Elastic Curves in Rolling Problems

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1 Introduction

Elastic curves have a long and rich history in the field of mathematics, and is still being studied by many scientists today. As these curves appear in many natural phenomena, their applicable potensiality hits a broad variety of modern sciences. To get a better understanding of how these curves behave, mathematicians have uncovered different problem in mathematics where these curves appear. Elastic curves can be formulated as a problem in the calculus of variations, as a solution to elliptic integrals, or as the differential equation describing a mathematical pendulum, to name a few.

A seemingly different problem relating to elastic curves, is know as the *rolling sphere problem* - first made famous by John Hammersley [10] in 1983. Loosely speaking, the problem is to roll the sphere from an initial contact configuration with the plane to a terminal contact configuration such that the curve traced out by its point of contact is as short as possible. Amazingly, the minimal curves that solve this problem are in fact elastic according to [3] and [11]. We will get to the formal statement of this problem later on.

In this text, we will look at how the elastic curves relates to some of the mathematical problems mentioned above. Our main focus will be on the relation to the rolling sphere problem and the rolling of a hyperboloid.

In section 2 we will recall some of the history regarding elastic curves. We will also recall some basic curve theory and properties regarding Jacobi elliptic functions - which are essential in deriving the intrinsic equations describing the behaviour of these curves. Next we will apply methods from the calculus of variations to obtain the intrinsic equation of elastic curves. In section 3 we will uncover the material or optimal control theory on Lie groups which we will apply in section 4, where we revisit the problem of elastic curves and study its relation to the rolling problems of both the sphere and the hyperboloid. Finally, in section 5 we will see how the different energy levels of the Hamiltonian system influences the optimal solutions. Furthermore, we will study some of the rolling motions along elastic curves with constant curvature.

2 Elastic Curves

Elastic curves are the main characters of this text. To build interest and to get a better understanding of how these curves appear, we present a short summary on their history, before we approach them in a mathematical manner.

2.1 History

Even though the modern version of the elastica is due to Euler, the history of elastic curves dates all the way back to 13th century with Jordanus de Nemore.

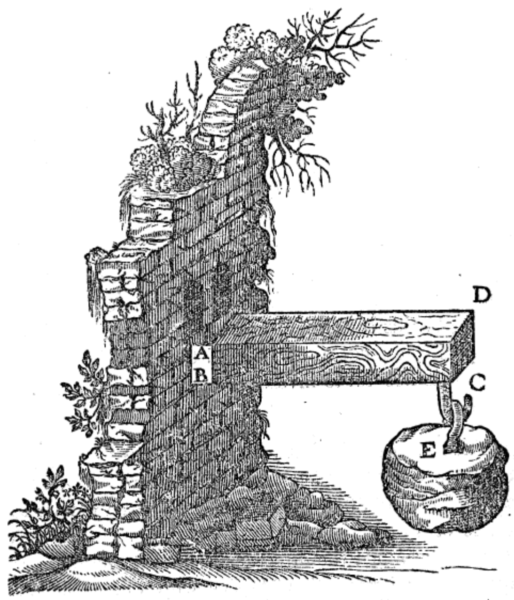
2.1.1 Early history

Jordanus de Nemore considered the shape of a uniform lamina resting at its center of gravity with its endpoints bending slightly down. By the theory of elasticity, this shape will resemble that of what we now call an *elastic curve*. de Nemore thought that the shape would become a circle if the weighted ends had sufficient weight to pull the them together. Though this proposed solution was incorrect, we now know that the circle is in fact a solution for another type of elastic curve.

Many years later, in 1638, Galileo also posed a problem regarding elastic curves. He formulated the following problem:

Considered a prismatic beam attached to a wall. How much weight is required to break this beam?

The following figure illustrates the setup:



(From the *Discorsi*, Leiden 1638.)

Figure 1: Galileo's problem.

Considering the beam as a compound lever with a fulcrum at B, Galileo derived several scaling relationships of this problem. This marks the first mathematical study of elasticity. Many scientists were inspired by Galileos work on elasticity in the coming decades.

It wasn't before Hookes law on the tension of springs and Christian Huygens treatment of involutes and evolutes of curves - leading to a better understanding of curvature - that inspired James Bernoulli to reformulate the problem of elastic curves. In 1691 James Bernoulli posed the following problem:

Consider a vertical lamina with a weight m attached at the top end of this lamina. For what value m will the endpoint become horizontal?

The following figure illustrates the setup:

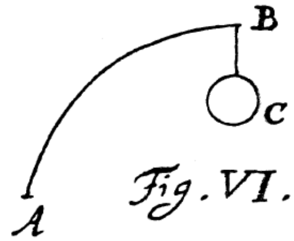


Figure 2: Bernoulli's problem.

The class of elastic curves which solves this particular problem, is now known as *rectangular elasticas*. This naming is due to the tangents of the endpoints being perpendicular with respect to each other. To get a grasp on the difficulty of the problem in these days, Huygens expressed himself in a letter to Leibniz, dated 16 November 1691, where he wrote:

"I cannot wait to see what Mr. Bernoulli the elder will produce regarding the curvature of the spring. I have not dared to hope that one would come out with anything clear or elegant here, and therefore I have never tried."

Even though James Bernoulli made great contribution to the problem of elastic curves, there was still lacking a great variety of elastic curves. It wasn't before his nephew, Daniel Bernoulli, proposed the general version of the problem to Leonard Euler that we obtained the complete classification of elastic curves.

2.1.2 Euler elastica problem

Daniel thought that Euler and his newly developed theory on calculus of variations might do the job. In 1743 Daniel wrote to Euler suggesting that the total energy of the elastic curves were proportional to the magnitude of

$$E = \int \frac{ds}{r^2},$$

where r is the radius of curvature. With this in hand, Euler could use his apparatus on calculus of variations to minimize the functional E. Eulers formulation of the problem of elastic curves goes as follows:

"That among all curves of the same length which not only pass through the points A and B, but are also tangent to given straight lines at these points, that curve be determined in which the value of

$$\int_A^B \frac{ds}{r^2}$$

be a minimum."

Euler published his treatment of elastic curves as an appendix to his landmark book [8] on variational techniques in 1744. Euler discovered that there is an infinite number of elastic curves solving this variational problem. By deriving the ODEs of the problem, he managed to classify all of these into 9 different types, depending on two parameters - which we will discuss in section 5.1.

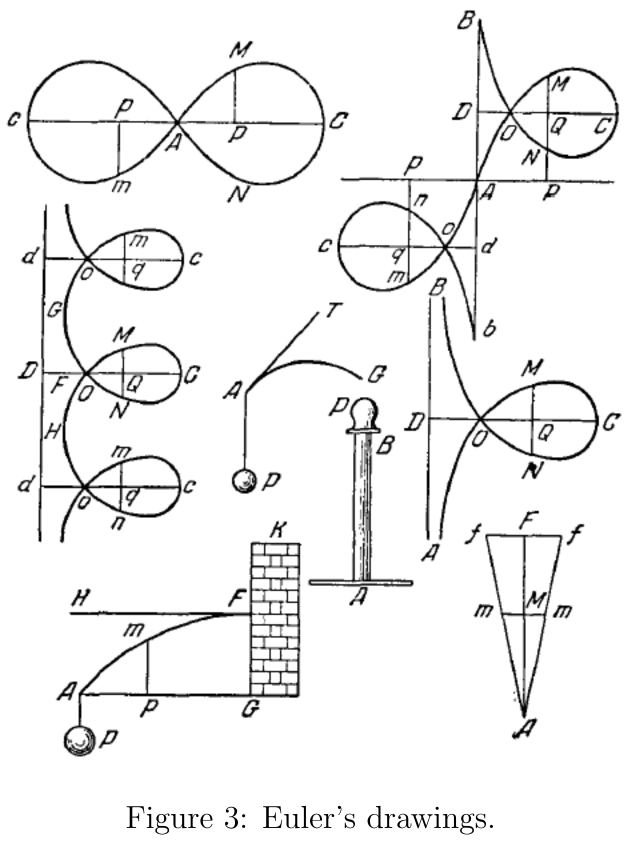


Figure 3: Euler's drawings.

Due to his detailed description on the solutions of elastic curves, the general statement of this problem, as quoted above, is now often referred to as *the Euler elastica problem*.

2.1.3 Elastica and the pendulum equation

Another important insight in the study of elastic curves, involves its relation to the pendulum equation. It is not entirely clear when the analogy between the pendulum and the elastica was established, but Kirchhoff established an analogy within the context of his study [14] of a spinning top and a twisted rod. The spinning top and twisted rod are three-dimensional generalisations of the pendulum and elastica respectively. This relationship is also attributed the German physicist and mathematician, Max Born. In his 1906 Ph.D. thesis [4] "Stability of elastic lines in the plane and the space", he discovered that the differential equation describing the elastica resembles that of a mathematical pendulum.

The kinetic analogy of the pendulum is really helpful in understanding the classifications of the elastic curves. In most literature on periodic systems, the mechanics of the swinging pendulum is the most basic and standard example. First of all, this analogy suggests periodicity in elastic curves. Moreover, we have a connection between the curvature of the elastic curve and the swing angle of the pendulum. More specifically, the curvature of the elastic curve corresponds to the angular momentum of the pendulum. Considering the pendulum equation

$$\ddot{\theta} + \frac{g}{r} \sin \theta = 0,$$

where θ is the angle that the pendulum makes with the vertical axis, g is the gravitational constant, and r the radius of the pendulum, we see that only the swing-height of the pendulum will infect the fundamental solution:

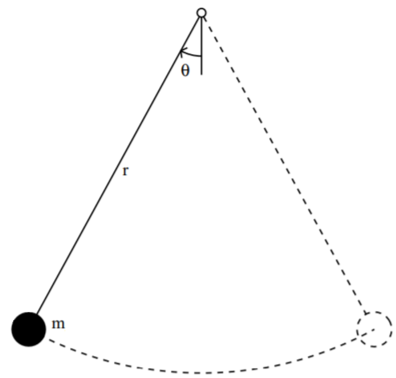


Figure 4: Simple pendulum.

Thus, the family of solutions to the problem of elastic curves can be characterized by a single scalar parameter. We will discuss this relation in more depth in section 4.1.5 and 5.1.

2.1.4 Elastica and the rolling sphere

In more recent years, a new connection to the elastica was discovered. A. Arthur and G.R. Walsh [3] (1986) and V. Jurdjevic [11] (1993) independently discovered that the solution set of the rolling sphere problem coincides with that of elastic curves. The rolling sphere problem says the following:

"Consider a ball rolling on a horizontal plane without slipping or twisting. The problem is to roll the ball from an initial contact configuration (defined by contact point of the ball with the plane, and orientation of the ball in the 3-space) to a terminal contact configuration, so that the curve traced by the contact point in the plane is the shortest possible."

The problem of the rolling sphere has been studied by many mathematicians in recent years. Regarding the three model spaces in Riemannian geometry, it is natural to ask a similar question about the hyperbolic space. Is it possible to roll a hyperbolic space from an initial configuration to a terminal configuration, where no slipping nor twisting is allowed? And, if such solutions exists, what can we say about the optimal solutions? Answers to these questions can be found in the recent work [13] by Jurdjevic and J.

Zimmerman. However, in this text we will show a direct passage between these problems via the pendulum analogy.

For more on the history of elastic curves, the reader is referred to [22], [20], [33]-[35], and [28]. For now, we are going to recall some curve theory to better understand the structure of the problem regarding elastic curves.

2.2 Preliminaries

To better understand the structure of the problem on elastic curves, we need some more terminology regarding curves and how they behave under certain conditions. In deriving the closed-form solutions, we also need some properties regarding Jacobi elliptic functions. Hopefully, the following subsections will strengthen the readers intuition on elastic curves. By recalling some basic theory on curves and curvature, we are of to a gentle start and set the stage for the material that is about to be covered.

2.2.1 Basic Curve Theory

In this section we will define some of the fundamental concepts regarding curves and curvature. Moreover, we will introduce the Frenet-Serret apparatus which will be useful in the variational analysis of the elastic curves that we will discuss in section 2.4. In this text we will be working with regular curves in most of our calculations.

Definition 2.2.1. Let k be a non-negative integer and $C^k([t_0, t_1])$ the space of continuously differentiable functions of order k on $[t_0, t_1]$. A **k-regular curve** in $\mathbb{R}^3$ is a map $\gamma : [t_0, t_1] \to \mathbb{R}^3$ such that $\gamma \in C^k([t_0, t_1])$ and

$$\frac{d\gamma}{dt} \neq 0$$

for all $t \in [t_0, t_1]$.

It is well known that the length of a curve does not depend on its parametrization and, moreover, that any regular curve can be parametrized by arc length, at least locally. All curves in this text will be parametrized by arc length and is therefore of unitary speed. We define this notion by the following:

Definition 2.2.2. A curve $\gamma : [t_0, t_1] \to \mathbb{R}^3$ is said to be a **unit speed curve** if

$$\left\| \frac{d\gamma}{dt} \right\| = 1$$

for all $t \in [t_0, t_1]$, where $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^3$.

Definition 2.2.3. Let $\gamma : [t_0, t_1] \to \mathbb{R}^3$ be a regular curve in $\mathbb{R}^3$. The **length** of γ is defined by

$$\ell(\ell) := \int_{t_0}^{t_1} \left\| \frac{d\gamma}{dt} \right\| dt$$

Remark 2.2.4. Note that, for unitary speed curves, we have that the length of these curves are the same as the length of their time-domain. That is,

$$\ell = t_1 - t_0,$$

for any real numbers $t_1 > t_0$.

▲

Definition 2.2.5. Let $\gamma : [t_0, t_1] \to \mathbb{R}^3$ be a k-regular unit speed curve. The **curvature** $\kappa : [t_0, t_1] \to \mathbb{R}$ of γ is define by

$$\kappa(t) := \|\gamma''(t)\|.$$

We denote by $T(t) = \gamma'(t)$ the tangent vector of $t \mapsto \gamma(t)$. Assuming that our curve has non-vanishing curvature $\kappa \not\equiv 0$, we can construct two other vector fields which we will now define.

Definition 2.2.6. Let $\gamma : [t_0, t_1] \to \mathbb{R}^3$ be a k-regular curve with non-vanishing curvature.

(a) The **principal normal vector field** of γ is the vector field

$$N(t) := \frac{T'(t)}{\kappa(t)}.$$

(b) The **binormal vector field** of γ is the vector field

$$B(t) := T(t) \times N(t).$$

(c) The **torsion** of γ is a function $\tau : [t_0, t_1] \to \mathbb{R}$ defined by

$$\tau(t) := \langle B'(t), N(t) \rangle.$$

Note that all vectors in the definition above is of unit length. Thus, for a curve with non-vanishing curvature $\kappa \not\equiv 0$, we obtain a well-defined orthonormal frame

$$\{T(t), N(t), B(t)\} \tag{2.1}$$

along γ for almost all $t \in [t_0, t_1]$.

Definition 2.2.7. The orthonormal frame in (2.1) is known as a **Frenet-Serret frame** along γ. Together with the functions $\kappa(t)$ and $\tau(t)$ we have the **Frenet-Serret aparatus**

$$\{\kappa(t), \tau(t), T(t), N(t), B(t)\}$$

along γ.

By the Frenet-Serret aparatus we easily obtain the Frenet-Serret equations

$$\frac{d\gamma}{dt} = T, \ \frac{dT}{dt} = \kappa N, \ \frac{dN}{dt} = -\kappa T + \tau B, \text{ and } \frac{dB}{dt} = -\tau N, \tag{2.2}$$

by using the fact that

$$v = \langle T, v \rangle T + \langle N, v \rangle N + \langle B, v \rangle B$$

for any vector v along γ. Worth mentioning is the known fact that any curve $\gamma : [t_0, t_1] \to \mathbb{R}^3$ can be totally determined, up to some isomorphism, by the functions $\kappa(t)$ and $\tau(t)$ under the requirement that $\kappa \not\equiv 0$. We will make use of this fact in later calculations. For more on curve theory, see Millman and Parkers - *Elements of Differential Geometry* [25].

2.2.2 Jacobi Elliptic Functions

Before we proceed in deriving the family of elastic curves and, moreover, deriving the parametric equations of these curves, we need to justify some basic properties regarding Jacobi elliptic functions.

The well understood functions $\cos\theta$ and $\sin\theta$ are defined on the unit circle. The Jacobi elliptic functions are defined in a similar fashion on the ellipse.

Let a and b be real numbers. Then, the general formula for an ellipse is given by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,$$

with parameters a and b. Introducing polar coordinates, $x = r\sin\theta$ and $y = r\sin\theta$, the above equation yields

$$r(a, b, \theta) = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}},$$

describing the radius of the ellipse with respect to the parameters a and b and the angle θ. In the unit ellipse we let $a = 1$ and $b \geq 1$, such that the above equation becomes

$$r(b, \theta) = \frac{1}{\sqrt{\cos^2\theta + \frac{1}{b^2}\sin^2\theta}}.$$

Making the substitution $p^2 = 1 - 1/b^2$ we have that $0 \leq p \leq 1$ and the above equation becomes

$$r(p, \theta) = \frac{1}{\sqrt{1 - p^2\sin^2\theta}}. \tag{2.3}$$

By this, we can compute the angular arc length α of the ellipse via the integral

$$\alpha(p, \phi) = \int_0^\phi r(p, \theta)\ d\theta. \tag{2.4}$$

Definition 2.2.8. The integral in (2.4) is referred to as an **elliptic integral of the first kind** where $0 \leq p \leq 1$ is called the **elliptic modulus**.

The geometric interpretation of the elliptic modulus p is best understood via formula (2.3). When $p \to 0$ we have that $r \to 1$ yielding a circle of constant radius 1 - the unit circle. For $0 < p < 1$ we have π-periodic radius. And when $p \to 1$ we get discontinuities at $\theta = n\pi + \pi/2$ for $n \in \mathbb{Z}$.

Thus, for some fixed elliptic modulus $0 \leq p \leq 1$, denoting the inverse of the functional α by ϕ, we have that the *Jacobi elliptic functions* are defined by

$$\begin{cases} \mathrm{am}(\alpha, p) := \phi \\ \mathrm{sn}(\alpha, p) := \sin(\phi), \\ \mathrm{cn}(\alpha, p) := \cos(\phi), \\ \mathrm{dn}(\alpha, p) := \sqrt{1 - p^2 \sin^2 \phi}, \end{cases}$$

where, obviously, $\mathrm{am}(\alpha, p)$ denotes the *Jacobi amplitude*, sn denotes the *elliptic sine*, cn denotes the *elliptic cosine*, and dn denotes the *elliptic delta*. Note that whenever $p = 0$, sn and cn are equivalent to sin and cos, respectively, and when $p = 1$, sn and cn are equivalent to tanh and sech.

Similar to the identities of trigonometric functions, the elliptic functions satisfy the following useful relations:

$$\mathrm{sn}^2(\alpha, p) + \mathrm{cn}^2(\alpha, p) = 1 \tag{2.5}$$

and

$$p^2 \mathrm{sn}^2(\alpha, p) + \mathrm{dn}^2(\alpha, p) = 1. \tag{2.6}$$

For more details on Jacobi elliptic functions, the reader is referred to [18]. We are now ready to delve into the theory of the family of curves which is the topic of this section.

2.3 The Problem of Elastic Curves

As we now have a better understanding on the theory of curves, we can make a precise definition on the family of elastic curves. The most general form of the problem on elastic curves, as formulated by Euler, states the following:

Problem 1 (Elastic curves)**.** *Suppose that p_0 and p_1 are arbitrary distinct points in $\mathbb{R}^2$ and that v_0 and v_1 are fixed tangent vectors of unit length at p_0 and p_1, respectively. Find a unitary speed curve $\gamma : [t_0, t_1] \to \mathbb{R}^2$ of fixed length ℓ with boundary conditions $\gamma(t_i) = p_i$ and $\dot{\gamma}(t_i) = v_i$ such that*

$$E[\kappa(t)] = \frac{1}{2} \int_{t_0}^{t_1} k^2(t) \; dt$$

is minimized over all such curves, where $k(t)$ denotes the curvature of $\gamma(t)$.

Remark 2.3.1. Note that the unitary speed and fixed length condition of the problem above implies that the length ℓ of the elastic curve γ is given by:

$$\ell(\gamma) = t_1 - t_0.$$

Note also that if we drop the fixed length length condition in this problem, the resulting curve is usually referred to as a *free elastica*. We will make further remarks on this type of elastic curves in section 2.4.1.

▲

Definition 2.3.2. A curve that solves Problem 1 on elastic curves, is called an **Euler elastica** or simply an **elastica**.

When completely unconstrained, the elastica will assume the shape of a straight line, in which the curvature is everywhere zero, and thus the total bending energy is also zero. When constrained, the bending energy will tend to the minimum possible under the constraints. The problem of the elastica is related to many other optimization problems. Thus the presence of elastic curves arises in many natural phenomena. We will only give a few examples:

Example 2.3.3. Imagine a robot that has to move some load m from one initial point $p_0 \in \mathbb{R}^3$ to another terminal point $p_1 \in \mathbb{R}^3$ with initial velocity v_0 and terminal velocity v_1. The amount of work, or energy, required of the robot to preform this task, is the total amount of force applied to the load under the motion. The force at any instant is given by

$$F(t) = ma(t)$$

where $a(t)$ is the acceleration of the load under the motion. As acceleration is the rate of change in velocity, the optimal path, that minimizes the total amount of work, in which the robot moves the load from p_0 to p_1 is via an elastic curve that minimizes the curvature. This example generalizes to any setting where the goal is to minimize the acceleration along a curve.

Elastic curves also appear in the shape of the capillary [24], and in three-dimensions it resembles the shape of a helix - the main structure of DNA molecules.

■

2.4 Variational Analysis

We will now attempt to solve the problem of elastic curves (Problem 1). We proceed by deriving the intrinsic equations in the general case of three space dimensions. In this way, we will obtain useful relations that we will use in further calculations. Then we use the fact that the torsion $\tau \equiv 0$ for planar curves to obtain the planar elasticas.

Let $\gamma : [t_0, t_1] \to \mathbb{R}^3$ be a unit speed curve in $\mathbb{R}^3$. Assume that the curvature κ of γ in non-vanishing such that we have a well-defined orthonormal Frenet-Serret frame $\{T, N, B\}$ along the curve γ for all $t \in [t_0, t_1]$. Given two distinct points $p_0, p_1 \in \mathbb{R}^3$, and two corresponding tangent vectors v_0 and v_1, define the set of curves connecting p_0 with p_1 by:

$$\Omega := \{\gamma \subset \mathbb{R}^3 : \gamma(t_i) = p_i, \text{ and } \gamma'(t_i) = v_i, \text{ for } i = 1, 2\}.$$

Denote by Ω_u the set of unit speed curves in Ω. That is,

$$\Omega_u := \{\gamma \in \Omega : \|\gamma'(t)\| = 1\}.$$

The problem of elastica (Problem 1) is defined to minimize the functional $F : \Omega_u \to \mathbb{R}$ defined by

$$F(\gamma) := \int_{t_0}^{t_1} \kappa^2(t)\ dt,$$

where γ has fixed length and boundary conditions.

To apply the method of Lagrange multipliers, we define the functional $F^\lambda : \Omega \to \mathbb{R}$ by

$$F^\lambda(\gamma) := \frac{1}{2}\int_\gamma \big(\|\gamma''\|^2 + \Lambda(t,\lambda)(\|\gamma'\|^2 - 1)\big)\, dt.$$

The Lagrange multiplier principle tells us that a minimum of F over Ω_u is a stationary point for F^λ for some $\Lambda(t,\lambda)$, where $\Lambda(t,\lambda)$ is a pointwise multiplier constraining the speed. We will see in the following subsection how the function $\Lambda(t,\lambda)$ depends on the parameter λ.

2.4.1 Intrinsic equation

Assume that γ is an extremum of F^λ. Then, if X is some arbitrary vector field along γ, we have that the first variation of the functional F^λ is given by

$$\frac{d}{d\epsilon} F^\lambda(\gamma + \epsilon X)\Big|_{\epsilon=0} = 0,$$

since γ is assumed to be a stationary point of the functional F^λ. By this, we have that

$$\begin{aligned}
0 &= \frac{d}{d\epsilon}\left(\frac{1}{2}\int_{t_0}^{t_1} \big[\|(\gamma+\epsilon X)''\|^2 + \Lambda(t,\lambda)(\|(\gamma+\epsilon X)'\|^2 - 1)\big]\, dt\right)\Bigg|_{\epsilon=0} \\
&= \frac{1}{2}\int_{t_0}^{t_1} \left[\frac{\partial}{\partial\epsilon}\big((\gamma''+\epsilon X'')^2 + \Lambda(t,\lambda)(\gamma'+\epsilon X')^2 - \Lambda(t,\lambda)\big)\Big|_{\epsilon=0}\right] dt \\
&= \frac{1}{2}\int_{t_0}^{t_1} 2\gamma'' X'' + 2\Lambda(t,\lambda)\gamma' X'\ dt \\
&= \int_{t_0}^{t_1} \gamma'' X'' + \Lambda(t,\lambda)\gamma' X'\ dt.
\end{aligned}$$

Preforming integration by parts on the integrals $I_1 = \int_{t_0}^{t_1} \gamma'' X''\ dt$ and $I_2 = \int_{t_0}^{t_1} \Lambda\gamma' X'\ dt$ yields

$$I_1 = [\gamma'' X' - \gamma''' X]_{t_0}^{t_1} + \int_{t_0}^{t_1} \gamma'''' X\ dt \qquad \text{and} \qquad I_2 = [\Lambda\gamma' X]_{t_0}^{t_1} - \int_{t_0}^{t_1} (\Lambda\gamma')' X\ dt$$

such that

$$0 = I_1 + I_2 = (\gamma'' X' + (\Lambda\gamma' - \gamma''')X)\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} [\gamma'''' - (\Lambda\gamma')']\, X\ dt.$$

Denote by

$$E(\gamma) = \gamma'''' - (\Lambda\gamma')'$$

such that the above formula becomes

$$0 = (\gamma'' X' + (\Lambda\gamma' - \gamma''')X)\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} E(\gamma) X\ dt. \tag{2.7}$$

By the boundary conditions on the variational field X, $X(t_0) = X(t_1) = 0$, we have that $(\gamma'' X' + (\Lambda\gamma' - \gamma''')X)\Big|_{t_0}^{t_1} = 0$ such that

$$\int_{t_0}^{t_1} E(\gamma) X\ dt = 0.$$

Now, since the variational field X was arbitrary, the elastica must satisfy $E(\gamma) \equiv 0$ or equivalently

$$\gamma'''' - \frac{d}{dt}(\Lambda\gamma') \equiv 0$$

for some function $\Lambda(t, \lambda)$. Integrating this formula with respect to t, we obtain

$$\gamma''' - \Lambda\gamma' \equiv Y, \tag{2.8}$$

for some constant vector field Y.

We will now determine the constant vector field Y. Using the Frenet-Serret formulas (2.2), we get that

$$\gamma' = T \qquad \text{and} \qquad \gamma''' = (\kappa N)' = -\kappa^2 T + \kappa' N + \kappa\tau B.$$

Setting these relations back into equation (2.8) yields:

$$Y = -(\kappa^2 + \Lambda)T + \kappa' N + \kappa\tau B. \tag{2.9}$$

By differentiating this equation with respect to time, we obtain:

$$0 = -(3\kappa\kappa' + \Lambda')T + (\kappa'' - \kappa^3 - \Lambda\kappa - \kappa\tau^2)N + (2\kappa'\tau + \kappa\tau')B. \tag{2.10}$$

Now, since $\{T, N, B\}$ is an orthogonal system, we must have that all the scalar terms are zero. In particular, we have that $-3\kappa\kappa' - \Lambda' = 0$ which implies that

$$\Lambda(t, \lambda) = -\frac{3}{2}\kappa^2(t) + \frac{\lambda}{2}, \tag{2.11}$$

for some constant $\lambda \in \mathbb{R}$. Plugging this back into equation (2.9), we have that the constant vector field Y along γ is given by

$$Y := \frac{\kappa^2 - \lambda}{2}T + \kappa' N + \kappa\tau B. \tag{2.12}$$

By the above calculations, we have the following results:

Theorem 2.4.1. *An elastica with curvature κ and torsion τ satisfies the following relations:*

$$2\kappa'\tau + \kappa\tau' = 0 \tag{2.13}$$

and

$$\kappa'' + \frac{\kappa^3}{2} - \kappa\tau^2 - \frac{\lambda}{2}\kappa = 0, \tag{2.14}$$

for some constant $\lambda \in \mathbb{R}$.

Proof. Since $\{T, N, B\}$ is an orthogonal system, we must have that all the scalar terms in (2.10) are zero. In particular, we have that $2\kappa'\tau + \kappa\tau' = 0$ proving (2.13), and $\kappa'' - \kappa^3 - \Lambda\kappa - \kappa\tau^2 = 0$. Now, using the formula (2.11) for $\Lambda(t, \lambda)$, we obtain (2.14). □

Corollary 2.4.2. *An elastica with curvature κ and torsion τ satisfies*

$$\kappa^2\tau = c \tag{2.15}$$

for some constant $c \in \mathbb{R}$.

Proof. If we differentiate the quantity $\kappa^2\tau$, we obtain

$$\frac{d}{dt}(\kappa^2\tau) = \kappa(2\kappa'\tau + \kappa\tau'),$$

which is zero by equation (2.13). $\square$

Remark 2.4.3. Equation (2.14) is intrinsic to the curve and is usually referred to as the *elastica equation* in the literature. In the case where $\lambda = 0$, there is no restriction on the length of the curve making it tension free. Curves satisfying this relation is usually referred to as *free elastica* - as mentioned earlier in the text. A similar form of the free elastica equation was studied by Bernulli in his search for the rectangular elastica. In [7] we learn that it is actually possible to pass from the free elastica to the elastica with tension.

▲

Since the vector field Y is constant, we can write

$$4\|Y\|^2 = (\kappa^2 - \lambda)^2 + 4(\kappa')^2 + 4\kappa^2\tau^2 = a^2 \tag{2.16}$$

for some constant $a \in \mathbb{R}$. Observe that, if we differentiate equation (2.16), we obtain equation (2.14) by plugging in equation (2.13). Setting equation (2.15) into equation (2.16) and making the substitution $u = \kappa^2$, we can write equation (2.16) on the form

$$(u - \lambda)^2 + \frac{(u')^2}{u} + \frac{4c^2}{u} = a^2,$$

or equivalently

$$(u')^2 = P(u), \tag{2.17}$$

for a cubic polynomial

$$\begin{aligned} P(u) &= -u^3 + 2\lambda u^2 + (a^2 - \lambda^2)u - 4c^2 \\ &= u(a^2 - (u - \lambda)^2) - 4c^2. \end{aligned}$$

By a brief analysis of the polynomial $P(u)$, we observe that $P(0) = -4c^2 \leq 0$ with limits $P(u) \to -\infty$ as $u \to \infty$ and $P(u) \to \infty$ as $u \to -\infty$. By this, and the fact that $u = \kappa^2$ is a nonconstant solution of P such that $P(u) > 0$ for some value of u, we may assume that $P(u)$ have three real roots satisfying $-\alpha_1 \leq 0 \leq \alpha_2 \leq \alpha_3$. Thus we might write $P(u) = -(u+\alpha_1)(u-\alpha_2)(u-\alpha_3)$ such that equation (2.17) becomes

$$(u')^2 + (u+\alpha_1)(u-\alpha_2)(u-\alpha_3) = 0,$$

where the roots α_1, α_2, and α_3 of $P(u)$ are related to the coefficients of $P(u)$ by

$$\begin{aligned} 2\lambda &= \alpha_3 + \alpha_2 - \alpha_1, \\ a^2 - \lambda^2 &= \alpha_1\alpha_3 + \alpha_1\alpha_2 - \alpha_2\alpha_3, \text{ and} \\ 4c^2 &= \alpha_1\alpha_2\alpha_3. \end{aligned}$$

The solution of equation (2.17) is given by

$$u(t) = \alpha_3(1 - q^2\mathrm{sn}^2(rt, p)), \tag{2.18}$$

where $\mathrm{sn}(x, p)$ is the elliptic sine function with parameter p, and the variables p, q, and r are related to the roots α_1, α_2, and α_3 by

$$p^2 = \frac{\alpha_3 - \alpha_2}{\alpha_3 + \alpha_1}, \qquad q^2 = \frac{\alpha_3 - \alpha_2}{\alpha_3}, \qquad \text{and} \qquad r = \frac{1}{2}\sqrt{\alpha_3 + \alpha_1}.$$

Note that, by the assumption on the roots, we have $|p| < 1$ and $|q| < 1$. If we denote by $w^2 = \frac{p^2}{q^2}$ and $\kappa_0 = \sqrt{\alpha_3}$ the maximum curvature of γ, we can write equation (2.18) on the form

$$\kappa^2(t) = \kappa_0^2\left(1 - \frac{p^2}{w^2}\mathrm{sn}^2\left(\frac{\kappa_0}{2w}t, p\right)\right), \tag{2.19}$$

where the parameters p, w, and κ_0 is related to the constants λ and c by

$$2\lambda = \frac{\kappa_0^2}{w^2}(3w^2 - p^2 - 1).$$

and

$$4c^2 = \frac{\kappa_0^6}{w^4}(1 - w^2)(w^2 - p^2). \tag{2.20}$$

2.4.2 Classification

Since planar curves must satisfy $\tau \equiv 0$, we have that $c = 0$ by (2.15) because γ is regular (i.e. $\kappa(t) \not\equiv 0$). Therefore, whenever γ is a planar elastica, we must have either $w = \pm 1$ or $w = p$ by equation (2.20). Every solution corresponds to a point in the triangle $0 \leq p \leq w \leq 1$. The planar curves corresponds to two of the three edges of the triangle. The third edge of the triangle, $p = 0$ is made up of curves of constant curvature and torsion.

Now, using the relations (2.5) and (2.6) of the Jacobi elliptic functions, we have the following cases:

1 : For $w = p$ we have that $\kappa^2(t) = \kappa_0^2\left(1 - \mathrm{sn}^2\left(\frac{\kappa_0}{2p}t, p\right)\right) = \kappa_0^2\mathrm{cn}^2\left(\frac{\kappa_0}{2p}t, p\right)$ such that

$$\kappa(t) = \kappa_0 \mathrm{cn}\left(\frac{\kappa_0}{2p}t, p\right), \tag{2.21}$$

where the curvature oscillates between $-\kappa_0$ and κ_0:

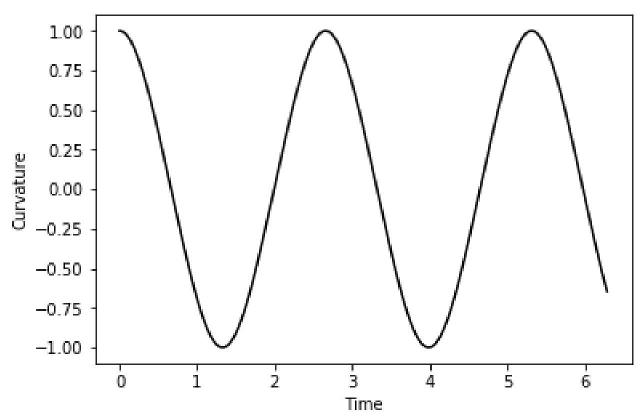


Figure 5: Curvature of inflectional elastica with $\kappa_0 = 1$ and $p = 0.2$

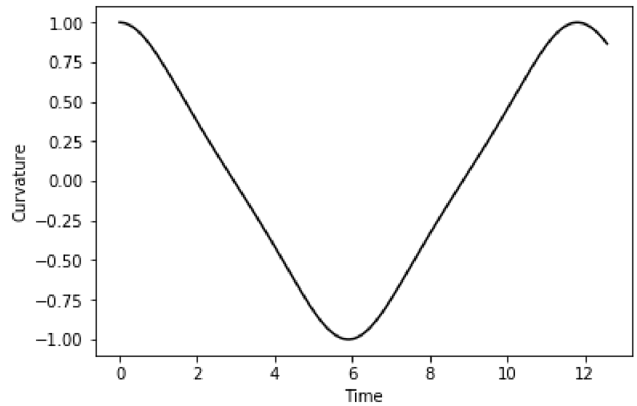


Figure 6: Curvature of inflectional elastica with $\kappa_0 = 1$ and $p = 1/\sqrt{2}$

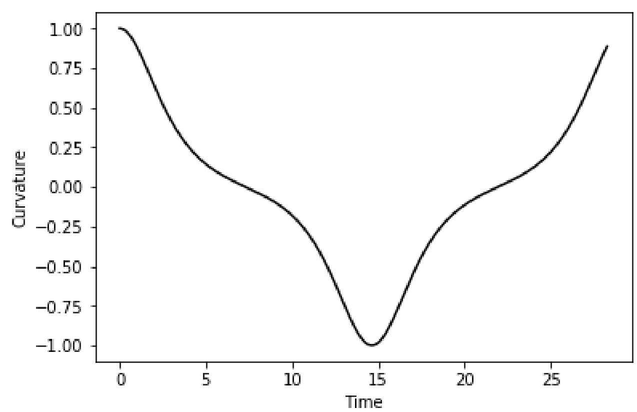


Figure 7: Curvature of inflectional elastica with $\kappa_0 = 1$ and $p = 0.99$

We refer to such solution as a *wave-like* or *inflectional* elastica.

2 : In the case where $w = p = 1$ we have that $\kappa^2(t) = \kappa_0^2 \left(1 - \mathrm{sn}^2\left(\frac{\kappa_0}{2}t, 1\right)\right) = \kappa_0^2\left(1 - \tanh^2\left(\frac{\kappa_0}{2}t\right)\right) = \kappa_0^2 \mathrm{sech}^2\left(\frac{\kappa_0}{2}t\right)$ such that

$$\kappa(t) = \kappa_0 \mathrm{sech}\left(\frac{\kappa_0}{2}t\right), \tag{2.22}$$

which has non-periodic curvature:

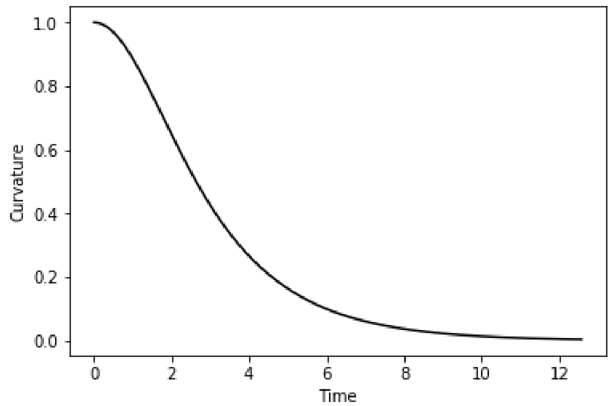


Figure 8: Curvature of critical elastica.

We refer to such solutions as *borderline* or *critical* elastica.

3 : For $w = \pm 1$ we have that $\kappa^2(t) = \kappa_0^2 \left(1 - p^2 \mathrm{sn}^2 \left(\frac{\kappa_0}{2} t, p\right)\right) = \kappa_0^2 \mathrm{dn}^2 \left(\frac{\kappa_0}{2} t, p\right)$ such that

$$\kappa(t) = \kappa_0 \mathrm{dn} \left(\frac{\kappa_0}{2} t, p\right), \tag{2.23}$$

where κ is non-vanishing:

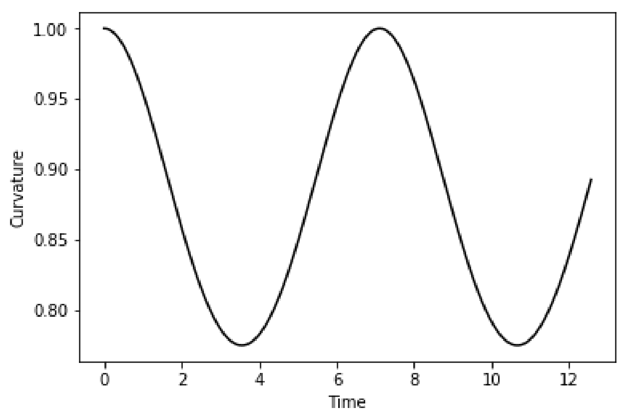


Figure 9: Curvature of non-inflectional elastica with $\kappa_0 = 1$ and $p = 0.4$

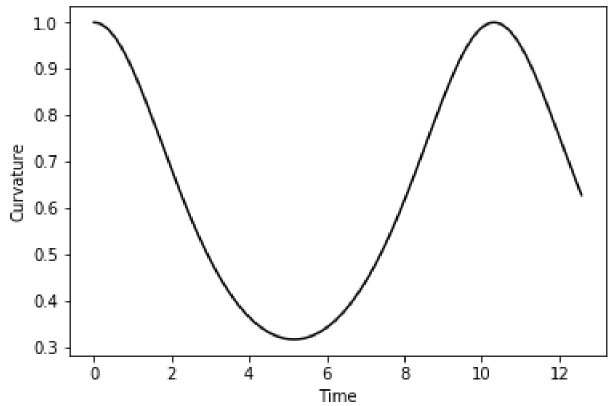


Figure 10: Curvature of non-inflectional elastica with $\kappa_0 = 1$ and $p = 0.9$

We refer to such solutions as *orbit-like* or *non-inflectional* elastica.

The naming of these solutions follows that of [16] and [22], respectively.

3 Optimal Control Problems on Lie Groups

Another approach to the solutions of the elastica can be obtained using a more modern tecnique. In this section we will introduce the theory of optimal control problems on Lie groups, which we will adapt in section 4 to derive both the pendulum equation and the intrinsic equation (2.14) of the elastica. A central result in this approach relies on Pontryagin maximum principle for invariant optimal control problems on smooth manifolds - we will discuss this further in section 3.3.

We start of by defining some basic concept from Lie theory. We will also introduce the notion of what it means to be an left-invariant optimal control problem on a Lie group. Next, we will lay the foundation for applying Pontryagin maximum principle on such problems by introducing Hamiltonian systems on T^*M for a smooth manifold M, and more specifically when M is a Lie group. Many interesting problems that arises in geometry or mechanics can be described by statespaces that have a Lie group structure - that is, a manifold that also have a group structure for which the group operation are smooth operators. Moreover, the cotangent bundle of Lie group has a particularly nice structure which we will uncover in section 3.4.1.

3.1 Control Theory on Lie Groups

Throughout this section, let (M, J) denote a smooth Riemannian manifold M with a Riemannian structure J.

Definition 3.1.1. Let $g : [t_0, t_1] \to M$ be a smooth curve in M and $V : [t_0, t_1] \to TM$ a smooth vector field along g. A smooth **dynamical system** on M is an ODE on the form

$$\dot{g}(t) = V(g(t)). \tag{3.1}$$

The solution of the dynamical system (3.1) above is usually called an *integral curve* of the vector field V. In order to control a dynamical system, we consider a collection of dynamical systems

$$\dot{g}(t) = \sum_{i=1}^{n} u_i(t) V_i(g(t)) \tag{3.2}$$

where the *control parameters* $u_i(t) : [t_0, t_1] \to U \subset \mathbb{R}$ are measurable and locally bounded.

Definition 3.1.2. Equations on the form (3.2) are called **kinematic equations**, and a collection $\Gamma \subset TM$ of vector fields $u_i V_i(g)$ is called a **control system** on M.

The problems that we are going to explore in section 4 will be described via Lie groups. Luckily for us, in the case where the underlying manifold M is a Lie group, we obtain some nice properties which is encoded in the associated Lie algebra of this group.

3.1.1 Lie groups and Lie algebras

Definition 3.1.3. A **Lie group** G is a group that is also a smooth manifold. That is, for all $g, h \in G$, the group operations:

- (multiplication) $(g, h) \mapsto gh$;
- (inversion) $g \mapsto g^{-1}$

are smooth functions.

The Lie groups in this text will be subgroups of the connected part of $GL(n)$, for some integer $n > 0$. That is, sets of $n \times n$-matrices with positive determinant, together with the usual matrix multiplication and inversion.

Definition 3.1.4. Let L be a linear space. A bilinear operator $[\cdot,\cdot]: L \times L \to L$ that

(a) is skew-symmetric:
$$[X,Y] = -[Y,X];$$

(b) satisfies the Jacobi identity:
$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0,$$

is called a **Lie bracket** (or **commutator**) on the linear space L.

Definition 3.1.5. A **Lie algebra** is a linear space that is endowed with a Lie bracket.

Since a Lie group $G \in GL_+(n)$ is also a smooth manifold, we can always define the tangent space T_gG for each $g \in G$. A linear space of special importance is the tangent space to a Lie group at its group identity element e.

Definition 3.1.6. The **Lie algebra** L **of the Lie group** G is defined to be the linear space
$$L := T_eG,$$
together with the Lie bracket $[\cdot,\cdot]: L \times L \to L$ defined by
$$[X,Y] = XY - YX.$$

All Lie algebras mentioned in this text will be the Lie algebra that is associated with a corresponding Lie group. That is, we will never work with Lie algebras in their own right.

A particularly nice property of Lie groups, is that we can understand most of its group structure only by looking at its Lie algebra, i.e., its tangent space at the identity element. This property is described through the concept of left-invariance of vector fields on Lie groups. We describe it in the following manner: Let $g \in G$ and consider the map $\ell_g : G \to G$ defined by

$$\ell_g(h) := gh.$$

Evaluating this map at the identity yields $\ell_g(e) = g$. By taking the differential of this map at the identity, we obtain a new map $d\ell_g : L \to T_gG$ defined by

$$d\ell_g(X) := gX.$$

Definition 3.1.7. Any vector field $V : G \to TG$ satisfying

$$V(g) = gX,$$

for $g \in G$ and $X \in L$, is said to be a **left-invariant** on the Lie group G.

Remark 3.1.8. We have a similar definition for right-invariant vector fields. Thus, all the following results also holds in this case simply by exchanging "left" by "right".

▲

By the property of left-invariant vector fields on Lie groups, we have that for all $g \in G$, any element V in the tangent space T_gG can be represented as a multiple of g and a linear combination of elements of its associated Lie algebra L. We summarize this in the following proposition:

Proposition 3.1.9 (Left-invariance of vector fields on Lie groups)**.** *Let G be an element of $GL_+(n)$ and L its corresponding Lie algebra. If $g \in G$, then*

$$T_gG = \{gX : X \in L\}.$$

By the result above, we have a nice and easy way to understand the structure of T_gG for any $g \in G$ through left translation. A direct consequence of this result is stated in the following corollary:

Corollary 3.1.10. *Let G and L be as above. Then the Cauchy problem*

$$\begin{cases} \dot{g} = gX, \\ g(0) = g_0 \end{cases} \tag{3.3}$$

for $g \in G$ and $X \in L$, is well-defined and has the solution

$$g(t) = g_0 \exp(tX).$$

3.1.2 Adjoint Maps on Lie Groups and Lie Algebras

Let G be a Lie group, and let

$$\phi : G \to \mathrm{Aut}(G)$$

be a map that sends the element $g \in G$ to the automorphism $\phi_g : G \to G$ defined by

$$\phi_g(h) = ghg^{-1}.$$

Denote by L the Lie algebra associated with the Lie group G.

Definition 3.1.11. We define the **adjoint map** of the Lie group G to be the derivative of the automorphism ϕ_g at the origin. That is,

$$\mathrm{Ad}_g := (d\phi_g)_e : L \to L,$$

where d denotes the differential operator.

By the definition of the Lie algebra, we have that $X \in L$ if and only if $e^X \in G$. Let $t \mapsto tX \in L$ be a curve in L for $t \in [0, t_1]$ such that $t \mapsto e^{tX} \in G$ is a curve in G. Then

$$\begin{aligned} \frac{d}{dt}(\phi_g(e^{tX}))\Big|_{t=0} &= \frac{d}{dt}(ge^{tX}g^{-1})\Big|_{t=0} \\ &= gXe^{tX}g^{-1}\Big|_{t=0} \\ &= gXg^{-1}. \end{aligned}$$

such that $\mathrm{Ad}_g : L \to L$ is defined by

$$\mathrm{Ad}_g(X) := gXg^{-1}.$$

By taking the derivative of the adjoint map $\mathrm{Ad} : G \to \mathrm{Aut}(L)$ at the identity element $e \in G$, we obtain the adjoint map $\mathrm{ad} : L \to \mathrm{Der}(L)$ of the Lie algebra L. We define this adjoint map by

$$\mathrm{ad}_X(Y) := [X, Y]$$

where $[\cdot,\cdot] : L \times L \to L$ is the Lie bracket associated with the Lie algebra L.

Definition 3.1.12. Suppose L is n-dimensional. We define the **adjoint operator** $\mathrm{ad} : L \to L$ by the following

$$\mathrm{ad}_{e_i} = \begin{pmatrix} \mathrm{ad}_{e_i}(e_1) & \cdots & \mathrm{ad}_{e_i}(e_n) \end{pmatrix}, \tag{3.4}$$

where $\mathrm{ad}_{e_i}(e_j) = [e_i, e_j]$ are column vectors and $[\cdot,\cdot]$ denotes the usual Lie bracket operator on L.

3.1.3 Semi-direct products of Lie groups

Definition 3.1.13. Suppose a Lie group G acts linearly on a linear space V. The **semi-direct product** of G and V is the Lie group H defined by

$$H = G \ltimes V = \{(g, v) : g \in G \text{ and } v \in V\}$$

with group operations

$$(g_1, v_1) \cdot (g_2, v_2) = (g_1 g_2, v_1 + g_1 v_2)$$

and

$$(g, v)^{-1} = (g^{-1}, -g^{-1} v).$$

A Lie group of special importance to us will be the special Euclidean group $SE(n)$. We define this group by the following:

Definition 3.1.14. The **special Euclidean group** $SE(n)$ is defined as the semi-direct product $SO(n) \ltimes \mathbb{R}^n$.

Remark 3.1.15. It will be convenient for us to represent the group $SE(n)$ as a subgroup of $GL(n+1)$. Consider the embedding $SO(n) \ltimes \mathbb{R}^n \hookrightarrow GL(n+1)$ defined by

$$(R, \gamma) \mapsto \begin{pmatrix} R & p^\top \\ 0 & 1 \end{pmatrix},$$

where $R \in SO(n)$, $p \in \mathbb{R}^n$, and the group operations are defined by:

$$\begin{pmatrix} R_1 & p_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_2 & p_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_1 R_2 & R_1 p_2 + p_1 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} R^{-1} & -R^{-1} p \\ 0 & 1 \end{pmatrix}.$$

▲

The special Euclidean group $SE(n)$ acts as an isometry group of $\mathbb{R}^n$. That is, any element $\phi \in SE(n)$ acts on vectors $v \in \mathbb{R}^n$ by

$$\phi(v) = R_2 v + p_2$$

where $R_2 \in SO(n)$ is a rotation and $p_2 \in \mathbb{R}^n$ is a translation. By this, we can check the group operations by the following calculations. Acting by another isometry $\psi \in SE(n)$ on the already transformed vector $\phi(v) \in \mathbb{R}^n$ we obtain

$$\psi(R_2 v + p_2) = R_1 R_2 v + R_1 p_2 + p_1,$$

where $R_1 \in SO(3)$ and $p_1 \in \mathbb{R}^n$, which justifies the group multiplication. For the inverse operation, consider the matrix equation

$$\begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix},$$

for some $A \in SO(n)$ and $b \in \mathbb{R}^n$. By the group multiplication, this equation suggests that $RA = I$ and $p + Rb = 0$ such that $A = R^{-1}$ and $b = -R^{-1} p$.

Later, in section 4.1, we are going to revisit the problem of elastia as an optimal control problem on the Lie group $SE(2)$. In doing so, we will need to know how curves in $SE(2)$ behave. By the identification $SE(2) = SO(2) \ltimes \mathbb{R}^2$ and the definition of its corresponding Lie algebra $se(2) \simeq so(2) \times \mathbb{R}^2$, this boils down to understanding the structure of the Lie algebra of $SO(2)$. We consider this in the following proposition:

Proposition 3.1.16. *The Lie algebra $so(n)$ of the Lie group $SO(n)$ is the set of $n \times n$ skew-symmetric matrices.*

Proof. Consider a smooth curve $R : [0, t_1] \to SO(n)$ for some $t_1 > 0$ satisfying the initial conditions

$$R(0) = I \qquad \text{and} \qquad \dot{R}(0) = X,$$

where X is an element in the Lie algebra $T_I SO(n) = so(n)$ of the Lie group $SO(n)$. Since $R(t) \in SO(n)$ for all $t \in [0, t_1]$, we must have, by the orthogonality condition, that

$$R^\top(t)R(t) = I$$

holds for all $t \in [0, t_1]$. Differentiating this equation with respect to t yields

$$\dot{R}^\top(t)R(t) + R^\top(t)\dot{R}(t) = 0$$

for all $t \in [0, t_1]$. Now, evaluating this equation at $t = 0$ yields

$$X^\top + X = 0,$$

which implies that the Lie algebra $so(n)$ is the set of $n \times n$ skew-symmetric matrices. $\square$

Another Lie group of importance is the semi-direct product $SO_+(2,1) \ltimes \mathbb{R}^{2,1}$ where $\mathbb{R}^{2,1}$ denotes the Euclidean space $\mathbb{R}^3$ under the metric

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.5}$$

and $SO_+(2,1)$ denotes the group of orientation preserving rotations of $\mathbb{R}^3$ that preserve the metric J. The group $SO_+(2,1)$ is a subgroup of $O_+(2,1)$ and the semi-direct product $O_+(2,1) \ltimes \mathbb{R}^{2,1}$ is know as the *Poincaré group*. We are mostly interested in the group $SO_+(2,1) \ltimes \mathbb{R}^{2,1}$, so we will denote it by

$$SP(3) = SO_+(2,1) \ltimes \mathbb{R}^{2,1} \tag{3.6}$$

and refer to it as the *special Poincaré group*. In section 4.2, when we are considering the rolling problems, we need to understand the behaviour of curves in $SP(3)$. This boils down to understanding the structure of the Lie algebra of $SO_+(2,1)$. We justify this structure in the following proposition:

Proposition 3.1.17. *The Lie algebra $so(2,1)$ of the Lie group $SO_+(2,1)$ is given by*

$$so(2,1) = \left\{ \begin{pmatrix} A & u \\ u^\top & 0 \end{pmatrix} \in gl(3) : A \in so(2),\ u \in \mathbb{R}^2 \right\}.$$

Proof. Consider a curve $R : [0, t_1] \to SO_+(2,1)$ for some $t_1 > 0$ satisfying the initial conditions

$$R(0) = I \qquad \text{and} \qquad \dot{R}(0) = X,$$

where X is an element in the Lie algebra $T_I SO_+(2,1) = so(2,1)$ of the Lie group $SO_+(2,1)$. Since $R(t) \in SO_+(2,1)$ for all $t \in [0, t_1]$, we must have, by the orthogonality condition, that

$$R^\top(t) J R(t) = I$$

holds for all $t \in [0, t_1]$, where J denotes the metric on $\mathbb{R}^3$ defined in (3.5). Differentiating the equation above yields

$$\dot{R}^\top(t) J R(t) + R^\top(t) J \dot{R}(t) = 0$$

for all $t \in [0, t_1]$. Evaluating this equation at $t = 0$ yields

$$X^\top J + JX = 0. \tag{3.7}$$

Now, suppose $X \in so(2,1)$ is given by

$$X = \begin{pmatrix} A & u \\ v^\top & w \end{pmatrix}$$

for some $A \in gl(2)$, $u, v \in \mathbb{R}^2$, and $w \in \mathbb{R}$. Plugging this back into equation (3.7), we obtain

$$\begin{pmatrix} A^\top & v \\ u^\top & w \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} A & u \\ v^\top & w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

implying $A^\top + A = 0$ such that $A \in so(2)$, $u = v$, and $w = 0$ which was what we wanted to show. $\square$

For more details on semi-Riemannian geometry, the reader is referred to [26].

3.1.4 Left-invariant control systems on Lie groups

Similar to the definition of a control system, we can now use the notion of left-invariance to define another class of control systems on Lie groups that we refer to as left-invariant control systems. We define it by the following:

Definition 3.1.18. A **left-invariant control system** Γ on a Lie group G is an arbitrary collection of left-invariant vector fields on G.

Remark 3.1.19. There is a slight difference between this definition and that of Definition 3.1.2. By left-invariance of vector fields on a Lie group, we actually have that

$$\Gamma \subset L, \tag{3.8}$$

where L denotes the associated Lie algebra of G. Usually, we write these control systems on the form:

$$\Gamma = \left\{ \sum_{i=1}^{n} u_i X_i : u_i \in U \subset \mathbb{R}^n \right\}, \tag{3.9}$$

where $X_i \in L$.

▲

Throughout this text, we will write left-invariant control systems on the form (3.8) or (3.9), i.e., as a collection of left-invariant vector fields, and we write the kinematic equation on the form:

$$\dot{g}(t) = \sum_{i=1}^{n} u_i(t) g(t) X_i(g(t)), \tag{3.10}$$

with $u(t) \in U$, $g(t) \in G$, and $X_i(t) \in L$ for $t \in [0, t_1]$.

Definition 3.1.20. A **trajectory** of a left-invariant control system Γ on a Lie group G, is a continuous curve $g : [0, T] \to G$ such that there exists a partition

$$0 = t_0 < t_1 < \cdots < t_n = T$$

and left-invariant vector fields

$$X_1, \ldots, X_n \in \Gamma$$

such that the restriction of $g(t)$ to each of the open intervals (t_{i-1}, t_i) is differentiable and

$$\dot{g}(t) = g(t)X_i$$

for $t \in (t_{i-1}, t_i)$ and $i = 1, \ldots, N$.

Definition 3.1.21. Let G denote a Lie group. For any time $t_1 \geq 0$ and any $g \in G$, we have the following definitions:

(a) The **reachable set for time** t_1 of a left-invariant control system $\Gamma \subset L$ from the point g is the set $\mathcal{A}_\Gamma(g, t_1)$ of all points that can be reached from g in exactly t_1 units of time. That is,

$$\mathcal{A}_\Gamma(g, t_1) = \{g(t_1) : g(\cdot) \text{ is a trajectory of } G \text{ and } g(0) = g\}.$$

(b) The **reachable set for time not greater that** $t_1 \geq 0$ is defined as

$$\mathcal{A}_\Gamma(g, \leq t_1) = \bigcup_{0 \leq t \leq t_1} \mathcal{A}_\Gamma(g, t).$$

(c) The **reachable** (or **attainable**) **set** of a left-invariant control system Γ from a point $g \in G$ is the set $\mathcal{A}_\Gamma(g)$ of all terminal points $g(t_1)$ for $t_1 \geq 0$ of all trajectories of Γ starting at g. That is,

$$\mathcal{A}_\Gamma(g) \bigcup_{t_1 \geq 0} \mathcal{A}_\Gamma(g, t_1).$$

Usually, we denote the reachable sets $\mathcal{A}_\Gamma(g, t_1)$ and $\mathcal{A}_\Gamma(g)$ by $\mathcal{A}(g, t_1)$ and

$\mathcal{A}(X)$, respectively, if there is no room for confusion.

Definition 3.1.22. Let $g_0, g_1 \in G$. A left-invariant control system $\Gamma \subset L$ on G is said to be **controllable** if the point g_1 can be reached from g_0 along a trajectory of Γ. That is,

$$g_1 \in \mathcal{A}(g_0)$$

for any $g_0, g_1 \in G$. Or, in other words, if

$$\mathcal{A}(g) = G$$

for any $g \in G$.

To determine whether a left-invariant control system Γ is controllable, we have a central result in control theory due to Rashevsky-Chow which states the following:

Theorem 3.1.23 (Rashevsky-Chow)**.** *Let Γ be a left-invariant control system on the form* (3.9)*. If we can generate the whole Lie algebra by using the Lie bracket on the elements $X_1, \ldots, X_n$, then Γ is controllable.*

Definition 3.1.24. An **optimal control problem** on G is given by

$$\begin{cases} \dot{g} = \sum_{i=1}^{n} u_i(t) g(t) X_i(g(t)), \qquad g \in M, \qquad u \in U \subset \mathbb{R}^n, \\ g(0) = g_0, \qquad g(t_1) = g_1, \\ E(u) = \int_0^{t_1} \varphi(g(t), u(t)) dt \to \min, \end{cases} \tag{3.11}$$

where $f(g, u)$ and $\varphi(g, u)$ are smooth functions, and the admissible controls $u(t)$ are measurable and locally bounded.

Remark 3.1.25. In order to compare the controls $t \mapsto u(t) \in U$, we introduce the cost functional

$$E(u) = \int_0^{t_1} \varphi(g(t), u(t)) dt.$$

The problem is to minimize the functional E over all control functions $t \mapsto u(t)$ for which the corresponding solution to the Cauchy problem above satisfies the boundary condition $g(t_1) = g_1$.

▲

3.2 Hamiltonian Systems on T^*M

In the Hamiltonian formulation of classical mechanics, we view the set of all possible configurations of a dynamical system as a smooth manifold M, where its associated cotangent bundle T^*M describes the phase space of the system.

Let $p \in M$ and $x(p) = (x^1, \ldots, x^n)$ be local coordinates on a smooth manifold M. Then the collection $\{\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p\}$ form a basis for the tangent space T_pM, and $\{dx^1|_p, \ldots, dx^n|_p\}$ forms a basis for its dual - the cotangent space T_p^*M. By this, we have that any covector $\lambda_p \in T_p^*M$ can be uniquely written as a linear combination on the form

$$\lambda_p = \sum_{i=1}^{n} a_i(p)(dx^i)_p,$$

for some functions $a_i : M \to \mathbb{R}$. Thus any coordinates $(x^1, \ldots, x^n)$ on M gives rise to the *canonical coordinates* $(a, x) = (a_1, \ldots, a_n, x^1, \ldots, x^n)$ on the cotangent bundle T^*M.

In optimal control problems we would like to study trajectories on the cotangent bundle T^*M. To study the behaviour of these trajectories, we need to look at the vector fields in the tangent bundle of T^*M. That is, to understand the structure of $T(T^*M)$. To do so, we start of by defining a 1-form on T^*M.

3.2.1 The Liouville form and the symplectic form

Consider the canonical projection $\pi : T^*M \to M$ that maps (λ, p) to its base point $p \in M$ for all $\lambda_p \in T_p^*M$. The differential, or pushforward, of this map will be a map $\pi_* : T(T^*M) \to TM$. We define this differential pointwise by

$$\pi_*|_{(\lambda,p)} : T_{(\lambda,p)}(T^*M) \to T_pM$$

that pushforward vectors in $T_{(\lambda,p)}(T^*M)$ to vectors in T_pM. To simplify notation, we will from now on write the element $(\lambda, p) \in T^*M$ simply as λ. By the differential map π_*, we can now define the Liouville form - also known as the *tautological 1-form* in the literature.

Definition 3.2.1. Given $X_\lambda \in T_\lambda(T^*M)$, we define the **tautological 1-form** $\tau \in \Lambda^1(T^*M)$ pointwise at $\lambda \in T^*M$ as the map $\tau_\lambda : T_\lambda(T^*M) \to \mathbb{R}$ given by

$$\langle \tau_\lambda, X_\lambda \rangle := \langle \lambda, \pi_* X_\lambda \rangle.$$

Remark 3.2.2. The definition above tells us to project the vector $X_\lambda \in T_\lambda(T^*M)$ to the vector $\pi_* X_\lambda \in T_pM$, and the act by the covector $\lambda \in T_p^*M$. That is,

$$\tau_\lambda := \lambda \circ \pi_*.$$

We refer to the Liouville form τ_λ as the tautological form because its representation in canonical coordinates is the same as that for the base form λ. Indeed, in canonical coordinate (a, x) on T^*M, we have that

$$\lambda = \sum_{i=1}^{n} a_i dx^i \qquad \text{and} \qquad X_\lambda = \sum_{i=1}^{n} \left(\alpha_i \frac{\partial}{\partial a_i} + \beta_i \frac{\partial}{\partial x^i} \right).$$

The canonical projection written in canonical coordinates $\pi : (a, x) \mapsto x$ is a linear mapping for which its differential acts as

$$\pi_* \left(\frac{\partial}{\partial a_i} \right) = 0 \qquad \text{and} \qquad \pi_* \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i}$$

for $i = 1, \dots, n$ such that

$$\pi_* X_\lambda = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x^i}.$$

By this, we have that

$$\langle \tau_\lambda, X_\lambda \rangle = \langle \lambda, \pi_* X_\lambda \rangle = \sum_{i=1}^{n} a_i \beta_i.$$

But $\beta_i = \langle dx^i, X_\lambda \rangle$, which implies that the form τ is expressed as

$$\tau_\lambda = \sum_{i=1}^{n} a_i dx^i$$

in canonical coordinates (a, x).

▲

Another important operator in symplectic geometry is that of a symplectic form. We define it by the following:

Definition 3.2.3. We define the **symplectic form** $\sigma \in \Lambda^2(T^*M)$ to be the differential of the tautological 1-form $\tau \in \Lambda^1(T^*M)$. That is,

$$\sigma := d\tau,$$

where τ is the tautological 1-form defined above.

Since $\tau = \sum_{i=1}^{n} a_i dx^i$, we have that

$$\begin{aligned} \sigma &= \sum_{i=1}^{n} d(a_i dx^i) = \sum_{i=1}^{n} d(a_i \wedge dx^i) = \sum_{i=1}^{n} \big(da_i \wedge dx^i + a_i \wedge d(dx^i) \big) \\ &= \sum_{i=1}^{n} da_i \wedge dx^i, \end{aligned}$$

in canonical coordinates. This expression shows that the bilinear skew-symmetric symplectic form is nondegenerate. That is, the map

$$\sigma_\lambda : T_\lambda(T^*M) \times T_\lambda(T^*M) \to \mathbb{R}$$

has no kernel such that $\sigma(X_\lambda, \cdot) = 0$ if and only if $X_\lambda = 0$. Moreover, we have that σ is closed because $d \circ d = 0$.

Definition 3.2.4. A smooth manifold for which a symplectic form is defined is said to be a **symplectic manifold**.

Since the tautological 1-form τ is an element of $\Lambda(T^*M)$, we have that the symplectic 2-form σ belongs to $\Lambda^2(T^*M)$. This makes T^*M a symplectic manifold by the definition above.

3.2.2 Hamiltonian Vector fields

Due to the symplectic structure $\sigma \in \Lambda^2(T^*M)$, we are now ready to uncover the Hamiltonian formalism on T^*M. We start by introducing the Hamiltonian:

Definition 3.2.5. Any smooth function on a symplectic manifold is called a **Hamiltonian**.

Since (T^*M, σ) is a symplectic manifold, we have that any function $h \in C^\infty(T^*M)$ is a Hamiltonian. We will now see that the Hamiltonian function h induces a special vector field on the symplectic manifold T^*M. This vector field is known as the *Hamiltonian vector field* associated with the Hamiltonian function. We define it in the following manner: First we define a 1-form on T^*M by taking the differential of the Hamiltonian function. That is, $dh \in \Lambda^1(T^*M)$ for some Hamiltonian function $h \in C^\infty(T^*M)$. Now, using the symplectic form $\sigma \in \Lambda^2(T^*M)$, we might define another 1-form on T^*M by a contraction of σ with a vector field $V \in \mathfrak{X}(T^*M)$. That is, $\sigma(V, \cdot) = i_V\sigma \in \Lambda^1(T^*M)$. By this, the Hamiltonian vector field is defined by the following:

Definition 3.2.6. The **Hamiltonian vector field** corresponding to a Hamiltonian function $h \in C^\infty(T^*M)$ is defined to be the vector field $H \in \mathfrak{X}(T^*M)$ which satisfy

$$dh = -i_H\sigma.$$

Remark 3.2.7. To clearify the above definition, we have that $h : T^*M \to \mathbb{R}$, and the corresponding vector field is a function $H : T^*M \to T(T^*M)$. Now $dh : T(T^*M) \to T(\mathbb{R}) \simeq \mathbb{R}$. That is, $dh \in T^*(T^*M)$ or $dh \in \Lambda(T^*M)$.

Since the symplectic form σ is nondegenerate, we have that the mapping

$$T_\lambda(T^*M) \xrightarrow{\sigma_\lambda} T_\lambda(T^*M)$$

is a linear isomorphism. We can therefore say that the Hamiltonian vector field H exists and is uniquely determined by the Hamiltonian function h.

▲

Now, in canonical coordinates (a, x) on T^*M, the differential of the Hamiltonian function $h = h(a, x) \in C^\infty(T^*M)$ reads

$$dh = \sum_{i=1}^{n} \left(\frac{\partial h}{\partial a_i} da_i + \frac{\partial h}{\partial x^i} dx^i \right).$$

Thus, by the definition of the symplectic form, we have the Hamiltonian vector field H given by

$$H = \sum_{i=1}^{n} \left(\frac{\partial h}{\partial a_i} \frac{\partial}{\partial x^i} - \frac{\partial h}{\partial x^i} \frac{\partial}{\partial a_i} \right)$$

Now, we have the Hamiltonian system of ODEs $\frac{d\lambda}{dt} = H(\lambda)$ on the form

$$\frac{d\lambda}{dt} = \left(\frac{da}{dt}, \frac{dx}{dt} \right) = \left(-\frac{\partial h}{\partial x}, \frac{\partial h}{\partial a} \right) = H(\lambda),$$

such that

$$\frac{da}{dt} = -\frac{\partial h}{\partial x} \qquad \text{and} \qquad \frac{dx}{dt} = \frac{\partial h}{\partial a} \tag{3.12}$$

in canonical coordinates (a, x) on T^*M where $x \in M$ and $a \in \Lambda^1(M)$.

Definition 3.2.8. A system on the form (3.12) is refered to as a **Hamiltonian System**. The first equation is referred to as the **vertical part** and the second equation as the **horizontal part**.

3.3 Pontryagin Maximum Principle on Smooth Manifolds

Consider the optimal control problem that we defined in (3.11):

$$\begin{cases} \dot{g} = \sum_{i=1}^{n} u_i(t)g(t)X_i(g(t)), \qquad g \in M, \qquad u \in U \subset \mathbb{R}^n, \\ g(0) = g_0, \qquad g(t_1) = g_1, \\ E(u) = \int_0^{t_1} \varphi(g(t), u(t))dt \to \min, \end{cases}$$

for some smooth function $\varphi(g, u)$.

Let $\lambda \in T^*M$ be a covector, $\nu \in \mathbb{R}$ a non-positive parameter, and $u \in U$ a control parameter. We define a *family of Hamiltonians* associated with an optimal control problem on the form (3.11) by

$$h_u^\nu(\lambda) = \left\langle \lambda, \sum_{i=1}^{n} u_i(t)g(t)X_i(g(t)) \right\rangle + \nu\varphi(g, u). \tag{3.13}$$

The Pontryaginal maximum principle gives us the necessary condition of optimality for optimal control problems on smooth manifolds. We state it by the following Theorem:

Theorem 3.3.1 (Pontryagin maximum principle on smooth manifolds)**.** *If* $\tilde{u} : [0, t_1] \to \mathbb{R}^n$ *is an optimal control of problem* (3.11)*, then there exist a non-trivial Lipschitzian curve* $\lambda : [0, t_1] \to T^*_{\tilde{g}(t)}M$ *and non-positive constant* $\nu \in \mathbb{R}$ *such that*

$$\dot{\lambda} = H^\nu_{\tilde{u}(t)}(\lambda) \qquad \textit{and} \qquad h^\nu_{\tilde{u}(t)}(\lambda) = \max_{u \in U} h^\nu_u(\lambda) \tag{3.14}$$

holds for all $t \in [0, t_1]$.

Proof. See [1] section 12. □

Remark 3.3.2. This theorem tells us that solving the problem (3.11), that is, to minimize the functional $E(u)$, is equivalent to finding the maximum over the Hamiltonian functions h_u^ν associated with the optimal control problem (3.11). If we were to consider a problem on the form (3.11) where we wanted to maximise the functional instead of minimizing it, we only need to reverse the property of ν. That is, to require ν to be a non-negative real number. That being said, we will only consider minimization problems in this text.

▲

There are two distinct possibilities for the constant $\nu \leq 0$ in the Pontryaginal maximum principle (PMP) above. We define them by the following:

Definition 3.3.3. Let $\lambda_t \in T^*_{\tilde{g}(t)}M$ denote the Lipschitzian curve obtained in Theorem 3.3.1 above.

(a) (Normal case) If $\nu \neq 0$, then λ_t is called a **normal extremal** of the Hamiltonian.

(b) (Abnormal case) If $\nu = 0$, then λ_t is called a **abnormal extremal** of the Hamiltonian.

Remark 3.3.4. Since the pair (ν, λ_t) can be scaled by a positive real number, we can always normalize $\nu < 0$, such that it suffices to consider $\nu = -1$ in the normal case. Thus we might always assume that either $\nu = 0$ or $\nu = -1$.

▲

3.4 Hamiltonian Systems on The Cotangent Bundle of a Lie Group G

The problems that we are going to explore in section 4 are described as left-invariant optimal control problems on Lie groups. To apply the PMP, we will like to know how we can write the Hamiltonian system on the cotangent bundle of a Lie group. As mentioned before, the cotangent bundle of a Lie group has a particularly nice structure obtained through what we call a trivialization. We define this concept in the following subsection.

3.4.1 Trivialization of T^*G

Consider first an n-dimensional smooth manifold M and a n-dimensional linear space V.

Definition 3.4.1. A **trivialization** of the cotangent bundle T^*M is a diffeomorphism $\phi : V \times M \to T^*M$ such that

(a) $\phi(v,p) \in T_p^*M$ for $v \in V$ and $p \in M$,

(b) $\phi(\cdot,p) : V \to T_p^*M$ is a linear isomorphism for all $p \in M$.

By the definition above, the linear space V is identified with the vertical fiber T_p^*M of the cotangent bundle T^*M for all $p \in M$.

Note that, in general, the cotangent bundle of a smooth manifold is not trivial. That is, it cannot be identified as a product of a linear space and its base space. However, the cotangent bundle of a Lie group G has a natural trivialisation on the form $L^* \times G$ where L denotes its corresponding Lie algebra and L^* its dual. We will apply this trivialization in order to write the Hamiltonian system of the PMP for optimal control problems on Lie groups.

Thus for any manifold M that admits a natural trivialization of the cotangent bundle $V \times M \simeq T^*M$, we have the following identifications for all $(v,p) \in T^*M$:

$$\begin{aligned} T_{(v,p)}(V \times M) &\simeq T_vV \oplus T_pM \simeq V \times T_pM, \\ T^*_{(v,p)}(V \times M) &\simeq T^*_vV \oplus T^*_pM \simeq V^* \times T^*_pM. \end{aligned}$$

By this, we have that any tangent vector $X \in T_{(v,p)}(V \times M)$ and covector $\omega \in T^*_{(v,p)}(V \times M)$ can be decomposed into their *vertical* and *horizontal* parts. That is, for $X \in T_{(e,p)}(E \times M)$ and $\omega \in T_{(e,p)}(E \times M)$ we have that

$$X = X_v + X_h \qquad \text{and} \qquad \omega = \omega_v + \omega_h,$$

where $X_v \in V$, $X_h \in T_pM$, $\omega_v \in V^*$, and $\omega_h \in T_p^*M$.

Now, let G denote a Lie group and L its associated Lie algebra. Then the cotangent bundle T^*G has the natural trivialization

$$\phi : L^* \times G \to T^*G,$$

where L^* denotes the dual space of the Lie algebra L. This trivialization is defined by the following: let $\omega_e \in L^*$ and $g \in G$, then $(\omega_e, g) \xrightarrow{\phi} \omega_g$ where

$\omega_g \in \Lambda^1(G)$ is the left-invariant 1-form on G obtained by left translations of $\omega_e \in L^*$ by $g \in G$ such that

$$\langle \omega_g, gX \rangle = \langle \omega_e, X \rangle$$

holds for all $g \in G$ and $X \in L$.

To make use of this trivialization, we will now compute the pullback of the tautological 1-form $\tau \in \Lambda^1(T^*G)$, the symplectic 2-form $\sigma \in \Lambda^2(T^*G)$, and the Hamiltonian vector field $H \in \mathfrak{X}(T^*G)$ to the trivialized cotangent bundle $L^* \times G$. In doing so, we denote by $\phi_* : T(L^* \times G) \to T(T^*G)$ the differential of the trivialization $\phi : L^* \times G \to T^*G$, and by $\phi^* : T^*(T^*G) \to T^*(L^* \times G)$ the codifferential that pulls back k-forms on T^*G to k-forms on $L^* \times G$.

3.4.2 Tautological form on $L^* \times G$

For the tautological 1-form $\phi^*\tau \in \Lambda^1(L^* \times G)$, take any point $(\omega_e, g) \in L^* \times G$ and a tangent vector $(\xi, gX) \in L^* \oplus T_gG \simeq T_{(\omega_e,g)}(L^* \times G)$ such that

$$\left\langle (\phi^*\tau)_{(\omega_e,g)}, (\xi, gX) \right\rangle = \left\langle \tau_{\omega_g}, \phi_{*,(\omega_e,g)}(\xi, gX) \right\rangle .$$

Note that $\pi_*(\phi_{*,(\omega_e,g)}(\xi, gX)) = gX$, thus we might write

$$\left\langle \tau_{\omega_g}, \phi_{*,(\omega_e,g)}(\xi, gX) \right\rangle = \left\langle \omega_g, \pi_*(\phi_{*,(\omega_e,g)}(\xi, gX)) \right\rangle = \langle \omega_g, gX \rangle = \langle \omega_e, X \rangle,$$

such that

$$\left\langle (\phi^*\tau)_{(\omega_e,g)}, (\xi, gX) \right\rangle := \langle \omega_e, X \rangle. \tag{3.15}$$

3.4.3 Symplectic form on $L^* \times G$

For the symplectic 2-form $\phi^*\sigma \in \Lambda^2(L^* \times G)$, take any point $(\omega_e, g) \in L^* \times G$ and tangent vectors $(\xi, gX), (\eta, gY) \in L^* \oplus T_gG$. We will like to compute $\left\langle (\phi^*\sigma)_{(\omega_e,g)}, ((\xi, gX), (\eta, gY)) \right\rangle$. Since the pullback commutes with the differential, we have that $\phi^*\sigma = \phi^* d\tau = d\phi^*\tau$ such that

$$\left\langle (\phi^*\sigma)_{(\omega_e,g)}, ((\xi, gX), (\eta, gY)) \right\rangle = \left\langle (d\phi^*\tau)_{(\omega_e,g)}, ((\xi, gX), (\eta, gY)) \right\rangle .$$

Remark 3.4.2. Note that, if α is a 1-form on a vector space containing v_1 and v_2. Then the exterior derivative $d\alpha$ is a 2-form on the same vector space satisfying the the relation

$$\langle (d\alpha, (v_1, v_2)\rangle = v_1\langle \alpha, v_2 \rangle - v_2\langle \alpha.v_1 \rangle - \langle \alpha, [v_1, v_2]\rangle.$$

$\blacktriangle$

Using this observation on the expression above, we get that

$$\big\langle (d\phi^*\tau)_{(\omega_e,g)}, ((\xi, gX), (\eta, gY))\big\rangle = \left\{ \begin{array}{l} (\xi, gX)\langle(\phi^*\tau)_{(\omega_e,g)}, (\eta, gY)\rangle \\ -(\eta, gY)\langle(\phi^*\tau)_{(\omega_e,g)}, (\xi, gX)\rangle \\ -\langle(\phi^*\tau)_{(\omega_e,g)}, [(\xi, gX), (\eta, gY)]\rangle \end{array} \right. .$$

Now, by the pullback of the tautological 1-form in equation (3.15), we have that

$$\begin{aligned} \left. \begin{array}{r} (\xi, gX)\langle(\phi^*\tau)_{(\omega_e,g)}, (\eta, gY)\rangle \\ -(\eta, gY)\langle(\phi^*\tau)_{(\omega_e,g)}, (\xi, gX)\rangle \\ -\langle(\phi^*\tau)_{(\omega_e,g)}, [(\xi, gX), (\eta, gY)] \end{array} \right\} &= (\xi, X)\langle\omega_e, Y\rangle - (\eta, Y)\langle\omega_e, X\rangle - \langle\omega_e, [X, Y]\rangle \\ &= \langle\xi, Y\rangle - \langle\eta, X\rangle - \langle\omega_e, [X, Y]\rangle, \end{aligned}$$

such that

$$\big\langle (\phi^*\sigma)_{(\omega_e,g)}, ((\xi, gX), (\eta, gY))\big\rangle := \langle\xi, Y\rangle - \langle\eta, X\rangle - \langle\omega_e, [X, Y]\rangle \qquad (3.16)$$

3.4.4 Hamiltonian system on $L^* \times G$

To define the Hamiltonian $h \in C^\infty(L^* \times G)$, fix $g \in G$ and consider $h(\cdot, g) : L^* \to \mathbb{R}$ with only $\omega_e \in L^*$ as a free variable. Decompose the associated vector field $H \in \mathfrak{X}(L^* \times G)$ into its vertical and horizontal parts such that

$$H(\omega_e, g) = (\xi, gX) \in T_{(\omega_e,g)}(L^* \times G) \simeq L^* \oplus T_gG,$$

with $\omega_e \in L^*$ and $g \in G$. We define this vector field using the pullback of σ such that

$$dh = -(\phi^*\sigma)(H, \cdot).$$

Since the Hamiltonian h does not depend on $g \in G$, we have that

$$dh = \frac{\partial h}{\partial \omega_e},$$

with $\frac{\partial h}{\partial \omega_e} \in (L^*)^* \simeq L$. Now, for an arbitrary tangent vector $(\eta, gY) \in L^* \oplus T_gG$, we have $\left\langle \frac{\partial h}{\partial \omega_e}, (\eta, gY) \right\rangle = \langle dh, (\eta, gY) \rangle = -\left\langle (\phi^*\sigma)_{(\omega_e, g)}, ((\xi, gX), (\eta, gY)) \right\rangle$ such that

$$\left\langle \frac{\partial h}{\partial \omega_e}, (\eta, gY) \right\rangle = -\langle \xi, Y \rangle + \langle \eta, X \rangle + \langle \omega_e, [X, Y] \rangle \tag{3.17}$$

We compute the vertical part of H by setting $Y = 0$ in (3.17) such that

$$\left\langle \frac{\partial h}{\partial \omega_e}, (\eta, 0) \right\rangle = \left\langle \eta, \frac{\partial h}{\partial \omega_e} \right\rangle = \langle \eta, X \rangle$$

for all $\eta \in L^*$. Thus

$$X = \frac{\partial h}{\partial \omega_e}. \tag{3.18}$$

We compute the horizontal part of H by setting $\eta = 0$ in (3.17) such that $-\langle \xi, Y \rangle + \langle \omega_e, [X, Y] \rangle = \langle dh, (0, gY) \rangle = 0$ and

$$\langle \xi, Y \rangle = \langle \omega_e, [X, Y] \rangle.$$

Denote by $\mathrm{ad}_X : L \to L$ the adjoint action of X on L defined by

$$\mathrm{ad}_X(Y) = [X, Y].$$

Now $\langle \omega_e, [X, Y] \rangle = \langle \omega_e, \mathrm{ad}_X(Y) \rangle = \langle (\mathrm{ad}_X)^* \omega_e, Y \rangle$ where the last equality follows by the definition of the adjoint map $(\mathrm{ad}_X)^* : L^* \to L^*$ to the linear adjoint operator $\mathrm{ad}_X : L \to L$. Thus $\langle \xi, Y \rangle = \langle (\mathrm{ad}_X)^* \omega_e, Y \rangle$ holds for all $Y \in L$ such that

$$\xi = (\mathrm{ad}_X)^* \omega_e = \left(\mathrm{ad}_{\frac{\partial h}{\partial \omega_e}} \right)^* \omega_e. \tag{3.19}$$

By this, we have that the Hamiltonian system on the trivialized cotangent bundle $T^*G \simeq L^* \times G$ for a left-invariant Hamiltonian $h(\cdot, X) : L^* \to \mathbb{R}$ has the form

$$\frac{d\omega_e}{dt} = \left(\mathrm{ad}_{\frac{\partial h}{\partial \omega_e}} \right)^* \omega_e \qquad \text{and} \qquad \frac{dg}{dt} = g \frac{\partial h}{\partial \omega_e}, \tag{3.20}$$

where $\omega_e \in L^*$ and $g \in G$.

4 Elastic Curves and Rolling Manifolds

Now that we have uncovered some of the central ideas in optimal control theory, we are ready to apply our knowledge on the main problems of this text, namely, the problem of elastica and the problem of rolling manifolds. We will also look at how these problems are connected. We will view all of these problems as left-invariant optimal control problems on Lie groups.

To adapt the optimal control theory, we will start by lifting the problems onto the trivialized cotangent bundle - the phase space in classical mechanics - of its corresponding configurations space. Next, we will apply Pontryagin maximum principle to obtain the extremal curves in the trivialized bundle. Finally, we project the extremal curves onto the state space to obtain the minimal curves as solutions to our problems.

We will start by revisiting the problem of elastica. The problem of the rolling sphere is widely treated in [12], so our main goal of this section will be to adapt similar methods in describing both the rolling of a sphere and the rolling of a hyperboloid.

4.1 Elastica as an Optimal Control Problem

We will now return to the problem of elastic curves (Problem 1) which is one of the main characters of this text. In this section we will see how the optimal control approach towards a solution of elastica resembles the equation of a mathematical pendulum. This relationship was first hinted by Kirchhoff [14] and is therefore usually referred to as *Kirchhoffs kinetic analogy* in the literature.

Let $\gamma : [t_0, t_1] \to \mathbb{R}^2$ denote an elastic curve that solves Problem 1. Let $\theta : [t_0, t_1] \to S^1$ denote the angle between the velocity vector $\dot{\gamma}(t) = (\dot{x}(t), \dot{y}(t))$ and the positive direction of the x-axis as illustrated in the following figure:

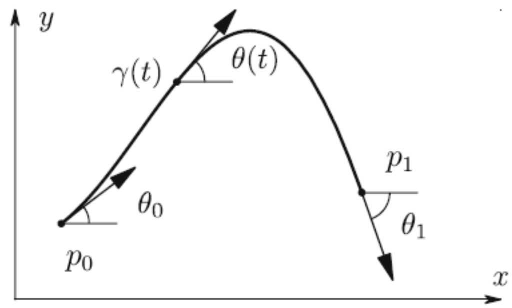


Figure 11: Problem or elastica.

As usual, we assume that γ is a unitary speed regular curve. Then the curvature equals, up to sign, to the angular velocity $\dot{\theta}$. That is,

$$\kappa = \pm\dot{\theta} \qquad \text{or} \qquad \kappa^2 = \dot{\theta}^2.$$

4.1.1 Kinematic equations

To describe the problem of elastica as an optimal control problem on a Lie group, we need to reformulate the problem as a dynamical system. To totally describe the dynamics of the elastica, we need to know $\gamma(t) = (x(t), y(t))$ and $\dot{\gamma}(t) = (\dot{x}(t), \dot{y}(t))$ for each $t \in [t_0, t_1]$. $\gamma(t)$ is easily described as a point in $\mathbb{R}^2$ for each $t \in [t_0, t_1]$. A good candidate for describing $\dot{\gamma}(t)$, will be an element of the group $SO(2)$ - the group of orientation preserving rotations of $\mathbb{R}^2$. We can regard the element $R(t) \in SO(2)$ as the rotation by the angle θ that the tangent vector $\dot{\gamma}(t)$ makes with the positive x-axis for each $t \in [t_0, t_1]$. Thus we can totally describe the dynamics of the elastic curve by a curve $t \mapsto g(t) = (R(t), \gamma(t)) \in SO(2) \times \mathbb{R}^2$.

As topological spaces, we have that $SO(2) \times \mathbb{R}^2$ is homeomorphic to the special Euclidean group $SE(2)$ - the group of orientation preserving isometries of $\mathbb{R}^2$. Following Remark 3.1.15, we will embed $SE(2)$ into $GL(3)$ and denote this subgroup by G. In this way, the curve $t \mapsto g(t) = (R(t), \gamma(t)) \in SO(2) \times \mathbb{R}^2$ is represented as

$$t \mapsto \begin{pmatrix} R(t) & \gamma(t)^\top \\ 0 & 1 \end{pmatrix} \in G \subset GL(3).$$

The problem can now be considered as a left-invariant optimal control problem on the group of planar Euclidean isometries, $SE(2)$.

Now, taking the derivative of g with respect to time yields:

$$\begin{aligned}
\dot{g} &= \frac{d}{dt}\begin{pmatrix}\cos\theta & -\sin\theta & x\\ \sin\theta & \cos\theta & y\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}-\sin\theta\,\dot{\theta} & -\cos\theta\,\dot{\theta} & \dot{x}\\ \cos\theta\,\dot{\theta} & -\sin\theta\,\dot{\theta} & \dot{y}\\ 0 & 0 & 0\end{pmatrix}\\
&= \begin{pmatrix}-\sin\theta\,u & -\cos\theta\,u & \cos\theta\\ \cos\theta\,u & -\sin\theta\,u & \sin\theta\\ 0 & 0 & 0\end{pmatrix}\\
&= \underbrace{\begin{pmatrix}\cos\theta & -\sin\theta & x\\ \sin\theta & \cos\theta & y\\ 0 & 0 & 1\end{pmatrix}}_{g}\underbrace{\begin{pmatrix}0 & -u & 1\\ u & 0 & 0\\ 0 & 0 & 0\end{pmatrix}}_{X_u},
\end{aligned}$$

where X_u is an element in the Lie algebra L associated with the Lie group G and $u = \dot{\theta}$. The equation $\dot{g} = gX_u$ is consistent with the left-invariant structure of vector fields on Lie groups.

By the identification $G \simeq SO(2) \times \mathbb{R}^2$, we have that the Lie algebra L of G can be identified with $so(2) \times \mathbb{R}^2$ because $L = T_eG \simeq T_e(SO(2) \times \mathbb{R}^2) \simeq so(2) \times \mathbb{R}^2$. By this, we have that the Lie algebra L of G is determined by $L = \text{span}\{E_{21} - E_{12}, E_{13}, E_{23}\}$, where

$$\begin{aligned}
E_{21} - E_{12} &= \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\\
E_{13} &= \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \text{ and}\\
E_{23} &= \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}.
\end{aligned}$$

Let us denote by $e_1 = E_{21} - E_{12}$, $e_2 = E_{13}$, and $e_3 = E_{23}$ the basis elements of $L = se(2)$. In this notation, we might write the left-invariant vector field $X_u \in L$ as $X_u = (ue_1 + e_2)$ and we have the kinematic equation

$$\dot{g}(t) = g(t)(u(t)e_1 + e_2), \tag{4.1}$$

describing the dynamics of the elastic curve γ. By this, we might restate the problem of elastica as a left-invariant optimal control problem on the Lie group G in the following manner:

Problem 2 (Elastica)**.** *Consider a curve $g : [t_0, t_1] \to G$ satisfying the kinematic equation* (4.1) *and the boundary conditions $g(t_i) = g_i$ for $i \in \{1, 2\}$. Find the optimal control $u : [t_0, t_1] \to \mathbb{R}$ for which*

$$\frac{1}{2}\int_{t_0}^{t_1} u^2(t)\ dt \to \min .$$

4.1.2 Controllability

The element $ue_1 + e_2$ generates the subspace $\Gamma = \operatorname{span}\{ue_1, e_2\} \subsetneqq L$ where $u \in \mathbb{R}$. Since $[ue_1, e_2] = u(e_1e_2 - e_2e_1) = ue_3$, controllability follows by Rashevsky-Chow Theorem 3.1.23.:

4.1.3 Hamiltonian system

We now turn our attention towards the Hamiltonians for the PMP on T^*G. By the bracket operations on L, we have $[e_1, e_3] = -e_2$ and $[e_2, e_3] = 0$ such that

$$\left.\begin{aligned} \operatorname{ad}(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \operatorname{ad}(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \text{and} \\ \operatorname{ad}(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}\right\} \tag{4.2}$$

where $\operatorname{ad} : L \to L$ is the adjoint operator defined in (3.4).

Let L^* denote the dual space of the Lie algebra L and denote by ω^1, ω^2, and ω^3 its basis elements such that

$$L^* = \operatorname{span}\{\omega^1, \omega^2, \omega^3\} \qquad \text{and} \qquad \langle \omega^i, e_j \rangle = \delta^i_j.$$

We write elements of the Lie algebra $X \in L$ as column vectors

$$X = \sum_{i=1}^{3} X^i e_i = \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix},$$

and elements of its dual $a \in L^*$ as row vectors

$$a = \sum_{i=1}^{3} a_i \omega^i = (a_1 \ a_2 \ a_3).$$

Now, as all the necessary tools are established, we are ready to describe the extremal trajectories. The family of Hamiltonian functions of the PMP is given by

$$h_u^\nu(a) = \langle a, ue_1 + e_2 \rangle + \frac{\nu}{2} u^2,$$

where $a \in L^*$, $u : [t_0, t_1] \to \mathbb{R}$ is a control, and $\nu \in \mathbb{R}$ a parameter. To compute the vertical part of (3.20), we first note that

$$\frac{\partial h}{\partial a} = ue_1 + e_2.$$

To compute the map $\mathrm{ad}_{\frac{\partial h}{\partial a}} : L \to L$, we evaluate it at an arbitrary element $Y \in L$. This yields

$$\begin{aligned} \mathrm{ad}_{\frac{\partial h}{\partial a}}(Y) &= [ue_1 + e_2, Y] \\ &= u[e_1, Y] + [e_2, Y] \\ &= u \ \mathrm{ad}_{e_1} Y + \mathrm{ad}_{e_2} Y \\ &= u(\mathrm{ad}_{e_1} + \mathrm{ad}_{e_2})(Y), \end{aligned}$$

such that

$$\mathrm{ad}_{\frac{\partial h}{\partial a}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -u \\ -1 & u & 0 \end{pmatrix},$$

by (4.2). Now, by the definition of the dual to the linear adjoint operator $\mathrm{ad}_{\frac{\partial h}{\partial a}} : L \to L$, we have

$$\left(\mathrm{ad}_{\frac{\partial h}{\partial a}}\right)^* a = a \left(\mathrm{ad}_{\frac{\partial h}{\partial a}}\right).$$

Evaluating the right hand side of this equation yields

$$\begin{aligned} a\left(\mathrm{ad}_{\frac{\partial h}{\partial a}}\right) &= (a_1\ a_2\ a_3)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -u \\ -1 & u & 0 \end{pmatrix} \\ &= (-a_3 \quad ua_3 \quad -ua_2), \end{aligned}$$

such that the Hamiltonian system (3.20) takes the form

$$\text{Vertical part:} \quad \begin{cases} \dot{a}_1 = -a_3 \\ \dot{a}_2 = ua_3 \\ \dot{a}_3 = -ua_2 \end{cases}, \qquad \text{Horizontal part:} \quad \dot{g} = gX_u.$$

Remark 4.1.1. Notice that the subsystem for the vertical coordinates is independent of the horizontal coordinates. This is a corollary of the left-invariant symmetry of the system and of appropriate choice of the coordinates (a_1, a_2, a_3) (see [1]).

▲

4.1.4 Solutions

In the abnormal case, the maximality over the Hamiltonians yields

$$h_u^0(a) = \max_{u\in\mathbb{R}}(ua_1 + a_2) < \infty$$

such that $\partial_u h_u^0(a) = 0$ and

$$a_1 \equiv 0.$$

By this, we have that $\dot{a}_1 = 0$ such that $a_3 = 0$. Now $\dot{a}_2 = ua_3 = 0$ such that a_2 is constant and $\dot{a}_3 = 0 = -ua_2$ such that u must be identically 0 by the nontriviality $(\nu, \lambda_t) \neq 0$ of the PMP. Since the abnormal extremal control is identically zero, the energy functional $E = 0$, the absolute minimum, and the elastic curve must be a straight line. Notice that these controls are singular since they are not uniquely determined by the maximality condition of PMP.

In the normal case, $\nu = -1$, the maximality condition of the PMP yields

$$h_u^{-1}(a) = \max_{u\in\mathbb{R}} \left(ua_1 + a_2 - \frac{1}{2}u^2 \right) < \infty,$$

such that $\partial_u h_u^{-1}(a) = a_1 - u = 0$ and

$$a_1 = u.$$

So the corresponding normal Hamiltonian of the PMP is

$$h = \frac{1}{2}a_1^2 + a_2 \tag{4.3}$$

and the vertical subsystem of the Hamiltonian system (3.20) of the PMP reads

$$\begin{cases} \dot{a}_1 = -a_3, \\ \dot{a}_2 = a_1 a_3, \\ \dot{a}_3 = -a_1 a_2, \end{cases} \tag{4.4}$$

Note that $\frac{d}{dt}(a_2^2 + a_3^2) = 2(a_2\dot{a}_2 + a_3\dot{a}_3) = 0$ such that

$$a_2^2 + a_3^2 = A^2, \tag{4.5}$$

for some constant $A \in \mathbb{R}$. By this observation, we can make the following change of coordinates

$$a_2 = \pm A\cos\varphi \qquad \text{and} \qquad a_3 = \pm A\sin\varphi,$$

for some $\varphi : [t_0, t_1] \to S^1$. Any choice of the variables a_2 and a_3 above, solves system (4.4) up to a sign. However, we will now consider the case where

$$a_2 = A\cos\varphi \qquad \text{and} \qquad a_3 = -A\sin\varphi.$$

By this choice, the vertical subsystem (4.4) of the PMP becomes

$$\begin{cases} \dot{a}_1 = A\sin\varphi, \\ \dot{a}_2 = -a_1 A\sin\varphi, \\ \dot{a}_3 = -a_1 A\cos\varphi, \end{cases}$$

implying $a_1 = \dot{\varphi}$. Note that, by choosing the coordinates $a_2 = -A\cos\varphi$ and $a_3 = A\sin\varphi$ we will also obtain $a_1 = \dot{\varphi}$. Since $a_1 = u = \dot{\theta}$, we have $\varphi = \theta + \psi$ for some $\psi \in S^1$ and the vertical subsystem (4.4) yields the equation:

$$\ddot{\theta} - A\sin(\theta + \psi) = 0.$$

Moreover, choosing a_2 and a_3 with equal sign will lead to $a_1 = -\dot{\varphi}$. By this, the vertical subsystem (4.4) yields the equation:

$$\ddot{\theta} + A\sin(\theta - \phi) = 0,$$

for some constant shift in angle $\phi \in S^1$. By the symmetry of the problem, we might set ψ and $-\phi$ equal to φ. Thus the optimal control $u = \dot{\theta}$ that solves the problem of the elastica, satisfies the general formula for a mathematical pendulum

$$\ddot{\theta} \pm A\sin(\theta + \varphi) = 0. \tag{4.6}$$

4.1.5 The pendulum equation in elastic curves

As mentioned earlier, Kirchhoff discovered a relation between the elastic curves and the pendulum equation. By this observation we refer to the pendulum equation in the context of elastica as Kirchhoffs kinetic analogy. We will now uncover this relation in a similar fashion.

Consider the pendulum equation (4.6) with positive sign. A similar relation holds for the negative case. Differentiating this with respect to time yields

$$\dddot{\theta} + \dot{\theta}A\cos\theta = 0. \tag{4.7}$$

On the other hand, multiplying (4.6) by $\dot{\theta}$ and taking the integral yields

$$\frac{1}{2}\dot{\theta}^2 - A\cos\theta = B, \tag{4.8}$$

for some constant $B \in \mathbb{R}$. Inserting (4.7) into (4.8) we obtain

$$\dddot{\theta} + \frac{1}{2}\dot{\theta}^3 - B\dot{\theta} = 0. \tag{4.9}$$

This equation should look familiar to us.

To link equation (4.6) with the solutions (2.21)-(2.23), recall the fact that $\kappa = \dot{\theta}$ and $\tau \equiv 0$ for planar curves. By this, equation (4.9) yields the intrinsic equation (2.14) of the elastica with $B = \lambda/2$. Multiplying (2.14) with $8\kappa'$ on both sides and taking the integral yields (2.16), which is the equation we obtained from the variational analysis of the elastica. We will return to this relation later on in section 5.1.

4.2 Euclidean Rollings

As mentioned in section 2.1.4, A. Arthur and G.R. Walsh [3] (1986) and V. Jurdjevic [11] (1993) independently discovered that the solution set of the rolling sphere problem coincides with that of elastic curves. Furthermore, Jurdjevic and J. Zimmerman [13] (2008) generalized this result relating the elastic curves on the Riemannian model spaces to rolling problems on the corresponding model spaces - where *euclidean rolling* refer to the case where the stationary manifold is Euclidean. From now on, when we say Euclidean rolling "problems" we refer to both the rolling of the sphere and the rolling of the hyperboloid. Moreover, we will adapt the term *pseudo-sphere* as a reference to the hyperboloid. Thus, when we speak of "spheres" we refer to both the sphere and the hyperboloid.

In this section, we will show how the Euclidean rolling problems are related to the elastica by extracting the general pendulum equation (4.6) from the Hamiltonian system of the rolling sphere problems. We approach this in a similar fashion as we did for the elastica in the previous section.

Before we start, we need a proper definition of what it means to roll a manifold over a stationary plane where no "slipping" nor "twisting" between the two is allowed.

4.2.1 Definition of rolling

To treat both the rolling of the sphere and the rolling of the hyperboloid at once, let $\mu \in \{-1, 1\}$ and denote by M_μ the rolling manifold where

$$M_{-1}(c) := H_1^2(c) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -(x_1-c_1)^2-(x_2-c_2)^2+(x_3-c_3)^2 = 1\},$$

denotes the two-sheeted *unit hyperboloid* with center $c = (c_1, c_2, c_3) \in \mathbb{R}^3$, and

$$M_1(c) := S_1^2(c) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1 - c_1)^2 + (x_2 - c_2)^2 + (x_3 - c_3)^2 = 1\},$$

denotes the *unit sphere* with center $c = (c_1, c_2, c_3) \in \mathbb{R}^3$. Since we are only considering a two-sheeted hyperboloid in this text, the tangent space will only intersect the hyperboloid in one point. Denote by $\hat{M}$ the stationary plane which we will now define. To allow an arbitrary initial contact point $p \in M_\mu \cap \hat{M} \subset \mathbb{R}^3$ and orientation of M_μ, we define the stationary plane $\hat{M}$ as the affine tangent space of M_μ at $p \in M_\mu$. That is,

$$\hat{M} := T_p M_\mu = \{p + X \in \mathbb{R}^3 : X \in T_p M_\mu\}.$$

The notion of a "rolling manifold" might be a bit vague. To make this notion clearer, we define the rolling of M_μ in the following manner:

Let us denote by $SO_\mu(3)$ where $SO_{-1}(3) = SO_+(2,1)$ and $SO_1(3) = SO(3)$. We know that the group $SO_\mu(3)$ acts transitively on orthonormal bases of M_μ. That is, given any $p, q \in M_\mu$ and orthonormal bases $\{X_1, X_2\}$ for $T_p M_\mu$ and $\{Y_1, Y_2\}$ for $T_q M_\mu$, there exists $R \in SO_\mu(3)$ such that $Rp = q$ and $R_* X_i = Y_i$ for $i = 1, 2$ (see [19] Proposition 3.3 & 3.6). Thus, given any initial point $p \in M_\mu$, we can determine any point $q \in M_\mu$ by an element R of $SO_\mu(3)$. Furthermore, a piecewise smooth curve $t \mapsto R(t)$ in $SO_\mu(3)$ defines a piecewise smooth curve $t \mapsto R(t)p$ in M_μ. Next, the corresponding contact point between M_μ and $\hat{M}$ can be identified by a point in $\mathbb{R}^2$. Now, regarding the rolling process as a matching of contact points between the two manifolds M_μ and $\hat{M}$, we have that any state of this system can be determined by a point in the configuration space $SO_\mu(3) \times \mathbb{R}^2$.

The configuration space $SO_\mu(3) \times \mathbb{R}^2$ sits naturally inside of $SO_\mu(3) \times \mathbb{R}^3$ with one coordinate kept constant. Furthermore, we defined the semi-direct products $SO(3) \ltimes \mathbb{R}^3$ as the special Euclidean group and $SO_+(2,1) \ltimes \mathbb{R}^3$ as the special Poincaré group $SP(2,1)$ in section 3.1.3. Thus we might regard our configuration space as the semi-direct product $SO_\mu(3) \ltimes \mathbb{R}^2$ endowed with the group operations induces by $SE(3)$ or $SP(2,1)$, depending on the value of μ.

An element of $SO_\mu(3) \ltimes \mathbb{R}^2$ acts on $\mathbb{R}^3$ by orientation preserving isometries that preserve the structure of M_μ - as described in section 3.1.3. In the case $\mu = -1$, we have that $\mathbb{R}^{2,1} \to \mathbb{R}^{2,1}$ is an orientation preserving isomorphism of $\mathbb{R}^3$ under the metric

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\mathbb{R}^{2,1}$ denotes the pseudo-Riemannian manifold $(\mathbb{R}^3, J)$. By this, we have that an *isometric motion* of M_μ can be described by a continuous curve in the Lie group $SO_\mu(3) \ltimes \mathbb{R}^2$. We are now ready to formulate the notion of a rolling manifold.

Definition 4.2.1. Let M be a smooth manifold isometrically embedded into $\mathbb{R}^3$. A **rolling of M in $\mathbb{R}^3$** is an isometric motion of M in $\mathbb{R}^3$.

By the definition above, a smooth manifold M together with an isometric motion of M is referred to as a *rolling manifold*. To define a specific rolling process, we consider the following type of curves:

Definition 4.2.2. A piecewise smooth curve $t \mapsto \gamma(t)$ in a rolling manifold M is called a **rolling curve**.

Definition 4.2.3. Let $\gamma : [t_0, t_1] \to M$ be a rolling curve. A **rolling of M over $\hat{M}$ along γ** is an isometric motion $\iota : [t_0, t_1] \to \mathrm{Isom}(M)$ such that

(a) $\iota(t)\gamma(t) \in \hat{M}$ and

(b) $T_{\iota(t)\gamma(t)}(\iota(t)M) = T_{\iota(t)\gamma(t)}\hat{M}$

holds for each $t \in [t_0, t_1]$.

By the definition of rolling above, we might regard the rolling process as a matching of contact points between the two manifolds such that their corresponding tangent spaces are identical. The curve $\hat{\gamma} : [t_0, t_1] \to \hat{M}$ defined by $\hat{\gamma}(t) := \iota(t)\gamma(t)$ is usually referred to as the *development curve* of the rolling curve γ on the stationary manifold $\hat{M}$ - it is the curve in $\hat{M}$ that is traced out by the point of contact between the two manifolds under the rolling process.

The map $\iota : [t_0, t_1] \to \mathrm{Isom}(M)$ in the definition above describes a rolling process in generality. To restrict the rolling process where no slipping nor twisting between the two manifolds is allowed, we need some additional restrictions on the curve $t \mapsto \iota(t)$.

Definition 4.2.4. Let $\iota : [t_0, t_1] \to \mathrm{Isom}(M)$ be a rolling of M over $\hat{M}$. Then ι is a **rolling without slipping nor twisting** if

$$\iota(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t) \tag{4.10}$$

holds for all $t \in [t_0, t_1]$.

In short, we shall refer to the map $\iota : [t_0, t_1] \to \mathrm{Isom}(M)$ satisfying (4.10) in the definition above simply as a *rolling map* - when there's no confusion of the manifolds involved in the rolling process. We state the Euclidean rolling sphere problem by the following:

Problem 3 (Euclidean rolling)**.** *Given $g_0, g_1 \in SO_\mu(3) \ltimes \mathbb{R}^2$, find a rolling map $g : [t_0, t_1] \to SO_\mu(3) \ltimes \mathbb{R}^2$, with $t \mapsto (R(t), x(t), y(t))$ such that $g(t_i) = g_i$ and*

$$\ell = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2} dt \to \min,$$

for $i \in \{0, 1\}$, where $\hat{\gamma}(t) = (x(t), y(t))$ is the development curve in $\hat{M} \simeq \mathbb{R}^2$ traced out by the contact point and $R(t) \in SO_\mu(3)$ is the orientation matrix.

Note that the definition of rolling above only covers the no slip condition. Since the codimension of the rolling manifold is 1 in our case, the no twist condition comes for free. Indeed, we have the following lemma:

Lemma 4.2.5. *Let $\gamma : [t_0, t_1] \to M_\mu$ be a rolling curve and $Y \in T_{\gamma(t)}M_\mu$. If $g : [t_0, t_1] \to SO_\mu(3) \ltimes \mathbb{R}^2$ is a rolling map, then $\dot{Y} \in (T_{g(t)\gamma(t)}M_\mu)^\perp$. In particular,*

$$(\dot{R}(t)R^{-1}(t))\hat{X} \in \operatorname{span}\{e_3\}, \tag{4.11}$$

for all $\hat{X} \in T_{\hat{\gamma}(t)}\hat{M}$.

Proof. Consider a tangent vector $Y \in T_pM_\mu$ for some $p \in M_\mu$. Now, the motion of Y under the action of $g(t)$ on M_μ determines a family of vectors Y_t defined by

$$Y_t := g(t)_*Y = R(t)Y,$$

such that $Y_t \in T_{g(t)p}(g(t)M)$ for each $t \in [t_0, t_1]$. Fix some $T \in [t_0, t_1]$ and pick some tangent vector $\hat{X} \in T_{\hat{\gamma}(T)}\hat{M} = T_{g(T)\gamma(T)}(g(T)M)$. Now, if we let $Y = (g(T)_*)^{-1}\hat{X} = R^{-1}(T)\hat{X}$, we have that $\hat{X} = g(T)_*Y = R(T)Y = Y_T$ and

$$\dot{Y}_t|_{t=T} = \dot{R}(T)Y = \dot{R}(T)R^{-1}(T)\hat{X}.$$

Thus $\dot{R}(t)R^{-1}(t)\hat{X} \subset \operatorname{span}\{e_3\}$ for all $\hat{X} \in T_{\hat{\gamma}(t)}\hat{M}$, which was what we wanted to show. □

4.2.2 Kinematic equations

Before we start to describe the kinematic equations of the rolling sphere problem, the reader might want to recall the structure of the Lie algebra $so(2,1)$ and $so(3)$ described in section 3.1.3. The dynamics of the rolling spheres can be described by the following ODEs:

Proposition 4.2.6. *If $R : [t_0, t_1] \to SO_\mu(3)$ and $\gamma : [t_0, t_1] \to \mathbb{R}^2$ solves the following system of ODEs:*

$$\left.\begin{array}{l} \dot{x} = u_1 \\ \dot{y} = u_2 \\ \dot{R} = R \begin{pmatrix} 0 & 0 & -\mu u_1 \\ 0 & 0 & -\mu u_2 \\ u_1 & u_2 & 0 \end{pmatrix} \end{array}\right\}, \tag{4.12}$$

for some piecewise smooth functions $u_1, u_2 : [t_0, t_1] \to \mathbb{R}$, then $\iota = (R^{-1}, \gamma) : [t_0, t_1] \to SO_\mu(3) \ltimes \mathbb{R}^3$ is a rolling map satisfying $\iota(0) = (I, 0)$.

Proof. The condition of rolling without slipping nor twisting link the translational motion of the spheres with their rotational motion - a rotation generates a translation and vice versa. So we might define the controls $u_1, u_2 : [t_0, t_1] \to \mathbb{R}$ generating the motion by the two first equations. That is,

$$u_1 = \dot{x} \qquad \text{and} \qquad u_2 = \dot{y}.$$

We will now like to show how these controls are related to the rotational matrix $R \in SO_\mu(3)$. By Proposition 3.1.16 and 3.1.17, together with their left-invariant structure, we might write

$$\dot{R} = R \begin{pmatrix} 0 & -a_3 & a_1 \\ a_3 & 0 & a_2 \\ -\mu a_1 & -\mu a_2 & 0 \end{pmatrix}$$

for some piecewise smooth functions $a_1, a_2, a_3 : [t_0, t_1] \to \mathbb{R}$. Fix $t \in [t_0, t_1]$ and denote by $c(t) = (x(t), y(t), 1)$ the center of the sphere, and $p(t)$ an arbitrary point on the sphere with respect to the stationary frame $\mathcal{E} = \text{span}\{e_1, e_2, e_3\}$ of the ambient space. Moreover, define $q(t) = p(t) - c(t)$ such that

$$\dot{q}(t) = \dot{p}(t) - \dot{c}(t).$$

Following the notation of [2], let Q denote the point $q(t)$ with respect to the moving frame $\mathcal{F} = \text{span}\{f_1, f_2, f_3\}$ attached to the center of the sphere, we have that

$$q(t) = QR(t).$$

Now, let us assume that $p(t)$ is the point of contact. Then the rolling without slipping nor twisting condition implies $\dot{p}(t) = 0$. By this, we have that

$$\dot{c} = -\dot{q} = -Q\dot{R} = -QR \begin{pmatrix} 0 & -a_3 & a_1 \\ a_3 & 0 & a_2 \\ -\mu a_1 & -\mu a_2 & 0 \end{pmatrix} = -q(t) \begin{pmatrix} 0 & -a_3 & -\mu a_1 \\ a_3 & 0 & -\mu a_2 \\ a_1 & a_2 & 0 \end{pmatrix}.$$

Moreover, since $p(t)$ is assumed to be the point of contact, we have that $q(t) = -e_3$ and the above equation reads

$$(u_1, u_2, 0) = (0, 0, 1) \begin{pmatrix} 0 & -a_3 & a_1 \\ a_3 & 0 & a_2 \\ -\mu a_1 & -\mu a_2 & 0 \end{pmatrix} = (-\mu a_1, -\mu a_2, 0),$$

such that $a_1 = -\mu u_1$, $a_2 = -\mu u_2$.

It remains to show that $a_3 \equiv 0$. If $t \mapsto (R^{-1}(t), \gamma(t))$ is a rolling map, the no-twist condition reads

$$\frac{dR^{-1}(t)}{dt}(R^{-1}(t))^{-1}\hat{X} = \frac{dR^{-1}(t)}{dt}R(t)\hat{X} \in \operatorname{span}\{e_3\} \tag{4.13}$$

for all $\hat{X} \in T_{\gamma(t)}\hat{M}$ by Lemma 4.2.5. Differentiating the orthogonality condition for $R(t) \in SO_\mu(3)$ we obtain

$$\frac{dR^{-1}}{dt} = R^{-1}\dot{R}R^{-1}$$

such that condition (4.13) is equivalent to

$$R^{-1}(t)\dot{R}(t)\hat{X} \in \operatorname{span}\{e_3\}$$

for all $\hat{X} \in T_{\hat{\gamma}(t)}\hat{M}$. Now, by our assumption above, we have that

$$R^{-1}\dot{R} = \begin{pmatrix} 0 & -a_3 & u_1 \\ a_3 & 0 & u_2 \\ u_1 & u_2 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & -a_3 & u_1 \\ a_3 & 0 & u_2 \\ u_1 & u_2 & 0 \end{pmatrix} e_i \in \operatorname{span}\{e_3\}$$

for $i \in \{1, 2\}$ if and only if $a_3 \equiv 0$. $\square$

Similarly, as for the problem of elastica, we embed our configuration space $SO_\mu(3) \ltimes \mathbb{R}^2$ into $GL(6)$ and denote this subgroup by G_μ. We achieve this by the following diffeomorphism:

$$(R(t), \gamma(t)) \hookrightarrow \begin{pmatrix} R(t) & 0 \\ 0 & \begin{matrix} 1 & 0 & x(t) \\ 0 & 1 & y(t) \\ 0 & 0 & 1 \end{matrix} \end{pmatrix} \in G_\mu \subset GL(6). \tag{4.14}$$

In this setting, we might view the problem of rolling the sphere as a left-invariant optimal control problem on the Lie group G_μ.

Now, taking into account the embedding $SO_\mu(3) \ltimes \mathbb{R}^2 \hookrightarrow GL(6)$, we can write the kinematic equation of the rolling sphere problem on the form:

$$\dot{g} = \begin{pmatrix} \dot{R} & 0 \\ 0 & \begin{matrix} 1 & 0 & \dot{x} \\ 0 & 1 & \dot{y} \\ 0 & 0 & 1 \end{matrix} \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & \begin{matrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{matrix} \end{pmatrix} \begin{pmatrix} \begin{matrix} 0 & 0 & -\mu u_1 \\ 0 & 0 & -\mu u_2 \\ u_1 & u_2 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 0 & u_1 \\ 0 & 0 & u_2 \\ 0 & 0 & 0 \end{matrix} \end{pmatrix},$$

using the relation in equation (4.12). Writing this equation in terms of the basis elements of $GL(6)$, we obtain

$$\dot{g} = g\left(u_1(E_{31} - \mu E_{13} + E_{46}) + u_2(E_{32} - \mu E_{23} + E_{56})\right).$$

The Lie algebra L_μ associated with the Lie group G_μ is given by

$$L_\mu = \operatorname{span}\{E_{32} - \mu E_{23}, \mu E_{13} - E_{31}, E_{21} - E_{12}, E_{46}, E_{56}\},$$

which is consistent with what we obtained in Proposition 3.1.16 and 3.1.17, depending on the value of μ. If we denote by $e_1^\mu = E_{32} - \mu E_{23}$, $e_2^\mu = \mu E_{13} - E_{31}$, $e_3 = E_{21} - E_{12}$, $e_4 = E_{46}$, and $e_5 = E_{56}$ the basis elements of L_μ, we might write the vector field $X_u^\mu \in L_\mu$ as $X_u^\mu = u_1(e_2^\mu + e_4) + u_2(e_1^\mu + e_5)$. In this way, the kinematic equation of the rolling sphere becomes

$$\dot{g} = gX_u^\mu = g(u_1(e_4 - e_2^\mu) + u_2(e_1^\mu + e_5)). \tag{4.15}$$

By this, we can rephrase the rolling sphere problem as a left-invariant optimal control problem on the Lie group $G_\mu \subset GL(6)$ in the following manner:

Problem 4 (Euclidean Rolling)**.** *Let $g : [t_0, t_1] \to G_\mu$ be a rolling map satisfying the boundary conditions $g(t_i) = g_i$ for $i \in \{0, 1\}$. Find the optimal control $u : [t_0, t_1] \to \mathbb{R}^2$ in (4.15) such that*

$$E(u) = \frac{1}{2}\int_{t_0}^{t_1} u^2(t)\ dt \to \min.$$

Remark 4.2.7. The problem was originally stated as to minimize a length functional $\ell(u)$ on $\mathbb{R}^2$. In the problem above, we have introduced a new functional $E(u)$ on $\mathbb{R}^2$ that minimizes the energy. The minimization of this functional is equivalent to that of the original problem. Indeed, by Cauchy-Schwartz, we have that

$$\ell^2(u) = \left(\int_{t_0}^{t_1} \|u(t)\| dt\right)^2 \leq \int_{t_0}^{t_1} \|u(t)\|^2 dt,$$

where we have equality if and only if u is a constant function. The functional E is smooth and easier to work with than the functional ℓ.

▲

4.2.3 Controllability

The vector field X_u^μ in (4.15) defines a subspace $\Gamma = \mathrm{span}\{u_1(e_2^\mu + e_4), u_2(e_1^\mu + e_5)\} \subsetneq L_\mu$ that is bracket generating by the following bracket operations:

$$[e_1^\mu, e_2^\mu] = \mu e_3, \ [e_1^\mu, e_3] = -e_2^\mu, \ [e_2^\mu, e_3] = e_1^\mu, \text{ and } \mathrm{ad}_{e_4} = \mathrm{ad}_{e_5} = 0. \tag{4.16}$$

Thus, the system is controllable by the Rashevski-Chow Theorem 3.1.23.

4.2.4 Hamiltonian system

We now turn our attention towards the Hamiltonians for the PMP on T^*G. By the bracket operations in (4.16), we have the following nonzero adjoint operators:

$$\left.\begin{aligned} \mathrm{ad}_{e_1^\mu} = \begin{pmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & \mu & 0 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_{e_2^\mu} = \begin{pmatrix} \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -\mu & 0 & 0 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}, \\ \text{and} \qquad \mathrm{ad}_{e_3} = \begin{pmatrix} \begin{matrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}\right\} \tag{4.17}$$

where $\mathrm{ad}: L_\mu \to L_\mu$ is the adjoint operator defined in (3.4).

As before, let L_μ^* denote the dual space of the Lie algebra L_μ and denote by $\omega^1, \dots, \omega^5$ its basis elements such that

$$L_\mu^* = \operatorname{span}\{\omega^1, \dots, \omega^5\} \qquad \text{and} \qquad \langle \omega^i, e_j \rangle = \delta^i_j.$$

Furthermore, we write the elements X of the Lie algebra L_μ as column vectors:

$$X = \sum_{i=1}^{5} X^i e_i = \begin{pmatrix} X^1 \\ \vdots \\ X^5 \end{pmatrix},$$

and elements a of the corresponding dual space L_μ^* as row vectors:

$$a = \sum_{i=1}^{5} a_i \omega^i = (a_1 \ \cdots \ a_5).$$

The Hamiltonians of the PMP now takes the form

$$\begin{aligned} h_u^{\nu,\mu}(a) &= \langle a, u_1(e_4 - e_2^\mu) + u_2(e_1^\mu + e_5) \rangle + \frac{\nu}{2}(u_1^2 + u_2^2) \\ &= u_1(a_4 - a_2^\mu) + u_2(a_1^\mu + a_5) + \frac{\nu}{2}(u_1^2 + u_2^2), \end{aligned}$$

such that

$$\frac{\partial h}{\partial a} = u_1(e_4 - e_2^\mu) + u_2(e_1^\mu + e_5).$$

To determine the operator $\mathrm{ad}_{\frac{\partial h}{\partial a}}$, we evaluate it at an arbitrary element $Y \in L$. This yields

$$\begin{aligned} \mathrm{ad}_{\frac{\partial h}{\partial a}}(Y) &= \left[\frac{\partial h}{\partial a}, Y\right] \\ &= [u_1(e_4 - e_2^\mu) + u_2(e_1^\mu + e_5), Y] \\ &= u_1[e_4, Y] - u_1[e_2^\mu, Y] + u_2[e_1^\mu, Y] + u_2[e_5, Y] \\ &= u_1 \mathrm{ad}_{e_4}(Y) - u_1 \mathrm{ad}_{e_2^\mu}(Y) + u_2 \mathrm{ad}_{e_1^\mu}(Y) + u_2 \mathrm{ad}_{e_5}(Y) \\ &= (u_1(\mathrm{ad}_{e_4} - \mathrm{ad}_{e_2^\mu}) + u_2(\mathrm{ad}_{e_1^\mu} + \mathrm{ad}_{e_5}))(Y), \end{aligned}$$

such that

$$\begin{aligned}
\mathrm{ad}_{\frac{\partial h}{\partial a}} &= u_1(\mathrm{ad}_{e_4} - \mathrm{ad}_{e_2^\mu}) + u_2(\mathrm{ad}_{e_1^\mu} + \mathrm{ad}_{e_5}) \\
&= \begin{pmatrix} 0 & 0 & -u_1 & & \\ 0 & 0 & -u_2 & 0 & \\ \mu u_1 & \mu u_2 & 0 & & \\ & & & & \\ & 0 & & 0 & \\ & & & & \end{pmatrix},
\end{aligned}$$

by equation (4.16) and (4.17). Now, by the definition of the adjoint to the linear adjoint operator $\mathrm{ad}_{\frac{\partial h}{\partial a}} : L \to L$, we have

$$\left\langle \left(\mathrm{ad}_{\frac{\partial h}{\partial a}}\right)^*,\ a \right\rangle = \left\langle a,\ \mathrm{ad}_{\frac{\partial h}{\partial a}} \right\rangle.$$

Evaluating the right hand side of this equation yields

$$\begin{aligned}
\left\langle a,\ \mathrm{ad}_{\frac{\partial h}{\partial a}} \right\rangle &= (a_1^\mu,\ a_2^\mu,\ a_3,\ a_4,\ a_5) \begin{pmatrix} 0 & 0 & -u_1 & & \\ 0 & 0 & -u_2 & 0 & \\ \mu u_1 & \mu u_2 & 0 & & \\ & & & & \\ & 0 & & 0 & \\ & & & & \end{pmatrix} \\
&= (\mu u_1 a_3,\ \mu u_2 a_3,\ -u_1 a_1^\mu - u_2 a_2^\mu,\ 0,\ 0).
\end{aligned}$$

So the Hamiltonian system (3.20) of the PMP takes the following form:

$$\begin{cases} \dot{a}_1^\mu = \mu u_1 a_3 \\ \dot{a}_2^\mu = \mu u_2 a_3 \\ \dot{a}_3 = -u_1 a_1^\mu - u_2 a_2^\mu \\ \dot{a}_4 = 0 \\ \dot{a}_5 = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{x} = u_1 \\ \dot{y} = u_2 \\ \dot{R} = R \begin{pmatrix} 0 & 0 & -\mu u_1 \\ 0 & 0 & -\mu u_2 \\ u_1 & u_2 & 0 \end{pmatrix} \end{cases}.$$

4.2.5 Solutions

Consider first the abnormal case $\nu = 0$:

$$h_u^0(a) = \max_{(u_1,u_2)\in\mathbb{R}^2} (u_1(a_4 - a_2^\mu) + u_2(a_1^\mu + a_5)) < \infty,$$

which implies that $a_4 - a_2^\mu \equiv 0$ and $a_1^\mu + a_5 \equiv 0$ such that

$$\begin{cases} a_1^\mu = -a_5 \equiv \text{constant}, \\ a_2^\mu = a_4 \equiv \text{constant}, \end{cases} \quad \text{and} \quad \begin{cases} \dot{a}_1^\mu = 0 = \mu u_1 a_3, \\ \dot{a}_2^\mu = 0 = \mu u_2 a_3. \end{cases}$$

Now $0 = (\dot{a}_1^\mu)^2 + (\dot{a}_2^\mu)^2 = (\mu u_1 a_3)^2 + (\mu u_2 a_3)^2 = (u_1^2 + u_2^2)a_3^2$. Assuming nontrivial solutions $(u_1, u_2) \not\equiv (0, 0)$, we must have $a_3 \equiv 0$ such that

$$\dot{a}_3 = -u_1 a_1^\mu - u_2 a_2^\mu = 0.$$

Thus the abnormal optimal controls (u_1, u_2) must be constant and the corresponding curve $\hat{\gamma}(t) = (x(t), y(t))$ is a straight line with orientation matrix $R(t)$ given by

$$R^\mu(t) = \exp\left(t\begin{pmatrix} 0 & 0 & -\mu u_1 \\ 0 & 0 & -\mu u_2 \\ u_1 & u_2 & 0 \end{pmatrix}\right).$$

For the normal case $\nu = -1$, we have

$$h_u^{-1}(a) = \max_{(u_1,u_2)\in\mathbb{R}^2}\left(u_1(a_4 - a_2^\mu) + u_2(a_1^\mu + a_5) - \frac{1}{2}(u_1^2 + u_2^2)\right) < \infty,$$

which implies that

$$u_1 = a_4 - a_2^\mu \qquad \text{and} \qquad u_2 = a_1^\mu + a_5.$$

For these controls, the vertical subsystem of the Hamiltonian system of the PMP takes the form

$$\begin{cases} \dot{a}_1^\mu = \mu(a_4 - a_2^\mu)a_3, \\ \dot{a}_2^\mu = \mu(a_1^\mu + a_5)a_3, \\ \dot{a}_3 = -(a_4 - a_2^\mu)a_1^\mu - (a_1^\mu + a_5)a_2^\mu, \\ \dot{a}_4 = 0, \\ \dot{a}_5 = 0. \end{cases}$$

By making the following change of variables $b_1^\mu = a_4 - a_2^\mu = u_1$, $b_2^\mu = a_1^\mu + a_5 = u_2$, and $b_3^\mu = a_3$ we have that $\dot{b}_1^\mu = \dot{a}_4 - \dot{a}_2^\mu = -\mu(a_1^\mu + a_5)a_3 = -\mu b_2 b_3$, $\dot{b}_2 = \dot{a}_1^\mu + \dot{a}_5 = \mu(a_4 - a_2^\mu)a_3 = \mu b_1 b_3$, and $\dot{b}_3^\mu = \dot{a}_3 = -(a_4 - a_2^\mu)a_1^\mu - (a_1^\mu + a_5)a_2^\mu = a_5 b_1^\mu - a_4 b_2^\mu$, such that the vertical subsystem becomes

$$\begin{cases} \dot{b}_1^\mu = -\mu b_2^\mu b_3, \\ \dot{b}_2^\mu = \mu b_1^\mu b_3, \\ \dot{b}_3^\mu = a_5 b_1^\mu - a_4 b_2^\mu, \end{cases} \tag{4.18}$$

where we have used the change of variables to obtain an expression only containing the constants a_4 and a_5 in the last equation.

Note that $\frac{d}{dt}((b_1^\mu)^2 + (b_2^\mu)^2) = 2(b_1^\mu \dot{b}_1^\mu + b_2^\mu \dot{b}_2^\mu) = 0$ which implies that $(b_1^\mu)^2 + (b_2^\mu)^2 = A^2$ for some constant $A \in \mathbb{R}$. Indeed, by the unitary speed condition of the problem, we have that $1 = \dot{x}^2(0) + \dot{y}^2(0) = u_1^2(0) + u_2^2(0) = b_1^2(0) + b_2^2(0)$ such that $A = 1$. By this observation, we can make the following change in coordinates:

$$b_1^\mu = \pm \cos \varphi^\mu \qquad \text{and} \qquad b_2^\mu = \pm \sin \varphi^\mu$$

for some function $\varphi^\mu : [t_0, t_1] \to S^1$. Consider the case where $b_1^\mu = \cos \varphi^\mu$ and $b_2^\mu = \sin \varphi^\mu$. By these coordinates the vertical subsystem (4.18) becomes:

$$\begin{cases} \dot{b}_1^\mu = -\mu \sin \varphi^\mu \; b_3^\mu, \\ \dot{b}_2^\mu = \mu \cos \varphi^\mu \; b_3^\mu, \\ \dot{b}_3^\mu = \dot{a}_3. \end{cases}$$

Define $\varphi^\mu := \mu\theta$ for some function $\theta : [t_0, t_1] \to S^1$. Now, the above system have a solution for $b_3^\mu = \dot{\varphi}^\mu = \mu\dot{\theta}$. By this, the vertical subsystem (4.18) yields the equation:

$$\mu\ddot{\theta} = a_5 \cos(\mu\theta) - a_4 \sin(\mu\theta). \tag{4.19}$$

If we set $a_4 = B \cos \psi$ and $a_5 = B \sin \psi$ for some constants $B \in \mathbb{R}$ and $\psi \in S^1$, equation (4.19) becomes

$$\ddot{\theta} + B \sin(\theta - \psi) = 0,$$

for $\mu = 1$, and

$$\ddot{\theta} + B \sin(\theta + \psi) = 0,$$

for $\mu = -1$. Furthermore, by the symmetries of the problem, we might set $\psi = 0$ and $B = \pm 1$. In conclusion, the rolling sphere problem satisfy the general equation of a mathematical pendulum (4.6), which we obtained for the elastica problem. That is, the development curves of the rolling spheres traces an elastica in the plane. This also concludes a similarity between the rolling sphere and the rolling hyperboloid. We will discuss some of these solutions in section 5.2.

5 Analysis

As we have just seen that the optimal rolling of the spheres traces an elastica in the stationary manifold, we will now consider the different energy levels of the pendulum and see how they effect the optimal solutions. Finally, we will consider some of the rolling motions of the sphere along rolling curves of constant curvature.

5.1 Elastic Curves and The Pendulum Equation

Note that, by the kinetic analogy of the pendulum equation (4.6), the constant A is given by

$$A = \frac{g}{L},$$

where g is the constant of gravitational acceleration and L is the length of the pendulum.

We will now consider the possible solutions of equation (4.6). In the case where $A = 0$, we have that $\ddot{\theta} = 0$ such that $\dot{\theta} = C$ for some constant $C \in \mathbb{R}$. Since $\kappa = \dot{\theta}$, the curvature of the elastica must be constant. That is, the optimal trajectories must either be straight lines or circular arcs:

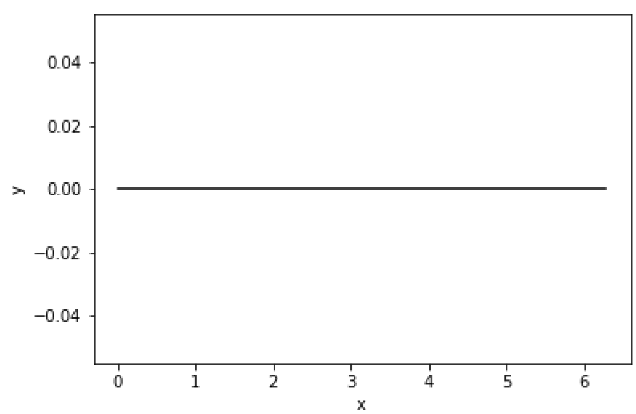


Figure 12: Straight line elastica.

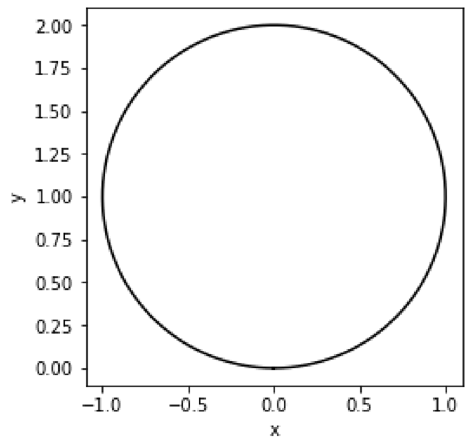


Figure 13: Circular elastica with $\kappa_0 = 1$.

By the kinetic analogy, we can look at this case as a pendulum in the absence of gravity at a fixed angle (Figure 12) or in continuous motion (Figure 13).

Consider now the case where $A > 0$. Since the physical nature of the problem is invariant under translations and rotations, we can apply homotheties and rotations to obtain $\pm A = 1$ and $\varphi = 0$ depending on the sign chosen in (4.6). By this, the angle θ satisfies the standard equation of the mathematical pendulum

$$\ddot{\theta} + \sin\theta = 0,$$

with $\dot{\theta} = \kappa$ and $\kappa' = -\sin\theta$, where κ denotes the curvature of the elastica. The different solutions of the pendulum equation depends on the different energy levels of the dynamical system describing the motion. Multiplying the pendulum equation above by $\dot{\theta}$ and taking the integral yields the conservation law

$$E = \frac{\dot{\theta}^2}{2} - \cos\theta,$$

which takes values in $[-1, \infty)$. Note that, in Hamiltonian formalisation of classical mechanics, this energy integral is described by the Hamiltonian function. We have the following possible cases for the total energy:

1. $E = -1$: By the energy equation above, we must have that $\dot{\theta} \equiv 0$ and $\theta \equiv 0$. So this case leads to straight line elasticas as in figure 12. By

the kinetic analogy, this corresponds to the case where the pendulum is at rest at a stable equilibrium.

2. $E \in (-1, 1)$: By the energy equation above, we must have that $-\pi < \theta < \pi$. At $\theta = 0$, we have that $\cos\theta = 1$ such that $\dot{\theta}^2 > 0$. That is, $\dot{\theta}$ takes on both signs. So this case leads to the following inflectional elasticas determined by (2.21):

   (a) For $p \in (0, 1/2)$, we have:

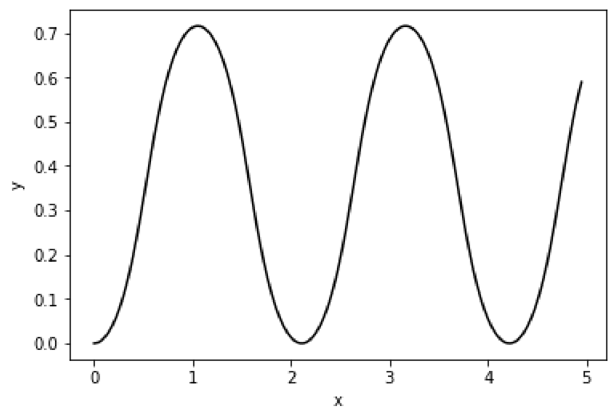


Figure 14: Inflectional elastica - sinusoidal.

   Note that whenever $p \to 0$, the elastica tends to a sinusoidal wave - by the definition of the Jacobi elliptic function $\text{cn}(\alpha, p)$.

   (b) For $p = 1/2$, we have rectangular elasticas:

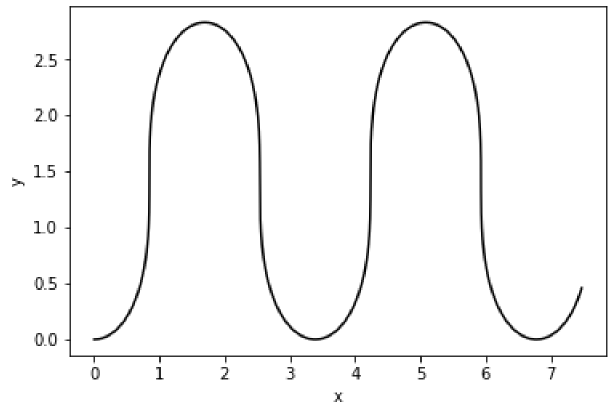


Figure 15: Inflectional elastica - rectangular.

(c) For $p \in (1/2, 0.826)$, we have:

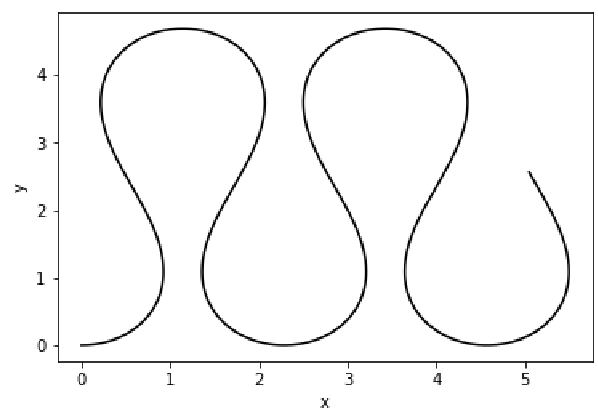


Figure 16: Inflectional elastica.

(d) For $p \approx 0.826$, we have the figure eight elastica:

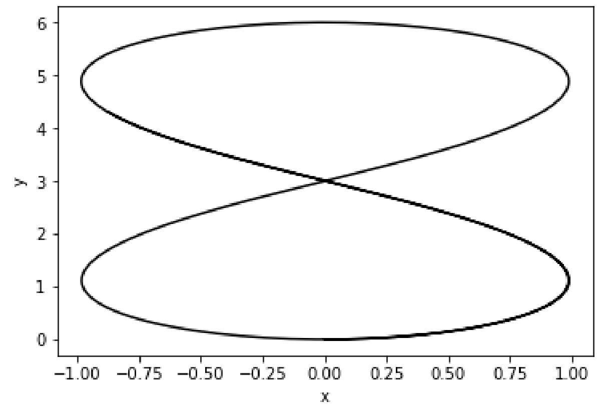


Figure 17: Inflectional elastica - figure eight.

Together with the circle, these are the only closed elasticas.

(e) For $p \in (0.826, 1)$, we have self-intersecting elasticas:

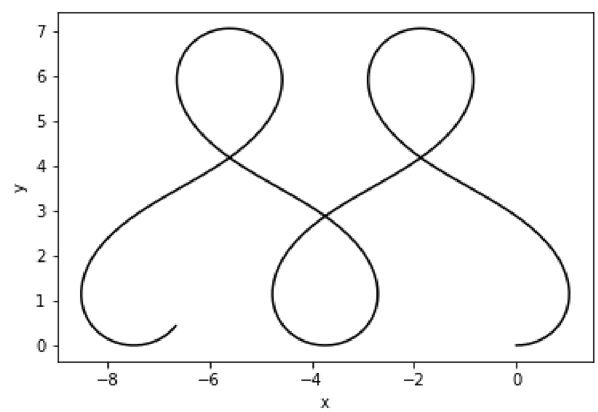


Figure 18: Inflectional elastica - self-intersecting.

By the kinetic analogy, these elasticas have inflections at points where $\dot{\theta} = 0$.

3. $E = 1$ and

   (a) $\theta \neq \pm\pi$: When $\theta \to \pm\pi$, we have that $\cos\theta \to -1$ such that $\dot{\theta} \to 0$. When $\theta = 0$, we have that $\cos\theta = 1$ such that $\dot{\theta} = 2$. So this case leads to critical elasticas determined by (2.22):

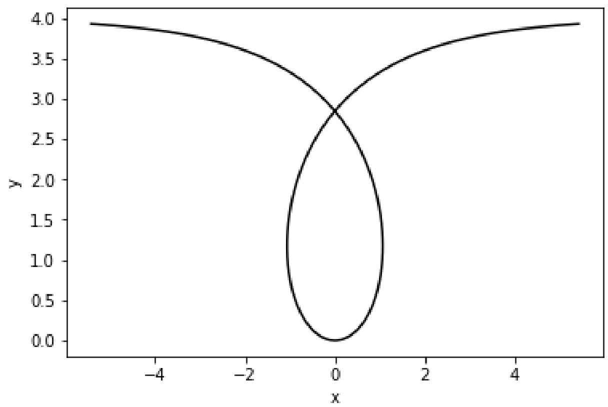


Figure 19: Critical elastica.

By the kinetic analogy, this is equivalent to the case where we start the pendulum as close to the maximal angle $\pm\pi$ as possible - the unstable equilibrium. In this case, the pendulum will swing once and tend towards its initial position in infinite time.

(b) $\theta = \pm\pi$: When $\theta = \pm\pi$, we have that $\cos\theta = -1$ such that $\dot{\theta} \equiv 0$. So this case leads to straight line elasticas, as in Figure 12. By the kinetic analogy, this is equivalent to the case where we start the pendulum at the maximum angle $\pm\pi$. In this case, the pendulum is at rest at an unstable equilibrium.

4. $E \in (1, \infty)$: We have that $\cos\theta = 1$ for $\theta = 0$, thus $\dot{\theta} > 2$ for all $t \in [t_0, t_1]$. That is, we have non-vanishing curvature. So this case leads to non-inflectional elasticas determined by (2.23):

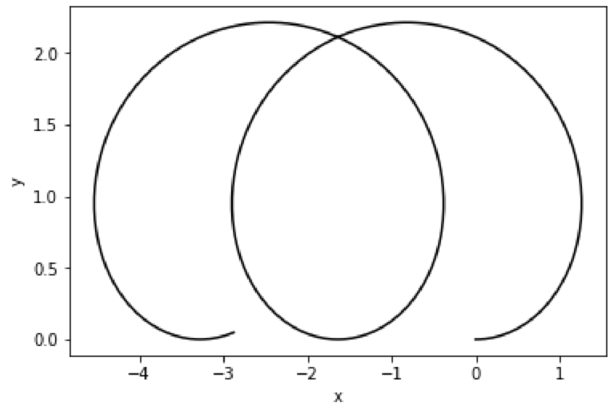


Figure 20: Non-inflectional elastica.

By the kinetic analogy, the pendulum rotates non-uniformly in the clockwise direction whenever $\dot{\theta} < 0$ and counter-clockwise whenever $\dot{\theta} > 0$.

5.2 Rolling Along Constant Curvature Elasticas

Even though the rolling sphere problems are solvable and we know that the optimal solutions are given by an eastic curve, it is still very hard to pick an optimal elastic curve that minimizes the length functional on the sphere. In fact, the problem of finding the optimal elastica in its own right still remains open. Resent work regarding both the local and global optimality, together with a precise description of conjugate points, can be found in [28] and [29].

Constructive proofs on the controllability of the Euclidean rolling sphere problems are given in [15] and [23], using piecewise constant controls. Though these rolling motions are far from optimal in most cases, they, at least, show us how any configuration can be reached by a rolling motion of the sphere. These proofs boils down to generating the restricted motions of slipping and twisting.

We will now consider some cases where we can preform the rolling along a constant curvature elastica. That is, straight lines or circular arcs. We will study which configurations can be obtained, and look at the controllability using a composition of these curves.

Before we start, recall the kinematic equations (4.12) of the Euclidean rolling sphere with $\mu = 1$:

$$\begin{cases} \dot{x}(t) = \cos(\theta(t)) \\ \dot{y}(t) = \sin(\theta(t)) \end{cases} \quad \text{and} \quad \dot{R}(t) = R(t) \begin{pmatrix} 0 & 0 & -\cos(\theta(t)) \\ 0 & 0 & -\sin(\theta(t)) \\ \cos(\theta(t)) & \sin(\theta(t)) & 0 \end{pmatrix},$$

where $R \in SO(3)$ and the angle $\theta(t)$ satisfies the pendulum equation

$$\ddot{\theta} + \sin\theta = 0.$$

From now on, following the notation from section 4.2, we denote by $\hat{x}$, $\hat{y}$, and $\hat{\kappa}_0$ variables related to the stationary plane $\hat{M}$ and x, y, z, and κ_0 the variables related to the rolling sphere M. By this, the two first kinematic equations determines the development curve $t \mapsto \hat{\gamma}(t)$ under the rolling, and

$$\theta(t) = \hat{\kappa}_0 t + \theta_0, \tag{5.1}$$

where $\hat{\kappa}_0$ denotes the curvature of the development curve and the constant of integration determines the direction of motion. For simplicity, we assume that we start our rolling in the direction of the positive x-axis, such that $\theta_0 = 0$, and with initial orientation $R(0) = I$.

5.2.1 Rolling along straight lines

First we consider the case of vanishing curvature. That is, $\dot{\theta} \equiv \hat{\kappa}_0 \equiv 0$ for which the elastic curve is just a straight line. In this case, the kinematic equations becomes:

$$\begin{cases} \dot{\hat{x}}(t) \equiv 1, \\ \dot{\hat{y}}(t) \equiv 0, \\ \dot{R}(t) = R(t) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{cases}$$

Integration of the two first equations gives

$$\begin{cases} \hat{x}(t) = t \\ \hat{y}(t) = 0 \end{cases},$$

where the constant of integration is set to zero by the initial condition $(\hat{x}(0), \hat{y}(0)) = (0, 0)$. Moreover, the orientation matrix is given by

$$R(t) = \exp\left(t\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{pmatrix}.$$

For obvious reasons, the possible configurations of rolling along straight lines are quite restricted. As an example, we have that the attainable set for which the terminal orientation of the sphere is given by $R(t_1) = I$, is the family of concentric circles of radius $2n\pi$ about the origin:

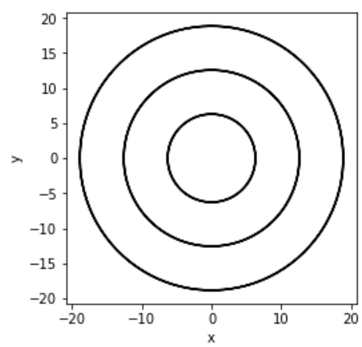


Figure 21: Concentric circles.

5.2.2 Rolling along circular arcs

Let us now consider a rolling of the sphere along circular arcs. That is, with $\hat{\kappa}_0$ a non-negative real constant. As before, let us assume that we start our rolling at $(0, 0) \in \mathbb{R}^2$ in the direction of the positive x-axis, such that $\theta_0 = 0$, and with an initial orientation $R(0) = I$. In this setting, equation (5.1) reads

$$\theta(t) = \hat{\kappa}_0 t,$$

and the kinematic equations of the rolling sphere becomes:

$$\begin{cases} \dot{\hat{x}}(t) = \cos(\hat{\kappa}_0 t) \\ \dot{\hat{y}}(t) = \sin(\hat{\kappa}_0 t) \\ \dot{R}(t) = R(t)\begin{pmatrix} 0 & 0 & -\cos(\hat{\kappa}_0 t) \\ 0 & 0 & -\sin(\hat{\kappa}_0 t) \\ \cos(\hat{\kappa}_0 t) & \sin(\hat{\kappa}_0 t) & 0 \end{pmatrix} \end{cases} \tag{5.2}$$

Integration of the two first equations yields the development curve:

$$\hat{\gamma}(t) = (\hat{x}(t), \hat{y}(t)) = \frac{1}{\hat{\kappa}_0}(\sin(\hat{\kappa}_0 t), 1 - \cos(\hat{\kappa}_0 t)), \tag{5.3}$$

where the constants of integration follows by the initial condition $(\hat{x}(0), \hat{y}(0)) = (0, 0)$.

5.2.3 Parametric equation for circular rolling curves

To work with rolling motions along circular rolling curves, we need a way to describe the chosen rolling curve $t \mapsto \gamma(t)$ on M. To get a parametric equation for the rolling curve, we first work out how the curvature of the rolling curve $t \mapsto \gamma(t)$ relates to the curvature of the development curve $t \mapsto \hat{\gamma}(t)$, which we already have defined in (5.3). We state this relationship by the following theorem:

Theorem 5.2.1. *If κ_0 denotes the curvature of a circular rolling curve and $\hat{\kappa}_0$ denotes the curvature of the development curve, then*

$$\kappa_0^2 = \hat{\kappa}_0^2 + 1 \tag{5.4}$$

where $1 < \kappa_0 < \infty$*.*

Proof. With initial conditions as described above, we can view the rolling of the sphere as a rolling of a cone as in the following illustration:

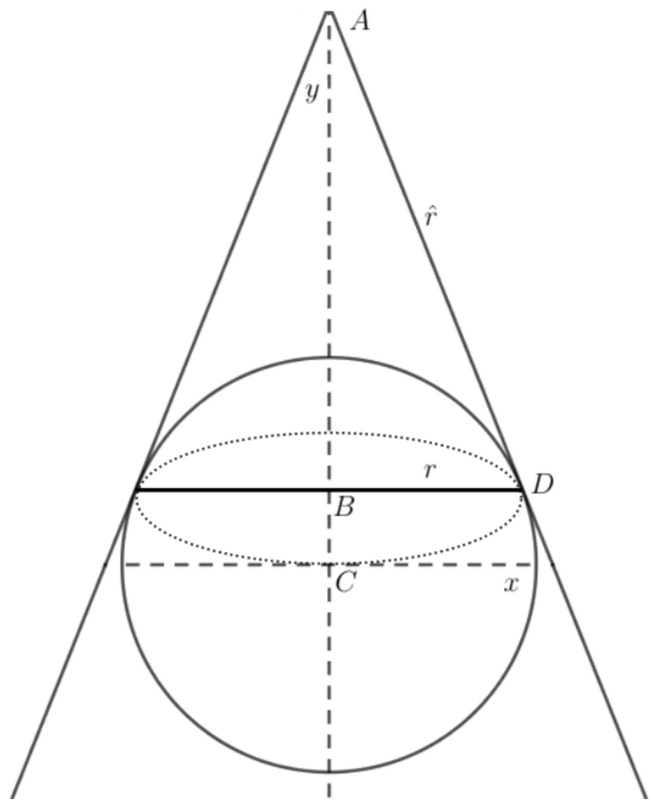


Figure 22: Cross-section of sphere in cone.

By the figure above, we have that the radius r of the rolling curve is given by the length of BD and the radius $\hat{r}$ of the development curve is given by the length of AD. Note that as $r \to 0$, we have that $\hat{r} \to 0$, and as $r \to 1$, the cone tends to a cylinder - for which the rolling curve becomes a great circle and the development curve a straight line - which justifies the restriction $1 < \kappa_0 < \infty$.

By the geometric equivalence $\triangle ADC \sim \triangle ABD \sim \triangle DBC$, we extract the following relationship between r and $\hat{r}$:

$$r^2 = \frac{\hat{r}^2}{1+\hat{r}^2} \qquad \text{and} \qquad \hat{r}^2 = \frac{r^2}{1-r^2}. \tag{5.5}$$

By definition of curvature, we have that $\kappa_0 = 1/r$ and $\hat{\kappa}_0 = 1/\hat{r}$. Inserting this into the equation above yields (5.4). $\square$

A direct consequence of the result above states the following:

Corollary 5.2.2. *Circular rolling curves with curvature*

$$\kappa_0 = \frac{n}{\sqrt{n^2 - 1}}$$

returns the unit sphere to its initial configuration for all integers $n \geq 2$.

Proof. Writing equation (5.4) on the form (5.5), we have that the circumference of the rolling curve and the development curve is given by

$$C_{\kappa_0} = \frac{2\pi\hat{r}}{\hat{r}^2 + 1} \qquad \text{and} \qquad C_{\hat{\kappa}_0} = 2\pi\hat{r},$$

respectively. We need to check for which integer n the equation

$$nC_{\kappa_0} = C_{\hat{\kappa}_0}$$

holds. This equation leads to

$$\hat{r} = \sqrt{n^2 - 1}$$

for $n \geq 2$. Taking into account $\hat{\kappa}_0 = 1/\hat{r}$ and (5.4) the result follows. $\square$

We will now like to derive the parametric equation of the rolling curve. By the rolling cone analogy above and the assumption of initial rolling direction $\theta_0 = 0$, we have that the rolling curve can be determined by the intersection of the sphere and the plane spanned by the x-axis and the vector $\vec{OP}$, where $P(y, z)$ is defined for $0 < y \leq 1$ and $0 < z < 2$:

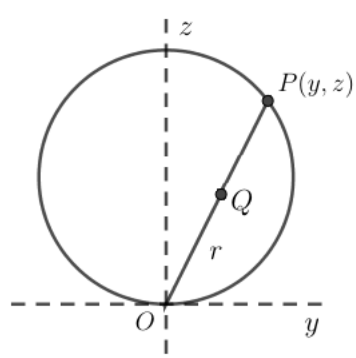


Figure 23: Cross-section of sphere.

By inspection of this figure, we have that

$$z^2 + y^2 = 4r^2$$

and

$$y(z) = \sqrt{1-(z-1)^2} = \sqrt{z(2-z)},$$

for $0 < z < 2$. By these two equation, we get that

$$z(r) = 2r^2. \tag{5.6}$$

We can parametrize the rolling curve by

$$\gamma(t) = Q + r(\cos t)v_1 + r(\sin t)v_2,$$

where v_1 and v_2 are unit vectors that span the plane containing the circle and Q denotes the center of the circle. If we let $v_1 = (1,0,0)$ and $v_2 = \vec{OP}/\|\vec{OP}\|$ span the plane that contains the rolling curve, we have that $Q = rv_2$ and the above equation reads

$$\gamma(t) = r(\cos t)v_1 + r(\sin t + 1)v_2$$

Now, v_2 is given by

$$\begin{aligned} v_2 &= (0, \sqrt{z(2-z)}, z)/\sqrt{z(2-z)+z^2} \\ &= \frac{1}{\sqrt{2z}}(0, \sqrt{z(2-z)}, z) \\ &= (0, \sqrt{1-r^2}, r), \end{aligned}$$

using relation (5.6). Putting all this together yields the equation

$$\gamma(t) = r(\cos t, \sqrt{1-r^2}(\sin t + 1), r(\sin t + 1)).$$

To make the parametrization coincide with our initial condition $\gamma(0) = (0,0,0)$, we make the shift in parameter $t \mapsto t - \pi/2$ such that

$$\gamma(t) = r(\sin t, \sqrt{1-r^2}(1-\cos t), r(1-\cos t)).$$

This parametrization has constant speed equal to r. So, by redefining t by $t \mapsto \kappa_0 t$ and by using (5.4) we obtain the unitary speed parametrization

$$\gamma(t) = \frac{1}{\kappa_0^2}(\kappa_0 \sin(\kappa_0 t), \hat{\kappa}_0(1-\cos(\kappa_0 t)), 1-\cos(\kappa_0 t)) \tag{5.7}$$

for the rolling curve. We summarize this construction in the following result:

Proposition 5.2.3. *Given any $P = (p_1, p_2, p_3) \in M \setminus (0, 0, 2)$ and $v_1 \in \mathbb{R}^2$, there exists a unique circular rolling curve $t \mapsto \gamma(t)$ satisfying $\gamma(0) = (0, 0, 0)$, $\dot{\gamma}(0) = v_1$, and $\gamma(t_1) = P$ for some $t_1 > 0$.*

Proof. We have that $r = \sqrt{p_1^2 + p_2^2 + p_3^2}/2 < 1$ which uniquely defines the curvature $\kappa_0 = 1/r > 1$. By the initial conditions $\dot{\gamma} = v_1$, we have that $\gamma(t) \in \operatorname{span}\{v_1, v_2\} \subset \mathbb{R}^3$ for all t, where $v_2 = \vec{OP}$. The result now follows by the fundamental theorem of planar curves together with the initial condition $\gamma(0, 0) = (0, 0, 0)$. □

5.2.4 Attainable sets

Rolling along circular arcs will not lead to optimal solution in most cases. However, we will now study some configurations that can easily be obtained using these elastic curves.

To get a better understanding of where these curves will bring the sphere under a rolling, we can look at how the reachable points in $\hat{M}$ distributes. If we fix t, say $t = 2\pi$, the reachable points in $\hat{M}$ forms a periodic spiral as we vary $\hat{\kappa}$:

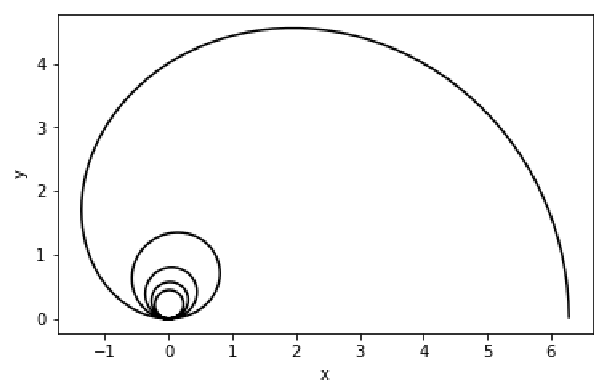


Figure 24: Periodic spiral for $0 < \hat{\kappa}_0 \leq 5$.

It is obvious from equation (5.7) that any rolling curve $t \mapsto \gamma(t)$ determined by $1 < \kappa < \infty$ will, eventually, return the sphere to its initial position. By Corollary 5.2.2 we know which values of κ_0 the sphere also returns to its initial orientation. Moreover, we can get an exact description of the possible configurations of the sphere along rolling curves with curvature $\kappa_0 = n/\sqrt{n^2-1}$ for any $t = 2\pi m/\kappa_0$, where m is an integer determined by $1 \leq m \leq n$. In fact, recalling the sphere in cone analogy in Figure 22, we have that for each n that the initial contact point $P \in M$ gets matched with $\hat{\gamma}(2\pi m/\kappa_0) \in \hat{M}$ for each integer $1 \leq m \leq n$, and the corresponding orientation of the sphere is given by $R_z(2\pi/m)$.

We will now see that Corollary 5.2.2 is a special case of the following result:

Proposition 5.2.4. *A rolling curve $t \mapsto \gamma(t)$ determined by $1 < \kappa_0 < \infty$ have periodic contact configuration if and only if*

$$\kappa_0 = \frac{n}{\sqrt{n^2 - m^2}}, \tag{5.8}$$

for $1 \leq m < n$.

Proof. The circumference of the rolling circle is given by $C_{\kappa_0} = 2\pi/\kappa_0$. So the initial contact point will be matched with $\hat{\gamma}(C_{\kappa_0} n)$ for each $n \geq 1$. Now, suppose $\hat{\gamma}(C_{\kappa_0} n_1) = \hat{\gamma}(C_{\kappa_0} n_2)$ for some integers $n_1 \neq n_2$. Then $\hat{x}(C_{\kappa_0} n_1) = \hat{x}(C_{\kappa_0} n_2)$ and $\hat{y}(C_{\kappa_0} n_1) = \hat{y}(C_{\kappa_0} n_2)$ if and only if

$$\sin\left(2\pi\frac{\hat{\kappa}_0 n_1}{\kappa_0}\right) = \sin\left(2\pi\frac{\hat{\kappa}_0 n_2}{\kappa_0}\right) \qquad \text{and} \qquad \cos\left(2\pi\frac{\hat{\kappa}_0 n_1}{\kappa_0}\right) = \cos\left(2\pi\frac{\hat{\kappa}_0 n_2}{\kappa_0}\right).$$

These equations holds if and only if

$$\frac{\hat{\kappa}_0 n_1}{\kappa_0} = m_1 \qquad \text{and} \qquad \frac{\hat{\kappa}_0 n_2}{\kappa_0} = m_2,$$

for some $m_1, m_2 \in \mathbb{Z}$, or, equivalently,

$$\frac{\sqrt{\kappa_0^2 - 1}}{\kappa_0} = \frac{\bar{m}}{\bar{n}}$$

for some integers $\bar{n} > \bar{m} \geq 1$, using (5.4). The above equation yields (5.8) with $m = \bar{m}$ and $n = \bar{n}$. $\square$

By the result above, we might ask ourself what happens under a rolling motion where the rolling curve $t \mapsto \gamma(t)$ is determined by $\kappa_0 \neq n/\sqrt{n^2 - m^2}$. We conjecture the following:

Conjecture 5.2.5. *Let $P \in M$ and $t \mapsto \gamma(t)$ be a rolling curve determined by $\kappa_0 \neq n/\sqrt{n^2 - m^2}$ for some integers $1 \leq m < n$ such that $\gamma(t_1) = P$ for some $t_1 > 0$. Then, given any $\epsilon > 0$ and $\hat{P} \in \mathrm{Im}(\hat{\gamma})$, there exists an integer $k \geq 1$ such that*

$$\left\| \hat{P} - \hat{\gamma}\left(t_1 + \frac{2\pi k}{\kappa_0}\right) \right\| < \epsilon.$$

That is, we can get arbitrarily close to the terminal contact configuration by rolling the sphere k times.

As we do not have periodicity in the contact configurations, we might argue that the above statement is true. However, as $\kappa_0 \to 1$ we have that $\hat{\kappa}_0 \to \infty$ and the development curve tends towards a straight line. Can we still find k in this setting? Unfortunately, we will have to leave this as an open question for further research.

5.2.5 Constructive proof on the controllability of the rolling sphere problem

To round this text of, we introduce a simple constructive proof on the controllability of the rolling sphere problem. That is, to show that any point on the sphere can be matched with any point in the plane under a rolling without slipping nor twisting.

Let $P = (p_1, p_2, p_3)$ be the point on the sphere M that we want to match with $\hat{P} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$ in the plane $\hat{M}$.

1. Given a point $P \in M$, we have that $r = \sqrt{p_1^2 + p_2^2 + p_3^2}$ and $\kappa_0 = 1/r$, which uniquely defines a rolling curve $t \mapsto \gamma(t)$ through $(0, 0, 0)$ and P determined by equation (5.7). Now, solving $\gamma_i(t_1) = p_i$ for some $i \in \{1, 2, 3\}$, we get $t_1 > 0$ such that $\gamma(t_1) = P$. This rolling curve brings the point P to $\hat{\gamma}(t_1) \in \hat{M}$, where $t \mapsto \hat{\gamma}(t)$ denotes the corresponding development curve determined by (5.3).

2. (a) In the case where $\|\hat{\gamma}(t_1) - \hat{P}\| = 2\pi m$ for some integer $m > 0$, roll M in a straight line of length $2\pi m$ towards $\hat{P}$ to obtain the terminal orientation.

   (b) In the case where $\|\hat{\gamma}(t_1) - \hat{P}\| \neq 2\pi m$, define two circles

$$\hat{O}_1 = \{(x, y) \in \hat{M} : (x - \hat{x}(t_1))^2 + (y - \hat{y}(t_1))^2 = (2\pi)^2\}$$

   and

$$\hat{O}_2(n) = \{(x, y) \in \hat{M} : (x - \hat{p}_1(t_1))^2 + (y - \hat{p}_2(t_1))^2 = (2\pi n)^2\}$$

   for some integer $n \geq 1$. Now, for some n we must have that

$$\hat{O}_1 \cap \hat{O}_2(n) = \{\hat{Q}_1, \hat{Q}_2\} \subset \hat{M}.$$

   Choose $i \in \{1, 2\}$ such that $\|\hat{\gamma}(t) - \hat{Q}_i\| \to \min$. Roll the sphere in a straight line towards $\hat{Q}_i$ in 2π units of time and follow up with a rolling of the sphere in a straight line towards $\hat{P}$ in $2\pi n$ units of time to obtain the terminal positioning of the sphere.

3. If we need to preform a twist by ϕ radians, we might adapt the method described in [15]:

$$R_z(\phi) = R_x(\pi/2)R_y(\phi/2)R_x(-\pi)R_y(-\phi/2)R_x(\pi/2),$$

   where

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \qquad \text{and} \qquad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

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I will also like to thank the mathematics department at the University of Bergen. I did not foresee the journey that my studies at this department have put me through. Marking my highlights at a semester abroad studying mathematics at University of California Berkeley, and a week spent at the Fields institute, Toronto Canada - where I met with one of the main contributor of the material covered in this text, Velimir Jurdjevic. I'm also very grateful for the flexibility and freedom your study programs have offered me, making it possible for me to choose courses of own interest.

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