at right angles to each other. This is consistent with our intuition, but in a general scalar product space this may not be true.

Example 2.8. Let $V = \mathbb{R}^2$, $g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and its signature is therefore $(1, 1)$. Let $u = \begin{bmatrix} x \\ y \end{bmatrix}$, we have that $\|u\|^2 = x^2 - y^2$ and

$$\begin{aligned} \text{Space}(V, g) &= \{ \begin{bmatrix} x \\ y \end{bmatrix} \in V |\ x^2 - y^2 > 0 \} \\ \text{Null}(V, g) &= \{ \begin{bmatrix} x \\ y \end{bmatrix} \in V |\ x^2 - y^2 = 0 \} \\ \text{Time}(V, g) &= \{ \begin{bmatrix} x \\ y \end{bmatrix} \in V |\ x^2 - y^2 < 0 \} \end{aligned}$$

The two vectors $v = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$ and $w = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$ are orthogonal to each other, since
$v^T g w = \begin{bmatrix} 1 & \lambda \end{bmatrix} \begin{bmatrix} \lambda \\ -1 \end{bmatrix} = 0.$

Definition 2.9. Let (V, g) be a scalar product space and W a subspace of V. We define the set

$$W^{\perp} = \{x \in V \mid g(x, y) = 0, \ \forall \, y \in W\}.$$

Also W is said to be a *nondegenerate* if $g|_W$ is nondegenerate.

Proposition 2.10. *[22]*
Let (V, g) be a scalar product space and W a subspace of V. The subspace W is nondegenerate if and only if V is the direct sum of W and $W^{\perp}$ i.e $V = W \oplus W^{\perp}$.

2.1.2 Dual space

The dual space of a vector space will be important in this thesis so we give a short review.

Definition 2.11. Let V be a vector space. The *dual space of* V, denoted V^* is the set of all linear transformations from V to $\mathbb{R}$ i.e

$$V^* = \mathrm{Hom}(V, \mathbb{R}).$$

The dual space of V is again a vector space and we call an element in the dual space a *covector*. In fact it is of same dimension as V.

Definition 2.12. Let $\{e_i\}_{i=1}^n$ be a basis for V. We define its *dual basis* as the set of covectors given by

$$\alpha^j(v_i) = \delta_i^j = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Proposition 2.13. *[26]*
The covectors $\{\alpha^i\}_{i=1}^n$ forms a basis for V^*.

All finite dimensional vector space are isomorphic, but the scalar product gives us extra structure to define a special isomorphism between a vector space and its dual.

Definition 2.14. (Musical isomorphisms)
Let (V, g) be a scalar product space and V^* be the dual space of V. Let $x \in V$ and $f \in V^*$, we define two maps called flat and sharp

$$\begin{aligned} \flat : \quad & V \to V^* \\ & x \mapsto \flat x = g(x, \cdot) \end{aligned}$$

$$\begin{aligned} \sharp : \quad & V^* \to V \\ & f \mapsto \sharp f \end{aligned}$$

where $\sharp f$ is such that $f(x) = g(\sharp f, x) \quad \forall \, x \in V$.

These two maps are isomorphisms and in fact they are inverse of each other such that

$$\sharp \circ \flat = \mathrm{I}_V$$
$$\flat \circ \sharp = \mathrm{I}_{V^*}$$

If $x = x^i e_i$ and $f = f_i \alpha^i$ we have that

$$\flat x = g_{ij} x^i \alpha^j \quad \text{and} \quad \sharp f = g^{ij} f_i e_j \quad \text{where } (g^{ij}) \text{ is the inverse of } (g_{ij}).$$

Proposition 2.15. *Let (V, g) be a scalar product space with basis $\{e_i\}_{i=1}^n$ and let $\{\alpha^i\}_{i=1}^n$ be its dual basis. If $f : I \to V^*$ is curve in V^* such that $t \mapsto f_i(t)\alpha^i$, where the component functions $f_i : I \to \mathbb{R}$ are differentiable, then*

$$\frac{d}{dt}(\sharp f) = \sharp(\frac{df}{dt}) \tag{2.1.2}$$

Proof. We compute both sides of (2.1.2)

$$\frac{d}{dt}(\sharp f) = \frac{d}{dt}\Big[g^{ij} f_i(t) e_j\Big] = g^{ij} \dot{f}_j(t) e_i$$

$$\sharp(\frac{df}{dt}) = \sharp\Big[\dot{f}_i(t)\alpha^i\Big] = g^{ij} \dot{f}_j(t) e_i$$

■

There is a natural way to make the dual space of any scalar product space into a scalar product space.

Definition 2.16. We define a scalar product on V^* induced by g, denoted g^* as

$$g^*(f, h) := g(\sharp f, \sharp g)$$

and the matrix of this scalar product is simply $g^* = (g^{ij}) = g^{-1}$.

2.1.3 Tensors

Tensors is a integral part of the machinery we will use in this thesis. We give basic definitions and results.

Definition 2.17. Let V be a vector space and V^* be its dual. A *k-contravariant, l-covariant tensor* over V is a multilinear map

$$\overbrace{V^* \times \cdots \times V^*}^{k} \times \overbrace{V \times \cdots \times V}^{l} \xrightarrow{\sim} \mathbb{R}$$

and we say that it is a $\binom{k}{l}$ tensor. The set of all $\binom{k}{l}$ tensors is denoted $T_l^k(V)$.

Remark 2.18. It is obvious that a covector $V^* \ni \omega : V \xrightarrow{\sim} \mathbb{R}$ is a $\binom{0}{1}$ tensor, but a vector $v \in V$ can be viewed as a $\binom{1}{0}$ tensor. By defining the action of v on ω as $\omega(v)$ i.e $v(\omega) = \omega(v)$. In this sense we have that $V \ni v : V^* \xrightarrow{\sim} \mathbb{R}$.

We can make $T_l^k(V)$ into a vector space by defining addition and scalar multiplication pointwise.

Definition 2.19. Let $\{e_i\}_{i=1}^n$ be a a basis for V and $\{\alpha^i\}_{i=1}^n$ be its dual basis. For $T \in T_l^k(V)$ we define the numbers

$$T^{i_1 \dots i_k}_{j_1 \dots j_l} = T(\alpha^{i_1}, \dots, \alpha^{i_k}, e_{j_1}, \dots, e_{j_l})$$

and we call them the *components of* T.

We can define a multiplication on tensors, called tensor multiplication.

Definition 2.20. Let $T \in T_l^k(V)$ and $S \in T_q^p(V)$. We define their *tensor product* as the $\binom{k+p}{l+q}$ tensor given by

$$(T \otimes S)(\omega^1, \dots, \omega^{k+p}, v^1, \dots, v^{l+q})$$

$$= T(\omega^1, \dots, \omega^k, v^1, \dots, v^l) S(\omega^{k+1}, \dots, \omega^{k+p}, v^{l+1}, \dots, v^{l+q}).$$

If T has components $T^{i_1 \dots i_k}_{j_1 \dots j_l}$ and S has components $S^{r_1 \dots r_p}_{s_1 \dots s_q}$, then $T \otimes S$ has components

$$T^{i_1 \dots i_k}_{j_1 \dots j_l} S^{r_1 \dots r_p}_{s_1 \dots s_q}.$$

Proposition 2.21. *[17]*
Let $\{e_i\}_{i=1}^n$ *be a a basis for* V *and* $\{\alpha^i\}_{i=1}^n$ *be its dual basis. Then*

$$e_{i_1} \otimes \dots \otimes e_{i_k} \otimes \alpha^{j_1} \otimes \dots \otimes \alpha^{j_l} \tag{2.1.3}$$

is a basis for $T_l^k(V)$. *Hence* $\dim T_l^k(V) = n^{k+l}$.

By proposition 2.21 we have that any tensor can be written as a linear combination of (2.1.3). Therefore if $T \in T_l^k(V)$ has components $T^{i_1 \dots i_k}_{j_1 \dots j_l}$, then

$$T = T^{i_1 \dots i_k}_{j_1 \dots j_l} e_{i_1} \otimes \dots \otimes e_{i_k} \otimes \alpha^{j_1} \otimes \dots \otimes \alpha^{j_l}$$

and we write $T = (T^{i_1 \dots i_k}_{j_1 \dots j_l})$.

There is one more algebraic operation we can do on tensors to get a new one, called contraction.

Definition 2.22. Let $T = (T^{i_1 \dots i_k}_{j_1 \dots j_l}) \in T_l^k(V)$. By setting one upper index and one lower index as equal and then summing over them, we preform an operation called *contraction.*

Example 2.23. Let $T = (T^i_j) \in T_1^1(V)$. Preforming contraction over the indices, we get

$$T^i_i = \sum_{i=j} T^i_j.$$

This analogous to the trace of a matrix and as we know is independent of a choice of basis.

Proposition 2.24. *[19]*
The contraction of a $\binom{k}{l}$ *tensor is a* $\binom{k-1}{l-1}$ *tensor.*

We can combine tensor multiplication and contraction, to get a new operation on tensors called contracted multiplication.

Definition 2.25. Let $T \in T^k_l(V)$ and $S \in T^p_q(V)$. By first multiplying T and S and then contracting over a upper and a lower index we get a new $\binom{k+p-1}{l+q-1}$ tensor. This operation is called *contracted multiplication.*

We give a few examples of tensors and various operation on them.

Example 2.26.

i) The scalar product g on a vector space V is a $\binom{0}{2}$ tensor by definition. If $\{e_i\}_{i=1}^n$ is a orthonormal basis for V and $\{\alpha^i\}_{i=1}^n$ its dual, then $g = g_{ij}\alpha^i \otimes \alpha^j$. Likewise g^* is a $\binom{2}{0}$ tensor, $g^* = g^{ij} e_i \otimes e_j$.

ii) If $V \ni x = x^i e_i$, then x is $\binom{1}{0}$ tensor. We can multiply x with g and we get $x \otimes g = x^l g_{ij} e_l \otimes \alpha^i \otimes \alpha^j$ which is a $\binom{1}{2}$ tensor. By contracting, we get a $\binom{0}{1}$ tensor or a covector, in coordinates the covector has the components $g_{ij}x^i\alpha^j$. In fact this is nothing more than applying the isomorphism $\flat$ to the vector x since if $y = y^j e_j$ then

$$\flat(x)(y) = \flat(x)(y^j e_j) = g_{ij}x^i y^j = g_{ij}x^i y^k \delta_{jk} = (g_{ij}x^i\alpha^j)(y^k e_k) = (g_{ij}x^i\alpha^j)(y).$$

A similar argument shows that $\sharp f$ for $f = f_i\alpha^i \in V^*$ is nothing more than than the contracted multiplication of g^* with f i.e $\sharp f = g^{ij} f_i e_j$.

We now give the basic definitions, that allows us to use smooth differential forms later in this thesis.

Definition 2.27. Let T be a contravariant or a covariant tensor of order at least 2.

- T is said to be *symmetric* in two of its arguments if transposing them leaves its value unchanged.
- T is said to be *anti-symmetric* in two of its argument if transposing them changes the sign of its value.

$T \in T^k_l(V)$ is said to be *completely symmetric* if it is symmetric in any two of its argument. $T \in T^k_l(V)$ is called *completely anti-symmetric* if it is anti-symmetric in any two of its arguments.

We will now consider all completely anti-symmetric l-covariant tensors.

Definition 2.28. Let $A_l(V)$ denote the set all completely anti-symmetric l-covariant tensors i.e

$$A_l(V) = \left\{ T \in T^0_l(V) \;\middle|\; T \text{ is completely anti-symmetric} \right\}.$$

Under pointwise addition and scalar multiplication, $A_l(V)$ is vector space. We can define a product such that the product of two completely anti-symmetric tensor, is also a completely anti-symmetric tensor.

Definition 2.29. Let S_l denote the group of all permutations of the set $\{1, \ldots, l\}$. Let $f \in T_l^0(V)$, $\{v_1, \ldots, v_l\}$ be vectors in V and $\sigma \in S_l$, we define a new l-covariant tensor σf by

$$(\sigma f)(v_1, \ldots, v_l) = f(v_{\sigma(1)}, \ldots, v_{\sigma(l)}).$$

Let $f \in A_l(V)$ and $g \in A_m(V)$, we define their *wedge product* as

$$f \wedge g = \frac{1}{l!m!} \sum_{\sigma \in S_{l+m}} (\operatorname{sgn} \sigma) \sigma(f \otimes g) \in A_{l+m}(V).$$

We list some basic properties of the wedge product

Proposition 2.30. *[26]*
Let $f \in A_l(V)$, $g \in A_m(V)$ and $h \in A_r(V)$. Then

i) $f \wedge g = (-1)^{lm} g \wedge f$

ii) $(f \wedge g) \wedge h = f \wedge (g \wedge h)$

We said that $A_l(V)$ is a vector space, we now give a basis for $A_l(V)$.

Proposition 2.31. *[26]*
Let $\{e_i\}_{i=1}^n$ be a basis for V and $\{\alpha^i\}_{i=1}^n$ be its dual basis. Then the set of all k-covariant tensors on the form

$$\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_l} \quad (i_1 < \cdots < i_l)$$

is a basis for $A_l(V)$. Hence we have that $\dim A_l(V) = \binom{n}{k}$ *and if $l > n$ then* $\dim A_l(V) = 0$.

2.1.4 Algebras

Definition 2.32. Let V be a vector space. An *algebra* is a tuple $(V, \cdot)$ where $\cdot : V \times V \xrightarrow{\sim} V$ is a bilinear map. We define the *dimension of* $(V, \cdot)$ as the dimension of V. Moreover an algebra is

Associative	if	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
Unital	if	$\exists\, \mathbb{1} \in V$ such that $\mathbb{1} \cdot y = y = y \cdot \mathbb{1}$, $\forall\, y \in V$
Symmetric	if	$x \cdot y = y \cdot x$
Anti-symmetric	if	$x \cdot y = -y \cdot x$
Graded	if	$V = \bigoplus_{k=0}^{\infty} V^k$ and $\cdot : V^k \times V^l \xrightarrow{\sim} V^{l+k}$ and V^k is a subspace of V.

We give some examples.

Example 2.33.

i) All fields are algebras by definition.

ii) The set of all $n \times n$ matrices $M_n(\mathbb{R})$, with the usual matrix multiplication is an associative and unital algebra. Where the identity element is $\mathbb{1} = I_n$.

iii) The set of all continuous functions on $[0,1]$, with pointwise multiplication is an associative, unital and symmetric algebra.

iv) The vector space $T(V) \coloneqq \bigoplus_{k=0}^{\infty} T_0^k(V)$, with tensor multiplication is a graded algebra. This algebra is called the tensor algebra.

Definition 2.34. Let $(V,\cdot)$ be an algebra and $\{e_i\}_{i=1}^n$ a basis for V. We define the *structure constants* of $(V,\cdot)$ with respect to the basis $\{e_i\}_{i=1}^n$, as the scalars $f_{ij}^k \in \mathbb{R}$ given by

$$e_i \cdot e_j = f_{ij}^k e_k$$

Since the the product is bilinear, we have that the structure constants completely determines the product of any two vectors in V. For $x = x^i e_i \in V$ and $y = y^i e_i \in V$ we have that

$$x \cdot y = x^i y^j e_i \cdot e_j = x^i y^j f_{ij}^k e_k.$$

Just as for vector spaces, we want to know when two algebras are equivalent.

Definition 2.35. Let $(V,\cdot)$ and $(W,\star)$ be two algebras and $\phi : V \xrightarrow{\sim} W$ be vector space homomorphism. We say that ϕ is a an *algebra homomorphism* if

$$\phi(x \cdot y) = \phi(x) \star \phi(y) \quad \forall\, x,y \in V.$$

An *algebra isomorphism* is an injective and surjective algebra homomorphism.

Example 2.36. Let $V = \mathbb{R}^2$ and let $\{e_i\}_{i=1}^2$ be the standard basis. We define the product on $\mathbb{R}^2$ by the table

$\downarrow \cdot \rightarrow$	e_1	e_2
e_1	e_1	e_2
e_2	e_2	$-e_1$

where the leftmost column is the first argument and the top row is the second argument in the product. The structure constants are therefore

f_{ij}^k	$i,j=1$	$i=1,j=2$	$i=2,j=1$	$i,j=2$
$k=1$	1	0	0	-1
$k=2$	0	1	1	0

Let $x = x^i e_i, y = y^i e_i \in \mathbb{R}^2$, then

$$x \cdot y = x^i y^j f_{ij}^k e_k = (x^1 y^1 - x^2 y^2) e_1 + (x^1 y^2 + x^2 y^1) e_2.$$

The vector space $\mathbb{R}^2$ equipped with the product described above is isomorphic to $\mathbb{C}$ with the usual complex multiplication, where the role of the imaginary unit is played by e_2 and the real unit played by e_1.

Definition 2.37. Let $(V,\cdot)$ be an algebra and U, W be two subsets of V. We introduce the notation

$$U \cdot W = \operatorname{span}\big\{x \cdot y \;\big|\; x \in U \text{ and } y \in W\big\}.$$

We say that $(U,\cdot)$ is a *subalgebra* of $(V,\cdot)$ if U is a subspace of V and

$$U \cdot U \subseteq U.$$

We say that $(U, \cdot)$ is an *ideal* of $(V, \cdot)$ if U is a subspace of V and

$$U \cdot V \subseteq U.$$

The *center of an algebra*, denoted $\mathscr{Z}(V, \cdot)$ is the subalgebra given by

$$\mathscr{Z}(V, \cdot) = \big\{x \in V \;\big|\; x \cdot y = 0 \quad \forall\, y \in V\big\}.$$

There are two kinds of algebras that are of special interest for us, namely Lie algebras and Clifford algebras.

Definition 2.38. Let $(\mathfrak{g}, [\cdot, \cdot])$ be an algebra. We say that $(\mathfrak{g}, [\cdot, \cdot])$ is a *Lie algebra* if $\forall\, x, y, z \in \mathfrak{g}$ we have that

- $[x, y] = -[y, x]$ i.e $[\cdot, \cdot]$ is anti-symmetric
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity)

We call $[\cdot, \cdot]$, the Lie bracket of $\mathfrak{g}$.

Example 2.39.

i) Let $(V, \cdot)$ be an associative algebra. We can make $(V, \cdot)$ into a Lie algebra by defining the Lie bracket as
$$[x, y] = x \cdot y - y \cdot x \quad \forall\, x, y \in V.$$

ii) Let V be n-dimensional vector space and let $\mathrm{End}(V)$ be the space of all endomorphisms on V. Under composition of maps, we have that $\mathrm{End}(V)$ is an associative algebra and can therefore be made into a Lie algebra by defining the bracket as

$$[A, B] = A \circ B - B \circ A.$$

This Lie algebra will be denoted $\mathfrak{gl}(V)$. Since all finite dimensional vector spaces are isomorphic, we will also denote it as $\mathfrak{gl}(n, \mathbb{R})$.

Definition 2.40. Let $\mathfrak{g}$ be a Lie algebra. We define its *lower central series* as the sequence with elements given by
$$\mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i] \text{ where } \mathfrak{g}_1 = \mathfrak{g}.$$

and its *upper central series* as the sequence with elements given by

$$\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}^i] \text{ where } \mathfrak{g}^1 = \mathfrak{g}.$$

The Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if its lower central series eventually is zero and *solvable* if its upper central series eventually is zero. Moreover we say that $\mathfrak{g}$ is *k-step nilpotent* if $\mathfrak{g}_i = 0$ when $i \geq k$ and *k-step solvable* if $\mathfrak{g}^i = 0$ when $i \geq k$.

Notice that any nilpotent Lie algebra is also a solvable algebra.

Definition 2.41. A Lie algebra $\mathfrak{g}$ is said to be *abelian* if its center is the whole of $\mathfrak{g}$ i.e $[\mathfrak{g}, \mathfrak{g}] = 0$. A *simple* Lie algebra $\mathfrak{g}$ is a Lie algebra which contains no proper ideals and is not abelian. If a Lie algebra is the direct sum of simple Lie algebras, then it is called *semisimple* and if it is a direct sum of simple and abelian Lie algebras it is called *reductive.*

We now give the basic notions and definitions of Clifford algebras needed in this thesis.

Definition 2.42. Let V be a vector space. A *quadratic form on* V is map $q : V \to \mathbb{R}$ such that

i) $q(ax) = a^2 q(x)$ for all $x \in V$ and $a \in \mathbb{R}$.

ii) the symmetric form $(x, y) \mapsto q(x + y) - q(x) - q(y)$ is bilinear

We say that the quadratic form is *nondegenerate* if $(x, y) \mapsto q(x + y) - q(x) - q(y)$ is nondegenerate. The tuple (V, q) is called a *quadratic space* and when the quadratic form is nondegenerate we call (V, q) a *nondegenerate quadratic space.*

Any scalar product space (V, g) can be made into a nondegenerate quadratic space by simply setting $q(x) = g(x, x)$. The associated bilinear form is nondegenerate, since $(x, y) \mapsto 2g(x, y)$. Hence any scalar product space can be made into a nondegenerate quadratic space. A nondegenerate quadratic space can be made into a scalar product space by setting $g(x, y) = q(x + y) - q(x) - q(y)$. We will only consider quadratic spaces arising from scalar product spaces, so let $q(x) = g(x, x)$ unless said otherwise.

Definition 2.43. Let (V, g) be a scalar product space and $T(V) = \bigoplus_{k=0}^{\infty} T^k(V)$ be the tensor algebra of V. Let $I(V)$ be the two sided ideal, generated by all elements of the form

$$x \otimes x + q(x) \quad \text{for} \quad x \in V.$$

The Clifford algebra associated with (V, g), denoted $\mathbf{Cl}(V, g)$ is defined to be

$$\mathbf{Cl}(V, g) = T(V)/I(V)$$

and multiplication will be written as a juxtaposition of elements.

Alternatively we can define the Clifford algebra associated with (V, g) as the real unital associative algebra, containing isomorphic copies of $\mathbb{R}$ and V as subspaces such that V and $\{1\}$ generates the algebra and

$$xx = -g(x)\mathbb{1} \qquad \text{for all } x \in V.$$

We list some properties of Clifford algebras.

Proposition 2.44. *[23]*
Let $x, y \in V$. Then, in $\mathbf{Cl}(V, g)$

$$g(x, y) = -\frac{1}{2}(xy + yx)$$

and in particular, x and y are orthogonal to each other if and only if $xy = -yx$.

Proposition 2.45. *Let* (V, g) *be a scalar product space with signature* (r, s) *and* $\{e_i\}_{i=1}^{n=r+s}$ *be an orthonormal basis for* (V, g)*. Then*

$$e_i e_i = \begin{cases} -\mathbb{1} & \text{for} \quad 1 \leq i \leq r \\ \mathbb{1} & \text{for} \quad r < i \leq n \end{cases}$$

and

$$e_i e_j = -e_j e_i \quad \text{for } i \neq j$$

Proof. By definition we have that $e_i e_i = -q(e_i)\mathbb{1} = -\|e_i\|^2\mathbb{1}$. Since e_i is orthogonal to e_j, we can use proposition 2.44 and we have that $e_i e_j = -e_j e_i$. ■

Proposition 2.46. *[23]*
Let (V, g) *be a scalar product space of dimension* n *and signature* (r, s)*. Then the Clifford algebra associated with* (V, g) *is of dimension* 2^n *or* 2^{n-1}*. The* 2^{n-1} *case corresponds to when* $r - s + 1$ *is divisible by* 4*.*

Proposition 2.47. *[15]*
Let (V, g) *be a scalar product space and let* $K : V \to \mathcal{A}$ *be a linear map from* V *into a unital associative algebra* $\mathcal{A}$*, such that*

$$K(x)K(x) = -q(x)\mathbb{1} \quad \text{for all } x \in V.$$

Then K *can be extended uniquely into a algebra homomorphism* $\tilde{K} : \mathbf{Cl}(V, g) \to \mathcal{A}$*. Furthermore* $\mathbf{Cl}(V, g)$ *is the unique associative algebra with this property.*

2.1.5 Algebras Representations

For each element in a Lie algebra, we can associate an endomorphism over a vector space. In this way we can think of elements of the algebra as actions over a vector space. This leads us to the definition of representations.

Definition 2.48. Let $\mathfrak{g}$ be a Lie algebra and U a vector space. A *Lie algebra representation* of the Lie algebra $\mathfrak{g}$ over U is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(U)$$

Where $\mathfrak{gl}(U)$ is the Lie algebra of endomorphism of U. The vector space U is called the *representation space* of $\mathfrak{g}$ or a $\mathfrak{g}$*-module* and the representation over U is called *faithful* if ρ is a Lie algebra isomorphism.

Among the representations of Lie algebras, there is one of special interest namely, the adjoint representation.

Definition 2.49. Let $\mathfrak{g}$ be a Lie algebra. The *adjoint* representation of $\mathfrak{g}$ is the map

$$\begin{aligned} \mathrm{ad} : & \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \\ & x \mapsto \mathrm{ad}_x = [x, \cdot] \end{aligned}$$

such that $\mathrm{ad}_x(y) = [x, y]$ for all $x, y \in \mathfrak{g}$.

The representation space of the adjoint representation is $\mathfrak{g}$ considered as a vector space. The kernel of the adjoint representation

$$\text{Ker}(\text{ad}) = \{x \in \mathfrak{g} \mid \text{ad}_x = 0\}$$

is clearly the center of $\mathfrak{g}$ and hence the adjoint representation is faithful if the $\mathcal{Z}(\mathfrak{g}) = 0$. Hence we have that the adjoint representation of any simple Lie algebra is faithful and that the adjoint representation of any abelian Lie algebra is never faithful.

Just as for Lie algebras, if we have an associative algebra we can associate each element of the algebra with an endomorphism over a vector space.

Definition 2.50. Let $(V, \cdot)$ be an associative algebra and U a vector space. A *representation* of the algebra $(V, \cdot)$ over U is an algebra homomorphism

$$\rho : V \to \text{End}(U)$$

Here we consider $\text{End}(U)$ as an algebra under compositions of maps. The vector space U is called the *representation space* of $(V, \cdot)$ or a $(V, \cdot)$-*module* and the representation over U is called *faithful* if ρ is algebra isomorphism.

Out of all associative algebras, the Clifford algebras are of spesial interest for us, as they are intimately related with what is called semi-H-type algebras.

Proposition 2.51. *Let (V, g) be a scalar product space, U a vector space and assume that there exist a linear map $K : V \to \text{End}(U)$ such that*

$$K(x) \circ K(x) = -\|x\|^2 \text{I}_U \qquad \textit{for all } x \in V.$$

Then K can be extended to a representation of the Clifford algebra $\mathbf{Cl}(V, g)$ over U.

Proof. Seeing as $\text{End}(U)$ is a unital associative algebra, with unit I_U, the results follows by setting $\mathcal{A} = \text{End}(U)$ in proposition 2.47. ∎

Definition 2.52. Let (V, g_V) and (U, g_U) be two scalar product spaces. Moreover let ρ be a representation of $\mathbf{Cl}(V, g_v)$ over U, we say that ρ is an *admissible* $\mathbf{Cl}(V, g_V)$ *representation* if

$$g_U(\rho_z v, w) = -g_U(v, \rho_z w) \quad \text{for all } z \in V \text{ and } v, w \in U.$$

We also say that (U, g_U) is a an *admissible* $\mathbf{Cl}(V, g_V)$ *module.*

Proposition 2.53. *[4]*
Let (V, g) be a scalar product space. For all signatures of g there exist an admissible $\mathbf{Cl}(V, g)$ module.

2.2 Manifolds

This section is devoted to the theory of semi-Riemannian manifolds and sub-semi-Riemannian manifolds . We will assume that the reader is familiar with the basic theory of smooth manifolds and vector bundles.

2.2.1 Tensor Fields

Let M be a smooth manifold and let $\mathfrak{X}(M)$ denote the set of smooth vector fields on M and $\mathfrak{X}^*(M)$ the set of smooth differential 1-forms. A module over a ring, has formally the same definition as a vector space, the only difference being that the set of scalars belongs to a ring. The set of smooth functions on a smooth manifold $C^\infty(M)$ is a ring and the set of smooth vector fields is a vector space over $\mathbb{R}$ and a module over $C^\infty(M)$. The set of smooth differential 1-forms $\mathfrak{X}^*(M)$ is the dual module of $\mathfrak{X}(M)$. Analogous to a tensor over a vector space, we can define a similar object over a module.

Definition 2.54. A $\binom{k}{l}$ *tensor field* is a tensor over the $C^\infty(M)$ module $\mathfrak{X}(M)$ i.e a multilinear $C^\infty(M)$ function

$$T : \overbrace{\mathfrak{X}^*(M) \times \cdots \times \mathfrak{X}^*(M)}^{k} \times \overbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}^{l} \xrightarrow{\sim} C^\infty(M)$$

The set of all $\binom{k}{l}$ tensor fields will be denoted as $\mathfrak{T}^k_l(M)$. Just at for tensors over a vector space, a $\binom{k}{l}$ tensor is said to be of type *k-contravariant* and *l-covariant*.

We can let vector fields act on 1-forms by letting $X(\theta) := \theta(X)$, in this sense we have that a vector field $X \in \mathfrak{X}(M)$ is therefore a $\binom{1}{0}$ tensor field and a 1-form $\theta \in \mathfrak{X}^*(M)$ is $\binom{0}{1}$ tensor field. Intuitively a tensor field is a way to smoothly assign a $\binom{k}{l}$ tensor over T_pM for all $p \in M$. The tensor over T_pM assigned at point $p \in M$ by a tensor field $T \in \mathfrak{T}^k_l(M)$, is called the value of T at p and will be denoted as T_p.

We define the tensor product of two tensor fields, symmetric and anti-symmetric tensor fields as in section 2.1.3. Therefore we have that if $T \in \mathfrak{T}^k_l(M)$ and $S \in \mathfrak{T}^p_q(M)$ then $T \otimes S$ is a $\binom{k+p}{l+q}$ tensor field, T is completely symmetric if it is symmetric in any of its two arguments and S is completely anti-symmetric if it is anti-symmetric in any of its two arguments.

Definition 2.55. Let T be a $\binom{0}{2}$ tensor field. Then T is said to be a nondegenerate if

$$T(X, Y) = 0 \text{ for all } Y, \text{ implies } X = 0.$$

If a $\binom{0}{2}$ tensor field is nondegenerate, then its values T_p are nondegenerate bilinear forms over T_pM.
Just as for tensors over vector spaces, we can define the components of a tensor field.

Definition 2.56. If $(U, x^1, \ldots, x^n)$ is a chart on M, then the *components of* $T \in \mathfrak{T}^k_l(M)$ *relative to* $(U, x^1, \ldots, x^n)$ are defined to be the $C^\infty(M)$ functions

$$T^{i_1 \ldots i_k}_{j_1 \cdots j_l} := T(dx^{i_1}, \ldots, dx^{i_k}, \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_l}})$$

With the notion of tensor field components we can define contraction of tensor field and contracted multiplication of two tensor fields as in section 2.1.3.

Proposition 2.57. *[22]*
Let $(U, x^1, \ldots, x^n)$ *be a chart on* M*. If* $T \in \mathfrak{T}^k_l(M)$*, then on* U

$$T = T^{i_1 \ldots i_k}_{j_1 \cdots j_l} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l}$$

If $T \in \mathfrak{T}^0_2(M)$ and $(U, x^1, \ldots, x^n)$ is a chart on M. Then T can be written as a matrix with $C^\infty(U)$ functions as elements

$$(T_{ij}) = \begin{bmatrix} T_{11} & \ldots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \ldots & T_{nn} \end{bmatrix}$$

Moreover if T is nondegenerate, then matrix (T_{ij}) will be invertible.

Proposition 2.58. *[16]*
Let

$$T : \mathfrak{X}^*(M)^k \times \mathfrak{X}(M)^l \xrightarrow{\sim} \mathfrak{X}(M) \tag{2.2.1}$$

be a $C^\infty(M)$ multilinear function and let $\theta, \theta^1, \ldots, \theta^k \in \mathfrak{X}^*(M)$ and $X^1, \ldots, X^l \in \mathfrak{X}(M)$. Then map given by

$$\overline{T}(\theta, \theta^1, \ldots, \theta^k, X^1, \ldots, X^l) := \theta(T(\theta^1, \ldots, \theta^k, X^1, \ldots, X^l)) \tag{2.2.2}$$

is a $\binom{k+1}{l}$ tensor field.

Definition 2.59. Let T and $\overline{T}$ be as in proposition 2.58. Then $\overline{T}$ is said to be *induced by T*.

2.2.2 Connections

Let $\gamma : \mathbb{R} \supset I \to M$ be a smooth curve in (M, g), then its velocity at time $t_0 \in I$ is defined to be tangent vector given by

$$\dot{\gamma}(t_0) = \gamma_{*,t_0}(\frac{d}{dt}\Big|_{t_0}) \in T_{\gamma(t_0)}M$$

If we would like to compute acceleration of this curve we would have to take the difference of two tangent vectors belonging to two different tangent spaces, which makes no sense. A linear connection gives us canonical way to compute the acceleration of curves and in general taking the derivatives of vector fields along vector fields.

Definition 2.60. A *linear connection on a manifold M* is a map
$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ written $(X, Y) \mapsto \nabla_X Y$ such that

i) ∇ is $C^\infty(M)$ linear in the first argument and $\mathbb{R}$ linear in the second argument.

ii) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$

We say that $\nabla_X Y$ is the *covariant derivative of Y along X*.

Let U be an open subset of M and ∇ be a linear connection on M. If $\{E_1, \ldots, E_n\}$ is a local frame on U and $X = X^i E_i$ and $Y = Y^i E_i$ then

$$\nabla_X Y = X(Y^j)E_j + X^i Y^j \nabla_{E_i} E_j \tag{2.2.3}$$

Moreover since $\nabla_{E_i} E_j \in \mathfrak{X}(U)$ we can write $\nabla_{E_i} E_j$ as linear combination of $\{E_1, \ldots, E_n\}$ i.e $\nabla_{E_i} E_j = \Gamma^k_{ij} E_k$ such that equation (2.2.3) becomes

$$\nabla_X Y = X(Y^j)E_j + X^i Y^j \Gamma_{ij}^k E_k$$

The n^3 smooth functions Γ_{ij}^k are called the *Christoffel symbols of* ∇ with respect to the local frame $\{E_1, \ldots, E_n\}$.

We can use a linear connection on a manifold to define the acceleration of a smooth curve. To do so, we need some definitions and results.

Definition 2.61. Let $\gamma : I \to M$ be a smooth curve. A *vector field along* γ is a map $X : I \to TM$ such that $X(t) \in T_{\gamma(t)}M$ for all $t \in I$. The set of all vector fields along γ, will be denoted $\mathfrak{T}(\gamma)$.

Proposition 2.62. *[16]*
Let ∇ *be a linear connection on* M*. For each curve* $\gamma : I \to M$*,* ∇ *determines a unique operator*

$$D_t : \mathfrak{T}(\gamma) \to \mathfrak{T}(\gamma)$$

such that

i) $D_t(aX + bY) = aD_t(X) + bD_t(Y)$ *for* $a, b \in \mathbb{R}$

ii) $D_t(fX) = \dot{f}X + fD_t(X)$ *for* $f \in C^\infty(I)$

iii) If X *is extendible, then for any extension* $\tilde{X}$ *of* X*,* $D_t(X) = \nabla_{\dot{\gamma}}\tilde{X}$

For any $X \in \mathfrak{T}(\gamma)$*,* D_tX *is called* the covariant derivative of X along γ*.*

Equipped with a linear connection, we can now define acceleration of smooth curves and geodesics.

Definition 2.63. Let ∇ be a linear connection on M and γ be a smooth curve in M. We define the acceleration of γ as the covariant derivative of $\dot{\gamma}$ along γ i.e $D_t\dot{\gamma}$. A *geodesic* with respect to ∇ is smooth curve $\gamma : I \to M$ such that $D_t\dot{\gamma} = 0$ i.e vanishing acceleration.

Proposition 2.64. *Let* $(U, x^1, \ldots, x^n)$ *be a chart on* M *and* ∇ *a linear connection on* M*. Moreover let* $\gamma(t) = (x^1, \ldots, x^n)$ *be a geodesic, then* γ *satisfies*

$$\ddot{x}^k + \dot{x}^i\dot{x}^j\Gamma_{ij}^k = 0$$

Proof. We compute $D_t\dot{\gamma}$. The velocity vector field is given by $\dot{\gamma} = \dot{x}^i\frac{\partial}{\partial x^i}$ and therefore we have that

$$D_t(\dot{x}^i(t)\frac{\partial}{\partial x^i}) = (\ddot{x}^k + \dot{x}^i\dot{x}^j\Gamma_{ij}^k)\frac{\partial}{\partial x^k} = 0 \implies \ddot{x}^k + \dot{x}^i\dot{x}^j\Gamma_{ij}^k = 0$$

■

Proposition 2.65. *[16]*
Let M *be a manifold equipped with a linear connection* ∇*. For any* $p \in M$*, any* $V \in T_pM$*, and any* $t_0 \in \mathbb{R}$*, there exist an open interval* $I \subset \mathbb{R}$ *containing* t_0 *and a geodesic* $\gamma : I \to M$ *satisfying* $\gamma(t_0) = p, \dot{\gamma}(t_0) = V$*. Any two such geodesics agree on their common domain.*

2.2.3 Semi-Riemannian Manifolds

Definition 2.66. A *semi-Riemannian metric* g on a smooth manifold M is a nondegenerate symmetric $\binom{0}{2}$ tensor field

$$g : \mathfrak{X}(M)^2 \xrightarrow{\sim} C^\infty(M)$$

such that (T_pM, g_p) is a scalar product space and the signature of g_p is the same for all $p \in M$. A *semi-Riemannian manifold* is smooth manifold equipped with a semi-Riemannian metric and will be denoted (M, g).

We can think of a metric on a smooth manifold as a scalar product on smooth vector fields, but instead of assigning each pair of vector fields with a real number, we assign them a $C^\infty(M)$ function.

Definition 2.67. Let U be an open subset of M and $\{E_i\}_{i=1}^n$ a frame on U. We say that $\{E_i\}_{i=1}^n$ is an *orthonormal frame* if for all $p \in U$ we have that $\{E_i(p)\}_{i=1}^n$ is an orthonormal basis for T_pM.

Definition 2.68. Let (M, g_M) and (N, g_N) be two semi-Riemannian manifolds. An isometry from M to N is diffeomorphism $\phi : M \xrightarrow{C^\infty} N$ such that

$$g_N(\phi_*(X), \phi_*(Y)) = g_M(X, Y) \text{ for all } X, Y \in \mathfrak{X}(M).$$

If there exist an isometry between two semi-Riemannian manifolds they are said to be *isometric*. A map $\phi : M \to N$ is called a *local isometry* if for each $p \in M$ there exist an open subset U of M, such that $\phi|_U$ is an isometry onto an open set of N. If there exist a local isometry, we say that M and N are *locally isometric*.

Given a metric on M, we can use this metric to define an isomorphism between $\mathfrak{X}(M)$ and $\mathfrak{X}^*(M)$. This is analogous to the musical isomorphism on vector space.

Definition 2.69. Let (M, g) be a semi-Riemannian metric, we define the maps

$$\begin{array}{lll} \flat : \mathfrak{X}(M) \longrightarrow \mathfrak{X}^*(M) & \text{such that} & X \mapsto g(X, \cdot) \\ \sharp : \mathfrak{X}^*(M) \to \mathfrak{X}(M) & \text{such that} & \theta \mapsto \sharp\theta \end{array}$$

where $\sharp\theta$ is such that $g(\sharp\theta, X) = \theta(X)$ for all $X \in \mathfrak{X}(M)$.

If $(U, x^1, \ldots, x^n)$ is chart on (M, g), then we can write

$$g = g_{ij} dx^i \otimes dx^j$$

Moreover if $X = X^i \frac{\partial}{\partial x^i}$ and $\theta = \theta_i dx^i$, then $\flat X = g_{ij} X^i dx^j$ and $\sharp\theta = g^{ij}\theta_i \frac{\partial}{\partial x^j}$, where $(g^{ij}) = (g_{ij})^{-1}$.

Definition 2.70. Let $\gamma : I \to \mathbb{R}$ be a smooth curve in (M, g) we define its *arc length* as

$$L(\gamma) = \int_I |\dot{\gamma}| dt$$

The arc length of a curve, could be zero or positive.

Example 2.71. Let $M = \mathbb{R}^2$ with chart $(\mathbb{R}^2, x, y)$ and let $g = dx \otimes dx - dy \otimes dy$. Then we have that

$$g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = 1 \quad g(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = -1 \quad \text{and} \quad g(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = 0$$

Let γ be a smooth curve such that $t \mapsto (t, \epsilon t)$ for $t \in [0,1]$ and $\epsilon \in [-1,1]$. Then its velocity vector field is $\dot{\gamma} = \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial y}$ and $\|\dot{\gamma}\|^2 = 1 - \epsilon^2$. The arc length of γ is therefore

$$L(\gamma) = \int_0^1 \sqrt{1-\epsilon^2} dt = \sqrt{1-\epsilon^2}$$

When $\epsilon = \pm 1$ we have that the arc length of γ is zero. This is because $g(\dot{\gamma}, \dot{\gamma}) = 0$ i.e $\dot{\gamma}(t_0) \in \text{Null}(T_{\gamma(t_0)}\mathbb{R}^2, g_{\gamma(t_0)})$ for all $t_0 \in [0,1]$.

Among all linear connections on (M, g) we choose a very special one called the Levi-Civita connection.

Proposition 2.72. *[22]*
On a semi-Riemannian manifold there is a unique connection ∇ *such that*

i) $[X, Y] = \nabla_X Y - \nabla_Y X$,

ii) $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,

for all $X, Y, Z \in \mathfrak{X}(M)$. *This unique connection is called the Levi-Civita connection and is characterized by the* Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) = & \; Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ & -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned} \tag{2.2.4}$$

From now on, if we talk about a linear connection on a semi-Riemannian manifold we will always mean the Levi-Civita connection, unless said otherwise.

We are now ready to define a semi-Riemannian geodesic.

Definition 2.73. Let (M, g) be a semi-Riemannian manifold and let ∇ be the Levi-Civita connection. A smooth curve is a *semi-Riemannian geodesic* if it is a geodesic with respect to the Levi-Civita connection.

We want to find properties of semi-Riemannian manifolds that are invariant under local isometries, this leads us to define curvature.

Definition 2.74. Let (M, g) be a semi-Riemannian manifold. The map

$$R : \mathfrak{X}(M)^3 \longrightarrow \mathfrak{X}(M) \quad \text{given by}$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is called the *Riemann curvature endomorphism.*

This map measures how much $\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z$ and $\nabla_{[X,Y]} Z$ fails to commute.

Proposition 2.75. *[16]*
The Riemann curvature endomorphism induces a $\binom{1}{3}$ *tensor field on* M.

Definition 2.76. Let (M, g) be a semi-Riemannian manifold and R the Riemann endomorphism. The $\binom{1}{3}$ tensor field induced by the Riemann endomorphism is called the *Riemann curvature tensor* and will be denoted also by R.

Since the Riemann curvature tensor is $\binom{1}{3}$ tensor field, we can lower the contravariant index to get a 4-covariant tensor field. This tensor field will be denoted as Rm and its action on vector fields are given by

$$Rm(W, X, Y, Z) = g(R(X, Y)Z, W)$$

Let $(U, x^1, \dots, x^n)$ be a chart on a semi-Riemannian manifold (M, g). Then the components of the Riemannian curvature tensor R^i_{jkl} relative to this chart are given by

$$R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})\frac{\partial}{\partial x^l} = R^i_{jkl}\frac{\partial}{\partial x^i}$$

Proposition 2.77. *[16][22]*
Let (M, g) *be a semi-Riemannian manifold. Then*

1. $R(X, Y)Z = -R(Y, X)Z$
2. $Rm(W, X, Y, Z) = -Rm(Z, X, Y, W)$
3. $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$
4. $Rm(W, X, Y, Z) = Rm(Y, Z, W, X)$

For any smooth vector fields $X, Y, Z, W \in \mathfrak{X}(M)$.

Definition 2.78. We define two new tensor fields from the Riemann curvature tensor.

$$Ric = R_{ij}dx^i \otimes dx^j \quad \text{where } R_{ij} = R^k_{kij}.$$
$$S = R^i_i \quad \text{where } R^i_i = g^{ij}R_{ij}.$$

Ric is called the *Ricci curvature tensor* and S is called the *Scalar curvature.*

Proposition 2.79. *Let* U *be an open subset of* (M, g) *and* $\{E_i\}_{i=1}^n$ *be an orthonormal frame on* U. *Then*

$$Ric(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(E_i, X)Y, E_i)$$

and

$$S = \sum_{i=1}^{n} Ric(E_i, E_i)\epsilon_i$$

where $\epsilon_i = g(E_i, E_i)$.

Proof. Let $\{F^i\}_{i=1}^n$ be the dual frame of $\{E_i\}_{i=1}^n$, $X = X^i E_i$ and $Y = Y^i E_i$. Then

$$Ric(X,Y) = R^k_{kij}(F^i \otimes F^j)(X,Y) = R^k_{kij} X^i Y^j$$

Since

$$R^k_{kij} X^i Y^j = \sum_{k=1}^n F^k(R(E_k,X)Y) \quad \text{and} \quad R(E_k,X)Y = \sum_{m=1}^n \epsilon_m g(R(E_k,X)Y,E_m)E_m$$

we have that

$$Ric(X,Y) = \sum_{k=1}^n F^k\left[\sum_{m=1}^n \epsilon_m g(R(E_k,X)Y,E_m)E_m\right] = \sum_{k=1}^n \epsilon_k g(R(E_k,X)Y,E_k).$$

Since $\{E_i\}_{i=1}^n$ is an orthonormal frame, we have that $g_{ij} = g^{ij} = \delta_{ij}\epsilon_i$ such that

$$S = \sum_{i=1}^n \sum_{j=1}^n g^{ij} R_{ij} = \sum_{i=1}^n \sum_{j=1}^n \delta_{ij}\epsilon_i R_{ij} = \sum_{i=1}^n Ric(E_i,E_i)\epsilon_i$$

∎

As we will see, the Riemann curvature tensor, Ricci curvature tensor and Scalar curvature are invariant under local isometries. Another such invariant is sectional curvature.

Definition 2.80. Let (M,g) be a semi-Riemannian manifold and $p \in M$. Moreover let $\Pi = \text{span}\{x,y\}$ be a nondegenerate two dimensional subspace of (T_pM, g_p), we define

$$K(\Pi) = \frac{g_p(R_p(x,y)y,x)}{g_p(x,x)g_p(y,y) - g_p(x,y)^2}$$

to be the *sectional curvature of* Π.

Proposition 2.81. *[22]*
Let (M,g) be a semi-Riemannian manifold and $p \in M$. Moreover let Π be a nondegenerate subspace of (T_pM, g_p). Then the sectional curvature of Π does not depend on choice of basis.

A nice interpretation of the scalar curvature is that, if $\{E_i\}$ is an orthonormal frame on U, then

$$S_p = 2\sum_{i<j} K(\Pi_{ij}) \quad \text{where } \Pi_{ij} = \text{span}\{E_i(p), E_j(p)\} \subset T_pM.$$

Proposition 2.82. *[16]*
The Riemann curvature endomorphism and the Riemann curvature tensors are local isometry invariants. In other words, if $\phi : M \to N$ is a local isometry then

$$\phi^*(Rm_N) = Rm_M$$

and

$$R_N\big(\phi_*(X), \phi_*(Y)\big)\phi_*(Z) = \phi_*(R_M(X,Y)Z).$$

As a consequence of proposition 2.82, we have that the Ricci curvature tensor, scalar curvature and sectional curvature are all local invariants of semi-Riemannian manifolds.

2.2.4 Semi-Riemannian submanifolds

Definition 2.83. A subset M_0 of a smooth manifold M, is said to be a *submanifold of* M if

i) M_0 is a topological subspace of M

ii) The inclusion map is smooth and an immersion i.e ι_* is injective.

Let $p \in M_0$ and since $\iota_{*,p} : T_pM_0 \to T_pM$ is injective we can consider T_pM_0 as a vector subspace of T_pM.

Definition 2.84. Let M_0 be a submanifold of M and let g be a semi-Riemannian metric on M. If the pullback of g is a metric tensor on M_0, then we say that (M_0, ι^*g) is a *semi-Riemannian submanifold* of (M, g).

If we consider T_pM_0 as a vector subspace of T_pM, then ι^*g_p is simply the restriction of g_p to T_pM_0. Moreover since ι^*g is a metric tensor by definition, we have that T_pM_0 is a nondegenerate subspace of (T_pM, g_p).

If M_0 is a semi-Riemannian submanifold of (M, g) we would like to compare their Levi-Civita connection and Riemann curvature tensors. So from now, let ∇^0, ∇ be the Levi-Civita connection of M_0 and M respectively.

Definition 2.85. We define the *ambient tangent bundle over* M_0 as

$$TM|_{M_0} = \coprod_{p \in M_0} T_pM$$

and is a smooth vector bundle over M_0.

Given any smooth vector field on M, we can restrict it to get a smooth section of the ambient tangent bundle over M_0. Likewise given a smooth section of the ambient tangent bundle over M_0, we can extended it to be a smooth vector field on M. At each point $p \in M_0$, we have that $T_pM = T_pM_0 \oplus (T_pM_0)^\perp$, since (T_pM_0, g_p) is a nondegenerate subspace of (T_pM, g_p).

Definition 2.86. We define the *normal bundle over* M_0 as

$$NM_0 = \coprod_{p \in M_0} (T_pM_0)^\perp$$

and is a smooth vector bundle over M_0. We also define two projections from the ambient tangent bundle;

$$\pi^\top : TM|_{M_0} \longrightarrow TM_0,$$

$$\pi^\perp : TM|_{M_0} \longrightarrow NM_0.$$

We say that a smooth section X of the ambient tangent bundle is *normal* to M_0 if for all $p \in M$ we have that $X_p \in T_pM_0^\perp$. Likewise a smooth section Y of the ambient tangent bundle is *tangent* to M_0 if for all $p \in M$ we have that $Y_p \in (T_pM_0)$.

These two projection maps smooth sections to smooth sections. We are now ready to define the second fundamental form.

Definition 2.87. Let $X, Y \in \mathfrak{X}(M_0)$ and $\tilde{X}, \tilde{Y}$ be extensions of X, Y to M. Then we define the *second fundamental form* as

$$II(X,Y) = \pi^{\perp}(\nabla_{\tilde{X}}\tilde{Y})$$

Proposition 2.88. *[16]*
The second fundamental form is

1. *Independent of extensions of X and Y*
2. *Bilinear over $C^{\infty}(M_0)$*
3. *Symmetric in X and Y*

Proposition 2.89. *[16]*
Let $X, Y \in \mathfrak{X}(M_0)$ be extended arbitrary to vector fields $\tilde{X}, \tilde{Y}$ on M, then the following formula holds along M_0

$$\nabla_{\tilde{X}}\tilde{Y} = \nabla^0_X Y + II(X,Y)$$

The second fundamental form also play a role when comparing the curvature tensors of M_0 and M.

Proposition 2.90. *[16][22]*
Let M_0 be a semi-Riemannian submanifold of (M, g), with Rm^0 and Rm as their $\binom{0}{4}$ Riemann curvature tensor. Then for any smooth sections W, X, Y, Z of the ambient tangent bundle, tangent to M_0 we have

$$Rm^0(W,X,Y,Z) = Rm(W,X,Y,Z) - g(II(X,W), II(Y,Z)) + g(II(X,Z), II(Y,W)).$$

There are some special submanifolds, that are of special interest, namely totally geodesic submanifold.

Definition 2.91. A semi-Riemannian submanifold M_0 of (M, g) is *totally geodesic* provided its second fundamental form vanishes i.e $II = 0$.

Alternative we say that M_0 is a totally geodesic submanifold of M, if for all $p \in M_0$ and for all geodesics in M tangent to M_0, the geodesics are curves in M_0.

Proposition 2.92. *[22]*
If M_0 is a semi-Riemannian submanifold of (M, g), then the following is equivalent

1. *M_0 is totally geodesic.*
2. *Every geodesic of M_0 is also a geodesic of M.*

Hence we can find geodesics in M, by finding them in the totally geodesic submanifold M_0. Moreover since the second fundamental form vanishes we have that
$Rm(W,X,Y,Z) = Rm^0(W,X,Y,Z)$ for sections of the ambient tangent space tangent to M_0.

Example 2.93. Let $M = \mathbb{R}^3$ with chart $(\mathbb{R}^3, x, y, z)$ and let $M_0 = \{p \in M \mid p = (x, y, 0)\}$. We choose the Euclidean on metric M, given by

$$g = dx \otimes dx + dy \otimes dy + dz \otimes dz.$$

Since M_0 is clearly a submanifold of M and the restriction of g to M_0 is a metric tensor on M_0, we have that M_0 is semi-Riemannian submanifold of M. Moreover any geodesic in M is just a straight line and any geodesic of M tangent to M_0 is a straight line in M_0. Conversely any geodesic in M_0 is a straight line in M_0 and therefore a geodesic in M. Hence M_0 is a totally geodesic submanifold of M.

2.2.5 Sub-semi-Riemannian manifolds

Roughly speaking a sub-semi-Riemannian manifold is a semi-Riemannian manifold with some constraints such that the possible directions of motion are limited.

Definition 2.94. Let M be a smooth manifold and let D be a subbundle of the tangent bundle TM. A *sub-semi-Riemannian* metric on M is a $C^\infty(M)$ bilinear, symmetric and nondegenerate map

$$g : \Gamma(D) \times \Gamma(D) \to C^\infty(M)$$

such that (D_p, g_p) is a scalar product space and the signature of g_p is the same for all $p \in M$. A *sub-semi-Riemannian manifold* is the triple (M, D, g) and D is called the *distribution* of (M, D, g).

The definition of a sub-semi-Riemannian metric mirrors the definitions of a tensor field and semi-Riemannian metric. The key difference is that the fibers of the subbundle are vector subspaces of T_pM i.e $D_p \subset T_pM$ and that the scalar product assigned to $p \in M$ is scalar product on this vector subspace and not the whole of T_pM.

Definition 2.95. Let X be a vector field on M. Then X is said to be *horizontal* if for all $p \in M$ we have that $X_p \in D_p$. A piecewise smooth curve $\gamma : I \to M$ is said to be *horizontal* if its velocity $\dot\gamma$ is be horizontal whenever it exist. We define the *arc length* of a smooth horizontal curve as

$$L(\gamma) = \int_I |\dot\gamma(t)| dt, \quad \dot\gamma(t) \in D_{\gamma(t)}.$$

Definition 2.96. A distribution D is called *bracket generating* if for any $p \in M$ there exist an open neighborhood of p $N(p)$ and a local frame $E = \{E_i\}_{i=1}^r$ of D on $N(p)$, such that

$$T_pM = E_p + [E, E]_p + [E, [E, E]]_p + \ldots$$

Proposition 2.97. *[21]*
Let (M, D, g) be a sub-semi-Riemannian manifold, D be bracket generating and M connected, then any two points of M can be joined by a piecewise smooth horizontal curve.

We wish to define sub-semi-Riemannian geodesics, to do so we need some more definitions and results.

Definition 2.98. Let M be a smooth manifold. A *cometric on* M is a symmetric nondegenerate $\binom{2}{0}$ tensor field on M denoted by g^* such that (T_p^*M, g_p^*) is a scalar product space and the signature of g_p^* is the same for all $p \in M$.

Given a cometric we can define a map analogous to the $\sharp$ map defined in section 2.2.3.

Definition 2.99. Let M be a smooth manifold with cometric g^*. We define the map

$$\beta : \mathfrak{X}^*(M) \longrightarrow \mathfrak{X}(M)$$

such that $\omega(\beta(\theta)) = g^*(\omega, \theta)$ for all $\omega, \theta \in \mathfrak{X}^*(M)$.

If (M, D, g) is a sub-semi-Riemannian manifold, then there exist a unique cometric such that

1. $\mathrm{Im}(\beta) = \Gamma(D)$
2. $\theta(X) = g(\beta(\theta), X)$ for all $\theta \in \mathfrak{X}^*(M)$ and $X \in \Gamma(D)$.

Definition 2.100. Let (M, D, g) be a sub-semi-Riemannian manifold with cometric g^* as in definition 2.2.5. The *Hamiltonian* is given by

$$H(p, \omega_p) = \frac{1}{2} g_p^*(\omega_p, \omega_p) \quad \text{for } p \in M \text{ and } \omega_p \in T_p^*M.$$

If D is spanned by an orthonormal frame $\{E_1, \ldots, E_r\}$ we can write the Hamiltonian easier.

Proposition 2.101. *Let $\{E_1, \ldots, E_r\}$ be an orthonormal frame for the distribution D. Then*

$$H(p, \omega) = \frac{1}{2} \sum_{i=1}^{r} \omega(E_i)^2 \epsilon_i \tag{2.2.5}$$

Proof. Since $\mathrm{Im}(\beta) = \Gamma(D)$, we have that $\beta(\omega) = \sum_{i=1}^{r} a^i E_i$. Hence the Hamiltonian is

$$H(p, \omega) = \frac{1}{2} g^*(\omega, \omega) = \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} a^i a^j g(E_i, E_j) =$$
$$\frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} a^i a^j \epsilon_{ij} = \frac{1}{2} \sum_{i=1}^{r} (a^i)^2 \epsilon_i$$

Moreover we have that

$$\omega(E_i) = g(E_i, \beta(\omega)) = \sum_{j=1}^{r} a^j g(E_i, E_j) = \sum_{j=1}^{r} a^j \epsilon_{ij} = a^i \epsilon_i$$

or equivalently

$$a^i = \omega(E_i) \epsilon_i$$

■

Choosing the coordinates $(U, x^1, \ldots, x^n)$ on M, we have that any smooth one-form $\theta \in \mathfrak{X}^*(U)$ can be written as $\theta = \sum y_i dx^i$, where y_i are smooth functions on U. Having chosen coordinates on M, it induces coordinates $(T^*U, x^1, \ldots, x^n, y_1, \ldots, y_n)$ on T^*M, called canonical coordinates, making the cotangent bundle into an even dimensional manifold. We can make the cotangent bundle of any manifold into a symplectic manifold by defining the symplectic form as the two-form

$$-d\theta = \sum_{i=1}^{n} dx^i \wedge dy_i.$$

Given a function f on the cotangent bundle, then there exists a unique vector field denoted X_f such that

$$df(\cdot) = \omega(X_f, \cdot)$$

This vector field X_f is called the Hamiltonian vector field of f. Computing the integral curves of this vector field gives us the following $2n$ ODE's:

$$\begin{aligned} \dot{x}^i &= \frac{\partial f}{\partial y_i} \\ \dot{y}_i &= -\frac{\partial f}{\partial x^i} \end{aligned} \tag{2.2.6}$$

Seeing as the Hamiltonian is a function on the cotangent bundle, we have that it also must satisfy equations 2.2.6. For more information on symplectic manifolds and Hamiltonian functions see [1] [21]

Definition 2.102. The Hamiltonian $H(p, \omega_p)$ generates $2n$ equations, called the *sub-semi-Riemannian geodesic equations* :

$$\begin{aligned} \dot{x}^i &= \frac{\partial H}{\partial y_i}, \\ \dot{y}_i &= -\frac{\partial H}{\partial x^i}. \end{aligned} \tag{2.2.7}$$

The solutions of the equations (2.2.7) will be the integral curves of the Hamiltonian vector field associated to the Hamiltonian.

Definition 2.103. Let $\gamma = (x(t), y(t))$ be a solution to the geodesic equations and let $\pi : T^*M \to M$ be the projection down to the manifold. Then we define the *sub-semi-Riemannian geodesics* on M as the projected integral curves of X_H i.e $\pi \circ \gamma$.

2.3 Lie Theory

We will now introduce Lie groups, the Lie algebra of a Lie group and the Lie exponential map.

2.3.1 Lie groups

Definition 2.104. A *Lie group* is smooth manifold G together with two smooth maps

$$\begin{aligned} \mu: \quad G \times G &\to G \\ (a, b) &\mapsto \mu(a, b) = ab \end{aligned} \tag{2.3.1}$$

called *multiplication*

$$\begin{aligned} \phi: \quad G &\to G \\ a &\mapsto \phi(a) = a^{-1} \end{aligned} \tag{2.3.2}$$

called *inversion*, that satisfies the group axioms.

Any smooth manifold with these two maps, makes the manifold into a group. We will denote the identity element of a Lie group by e.

Example 2.105. Let $M(n, \mathbb{R})$ be the set of all square matrices with real entries. Since $\det : M(n, \mathbb{R}) \to \mathbb{R}$ is a continuous map, we have that $\det^{-1}(\mathbb{R} - \{0\})$ is a open subset of $\mathbb{R}^{n^2}$. This is in fact the set of all invertible matrices and under the usual matrix multiplication, we have that it becomes a Lie group. This Lie group will be denoted by $GL(n, \mathbb{R})$. Also for any finite dimensional vector space V of dimension n, we have that the set of automorphism on V is isomorphic to $GL(n, \mathbb{R})$. To see this just choose a basis on V and then we can express any automorphism as invertible matrix. The set of all automorphism of a vector space, will be denoted as $GL(V)$.

Definition 2.106. Given two Lie groups G and H, a map $\psi : G \to H$ such that

$$\psi(ab) = \psi(a)\psi(b)$$

and ψ smooth map, is called a *Lie group homomorphism.* If ψ is bijective and smooth i.e an diffeomorphism, we call it a *Lie group isomorphism.*

Definition 2.107. Let G be a Lie group. A *Lie subgroup of* G is pair (H, ϕ) such that

1. H is a Lie group
2. H is a submanifold of G
3. $\phi : H \to G$ is a Lie group homomorphism

Since H is a submanifold of G we know that the inclusion map $\iota : H \to G$ is smooth and that its differential is injective. We can now define Lie groups representations.

Definition 2.108. Given a Lie group G, a Lie group homomorphism $\rho : G \to GL(V)$ is called a *representation* of G . It maps elements of G into linear invertible maps from V to V, i.e for all $g \in G$, $g \mapsto \rho(g) : V \to V$. The map $\rho(g) : V \to V$, is denoted ρ_g and V is called the *representation space.*

Among the all representations of Lie groups, the adjoint representation is of special interest.

Definition 2.109. Let $a : G \times G \to G$ be the map given by $a(g, h) = ghg^{-1}$ and let $a_g : G \to G$ be the map given by $a_g(h) = a(g, h) = ghg^{-1}$, for a fixed $g \in G$. Then a_g is called an *inner automorphism* and it is a Lie group diffeomorphism.

Definition 2.110. Let G be a Lie group and M be smooth manifold. A map $\theta : G \times M \to M$ is called a *left action of G on M* if

$$\theta(gh, p) = \theta(g, \theta(h, p)) \text{ and } \theta(e, p) = p \text{ for all } g, h \in G \text{ and for all } p \in M.$$

For all $g \in G$, let $\theta_g : M \to M$ denote the map given by $p \mapsto \theta(g, p)$.

Proposition 2.111. *[27]*
Let $\theta : G \times M \to M$ be a left action of G on M. Let $p_0 \in M$ be a fixed point of θ_g, i.e $\theta(g, p_0) = \theta_g(p_0) = p_0$ for all $g \in G$. Then the map

$$\phi : G \to GL(T_{p_0}M) \text{ given by}$$

$$\phi(g) = (\theta_g)_{*,p_0}$$

is a linear representation of G.

Two important remarks are that the inner automorphism a_g is a left action of G on G and that identity element of G is a fixed point of a_g.

Definition 2.112. Let G be Lie group. The *Adjoint representation* of G denoted Ad, is given by

$$\text{Ad}_g = (a_g)_{*,e} \in GL(T_eG).$$

Proposition 2.111 tells us that the Adjoint representation is in fact a Lie group representation.

2.3.2 Lie Algebra of a Lie group

If G is a Lie group, then from the theory of smooth manifolds, we know that $\mathfrak{X}(G)$ is a Lie algebra. It turns out that any Lie group has Lie algebra, that is a subalgebra of $\mathfrak{X}(G)$ and isomorphic to the tangent space of the Lie group at the identity.

Definition 2.113. Let G be a Lie group and $a \in G$. The diffeomorphism $L_a : G \to G$ such that $\forall g \in G$ we have that $L_a(g) = \mu(a, g) = ag$, is called *left translation by a*.

This map moves each element of $g \in G$ to $ag \in G$, giving us a tool to move around in the Lie group.

Definition 2.114. Let G be a Lie group. A *left invariant vector field* is a vector field such that

$$(L_g)_{*,p}(X_p) = X_{gp} \text{ for all } g, p \in G.$$

Therefore we se that left translation on the Lie group just permutes the tangent vectors constituting the left invariant vector field X. An important remark is that we did not specify whether the left invariant vector field is smooth or not.

Proposition 2.115. *[26][27] Any left invariant vector field is smooth.*

Since any left invariant vector field is smooth we have that the set of left invariant vector fields is a vector subspace of $\mathfrak{X}(G)$.

Proposition 2.116. *[26][27]*
Let X, Y be two left invariant vector filed on G, then so is $[X, Y]$.

Hence the set of left invariant vector fields on a Lie group is subalgebra of $\mathfrak{X}(M)$. We denote the set of left invariant vector fields on a Lie group as $\mathfrak{g}$ or Lie(G). In fact $\mathfrak{g}$ is isomorphic to T_eG as vector spaces.

Proposition 2.117. *[27]*
The vector space $\mathfrak{g}$ is isomorphic to T_eG under the map

$$\begin{aligned} \alpha : \quad \mathfrak{g} &\to T_eG \\ X &\mapsto X_e \end{aligned} \tag{2.3.3}$$

Hence $\dim(\mathfrak{g}) = \dim(T_eG)$.

Definition 2.118. Let G be a Lie group. We define *the Lie algebra of G* as $\mathfrak{g}$; the set of left invariant vector field with the usual vector field bracket. Moreover we say that G is a *nilpotent Lie group* if its Lie algebra is nilpotent and G is a *solvable Lie group* if its Lie algebra is solvable.

In fact given any finite dimensional real Lie algebra $\mathfrak{g}$, then there exists a simply connected Lie group with $\mathfrak{g}$ as its Lie algebra. See [27].

Example 2.119. The Lie algebra of $GL(V)$ is $\mathfrak{gl}(V)$. To see this, we use that $GL(V) \cong GL(n, \mathbb{R})$ and that $\mathfrak{gl}(V) \cong \mathfrak{gl}(n, \mathbb{R})$. Since $GL(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$, we have that the tangent space at any point of $GL(n, \mathbb{R})$ can be identified with $\mathbb{R}^{n^2}$. In particular we have that

$$T_{\mathrm{I}}GL(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$$

By identifying $\mathbb{R}^{n^2}$ with $M(n, \mathbb{R})$, we have that $\mathfrak{gl}(n, \mathbb{R}) \cong T_{\mathrm{I}}GL(n, \mathbb{R}) \cong \mathfrak{g}$

Proposition 2.120. *[27]*
Let G and H be two Lie groups, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. If $\phi : G \to H$ is a Lie group homomorphism, then

$$\phi_{*,e} : \mathfrak{g} \to \mathfrak{h}$$

is a algebra homomorphism.

We can now give another description of the adjoint representation of a Lie group. We have that

$$\mathrm{Ad} : G \to GL(\mathfrak{g})$$

and moreover by proposition 2.120 we have that

$$\mathrm{Ad}_{*,e} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is an algebra homomorphism, hence $\mathrm{Ad}_{*,e}$ is actually a Lie algebra representation.

Proposition 2.121. *[27]*
Let G be Lie group with Lie algebra $\mathfrak{g}$ and let $X \in \mathfrak{g}$, then

$$\mathrm{Ad}_{*,e}(X) = [X, \cdot]$$

Therefore we that $\mathrm{Ad}_{*,e}$ is actually the adjoint representation ad of $\mathfrak{g}$.

Proposition 2.122. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. If G is simply connected and $\rho : \mathfrak{g} \to \mathfrak{h}$ is an algebra homomorphism, then there exists a unique Lie group homomorphism $\phi : G \to H$ such that $\phi_{*,e} = \rho$.*

2.3.3 The Lie exponential map

Definition 2.123. Let G be a Lie group. A Lie group homomorphism $\phi : \mathbb{R} \to G$ is called a *1-parameter subgroup of* G, where $\mathbb{R}$ is considered to be a Lie group under addition. Hence

$$\phi(s + t) = \phi(s)\phi(t) \text{ for all } s, t \in \mathbb{R}$$

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Since Lie($\mathbb{R}$) is the one dimensional Lie algebra spanned by $\frac{d}{dt}$, we have that any left invariant vector field can be written as $s\frac{d}{dt}$, for $s \in \mathbb{R}$. Now if $X \in \mathfrak{g}$ then the map

$$s\frac{d}{dt} \mapsto sX \tag{2.3.4}$$

is algebra homomorphism from the Lie algebra of $\mathbb{R}$ to $\mathfrak{g}$. Hence by propostion 2.122 there exist a unique 1-parameter group ϕ of G such that its differential at 0 is given by (2.3.4)

Definition 2.124. Let G be Lie group and $\mathfrak{g}$ its Lie algebra. For all $X \in \mathfrak{g}$ we define the maps

$$\begin{aligned} \phi_X : \quad & \mathrm{Lie}(\mathbb{R}) \to \mathfrak{g} \\ & s\tfrac{d}{dt} \mapsto sX \end{aligned}$$

and

$$\exp_X : \mathbb{R} \to G \qquad \text{such that } (\exp_X)_{*,0} = \phi_X$$

We define the *Lie exponential map* as the map given by

$$\begin{aligned} \exp : \quad & \mathfrak{g} \to G \\ & X \mapsto \exp_X(1) \end{aligned}$$

When we want to be explicit we write $\exp_G$ for the exponential map of the Lie group G. We list some properties of the Lie exponential map.

Proposition 2.125. *[27]*
Let $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Then

1. $\exp(tX) = \exp_X(t)$
2. $\exp(t_1X + t_2X) = \exp(t_1X)\exp(t_2X)$

3. $\exp(-tX) = \exp(tX)^{-1}$

4. $\exp$ *is smooth and* $(\exp)_{*,e}$ *is the identity map on* $\mathfrak{g}$*. Hence* $\exp$ *is a diffeomorphism of a neighbourhood about* $0 \in \mathfrak{g}$ *to a neighbourhood about* $e \in G$*.*

Infact when a connected Lie group is nilpotent, we have that the Lie exponential map is diffeomorphism from the Lie algebra onto the Lie group.

Proposition 2.126. *[13]*
Let G *be a connected nilpotent Lie group and* $\mathfrak{g}$ *be its Lie algebra. Then the Lie exponential map is a diffeomorphism from* $\mathfrak{g}$ *onto* G*.*

Proposition 2.127. *[27]*
Let A *be subgroup of the Lie group* G *and let* $\mathfrak{a}$ *be a subspace of* $\mathfrak{g}$*. Let* U *be a neighborhood of* $0 \in \mathfrak{g}$ *diffeomorphic under the Lie exponential map with a neighborhood* U *of* $e \in G$*. Suppose that*

$$\exp(A \cap U) = G \cap V$$

Then A *with the subspace topology is a Lie subgroup of* G*,* $\mathfrak{a}$ *is an subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{a}$ *is the Lie algebra of* A*.*

Proposition 2.128. *[17]*
For any $X \in \mathfrak{gl}(n, \mathbb{R})$*, the series*

$$\sum_{k=0}^{\infty} \frac{X^k}{k!} = I_n + X + \frac{X^2}{2} + \cdots$$

converges to an element in $GL(n, \mathbb{R})$ *denoted* $\exp_{GL}(X)$*. Moreover, the unique one-parameter subgroup of* $GL(n, \mathbb{R})$ *generated by* X *is* $\exp_X(t) = \exp_{GL}(tX)$*.*

Hence by defintion, we have that the Lie exponential map of $GL(n, \mathbb{R})$ is given by matrix exponentiating i.e $\exp(X) = \exp_X(1) = \exp_{GL}(X)$.

There is a formula called the *Baker-Campbell-Hausdorff formula*, that says that the product of two exponentials can be written as the exponential of a series with terms in the Lie algebra. If $X, Y \in \mathfrak{g}$ then

$$\exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[X,Y],Y] + \dots)$$

For more information and proof of this formula see [12] [13] [25] [27].

Proposition 2.129. *[27]*
Let ϕ *be a Lie group homomorphism from* G *to* H*. Then*

$$\phi(\exp_G(X)) = \exp_H(\phi_{*,e}(X)) \text{ for all } X \in \mathfrak{g}.$$

In other words, the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{\ \phi\ } & H \\ {\scriptstyle \exp_G}\uparrow & & \uparrow{\scriptstyle \exp_H} \\ \mathfrak{g} & \xrightarrow[\ \phi_{*,e}\ ]{} & \mathfrak{h} \end{array}$$

Corollary 2.130. *By applying proposition 2.129 we have the following commutative diagram*

$$\begin{array}{ccc} G & \xrightarrow{\ Ad\ } & GL(\mathfrak{g}) \\ {\scriptstyle \exp_G}\uparrow & & \uparrow{\scriptstyle \exp_{GL}} \\ \mathfrak{g} & \xrightarrow[\ ad\ ]{} & \mathfrak{gl}(\mathfrak{g}) \end{array}$$

i.e

$$Ad_{\exp_G(X)} = \exp_{GL}(ad_X) \textit{ for all } X \in \mathfrak{g}.$$

Corollary 2.131. *By applying proposition 2.129 we have a commutative diagram*

$$\begin{array}{ccc} G & \xrightarrow{\ a_g\ } & G \\ {\scriptstyle \exp_G}\uparrow & & \uparrow{\scriptstyle \exp_G} \\ \mathfrak{g} & \xrightarrow[\ Ad_g\ ]{} & \mathfrak{g} \end{array}$$

i.e

$$g \exp_G(X) g^{-1} = a_g(\exp_G(X)) = \exp_G(Ad_g(X)) \textit{ for all } X \in \mathfrak{g} \textit{ and } g \in \mathbf{G}.$$

3 Semi-H-Type Groups

3.1 Semi-H-type Lie algebra

Let $\mathfrak{n}$ be a finite 2-step nilpotent Lie algebra, endowed with a scalar product g. We denote its center by $\mathfrak{z}$ and its orthogonal complement by $\mathfrak{v} = \mathfrak{z}^{\perp}$, such that $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$. Hence we have that

$$[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{z} \quad \text{and} \quad [\mathfrak{n}, \mathfrak{z}] = 0$$

Moreover we assume that the restriction of the scalar product to $\mathfrak{z}$ is nondegenerate. We define the linear map $J : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$, such that for any $v_1, v_2 \in \mathfrak{v}$ and any $z \in \mathfrak{z}$ we have that

$$g(J_z v_1, v_2) = g(z, [v_1, v_2]) \tag{3.1.1}$$

Definition 3.1. If the J-map satisfy the condition

$$J_z^2 v = -\|z\|^2 v \quad \forall z \in \mathfrak{z} \text{ and } \forall v \in \mathfrak{v} \tag{3.1.2}$$

we call $(\mathfrak{n}, [\cdot, \cdot], g, J)$ a *semi-H-type algebra.*

By definition we have that for all $z \in \mathfrak{z}$, J_z is skew symmetric with respect to the scalar product g, i.e $g(J_z v_1, v_2) = -g(v_1, J_z v_2)$.

Proposition 3.2. *If $\mathfrak{n}$ is semi-H-type Lie algebra and $z \in \mathfrak{z}$ such that $\|z\|^2 = 1$, then J_z is a isometry.*

Proof. For J_z to be an isometry we need to check that $g(J_z v_1, J_z v_2) = g(v_1, v_2)$ for any $v_1, v_2 \in \mathfrak{v}$.

$$g(J_z v_1, J_z v_2) = g(z, [v_1, J_z v_2]) = -g(J_z^2 v_1, v_2) = \|z\|^2 g(v_1, v_2) = g(v_1, v_2)$$

■

Proposition 3.3. *The J map satisfy the identity*

$$J_{z_1} J_{z_2} + J_{z_2} J_{z_1} = -2g(z_1, z_2) I_{\mathfrak{v}} \quad \forall z_1, z_2 \in \mathfrak{z} \tag{3.1.3}$$

Proof. By associativity of composition and the distributive property of homomorphisms we have that

$$J_{z_1+z_2}^2 = J_{z_1}^2 + J_{z_1} J_{z_2} + J_{z_2} J_{z_1} + J_{z_2}^2$$

rearranging the terms and using (3.1.2) we get that

$$J_z J_{z'} + J_{z'} J_z = -\|z_1 + z_2\|^2 I_{\mathfrak{v}} + \|z_1\|^2 I_{\mathfrak{v}} + \|z_2\|^2 I_{\mathfrak{v}} = -2g(z_1, z_2) I_{\mathfrak{v}}$$

■

As a consequence of proposition (3.3) we have the following identities

$$g(J_z v_1, J_z v_2) = \|z\|^2 g(v_1, v_2) \tag{3.1.4}$$

$$g(J_{z_2} v, J_{z_2} v) = \|v\|^2 g(z_1, z_2) \tag{3.1.5}$$

$$[v, J_z v] = \|v\|^2 z \tag{3.1.6}$$

Proposition 3.4. *Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be semi-H-type Lie algebra. Then $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$.*

Proof. Let $v \in \mathfrak{v}$, such that $\|v\|^2 = \pm 1$. We show that $\mathrm{ad}_v : \mathfrak{v} \to \mathfrak{z}$ is surjective. We have that

$$g(z_1, \mathrm{ad}_v(J_{z_2}v)) = g(z_1, [v, J_{z_2}v]) = \pm g(z_1, z_2) \quad \text{for all } z_1, z_2 \in \mathfrak{z}$$

and hence $\mathrm{ad}_v(J_{z_2}v) = \pm z_2$ by the nondegenerate property of the scalar product. ∎

Proposition 3.5. *[4]*
Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be semi-H-type Lie algebra. If $g_{\mathfrak{z}}$ has signature (ρ, κ) with $\rho \geq 1$, then $g_{\mathfrak{v}}$ is neutral i.e its signature is $(\frac{n}{2}, \frac{n}{2})$.

Now let $\beta = \{V_1, \ldots, V_n, Z_1, \ldots, Z_m\}$ be an ordered orthonormal basis for $\mathfrak{n}$ such that

$$\|V_i\|^2 = \begin{cases} 1 & \text{if} \quad 1 \leq i \leq r \\ -1 & \text{if} \quad r < i \leq n \end{cases}$$

$$\|Z_\alpha\|^2 = \begin{cases} 1 & \text{if} \quad 1 \leq \alpha \leq \rho \\ -1 & \text{if} \quad \rho < \alpha \leq m \end{cases}$$

Then we have that the structure constants C_{ij}^α of $\mathfrak{n}$ with respect to β are given by

$$[V_i, V_j] = \sum_{\alpha=1}^{m} C_{ij}^\alpha Z_\alpha$$

Definition 3.6. We define the *Clifford constants* with respect to β, as the numbers D_{ij}^α given by

$$J_{Z_\alpha} V_i = \sum_{j=1}^{n} D_{ij}^\alpha V_j \tag{3.1.7}$$

Proposition 3.7. *The structure constants and the J-map constants with respect to β are related in the following way*

$$D_{ij}^\alpha = C_{ij}^\alpha \epsilon_\alpha \epsilon_j$$

Proof. The results follows from

$$g(Z_\alpha, [V_i, V_j]) = C_{ij}^\alpha \epsilon_\alpha$$

$$g(J_{Z_\alpha} V_i, V_j) = D_{ij}^\alpha \epsilon_j$$

∎

Since $(\mathfrak{z}, g_{\mathfrak{z}})$ is a scalar product space and we have that $J_z^2 = -\|z\|^2 \mathrm{I}_{\mathfrak{v}}$ by definition, proposition 2.51, tells us that J-map can be extended to a representation of the Clifford algebra $\mathbf{Cl}(\mathfrak{z}, g_{\mathfrak{z}})$ over $\mathfrak{v}$. Moreover since the J-map is skew symmetric with respect to the scalar product i.e

$$g(J_z v_1, v_2) = -g(v_1, J_z v_2)$$

we have that Clifford representation induced by the J-map is admissible i.e $(\mathfrak{v}, g_{\mathfrak{v}})$ is an admissible $\mathbf{Cl}(\mathfrak{z}, g_{\mathfrak{z}})$ module. Conversely if $K : \mathbf{Cl}(V, g_v) \to \mathrm{End}(U)$ is a representation of the Clifford algebra $\mathbf{Cl}(V, g_v)$, we can restrict it to V and it will satisfy

$$K(v)K(v) = -\|x\|^2 \mathrm{I}_U \quad \text{for all } v \in V.$$

If we equip U with a scalar product g_u and require that (U, g_u) is an admissible $\mathbf{Cl}(V, g_v)$ module i.e

$$g_u(K(v)u_1, u_2) = -g_u(u_1, K(x)u_2)$$

We can then construct a semi-H-type Lie algebra, by defining $\mathfrak{n} = U \oplus V$ with the the scalar product $g = g_u + g_v$ and defining the Lie bracket by

$$g(v, [u_1, u_2]) = g(K(v)u_1, u_2)$$

Therefore we have that the semi-H-type Lie algebras are in one-to-one correspondence with the admissible representation of Clifford algebras.

Proposition 3.8. *[4]*
Let $(\mathfrak{z}, g_{\mathfrak{z}})$ be scalar product space and let $\mathfrak{v}$ be a $\mathbf{Cl}(\mathfrak{z}, g_{\mathfrak{z}})$ module. Then $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ can be endowed with a semi-H-type Lie algebra structure if and only if there exist a scalar product $g_{\mathfrak{v}}$ on $\mathfrak{v}$ such that $(\mathfrak{v}, g_{\mathfrak{v}})$ is an admissible $\mathbf{Cl}(\mathfrak{z}, g_{\mathfrak{z}})$ module.

For a complete classification of semi-H-type Lie algebras see [10] [11].

3.2 Semi-H-Type Groups

From section 2.3.2 we know that any Lie algebra is associated with connected and simply connected Lie group of the same dimension.

Definition 3.9. Let $\mathfrak{n}$ be a semi-H-type algebra and N the simply connected Lie group associated with $\mathfrak{n}$. Then N is called a *semi-H-type group.*

By proposition 2.126 we have that the Lie exponential map of every, simply connected, nilpotent group is a global diffeomorphism. Hence $\exp : \mathfrak{n} \to N$ is a global diffeomorphism. We can use the Lie exponential map and the basis β to define global coordinates on N as follows:

$$(v^1, \ldots, v^n, z^1, \ldots, z^m) \xmapsto{\exp^{-1}} \sum_{i=1}^{n} v^i V_i + \sum_{\alpha=1}^{m} z^\alpha Z_\alpha$$

By use of the Baker-Campbell-Hausdorff formula we can also give the group law.

Proposition 3.10. *Let N be semi-H-type group. If $p = \exp(x)$ and $q = \exp(y)$ with $x, y \in \mathfrak{n}$, then the group law is given by*

$$pq = \exp(x)\exp(y) = \exp(x + y + \frac{1}{2}[x, y]). \tag{3.2.1}$$

Proof. The proof follows immediately by applying Baker-Campbell-Hausdorff formula. ∎

As consequence of the proposition above we see that

$$e = \exp(0) = (0, \dots, 0)$$

is the identity element of a semi-H-type group. If $p = (v^1, \dots, v^n, z^1, \dots, z^m)$ and $q = (u^1, \dots, u^n, t^1, \dots, t^m)$, then their product are given in coordinates by

$$pq = \left(v^1 + u^1, \dots, v^n + u^n, z^1 + t^1 + \frac{1}{2}\sum_{i,j=1}^{n} v^i u^j C_{ij}^1, \dots, z^m + t^m + \frac{1}{2}\sum_{i,j=1}^{n} v^i u^j C_{ij}^m\right) \quad (3.2.2)$$

Proposition 3.11. *The left invariant vector fields on N are given by*

$$V_i = \frac{\partial}{\partial v^i} + \frac{1}{2}\sum_{j=1}^{n}\sum_{\alpha=1}^{m} v^j C_{ji}^{\alpha} \frac{\partial}{\partial z^\alpha} \quad (3.2.3)$$

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} \quad (3.2.4)$$

Proof. To find all left invariant vector fields on N, we differentiate (3.2.2) with respect to q to find $(L_g)_{*,e}$, we get that

$$(L_g)_{*,e} = \begin{bmatrix} I_n & 0 \\ \frac{1}{2}A & I_m \end{bmatrix}$$

where

$$A = \begin{bmatrix} \sum_{i=1}^n v^i C_{i1}^1 & \dots & \sum_{i=1}^n v^i C_{in}^1 \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n v^i C_{i1}^m & \dots & \sum_{i=1}^n v^i C_{in}^m \end{bmatrix}$$

Letting $(L_g)_{*,e}$ act on the vectors $\{\frac{\partial}{\partial v^i}\big|_e\}$ and $\{\frac{\partial}{\partial z^\alpha}\big|_e\}$ we get that

$$V_i = \frac{\partial}{\partial v^i} + \frac{1}{2}\sum_{j=1}^{n}\sum_{\alpha=1}^{m} v^j C_{ji}^{\alpha} \frac{\partial}{\partial z^\alpha} \quad (3.2.5)$$

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} \quad (3.2.6)$$

■

We can make N into a semi-Riemannian manifold, by defining a metric on N. We choose to endow N with a left invariant metric and to do so, we use the scalar product on $\mathfrak{n}$ and left translations as follows

$$g_p(X_p, Y_p) = g((L_{p^{-1}})_{*,p}X_p, (L_{p^{-1}})_{*,p}Y_p) \quad \text{for } X, Y \in \mathfrak{X}(N)$$

making N a Lie group endowed with a metric tensor.

3.3 The Levi-Civita Connection

The Levi-Civita connection on a semi-H-type group, is completely determined by its values on the left invariant vector fields in $\mathfrak{n}$.

Proposition 3.12. *Let N be a semi-H-type group. Then the Levi-Civita connection is given by*

$$\nabla_{(v_1+z_1)}(v_2+z_2) = \frac{1}{2}[v_1, v_2] - \frac{1}{2}J_{z_1}v_2 - \frac{1}{2}J_{z_2}v_1 \tag{3.3.1}$$

for all $v_1, v_2 \in \mathfrak{v}$ and $z_1, z_2 \in \mathfrak{z}$.

Proof. For any left invariant metric g on a Lie group, the Koszul formula reads

$$2g(\nabla_X Y, Z) = g([X,Y], Z) + g([Z,X], Y) + g([Z,Y], X) \tag{3.3.2}$$

Let $v_1, v_2 \in \mathfrak{v}$, $z_1, z_2 \in \mathfrak{z}$ and let $w \in \mathfrak{n}$. Using (3.3.2) we have that

$$\begin{aligned} 2g(\nabla_{v_1}z_1, w) = -g(J_{z_1}v_1, w) &\implies \nabla_{v_1}z_1 = -\tfrac{1}{2}J_{z_1}v_1 \\ 2g(\nabla_{z_1}v_1, w) = -g(J_{z_1}v_1, w) &\implies \nabla_{z_1}v_1 = -\tfrac{1}{2}J_{z_1}v_1 \\ 2g(\nabla_{v_1}v_2, w) = g([v_1, v_2], w) &\implies \nabla_{v_1}v_2 = \tfrac{1}{2}[v_1, v_2] \\ 2g(\nabla_{z_1}z_2, w) = 0 &\implies \nabla_{z_1}z_2 = 0 \end{aligned}$$

■

3.4 Riemann Curvature Tensor

Having computed the Levi-Cevita connection, we can now compute the Riemann curvature endomorphism.

Proposition 3.13. *Let N be a semi-H-type group. Then the Riemann curvature endomorphism is given by*

$$\begin{aligned} R(v_1+z_1, v_2+z_2)(v_3+z_3) = &-\frac{1}{4}J_{[v_2,v_3]}v_1 + \frac{1}{4}J_{[v_1,v_3]}v_2 + \frac{1}{2}J_{[v_1,v_2]}v_3 \\ &+\frac{1}{4}[v_2, J_{z_1}v_3] - \frac{1}{4}[v_1, J_{z_2}v_3] + \frac{1}{4}J_{z_1}J_{z_2}v_3 \\ &-\frac{1}{4}J_{z_2}J_{z_1}v_3 - \frac{1}{4}[v_1, J_{z_3}v_2] + \frac{1}{4}[v_2, J_{z_3}v_1] \\ &+\frac{1}{4}J_{z_1}J_{z_3}v_2 - \frac{1}{4}J_{z_2}J_{z_3}v_1 \end{aligned}$$

for all $v_1, v_2, v_3 \in \mathfrak{v}$ and $z_1, z_2, z_3 \in \mathfrak{z}$.

Proof. Since the Riemann curvature endomorphism is linear we have that

$$\begin{aligned} R(v_1+z_1, v_2+z_2)(v_2+z_3) = &R(v_1,v_2)v_3 + R(z_1,v_2)v_3 + R(v_1,z_2)v_3 + R(z_1,z_2)v_3 \\ &+R(v_1,v_2)z_3 + R(z_1,v_2)z_3 + R(v_1,z_2)z_3 + R(z_1,z_2)z_3 \end{aligned}$$

and we compute each term by using proposition 3.12.

$$R(z_1, z_2)z_3 = 0$$

$$R(v_1, z_1)z_2 = -\frac{1}{4}J_{z_1}J_{z_2}v_1$$

$$R(v_1, v_2)z_1 = -\frac{1}{4}[v_1, J_{z_1}v_2] + \frac{1}{4}[v_2, J_{z_1}v_1]$$

$$R(v_1, z_1)v_2 = -\frac{1}{4}[v_1, J_{z_1}v_2]$$

$$R(z_1, z_2)v_1 = \frac{1}{4}J_{z_1}J_{z_2}v_1 - \frac{1}{4}J_{z_2}J_{z_1}v_1$$

$$R(v_1, v_2)v_3 = -\frac{1}{4}J_{[v_2,v_3]}v_1 + \frac{1}{4}J_{[v_1,v_3]}v_2 + \frac{1}{2}J_{[v_1,v_2]}v_3$$

The rest are given by $R(X, Y)Z = -R(Y, X)Z$. ■

3.5 The Ricci Tensor and Scalar Curvature

Proposition 3.14. *Let N be a semi-H-type group. Then the Ricci tensor is given by*

$$Ric(v_1 + z_1, v_2 + z_2) = -\frac{m}{2}g(v_1, v_2) + \frac{n}{4}g(z_1, z_2)$$

for all $v_1, v_2 \in \mathfrak{n}$ and $z_1, z_2 \in \mathfrak{z}$.

Proof. We compute the Ricci tensor using the ordered orthonormal basis β and by proposition 2.79 we have that

$$Ric(x, y) = \sum_{i=1}^{n} g(R(V_i, x)y, V_i)\epsilon_i + \sum_{\alpha=1}^{m} g(R(Z_\alpha, X)y, Z_\alpha)\epsilon_\alpha \tag{3.5.1}$$

There are three different cases:

Case I: $x, y \in \mathfrak{v}$
We have that

$$g(R(V_i, x)y, V_i) = g(-\frac{1}{4}J_{[x,y]}V_i + \frac{1}{4}J_{[V_i,y]}x + \frac{1}{2}J_{[V_i,x]}y, V_i) = -\frac{3}{4}g([x, V_i], [y, V_i])$$

and

$$g(R(Z_\alpha, x)y, Z_\alpha) = g(-R(x, Z_\alpha)y, Z_\alpha) = \frac{1}{4}([x, J_{Z_\alpha}y], Z_\alpha) = \frac{1}{4}g(J_{Z_\alpha}x, J_{Z_\alpha}y).$$

Hence (3.5.1) becomes

$$Ric(x, y) = -\frac{3}{4}\sum_{i=1}^{n} g([x, V_i], [y, V_i])\epsilon_i + \frac{1}{4}\sum_{\alpha=1}^{m} g(J_{Z_\alpha}x, J_{Z_\alpha}y)\epsilon_\alpha \tag{3.5.2}$$

The first sum in (3.5.2) can be written as

$$\sum_{i=1}^{n} \epsilon_i g([x,V_i],[y,V_i]) = \sum_i \sum_\alpha \epsilon_i \epsilon_\alpha g(g([x,V_i],Z_\alpha)Z_\alpha,[y,V_i]) =$$
$$\sum_\alpha \epsilon_\alpha g(Z_\alpha,[y,\sum_i \epsilon_i g(J_{Z_\alpha}x,V_i)V_i]) =$$
$$\sum_\alpha \epsilon_\alpha g(J_{Z_\alpha}y, J_{Z_\alpha}x).$$

Here we used that $[x,V_i] = \sum_\alpha g([x,V_i],Z_\alpha)Z_\alpha\epsilon_\alpha$ and that $J_{Z_\alpha}x = \sum_i g(J_{Z_\alpha}x,V_i)V_i\epsilon_i$. Therefore we have that (3.5.2) becomes

$$Ric(x,y) = -\frac{1}{2}\sum_\alpha \epsilon_\alpha g(J_{Z_\alpha}x, J_{Z_\alpha}y) = -\frac{1}{2}\sum_\alpha \epsilon_\alpha\epsilon_\alpha g(x,y) = -\frac{m}{2}g(x,y). \tag{3.5.3}$$

Case II: $x, y \in \mathfrak{z}$
We have that

$$g(R(V_i,x)y,V_i) = -\frac{1}{4}g(J_xJ_yV_i,V_i) = \frac{1}{4}\epsilon_i g(x,y)$$

and

$$g(R(Z_\alpha,x)y,Z_\alpha) = 0.$$

Hence (3.5.1) becomes

$$Ric(x,y) = \frac{1}{4}\sum_{i=1}^{n}\epsilon_i\epsilon_i g(x,y) = \frac{n}{4}g(x,y). \tag{3.5.4}$$

Case III: $x \in \mathfrak{z}$ and $y \in \mathfrak{v}$
We have that

$$R(V_i,x,y) = -\frac{1}{4}[V_i,J_xy] \in \mathfrak{z}$$

and

$$R(Z_\alpha,x,y) = \frac{1}{4}J_{Z_\alpha}J_xy - \frac{1}{4}J_xJ_{Z_\alpha}y \in \mathfrak{v}$$

Since $g(R(V_i,x,y),Z_\alpha) = 0$ and $g(R(Z_\alpha,x,y),Z_\alpha) = 0$, we get that equation (3.5.1) becomes

$$Ric(x,y) = 0 \tag{3.5.5}$$

The Ricci tensor is therefore

$$Ric(v_1+z_1,v_2+z_2) = -\frac{m}{2}g(v_1,v_2) + \frac{n}{4}g(z_1,z_2)$$

■

Proposition 3.15. *Let N be a semi-H-type group. Then the scalar curvature is given by*

$$S = -\frac{mn}{4}$$

Proof. By proposition 2.79 and proposition 3.14 we have that

$$S = \sum_{i=1}^{n} Ric(V_i, V_i)\epsilon_i + \sum_{\alpha=1}^{m} Ric(Z_\alpha, Z_\alpha)\epsilon_\alpha = -\sum_{i=1}^{n} \frac{m}{2} + \sum_{\alpha=1}^{m} \frac{n}{4} = -\frac{mn}{4}$$

■

3.6 Sectional Curvature

Proposition 3.16. *Let N be semi-H-Type group. The sectional curvature of the coordinate planes are given by*

$$K(\mathbb{R}V_i \oplus \mathbb{R}V_j) = -\frac{3}{4}\|[V_i, V_j]\|^2 \epsilon_i \epsilon_j \quad \text{for } i \neq j$$

$$K(\mathbb{R}Z_\alpha \oplus \mathbb{R}Z_\lambda) = 0 \quad \text{for } \alpha \neq \lambda$$

$$K(\mathbb{R}V_i \oplus \mathbb{R}Z_\alpha) = \frac{1}{4}$$

Hence any semi-H-type group has nonconstant sectional curvature.

Proof. We compute the sectional curvatures according to the definition and proposition 4.9

$$K(\mathbb{R}V_i \oplus \mathbb{R}V_j) = \frac{g(R(V_i, V_j)V_j, V_i)}{\epsilon_i \epsilon_j} = \frac{3}{4} g(J_{[V_i, V_j]} V_j, V_i)\epsilon_i \epsilon_j =$$

$$-\frac{3}{4}\|[V_i, V_j]\|^2 \epsilon_i \epsilon_j$$

$$K(\mathbb{R}Z_\alpha \oplus \mathbb{R}Z_\lambda) = \frac{g(R(Z_\alpha, Z_\lambda)Z_\lambda, Z_\alpha)}{\epsilon_\alpha \epsilon_\lambda} = g(0, Z_\alpha)\epsilon_\alpha \epsilon_\lambda = 0$$

$$K(\mathbb{R}V_i \oplus \mathbb{R}Z_\alpha) = \frac{g(R(V_i, Z_\alpha)Z_\alpha, V_i)}{\epsilon_i \epsilon_\alpha} = -\frac{1}{4} g(J_{Z_\alpha}^2 V_i, V_i)\epsilon_i \epsilon_\alpha = \frac{1}{4}\epsilon_i^2 \epsilon_\alpha^2 = \frac{1}{4}$$

■

3.7 Semi-Riemannian Geodesics

Theorem 3.17. *The semi-Riemannian geodesics in the semi-H-type group N, passing through the identity at time $t = 0$ with velocity $\dot{\gamma}(0) = V_0 + Z_0$, are given by*

- *If Z_0 is spacelike:*

$$v = \frac{1}{|Z_0|^2}\left[(1 - \cos(|Z_0|t))J_{Z_0}V_0 + |Z_0| \sin(|Z_0|t)V_0\right]$$

$$z = \left[t + \frac{1}{2}\frac{\|V_0\|^2}{|Z_0|^2}\left(t - \frac{\sin(|Z_0|t)}{|Z_0|}\right)\right]Z_0$$

- *If Z_0 is null:*

$$v = tV_0 + \frac{1}{2}t^2 J_{Z_0} V_0$$
$$z = (t + \frac{1}{12}t^3 \|V_0\|^2) Z_0$$

- *If Z_0 is timelike:*

$$v = \frac{1}{|Z_0|^2}\Big[(\cosh(|Z_0|t) - 1) J_{Z_0} V_0 + |Z_0| \sinh(|Z_0|t) V_0\Big]$$
$$z = \left[t + \frac{1}{2}\frac{\|V_0\|^2}{|Z_0|^2}\left(\frac{\sinh(|Z_0|t)}{|Z_0|} - t\right)\right] Z_0$$

Proof. Let $\gamma : \mathbb{R} \to N$ be a curve such that $\gamma(0) = e$, then velocity of the curve is given by

$$\dot{\gamma}(t) = \sum_{i=1}^{n} \dot{v}^i \frac{\partial}{\partial v^i} + \sum_{\alpha=1}^{m} \dot{z}^\alpha \frac{\partial}{\partial z^\alpha}$$

Using equations (3.2.3) and (3.2.4) we have that

$$\dot{\gamma}(t) = \sum_{i=1}^{n} \dot{v}^i V_i + \sum_{\alpha=1}^{m} \left[\dot{z}^\alpha + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \dot{v}^i v^j C_{ij}^\alpha\right] Z_\alpha$$

We define two vectors in $\mathfrak{n}$, given by

$$v = \sum_{i=1}^{n} v^i V_i \quad \text{and} \quad z = \sum_{\alpha=1}^{m} z^\alpha Z_\alpha.$$

By using these two vectors we can express the velocity of the curve γ as

$$\dot{\gamma}(t) = \dot{v} + \dot{z} + \frac{1}{2}[\dot{v}, v]$$

If the curve γ is to be a geodesic we require that $D_t\dot{\gamma} = 0$. Hence we must find the acceleration vector field along γ and therefore compute

$$D_t(\dot{v} + \dot{z} + \frac{1}{2}[\dot{v}, v]) = D_t\dot{v} + D_t\dot{z} + \frac{1}{2}D_t[\dot{v}, v]$$

By using proposition 3.12 we have that

$$D_t\dot{v} = \ddot{v} - \frac{1}{2}J_{(\dot{z}+\frac{1}{2}[\dot{v},v])}\dot{v}$$

$$D_t\dot{z} = \ddot{z} - \frac{1}{2}J_{\dot{z}}\dot{v}$$

$$D_t([\dot{v}, v]) = [\ddot{v}, v] - \frac{1}{2}J_{[\dot{v},v]}\dot{v}$$

and hence we have that the acceleration vector field is given by

$$D_t\dot{\gamma} = \ddot{v} - J_{\dot{z}+\frac{1}{2}[\dot{v},v]}\dot{v} + \ddot{z} + \frac{1}{2}[\ddot{v}, v].$$

The semi-Riemannian geodesic equations for N, are therefore

$$\ddot{v} - J_{\dot{z}+\frac{1}{2}[\dot{v},v]}\dot{v} = 0 \tag{3.7.1}$$

$$\ddot{z} + \frac{1}{2}[\ddot{v}, v] = 0. \tag{3.7.2}$$

Since $\frac{d}{dt}[\dot{v}, v] = [\ddot{v}, v]$, equation (3.7.2) tells us that

$$\dot{z} + \frac{1}{2}[\dot{v}, v] = K$$

and if we assume that $\gamma(0) = e$ and that $\dot{\gamma}(0) = V_0 + Z_0$ we have that

$$K = \dot{z}(0) + \frac{1}{2}[\dot{v}(0), v(0)] = Z_0 + \frac{1}{2}[V_0, 0] = Z_0$$

and therefore

$$\dot{z} = \frac{1}{2}[v, \dot{v}] + Z_0 \tag{3.7.3}$$

Equation (3.7.2) becomes

$$J_{Z_0}\dot{v} = \ddot{v}$$

Using the fact that the solution of the IVP problem $\dot{x} = Ax, x(0) = x_0$ where A is any constant $n \times n$ matrix, is $x = \exp_{GL}(tA)x_0$, we get that

$$\dot{v} = \exp_{GL}(tJ_{Z_0})V_0 \tag{3.7.4}$$

By using the matrix exponential, we have that

$$\exp_{GL}(tJ_{Z_0}) = \sum_{k=0}^{\infty}\frac{(tJ_{Z_0})^k}{k!} = \sum_{k=0}^{\infty}\frac{(tJ_{Z_0})^{2k}}{(2k)!} + \sum_{k=0}^{\infty}\frac{(tJ_{Z_0})^{2k+1}}{(2k+1)!} =$$

$$\sum_{k=0}^{\infty}\frac{t^{2k}(J_{Z_0})^{2k}}{(2k)!} + \sum_{k=0}^{\infty}\frac{t^{2k+1}(J_{Z_0})^{2k}J_{Z_0}}{(2k)!} \tag{3.7.5}$$

where J_{Z_0} to the power of zero is identity i.e $J_{Z_0}^0 = I$.

We need to consider three cases:

CASE I: $\mathbf{Z_0}$ is spacelike

In the case that Z_0 is spacelike we have that $\|Z_0\|^2 > 0$ and $|Z_0|^2 = \|Z_0\|^2$. Using that $J_{Z_0}^2 = -\|Z_0\|^2 I = -|Z_0|^2 I$, equation (3.7.5) reads

$$\sum_{k=0}^{\infty} \frac{t^{2k}(-1)^k |Z_0|^{2k} I}{(2k)!} + \frac{1}{|Z_0|} \sum_{k=0}^{\infty} \frac{t^{2k+1}(-1)^k |Z_0|^{2k+1} J_{Z_0}}{(2k)!} =$$

$$\cos(|Z_0|t)I + \frac{1}{|Z_0|}\sin(|Z_0|t)J_{Z_0} \tag{3.7.6}$$

Inserting equation (3.7.6) into equation (3.7.4)

$$\dot{v} = \cos(|Z_0|t)V_0 + \frac{1}{|Z_0|}\sin(|Z_0|t)J_{Z_0}V_0$$

and then integrating, we get that

$$v = \frac{1}{|Z_0|^2}\Big[(1-\cos(|Z_0|t))J_{Z_0}V_0 + |Z_0|\sin(|Z_0|t)V_0\Big]$$

We compute the Lie bracket of $\dot{v}$ and v:

$$[v, \dot{v}] = \frac{\|V_0\|^2}{|Z_0|^2}\Big[1-\cos(|Z_0|t)\Big]Z_0 \tag{3.7.7}$$

and by inserting (3.7.7) into equation (3.7.3) we have that

$$\dot{z} = \Big[1 + \frac{1}{2}\frac{\|V_0\|^2}{|Z_0|^2}\Big(1-\cos(|Z_0|t)\Big)\Big]Z_0. \tag{3.7.8}$$

We find z by integrating equation (3.7.8) and get that

$$z = \Big[t + \frac{1}{2}\frac{\|V_0\|^2}{|Z_0|^2}\Big(t - \frac{\sin(|Z_0|t)}{|Z_0|}\Big)\Big]Z_0.$$

CASE II: $\mathbf{Z_0}$ is null

In the case that Z_0 is null we have that $\|Z_0\|^2 = 0$ and equation (3.7.5) reads

$$\exp_{GL}(tJ_{Z_0}) = I + tJ_{Z_0}$$

and equation (3.7.4) becomes

$$\dot{v} = V_0 + tJ_{Z_0}V_0 \implies v = tV_0 + \frac{1}{2}t^2 J_{Z_0}V_0.$$

Equation (3.7.3) therefore becomes

$$\dot{z} = (1 + \frac{1}{4}t^2\|V_0\|^2)Z_0 \implies z = (t + \frac{1}{12}t^3\|V_0\|^2)Z_0$$

CASE III: $\mathbf{Z_0}$ is timelike

In the case that Z_0 is timelike we have that $\|Z_0\|^2 < 0$ and $|Z_0|^2 = -\|Z_0\|^2$. Using that $J_{Z_0}^2 = -\|Z_0\|^2 I = |Z_0|^2 I$, equation (3.7.5) reads

$$\sum_{k=0}^{\infty} \frac{t^{2k}|Z_0|^{2k} I}{(2k)!} + \frac{1}{|Z_0|} \sum_{k=0}^{\infty} \frac{t^{2k+1}|Z_0|^{2k+1} J_{Z_0}}{(2k)!} =$$

$$\cosh(|Z_0|t) I + \frac{1}{|Z_0|} \sinh(|Z_0|t) J_{Z_0} \tag{3.7.9}$$

Inserting equation (3.7.9) into equation (3.7.4)

$$\dot{v} = \cosh(|Z_0|t) V_0 + \frac{1}{|Z_0|} \sinh(|Z_0|t) J_{Z_0} V_0$$

and then integrating, we get that

$$v = \frac{1}{|Z_0|^2} \Big[(\cosh(|Z_0|t) - 1) J_{Z_0} V_0 + |Z_0| \sinh(|Z_0|t) V_0 \Big]$$

We compute the Lie bracket of $\dot{v}$ and v:

$$[v, \dot{v}] = \frac{\|V_0\|^2}{|Z_0|^2} \Big[\cosh(|Z_0|t) - 1 \Big] Z_0 \tag{3.7.10}$$

and by inserting (3.7.10) into equation (3.7.3) we have that

$$\dot{z} = \left[1 + \frac{1}{2} \frac{\|V_0\|^2}{|Z_0|^2} \Big(\cosh(|Z_0|t) - 1 \Big) \right] Z_0. \tag{3.7.11}$$

We find z by integrating equation (3.7.11) and get that

$$z = \left[t + \frac{1}{2} \frac{\|V_0\|^2}{|Z_0|^2} \left(\frac{\sinh(|Z_0|t)}{|Z_0|} - t \right) \right] Z_0.$$

■

By theorem 3.17 we see that we can divide the geodesics into more cases.

CASE Ia) $\mathbf{Z_0}$ is spacelike and $\mathbf{V_0}$ is spacelike

$$v = \frac{1}{|Z_0|^2} \Big[(1 - \cos(|Z_0|t)) J_{Z_0} V_0 + |Z_0| \sin(|Z_0|t) V_0 \Big]$$

$$z = \left[t + \frac{1}{2} \frac{|V_0|^2}{|Z_0|^2} \left(t - \frac{\sin(|Z_0|t)}{|Z_0|} \right) \right] Z_0$$

CASE Ib) $\mathbf{Z_0}$ is spacelike and $\mathbf{V_0}$ is null

$$v = \frac{1}{|Z_0|^2}\left[(1-\cos(|Z_0|t))J_{Z_0}V_0 + |Z_0|\sin(|Z_0|t)V_0\right]$$
$$z = tZ_0$$

CASE Ic) $\mathbf{Z_0}$ is spacelike and $\mathbf{V_0}$ is timelike

$$v = \frac{1}{|Z_0|^2}\left[(1-\cos(|Z_0|t))J_{Z_0}V_0 + |Z_0|\sin(|Z_0|t)V_0\right]$$
$$z = \left[t - \frac{1}{2}\frac{|V_0|^2}{|Z_0|^2}\left(t - \frac{\sin(|Z_0|t)}{|Z_0|}\right)\right]Z_0$$

CASE IIa) $\mathbf{Z_0}$ is null and $\mathbf{V_0}$ is spacelike

$$v = tV_0 + \frac{1}{2}t^2 J_{Z_0}V_0$$
$$z = (t + \frac{1}{12}t^3|V_0|^2)Z_0$$

CASE IIb) $\mathbf{Z_0}$ is null and $\mathbf{V_0}$ is null

$$v = tV_0 + \frac{1}{2}t^2 J_{Z_0}V_0$$
$$z = tZ_0$$

CASE IIc) $\mathbf{Z_0}$ is null and $\mathbf{V_0}$ is timelike

$$v = tV_0 + \frac{1}{2}t^2 J_{Z_0}V_0$$
$$= (t - \frac{1}{12}t^3|V_0|^2)Z_0$$

CASE IIIa) $\mathbf{Z_0}$ is timelike and $\mathbf{V_0}$ is spacelike

$$v = \frac{1}{|Z_0|^2}\Big[(\cosh(|Z_0|t) - 1)J_{Z_0}V_0 + |Z_0|\sinh(|Z_0|t)V_0\Big]$$
$$z = \left[t + \frac{1}{2}\frac{|V_0|^2}{|Z_0|^2}\left(\frac{\sinh(|Z_0|t)}{|Z_0|} - t\right)\right]Z_0$$

CASE IIIb) $\mathbf{Z_0}$ is timelike and $\mathbf{V_0}$ is null

$$v = \frac{1}{|Z_0|^2}\Big[(\cosh(|Z_0|t) - 1)J_{Z_0}V_0 + |Z_0|\sinh(|Z_0|t)V_0\Big]$$
$$z = tZ_0$$

CASE IIIc) $\mathbf{Z_0}$ is timelike and $\mathbf{V_0}$ is timelike

$$v = \frac{1}{|Z_0|^2}\Big[(\cosh(|Z_0|t) - 1)J_{Z_0}V_0 + |Z_0|\sinh(|Z_0|t)V_0\Big]$$
$$z = \left[t - \frac{1}{2}\frac{|V_0|^2}{|Z_0|^2}\left(\frac{\sinh(|Z_0|t)}{|Z_0|} - t\right)\right]Z_0$$

Definition 3.18. Let $\gamma : \mathbb{R} \to N$ be the geodesic in N such that $\gamma(0) = e$ and $\dot{\gamma} = V_0 + Z_0$. We define

$$\mathfrak{n}_0 = \mathbb{R}V_0 \oplus \mathbb{R}J_{Z_0}V_0 \oplus \mathbb{R}Z_0$$

Theorem 3.19. *The semi-Riemannian geodesics $\gamma : \mathbb{R} \to N$, such that $\gamma(0) = e$ and $\dot{\gamma}(0) = V_0 + Z_0$, lies in the submanifold $N_0 = \exp(\mathfrak{n}_0)$. If V_0 and Z_0 are not null vectors, then N_0 is a three dimensional semi-Riemannian totally geodesically submanifold and a semi-H-type group.*

Proof. We start by proving that $\mathfrak{n}_0$ is a subalgebra of $\mathfrak{n}$. This is seen by noticing that

$$[V_0, J_{Z_0}V_0] = \|V_0\|^2 Z_0 \qquad [V_0, Z_0] = 0 \qquad [J_{Z_0}V_0, Z_0] = 0$$

and hence $\mathfrak{n}_0$ is a subalgebra of $\mathfrak{n}$. Since

$$\exp(x)\exp(y) = \exp(x + y + \frac{1}{2}[x, y])$$

we have that $N_0 = \exp(\mathfrak{n}_0)$ is a subgroup of N and by proposition 2.127 we have that N_0 is a Lie subgroup of N with $\mathfrak{n}_0$ as its Lie algebra. Hence N_0 is a submanifold of N and by theorem 3.17 we have that any semi-Riemannian geodesic lies in N_0.

Assume that V_0 and Z_0 are not null vectors. We want to show that $\mathfrak{n}_0$ with the inherited scalar product, Lie bracket and J-map is a semi-H-type Lie algebra. Let

$$\mathfrak{v}_0 = \mathbb{R}V_0 \oplus \mathbb{R}J_{Z_0}V_0 \quad \text{and} \quad \mathfrak{z}_0 = \mathbb{R}Z_0.$$

By hypothesis we have that the scalar product on $\mathfrak{n}_0$ and $\mathfrak{z}_0$ is nondegenerate and therefore we have that $\mathfrak{n}_0$ is subalgebra of $\mathfrak{n}$, with center $\mathfrak{z}_0$ and orthogonal complement $\mathfrak{v}_0$. Moreover we have that

$$J_{Z_0}^2 V_0 = -\|Z_0\|^2 V_0 \qquad J_{Z_0}^2 J_{Z_0} V_0 = -\|Z_0\|^2 J_{Z_0} V_0$$

and hence by linearity we have that $J_z^2 = -\|z\|^2 \mathrm{I}_{\mathfrak{v}_0}$ for all $z \in \mathfrak{z}_0$. We conclude therefore that $\mathfrak{n}_0$ is a semi-H-type Lie algebra and the associated simply connected Lie group is a semi-H-type group. By defining the metric on N_0 in the same way we did in section 3.2, we have that N_0 is semi-Riemannian submanifold of N.

To show that N_0 is a totally geodesic semi-Riemannian submanifold we calculate the second fundamental form. Let $v_1, v_2 \in \mathfrak{v}_0$ and $z_1, z_2 \in \mathfrak{z}_0$, then

$$\nabla_{v_1} v_2 = \frac{1}{2}[v_1, v_2] \qquad \nabla_{v_1} z_1 = -\frac{1}{2} J_{z_1} v_1 = \nabla_{z_1} v_1 \qquad \nabla_{z_1} z_2 = 0$$

Hence we have that the second fundamental form vanishes and by 2.92 we have that N_0 is a totally geodesic semi-Riemannian submanifold of N.

■

Proposition 3.20. *Let $N_0 = \exp(\mathfrak{n}_0)$ such that V_0 and Z_0 are not null vectors. Then*

$$K(\mathbb{R}V_0 \oplus \mathbb{R}J_{Z_0}V_0) = -\frac{3}{4}$$

$$K(\mathbb{R}V_0 \oplus \mathbb{R}Z_0) = \frac{1}{4}$$

$$K(\mathbb{R}J_{Z_0}V_0 \oplus \mathbb{R}Z_0) = \frac{1}{4}$$

Proof. Since the second fundamental form vanishes, we have that the Levi-Civita connection of N_0 is given by proposition 3.12 and hence the Riemann curvature endomorphism is the same as for N_0

$$K(\mathbb{R}V_0 \oplus \mathbb{R}J_{Z_0}V_0) = \frac{g(R(V_0, J_{Z_0}V_0)J_{Z_0}V_0, V_0)}{\|V_0\|^4\|Z_0\|^2} = \frac{3}{4}\frac{g(\|V_0\|^2 J_{Z_0}^2 V_0, V_0)}{\|V_0\|^4\|Z_0\|^2} = -\frac{3}{4}$$

$$K(\mathbb{R}V_0 \oplus \mathbb{R}J_{Z_0}V_0) = \frac{g(R(V_0, Z_0)Z_0, V_0)}{\|V_0\|^2\|Z_0\|^2} = -\frac{1}{4}\frac{g(J_{Z_0}^2 V_0, V_0)}{\|V_0\|^2\|Z_0\|^2} = \frac{1}{4}$$

$$K(\mathbb{R}J_{Z_0}V_0 \oplus \mathbb{R}Z_0) = \frac{g(R(J_{Z_0}V_0, Z_0)Z_0, J_{Z_0}V_0)}{\|V_0\|^2\|Z_0\|^4} = -\frac{1}{4}\frac{g(J_{Z_0}^2 J_{Z_0}V_0, J_{Z_0}V_0)}{\|V_0\|^2\|Z_0\|^4} = \frac{1}{4}$$

■

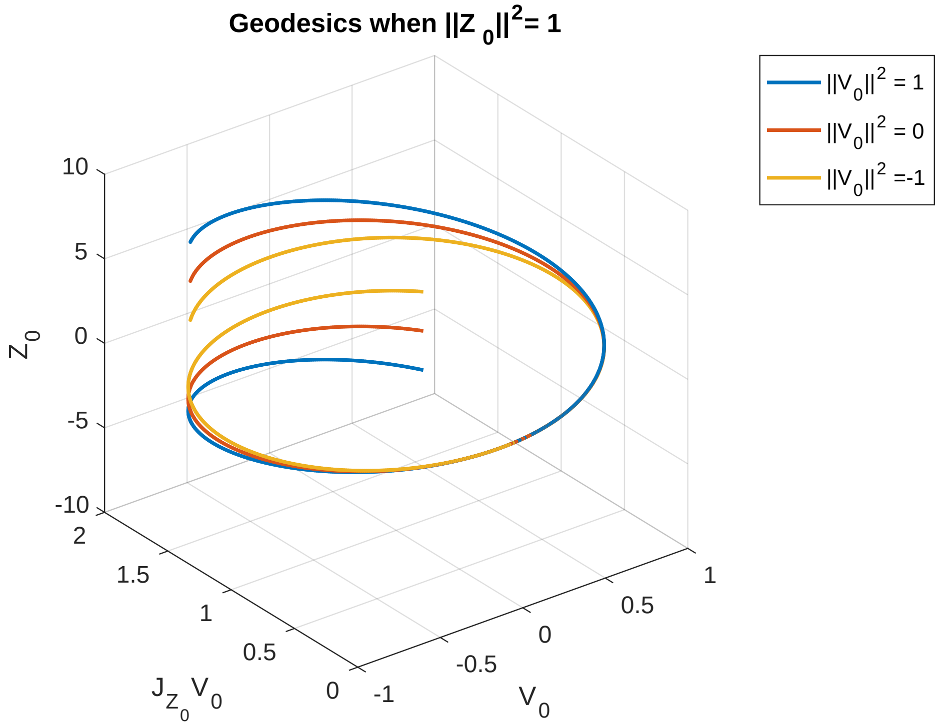

Figure 1: Plot showing the semi-Riemannian geodesics of a semi-H-type group, when $\|Z_0\|^2 = 1$ and $\|V_0\|^2$ takes the values $1, 0, -1$.

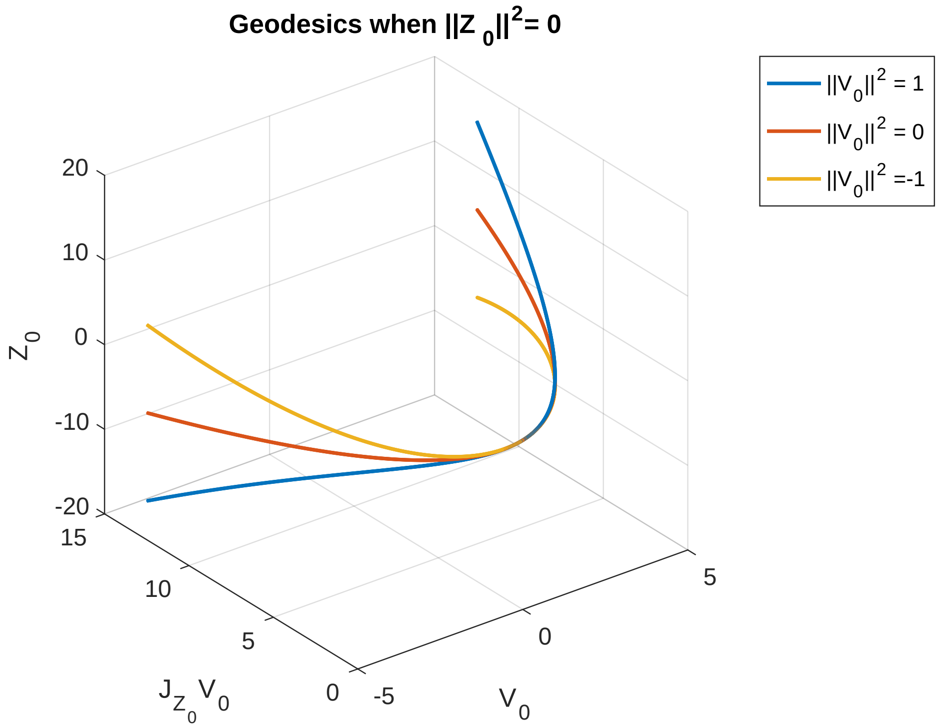

Figure 2: Plot showing the semi-Riemannian geodesics of a semi-H-type group, when $\|Z_0\|^2 = 0$ and $\|V_0\|^2$ takes the values $1, 0, -1$.

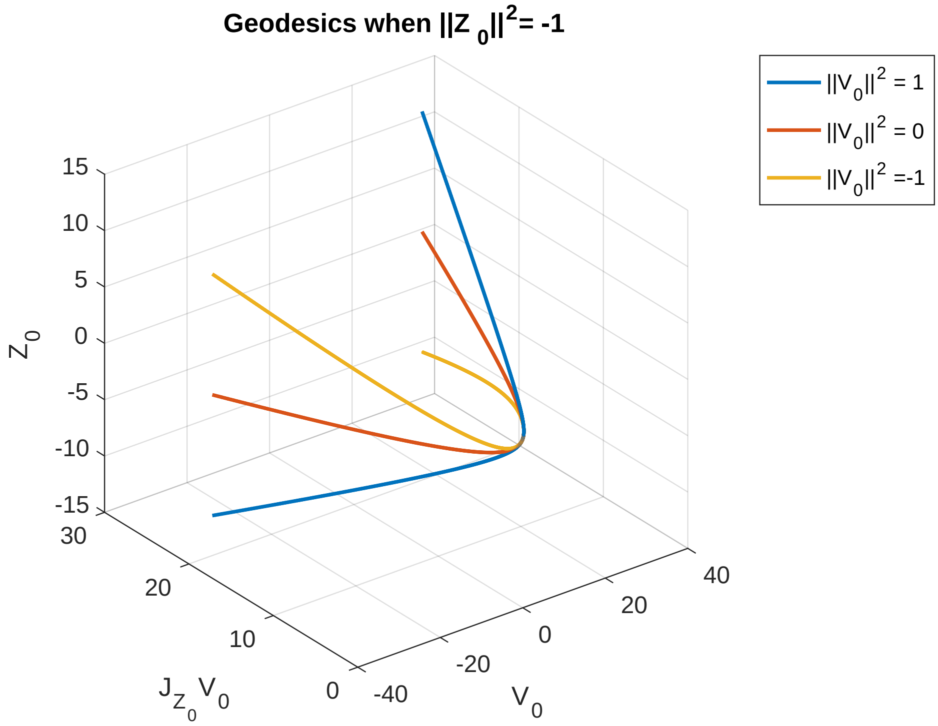

Figure 3: Plot showing the semi-Riemannian geodesics of a semi-H-type group, when $\|Z_0\|^2 = -1$ and $\|V_0\|^2$ takes the values $1, 0, -1$.

3.8 Sub-Semi-H-Type Groups

Definition 3.21. Let $\mathfrak{n}$ be a semi-H-type Lie algebra and let N be the semi-H-type group associated with $\mathfrak{n}$. We define the distributions

$$\mathcal{V}(p) = L_{p,*}(\mathfrak{v}) \quad \text{and} \quad \mathcal{Z}(p) = L_{p,*}(\mathfrak{z})$$

and a left invariant metric on $\mathcal{V}$ given by

$$g_p(X_p, Y_p) = g_{\mathfrak{v}}((L_{p^{-1}})_{*,p}X_p, (L_{p^{-1}})_{*,p}Y_p) \quad \text{for any } X, Y \in \mathcal{V}.$$

Then the tripple $(N, \mathcal{V}, g)$ is called a *sub-semi-H-type group.*

The distribution $\mathcal{V}$ is bracket generating since $\mathfrak{n} = \mathfrak{v} \oplus [\mathfrak{v}, \mathfrak{v}]$ and hence we have that by proposition 2.97 we have that any two points of N can be joined by a horizontal piecewise smooth curve.

Proposition 3.22. *The hamiltonian of* $(N, \mathcal{V}, g)$ *is given by*

$$H(p,\omega) = \frac{1}{2}\sum_{i=1}^{n} \eta_i^2 \epsilon_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\alpha=1}^{m} \eta_i \mu_\alpha C_{ji}^{\alpha} v^j \epsilon_i + \frac{1}{8}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\sum_{\alpha=1}^{m} \mu_\alpha C_{ji}^{\alpha} v^j\right]^2 \epsilon_i$$

Proof. We wish now to compute the Hamiltonian associated with $\mathcal{V}$. By using that $\{V_1, \ldots, V_n\}$ is an orthonormal frame for $\mathcal{V}$ and proposition 2.101 we have that

$$\mathcal{H}(p,\omega) = \frac{1}{2}\sum_{i=1}^{n} \omega(V_i)^2 \epsilon_i$$

Let $\omega = \sum\limits_{i=1}^{n} \eta_i dv^i + \sum\limits_{\alpha=1}^{m} \mu_\alpha dz^\alpha$ be a one-form on N. Using equation (3.2.3) in proposition 3.11, we get that

$$\omega(V_i) = \eta_i + \frac{1}{2}\sum_{j=1}^{n}\sum_{\alpha=1}^{m} \mu_\alpha C_{ji}^{\alpha} v^j$$

and therefore

$$\omega(V_i)^2 = \eta_i^2 + \eta_i \sum_{j=1}^{n}\sum_{\alpha=1}^{m} \mu_\alpha C_{ji}^{\alpha} v^j + \frac{1}{4}\left[\sum_{j=1}^{n}\sum_{\alpha=1}^{m} \mu_\alpha C_{ji}^{\alpha} v^j\right]^2.$$

We get that the Hamiltonian is

$$H(p,\omega) = \frac{1}{2}\sum_{i=1}^{n}\omega(V_i)^2\epsilon_i = \frac{1}{2}\sum_{i=1}^{n}\eta_i^2\epsilon_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\eta_i\mu_\alpha C_{ji}^{\alpha}v^j\epsilon_i + \frac{1}{8}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\mu_\alpha C_{ji}^{\alpha}v^j\right]^2\epsilon_i$$

■

Proposition 3.23. *The sub-semi-Riemannian geodesic equations for* $(N, \mathcal{D}, g)$ *are*

$$\dot{v}^k = \eta_k\epsilon_k + \frac{1}{2}\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\mu_\alpha C_{jk}^{\alpha}v^j\epsilon_k$$

$$\dot{\eta}_k = \frac{1}{2}\sum_{i=1}^{n}\sum_{\alpha=1}^{m}\eta_i\mu_\alpha C_{ik}^{\alpha}\epsilon_i + \frac{1}{4}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\mu_\alpha C_{ji}^{\alpha}v^j\right]\left[\sum_{\alpha=1}^{m}\mu_\alpha C_{ik}^{\alpha}\right]\epsilon_i$$

$$\dot{z}^\lambda = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\eta_i C_{ji}^{\lambda}v^j\epsilon_i + \frac{1}{4}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\mu_\alpha C_{ji}^{\alpha}v^j\right]\left[\sum_{j=1}^{n}C_{ji}^{\alpha}v^j\right]\epsilon_i$$

$$\dot{\mu}_\lambda = 0$$

Proof. Since the sub-semi-Riemannian equations are given by

$$\begin{aligned} \dot{v}^k &= \tfrac{\partial\mathcal{H}}{\partial\eta_k} \qquad & \dot{\eta}_k &= -\tfrac{\partial\mathcal{H}}{\partial v^k} \\ \dot{z}^\lambda &= \tfrac{\partial\mathcal{H}}{\partial\mu_\lambda} \qquad & \dot{\mu}_\lambda &= -\tfrac{\partial\mathcal{H}}{\partial z^\lambda} \end{aligned} \tag{3.8.1}$$

we need only compute the partial derivatives of the Hamiltonian. ■

Definition 3.24. Let $\beta^* = \{\Lambda^1, \ldots, \Lambda^n, \Gamma^1, \ldots, \Gamma^m\}$ be the dual basis of β. We define the vectors v and z in $\mathfrak{v}$ and $\mathfrak{z}$ respectively, given by

$$v = \sum_{i=1}^{n} v^i V_i \quad \text{and} \quad z = \sum_{\alpha=1}^{m} z^\alpha Z_\alpha$$

and η and μ be the covectors in $\mathfrak{v}^*$ and $\mathfrak{z}^*$ respectively, given by

$$\eta = \sum_{i=1}^{n}\eta_i\Lambda^i \quad \text{and} \quad \mu = \sum_{\alpha=1}^{m}\mu_\alpha\Gamma^\alpha.$$

Moreover we define

$$x := \sharp\mu = \sum_{\alpha=1}^{m} \epsilon_\alpha \mu_\alpha Z_\alpha \qquad u := \sharp\eta = \sum_{i=1}^{n} \epsilon_i \eta_i V_i.$$

Following the definition from above we have that

$$(J_x v)^i = \sum_{j=1}^{n}\sum_{\alpha=1}^{m} \mu_\alpha v^j C_{ji}^{\alpha} \epsilon_i$$

$$(J_x u)^i = \sum_{j=1}^{n}\sum_{\alpha=1}^{m} \mu_\alpha \eta_j C_{ji}^{\alpha} \epsilon_j \epsilon_i$$

$$[v, u]^\alpha = \sum_{i=1}^{n}\sum_{j=1}^{n} v^j \eta_i \epsilon_i C_{ij}^{\alpha}$$

and

$$[v, J_x v]^\alpha = \sum_{i=1}^{n}\sum_{j=1}^{n} v^j (J_x v)^i C_{ji}^{\alpha}$$

We can now rewrite the sub-semi-Riemannian geodesic equations as

$$\dot{v}^k = u^k + \frac{1}{2}(J_x v)^k$$

$$\dot{\eta}_k = \frac{1}{2}(J_x u)^k \epsilon_k + \frac{1}{4}(J_x^2 v)^k \epsilon_k$$

$$\dot{z}^\lambda = \frac{1}{2}[v, u]^\lambda + \frac{1}{4}[v, J_x v]^\lambda$$

$$\dot{\mu}_\lambda = 0$$

By using proposition 2.15 we have that the sub-semi-Riemannian geodesic equation are given by

$$\dot{v} = u + \frac{1}{2} J_x v \tag{3.8.2}$$

$$\dot{u} = \frac{1}{2} J_x (u + \frac{1}{2} J_x v) \tag{3.8.3}$$

$$\dot{z} = \frac{1}{2}[v, u + \frac{1}{2} J_x v] \tag{3.8.4}$$

$$\dot{x} = 0 \tag{3.8.5}$$

Theorem 3.25. *The geodesics $\gamma(t)$ of the sub-semi-H-type groups $(N, \mathcal{V}, g)$, passing through the identity at time $t = 0$ with initial velocity $\dot{\gamma}(0) = V_0$ and initial covector μ_0 are given by*

- *If* $\|x_0\|^2 > 0$*:*

$$v = \frac{1}{|x_0|^2}\Big[(1-\cos(|x_0|t))J_{x_0}V_0 + |x_0|\sin(|x_0|t)V_0\Big]$$
$$z = \left[\frac{1}{2}\frac{\|V_0\|^2}{|x_0|^2}\left(t - \frac{\sin(|x_0|t)}{|x_0|}\right)\right]x_0$$

- *If* $\|x_0\|^2 = 0$*:*

$$v = tV_0 + \frac{1}{2}t^2 J_{x_0}V_0$$
$$z = \frac{\|V_0\|^2}{12}t^3 x_0$$

- *If* $\|x_0\|^2 < 0$*:*

$$v = \frac{1}{|x_0|^2}\Big[(\cosh(|x_0|t)-1)J_{x_0}V_0 + |x_0|\sinh(|x_0|t)V_0\Big]$$
$$z = \left[\frac{1}{2}\frac{\|V_0\|^2}{|x_0|^2}\left(\frac{\sinh(|x_0|t)}{|x_0|} - t\right)\right]x_0$$

where $x_0 = \sharp\mu_0$*.*

Proof. Equation (3.8.5) implies that

$$x = x(0) = \sharp(\mu(0)) = \sharp(\mu^0) = x_0$$

and hence $\frac{d}{dt}(J_x v) = J_x\dot{v}$. Now if we combine (3.8.2) and (3.8.3) we get

$$\dot{u} = \frac{1}{2}J_x\dot{v}. \tag{3.8.6}$$

Taking the derivative of (3.8.2) and using (3.8.6), we have that

$$\ddot{v} = J_x\dot{v}$$

Inserting (3.8.2) into (3.8.4) gives us

$$\dot{z} = \frac{1}{2}[v,\dot{v}]$$

Hence we must solve

$$\ddot{v} = J_{x_0}\dot{v} \tag{3.8.7}$$
$$\dot{z} = \frac{1}{2}[v,\dot{v}] \tag{3.8.8}$$

Using the fact that the solution of the IVP problem $\dot{y} = Ay, x(0) = y_0$ where A is any constant $n \times n$ matrix, is $x = \exp_{GL}(tA)y_0$, we get that equation (3.8.7) gives us

$$\dot{v} = \exp_{GL}(tJ_{x_0})V_0 \tag{3.8.9}$$

where

$$\exp_{GL}(tJ_{x_0}) = \sum_{k=0}^{\infty} \frac{(tJ_{x_0})^k}{k!} = \sum_{k=0}^{\infty} \frac{(tJ_{x_0})^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(tJ_{x_0})^{2k+1}}{(2k+1)!} =$$
$$\sum_{k=0}^{\infty} \frac{t^{2k}(J_{x_0})^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{t^{2k+1}(J_{x_0})^{2k}J_{x_0}}{(2k)!} \tag{3.8.10}$$

CASE I: $\|\mathbf{x_0}\|^2 > \mathbf{0}$

In the case that x_0 is spacelike we have that $\|x_0\|^2 > 0$ and $|x_0|^2 = \|x_0\|^2$. Using that $J_{x_0}^2 = -\|x_0\|^2 I = -|x_0|^2 I$, equation (3.8.10) reads

$$\sum_{k=0}^{\infty} \frac{t^{2k}(-1)^k|x_0|^{2k}I}{(2k)!} + \frac{1}{|x_0|}\sum_{k=0}^{\infty} \frac{t^{2k+1}(-1)^k|x_0|^{2k+1}J_{x_0}}{(2k)!} =$$
$$\cos(|x_0|t)I + \frac{1}{|x_0|}\sin(|x_0|t)J_{x_0} \tag{3.8.11}$$

Inserting equation (3.8.11) into equation (3.8.9)

$$\dot{v} = \cos(|x_0|t)V_0 + \frac{1}{|x_0|}\sin(|x_0|t)J_{x_0}V_0$$

and then integrating, we get that

$$v = \frac{1}{|x_0|^2}\Big[(1-\cos(|x_0|t))J_{x_0}V_0 + |x_0|\sin(|x_0|t)V_0\Big]$$

We compute the Lie bracket of v and $\dot{v}$:

$$[v, \dot{v}] = \frac{\|V_0\|^2}{|x_0|^2}\Big[1-\cos(|x_0|t)\Big]x_0 \tag{3.8.12}$$

and by inserting (3.8.12) into equation (3.8.8) we have that

$$\dot{z} = \frac{1}{2}\frac{\|V_0\|^2}{|x_0|^2}\Big[1-\cos(|x_0|t)\Big]x_0 \tag{3.8.13}$$

We find z by integrating equation (3.8.13) and get that

$$z = \frac{1}{2}\frac{\|V_0\|^2}{|x_0|^2}\Big[t - \frac{\sin(|x_0|t)}{|x_0|}\Big]x_0$$

CASE II: $\|\mathbf{x_0}\|^2 = \mathbf{0}$

In the case that x_0 is null we have that $\|x_0\|^2= 0$ and equation (3.7.5) reads

$$\exp_{GL}(tJ_{x_0}) = I + tJ_{x_0}$$

and equation (3.8.9) becomes

$$\dot{v} = V_0 + tJ_{x_0}V_0 \implies v = tV_0 + \frac{1}{2}t^2 J_{x_0}V_0.$$

We compute the Lie bracket of v and $\dot{v}$:

$$[v, \dot{v}] = \frac{1}{2}t^2\|V_0\|^2 x_0 \tag{3.8.14}$$

Equation (3.8.8) therefore becomes

$$\dot{z} = \frac{1}{4}t^2\|V_0\|^2 x_0 \implies z = \frac{1}{12}t^3\|V_0\|^2 x_0$$

CASE III: $\|\mathbf{x_0}\|^2 < \mathbf{0}$

In the case that x_0 is timelike we have that $\|x_0\|^2< 0$ and $|x_0|^2= -\|x_0\|^2$. Using that $J_{x_0}^2 = -\|x_0\|^2 I = |x_0|^2 I$, equation (3.7.5) reads

$$\sum_{k=0}^{\infty}\frac{t^{2k}|x_0|^{2k}I}{(2k)!} + \frac{1}{|x_0|}\sum_{k=0}^{\infty}\frac{t^{2k+1}|x_0|^{2k+1}J_{x_0}}{(2k)!} =$$

$$\cosh(|x_0|t)I + \frac{1}{|x_0|}\sinh(|x_0|t)J_{x_0} \tag{3.8.15}$$

Inserting equation (3.8.15) into equation (3.8.9)

$$\dot{v} = \cosh(|x_0|t)V_0 + \frac{1}{|x_0|}\sinh(|x_0|t)J_{x_0}V_0$$

and then integrating, we get that

$$v = \frac{1}{|x_0|^2}\Big[(\cosh(|x_0|t) - 1)J_{x_0}V_0 + |x_0|\sinh(|x_0|t)V_0\Big]$$

We compute the Lie bracket of v and $\dot{v}$:

$$[v, \dot{v}] = \frac{\|V_0\|^2}{|x_0|^2}\Big[\cosh(|x_0|t) - 1\Big]x_0 \tag{3.8.16}$$

and by inserting (3.8.16) into equation (3.8.8) we have that

$$\dot{z} = \frac{1}{2}\frac{\|V_0\|^2}{|x_0|^2}\Big[\cosh(|x_0|t) - 1\Big]x_0 \tag{3.8.17}$$

We find z by integrating equation (3.8.17) and get that

$$z = \left[\frac{1}{2}\frac{\|V_0\|^2}{|x_0|^2}\left(\frac{\sinh(|x_0|t)}{|x_0|} - t\right)\right]x_0.$$

■

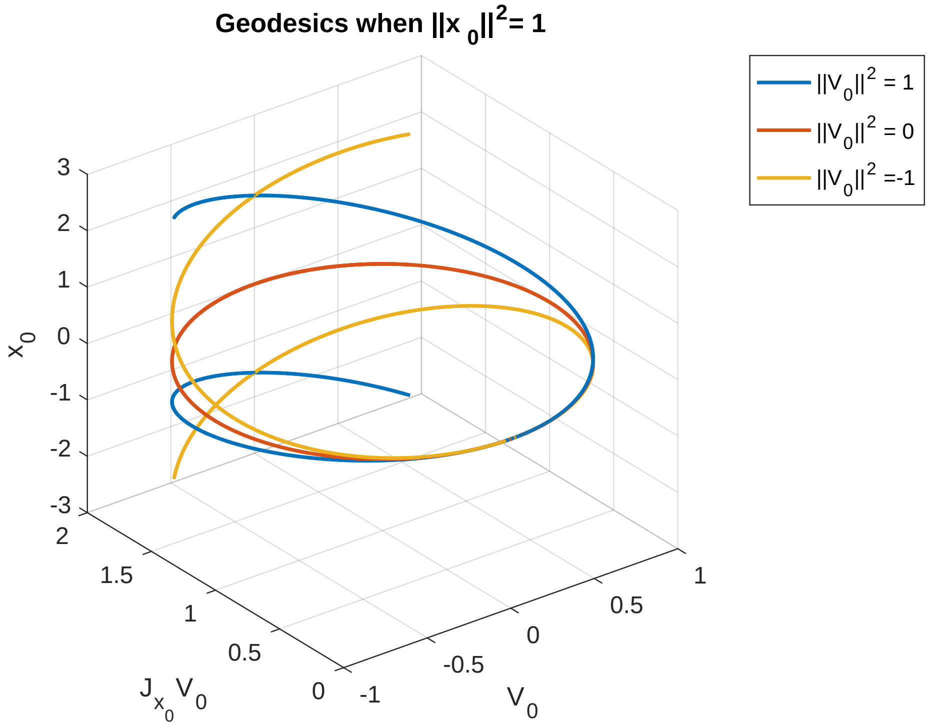

Figure 4: Plot showing the sub-semi-Riemannian geodesics of a semi-H-type group, when $\|x_0\|^2 = 1$ and $\|V_0\|^2$ takes the values $1, 0, -1$.

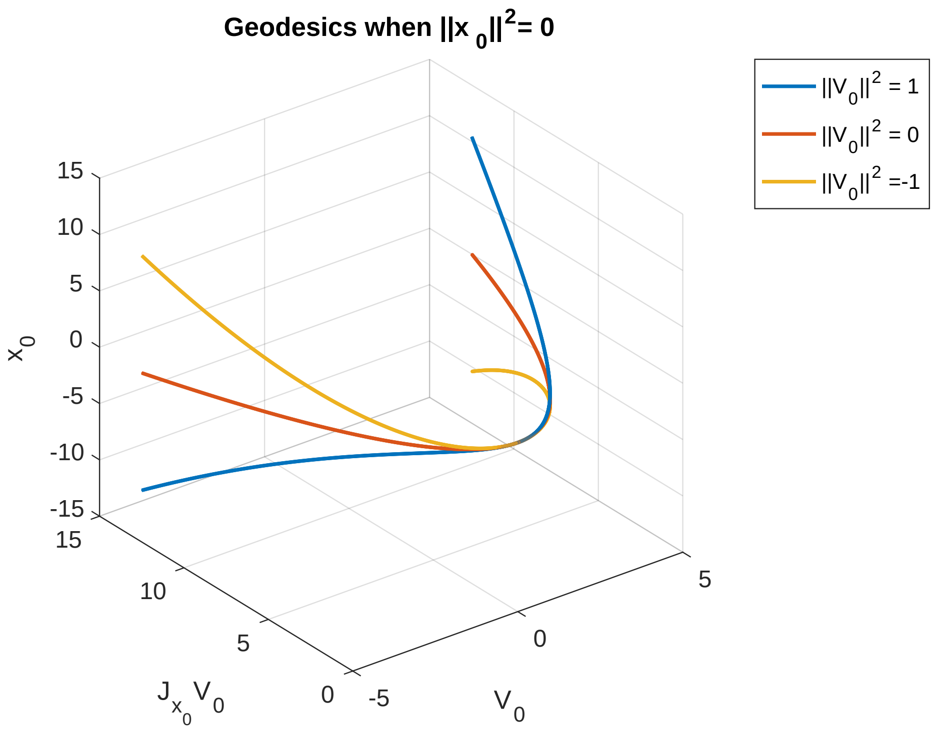

Figure 5: Plot showing the sub-semi-Riemannian geodesics of a semi-H-type group, when $\|x_0\|^2 = 0$ and $\|V_0\|^2$ takes the values $1, 0, -1$.

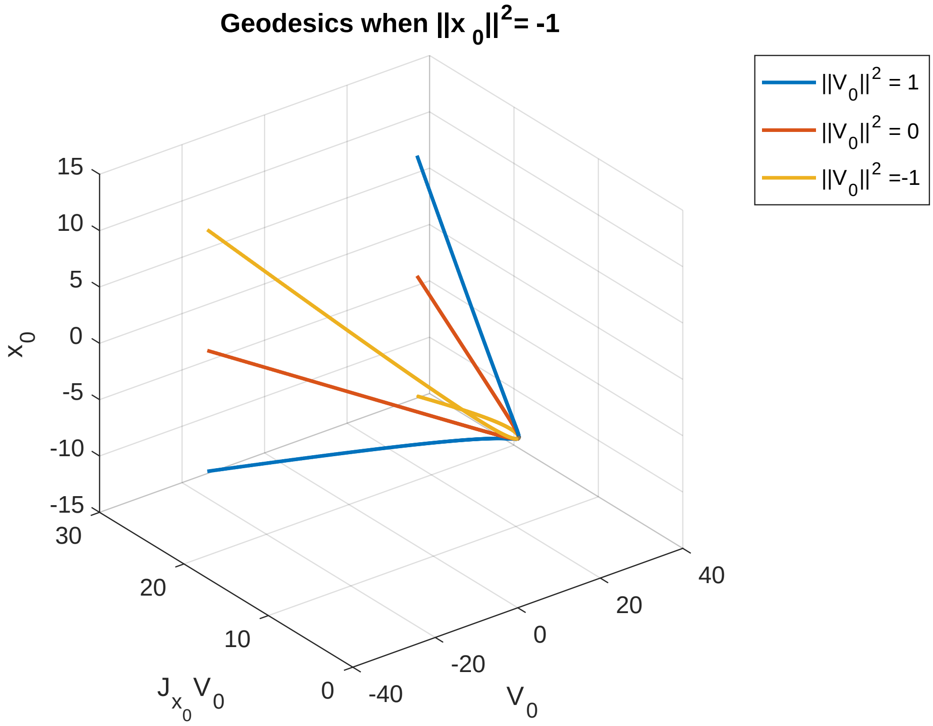

Figure 6: Plot showing the sub-semi-Riemannian geodesics of a semi-H-type group, when $\|x_0\|^2 = -1$ and $\|V_0\|^2$ takes the values $1, 0, -1$.

4 Semi-Damek-Ricci Spaces

4.1 Semi-Damek-Ricci Lie algebras

Definition 4.1. Let $\mathfrak{n}$ be a semi-H-type Lie algebra and $\mathfrak{a}$ be a one dimensional Lie algebra endowed with a inner product $g_{\mathfrak{a}}$ such that if $\mathfrak{a} = \mathbb{R}H$ we have that $g_{\mathfrak{a}}(H,H) = 1$. We define the Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ of dimension $n+m+1$ with the relations

$$[H,v] = \frac{1}{2}v \quad \text{and} \quad [H,z] = z \tag{4.1.1}$$

for all $v \in \mathfrak{v}$ and $z \in \mathfrak{z}$ and call $\mathfrak{s}$ a *semi-Damek-Ricci Lie algebra.*

From the definition above we have that $\mathfrak{s}$ is endowed with the scalar product $g_{\mathfrak{s}} = g_{\mathfrak{n}} + g_{\mathfrak{a}}$ and that any vector $x \in \mathfrak{s}$ can be written uniquely as

$$x = v + z + sH \qquad \text{with some } v \in \mathfrak{v}, z \in \mathfrak{z} \text{ and } s \in \mathbb{R}, \tag{4.1.2}$$

and hence $\Xi = \{V_1, \dots, V_n, Z_1, \dots, Z_m, H\}$ is a orthonormal basis for $\mathfrak{s}$. Moreover we have that $\mathfrak{s}$ is solvable since

$$[\mathfrak{s},\mathfrak{s}] = \mathfrak{n} \qquad [\mathfrak{n},\mathfrak{n}] = \mathfrak{z} \qquad [\mathfrak{z},\mathfrak{z}] = 0.$$

4.2 Semi-Damek-Ricci spaces

Definition 4.2. Let $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ be a semi-Damek-Ricci Lie algebra. The simply connected Lie group associated with $\mathfrak{a}$ will be denoted A and the simply connected Lie group associated with $\mathfrak{s}$ will be denoted S. We call S a *Semi-Damek-Ricci space.*

Since any one dimensional Lie algebra is abelian and hence a 1-step nilpotent Lie algebra, we have by proposition 2.126 that $\exp_{\mathfrak{a}} : \mathfrak{a} \to A$ is a diffeomorphism. Hence we can introduce coordinates on A as follows

$$(s) \xmapsto{\exp_{\mathfrak{a}}^{-1}} sH \qquad \text{for } s \in \mathbb{R}$$

and by the Baker-Campbell-Hausdorff formula we have that

$$\exp_{\mathfrak{a}}(sH)\exp_{\mathfrak{a}}(rH) = \exp_{\mathfrak{a}}((r+h)H).$$

Therefore we have that the group law on A is given by addition i.e if $p = (s)$ and $q = (r)$ then

$$pq = (s+r).$$

The Lie group S is in fact the semi-direct product of N and A i.e $S = N \rtimes A$, i.e any element of S can be written uniquely as a product of an element of N and an element of A, see [2] [8] [24]. Let $\exp_{\mathfrak{s}}$ and $\exp_{\mathfrak{n}}$ denoted the Lie exponential map of $\mathfrak{s}$ and $\mathfrak{n}$ respectively.

Proposition 4.3. *[2]*
The maps $\exp_{\mathfrak{s}} : \mathfrak{s} \to S$ *and* $\exp_{\mathfrak{n}} \times \exp_{\mathfrak{a}} : x + sH \mapsto \exp_{\mathfrak{n}}(x)\exp_{\mathfrak{a}}(sH)$ *are diffeomorphisms from* $\mathfrak{s}$ *to* S.

Proposition 4.4. *[2]*
The Lie exponential map $\exp_{\mathfrak{s}}$ *is given by*

$$\exp_{\mathfrak{s}}(V+Z+sH) = \begin{cases} \exp_{\mathfrak{n}}(V+Z)\exp_{\mathfrak{a}}(0) & \text{if } s=0 \\ \exp_{\mathfrak{n}}(\frac{2}{s}(e^{\frac{s}{2}}-1)V + \frac{1}{s}(e^{s}-1)Z)\exp_{\mathfrak{a}}(sH) & \text{if } s \neq 0 \end{cases}$$

From the above proposition we have that $\exp_{\mathfrak{s}}|_{\mathfrak{n}} = \exp_{\mathfrak{n}}$ and that $\exp_{\mathfrak{s}}|_{\mathfrak{a}} = \exp_{\mathfrak{a}}$.

We wish to give the group law on S, but to do so we need this next lemma.

Lemma 4.5. *For any* $s \in \mathbb{R}$, $V \in \mathfrak{v}$ *and* $Z \in \mathfrak{z}$ *we have that*

$$\exp_{\mathfrak{s}}(sH)\exp_{\mathfrak{s}}(V+Z)\exp_{\mathfrak{s}}(sH)^{-1} = \exp_{\mathfrak{s}}(e^{\frac{s}{2}}V + e^{s}Z).$$

Proof. By corollary 2.130 and 2.131 we have that

$$\exp_{\mathfrak{s}}(sH)\exp_{\mathfrak{s}}(V+Z)\exp_{\mathfrak{s}}(sH)^{-1} = \exp_{\mathfrak{s}}(\mathrm{Ad}_{\exp_{\mathfrak{s}}(sH)}(V+Z)) = \exp_{\mathfrak{s}}(\exp_{GL(\mathfrak{s})} s\,\mathrm{ad}_H(V+Z))$$

where $\exp_{GL(\mathfrak{s})}$ is the matrix exponential. By the commutators 4.1.1 we have that

$$\mathrm{ad}_H = \begin{bmatrix} \frac{1}{2}I_n & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and taking the matrix exponential we get

$$\exp_{GL(\mathfrak{s})} s\,\mathrm{ad}_H = \begin{bmatrix} e^{\frac{s}{2}}I_n & 0 & 0 \\ 0 & e^{s}I_m & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence we get

$$\exp_{\mathfrak{s}}(\exp_{GL(\mathfrak{s})} s\,\mathrm{ad}_H(V+Z)) = \exp_{\mathfrak{s}}(e^{\frac{s}{2}}V + e^{s}Z)$$

■

By use of lemma 4.5, we can now give the group law on S.

Proposition 4.6. *Let*

$$p = \exp_{\mathfrak{n}}(V+Z)\exp_{\mathfrak{a}}(sH)$$

and

$$q = \exp_{\mathfrak{n}}(V'+Z')\exp_{\mathfrak{a}}(s'H)$$

then their product is given by

$$pq = \exp_{\mathfrak{n}}(V + e^{\frac{s}{2}}V' + Z + e^{s}Z' + \frac{e^{\frac{s}{2}}}{2}[V,V'])\exp_{\mathfrak{a}}((s+s')H) \tag{4.2.1}$$

Proof. The result follows from lemma 4.5 and that

$$pq = \exp_{\mathfrak{s}}(V + Z)\exp_{\mathfrak{s}}(sH)\exp_{\mathfrak{s}}(V' + Z')\exp_{\mathfrak{s}}(s'H) =$$

$$\exp_{\mathfrak{s}}(V + Z)\exp_{\mathfrak{s}}(sH)\exp_{\mathfrak{s}}(V' + Z')\exp_{\mathfrak{s}}(sH)^{-1}\exp_{\mathfrak{s}}(sH)\exp_{\mathfrak{s}}(s'H)$$

■

As seen we have two diffeomorphism we can use to give coordinates on S, we choose to use $\exp_{\mathfrak{n}} \times \exp_{\mathfrak{n}}$. Therefore we define the coordinates on S as follows

$$(v^1, \dots, v^n, z^1, \dots, z^m, s) \xmapsto{(\exp_{\mathfrak{n}} \times \exp_{\mathfrak{a}})^{-1}} \sum_{i=1}^{n} v^i V_i + \sum_{\alpha=1}^{m} z^\alpha Z_\alpha + sH$$

If $p = (v^1, \dots, v^n, z^1, \dots, z^m, s)$ and $q = (u^1, \dots, u^n, y^1, \dots, y^m, r)$, then the group law (4.2.1) in coordinates is given by

$$pq = \left(v^1 + e^{\frac{s}{2}}u^1, \dots, v^n + e^{\frac{s}{2}}u^n, z^1 + e^s y^1 + \frac{e^{\frac{s}{2}}}{2}\sum_{i,j=1}^{n} v^i u^j C_{ij}^1, \dots, z^m + e^s y^m + \frac{e^{\frac{s}{2}}}{2}\sum_{i,j=1}^{n} v^i u^j C_{ij}^m, s + r\right) \tag{4.2.2}$$

Proposition 4.7. *The left invariant vector fields on S are given by*

$$V_i = e^{\frac{s}{2}}\frac{\partial}{\partial v^i} + \frac{e^{\frac{s}{2}}}{2}\sum_{j=1}^{n}\sum_{\alpha=1}^{m} v^j C_{ji}^\alpha \frac{\partial}{\partial z^\alpha}$$

$$Z_\alpha = e^s \frac{\partial}{\partial z^\alpha}$$

$$H = \frac{\partial}{\partial s}$$

Proof. We wish now to find all the left invariant vector fields on S, to do so we differentiate (4.2.2) with respect to q to find $(L_g)_{*,e}$, we get that

$$(L_g)_{*,e} = \begin{bmatrix} e^{\frac{s}{2}} I_n & 0 & 0 \\ \frac{e^{\frac{s}{2}}}{2} A & e^s I_m & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} \sum_{i=1}^n v^i C_{i1}^1 & \dots & \sum_{i=1}^n v^i C_{in}^1 \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n v^i C_{i1}^m & \dots & \sum_{i=1}^n v^i C_{in}^m \end{bmatrix}$$

Letting $(L_g)_{*,e}$ act on the vectors $\{\frac{\partial}{\partial v^i}\big|_e\}$, $\{\frac{\partial}{\partial z^\alpha}\big|_e\}$ and $\{\frac{\partial}{\partial s}\big|_e\}$ we get the wanted result. ■

We can make S into a semi-Riemannian manifold, by defining a metric on S. We choose to endow S with a left invariant metric and to do so, we use the scalar product on $\mathfrak{s}$ and left translations as follows

$$g_p(X_p, Y_p) = g((L_{p^{-1}})_{*,p}X_p, (L_{p^{-1}})_{*,p}Y_p) \quad \text{for } X, Y \in \mathfrak{X}(S)$$

making S a Lie group endowed with a metric tensor.

4.3 The Levi-Civita Connection

The Levi-Civita connection on a semi-Damek-Ricci space, is completely determined by its values on the left invariant vector fields in $\mathfrak{s}$.

Proposition 4.8. *Let S be a semi-H-type group. Then the Levi-Civita connection is given by*

$$\begin{aligned}\nabla_{(v_1+z_1+s_1H)}(v_2 + z_2 + s_2H) = \frac{1}{2}g(v_1, v_2)H + \frac{1}{2}[v_1, v_2] - \frac{1}{2}J_{z_1}v_1 \\ -\frac{1}{2}s_2v_1 - \frac{1}{2}J_{z_1}v_2 + g(z_1, z_2)H - s_2z_1\end{aligned}$$

for all $v_1, v_2 \in \mathfrak{v}$, $z_1, z_2 \in \mathfrak{z}$ and $s_1, s_2 \in \mathbb{R}$.

Proof. For any left invariant metric g on a Lie group, the Koszul formula reads

$$2g(\nabla_X Y, Z) = g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X) \tag{4.3.1}$$

Using (4.3.1) we have that

$$2g(\nabla_H X, Y) = -g(X, \operatorname{ad}_H Y) + g(Y, \operatorname{ad}_H X) = 0$$

since $(X, Y) \mapsto g(X, \operatorname{ad}_H Y)$ is symmetric in X and Y. Hence we have that $\nabla_H = 0$. Now let Let $v_1, v_2 \in \mathfrak{v}$, $z_1, z_2 \in \mathfrak{z}$, and let $w \in \mathfrak{s}$. We have that

$$\begin{aligned}
2g(\nabla_{v_1}z_1, w) = -g(J_{z_1}v_1, w) &\implies \nabla_{v_1}z_1 = -\tfrac{1}{2}J_{z_1}v_1 \\
2g(\nabla_{z_1}v_1, w) = -g(J_{z_1}v_1, w) &\implies \nabla_{z_1}v_1 = -\tfrac{1}{2}J_{z_1}v_1 \\
2g(\nabla_{v_1}v_2, w) = g([v_1, v_2], w) &\implies \nabla_{v_1}v_2 = \tfrac{1}{2}g(v_1, v_2)H + \tfrac{1}{2}[v_1, v_2] \\
2g(\nabla_{z_1}z_2, w) = 2g(g(z_1, z_2)H, w) &\implies \nabla_{z_1}z_2 = g(z_1, z_2)H \\
2g(\nabla_{z_1}H, w) = 2g(-z_1, w) &\implies \nabla_{z_1}H = -z_1 \\
2g(\nabla_{v_1}H, w) = g(-v_1, w) &\implies \nabla_{v_1}H = -\tfrac{1}{2}v_1
\end{aligned}$$

■

4.4 Riemann Curvature Tensor

Proposition 4.9. *Let S be a semi-Damek-Ricci space. Then the Riemann curvature endomorphism is given by*

$$
\begin{aligned}
R(v_1+z_1+s_1H, v_2+z_2+s_2H)(v_3+z_3+s_3H) &= -\frac{1}{4}g(v_2,v_3)v_1 - \frac{1}{4}J_{[v_2,v_3]}v_1 \\
+\frac{1}{4}g(v_1,v_3)v_2 + \frac{1}{4}J_{[v_1,v_3]}v_2 + \frac{1}{2}J_{[v_1,v_2]}v_3 + \frac{1}{2}g(v_1,J_{z_3}v_2)H - \frac{1}{4}[v_1,J_{z_3}v_2] &+ \frac{1}{4}[v_2,J_{z_3}v_1] + \frac{1}{2}s_3[v_1,v_2] \\
+\frac{1}{2}g(v_2,v_3)z_2 - \frac{1}{4}g(v_3,J_{z_2}v_1)H - \frac{1}{4}[v_1,J_{z_2}v_3] - \frac{1}{4}J_{z_2}J_{z_3}v_1 &- \frac{1}{2}g(z_2,z_3)v_1 + \frac{1}{4}s_3J_{z_2}v_1 \\
+\frac{1}{4}s_2g(v_1,v_3)H + \frac{1}{4}s_2[v_1,v_3] - \frac{1}{2}s_2J_{z_3}v_1 - \frac{1}{4}s_2s_3v_1 &- \frac{1}{2}g(v_2,v_3)z_1 + \frac{1}{4}g(v_3,J_{z_1}v_2)H \\
+\frac{1}{4}[v_2,J_{z_1}v_3] + \frac{1}{4}J_{z_1}J_{z_3}v_2 &+ \frac{1}{2}g(z_1,z_3)v_2 - \frac{1}{4}s_3J_{z_1}v_2 + \frac{1}{4}J_{z_1}J_{z_2}v_3 \\
-\frac{1}{4}J_{z_2}J_{z_1}v_3 - g(z_2,z_3)z_1 + g(z_1,z_3)z_2 &- \frac{1}{2}s_2J_{z_1}v_3 + s_2g(z_1,z_3)H - s_3z_1 \\
-\frac{1}{4}s_1g(v_2,v_3)H - \frac{1}{4}s_1[v_2,v_3] + \frac{1}{2}s_1J_{z_3}v_2 + \frac{1}{4}s_1s_3v_2 &+ \frac{1}{2}s_1J_{z_2}v_3 - s_1g(z_2,z_3)H + s_1s_3z_2
\end{aligned}
$$

for all $v_1,v_2,v_3\in\mathfrak{v}$, $z_1,z_2,z_3\in\mathfrak{z}$ and $s_1,s_2,s_3\in\mathbb{R}$.

Proof. Let $w\in\mathfrak{s}$. Then

$$R(H,H)w = [\nabla_H,\nabla_H]w - \nabla_{[H,H]}w = 0$$

$$R(H,z_2)w = [\nabla_H,\nabla_{z_2}]w - \nabla_{[H,z_2]}w = -\nabla_{z_2}w$$

$$R(H,v_2)w = [\nabla_H,\nabla_{v_2}]w - \nabla_{[H,v_2]}w = -\frac{1}{2}\nabla_{v_2}w$$

$$R(z_1,z_2)w = [\nabla_{z_1},\nabla_{z_2}]w - \nabla_{[z_1,z_2]}w = [\nabla_{z_1},\nabla_{z_2}]w$$

$$R(z_1,v_2)w = [\nabla_{z_1},\nabla_{v_2}]w - \nabla_{[z_1,v_2]}w = [\nabla_{z_1},\nabla_{v_2}]w$$

$$R(v_1,v_2)w = [\nabla_{v_1},\nabla_{v_2}]w - \nabla_{[v_1,v_2]}w$$

The rest can be found by anti-symmetry, see table 1.

We compute $R(v_1+z_1+s_1H, v_2+z_2+s_2H)(v_3+z_3+s_3H)$, by using linearity and proposition 4.8.

Case I : $w=v_3$

$$R(H,z_2)v_3 = \frac{1}{2}J_{z_2}v_3$$

$$R(H,v_2)v_3 = -\frac{1}{4}g(v_2,v_3)H - \frac{1}{4}[v_2,v_3]$$

$R(X,Y)w$	v_2	z_2	H
v_1	$[\nabla_{v_1}, \nabla_{v_2}]w - \nabla_{[v_1,v_2]}w$	$-[\nabla_{z_2}, \nabla_{v_1}]w$	$\frac{1}{2}\nabla_{v_1}w$
z_1	$[\nabla_{z_1}, \nabla_{v_2}]w$	$[\nabla_{z_1}, \nabla_{z_2}]w$	$\nabla_{z_1}w$
H	$-\frac{1}{2}\nabla_{v_2}w$	$-\nabla_{z_2}w$	0

Table 1: Table of the Riemann curvature endomorphism, when X, Y runs through $v_1, v_2 \in \mathfrak{v}$, $z_1, z_2 \in \mathfrak{z}$ and w is a arbitrary vector in $\mathfrak{s}$.

$$R(z_1, z_2)v_3 = \frac{1}{4}J_{z_1}J_{z_2}v_3 - \frac{1}{4}J_{z_2}J_{z_1}v_3$$

$$\begin{aligned} R(z_1, v_2)v_3 = -\frac{1}{2}g(v_2, v_3)z_1 + \frac{1}{4}g(v_3, J_{z_1}v_2)H \\ +\frac{1}{4}[v_2, J_{z_1}v_3] \end{aligned}$$

$$\begin{aligned} R(v_1, v_2)v_3 = -\frac{1}{4}g(v_2, v_3)v_1 - \frac{1}{4}J_{[v_2,v_3]}v_1 + \frac{1}{4}g(v_1, v_3)v_2 \\ +\frac{1}{4}J_{[v_1,v_3]}v_2 + \frac{1}{2}J_{[v_1,v_2]}v_3 \end{aligned}$$

Case II: $w = z_3$

$$R(H, z_2)z_3 = -g(z_2, z_3)H$$

$$R(H, v_2)z_3 = \frac{1}{2}J_{z_3}v_2$$

$$R(z_1, z_2)z_3 = -g(z_2, z_3)z_1 + g(z_1, z_3)z_2$$

$$R(z_1, v_2)z_3 = \frac{1}{4}J_{z_1}J_{z_3}v_2 + \frac{1}{2}g(z_1, z_3)v_2$$

$$R(v_1, v_2)z_3 = \frac{1}{2}g(v_1, J_{z_3}v_2)H - \frac{1}{4}[v_1, J_{z_3}v_2] + \frac{1}{4}[v_2, J_{z_3}v_1]$$

Case III: $w = H$

$$R(H, z_2)H = z_2$$

$$R(H, v_2)H = \frac{1}{4}v_2$$

$$R(z_1, z_2)H = 0$$

$$R(z_1, v_2)H = -\frac{1}{4}J_{z_1}v_2$$

$$R(v_1, v_2)H = \frac{1}{2}[v_1, v_2]$$

■

4.5 Ricci Tensor and Scalar Curvature

Proposition 4.10. *Let S be a semi-Damek-Ricci space. Then the Ricci tensor is given by*

$$Ric(x, y) = -(\frac{n}{4} + m)g(x, y)$$

Proof. From proposition 2.79 we have that

$$Ric(x, y) = \sum_{i=1}^{n} g(R(V_i, x)y, V_i)\epsilon_i + \sum_{\alpha=1}^{m} g(R(Z_\alpha, X)y, Z_\alpha)\epsilon_\alpha + g(R(H, x)y, H)$$

We compute this sum using proposition 4.8 and we consider six different cases:

Case I: $x, y \in \mathfrak{v}$
We have that

$$g(R(V_i, x)y, V_i) = -\frac{1}{4}g(x, y)\epsilon_i + \frac{1}{4}g(x, g(V_i, y)V_i) - \frac{3}{4}g([x, V_i], [y, V_i])$$

$$g(R(Z_\alpha, x)y, Z_\alpha) = -\frac{1}{2}g(x, y)\epsilon_\alpha + \frac{1}{4}g(x, y)\epsilon_\alpha$$

$$g(R(H, x)y, H) = -\frac{1}{4}g(x, y)$$

Hence

$$Ric(x, y) = -\frac{n}{4}g(x, y) + \frac{1}{4}g(x, y) - \frac{3m}{4}g(x, y) - \frac{m}{2}g(x, y) + \frac{m}{4}g(x, y) - \frac{1}{4}g(x, y) = \\ -\frac{n}{4}g(x, y) - mg(x, y) = -(\frac{n}{4} + m)g(x, y)$$

Here we used that

$$g(x, \sum_{i=1}^{n} g(V_i, y)\epsilon_i V_i) = g(x, y)$$

and that

$$\sum_{i=1}^{n} \epsilon_i g([x, V_i], [y, V_i]) = \sum_{\alpha=1}^{m} \epsilon_\alpha g(J_{Z_\alpha}y, J_{Z_\alpha}x) = \sum_{\alpha=1}^{m} g(x, y).$$

Case II: $x, y \in \mathfrak{z}$
We have that

$$g(R(V_i,x)y,V_i)=\frac{1}{4}g(x,y)\epsilon_i-\frac{1}{2}g(x,y)\epsilon_i$$

$$g(R(Z_\alpha,x)y,Z_\alpha)=-g(x,y)\epsilon_\alpha+g(x,g(Z_\alpha,y)Z_\alpha)$$

$$g(R(H,x)y,H)=-g(x,y)$$

Hence we have that

$$Ric(x,y)=\frac{n}{4}g(x,y)-\frac{n}{2}g(x,y)-mg(x,y)+g(x,y)-g(x,y)=\\ -\frac{n}{4}g(x,y)-mg(x,y)=-(\frac{n}{4}+m)g(x,y)$$

Here we used that

$$g(x,\sum_{\alpha=1}^{m}g(Z_\alpha,y)\epsilon_\alpha Z_\alpha)=g(x,y).$$

Case III: $x,y\in\mathfrak{a}$
Let $x=s_1H$ and $y=s_2H$ for $s_1,s_2\in\mathbb{R}$. We have that

$$g(R(V_i,s_1H)s_2H,V_i)=-\frac{1}{4}s_1s_2\epsilon_i$$

$$g(R(Z_\alpha,s_1H)s_1H,Z_\alpha)=-s_1s_2\epsilon_\alpha$$

$$g(R(H,s_1H)s_2,H)=0$$

Hence

$$Ric(x,y)=-\frac{n}{4}s_1s_2-ms_1s_2=-(\frac{n}{4}+m)s_1s_2=-(\frac{n}{4}+m)g(x,y).$$

Here we used that

$$g(x,y)=s_1s_2g(H,H)=s_1s_2.$$

Case IV: $x\in\mathfrak{v}$ and $y\in\mathfrak{z}$
We have that

$$g(R(V_i,x)y,V_i)=0 \qquad g(R(Z_\alpha,x)y,Z_\alpha)=0 \qquad g(R(H,x)y,H)=0$$

since

$$R(V_i,x)y=\frac{1}{2}g(V_i,J_yx)H-\frac{1}{4}[V_i,J_yx]+\frac{1}{4}[x,J_yV_i]\notin\mathfrak{v}.$$

$$R(Z_\alpha,x)y=\frac{1}{4}J_{Z_\alpha}J_yx+\frac{1}{2}g(Z_\alpha,y)x\notin\mathfrak{z}.$$

$$R(H,x)y=\frac{1}{2}J_yx\notin\mathfrak{a}$$

Hence $Ric(x, y) = 0$.

Case V: $x \in \mathfrak{v}$ and $y \in \mathfrak{a}$
Let $y = s_1 H$. Then

$$g(R(V_i, x)y, V_i) = 0 \qquad g(R(Z_\alpha, x)y, Z_\alpha) = 0 \qquad g(R(H, x)y, H) = 0$$

since

$$R(V_i, x)y = \frac{1}{2}s_1[V_i, x] \notin \mathfrak{v}$$

$$R(Z_\alpha, x)y = -\frac{1}{4}s_1 J_{Z_\alpha} x \notin \mathfrak{z}$$

$$R(H, x)y = \frac{1}{4}s_1 x \notin \mathfrak{a}$$

Hence $Ric(x, y) = 0$.

Case VI: $x \in \mathfrak{z}$ and $y \in \mathfrak{a}$
Let $y = s_1 H$. Then

$$g(R(V_i, x)y, V_i) = 0 \qquad g(R(Z_\alpha, x)y, Z_\alpha) = 0 \qquad g(R(H, x)y, H) = 0$$

since

$$R(V_i, x)y = \frac{1}{4}s_1 J_x V_i \quad \text{and} \quad g(J_x V_i, V_i) = 0$$

$$R(Z_\alpha, x)y = 0$$

$$R(H, x)y = s_1 x \notin \mathfrak{a}$$

Hence $Ric(x, y) = 0$.

■

Corollary 4.11. *Every semi-Damek-Ricci space is a Einstein manifold i.e*

$$Ric \propto g$$

Proposition 4.12. *Let S be a semi-Damek-Ricci space. Then the Scalar curvature is given by*

$$S = -(\frac{n}{4} + m)(m + n + 1)$$

Proof. By proposition 2.79 we have that

$$S = \sum_{i=1}^{n} Ric(V_i, V_i)\epsilon_i + \sum_{\alpha=1}^{m} Ric(Z_\alpha, Z_\alpha)\epsilon_\alpha + Ric(H, H) \tag{4.5.1}$$

and by using proposition 4.10 we get

$$S = -(\frac{n}{4} + m)\left[\sum_{i=1}^{n} \epsilon_i^2 + \sum_{\alpha=1}^{m} \epsilon_\alpha^2 + 1\right] = -(\frac{n}{4} + m)(n + m + 1)$$

■

4.6 Sectional Curvature

Proposition 4.13. *Let S be semi-Damek-Ricci space. The sectional curvature of coordinate planes are given by*

$$K(\mathbb{R}V_i \oplus \mathbb{R}V_j\}) = -\frac{1}{4} - \frac{3}{4}\|[V_i, V_j]\|^2\epsilon_i\epsilon_j \quad \textit{for } i \neq j$$

$$K(\mathbb{R}Z_\alpha \oplus \mathbb{R}Z_\lambda) = -1 \quad \textit{for } \alpha \neq \lambda$$

$$K(\mathbb{R}V_i \oplus \mathbb{R}Z_\alpha) = -\frac{1}{4}$$

$$K(\mathbb{R}V_i \oplus \mathbb{R}H) = -\frac{1}{4}$$

$$K(\mathbb{R}Z_\alpha \oplus \mathbb{R}H) = -1$$

Hence any semi-Damek-Ricci space has nonconstant sectional curvature.

Proof.

$$K(\text{span}\{V_i, V_j\}) = \frac{g(R(V_i, V_j)V_j, V_i)}{\epsilon_i\epsilon_j} = -\frac{1}{4}g(\epsilon_j V_i, V_i)\epsilon_i\epsilon_j + \frac{3}{4}g(J_{[V_i,V_j]}V_j, V_i) = -\frac{1}{4} - \frac{3}{4}\|[V_i, V_j]\|^2$$

$$K(\text{span}\{V_i, Z_\alpha\}) = \frac{g(R(V_i, Z_\alpha)Z_\alpha, V_i)}{\epsilon_i\epsilon_\alpha} = \frac{1}{4}g(\epsilon_\alpha V_i, V_i)\epsilon_i\epsilon_\alpha - \frac{1}{2}g(\epsilon_\alpha V_i, V_i)\epsilon_i\epsilon_\alpha = -\frac{1}{4}$$

$$K(\text{span}\{Z_\alpha, Z_\lambda\}) = \frac{g(R(Z_\alpha, Z_\lambda)Z_\lambda, Z_\alpha)}{\epsilon_\alpha\epsilon_\lambda} = -g(\epsilon_\lambda Z_\alpha, Z_\alpha)\epsilon_\alpha\epsilon_\lambda = -1$$

$$K(\text{span}\{V_i, H\}) = \frac{g(R(V_i, H)H, V_i}{\epsilon_i} = -\frac{1}{4}g(V_i, V_i)\epsilon_i = -\frac{1}{4}$$

$$K(\text{span}\{Z_\alpha, H\}) = \frac{g(R(Z_\alpha, H)H, Z_\alpha}{\epsilon_\alpha} = -g(Z_\alpha, Z_\alpha)\epsilon_\alpha = -1$$

∎

4.7 Semi-Riemannian Geodesics

The goal of this section is to find the semi-Riemannian geodesic in a semi-Damek-Ricci space. We start by finding the geodesic equations.

Theorem 4.14. *Let S be a semi-Damek-Ricci space and $\gamma : \mathbb{R} \to S$ be the geodesic such that $\gamma(0) = e$ and $\dot\gamma(0) = V_0 + Z_0 + s_0 H$. Then γ can be found by solving the system:*

$$\begin{cases} \ddot v = (\dot s I_{\mathfrak{v}} + e^s J_{Z_0})\dot v, \\ \dot z = \frac{1}{2}[v, \dot v] + e^{2s} Z_0, \\ \ddot s = -\frac{1}{2}e^s\|V_0\|^2 - e^{2s}\|Z_0\|^2. \end{cases}$$

Where $v = \sum_{i=1}^n v^i V_i$ and $z = \sum_{\alpha=1}^m z^\alpha Z_\alpha$.

Proof. We can describe the geodesic using the diffeomorphism $\exp_{\mathfrak{n}} \times \exp_{\mathfrak{a}}$ and a curve in $\mathfrak{s}$. Since $\mathfrak{s}$ is the direct sum of $\mathfrak{z}$, $\mathfrak{v}$ and $\mathfrak{a}$, we can write

$$\gamma(t) = (\exp_{\mathfrak{n}} \times \exp_{\mathfrak{a}})(v(t) + z(t) + s(t)H)$$

where $v : \mathbb{R} \to \mathfrak{v}$ and $z : \mathbb{R} \to \mathfrak{z}$ are vector valued curves and $s : \mathbb{R} \to \mathbb{R}$ is a real valued curve. The velocity of the curve is then given by

$$\dot{\gamma}(t) = \sum_{i=1}^{n} \dot{v}^i \frac{\partial}{\partial v^i} + \sum_{\alpha=1}^{m} \dot{z}^\alpha \frac{\partial}{\partial z^\alpha} + \dot{s}\frac{\partial}{\partial s}.$$

Using proposition 4.7 we have that

$$\dot{\gamma}(t) = \sum_{i=1}^{n} \dot{v}^i \left[e^{-\frac{s}{2}} V_i + \frac{e^{-s}}{2} \sum_{j=1}^{n} \sum_{\alpha=1}^{m} v^j C_{ij}^\alpha Z_\alpha \right] + e^{-s} \sum_{\alpha}^{m} \dot{z}^\alpha Z_\alpha + \dot{s}H =$$

$$e^{-\frac{s}{2}} \dot{v} + e^{-s}(\dot{z} + \frac{1}{2}[\dot{v}, v]) + \dot{s}H.$$

We introduce new variables, let

$$u = e^{-\frac{s}{2}} \dot{v} \quad \text{and} \quad x = e^{-s}(\dot{z} + \frac{1}{2}[\dot{v}, v])$$

such that the velocity becomes $\dot{\gamma}(t) = u + x + \dot{s}H$.

For $\gamma(t)$ to be a geodesic we need that the acceleration of the curve is zero, i.e $D_t\dot{\gamma}(t) = 0$. Hence we will need the covariant derivative of the basis vector fields along $\dot{\gamma}$, so we compute them now.

$$\nabla_{\dot{\gamma}} V_i = \frac{1}{2}[u, V_i] + \frac{1}{2} g(u, V_i)H - \frac{1}{2} J_x V_i$$

$$\nabla_{\dot{\gamma}} Z_\alpha = -\frac{1}{2} J_{Z_\alpha} u + g(x, Z_\alpha)H$$

$$\nabla_{\dot{\gamma}} H = -\frac{1}{2} u - x$$

We are ready to compute the acceleration.

$$D_t(u) = \sum_{i=1}^{n} D_t(u^i V_i) = \sum_{i=1}^{n} \dot{u}^i V_i + u^i \nabla_{\dot{\gamma}} V_i =$$

$$\sum_{i=1}^{n} \dot{u}^i V_i + u^i \left(\frac{1}{2}[u, V_i] + \frac{1}{2} g(u, V_i) - \frac{1}{2} J_x V_i \right) = \dot{u} + \frac{1}{2} g(u, u)H - \frac{1}{2} J_x u$$

$$D_t(x) = \sum_{\alpha=1}^{m} D_t(x^\alpha V_\alpha) = \sum_{\alpha=1}^{n} \dot{x}^\alpha Z_\alpha + x^\alpha \nabla_{\dot{\gamma}} Z_\alpha =$$

$$\sum_{\alpha=1}^{n} \dot{x}^\alpha Z_\alpha + x^\alpha \Big(-\frac{1}{2} J_{Z_\alpha} u + g(x, Z_\alpha) H \Big) = \dot{x} - \frac{1}{2} J_x u + g(x, x) H$$

$$D_t(\dot{s}H) = \ddot{s}H + \dot{s}\nabla_{\dot{\gamma}} H = \ddot{s}H - \dot{s}(\frac{1}{2}u + x)$$

By linearity of D_t we have that the acceleration is given by

$$D_t(\dot{\gamma}) = \dot{u} - \frac{1}{2}\dot{s}u - J_x u + \dot{x} - \dot{s}x + \frac{1}{2}g(u,u)H + g(x,x)H + \ddot{s}H$$

and the geodesic equations are

$$\dot{u} - \frac{1}{2}\dot{s}u = J_x u \tag{4.7.1}$$

$$\dot{x} = \dot{s}x \tag{4.7.2}$$

$$\ddot{s} = -\frac{1}{2}g(u,u) - g(x,x). \tag{4.7.3}$$

From equation (4.7.2) we have that

$$\dot{x} = \dot{s}x \implies x = e^s Z_0 \tag{4.7.4}$$

since $x(0) = Z_0$. Also from (4.7.1) we have that

$$\frac{d}{dt}g(u,u) = 2g(\dot{u}, u) = 2g(J_x u + \frac{\dot{s}}{2}u, u) = \dot{s}g(u,u)$$

and therefore $g(u,u) = e^s \|V_0\|^2$ and equation (4.7.3) reads

$$\ddot{s} = -\frac{1}{2}e^s\|V_0\|^2 - e^{2s}\|Z_0\|^2 \tag{4.7.5}$$

Substituting for u and x, gives the wanted result. ■

So to find the geodesics we need to solve the equation:

$$\ddot{v} = (\dot{s}I_{\mathfrak{v}} + e^s J_{Z_0})\dot{v}$$

and to do so, we need a proposition.

Proposition 4.15. *[20] [3]*
Let $A(t), U(t) \in M(n, \mathbb{R})$ such that

$$\dot{U} = AU, \qquad U(0) = I_n.$$

Then, if certain unspecified conditions of convergence are satisfied, $U(t)$ can be written in the form

$$U(t) = \exp_{GL}(\Omega(t)) \tag{4.7.6}$$

where

$$\Omega = \int_0^t A(t_1)dt_1 + \frac{1}{2}\int_0^t \int_0^{t_2} [A(t_1), A(t_2)]dt_2dt_1 + \tag{4.7.7}$$

$$\frac{1}{6}\int_0^t \int_0^{t_2} \int_0^{t_3} [A(t_1), [A(t_2), A(t_3)]] + [A(t_3), [A(t_2), A(t_1)]]dt_3dt_2dt_1 + \dots$$

The series Ω is called the Magnus expansion of A.

Before we continue with finding the geodesic, some remarks about the proposition above. Consider the initial value problem

$$\dot{y} = A(t)y, \qquad y(0) = y_0. \tag{4.7.8}$$

If $U(t)$ is the so called matrizant of y i.e $y(t) = U(t)y_0$, then $U(t)$ will satisfy the initial value problem

$$\dot{U} = A(t)U \quad U(0) = I_n$$

and by proposition 4.15 we have that

$$y(t) = \exp_{GL}(\Omega)y_0.$$

Also notice that if $[A(t_1), A(t_2)] = 0$ for all t_1, t_2, then we have that

$$U(t) = \exp_{GL}(\int_0^t A(t_1)dt_1)$$

and hence

$$y(t) = \exp_{GL}(\int_0^t A(t_1)dt_1)y_0.$$

Proposition 4.16. *Let $\gamma : \mathbb{R} \to S$ be a geodesic, passing through the identity at time zero and with initial velocity $\dot{\gamma}(0) = V_0 + Z_0 + s_0 H$. Then*

- *if $\|Z_0\|^2 > 0$:*

$$\dot{v} = e^{s(t)} \cos(|Z_0| f(t))V_0 + \frac{e^{s(t)}}{|Z_0|} \sin(|Z_0| f(t))J_{Z_0}V_0$$

- *if $\|Z_0\|^2 = 0$:*

$$\dot{v} = e^{s(t)}V_0 + e^{s(t)} f(t)J_{Z_0}V_0$$

- *if* $\|Z_0\|^2 < 0$*:*

$$\dot{v} = e^{s(t)}\cosh(|Z_0|f(t))V_0 + \frac{e^{s(t)}}{|Z_0|}\sinh(|Z_0|f(t))J_{Z_0}V_0$$

where $f(t) = \int_0^t e^{s(t_1)}dt_1$.

Proof. Let $A(t) = \dot{s}I_{\mathfrak{v}} + e^s J_{Z_0}$. We have that $[A(t_1), A(t_2)] = 0$ for all $t_1, t_2 \in \mathbb{R}$, so that the Magnus expansion of $A(t)$ becomes

$$\Omega = \int_0^t A(t_1)dt_1 = s(t)I_{\mathfrak{v}} + \int_0^t e^{s(t_1)}dt_1 J_{Z_0}$$

and since $[s(t)I_{\mathfrak{v}}, f(t)J_{Z_0}] = 0$ we have that

$$\exp_{GL}(\Omega) = \exp_{GL}(s(t)I_{\mathfrak{v}})\exp_{GL}(f(t)J_{Z_0}) = e^{s(t)}\exp_{GL}(f(t)J_{Z_0}).$$

Hence we have that

$$\dot{v} = e^{s(t)}\exp_{GL}(f(t)J_{Z_0})V_0.$$

We must consider three cases:

CASE I: $\mathbf{Z_0}$ is spacelike

In the case that Z_0 is spacelike we have that $\|Z_0\|^2 > 0$ and $|Z_0|^2 = \|Z_0\|^2$. Using that $J_{Z_0}^2 = -\|Z_0\|^2 I_{\mathfrak{v}} = -|Z_0|^2 I_{\mathfrak{v}}$, we have that

$$\exp_{GL}(f(t)J_{Z_0}) = \sum_{k=0}^{\infty}\frac{f(t)^{2k}(-1)^k|Z_0|^{2k}I_{\mathfrak{v}}}{(2k)!} + \frac{1}{|Z_0|}\sum_{k=0}^{\infty}\frac{f(t)^{2k+1}(-1)^k|Z_0|^{2k+1}J_{Z_0}}{(2k+1)!} =$$

$$\cos(|Z_0|f(t))I_{\mathfrak{v}} + \frac{1}{|Z_0|}\sin(|Z_0|f(t))J_{Z_0}. \tag{4.7.9}$$

Hence

$$\dot{v} = e^{s(t)}\cos(|Z_0|f(t))V_0 + \frac{e^{s(t)}}{|Z_0|}\sin(|Z_0|f(t))J_{Z_0}V_0.$$

CASE II: $\mathbf{Z_0}$ is null

In the case that Z_0 is null we have that $\|Z_0\|^2 = 0$ and $J_{Z_0}^2 = 0$. Therefore we have that

$$\exp_{GL}(f(t)J_{Z_0}) = I_{\mathfrak{v}} + f(t)J_{Z_0}$$

Hence

$$\dot{v} = V_0 + f(t)J_{Z_0}V_0.$$

CASE III: $\mathbf{Z_0}$ is timelike

In the case that Z_0 is timelike we have that $\|Z_0\|^2 < 0$ and $|Z_0|^2 = -\|Z_0\|^2$. Using that $J_{Z_0}^2 = -\|Z_0\|^2 I = |Z_0|^2 I_{\mathfrak{v}}$, we have that

$$\exp_{GL}(f(t)J_{Z_0}) = \sum_{k=0}^{\infty} \frac{f(t)^{2k}|Z_0|^{2k} I_{\mathfrak{v}}}{(2k)!} + \frac{1}{|Z_0|}\sum_{k=0}^{\infty} \frac{f(t)^{2k+1}|Z_0|^{2k+1} J_{Z_0}}{(2k+1)!} =$$

$$\cosh(|Z_0|f(t))I_{\mathfrak{v}} + \frac{1}{|Z_0|}\sinh(|Z_0|f(t))J_{Z_0}. \tag{4.7.10}$$

Hence

$$\dot{v} = e^{s(t)}\cosh(|Z_0|f(t))V_0 + \frac{e^{s(t)}}{|Z_0|}\sinh(|Z_0|f(t))J_{Z_0}V_0.$$

■

We can now find some geodesics for some particular values of $\|V_0\|^2, \|Z_0\|^2$ and s_0.

Proposition 4.17. *Let $\gamma : \mathbb{R} \to S$ be a geodesic such that $\gamma(0) = e$ and $\dot{\gamma}(0) = V_0 + Z_0 + s_0 H$, where V_0, Z_0 are null vectors. Then the geodesics are given by*

For $s_0 \neq 0$:

$$v = \frac{1}{s_0}(e^{s_0 t} - 1)V_0 + \frac{1}{2s_0^2}\big(e^{2s_0 t} - 2e^{s_0 t} + 1\big)J_{Z_0}V_0$$

$$z = \frac{1}{2s_0}(e^{2s_0 t} - 1)Z_0$$

$$s = s_0 t$$

For $s_0 = 0$:

$$v = tV_0 + \frac{1}{2}t^2 J_{Z_0}V_0$$

$$Z = tZ_0$$

$$s = 0$$

Proof. From theorem 4.14 we have that

$$\ddot{s} = 0 \implies s = s_0 t.$$

CASE I: $s_0 = 0$

We have that $f(t) = \int_0^t dt_1 = t$ and by proposition 4.16 we have that

$$\dot{v} = V_0 + tJ_{Z_0}V_0 \implies v = tV_0 + \frac{1}{2}t^2 J_{Z_0}V_0. \tag{4.7.11}$$

Since V_0 is a null vector, we have that $[V_0, J_{Z_0}V_0] = 0$ and hence

$$\dot{z} = Z_0 \implies z = Z_0 t.$$

CASE II: $s_0 \neq 0$

We have that $f(t) = \int_0^t e^{s_0 t_1} dt_1 = \frac{1}{s_0}(e^{s_0 t} - 1)$ and by proposition 4.16 we have that

$$\dot{v} = e^{s_0 t} V_0 + \frac{1}{s_0}(e^{2s_0 t} - e^{s_0 t}) J_{Z_0} V_0 \implies v = \frac{1}{s_0}(e^{s_0 t} - 1)V_0 + \frac{1}{2s_0^2}\big(e^{2s_0 t} - 2e^{s_0 t} + 1\big) J_{Z_0} V_0.$$

Again since V_0 is a null vector, we have that $[v, \dot{v}] = 0$ and

$$\dot{z} = e^{2s_0 t} Z_0 \implies z = \frac{1}{2s_0}(e^{2s_0 t} - 1) Z_0. \tag{4.7.12}$$

■

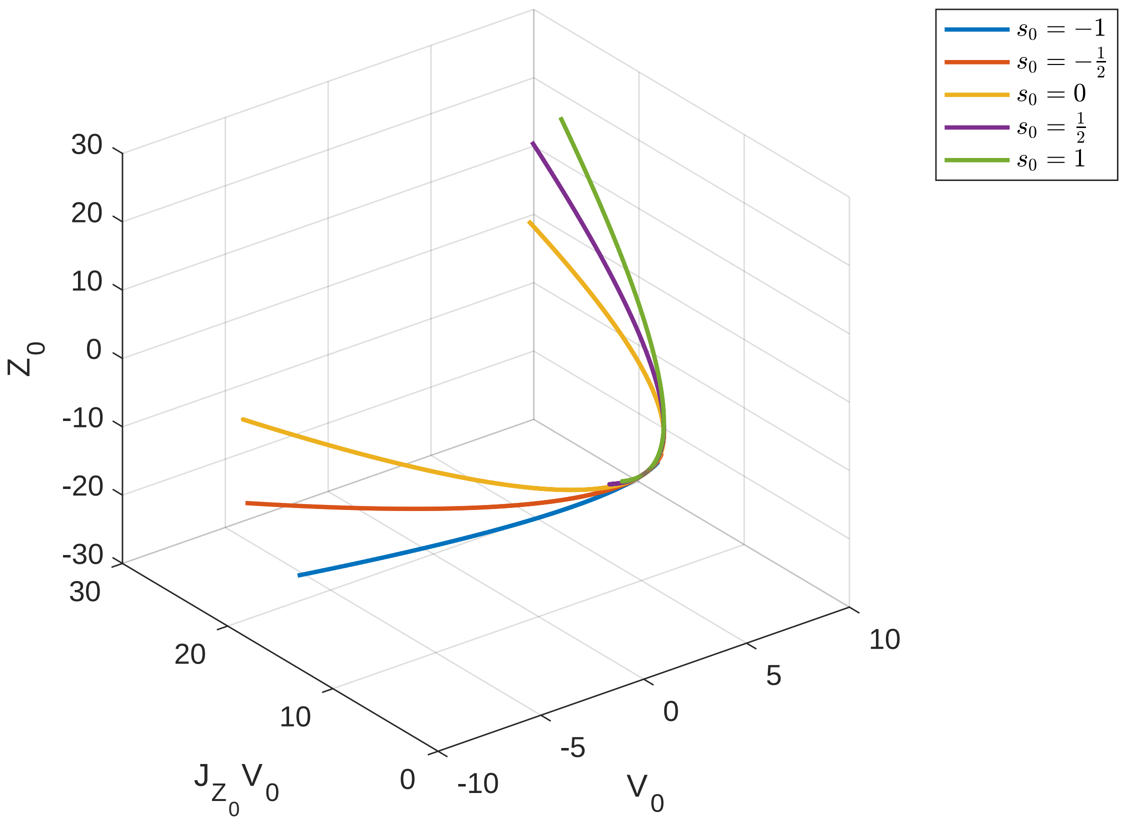

Figure 7: Plot showing the projection of the geodesics of a semi-Damek-Ricci space, down on $\mathfrak{n}_0$. V_0, Z_0 are null vectors and s_0 takes the values $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$.

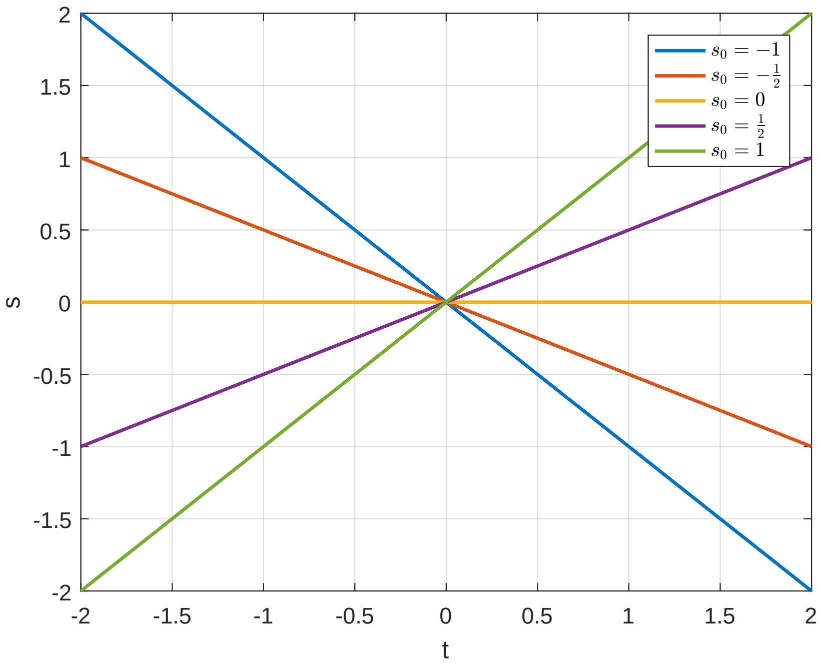

Figure 8: Plot of $s(t)$, when V_0, Z_0 are null vectors and s_0 takes the values $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$.

To find the rest of the geodesics we need to solve the initial value problem

$$\ddot{s} = -\frac{1}{2}e^s\|V_0\|^2 - e^{2s}\|Z_0\|^2 \qquad s(0) = 0, \dot{s}(0) = s_0 \tag{4.7.13}$$

when V_0 and Z_0 are not both zero, but no solution was found. But just as for semi-H-type groups, there exist totally geodesic submanifolds of semi-Damek-Ricci spaces and the hope is that the geodesic equations will be easier to either solve analytically or numerically.

Definition 4.18. Let $\gamma : \mathbb{R} \to S$ be the geodesic in S such that $\gamma(0) = e$ and $\dot{\gamma} = V_0 + Z_0 + s_0 H$. We define

$$\mathfrak{s}_0 = \mathfrak{n}_0 \oplus \mathfrak{a}.$$

Theorem 4.19. *The geodesic $\gamma : \mathbb{R} \to S$ in S such that $\gamma(0) = e$ and $\dot{\gamma} = V_0 + Z_0 + s_0 H$, where V_0, Z_0 and $s_0 H$ are not null vectors, lies in the totally geodesically four dimensional semi-Riemannian submanifold $S_0 = \exp_{\mathfrak{s}}(\mathfrak{s}_0)$. Moreover S_0 is a semi-Damek-Ricci space.*

Proof. We start the proof by showing that $\mathfrak{s}_0$ with inherited scalar product, Lie bracket and J-map is a semi-Damek-Ricci Lie algebra. From theorem 3.19 we have that $\mathfrak{n}_0$ is a semi-H-type Lie algebra. Hence we need only to check that $\mathfrak{s}_0$ is closed under the Lie brackets.
Since

$$[H, V_0] = \frac{1}{2}V_0 \qquad [H, J_{Z_0}V_0] = \frac{1}{2}J_{Z_0}V_0 \qquad [H, Z_0] = Z_0$$

we have that $\mathfrak{s}_0$ is a subalgebra of $\mathfrak{s}_o$ and a semi-Damek-Ricci Lie algebra. Therefore we have that the simply connected Lie group associated with $\mathfrak{s}_0$ is a semi-Damek-Ricci space.

By proposition 4.6 we have that $\exp_{\mathfrak{s}}(\mathfrak{s}_0)$ is a subgroup of S and since $\exp_{\mathfrak{s}}$ is a diffeomorphism we have by proposition 2.127 that S_0 is a Lie subgroup of S with $\mathfrak{s}_0$ as its Lie algebra. By defining the metric on S_0 in the same way we did in section 4.2, we have that S_0 is a semi-Riemannian submanifold of S.

To show that S_0 is a totally geodesic semi-Riemannian submanifold we calculate the second fundamental form. Let $v_1, v_2 \in \mathfrak{v}_0$, $z_1, z_2 \in \mathfrak{z}_0$ and $s_1, s_2 \in \mathbb{R}$, then

$$\nabla_{v_1} v_2 = \frac{1}{2} g(v_1, v_2)H + \frac{1}{2}[v_1, v_2] \qquad \nabla_{v_1} z_1 = -\frac{1}{2} J_{z_1} v_1 = \nabla_{z_1} v_1 \qquad \nabla_{z_1} z_2 = g(z_1, z_2)H$$

$$\nabla_{v_1}(s_1 H) = -\frac{1}{2} s_1 v_1 \qquad \nabla_{z_1}(s_1 H) = -s_1 z_1 \qquad \nabla_H(\cdot) = 0$$

Hence we have that the second fundamental form vanishes and by proposition 2.92 we have that S_0 is a totally geodesic semi-Riemannian submanifold of S.

■

The theorem tells us that we can find geodesics in S, by finding them in S_0, as long as none of V_0, Z_0 and $s_0 H$ are null vectors. If at least one of V_0, Z_0 and s_0 are zero, we have that the metric on S_0 is degenerate and S_0 is not a semi-Riemannian manifold. We conclude therefore that, there are geodesics in S, that cannot be found in the totally geodesic submanifold S_0. There are seven cases of geodesics that cannot be found in S_0, namely when the initial velocities are

V_0	Z_0	$s_0 H$
null	not null	not null
null	not null	null
null	null	not null
null	null	null
not null	not null	null
not null	null	not null
not null	null	null

Table 2: Cases of initial velocities of semi-Riemannian geodesics in S, that can not be found in S_0.

To visualize this we can plot the two dimensional surface $\|V_0\|^2 + \|Z_0\|^2 + s_0^2 = 0$ in $\mathbb{R}^3$, see figure 9. The seven cases in table 2, is the represented by all points on the three planes, given by $\|V_0\|^2 = 0, \|Z_0\|^2 = 0$ and $s_0 = 0$. Any point on $\|V_0\|^2 + \|Z_0\|^2 + s_0^2 = 0$ will represent null initial velocity, any point on the outside of the surface will represent spacelike initial velocity and any point on the inside of the surface will represent timelike initial velocity. The geodesics we found in proposition 4.17 have initial velocity lying on the intersection of the two planes $\|V_0\|^2 = 0$ and $\|Z_0\|^2 = 0$.

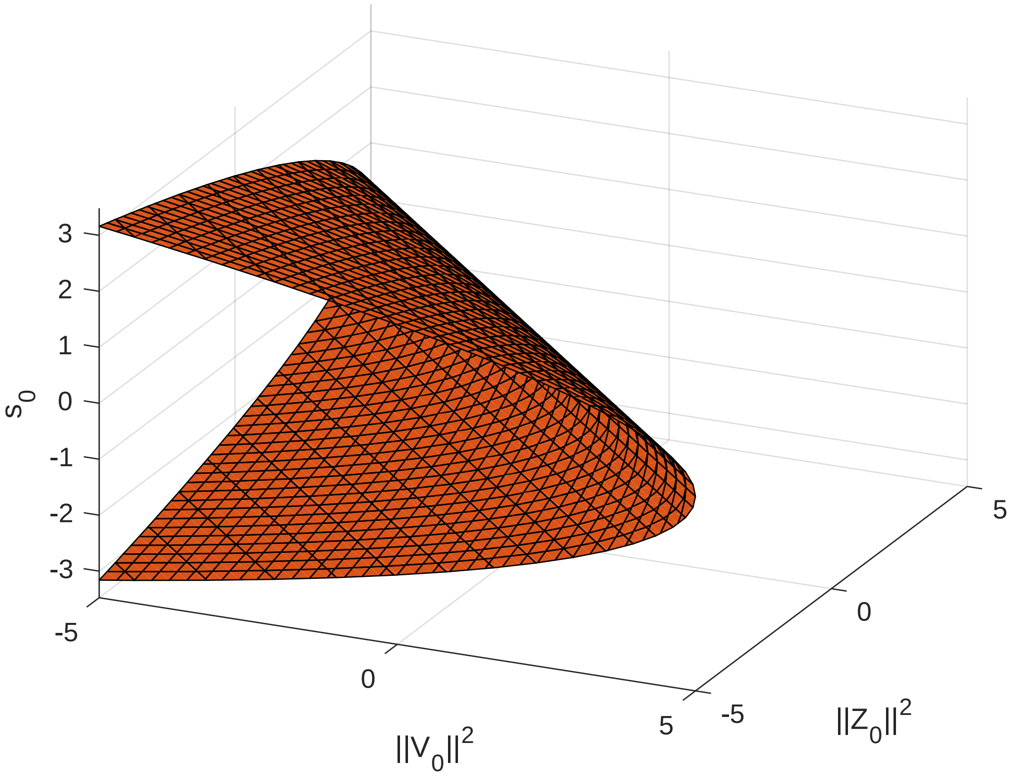

Figure 9: Plot of $\|V_0\|^2+\|Z_0\|^2+s_0^2 = 0$. Points on the surface represent geodesics with $\|\dot{\gamma}(0)\|^2 = 0$.

Since every geodesic $\gamma(t)$ in S, such that $\gamma(0) = e$ and $\dot{\gamma}(0) = V_0 + Z_0 + s_0 H$, where V_0, Z_0 and $s_0 H$ are not null vectors, lies in the totally geodesic submanifold $S_0 = \exp_{\mathfrak{s}}(\mathfrak{s}_0)$, we wish to find the geodesic equations in S_0. We introduce coordinates on S_0 as follows

$$(u^1, u^2, z, s) \xmapsto{(\exp_{\mathfrak{n}} \times \exp_{\mathfrak{a}})^{-1}} u^1 V_0 + u^2 J_{Z_0} V_0 + z Z_0 + sH$$

and since S_0 is a semi-Damek-Ricci space, we have by proposition 4.2.2 that left-invariant vector fields are given by

$$V_0 = e^{\frac{s}{2}} \frac{\partial}{\partial u^1} - \frac{e^{\frac{s}{2}}}{2} \|V_0\|^2 u^2 \frac{\partial}{\partial z}, \tag{4.7.14}$$

$$J_{Z_0} V_0 = e^{\frac{s}{2}} \frac{\partial}{\partial u^2} + \frac{e^{\frac{s}{2}}}{2} \|V_0\|^2 u^1 \frac{\partial}{\partial z}, \tag{4.7.15}$$

$$Z_0 = e^s \frac{\partial}{\partial z}, \tag{4.7.16}$$

$$H = \frac{\partial}{\partial s}. \tag{4.7.17}$$

Proposition 4.20. *Let $\gamma : \mathbb{R} \to S_0$ be the geodesic in S_0, such that $\gamma(0) = e$ and $\dot{\gamma}(0) = V_0 + Z_0 + s_0 H$. The geodesic equations in S_0 are*

$$\begin{cases} \ddot{u}^1 = \frac{\dot{s}}{2}\dot{u}^1 - \|Z_0\|^2 e^s \dot{u}^2 \\ \ddot{u}^2 = \frac{\dot{s}}{2}\dot{u}^2 + e^s \dot{u}^1 \\ \dot{z} = \frac{1}{2}\|V_0\|^2(u^1\dot{u}^2 - \dot{u}^1 u^2) + e^{2s} \\ \ddot{s} = -\frac{1}{2}e^s\|V_0\|^2 - e^{2s}\|Z_0\|^2. \end{cases}$$

Proof. Let the velocity of the geodesic be

$$\dot{\gamma}(t) = \dot{u}^1\frac{\partial}{\partial u^1} + \dot{u}^2\frac{\partial}{\partial u^2} + \dot{z}\frac{\partial}{\partial z} + \dot{s}\frac{\partial}{\partial s}$$

By using (4.7.14), (4.7.16) , (4.7.15) and (4.7.17), we can write the velocity as

$$\dot{\gamma}(t) = \dot{u}^1 e^{-\frac{s}{2}} V_0 + \dot{u}^2 e^{-\frac{s}{2}} J_{Z_0} V_0 + e^{-s}(\dot{z} + \frac{1}{2}\|V_0\|^2(\dot{u}^1 u^2 - \dot{u}^2 u^1))Z_0 + \dot{s}H$$

If the $\gamma(t)$ is to be the geodesic through the identity, with $\dot{\gamma}(t) = V_0 + Z_0 + s_0 H$ we need that $\dot{u}^1(0) = 1$, $\dot{u}^2(0) = 0$, $\dot{z}(0) = 1$ and $\dot{s}(0) = s_0$. We wish now to calculate the acceleration of $\gamma(t)$ and to make calculations easier, let

$$\begin{aligned} \mathcal{A} &= \dot{u}^1 e^{-\frac{s}{2}} \\ \mathcal{B} &= \dot{u}^2 e^{-\frac{s}{2}} \\ \mathcal{X} &= e^{-s}(\dot{z} + \frac{1}{2}\|V_0\|^2(\dot{u}^1 u^2 - \dot{u}^2 u^1)) \end{aligned}$$

such that $\dot{\gamma}(t) = \mathcal{A}(t)V_0 + \mathcal{B}(t)J_{Z_0}V_0 + \mathcal{X}(t)Z_0 + \dot{s}H$. Hence we need to compute $D_t(\dot{\gamma})$ and by linearity we can compute term by term.

$$D_t(AV_0) = \dot{\mathcal{A}}V_0 + \frac{\mathcal{A}^2}{2}\|V_0\|^2 H - \frac{\mathcal{AB}}{2}\|V_0\|^2 Z_0 - \frac{\mathcal{AX}}{2}J_{Z_0}V_0$$

$$D_t(\mathcal{B}U_2) = \dot{\mathcal{B}}J_{Z_0}V_0 + \frac{\mathcal{AB}}{2}\|V_0\|^2 Z_0 + \frac{\mathcal{B}^2}{2}\|V_0\|^2\|Z_0\|^2 H + \frac{\mathcal{BX}}{2}\|Z_0\|^2 V_0$$

$$D_t(\mathcal{X}Z_0) = \dot{\mathcal{X}}Z_0 - \frac{\mathcal{AX}}{2}J_{Z_0}V_0 + \frac{\mathcal{BX}}{2}\|Z_0\|^2 V_0 - \mathcal{X}^2\|Z_0\|^2 H$$

$$D_t(\dot{s}H) = \ddot{s}H - \dot{s}(\frac{\mathcal{A}}{2}V_0 + \frac{\mathcal{B}}{2}J_{Z_0}V_0 + \mathcal{X}Z_0)$$

Hence the geodesic equations are

$$\dot{\mathcal{A}} - \frac{\dot{s}}{2}\mathcal{A} = -\mathcal{B}\mathcal{X}\|Z_0\|^2 \tag{4.7.18}$$

$$\dot{\mathcal{B}} - \frac{\dot{s}}{2}\mathcal{B} = \mathcal{A}\mathcal{X} \tag{4.7.19}$$

$$\dot{\mathcal{X}} = \dot{s}\mathcal{X} \tag{4.7.20}$$

$$\ddot{s} = -\frac{\mathcal{A}^2}{2}\|V_0\|^2 - \frac{\mathcal{B}^2}{2}\|V_0\|^2\|Z_0\|^2 - \mathcal{X}^2\|Z_0\|^2 \tag{4.7.21}$$

From equation (4.7.20) we see that $\mathcal{X} = e^s$. By using equations (4.7.18) and (4.7.19) we have that

$$\frac{d}{dt}\left(\frac{\mathcal{A}^2}{2}\|V_0\|^2 + \frac{\mathcal{B}^2}{2}\|V_0\|^2\|Z_0\|^2\right) = \dot{s}\left(\frac{\mathcal{A}^2}{2}\|V_0\|^2 + \frac{\mathcal{B}^2}{2}\|V_0\|^2\|Z_0\|^2\right)$$

hence

$$\frac{\mathcal{A}^2}{2}\|V_0\|^2 + \frac{\mathcal{B}^2}{2}\|V_0\|^2\|Z_0\|^2 = \frac{1}{2}\|V_0\|^2 e^s$$

Inserting this into equation (4.7.21), we get that

$$\ddot{s} = -\frac{1}{2}e^s\|V_0\|^2 - e^{2s}\|Z_0\|^2. \tag{4.7.22}$$

Also since $\mathcal{X} = e^s$ we have that

$$e^s = e^{-s}(\dot{z} + \frac{1}{2}\|V_0\|^2(\dot{u}^1u^2 - \dot{u}^2u^1)) \implies \dot{z} = \frac{1}{2}\|V_0\|^2(u^1\dot{u}^2 - \dot{u}^1u^2) + e^{2s}$$

By substituting for $\mathcal{A}, \mathcal{B}$ and $\mathcal{X}$ in equations (4.7.18) and (4.7.19), we get the wanted result. ■

By comparing theorem 4.14 and proposition 4.20, we see that we must still solve the initial value problem (4.7.13), which we are not able to do. Hence we will solve the geodesic equations from proposition 4.20 numerically, to produce plots of the geodesics. See figure 10 and figure 11. This is done by using discretization and the forward Euler method, to iteratively compute the unknown functions u^1, u^2, z and s for given values of $\|V_0\|^2, \|Z_0\|^2$ and s_0.

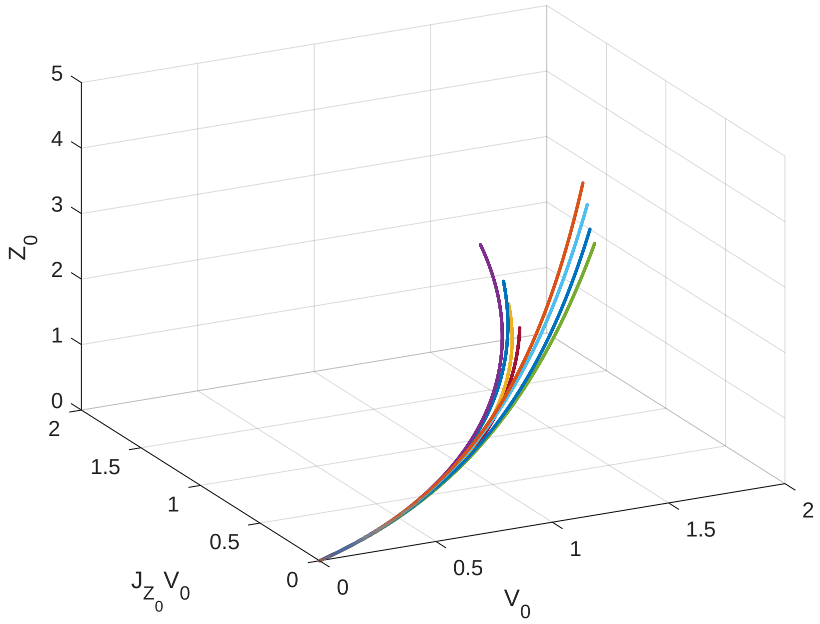

Figure 10: Plot showing the projection of geodesics of a semi-Damek-Ricci space, down on $\mathfrak{n}_0$, when found numerically. $\|V_0\|^2, \|Z_0\|^2$ takes on the values $-0.3, 0.3$ and s_0 takes on the values $-0.1, 0.1$.

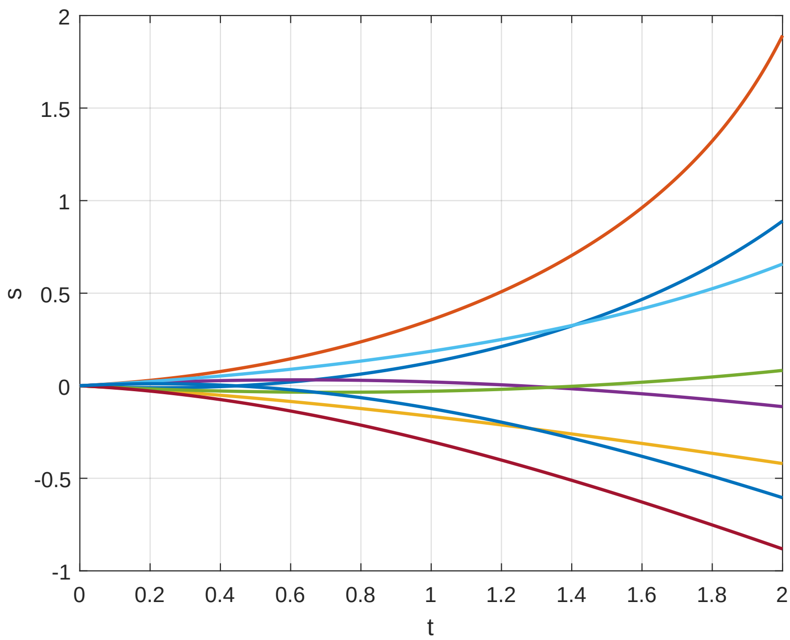

Figure 11: Plot of $s(t)$, when found numerically. $\|V_0\|^2, \|Z_0\|^2$ takes on the values $-0.3, 0.3$ and s_0 takes on the values $-0.1, 0.1$.

Even though we could not find all the geodesics, we know what they are when V_0, Z_0, s_0H are spacelike and we have that $\|\dot{\gamma}(0)\|^2 = 1$.

Proposition 4.21. *[2] [6]*
Let $\gamma : \mathbb{R} \to S$ be a geodesic with $\gamma(0) = e$ and $\dot{\gamma}(0) = V_0 + Z_0 + s_0H$, such that V_0, Z_0, s_0H are spacelike vectors and $\|\dot{\gamma}(0)\|^2 = 1$. Then

$$\gamma = (\exp_{\mathfrak{n}} \times \exp_{\mathfrak{a}})(\frac{2\theta(1-s_0\theta)}{\chi}V_0 + \frac{2\theta^2}{\chi}J_{Z_0}V_0 + \frac{2\theta}{\chi}Z_0 + \ln(\frac{1-\theta^2}{\chi}))$$

with

$$\theta(t) = \tanh(\frac{t}{2}) \quad \textit{and} \quad \chi = (1-s_0\theta)^2 + \|Z_0\|^2\theta^2.$$

The key statement in the proposition above is that V_0 and Z_0 are space like vectors, since in this case we have that the metric on S_0 is a Riemannian metric and we are in the classical Riemannian case.

Let D be the open unit disc in $\mathbb{C}^2$ i.e

$$D = \{z \in \mathbb{C}^2 \mid |z| < 1\}$$

and let

$$\mathfrak{S} = \{(z_1, z_2) \in \mathbb{C}^2 \mid \operatorname{Re}(z_2) > \frac{z_1\bar{z}_1}{4}\}.$$

Now let $\hat{V} = \frac{V_0}{|V_0|}$ and $\hat{Z} = \frac{Z_0}{|Z_0|}$. We define a bijection $\Phi : S_0 \to \mathfrak{S}$ given by

$$\Phi\Big((\exp_{\mathfrak{n}} \times \exp_{\mathfrak{a}})(a\hat{V} + bJ_{\hat{Z}}\hat{V} + c\hat{Z} + sH)\Big) \mapsto (a + ib, e^s + \frac{1}{4}(a^2 + b^2) + ic)$$

with inverse

$$\Phi^{-1}(z_1, z_2) = (\exp_{\mathfrak{n}} \times \exp_{\mathfrak{a}})(\operatorname{Re}(z_1)\hat{V} + \operatorname{Im}(z_1)J_{\hat{Z}}\hat{V} + \operatorname{Im}(z_2)\hat{Z} + \ln(\operatorname{Re}(z_2) - \frac{1}{4}|z_1|^2)H).$$

Now we define a map from $C : D \to \mathfrak{S}$ given by

$$C(z_1, z_2) = \Big(\frac{2z_1}{1+z_2}, \frac{1-z_2}{1+z_2}\Big).$$

Then $C : D \to \mathfrak{S}$ is biholomorphic i.e C is bijective, holomorphic and its inverse

$$C^{-1}(z_1, z_2) = \Big(\frac{z_1}{1+z_2}, \frac{1-z_2}{1+z_2}\Big)$$

is also holomorphic. Now we equip the open unit disk D with the metric given by

$$ds^2 = 4\frac{(1-|z|^2)dz \cdot d\bar{z} - \bar{z}dz \cdot zd\bar{z}}{(1-|z|^2)^2}$$

called the Bergman metric and we equip $\mathfrak{S}$ with the Riemannian metric such that C is an isometry. Then D and $\mathfrak{S}$ are two models of two-dimensional complex hyperbolic space, namely the unit disk model and the Siegel domain model respectively.

Now let $\alpha : \mathbb{R} \to D$ be a geodesic in unit disk model parametrized by arc length and $\alpha(0) = 0$, then the geodesic is on the form

$$\alpha(t) = \theta(t)z \qquad z \in \partial D.$$

Since C is an isometry, we have that the image of α under C will be a geodesic in the Siegel domain model, hence

$$\beta := C \circ \alpha : \mathbb{R} \to \mathfrak{S}$$

is a geodesic parametrized by arc length and passing through $(0,1)$ at time $t = 0$. Since Φ is also an isometry, see [2][6], we have that

$$\gamma = \Phi^{-1} \circ \beta : \mathbb{R} \to S_0$$

is a geodesic in S_0 parametrized by arc length and such that $\gamma(0) = e$. Now let

$$\alpha(t) = (z_1\theta, z_2\theta) \quad \text{such that } |z_1|^2 + |z_2|^2 = 1. \tag{4.7.23}$$

We have that

$$\beta(t) = \big(\beta^1(t), \beta^2(t)\big) = \left(\frac{2z_1\theta}{1 + z_2\theta}, \frac{1 - z_2\theta}{1 + z_2\theta}\right). \tag{4.7.24}$$

and

$$\gamma(t) = (\exp_{\mathfrak{n}} \times \exp_{\mathfrak{a}})\big(\operatorname{Re}(\beta^1)\hat{V} + \operatorname{Im}(\beta^1)J_{\hat{Z}}\hat{V} + \operatorname{Im}(\beta^2)\hat{Z} + \ln(\operatorname{Re}(\beta^2) - \frac{1}{4}|\beta^1|^2)\big). \tag{4.7.25}$$

If we require that $\dot{\gamma}(0) = V_0 + Z_0 + s_0H$, we have that

$$|V_0| = \dot{\beta}^1(0) = z_1 \qquad s_0 + i|Z_0| = \dot{\beta}^2(0) = -z_2.$$

By substituting for z_1 and z_2 in equations (4.7.24),(4.7.25) and then simplifying we get the wanted result.

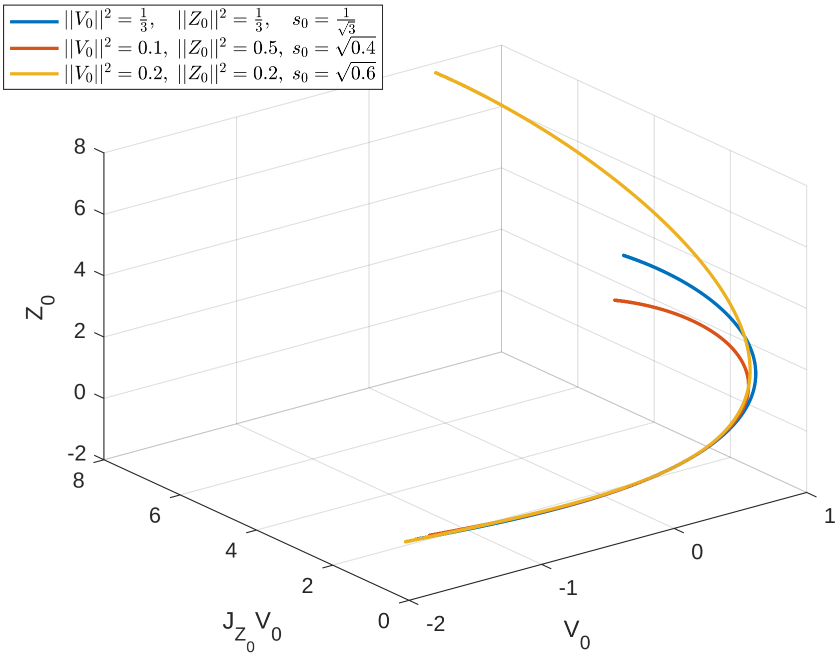

Figure 12: Plot showing the projection of geodesics of a semi-Damek-Ricci space, down on $\mathfrak{n}_0$. V_0 and Z_0 are space like vectors.

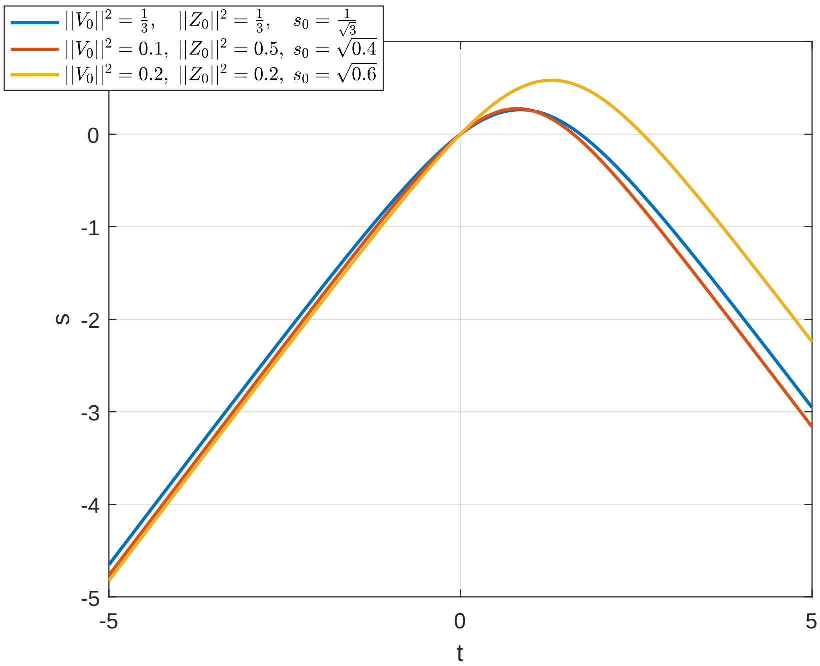

Figure 13: Plot of $s(t)$. $\|V_0\|^2$ and $\|Z_0\|^2$ are space like vectors.

5 Summary and Further Research

In chapter 3, we gave a complete description of the semi-Riemannian geodesics $\gamma : \mathbb{R} \to N$, passing trough the identity and with initial velocity $\dot{\gamma}(0) = V_0 + Z_0$. We proved that every semi-Riemannian geodesic lies in the submanifold $N_0 = \exp_{\mathfrak{n}}(\mathfrak{n}_0)$ and that N_0 is totally geodesic, when V_0 and Z_0 are not null vectors. We also computed the Levi-Civita connection and various curvatures of N. Moreover we gave a complete description of the sub-semi-Riemannian geodesics in sub-semi-H-type groups.

In chapter 4, we introduced the notion of a semi-Damek-Ricci space and gave a partial description of the semi-Riemannian geodesics $\gamma : \mathbb{R} \to S$, passing trough the identity and with initial velocity $\dot{\gamma}(0) = V_0 + Z_0 + s_0 H$. We proved that every semi-Riemannian geodesic such that $\gamma(0) = e$ and $\dot{\gamma}(0) = V_0 + Z_0 + s_0 H$, such that V_0, Z_0 and $s_0 H$ are not null vectors, lies in the totally geodesic semi-Riemannian submanifold $S_0 = \exp_{\mathfrak{s}}(\mathfrak{s}_0)$. We also computed the Levi-Civita connection and various curvatures.

We now list possible topics for further research concerning semi-H-type groups and semi-Damek-Ricci spaces:

- We found all the semi-Riemannian geodesic passing through the identity in semi-H-type groups, given an initial velocity. The natural next step is to consider the boundary value problem: Find the semi-Riemannian geodesics passing through the identity and a given point $p \in N$. A similar consideration can be made for semi-Damek-Ricci spaces.

- A complete description of the semi-Riemannian geodesics in semi-Damek-Ricci spaces. Seeing as we could not solve the second order autonomous given by

$$\ddot{s} = -\frac{1}{2} e^{s} \|V_0\|^2 - e^{2s} \|Z_0\|^2 \tag{5.0.1}$$

for any values of $\|V_0\|^2$ and $\|Z_0\|^2$, we could not give a complete description of the semi-Riemannian geodesics.

- Study the sub-semi-Riemannian manifold $(S, \mathcal{D}, g)$, where $\mathcal{D}(p) = L_{p,*}(\mathfrak{v} \oplus \mathfrak{a})$ and g is the left invariant metric defined on $\mathcal{D}$ by

$$g_p(X_p, Y_p) = g_{\mathfrak{v}\oplus\mathfrak{a}}((L_{p^{-1}})_{*,p} X_p, (L_{p^{-1}})_{*,p} Y_p) \quad \text{for any } X, Y \in \mathcal{D}.$$

In particular the sub-semi-Riemannian geodesic going through the identity, with initial velocity $V_0 + s_0 H \in \mathfrak{v} \oplus \mathfrak{a}$ and initial covector $\mu_0 \in \mathfrak{z}^*$.

- Finding the Isometry group of semi-H-type groups, that is finding all diffeomorphisms $\phi : N \to N$, such that ϕ preserves the left invariant metric on N. A similar problem can be considered for semi-Damek-Ricci spaces.

- Study the Jacobi vector fields and Killing vector fields of semi-H-type groups and semi-Damek-Ricci spaces.

- Classify semi-H-type groups and semi-Damek-Ricci spaces, according to when they are symmetric, natural reductive, weakly symmetric, commutative, geodesically orbital and harmonic. A similar classification was made in [2], in the Riemannian case.

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