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Properties of S_3-Symmetric Three-Higgs-Doublet Models

Author: Anton Kunčinas
Supervisor: Prof. Jörn Kersten
Co-supervisor: Prof. Per Osland

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Abstract

Some aspects of the S_3-symmetric three-Higgs-doublet models are analysed. A CP conserving potential with both real and complex vacuum configurations is considered. The S_3-symmetric potential is presented in the irreducible representation, which the main part of the thesis is based on, and in the Higgs basis. Hidden symmetries of the S_3-symmetric 3HDM potential are analysed: behaviour of the scalar potential under the subgroups of the $U(3)$ group, which result in Goldstone bosons, and the discrete $\mathbb{Z}_2$ symmetry required for a stable dark matter candidate. Possible models capable of accommodating the dark matter candidate are presented. Two vacuum configurations, one real and one complex, which can possibly accommodate the dark matter candidate, are further analysed. One of the vacuum configurations results in massless scalars and therefore the concept of soft symmetry breaking is applied. A numerical check of two vacuum configurations is performed based on the theoretical and experimental constraints.

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Abbreviations and Notations

2HDM - Two-Higgs-Doublet Model
3HDM - Three-Higgs-Doublet Model
CKM - Cabibbo–Kobayashi–Maskawa Matrix
CP - Charge-Parity
DM - Dark Matter
FCNC - Flavour-Changing Neutral Currents
IDM - Inert Doublet Model
NHDM - Multi-Higgs-Doublet Model
PMNS - Pontecorvo–Maki–Nakagawa–Sakata Matrix
SM - Standard Model
VEV - Vacuum Expectation Value

The trigonometric functions are denoted by:
$\mathrm{s}_\theta \equiv \sin(\theta)$,
$\mathrm{c}_\theta \equiv \cos(\theta)$,
$\mathrm{t}_\theta \equiv \tan(\theta)$.

Note:
Lagrangian $\equiv$ Lagrangian density
Natural units are used: $\hbar = c = 1$

Contents

Chapter 1

Introduction

The Standard Model (SM) of particle physics has been extensively tested for a few decades. The last missing piece, the Higgs boson, was discovered in 2012 with a combined mass of $m_h = 125.09 \pm 0.21(\text{stat.}) \pm 0.11(\text{syst.})$ GeV based on data from the ATLAS and CMS experiments [1–4]. This is undoubtedly a fascinating discovery in the field of particle physics and might be the final missing piece. Nevertheless, there is still no experimental verification that it is the only Higgs boson. Acknowledging the fact that the SM is the theory which describes an approximate observable world it is worth taking note of the fact that there might be physics beyond the SM. Some of the phenomena, which the SM does not account for are: neutrino oscillations, asymmetry of matter-antimatter, dark energy, gravity, *etc.* A particular physical phenomenon, which we are interested in and does not fit the frame of the SM is the absence of any Dark Matter (DM) candidate. As a consequence, the SM fails to describe nearly 85% of the matter. Extension of the Higgs sector could resolve some of the issues and is a common practice when models beyond the SM are constructed. Thus, we propose and are motivated that such extension could potentially solve several problems.

The SM uses the minimal Brout-Englert-Higgs mechanism [5–8], and a single complex $SU(2)$ doublet is considered. The simplest extension of the SM electroweak sector is the Two-Higgs-Doublet Model (2HDM) [9–15], where the second $SU(2)$ doublet is added to the SM-like $SU(2)$ doublet. Such extension predicts a rich scalar spectrum: two additional neutral states $h_{(1,2)}$, or three counting the SM-like Higgs boson h_{SM}, and a charged state $h^{\pm}$. The second $SU(2)$ doublet can be further constrained to result in a viable DM candidate. These are the so-called Inert Doublet Models (IDM) [15–19]. The 2HDM and IDM models have been extensively analysed and result in some interesting properties. There is no direct restriction on the amount of additional $SU(2)$ doublets. Two $SU(2)$ doublets can be combined with the SM-like one. Such extension results in a Three-Higgs-Doublet Model (3HDM). This model incorporates a rich spectrum of additional particles: the SM-like Higgs boson h_{SM} along with other neutral scalars $h_{1..4}$, and two additional charged scalars $h^{\pm}_{1,2}$ are present. Some of these particles can decouple from the visible matter, either a charged and two neutral scalars or two charged and four neutral states, see Refs. [20–23] for the current research on the 3HDM DM. The SM $SU(2)$ doublet can also be extended by other structures, *e.g.*, a singlet can be added. The singlet extended model is also capable of describing the DM [24–27]. Nevertheless, all these models are highly constrained by the $\rho \approx 1$ parameter [28, 29], which depends on the hypercharge of the Higgs structures.

Any model must specify interactions between the particles. In terms of the SM extended scalar sector, of interest are interactions between scalars and fermions, and scalars and gauge bosons. Of particular interest is the Yukawa Lagrangian. The extended scalar sector should be able to incorporate the experimental results: masses of the fermions, the Cabibbo–Kobayashi–Maskawa (CKM) matrix, and the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix. Generic Yukawa couplings might result in unacceptably large Flavour-Changing Neutral Currents (FCNC), which are not observed. Properties of the recently observed scalar particle are in agreement with those of the SM Higgs boson. Therefore, any non SM-like couplings involving the observed scalar are strongly constrained. The discovery of the Higgs boson motivates to constrain the extended scalar sector in a way that the SM-like Higgs boson couplings are in agreement.

We are interested in the 3HDM extension. Historically, the inspiration behind the 3HDM were three generations of fermions. The most general 3HDM scalar potential results in 54 real free parameters [30]. It is a tough task to analyse such models, especially since a lot of freedom is introduced. Thus, some additional symmetries may be imposed. A possible classification of the 3HDM based on additional symmetries was presented in Ref. [31]. One of the possibilities is to impose the discrete S_3 symmetry [32,33]. The S_3-symmetric scalar potential have been classified in terms of the minimization conditions and vacuum configurations in Ref. [34]. The thesis is based on the classification of the aforementioned article.

The thesis is organised as follows. In chapter 2 the general S_3-symmetric potential is presented and some of the properties of the potential are considered. In chapter 3 a specific complex vacuum configuration C-III-c is discussed, however, this configuration results in unrealistic states and the symmetry is softly broken. In the following chapter 4, one of the softly broken models C-III-c-ν^2 is further on analysed with tree-level couplings and constraints presented. Chapter 5 is devoted to yet another model, but this time the real vacuum configuration R-II-1a is considered. The two aforementioned vacuum configuration are numerically analysed in chapter 6 to answer the question if those contain possible DM candidates. Some technicalities are addressed in appendices: in Appendix A the scalar potential in different notations is presented, in Appendix B first derivatives of the scalar potential are considered, and in Appendix C some of the possible transformations between the generic and the Higgs basis can be found.

Chapter 2

The S_3-Symmetric Scalar Potential

In this chapter the basic building block, *i.e.* the S_3-symmetric scalar potential is presented. Mainly, the irreducible representation framework is used. We analyse possible vacuum configurations and try to verify consistency of vacuum configurations with those presented in Ref. [34]. In Ref. [34] it was mentioned that there exists a special direction of the scalar potential, when the $\lambda_4 = 0$ constraint is applied, which results in a continuous $SO(2)$ symmetry. We, however, considered the mass-squared matrices of all of the possible vacuum configurations and found additional hidden symmetries, see section 2.3. Other interesting properties of the scalar potential are also considered in this chapter.

2.1 The Scalar Potential

S_3 is a non-Abelian group and is the permutation group of three objects, in this case permutation of the three Higgs doublets $\{\phi_1,\ \phi_2,\ \phi_3\}$, where

$$\phi_i = \begin{pmatrix} \varphi_i^+ \\ \frac{1}{\sqrt{2}}\left(\rho_i + \eta_i + i\chi_i\right) \end{pmatrix}, \text{ for } i = \overline{1,3}, \tag{2.1.1}$$

where φ_i^+ is a complex field, and $\tilde{\eta}_i$ and $\tilde{\chi}_i$ are real fields, and, in general, the vacuum ρ_i is a complex value.

S_3 has two ^{1}D irreducible representations $\mathbf{1}_\mathrm{S}$ and $\mathbf{1}_\mathrm{A}$, and a ^{2}D doublet irreducible representation $\mathbf{2}$. We chose the following ^{3}D representation:

$$\begin{aligned} &\mathbf{2:} \qquad \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\phi_1 - \phi_2\right) \\ \frac{1}{\sqrt{6}}\left(\phi_1 + \phi_2 - 2\phi_3\right) \end{pmatrix}, \\ &\mathbf{1:} \qquad h_S = \frac{1}{\sqrt{3}}\left(\phi_1 + \phi_2 + \phi_3\right), \end{aligned} \tag{2.1.2}$$

where S in h_S indicates that it is an S_3 singlet.

In the new basis, the h_i fields are defined in the following way:

$$\begin{aligned} h_i &= \begin{pmatrix} h_i^+ \\ \frac{1}{\sqrt{2}}\left(w_i + \tilde{\eta}_i + i\tilde{\chi}_i\right) \end{pmatrix}, \quad i = \overline{1,2}, \\ h_S &= \begin{pmatrix} h_S^+ \\ \frac{1}{\sqrt{2}}\left(w_S + \tilde{\eta}_S + i\tilde{\chi}_S\right) \end{pmatrix}, \end{aligned} \tag{2.1.3}$$

where the vacuum values w_i and w_S can be complex.

In both representations the Vacuum Expectation Values (VEV) should satisfy constraint:

$$v = \sqrt{\rho_1^2 + \rho_2^2 + \rho_3^2} = \sqrt{w_1^2 + w_2^2 + w_S^2} \simeq 246 \text{ GeV}. \tag{2.1.4}$$

Transformation from one basis (2.1.1) to another (2.1.3) is given by the following S_3 ^{3}D representation:

$$\begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{2.1.5}$$

or equivalently:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix}. \tag{2.1.6}$$

The most general renormalizable $S_3 \otimes U(1)$ scalar potential can be written as [32,35–37]:

$$\begin{aligned} V =& V_2 + V_4, \\ V_2 =& \mu_1^2 \left(h_1^\dagger h_1 + h_2^\dagger h_2\right) + \mu_0^2 h_S^\dagger h_S, \\ V_4 =& \lambda_1 \left(h_1^\dagger h_1 + h_2^\dagger h_2\right)^2 + \lambda_2 \left(h_1^\dagger h_2 - h_2^\dagger h_1\right)^2 + \lambda_3 \left[\left(h_1^\dagger h_1 - h_2^\dagger h_2\right)^2 + \left(h_1^\dagger h_2 + h_2^\dagger h_1\right)^2\right] \\ &+ \lambda_4 \left[\left(h_S^\dagger h_1\right)\left(h_1^\dagger h_2 + h_2^\dagger h_1\right) + \left(h_S^\dagger h_2\right)\left(h_1^\dagger h_1 - h_2^\dagger h_2\right) + \text{h.c.}\right] \\ &+ \lambda_5 \left[\left(h_S^\dagger h_S\right)\left(h_1^\dagger h_1 + h_2^\dagger h_2\right)\right] + \lambda_6 \left[\left(h_1^\dagger h_S\right)\left(h_S^\dagger h_1\right) + \left(h_2^\dagger h_S\right)\left(h_S^\dagger h_2\right)\right] \\ &+ \lambda_7 \left[\left(h_S^\dagger h_1\right)^2 + \left(h_S^\dagger h_2\right)^2 + \text{h.c.}\right] + \lambda_8 \left(h_S^\dagger h_S\right)^2. \end{aligned} \tag{2.1.7}$$

Couplings μ and λ are assumed to be real provided that one is interested in the case when the Charge-Parity (CP) symmetry is not broken explicitly. Another possible way of writing down the scalar potential was presented by Derman [33,38], which is covered in Appendix A. We also present the S_3 scalar potential in matrix form in Appendix A.

2.2 Two Possible Choices: Real and Complex Vacua

The unique characteristic of different models are the vacuum configurations; different configurations result in different minimization conditions. Not all of the first derivatives of the potential result in a unique set. We are interested in defining different vacuum configurations and thus a way around is to consider derivatives with respect to VEVs, which are all independent. A more thorough guideline can be found in Ref. [34] in sections 3 to 5. If a vacuum configuration to be analysed is known and some of the VEVs vanish, the set of derivatives no longer spans the full set of minimization conditions. This is true if the VEVs are substituted before differentiating the scalar potential with respect to the fields. We take a look at all possible first derivatives of the potential with respect to different fields and VEVs. Derivatives are written down in Appendix B.

2.2.1 Real Vacua

First of all, we review possible real vacuum configurations. Those do not violate CP spontaneously. For convenience, one can verify minimization conditions by considering the following derivatives:

$$\left.\frac{\partial V}{\partial w_1}\right|_{\langle v\rangle}, \quad \left.\frac{\partial V}{\partial w_2}\right|_{\langle v\rangle}, \quad \left.\frac{\partial V}{\partial w_S}\right|_{\langle v\rangle}, \tag{2.2.1}$$

where $\langle v\rangle$ indicates that all of the fields are set to zero. The minimization conditions lead to the following relations[1]:

$$\mu_1^2 = -\left(\lambda_1 + \lambda_3\right)\left(w_1^2 + w_2^2\right) - \frac{1}{2} w_S \left[6\lambda_4 w_2 + \left(\lambda_5 + \lambda_6 + 2\lambda_7\right) w_S\right], \tag{2.2.2a}$$

[1] We consider that the minimization conditions are satisfied exactly and not in the limit.

$$\mu_1^2 = -\left(\lambda_1+\lambda_3\right)\left(w_1^2+w_2^2\right) - \frac{1}{2}\frac{w_S}{w_2}\left[3\lambda_4\left(w_1^2-w_2^2\right)+\left(\lambda_5+\lambda_6+2\lambda_7\right)w_2 w_S\right], \tag{2.2.2b}$$

$$\mu_0^2 = \frac{1}{2w_S}\left[\lambda_4\left(-3w_1^2 w_2+w_2^3\right) - w_S\left(\left(\lambda_5+\lambda_6+2\lambda_7\right)\left(w_1^2+w_2^2\right)+2\lambda_8 w_S^2\right)\right], \tag{2.2.2c}$$

It should be noted that additional conditions should be taken into account as eq. (2.2.2a) and eq. (2.2.2b) are not consistent for all possible values. For self-consistency the following relation should be satisfied:

$$\lambda_4 w_S\left(w_1^2-3w_2^2\right)=0. \tag{2.2.3}$$

As a result, possible solutions are $\lambda_4=0$ or $w_1=\pm\sqrt{3}w_2$, or $w_S=0$. An interesting case is that one of the solutions involves $w_S=0$. We take a closer look at what happens if we consider that one or several VEVs can acquire zero value.

If one of the VEVs acquires a zero value, the corresponding derivative vanishes automatically. For $w_1=0$, the self-consistency condition (2.2.3) is automatically satisfied as the μ_1^2 coupling is uniquely defined in this case. The choice of $w_2=0$ leads to additional minimization conditions in terms of eq. (2.2.3) as derivative with respect to $\tilde{\eta}_2$ does not vanish. In case of $w_S=0$, the non-vanishing derivative is the one with respect to the field $\tilde{\eta}_S$ and results in:

$$\lambda_4 w_2\left(3w_1^2-w_2^2\right)=0, \tag{2.2.4}$$

although it might seem that $w_S=0$ is a viable solution for eq. (2.2.3). As a result, vacuum configurations with $w_S=0$ should be supplemented by either $\lambda_4=0$, or $w_2=0$, or $w_2=\pm\sqrt{3}w_1$. The only non-trivial case when two of the VEVs acquire zero values is when $w_1=0$ and $w_S=0$. In this case additional condition is given by:

$$\lambda_4 w_2^3=0. \tag{2.2.5}$$

We expand the scalar potential with respect to different conditions. The different vacuum configurations are presented in Table 2.1. We find that all of the real vacuum configurations match those of Ref. [34]. It is worth mentioning that there is a special case of the R-III vacuum configuration $\{w_1,\, 0,\, w_S\}$, which was not mentioned in the original paper.

2.2.2 Complex Vacua

We take a look at a possible case when the vacuum can acquire complex values. We work in the following notation:

$$\{w_1,\, w_2,\, w_S\} \to \{\hat{w}_1 e^{i\sigma_1},\, \hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}, \tag{2.2.6}$$

where the hatted $\hat{w}_i$ value indicates the absolute value and σ_i stands for a phase. Because of the $U(1)_Y$ gauge invariance of the scalar potential it is always possible to rotate one of the phases away. For convenience, we rotated the S_3 singlet phase. We work with the following set of independent derivatives:

$$\left.\frac{\partial V}{\partial \hat{w}_1}\right|_{\langle v\rangle},\quad \left.\frac{\partial V}{\partial \hat{w}_2}\right|_{\langle v\rangle},\quad \left.\frac{\partial V}{\partial \hat{w}_S}\right|_{\langle v\rangle},\quad \left.\frac{\partial V}{\partial \sigma_1}\right|_{\langle v\rangle},\quad \left.\frac{\partial V}{\partial \sigma_2}\right|_{\langle v\rangle}. \tag{2.2.7}$$

Full equations can be found in Appendix B.

We start by analysing the most general case, *i.e.*, of the form of eq. (2.2.6). It leads to several possible vacuum configurations. The self-consistency condition requires that μ_1^2 values should coincide. The μ_1^2 coupling can be defined by taking a look at the derivative of the potential with respect to either $\hat{w}_1$ or $\hat{w}_2$. We can write down the self-consistency condition as:

$$\begin{aligned}
&-2\left(\lambda_2+\lambda_3\right)\hat{w}_2\left(\hat{w}_1^2-\hat{w}_2^2\right)\mathrm{s}^2_{(\sigma_1-\sigma_2)}\\
&+\frac{1}{2}\lambda_4\hat{w}_S\left[\mathrm{c}_{\sigma_2}\left(2\hat{w}_1^2-7\hat{w}_2\right)+\mathrm{c}_{(2\sigma_1-\sigma_2)}\left(\hat{w}_1^2-2\hat{w}_2\right)\right]\\
&-\lambda_7\hat{w}_2\hat{w}_S^2\left(\mathrm{c}_{2\sigma_1}-\mathrm{c}_{2\sigma_2}\right)=0,
\end{aligned} \tag{2.2.8}$$

Vacuum	$\{w_1,\, w_2,\, w_S\}$	Constraints
R-0	$\{0,\, 0,\, 0\}$	None
R-I-1	$\{0,\, 0,\, w_S\}$	$\mu_0^2 = -\lambda_8 w_S^2$
R-I-2a	$\{w,\, 0,\, 0\}$	$\mu_1^2 = -\left(\lambda_1+\lambda_3\right) w_1^2$
R-I-2b	$\{w,\, \sqrt{3}w,\, 0\}$	$\mu_1^2 = -\frac{4}{3}\left(\lambda_1+\lambda_3\right) w_2^2$
R-I-2c	$\{w,\, -\sqrt{3}w,\, 0\}$	$\mu_1^2 = -\frac{4}{3}\left(\lambda_1+\lambda_3\right) w_2^2$
R-II-1a	$\{0,\, w,\, w_S\}$	$\mu_0^2 = \frac{1}{2}\lambda_4 \frac{w_2^3}{w_S} - \frac{1}{2}\left(\lambda_5+\lambda_6+2\lambda_7\right) w_2^2 - \lambda_8 w_S^2,$ $\mu_1^2 = -\left(\lambda_1+\lambda_3\right) w_2^2 + \frac{3}{2}\lambda_4 w_2 w_S - \frac{1}{2}\left(\lambda_5+\lambda_6+2\lambda_7\right) w_S^2$
R-II-1b	$\{w,\, -w/\sqrt{3},\, w_S\}$	$\mu_0^2 = -4\lambda_4 \frac{w_2^3}{w_S} - 2\left(\lambda_5+\lambda_6+2\lambda_7\right) w_2^2 - \lambda_8 w_S^2,$ $\mu_1^2 = -4\left(\lambda_1+\lambda_3\right) w_2^2 - 3\lambda_4 w_2 w_S - \frac{1}{2}\left(\lambda_5+\lambda_6+2\lambda_7\right) w_S^2$
R-II-1c	$\{w,\, w/\sqrt{3},\, w_S\}$	$\mu_0^2 = -4\lambda_4 \frac{w_2^3}{w_S} - 2\left(\lambda_5+\lambda_6+2\lambda_7\right) w_2^2 - \lambda_8 w_S^2,$ $\mu_1^2 = -4\left(\lambda_1+\lambda_3\right) w_2^2 - 3\lambda_4 w_2 w_S - \frac{1}{2}\left(\lambda_5+\lambda_6+2\lambda_7\right) w_S^2$
R-II-2	$\{0,\, w,\, 0\}$	$\mu_1^2 = -\left(\lambda_1+\lambda_3\right) w_2^2,$ $\lambda_4 = 0$
R-II-3	$\{w_1,\, w_2,\, 0\}$	$\mu_1^2 = -\left(\lambda_1+\lambda_3\right)\left(w_1^2+w_2^2\right),$ $\lambda_4 = 0$
R-III	$\{w_1,\, w_2,\, w_S\}$	$\mu_0^2 = -\frac{1}{2}\left(\lambda_5+\lambda_6+2\lambda_7\right)\left(w_1^2+w_2^2\right) - \lambda_8 w_S^2,$ $\mu_1^2 = -\left(\lambda_1+\lambda_3\right)\left(w_1^2+w_2^2\right) - \frac{1}{2}\left(\lambda_5+\lambda_6+2\lambda_7\right) w_S^2,$ $\lambda_4 = 0$

Table 2.1: Possible real vacuum configurations. The classification is based on and adopts the notation of Ref. [34]. R stands for real vacuum configuration. The Roman numeral shows the total number of constraints. The last combination of numeral and letter is used to distinguish different configurations in the same category.

where we used the following symbols to denote trigonometric functions: $s_\xi \equiv \sin\xi$, $c_\xi \equiv \cos\xi$ and $t_\xi \equiv \tan\xi$.

Additional constraints arise from derivatives with respect to σ_1 and σ_2 due to the fact that those do not depend on the quadratic couplings μ_0^2 and μ_1^2:

$$-\left(\lambda_2+\lambda_3\right)\hat{w}_1^2\hat{w}_2^2 \mathrm{s}_{(2\sigma_1-2\sigma_2)} - \lambda_4\hat{w}_1^2\hat{w}_2\hat{w}_S \mathrm{s}_{(2\sigma_1-\sigma_2)} - \lambda_7\hat{w}_1^2\hat{w}_S^2 \mathrm{s}_{2\sigma_1} = 0, \quad (2.2.9a)$$

$$\left(\lambda_2+\lambda_3\right)\hat{w}_1^2\hat{w}_2^2 \mathrm{s}_{(2\sigma_1-2\sigma_2)} + \frac{1}{2}\lambda_4\hat{w}_2\hat{w}_S\left[\hat{w}_1^2 \mathrm{s}_{(2\sigma_1-\sigma_2)} - \left(2\hat{w}_1^2-\hat{w}_2^2\right)\mathrm{s}_{\sigma_2}\right] - \lambda_7\hat{w}_2^2\hat{w}_S^2 \mathrm{s}_{2\sigma_2} = 0. \quad (2.2.9b)$$

By taking a look at all of the possible cases in Table 2.2 we verify that our solutions coincide with the ones presented in Ref. [34], although some additional explanation is needed.

Vacuum	$\{w_1,\, w_2,\, w_S\}$	Constraints
C-I-a	$\{\hat{w}_1,\, \pm i\hat{w}_1,\, 0\}$	$\mu_1^2 = -2\left(\lambda_1-\lambda_2\right)\hat{w}_1^2$
C-III-a	$\{0,\, \hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$	$\mu_0^2 = -\frac{1}{2}\left(\lambda_5+\lambda_6-2\lambda_7\right)\hat{w}_2^2 - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1+\lambda_3\right)\hat{w}_2^2 - \frac{1}{2}\left(\lambda_5+\lambda_6-2\lambda_7-8\mathrm{c}_{\sigma_2}^2\lambda_7\right)\hat{w}_S^2,$ $\lambda_4 = \frac{4\mathrm{c}_{\sigma_2}\hat{w}_S}{\hat{w}_2}\lambda_7$

C-III-b	$\{\pm i\hat{w}_1,\, 0,\, \hat{w}_S\}$	$\mu_0^2 = -\frac{1}{2}\left(\lambda_5+\lambda_6-2\lambda_7\right)\hat{w}_1^2 - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1+\lambda_3\right)\hat{w}_1^2 - \frac{1}{2}\left(\lambda_5+\lambda_6-2\lambda_7\right)\hat{w}_S^2,$ $\lambda_4 = 0$
C-III-c	$\{\hat{w}_1 e^{i\sigma_1},\, \hat{w}_2 e^{i\sigma_2},\, 0\}$	$\mu_1^2 = -\left(\lambda_1+\lambda_3\right)\left(\hat{w}_1^2+\hat{w}_2^2\right),$ $\lambda_2+\lambda_3 = 0,$ $\lambda_4 = 0$
C-III-d	$\{\pm i\hat{w}_1,\, \hat{w}_2,\, \hat{w}_S\}$	$\mu_0^2 = \left(\lambda_2+\lambda_3\right)\frac{\left(\hat{w}_1^2-\hat{w}_2^2\right)^2}{\hat{w}_S^2} - \frac{\left(\hat{w}_1^2-\hat{w}_2^2\right)\left(\hat{w}_1^2-3\hat{w}_2^2\right)}{4\hat{w}_2\hat{w}_S}\lambda_4$ $-\frac{1}{2}\left(\lambda_5+\lambda_6\right)\left(\hat{w}_1^2+\hat{w}_2^2\right) - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1-\lambda_2\right)\left(\hat{w}_1^2+\hat{w}_2^2\right) - \frac{\hat{w}_S\left(\hat{w}_1^2-\hat{w}_2^2\right)}{4\hat{w}_2}\lambda_4 - \frac{1}{2}\left(\lambda_5+\lambda_6\right)\hat{w}_S^2,$ $\lambda_7 = \frac{\left(\hat{w}_1^2-\hat{w}_2^2\right)}{\hat{w}_S^2}\left(\lambda_2+\lambda_3\right) - \frac{\left(\hat{w}_1^2-5\hat{w}_2^2\right)}{4\hat{w}_2\hat{w}_S}\lambda_4$
C-III-e	$\{\pm i\hat{w}_1, -\hat{w}_2,\, \hat{w}_S\}$	$\mu_0^2 = \left(\lambda_2+\lambda_3\right)\frac{\left(\hat{w}_1^2-\hat{w}_2^2\right)^2}{\hat{w}_S^2} + \frac{\left(\hat{w}_1^2-\hat{w}_2^2\right)\left(\hat{w}_1^2-3\hat{w}_2^2\right)}{4\hat{w}_2\hat{w}_S}\lambda_4$ $-\frac{1}{2}\left(\lambda_5+\lambda_6\right)\left(\hat{w}_1^2+\hat{w}_2^2\right) - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1-\lambda_2\right)\left(\hat{w}_1^2+\hat{w}_2^2\right) + \frac{\hat{w}_S\left(\hat{w}_1^2-\hat{w}_2^2\right)}{4\hat{w}_2}\lambda_4 - \frac{1}{2}\left(\lambda_5+\lambda_6\right)\hat{w}_S^2,$ $\lambda_7 = \frac{\left(\hat{w}_1^2-\hat{w}_2^2\right)}{\hat{w}_S^2}\left(\lambda_2+\lambda_3\right) + \frac{\left(\hat{w}_1^2-5\hat{w}_2^2\right)}{4\hat{w}_2\hat{w}_S}\lambda_4$
C-III-f	$\{\pm i\hat{w}_1,\, i\hat{w}_2,\, \hat{w}_S\}$	$\mu_0^2 = -\frac{1}{2}\left(\lambda_5+\lambda_6-2\lambda_7\right)\left(\hat{w}_1^2+\hat{w}_2^2\right) - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1+\lambda_3\right)\left(\hat{w}_1^2+\hat{w}_2^2\right) - \frac{1}{2}\left(\lambda_5+\lambda_6-2\lambda_7\right)\hat{w}_S^2,$ $\lambda_4 = 0$
C-III-g	$\{\pm i\hat{w}_1,\, -i\hat{w}_2,\, \hat{w}_S\}$	$\mu_0^2 = -\frac{1}{2}\left(\lambda_5+\lambda_6-2\lambda_7\right)\left(\hat{w}_1^2+\hat{w}_2^2\right) - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1+\lambda_3\right)\left(\hat{w}_1^2+\hat{w}_2^2\right) - \frac{1}{2}\left(\lambda_5+\lambda_6-2\lambda_7\right)\hat{w}_S^2,$ $\lambda_4 = 0$
C-III-h	$\{\sqrt{3}\hat{w}_2 e^{i\sigma_2},\, \pm\hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$	$\mu_0^2 = -2\left(\lambda_5+\lambda_6-2\lambda_7\right)\hat{w}_2^2 - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -4\left(\lambda_1+\lambda_3\right)\hat{w}_2^2 - \frac{1}{2}\left(\lambda_5+\lambda_6-2\lambda_7-8\mathrm{c}_{\sigma_2}^2\lambda_7\right)\hat{w}_S^2,$ $\lambda_4 = \mp\frac{2\mathrm{c}_{\sigma_2}\hat{w}_S}{\hat{w}_2}\lambda_7$
C-III-i	$\{\sqrt{\frac{3(1+\mathrm{t}_{\sigma_1}^2)}{1+9\mathrm{t}_{\sigma_1}^2}}\hat{w}_2 e^{i\sigma_1},$ $\pm\hat{w}_2 e^{-i\arctan\left(3\mathrm{t}_{\sigma_1}\right)},\, \hat{w}_S\}$	$\mu_0^2 = \frac{16\left(1-3\mathrm{t}_{\sigma_1}^2\right)^2}{\left(1+9\mathrm{t}_{\sigma_1}^2\right)^2}\left(\lambda_2+\lambda_3\right)\frac{\hat{w}_2^4}{\hat{w}_S^2} \pm \frac{6\left(1-\mathrm{t}_{\sigma_1}^2\right)\left(1-3\mathrm{t}_{\sigma_1}^2\right)}{\left(1+9\mathrm{t}_{\sigma_1}^2\right)^{3/2}}\lambda_4\frac{\hat{w}_2^3}{\hat{w}_S}$ $-\frac{2\left(1+3\mathrm{t}_{\sigma_1}^2\right)}{\left(1+9\mathrm{t}_{\sigma_1}^2\right)}\left(\lambda_5+\lambda_6\right)\hat{w}_2^2 - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\frac{4\left(1+3\mathrm{t}_{\sigma_1}^2\right)}{\left(1+9\mathrm{t}_{\sigma_1}^2\right)}\left(\lambda_1-\lambda_2\right)\hat{w}_2^2 \mp \frac{\left(1-3\mathrm{t}_{\sigma_1}^2\right)}{2\sqrt{1+9\mathrm{t}_{\sigma_1}^2}}\lambda_4\hat{w}_2\hat{w}_S - \frac{1}{2}\left(\lambda_5+\lambda_6\right)\hat{w}_S^2,$ $\lambda_7 = -\frac{4\left(1-3\mathrm{t}_{\sigma_1}^2\right)}{\left(1+9\mathrm{t}_{\sigma_1}^2\right)}\left(\lambda_2+\lambda_3\right)\frac{\hat{w}_2^2}{\hat{w}_S^2} \mp \frac{\left(5-3\mathrm{t}_{\sigma_1}^2\right)}{2\sqrt{1+9\mathrm{t}_{\sigma_1}^2}}\lambda_4\frac{\hat{w}_2}{\hat{w}_S}$

C-IV-a	$\{\hat{w}_1 e^{i\sigma_1},\, 0,\, \hat{w}_S\}$	$\mu_0^2 = -\frac{1}{2}\left(\lambda_5+\lambda_6\right)\hat{w}_1^2 - \lambda_8 \hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1+\lambda_3\right)\hat{w}_1^2 - \frac{1}{2}\left(\lambda_5+\lambda_6\right)\hat{w}_S^2,$ $\lambda_4 = 0,$ $\lambda_7 = 0$
C-IV-b	$\{\hat{w}_1,\, \pm i\hat{w}_2,\, \hat{w}_S\}$	$\mu_0^2 = \left(\lambda_2+\lambda_3\right)\frac{\left(\hat{w}_1^2-\hat{w}_2^2\right)^2}{\hat{w}_S^2} - \frac{1}{2}\left(\lambda_5+\lambda_6\right)\left(\hat{w}_1^2+\hat{w}_2^2\right) - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1-\lambda_2\right)\left(\hat{w}_1^2+\hat{w}_2^2\right) - \frac{1}{2}\left(\lambda_5+\lambda_6\right)\hat{w}_S^2,$ $\lambda_4 = 0,$ $\lambda_7 = -\frac{\left(\hat{w}_1^2-\hat{w}_2^2\right)}{\hat{w}_S^2}\left(\lambda_2+\lambda_3\right)$
C-IV-c	$\{\sqrt{1+2\mathrm{c}_{\sigma_2}^2}\,\hat{w}_2,$ $\hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$	$\mu_0^2 = 2\mathrm{c}_{\sigma_2}^2\left(1+\mathrm{c}_{\sigma_2}^2\right)\left(\lambda_2+\lambda_3\right)\frac{\hat{w}_2^4}{\hat{w}_S^2} - \left(1+\mathrm{c}_{\sigma_2}^2\right)\left(\lambda_5+\lambda_6\right)\hat{w}_2^2 - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left[2\left(1+\mathrm{c}_{\sigma_2}^2\right)\lambda_1 - \left(2+3\mathrm{c}_{\sigma_2}^2\right)\lambda_2 - \mathrm{c}_{\sigma_2}^2\lambda_3\right]\hat{w}_2^2 - \frac{1}{2}\left(\lambda_5+\lambda_6\right)\hat{w}_S^2,$ $\lambda_4 = -2\mathrm{c}_{\sigma_2}\left(\lambda_2+\lambda_3\right)\frac{\hat{w}_2}{\hat{w}_S},$ $\lambda_7 = \mathrm{c}_{\sigma_2}^2\left(\lambda_2+\lambda_3\right)\frac{\hat{w}_2^2}{\hat{w}_S^2}$
C-IV-d	$\{\hat{w}_1 e^{i\sigma_1},\, \pm\hat{w}_2 e^{i\sigma_1},\, \hat{w}_S\}$	$\mu_0^2 = -\frac{1}{2}\left(\lambda_5+\lambda_6\right)\left(\hat{w}_1^2+\hat{w}_2^2\right) - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1+\lambda_3\right)\left(\hat{w}_1^2+\hat{w}_2^2\right) - \frac{1}{2}\left(\lambda_5+\lambda_6\right)\hat{w}_S^2,$ $\lambda_4 = 0,$ $\lambda_7 = 0$
C-IV-e	$\{\sqrt{-\frac{\mathrm{s}_{2\sigma_2}}{\mathrm{s}_{2\sigma_1}}}\,\hat{w}_2 e^{i\sigma_1},$ $\hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$	$\mu_0^2 = \frac{\mathrm{s}_{2(\sigma_1-\sigma_2)}^2}{\mathrm{s}_{2\sigma_1}^2}\left(\lambda_2+\lambda_3\right)\frac{\hat{w}_2^4}{\hat{w}_S^2} - \frac{1}{2}\left(1-\frac{\mathrm{s}_{2\sigma_2}}{\mathrm{s}_{2\sigma_1}}\right)\left(\lambda_5+\lambda_6\right)\hat{w}_2^2 - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left(1-\frac{\mathrm{s}_{2\sigma_2}}{\mathrm{s}_{2\sigma_1}}\right)\left(\lambda_1-\lambda_2\right)\hat{w}_2^2 - \frac{1}{2}\left(\lambda_5+\lambda_6\right)\hat{w}_S^2,$ $\lambda_4 = 0,$ $\lambda_7 = -\frac{\mathrm{s}_{2(\sigma_1-\sigma_2)}}{\mathrm{s}_{2\sigma_1}}\left(\lambda_2+\lambda_3\right)\frac{\hat{w}_2^2}{\hat{w}_S^2}$
C-IV-f	$\{\sqrt{2+\frac{\mathrm{c}_{(\sigma_1-2\sigma_2)}}{\mathrm{c}_{\sigma_1}}}\,\hat{w}_2 e^{i\sigma_1},$ $\hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$	$\mu_0^2 = -\frac{\left(\mathrm{c}_{(\sigma_1-2\sigma_2)}+3\mathrm{c}_{\sigma_1}\right)\mathrm{c}_{(\sigma_2-\sigma_1)}}{2\mathrm{c}_{\sigma_1}^2}\lambda_4\frac{\hat{w}_2^3}{\hat{w}_S}$ $-\frac{\mathrm{c}_{(\sigma_1-2\sigma_2)}+3\mathrm{c}_{\sigma_1}}{2\mathrm{c}_{\sigma_1}}\left(\lambda_5+\lambda_6\right)\hat{w}_2^2 - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\frac{\mathrm{c}_{(\sigma_1-2\sigma_2)}+3\mathrm{c}_{\sigma_1}}{\mathrm{c}_{\sigma_1}}\left(\lambda_1+\lambda_3\right)\hat{w}_2^2$ $-\frac{3\mathrm{c}_{2\sigma_1}+2\mathrm{c}_{2(\sigma_1-\sigma_2)}+\mathrm{c}_{2\sigma_2}+4}{4\mathrm{c}_{(\sigma_1-\sigma_2)}\mathrm{c}_{\sigma_1}}\lambda_4\hat{w}_2\hat{w}_S - \frac{1}{2}\left(\lambda_5+\lambda_6\right)\hat{w}_S^2,$ $\lambda_4 = -2\frac{\mathrm{c}_{\sigma_2-\sigma_1}}{\mathrm{c}_{\sigma_1}}\left(\lambda_2+\lambda_3\right)\frac{\hat{w}_2}{\hat{w}_S},$ $\lambda_7 = \frac{\mathrm{c}_{\sigma_2-\sigma_1}^2}{\mathrm{c}_{\sigma_1}^2}\left(\lambda_2+\lambda_3\right)\frac{\hat{w}_2^2}{\hat{w}_S^2}$
C-V	$\{\hat{w}_1 e^{i\sigma_1},\, \hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$	$\mu_0^2 = -\frac{1}{2}\left(\lambda_5+\lambda_6\right)\left(\hat{w}_1^2+\hat{w}_2^2\right) - \lambda_8\hat{w}_S^2,$ $\mu_1^2 = -\left(\lambda_1+\lambda_3\right)\left(\hat{w}_1^2+\hat{w}_2^2\right) - \frac{1}{2}\left(\lambda_5+\lambda_6\right)\hat{w}_S^2,$ $\lambda_2+\lambda_3 = 0,$ $\lambda_4 = 0,$ $\lambda_7 = 0$

Table 2.2: Possible complex vacuum configurations. The classification is based on and adopts the notation of Ref. [34]. C stands for complex vacuum configuration. The Roman numeral shows the total number of constraints. The last combination of numeral and letter is used to distinguish different configurations in the same category.

We note that not all of the possible vacuum configurations and constraints can be applied directly to minimize the scalar potential *a priori*. If we take into consideration vacuum configurations with additional constraints in terms of λ couplings, some caution is required. If one of the couplings depends explicitly on another, *e.g.*, in case of the C-III-a vacuum configuration one of the minimization constraints is:

$$\lambda_4 = \frac{4c_{\sigma_2}\hat{w}_S}{\hat{w}_2}\lambda_7, \tag{2.2.10}$$

such constraints cannot be applied to the potential before differentiating it. If one changed the form of the scalar potential before differentiating it, that would potentially lead to additional terms. These additional terms potentially change the scalar potential structure. This is not the case when $\lambda_2 + \lambda_3 = 0$.

2.3 Identifying the Goldstone States

We proceed to a general check of hidden symmetries of vacuum configurations presented in Table 2.1 and Table 2.2. We try to uncover hidden continuous symmetries of the scalar potential to identify additional Goldstone bosons. These Goldstone bosons are distinct from the longitudinal polarization components of the W and Z. The Goldstone bosons which are "eaten" by the three gauge bosons are called the would-be Goldstone bosons. Therefore we identify additional massless states as those which do not coincide with the would-be Goldstone bosons.

One way to determine if there is at least one additional massless state is to consider the determinant of the mass-squared matrix after identifying the would-be Goldstone bosons. If the determinant results in zero, this indicates that some of the scalar states are massless. We are, however, interested in the total number of massless states, and, in principle, we are not interested in fields expressed in terms of the mass-eigenstates and therefore adopt a more straightforward scheme. We found that exact S_3-symmetric models discussed in section 2.2 result in only neutral massless states. The additional charged massless scalars would result in a decent amount of issues, especially if additional charged massless scalars, not the would-be Goldstone bosons, coupled to photons.

The most general neutral mass-squared matrix $\mathcal{M}^0_{6\times 6}$ is of dim $= 6$. The determinant of $\mathcal{M}^0_{6\times 6}$ is a product of all eigenvalues:

$$\det\left(\mathcal{M}^0_{6\times 6}\right) = \prod_{i=1}^{6} m^2_{H_i}, \tag{2.3.1}$$

where $m^2_{H_i}$ are the corresponding masses squared of the scalar fields. The determinant $\det\left(\mathcal{M}^0_{6\times 6}\right)$ obviously results in zero as there is at least one would-be Goldstone boson present. The determinant can be used to find the eigenvalues of the mass-squared matrix by solving the characteristic equation:

$$\det\left(\mathcal{M}^0_{6\times 6} - \lambda\mathcal{I}_6\right) = 0, \tag{2.3.2}$$

where $\mathcal{I}_n$ is the identity matrix of dimension n. We are only interested in determining the total number of massless states and are not interested in explicit analytic expressions. The way around is to consider:

$$\begin{aligned}\det\left(\hat{\mathcal{M}}^0_{6\times 6} - \lambda\mathcal{I}_6\right) &= \left(m^2_{H_1} - \lambda\right)\left(m^2_{H_2} - \lambda\right) \ldots \left(m^2_{H_6} - \lambda\right) \\ &= \sum_{i=0}^{6}(-1)^i\lambda^i c_i,\end{aligned} \tag{2.3.3}$$

where $\hat{\mathcal{M}}^0_{6\times6}$ is a diagonalized mass-squared matrix and c_i are exponential Bell polynomials which satisfy the Cayley-Hamiltonian theorem [39, 40]:

$$c_0 = \det\left(\hat{\mathcal{M}}^0_{6\times6}\right), \tag{2.3.4a}$$

$$\begin{aligned} c_1 = \frac{1}{120}\Bigg[&\mathrm{tr}\left(\hat{\mathcal{M}}^0_{6\times6}\right)^5 - 10\,\mathrm{tr}\left(\hat{\mathcal{M}}^0_{6\times6}\right)^3 \mathrm{tr}\left((\hat{\mathcal{M}}^0_{6\times6})^2\right) + 15\,\mathrm{tr}\left(\hat{\mathcal{M}}^0_{6\times6}\right)\mathrm{tr}\left((\hat{\mathcal{M}}^0_{6\times6})^2\right)^2 \\ &+ 20\,\mathrm{tr}\left(\hat{\mathcal{M}}^0_{6\times6}\right)^2 \mathrm{tr}\left((\hat{\mathcal{M}}^0_{6\times6})^3\right) - 20\,\mathrm{tr}\left((\hat{\mathcal{M}}^0_{6\times6})^2\right)\mathrm{tr}\left((\hat{\mathcal{M}}^0_{6\times6})^3\right) \\ &- 30\,\mathrm{tr}\left(\hat{\mathcal{M}}^0_{6\times6}\right)\mathrm{tr}\left((\hat{\mathcal{M}}^0_{6\times6})^4\right) + 24\,\mathrm{tr}\left((\hat{\mathcal{M}}^0_{6\times6})^5\right)\Bigg], \end{aligned} \tag{2.3.4b}$$

$$\begin{aligned} c_2 = \frac{1}{24}\Bigg[&\mathrm{tr}\left(\hat{\mathcal{M}}^0_{6\times6}\right)^4 - 6\,\mathrm{tr}\left(\hat{\mathcal{M}}^0_{6\times6}\right)^2 \mathrm{tr}\left((\hat{\mathcal{M}}^0_{6\times6})^2\right) + 3\,\mathrm{tr}\left((\hat{\mathcal{M}}^0_{6\times6})^2\right)^2 \\ &+8\,\mathrm{tr}\left(\hat{\mathcal{M}}^0_{6\times6}\right)\mathrm{tr}\left((\hat{\mathcal{M}}^0_{6\times6})^3\right) - 6\,\mathrm{tr}\left((\hat{\mathcal{M}}^0_{6\times6})^4\right)\Bigg], \end{aligned} \tag{2.3.4c}$$

$$c_3 = \frac{1}{6}\left[\mathrm{tr}\left(\hat{\mathcal{M}}^0_{6\times6}\right)^3 - 3\,\mathrm{tr}\left(\hat{\mathcal{M}}^0_{6\times6}\right)\mathrm{tr}\left((\hat{\mathcal{M}}^0_{6\times6})^2\right) + 2\,\mathrm{tr}\left((\hat{\mathcal{M}}^0_{6\times6})^3\right)\right], \tag{2.3.4d}$$

$$c_4 = \frac{1}{2}\left[\mathrm{tr}\left(\hat{\mathcal{M}}^0_{6\times6}\right)^2 - \mathrm{tr}\left((\hat{\mathcal{M}}^0_{6\times6})^2\right)\right], \tag{2.3.4e}$$

$$c_5 = \mathrm{tr}\left(\hat{\mathcal{M}}^0_{6\times6}\right), \tag{2.3.4f}$$

$$c_6 = 1. \tag{2.3.4g}$$

Of particular interest are coefficients $c_{1..5}$ as the highest order polynomial among these, which satisfies $c_j = 0$, indicates that there are exactly j additional massless states. The coefficient c_0 is zero due to the would-be Goldstone boson and c_6 is of no particular interest as it does not depend on the mass-squared parameters. Therefore coefficients $c_{1..5}$ are checked.

The mass eigenvalues $\hat{\mathcal{M}}^0_{6\times6}$ are, in general, not known and therefore the following identity can be used:

$$\det\left(\mathcal{M}^0_{6\times6} - \lambda\mathcal{I}_6\right) = \det\left(\hat{\mathcal{M}}^0_{6\times6} - \lambda\mathcal{I}_6\right). \tag{2.3.5}$$

If a specific vacuum configuration results in at least two massless states, *i.e.*, $c_1 = 0$, the corresponding model is further analysed. Vacuum configurations, with at least $c_1 = 0$, are presented in Table 2.3.

The lowest non-vanishing $c_j \neq 0$ can be factorized to determine the number of mass-degenerate states. The power n, to which the $\mathcal{M}^0_{6\times6}$ elements of the c_j polynomial are raised, indicates that there are in total n mass-degenerate states. We are not interested in such states and therefore the general result is only presented in Table 2.5 without further discussion.

An interesting observation is that there seems to be some sort of correlation between the number of constraints in terms of λ_i and additional massless states. Moreover every vacuum configuration with at least one minimization condition given by $\lambda_i = 0$ results in massless states. One of the possible explanations is to take a look at hidden symmetries of the potential after applying the minimization conditions in terms of λ_i and relate those to the Goldstone theorem [41–43].

Consider that the hidden symmetry, at most, results in a $U(3)$ transformation U. Assume that the scalar potential transforms under the unitary transformation as $V \xrightarrow{U} V'$ and the $SU(2)$ doublets transform as $h_i' = U_{ij}h_j$ [2]. The scalar potential couplings μ_i^2 and λ_j are left intact after performing the aforementioned transformation U. In other words, the transformation U should leave the scalar potential invariant after changing back to the original basis $h_i' \to h_j$, so that $V' - V = 0$. This is not a property of the most general transformation U of the S_3-symmetric potential V (2.1.7).

An obvious question is which of the $SU(2)$ doublet combinations are invariant under U and if the most general form of the transformation matrix U can be simplified. The simplest way

[2]The S_3 singlet transforms as $h_S' = U_{3j}h_j$.

Vacuum	Vacuum configuration	Additional $m^0 = 0$	λ constraints
R-II-2	$\{0,\, w,\, 0\}$	1	$\lambda_4 = 0$
R-II-3	$\{\hat{w}_1,\, \hat{w}_2,\, 0\}$	1	$\lambda_4 = 0$
R-III	$\{\hat{w}_1,\, \hat{w}_2,\, w_S\}$	1	$\lambda_4 = 0$
C-III-b	$\{\pm i\hat{w}_1,\, 0,\, \hat{w}_S\}$	1	$\lambda_4 = 0$
C-III-c	$\{\hat{w}_1 e^{i\sigma_1},\, \hat{w}_2 e^{i\sigma_2},\, 0\}$	2	$\lambda_2 + \lambda_3 = 0,$ $\lambda_4 = 0$
C-III-f	$\{\pm i\hat{w}_1,\, i\hat{w}_2,\, \hat{w}_S\}$	1	$\lambda_4 = 0$
C-III-g	$\{\pm i\hat{w}_1,\, -i\hat{w}_2,\, \hat{w}_S\}$	1	$\lambda_4 = 0$
C-IV-a	$\{\hat{w}_1 e^{i\sigma_1},\, 0,\, \hat{w}_S\}$	2	$\lambda_4 = 0,$ $\lambda_7 = 0$
C-IV-b	$\{\hat{w}_1,\, \pm i\hat{w}_2,\, \hat{w}_S\}$	1	$\lambda_4 = 0$
C-IV-c	$\{\sqrt{1+2\mathrm{c}^2_{\sigma_2}}\hat{w}_2,$ $\hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$	1	$\lambda_4 = -2A\,(\lambda_2+\lambda_3),$ $\lambda_7 = A^2\,(\lambda_2+\lambda_3)$
C-IV-d	$\{\hat{w}_1 e^{i\sigma_1},\, \pm\hat{w}_2 e^{i\sigma_1},\, \hat{w}_S\}$	2	$\lambda_4 = 0,$ $\lambda_7 = 0$
C-IV-e	$\{\sqrt{-\frac{\mathrm{s}_{2\sigma_2}}{\mathrm{s}_{2\sigma_1}}}\hat{w}_2 e^{i\sigma_1},$ $\hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$	1	$\lambda_4 = 0$
C-IV-f	$\{\sqrt{2+\frac{\mathrm{c}_{(\sigma_1-2\sigma_2)}}{\mathrm{c}_{\sigma_1}}}\hat{w}_2 e^{i\sigma_1},$ $\hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$	1	$\lambda_4 = -2A\,(\lambda_2+\lambda_3),$ $\lambda_7 = A^2\,(\lambda_2+\lambda_3)$
C-V	$\{\hat{w}_1 e^{i\sigma_1},\, \hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$	3	$\lambda_2 + \lambda_3 = 0,$ $\lambda_4 = 0,$ $\lambda_7 = 0$

Table 2.3: Vacuum configurations with additional neutral massless states $m^0 = 0$ not counting the would-be Goldstone bosons. The coefficient A for the C-IV-c and C-IV-f cases is expressed in eq. (2.3.17).

to determine the form of U is to consider transformation of the quadratic terms V_2 of the scalar potential:

$$\mu_0^2 h_S^\dagger h_S \xrightarrow{U} \mu_0^2 (h_S^\prime)^\dagger h_S^\prime = \mu_0^2 h_S^\dagger h_S + \mu_0^2 \sum_{\substack{i,j=\{1,2,S\} \\ (i,j)\neq(S,S)}} \mathcal{G}_{ij} h_i^\dagger h_j, \tag{2.3.6a}$$

$$\mu_1^2 \left(h_1^\dagger h_1 + h_2^\dagger h_2\right) \xrightarrow{U} \mu_1^2 \left({h_1^\prime}^\dagger h_1^\prime + {h_2^\prime}^\dagger h_2^\prime\right) = \mu_1^2 \left(h_1^\dagger h_1 + h_2^\dagger h_2\right) + \mu_1^2 \sum_{\substack{i,j=\{1,2,S\} \\ (i,j)\neq(1,1) \\ (i,j)\neq(2,2)}} \mathcal{G}_{ij} h_i^\dagger h_j, \tag{2.3.6b}$$

where $\mathcal{G}_{ij}$ are coefficients which arise due to the U transformation, *e.g.*, $\mathcal{G}_{21} = U_{12}^* U_{11} + U_{22}^* U_{21}$. The quadratic terms remain invariant under the U transformation provided that the off-diagonal coefficients $\mathcal{G}_{ij}$ vanish, and is only true for quadratic terms as those depend on $h_i^\dagger h_i$. Consider transformation of the S_3 singlet h_S under U (2.3.6a). It follows that the unitary matrix U should not mix the S_3 doublet and singlet. If there is no mixing present, eq. (2.3.6b) results in $\mathcal{G}_{ij} = 0$. Thus, the U transformation, which is compatible with our primary assumption that it leaves the scalar potential invariant, is of the form:

$$\begin{pmatrix} h'_1 \\ h'_2 \\ h'_S \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & 0 \\ U_{21} & U_{22} & 0 \\ 0 & 0 & U_{33} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix}, \tag{2.3.7}$$

and the coefficients U_{ij} are such that U is unitary. In principle, this corresponds to the $U(2) \otimes U(1)$ transformation. The non-invariant terms under the U transformation are the ones which are multiplied by the following quartic couplings: $\{\lambda_2,\, \lambda_3,\, \lambda_4,\, \lambda_7\}$. We note that although terms multiplied by λ_2 and λ_3 are not invariant under the U transformation, there is a direction $\lambda_2 + \lambda_3 = 0$ of the scalar potential which is invariant under the $U(2)$ transformation. After substituting $\lambda_3 = -\lambda_2$, the resulting term in the primed basis is:

$$V'_{23} = -\lambda_2 \left[4 \left({h'_1}^\dagger h'_2 \right) \left({h'_2}^\dagger h'_1 \right) + \left({h'_1}^\dagger h'_1 - {h'_2}^\dagger h'_2 \right)^2 \right]. \tag{2.3.8}$$

It follows that a specific combination of the terms

$$\{\lambda_2 + \lambda_3,\, \lambda_4,\, \lambda_7\}, \tag{2.3.9}$$

might result in a non identity transformation U (2.3.7). The scalar potential results in the following dependence between the λ_i coupling of eq. (2.3.9), written in terms of the $SU(2)$ singlets:

$$(\lambda_2 + \lambda_3) \sim \{h_1^\dagger h_1,\, h_2^\dagger h_2,\, h_1^\dagger h_2\} + \text{h.c.} \;, \tag{2.3.10a}$$

$$\lambda_4 \sim \{h_S^\dagger h_1,\, h_S^\dagger h_2,\, h_1^\dagger h_1,\, h_2^\dagger h_2,\, h_1^\dagger h_2\} + \text{h.c.} \;, \tag{2.3.10b}$$

$$\lambda_7 \sim \{h_S^\dagger h_1,\, h_S^\dagger h_2\} + \text{h.c.} \;. \tag{2.3.10c}$$

It can be seen that the $SU(2)$ singlets multiplied by $(\lambda_2 + \lambda_3)$ and λ_7 are contained within the set (2.3.10b). Moreover, from Table 2.3 we see that there is a common feature of massless states associated with the $\lambda_4 = 0$ constraint; this is true with the only exception of vacuum configurations C-IV-c and C-IV-f. It turns out that if $\lambda_4 = 0$, the scalar potential is symmetric under $SO(2)$:

$$U = \begin{pmatrix} \mathrm{c}_{\theta_4} & \mathrm{s}_{\theta_4} & 0 \\ -\mathrm{s}_{\theta_4} & \mathrm{c}_{\theta_4} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.3.11}$$

In total, eight vacuum configurations fall into this category: {R-II-2, R-II-3, R-III, C-III-b, C-III-f, C-III-g, C-IV-b, C-IV-e}.

The constraints $\lambda_4 = 0$ and $\lambda_7 = 0$ result in an additional $U(1)$ symmetry alongside the $SO(2)$:

$$U = \begin{pmatrix} e^{i\theta_7} & 0 & 0 \\ 0 & e^{i\theta_7} & 0 \\ 0 & 0 & e^{i\theta'_7} \end{pmatrix}, \tag{2.3.12}$$

with either $\theta'_7 = 0$ or $\theta_7 = 0$. Due to the $U(1)$ invariance of the scalar potential, it makes little sense to consider both non-zero phases of the S_3 doublet and singlet. The case of $\lambda_4 = 0$ and $\lambda_7 = 0$ corresponds to vacuum configurations: {C-IV-a, C-IV-d}. The vacuum configuration C-IV-a is a special case of C-IV-d for $\hat{w}_2 = 0$. The choice of $\theta'_7 = 0$ leads to the fact that the σ_1 phase of the vacuum configuration C-IV-d $\{\hat{w}_1 e^{i\sigma_1},\, \pm\hat{w}_2 e^{i\sigma_1},\, \hat{w}_S\}$ can be rotated away by setting $\theta_7 = -\sigma_1$. Therefore both C-IV-a and C-IV-d become real.

Finally, when the minimization conditions $\lambda_4 = 0$, and $\lambda_2 + \lambda_3 = 0$, and $\lambda_7 = 0$ are applied, the scalar potential acquires an additional $SU(2)$ symmetry:

$$U = \begin{pmatrix} e^{i\varphi_1}\mathrm{c}_\theta & e^{i\varphi_2}\mathrm{s}_\theta & 0 \\ -e^{-i\varphi_2}\mathrm{s}_\theta & e^{-i\varphi_1}\mathrm{c}_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.3.13}$$

The only vacuum configuration which falls under this category is C-V. It follows that due to the $SU(2)$ transformation, the σ_i phases can be rotated away. As a result, C-V becomes real.

So far we have assumed that there is a unitary transformation U (2.3.7) that leaves the scalar potential invariant. This implies that the VEVs were considered to be independent as the whole $SU(2)$ doublets were transformed. To be more precise, the Goldstone theorem examines a continuous symmetry which is spontaneously broken by the ground state. The aforementioned transformations still hold true as the minimization conditions in terms of λ_i are characteristics of a specific model. The only models with additional Goldstone bosons which were not considered are: {C-III-c, C-IV-c, C-IV-f }.

Before we proceed, we need to define which parameters contribute to the mass-squared matrices. The mass-squared matrix parameters arise from either bilinear terms or quartic terms of the scalar potential:

$$\mathcal{M}^2_{\xi_i^*\xi_j} \propto V_2\left(\xi_i^*\xi_j\right), \tag{2.3.14a}$$

$$\mathcal{M}^2_{\xi_i^*\xi_j} \propto V_4\left(\hat{w}_k\hat{w}_l\xi_i^*\xi_j\right), \tag{2.3.14b}$$

where ξ are the gauge fields of eq. (2.1.3). Of particular interest are the λ terms (2.3.14b) as the self-consistency conditions result in some of them set to zero, see eq. (2.2.3) for the real case and eq. (2.2.8) for the complex case. Consider the R-I-1 model. Vacuum configuration is given by: $\{0,\, 0,\, w_S\}$. From eq. (2.3.14) it follows that the mass-squared terms can be written as:

$$\mathcal{M}^2 = \mu_0^2\mathcal{M}_0^2 + \sum_{i=5}^{8}\lambda_i\mathcal{M}_i^2. \tag{2.3.15}$$

The R-I-1 does not result in any minimization conditions in terms of λ_i and therefore there are no additional Goldstone bosons. On the other hand, the R-II-2 vacuum configuration is given by $\{0,\, w,\, 0\}$, and the model is constrained by $\lambda_4 = 0$. The general mass-squared matrix is given by:

$$\mathcal{M}^2 = \mu_1^2\mathcal{M}_0^2 + \sum_{i=1}^{7}\lambda_i\mathcal{M}_i^2, \tag{2.3.16}$$

and if not for the $\lambda_4 = 0$ constraint, there would have been an additional contribution in terms of the λ_4 coupling. However, the $\lambda_4 = 0$ constraint results in an additional $SO(2)$ symmetry and this leads to the Goldstone boson.

For the C-III-c vacuum configuration we get a continuous $SO(2)$ symmetry due to the minimization condition $\lambda_4 = 0$. This explains why there is one massless state. A closer inspection reveals that there are two massless states and not just one, see Table 2.3. Therefore another group of dimension dim $= 1$ should be broken for the Goldstone theorem to hold. The VEV of the S_3 singlet is given by $\langle h_S\rangle = 0$. This leads to the fact that although $\lambda_7 = 0$ is inconsistent with the C-III-c model, it is the VEV of the S_3 singlet which results in an additional $U(1)$ symmetry, with $U(1)$ given by eq. (2.3.12) with $\theta_7 = 0$. To distinguish this symmetry from the one which holds for the $SU(2)$ doublets transformations, we denote it $U(1)_w$.

In section 3.2 we discuss a possible solution to promote massless scalars to massive ones with soft symmetry breaking. Models with soft symmetry breaking parameters, which are discussed in section 3.2, "eat" the minimization condition $\lambda_2 + \lambda_3 = 0$ and thus the corresponding massless states are promoted to massive ones.

Other interesting cases are the vacuum configurations C-IV-c and C-IV-f. The minimization conditions in terms of quartic couplings $\lambda_4 = -2A\left(\lambda_2 + \lambda_3\right)$ and $\lambda_7 = A^2\left(\lambda_2 + \lambda_3\right)$ are more involved but both of these minimization conditions depend on the sum of couplings $\left(\lambda_2 + \lambda_3\right)$ and a proportionality coefficient:

$$A = \frac{c_{\sigma_2-\sigma_1}\hat{w}_2}{c_{\sigma_1}\hat{w}_S}. \tag{2.3.17}$$

The C-IV-c vacuum configuration is contained within the C-IV-f model in the limit $\sigma_1 = 0$. Therefore, we consider a single coefficient A for both cases. The minimization conditions can be expressed in a slightly different way:

$$\begin{aligned}\lambda_3 &= -\lambda_2 - \frac{1}{2A}\lambda_4, \\ \lambda_7 &= -\frac{A}{2}\lambda_4.\end{aligned} \tag{2.3.18}$$

This way the V'_{23} of eq. (2.3.8) is visible, which is $U(2)$ invariant. The terms which are not $U(2) \otimes U(1)$ invariant are multiplied by the λ_4 coupling. After analysing the expanded scalar potential in terms of the gauge fields and VEVs, it becomes obvious that the vacuum acquires an additional $U(1)_w$ symmetry. This is quite an involved statement but becomes more apparent when three facts are taken into consideration. First of all, both members of the S_3 doublets depend on a single $\hat{w}_2$. Secondly, the minimization conditions (2.3.18) depend on $\frac{\hat{w}_2}{\hat{w}_S}$. Finally, there are two constraint in terms of the λ_i couplings which result in a specific direction of the scalar potential. A more convincing statement is that models C-III-d/e[3], and C-III-h do not result in additional massless states as not all of the aforementioned conditions are satisfied. The minimization conditions for the C-III-d and C-III-e models can be expressed as $\lambda_3 = -\lambda_2 +$ terms with λ_4 and λ_7. Consider the C-III-d model. There is a direction:

$$(\lambda_2 + \lambda_3)\left(\hat{w}_1^2 - \hat{w}_2^2\right) + \lambda_4 \hat{w}_2 \hat{w}_S = 0, \quad \text{not satisfied,} \tag{2.3.19}$$

which results in an additional massless state. Although there exists such direction, it is not fixed by the minimization conditions. The other case is the C-III-h model. This time, an additional massless state would be present if the following constraint would have been satisfied:

$$4\left(\lambda_2 + \lambda_3\right) \mathrm{c}_{\sigma_2}^2 \hat{w}_2^2 - 4\lambda_7 \mathrm{c}_{\sigma_2}^4 \hat{w}_S^2 = 0, \quad \text{not satisfied.} \tag{2.3.20}$$

Therefore this explains why C-III-d, and C-III-e, and C-III-h share some of the properties with C-IV-c and C-IV-f, but only the last two vacuum configurations result in additional massless states. On the other hand, the C-IV-e model resembles the C-IV-f model. However, the C-IV-e model is supplemented by $\lambda_4 = 0$ and results in an additional $SO(2)$ symmetry.

The Goldstone theorem should be applicable to all of the discussed cases in this section. We take a look at which symmetries are broken and if the number of massless states is in agreement with the number of broken generators in Table 2.4.

2.4 Dark Matter Candidates

We are interested in a possible scalar DM candidate and thus take a look at which vacuum configurations might result in a plausible explanation. For the DM study we may[4] impose an additional $\mathbb{Z}_2$ symmetry:

$$\begin{pmatrix} h_{\text{active}} \\ h_{\text{inert}} \end{pmatrix} = \mathrm{diag}\left(1,\, -1\right) \begin{pmatrix} h_i \\ h_j \end{pmatrix}, \tag{2.4.1}$$

under which at least one of the $SU(2)$ scalar doublets is even, and such doublets are called active, while the other doublets are odd, and those are inert doublets, and would accommodate the DM. The $\mathbb{Z}_2$ symmetry prevents couplings between the SM particles and a single inert particle at tree-level. As a consequence, the lightest $\mathbb{Z}_2$-odd particle is stable and is a possible DM candidate.

All of the $SU(2)$ scalar doublets in the S_3 scalar potential come in pairs except for the $SU(2)$ doublets that are multiplied by λ_4. This is the only coupling which breaks the $\mathbb{Z}_2$ symmetry for h_2 and h_S. Provided that the DM candidate resides in h_2 or h_S it is a must to impose the constraint $\lambda_4 = 0$, but if the inert $SU(2)$ doublet corresponds to h_1, then the $\lambda_4 = 0$ constraint is not necessary.

Not all of the vacuum configurations with zero VEV components result in a possible DM candidate. The $\mathbb{Z}_2$ can be broken, based on the $\lambda_4 = 0$ constraint. Also we take a look at

[3] These two vacuum configurations can be analysed together as those differ only by the sign of $\langle h_2 \rangle$.

[4] It should be noted that, in principle, the $\mathbb{Z}_2$ symmetry might be a smaller symmetry and thus a subgroup of a larger symmetry under which the scalar potential is invariant.

Vacuum	Broken symmetries	Number of broken generators	Additional $m^0 = 0$	λ constraints
C-IV-c, C-IV-f	$U(1)_w$	1	1	$\lambda_4 = -2A\,(\lambda_2 + \lambda_3)$, $\lambda_7 = A^2\,(\lambda_2 + \lambda_3)$
R-II-2, R-II-3, R-III, C-III-b, C-III-f, C-III-g, C-IV-b, C-IV-e	$SO(2)$	1	1	$\lambda_4 = 0$
C-III-c	$SO(2) \otimes U(1)_w$	1+1	2	$\lambda_4 = 0$, $\lambda_2 + \lambda_3 = 0$
C-IV-a, C-IV-d	$SO(2) \otimes U(1)$	1+1	2	$\lambda_4 = 0$, $\lambda_7 = 0$
C-V	$SU(2)$	3	3	$\lambda_4 = 0$, $\lambda_2 + \lambda_3 = 0$, $\lambda_7 = 0$

Table 2.4: Comparison of the number of broken generators with the number of additional massless states. The $U(1)_w$ group indicates that additional symmetry is reached through the vacuum. The other cases result in a symmetry of the $SU(2)$ doublets.

the mass-squared matrices to determine if there is mixing present between the inert and active $SU(2)$ doublets. This is a trivial task and is based on determining if the mass-squared matrix is block-diagonal in the basis of the fields which correspond to the active-inert $SU(2)$ doublets. For simplicity, we write down all of the possible vacuum configurations and specify their properties. Results are presented in Table 2.5. It should be noted that only a basic check was performed and thus the mentioned DM candidates are possible but may not result in a viable model.

The only two vacuum configurations with a single inert doublet h_1 are R-II-1a and C-III-a. An interesting observation is that although vacua R-I-2b, and R-I-2c, and C-I-a involve zero VEVs, there is mixing present in the mass-squared matrices and therefore these models fail to describe DM. Cases R-II-2, and C-III-b, and C-IV-a result in one neutral inert massless scalar state. Another interesting observation is that if the h_2 or h_S $SU(2)$ doublets are the DM candidates, this results in a requirement that $\lambda_4 = 0$, and therefore at least one massless scalar state is present provided that the scalar potential is exactly S_3-symmetric.

Vacuum	Vacuum Configuration	Additional $m^0 = 0$	Mass-degenerate states	Symmetries	Possible DM candidate
R-I-1	$\{0,\, 0,\, w_S\}$		$1H^\pm$ and $2H^0$	$\not{\mathbb{Z}}_2$	
R-I-2a	$\{w,\, 0,\, 0\}$			$\not{\mathbb{Z}}_2$	
R-I-2b	$\{w,\, \sqrt{3}w,\, 0\}$				
R-I-2c	$\{w,\, -\sqrt{3}w,\, 0\}$				
R-II-1a	$\{0,\, w,\, w_S\}$				h_1
R-II-1b	$\{w,\, -w/\sqrt{3},\, w_S\}$				
R-II-1c	$\{w,\, w/\sqrt{3},\, w_S\}$				
R-II-2	$\{0,\, w,\, 0\}$	1		$SO(2)$	$h_1,\, h_S$

R-II-3	$\{w_1,\, w_2,\, 0\}$	1		$SO(2)$	h_S
R-III	$\{w_1,\, w_2,\, w_S\}$	1		$SO(2)$	
C-I-a	$\{\hat{w}_1,\, \pm i\hat{w}_1,\, 0\}$		$2H^0$		
C-III-a	$\{0,\, \hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$				h_1
C-III-b	$\{\pm i\hat{w}_1,\, 0,\, \hat{w}_S\}$	1		$SO(2)$	h_2
C-III-c	$\{\hat{w}_1 e^{i\sigma_1},\, \hat{w}_2 e^{i\sigma_2},\, 0\}$	2		$SO(2) \otimes U(1)_w$	h_S
C-III-d	$\{\pm i\hat{w}_1,\, \hat{w}_2,\, \hat{w}_S\}$				
C-III-e	$\{\pm i\hat{w}_1,\, -\hat{w}_2,\, \hat{w}_S\}$				
C-III-f	$\{\pm i\hat{w}_1,\, i\hat{w}_2,\, \hat{w}_S\}$	1		$SO(2)$	
C-III-g	$\{\pm i\hat{w}_1, -i\hat{w}_2,\, \hat{w}_S\}$	1		$SO(2)$	
C-III-h	$\{\sqrt{3}\hat{w}_2 e^{i\sigma_2},\, \pm\hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$				
C-III-i	$\{\sqrt{\frac{3(1+\mathrm{t}^2_{\sigma_1})}{1+9\mathrm{t}^2_{\sigma_1}}}\hat{w}_2 e^{i\sigma_1},$ $\pm\hat{w}_2 e^{-i\arctan(3\mathrm{t}_{\sigma_1})},\, \hat{w}_S\}$				
C-IV-a	$\{\hat{w}_1 e^{i\sigma_1},\, 0,\, \hat{w}_S\}$	2		$SO(2) \otimes U(1)$	h_2
C-IV-b	$\{\hat{w}_1,\, \pm i\hat{w}_2,\, \hat{w}_S\}$	1		$SO(2)$	
C-IV-c	$\{\sqrt{1+2\mathrm{c}^2_{\sigma_2}}\hat{w}_2,$ $\hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$	1		$U(1)_w$	
C-IV-d	$\{\hat{w}_1 e^{i\sigma_1},\, \pm\hat{w}_2 e^{i\sigma_1},\, \hat{w}_S\}$	2		$SO(2) \otimes U(1)$	
C-IV-e	$\{\sqrt{-\frac{\mathrm{s}_{2\sigma_2}}{\mathrm{s}_{2\sigma_1}}}\hat{w}_2 e^{i\sigma_1},$ $\hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$	1		$SO(2)$	
C-IV-f	$\{\sqrt{2+\frac{\mathrm{c}_{(\sigma_1-2\sigma_2)}}{\mathrm{c}_{\sigma_1}}}\hat{w}_2 e^{i\sigma_1},$ $\hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$	1		$U(1)_w$	
C-V	$\{\hat{w}_1 e^{i\sigma_1},\, \hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}$	3		$SU(2)$	

Table 2.5: Properties of different vacuum configurations. In the third column, the number of additional massless states is presented. In the fourth column, the number of mass-degenerate states is shown, *e.g.*, $2H^0$ indicates that there are in total two massive mass-degenerate pairs, values of which are not the same, *i.e.*, $m_{H_i} = m_{H_j} \neq 0$, and $m_{H_k} = m_{H_l} \neq 0$, and $m_{H_i} \neq m_{H_k}$. Same notation applies to the charged mass-degenerate states denoted by $1H^{\pm}$. In the fifth column, additional symmetries of the potential, after applying the minimization conditions in terms of λ_i, are presented, and $\not{\mathbb{Z}}_2$ indicates that the $\mathbb{Z}_2$ symmetry is broken. In the last column, inert $SU(2)$ doublets are written down. These doublets do not violate the $\mathbb{Z}_2$ symmetry and the mass-squared matrices do not mix with active $SU(2)$ doublets.

2.5 The Higgs Basis Transformation

Physical quantities are basis invariant and therefore a convenient choice would be to work in the so-called Higgs basis. In case of the 2HDM, the Higgs basis [44, 45] is defined as a basis in which one of the VEVs has a zero value. A more general definition of the Higgs basis, applicable to the multi-Higgs-doublet model, is that it is a basis in which only one VEV acquires a non-zero value:

$$\langle H_1 \rangle = \frac{v}{\sqrt{2}}, \quad \langle H_i \rangle = 0, \tag{2.5.1}$$

where $\langle H_i \rangle$ indicates the vacuum of the specific $SU(2)$ doublet, the H_i $SU(2)$ doublets indicate that those are in the Higgs basis, and the VEV of the H_1 doublet is real. Another property of the Higgs basis is that the would-be Goldstone bosons are isolated in the H_1 doublet.

Assume that the Higgs basis rotation from the generic basis h_i to the Higgs basis H_i is given by the following transformation:

$$H_i = \mathcal{R}_{ij} h_j, \tag{2.5.2}$$

or equivalently:

$$h_i = \mathcal{R}^*_{ji} H_j, \tag{2.5.3}$$

where the rotation matrix $\mathcal{R}$ is unitary. In the Higgs basis the S_3 symmetry is not explicit anymore and therefore we define the $SU(2)$ doublets as: $\{H_1,\, H_2,\, H_3\}$.

The scalar potential under eq. (2.5.3) transforms as[5]:

$$\left(h^\dagger_{\bar{i}} h_{\bar{j}}\right) = \mathcal{R}_{i\bar{i}} \mathcal{R}^*_{j\bar{j}} \left(H^\dagger_i H_j\right), \tag{2.5.4a}$$

$$\left(h^\dagger_{\bar{i}} h_{\bar{j}}\right)\left(h^\dagger_{\bar{k}} h_{\bar{l}}\right) = \frac{1}{2} c_{ijkl} \left[\mathcal{R}_{i\bar{i}} \mathcal{R}^*_{j\bar{j}} \mathcal{R}_{k\bar{k}} \mathcal{R}^*_{l\bar{l}} + \mathcal{R}_{k\bar{i}} \mathcal{R}^*_{l\bar{j}} \mathcal{R}_{i\bar{k}} \mathcal{R}^*_{j\bar{l}}\right] \left(H^\dagger_i H_j\right)\left(H^\dagger_k H_l\right), \tag{2.5.4b}$$

where additional terms of the quadratic transformation arise due to

$$\left(H^\dagger_i H_j\right)\left(H^\dagger_k H_l\right) = \left(H^\dagger_k H_l\right)\left(H^\dagger_i H_j\right), \tag{2.5.5}$$

and c_{ijkl} is a symmetry factor:

$$c_{ijkl} = \begin{cases} 1, & \text{if } i = k \text{ and } j = l \\ 2, & \text{otherwise} \end{cases}. \tag{2.5.6}$$

The scalar potential in the Higgs basis in the $SU(2)$-covariant form is given by [13, 46]:

$$V^{\mathrm{HB}} = Y_{ij} \left(H^\dagger_i H_j\right) + Z_{ijkl} \left(H^\dagger_i H_j\right)\left(H^\dagger_k H_l\right), \tag{2.5.7}$$

where due to hermiticity the following relations hold:

$$Y_{ij} = Y^*_{ji}, \tag{2.5.8a}$$

$$Z_{ijkl} = Z^*_{jilk}, \qquad Z_{ijkl} = Z_{klij}. \tag{2.5.8b}$$

The quadratic coupling in the Higgs basis are[6]:

$$\begin{aligned} Y_{ij} =& \frac{1}{2}\nu^2 \left(\mathcal{R}_{i\bar{1}} \mathcal{R}^*_{j\bar{2}} + \text{h.c.}\right) + \mu_0^2 \mathcal{R}_{i\bar{3}} \mathcal{R}^*_{j\bar{3}} + \mu_1^2 \left(\mathcal{R}_{i\bar{1}} \mathcal{R}^*_{j\bar{1}} + \mathcal{R}_{i\bar{2}} \mathcal{R}^*_{j\bar{2}}\right), \\ &+ \mu_2^2 \left(\mathcal{R}_{i\bar{1}} \mathcal{R}^*_{j\bar{1}} - \mathcal{R}_{i\bar{2}} \mathcal{R}^*_{j\bar{2}}\right) + \frac{1}{2}\mu_3^2 \left(\mathcal{R}_{i\bar{1}} \mathcal{R}^*_{j\bar{3}} + \text{h.c.}\right) + \frac{1}{2}\mu_4^2 \left(\mathcal{R}_{i\bar{2}} \mathcal{R}^*_{j\bar{3}} + \text{h.c.}\right), \end{aligned} \tag{2.5.9}$$

[5] We are not using the summation convention. The barred indices are not contracted with the un-barred.

[6] For completeness, we assume that soft symmetry breaking terms are added, see section 3.2.

and the quartic couplings are:

$$
\begin{aligned}
Z_{ijkl} =& \lambda_1 c_{ijkl} \left(\mathcal{R}_{i\bar{1}}\mathcal{R}^*_{j\bar{1}} + \mathcal{R}_{i\bar{2}}\mathcal{R}^*_{j\bar{2}}\right)\left(\mathcal{R}_{k\bar{1}}\mathcal{R}^*_{l\bar{1}} + \mathcal{R}_{k\bar{2}}\mathcal{R}^*_{l\bar{2}}\right) \\
&+ \lambda_2 c_{ijkl} \left(\mathcal{R}_{i\bar{1}}\mathcal{R}^*_{j\bar{2}} - \mathcal{R}_{i\bar{2}}\mathcal{R}^*_{j\bar{1}}\right)\left(\mathcal{R}_{k\bar{1}}\mathcal{R}^*_{l\bar{2}} - \mathcal{R}_{k\bar{2}}\mathcal{R}^*_{l\bar{1}}\right) \\
&+ \lambda_3 c_{ijkl} \Big[\mathcal{R}^*_{j\bar{1}}\left(\left(\mathcal{R}_{i\bar{1}}\mathcal{R}_{k\bar{1}} + \mathcal{R}_{i\bar{2}}\mathcal{R}_{k\bar{2}}\right)\mathcal{R}^*_{l\bar{1}} - \left(\mathcal{R}_{i\bar{1}}\mathcal{R}_{k\bar{2}} - \mathcal{R}_{i\bar{2}}\mathcal{R}_{k\bar{1}}\right)\mathcal{R}^*_{l\bar{2}}\right) \\
&\qquad + \mathcal{R}^*_{j\bar{2}}\left(\left(\mathcal{R}_{i\bar{1}}\mathcal{R}_{k\bar{1}} + \mathcal{R}_{i\bar{2}}\mathcal{R}_{k\bar{2}}\right)\mathcal{R}^*_{l\bar{2}} - \left(\mathcal{R}_{i\bar{2}}\mathcal{R}_{k\bar{1}} - \mathcal{R}_{i\bar{1}}\mathcal{R}_{k\bar{2}}\right)\mathcal{R}^*_{l\bar{1}}\right)\Big] \\
&+ \lambda_4 c_{ijkl} \Big[\mathcal{R}^*_{j\bar{1}}\Big(\left(\mathcal{R}_{i\bar{3}}\mathcal{R}_{k\bar{2}} + \mathcal{R}_{i\bar{2}}\mathcal{R}_{k\bar{3}}\right)\mathcal{R}^*_{l\bar{1}} + \left(\mathcal{R}_{i\bar{3}}\mathcal{R}_{k\bar{1}} + \mathcal{R}_{i\bar{1}}\mathcal{R}_{k\bar{3}}\right)\mathcal{R}^*_{l\bar{2}} \\
&\qquad\qquad + \left(\mathcal{R}_{i\bar{2}}\mathcal{R}_{k\bar{1}} + \mathcal{R}_{i\bar{1}}\mathcal{R}_{k\bar{2}}\right)\mathcal{R}^*_{l\bar{3}}\Big) \\
&\qquad + \mathcal{R}^*_{j\bar{2}}\Big(\left(\mathcal{R}_{i\bar{3}}\mathcal{R}_{k\bar{1}} + \mathcal{R}_{i\bar{1}}\mathcal{R}_{k\bar{3}}\right)\mathcal{R}^*_{l\bar{1}} - \left(\mathcal{R}_{i\bar{3}}\mathcal{R}_{k\bar{2}} + \mathcal{R}_{i\bar{2}}\mathcal{R}_{k\bar{3}}\right)\mathcal{R}^*_{l\bar{2}} \\
&\qquad\qquad + \left(\mathcal{R}_{i\bar{1}}\mathcal{R}_{k\bar{1}} - \mathcal{R}_{i\bar{2}}\mathcal{R}_{k\bar{2}}\right)\mathcal{R}^*_{l\bar{3}}\Big) \\
&\qquad + \mathcal{R}^*_{j\bar{3}}\left(\left(\mathcal{R}_{i\bar{2}}\mathcal{R}_{k\bar{1}} + \mathcal{R}_{i\bar{1}}\mathcal{R}_{k\bar{2}}\right)\mathcal{R}^*_{l\bar{1}} + \left(\mathcal{R}_{i\bar{1}}\mathcal{R}_{k\bar{1}} - \mathcal{R}_{i\bar{2}}\mathcal{R}_{k\bar{2}}\right)\mathcal{R}^*_{l\bar{2}}\right)\Big] \\
&+ \lambda_5 \frac{c_{ijkl}}{2}\left[\mathcal{R}_{i\bar{3}}\mathcal{R}^*_{j\bar{3}}\left(\mathcal{R}_{k\bar{1}}\mathcal{R}^*_{l\bar{1}} + \mathcal{R}_{k\bar{2}}\mathcal{R}^*_{l\bar{2}}\right) + \mathcal{R}_{k\bar{3}}\mathcal{R}^*_{l\bar{3}}\left(\mathcal{R}_{i\bar{1}}\mathcal{R}^*_{j\bar{1}} + \mathcal{R}_{i\bar{2}}\mathcal{R}^*_{j\bar{2}}\right)\right] \\
&+ \lambda_6 \frac{c_{ijkl}}{2}\left[\mathcal{R}_{i\bar{3}}\mathcal{R}^*_{l\bar{3}}\left(\mathcal{R}_{k\bar{1}}\mathcal{R}^*_{j\bar{1}} + \mathcal{R}_{k\bar{2}}\mathcal{R}^*_{j\bar{2}}\right) + \mathcal{R}_{k\bar{3}}\mathcal{R}^*_{j\bar{3}}\left(\mathcal{R}_{i\bar{1}}\mathcal{R}^*_{l\bar{1}} + \mathcal{R}_{i\bar{2}}\mathcal{R}^*_{l\bar{2}}\right)\right] \\
&+ \lambda_7 c_{ijkl}\left[\mathcal{R}_{i\bar{3}}\mathcal{R}_{k\bar{3}}\left(\mathcal{R}^*_{j\bar{1}}\mathcal{R}^*_{l\bar{1}} + \mathcal{R}^*_{j\bar{2}}\mathcal{R}^*_{l\bar{2}}\right) + \mathcal{R}^*_{j\bar{3}}\mathcal{R}^*_{l\bar{3}}\left(\mathcal{R}_{i\bar{1}}\mathcal{R}_{k\bar{1}} + \mathcal{R}_{i\bar{2}}\mathcal{R}_{k\bar{2}}\right)\right] \\
&+ \lambda_8 c_{ijkl}\mathcal{R}_{i\bar{3}}\mathcal{R}^*_{j\bar{3}}\mathcal{R}_{k\bar{3}}\mathcal{R}^*_{l\bar{3}}.
\end{aligned} \tag{2.5.10}
$$

Consider the most general vacuum configuration:

$$\{\hat{w}_1 e^{i\sigma_1},\, \hat{w}_2 e^{i\sigma_2},\, \hat{w}_S\}. \tag{2.5.11}$$

Such vacuum configuration is first rotated into an intermediate basis, where VEVs are expressed only in terms of the absolute value $\hat{w}_i$:

$$\begin{pmatrix} h_1' \\ h_2' \\ h_S' \end{pmatrix} = \mathrm{diag}\left(e^{-i\sigma_1},\, e^{-i\sigma_2},\, 1\right)\begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix}. \tag{2.5.12}$$

The next step is to rotate the modulus of VEVs. The most trivial approach would be to consider the Euler rotation matrices. In total, three components $\hat{w}_i$ need to be rotated into a single v (2.1.4). Such rotation can be performed in terms of just two angles. This indicates that at least one of the $\mathcal{R}$ components of eq. (2.5.2) will become zero. Based on a choice of the Euler rotation matrix, appropriate changes of the couplings given by eqs. (2.5.9, 2.5.10) are expected as there is a migrating zero. It follows that the Higgs basis is not unique and results in a spectrum of Higgs bases. Moreover, the Higgs basis is not well-defined as there is a freedom to redefine the $SU(2)$ doublets with vanishing VEVs, see Refs. [12, 13, 47]. In our case it is possible to rotate the H_2 and H_3 doublets by an additional $U(2)$ transformation. Such transformation could potentially simplify some of the couplings.

As stressed, the Higgs basis is not uniquely defined. One of the possibilities is to consider that we first rotate $\langle h_S \rangle$ into $\langle h_2 \rangle$ so it becomes $\langle h_2' \rangle$ and then $\langle h_2' \rangle$ into $\langle h_1 \rangle$. The Higgs basis rotation in this case is given by:

$$
\begin{aligned}
\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} &= \begin{pmatrix} c_{\beta_1} & s_{\beta_1} & 0 \\ -s_{\beta_1} & c_{\beta_1} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{\beta_2} & s_{\beta_2} \\ 0 & -s_{\beta_2} & c_{\beta_2} \end{pmatrix}\begin{pmatrix} e^{-i\sigma_1} & 0 & 0 \\ 0 & e^{-i\sigma_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\sigma_1} c_{\beta_1} & e^{-i\sigma_2} s_{\beta_1} c_{\beta_2} & s_{\beta_1} s_{\beta_2} \\ -e^{-i\sigma_1} s_{\beta_1} & e^{-i\sigma_2} c_{\beta_1} c_{\beta_2} & c_{\beta_1} s_{\beta_2} \\ 0 & -e^{-i\sigma_2} s_{\beta_2} & c_{\beta_2} \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix},
\end{aligned} \tag{2.5.13}
$$

The rotation of the $SU(2)$ doublets from the generic basis (2.1.3, 2.1.7) to the Higgs basis is therefore:

$$h_1 = e^{i\sigma_1}\left(H_1 c_{\beta_1} - H_2 s_{\beta_1}\right), \tag{2.5.14a}$$
$$h_2 = e^{i\sigma_2}\left(H_1 c_{\beta_2} s_{\beta_1} + H_2 c_{\beta_1} c_{\beta_2} - H_3 s_{\beta_2}\right), \tag{2.5.14b}$$
$$h_S = \left(H_1 s_{\beta_1} + H_2 c_{\beta_1}\right) s_{\beta_2} + H_3 c_{\beta_2}. \tag{2.5.14c}$$

The choice of the Euler rotation matrix results in $\mathcal{R}_{31} = 0$ and therefore h_1 does not contribute to H_3.

The Higgs basis rotation depends on whether one of the VEVs is zero. When $\hat{w}_2 \neq 0$, the β_i angles are given by:

$$c_{\beta_1} = \frac{\hat{w}_1}{v}, \tag{2.5.15a}$$
$$t_{\beta_2} = \frac{\hat{w}_S}{\hat{w}_2}, \tag{2.5.15b}$$

and if $\hat{w}_2 = 0$, *i.e.*, {R-I-1, R-I-2a, C-III-b, C-IV-a}, we get that:

$$t_{\beta_1} = \frac{\hat{w}_S}{\hat{w}_1}, \tag{2.5.16a}$$
$$\beta_2 = \frac{\pi}{2}. \tag{2.5.16b}$$

If both $\hat{w}_1$ and $\hat{w}_2$ vanish, *i.e.*, the R-I-1 model case, then the β_i angles are fixed:

$$\beta_1 = \beta_2 = \frac{\pi}{2}, \tag{2.5.17}$$

so that such rotation results in a translation of the h_S doublet into h_1.

Relations between the Higgs basis and the generic basis are presented in Appendix C.1. An interesting observation is that not all of the Z couplings depend on the λ_2 coupling. This is due to the basis transformation of eq. (2.5.14) $\mathcal{R}_{31} = 0$. The Z_{ijkl} couplings, which do not depend on the λ_2 coupling are: $\{Z_{1133}, Z_{2233}, Z_{3333}, Z_{1233}, Z_{1333}, Z_{2333}\}$. The latter three couplings are complex. Another interesting consequence of the choice of transformation (2.5.14) is that the couplings Z_{1233} and Z_{1332} become real.

For completeness, we take a look at a more general transformation, *i.e.*, when H_2 and H_3 doublets are transformed by $U(2)$. The most general $U(2)$ transformation is as follows:

$$U = e^{\frac{i\phi}{2}} \begin{pmatrix} e^{i\phi_1} c_\theta & e^{i\phi_2} s_\theta \\ -e^{-i\phi_2} s_\theta & e^{-i\phi_1} c_\theta \end{pmatrix}. \tag{2.5.18}$$

Promoting this to the Higgs basis transformation results in:

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = \begin{pmatrix} \mathcal{U}_{11} & \mathcal{U}_{12} & \mathcal{U}_{13} \\ \mathcal{U}_{21} & \mathcal{U}_{22} & \mathcal{U}_{23} \\ \mathcal{U}_{31} & \mathcal{U}_{32} & \mathcal{U}_{33} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix}, \tag{2.5.19}$$

where

$$\mathcal{U}_{11} = e^{-i\sigma_1} c_{\beta_1}, \tag{2.5.20a}$$
$$\mathcal{U}_{12} = e^{-i\sigma_2} c_{\beta_2} s_{\beta_1}, \tag{2.5.20b}$$
$$\mathcal{U}_{13} = s_{\beta_1} s_{\beta_2}, \tag{2.5.20c}$$
$$\mathcal{U}_{21} = -e^{\frac{1}{2}i(-2\sigma_1+\phi+2\phi_1)} c_\theta s_{\beta_1}, \tag{2.5.20d}$$
$$\mathcal{U}_{22} = e^{\frac{1}{2}i(\phi-2\sigma_2)}\left(e^{i\phi_1} c_{\beta_1} c_{\beta_2} c_\theta - e^{i\phi_2} s_{\beta_2} s_\theta\right), \tag{2.5.20e}$$

$$\mathcal{U}_{23} = e^{\frac{i\phi}{2}} \left(e^{i\phi_2} c_{\beta_2} s_\theta + e^{i\phi_1} c_{\beta_1} c_\theta s_{\beta_2} \right), \tag{2.5.20f}$$

$$\mathcal{U}_{31} = s_{\beta_1} s_\theta e^{\frac{1}{2}i(\phi - 2(\sigma_1+\phi_2))}, \tag{2.5.20g}$$

$$\mathcal{U}_{32} = -e^{\frac{1}{2}i(\phi - 2(\sigma_2+\phi_1+\phi_2))} \left(e^{i\phi_1} c_{\beta_1} c_{\beta_2} s_\theta + e^{i\phi_2} c_\theta s_{\beta_2} \right), \tag{2.5.20h}$$

$$\mathcal{U}_{33} = e^{\frac{i\phi}{2}} \left(e^{-i\phi_1} c_{\beta_2} c_\theta - e^{-i\phi_2} c_{\beta_1} s_{\beta_2} s_\theta \right). \tag{2.5.20i}$$

The resulting rotation from the generic basis to the Higgs basis is given by eq. (2.5.3) with $\mathcal{R}_{ji} \to \mathcal{U}_{ji}$. From the definition of the $\mathcal{U}_{ij}$ components (2.5.20) it follows that in this case all of the generic doublets h_i transform into all of the Higgs doublets H_j.

To emphasize the migrating zero feature of the Higgs basis transformation, when an additional $U(2)$ transformation of the H_2 and H_3 is not regarded, consider that $\langle h_2 \rangle$ is rotated into $\langle h_1 \rangle$ and $\langle h_S \rangle$ into $\langle h_1' \rangle$. In this case, the Higgs basis transformation is:

$$\begin{aligned}\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} &= \begin{pmatrix} c_{\beta_2} & 0 & s_{\beta_2} \\ 0 & 1 & 0 \\ -s_{\beta_2} & 0 & c_{\beta_2} \end{pmatrix} \begin{pmatrix} c_{\beta_1} & s_{\beta_1} & 0 \\ -s_{\beta_1} & c_{\beta_1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\sigma_1} & 0 & 0 \\ 0 & e^{-i\sigma_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\sigma_1} c_{\beta_1} c_{\beta_2} & e^{-i\sigma_2} s_{\beta_1} c_{\beta_2} & s_{\beta_2} \\ -e^{-i\sigma_1} s_{\beta_1} & e^{-i\sigma_2} c_{\beta_1} & 0 \\ -e^{-i\sigma_1} c_{\beta_1} s_{\beta_2} & -e^{-i\sigma_2} s_{\beta_1} s_{\beta_2} & c_{\beta_2} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix},\end{aligned} \tag{2.5.21}$$

and the transformation from the generic basis to the Higgs basis is given by:

$$h_1 = -e^{i\sigma_1} \left(-H_1 c_{\beta_1} c_{\beta_2} + H_2 s_{\beta_1} + H_3 c_{\beta_1} s_{\beta_2} \right), \tag{2.5.22a}$$

$$h_2 = e^{i\sigma_2} \left(H_1 s_{\beta_1} c_{\beta_2} + H_2 c_{\beta_1} - H_3 s_{\beta_1} s_{\beta_2} \right), \tag{2.5.22b}$$

$$h_S = H_1 s_{\beta_2} + H_3 c_{\beta_2}. \tag{2.5.22c}$$

The zero element is $\mathcal{R}_{23}$ and thus h_S does not depend on H_2. This transformation is equivalent to the Higgs basis transformation suggested in Ref. [48]:

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = \begin{pmatrix} \frac{\hat{w}_1}{N_1} & \frac{\hat{w}_2}{N_1} & \frac{\hat{w}_S}{N_1} \\ \frac{\hat{w}_2}{N_2} & \frac{-\hat{w}_1}{N_2} & 0 \\ \frac{\hat{w}_1}{N_3} & \frac{\hat{w}_2}{N_3} & \frac{X}{N_3} \end{pmatrix} \begin{pmatrix} e^{-i\sigma_1} & 0 & 0 \\ 0 & e^{-i\sigma_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix}, \tag{2.5.23}$$

where

$$N_1^2 = \hat{w}_1^2 + \hat{w}_2^2 + \hat{w}_S^2 \equiv v^2, \tag{2.5.24a}$$

$$N_2^2 = \hat{w}_1^2 + \hat{w}_2^2 \equiv w^2, \tag{2.5.24b}$$

$$N_3^2 = \hat{w}_1^2 + \hat{w}_2^2 + X^2, \tag{2.5.24c}$$

with

$$X = -\frac{w^2}{\hat{w}_S}. \tag{2.5.25}$$

Inverting eq. (2.5.23) results in transformation from the generic basis to the Higgs basis:

$$h_1 = \frac{e^{i\sigma_1}}{N_1 N_2 N_3} (N_2 N_3 \hat{w}_1 H_1 + N_1 N_3 \hat{w}_2 H_2 + N_1 N_2 \hat{w}_1 H_3), \tag{2.5.26a}$$

$$h_2 = \frac{e^{i\sigma_2}}{N_1 N_2 N_3} (N_2 N_3 \hat{w}_2 H_1 - N_1 N_3 \hat{w}_1 H_2 + N_1 N_2 \hat{w}_2 H_3), \tag{2.5.26b}$$

$$h_S = \frac{1}{N_1 N_3 \hat{w}_S} (N_3 \hat{w}_S^2 H_1 - N_1 w^2 H_3). \tag{2.5.26c}$$

Relations between the Higgs basis and the generic basis are presented in Appendix C.2.

2.5.1 Choice of the Higgs Basis

For completeness, we discuss general properties of a specific Higgs basis choice. Consider the following Euler rotation matrices:

$$\mathcal{R}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}_{\beta_x} & \mathrm{s}_{\beta_x} \\ 0 & -\mathrm{s}_{\beta_x} & \mathrm{c}_{\beta_x} \end{pmatrix}, \tag{2.5.27a}$$

$$\mathcal{R}_y = \begin{pmatrix} \mathrm{c}_{\beta_y} & 0 & \mathrm{s}_{\beta_y} \\ 0 & 1 & 0 \\ -\mathrm{s}_{\beta_y} & 0 & \mathrm{c}_{\beta_y} \end{pmatrix}, \tag{2.5.27b}$$

$$\mathcal{R}_z = \begin{pmatrix} \mathrm{c}_{\beta_z} & \mathrm{s}_{\beta_z} & 0 \\ -\mathrm{s}_{\beta_z} & \mathrm{c}_{\beta_z} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5.27c}$$

The Higgs basis transformation is thus:

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = \mathcal{R}_{\beta_i}\mathcal{R}_{\beta_j} \begin{pmatrix} e^{-i\sigma_1} & 0 & 0 \\ 0 & e^{-i\sigma_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix}. \tag{2.5.28}$$

We consider different combinations of $\mathcal{R}_{\beta_i}\mathcal{R}_{\beta_j}$ and in what specific direction of the scalar potential those result. Note that both H_2 and H_3 can further be mixed by a $U(2)$ transformation, but we do not consider this additional transformation.

$\mathcal{R}_{\beta_z}\mathcal{R}_{\beta_x}$ results in a rotation element $\mathcal{R}_{31} = 0$. This is the case of eq. (2.5.13). The Z_{ijkl} couplings then satisfy the following relations:

$$\mathrm{Im}\,(Z_{1112}) = -\,\mathrm{Im}\,(Z_{1222})\,, \tag{2.5.29a}$$
$$\mathrm{Im}\,(Z_{1123}) = -\,\mathrm{Im}\,(Z_{1231})\,, \tag{2.5.29b}$$
$$\mathrm{Im}\,(Z_{1223}) = \mathrm{Im}\,(Z_{1322})\,, \tag{2.5.29c}$$
$$\mathrm{Im}\,(Z_{1233}) = \mathrm{Im}\,(Z_{1332}) = 0. \tag{2.5.29d}$$

$\mathcal{R}_{\beta_x}\mathcal{R}_{\beta_z}$ results in a rotation element $\mathcal{R}_{13} = 0$. The Z_{ijkl} couplings then satisfy the following relations:

$$Z_{1233} = Z_{1332}, \tag{2.5.30a}$$
$$Z_{1223} = Z_{1322}, \tag{2.5.30b}$$
$$\mathrm{Im}\,(Z_{1123}) = -\,\mathrm{Im}\,(Z_{1231})\,. \tag{2.5.30c}$$

$\mathcal{R}_{\beta_y}\mathcal{R}_{\beta_z}$ results in a rotation element $\mathcal{R}_{23} = 0$. This is the case of eq. (2.5.21). The Z_{ijkl} couplings then satisfy the following relations:

$$Z_{1123} = Z^*_{1231}, \tag{2.5.31a}$$
$$Z_{1233} = Z_{1332}, \tag{2.5.31b}$$
$$\mathrm{Im}\,(Z_{1223}) = -\,\mathrm{Im}\,(Z_{1322})\,. \tag{2.5.31c}$$

$\mathcal{R}_{\beta_z}\mathcal{R}_{\beta_y}$ results in a rotation element $\mathcal{R}_{32} = 0$. The Z_{ijkl} couplings then satisfy the following relations:

$$\mathrm{Im}\,(Z_{1233}) = \mathrm{Im}\,(Z_{1332})\,, \tag{2.5.32a}$$
$$\mathrm{Im}\,(Z_{1123}) = -\,\mathrm{Im}\,(Z_{1231})\,, \tag{2.5.32b}$$

$$\mathrm{Im}\left(Z_{1223}\right) = \mathrm{Im}\left(Z_{1322}\right). \tag{2.5.32c}$$

$\mathcal{R}_{\beta_y}\mathcal{R}_{\beta_x}$ results in a rotation element $\mathcal{R}_{21} = 0$. The Z_{ijkl} couplings then satisfy the following relations:

$$\mathrm{Im}\left(Z_{1223}\right) = \mathrm{Im}\left(Z_{1322}\right) = 0, \tag{2.5.33a}$$
$$\mathrm{Im}\left(Z_{1233}\right) = \mathrm{Im}\left(Z_{1332}\right), \tag{2.5.33b}$$
$$\mathrm{Im}\left(Z_{1123}\right) = -\mathrm{Im}\left(Z_{1231}\right), \tag{2.5.33c}$$
$$\mathrm{Im}\left(Z_{1113}\right) = -\mathrm{Im}\left(Z_{1333}\right). \tag{2.5.33d}$$

$\mathcal{R}_{\beta_x}\mathcal{R}_{\beta_y}$ results in a rotation element $\mathcal{R}_{12} = 0$. The Z_{ijkl} couplings then satisfy the following relations:

$$\mathrm{Im}\left(Z_{1123}\right) = -\mathrm{Im}\left(Z_{1231}\right), \tag{2.5.34a}$$
$$\mathrm{Im}\left(Z_{1233}\right) = \mathrm{Im}\left(Z_{1332}\right), \tag{2.5.34b}$$
$$\mathrm{Im}\left(Z_{1223}\right) = \mathrm{Im}\left(Z_{1322}\right). \tag{2.5.34c}$$

The total number of free parameters can be counted:

$$Y^{\mathbb{R}} + 2Y^{\mathbb{C}} + Z^{\mathbb{R}} + 2Z^{\mathbb{C}} = 3 + 2\times 3 + 9 + 2\times 18 = 54, \tag{2.5.35}$$

where $Y^{\mathbb{R}}$, $Z^{\mathbb{R}}$ indicate real couplings Y_{ii}, and Z_{iiii}, and Z_{ijji}, and $Y^{\mathbb{C}}$, $Z^{\mathbb{C}}$ stand for complex couplings. Assume that the Higgs basis transformation is given by eq. (2.5.13). Constraints (2.5.29) result in:

$$Y^{\mathbb{R}} + Y^{\mathbb{C}} + Z^{\mathbb{R}} + Z^{\mathbb{C}} = 3 + 3 + 11 + 16 = 33, \tag{2.5.36}$$

counting the complex couplings as a single free parameter, *e.g.*, Z_{1113} and Z_{1113}^*. This coincides with the number of free parameters (33) for the general renormalizable $\mathcal{C}$-invariant potential in Ref. [30]:

$$\mathcal{N}_{\mathcal{C}} = \frac{1}{4}N\left(N^3 + 5N + 2\right), \tag{2.5.37}$$

where $\mathcal{N}_{\mathcal{C}}$ is the total number of free parameters and N indicates how many $SU(2)$ doublets are considered.

The number of free parameters in the Higgs basis by far surpasses the number of couplings in the generic basis. The exact S_3-symmetric scalar potential results in 2 quadratic and 8 quartic couplings, assuming that those are real. The soft breaking terms add another 4 quadratic couplings. All in all, there are (10+4) real parameters needed to specify the scalar potential. Some of the couplings in the generic basis could be promoted to complex ones: $\{\nu^2, \mu_3^2, \mu_4^2, \lambda_4, \lambda_7\}$. At most, this results in 12 parameters for the exact S_3 potential with complex couplings and 19 with softly broken ones. In both cases it is nowhere near the 54 free parameters. Therefore, a great amount of parameters in the Higgs basis are interdependent.

2.5.2 The Mass-Squared Matrices

We consider that the $SU(2)$ doublets in the Higgs basis are given by:

$$H_1 = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(v + \tilde{\eta}_1 + iG^0\right) \end{pmatrix}, \tag{2.5.38a}$$
$$H_2 = \begin{pmatrix} h_2^+ \\ \frac{1}{\sqrt{2}}\left(\tilde{\eta}_2 + i\tilde{\chi}_2\right) \end{pmatrix}, \tag{2.5.38b}$$
$$H_3 = \begin{pmatrix} h_3^+ \\ \frac{1}{\sqrt{2}}\left(\tilde{\eta}_3 + i\tilde{\chi}_3\right) \end{pmatrix}. \tag{2.5.38c}$$

The minimization conditions in the Higgs basis are drastically simplified due to a single VEV:

$$Y_{11} = -v^2 Z_{1111}, \tag{2.5.39a}$$
$$Y_{12} = -\frac{1}{2}v^2 Z_{1112}, \tag{2.5.39b}$$
$$Y_{12}^* = -\frac{1}{2}v^2 Z_{1112}^*, \tag{2.5.39c}$$
$$Y_{13} = -\frac{1}{2}v^2 Z_{1113}, \tag{2.5.39d}$$
$$Y_{13}^* = -\frac{1}{2}v^2 Z_{1113}^*. \tag{2.5.39e}$$

The charged mass-squared matrix without the would-be Goldstone boson is:

$$\mathcal{M}^2_{\text{Charged}} = \begin{pmatrix} Y_{22} + \frac{1}{2}v^2 Z_{1122} & Y_{23}^* + \frac{1}{2}v^2 Z_{1123}^* \\ Y_{23} + \frac{1}{2}v^2 Z_{1123} & Y_{33} + \frac{1}{2}v^2 Z_{1133} \end{pmatrix}, \tag{2.5.40}$$

with eigenvalues:

$$m^2_{H_1^\pm} = \frac{1}{4}\left[2Y_{22} + 2Y_{33} + v^2\left(Z_{1122} + Z_{1133}\right) - \Delta\right], \tag{2.5.41a}$$
$$m^2_{H_2^\pm} = \frac{1}{4}\left[2Y_{22} + 2Y_{33} + v^2\left(Z_{1122} + Z_{1133}\right) + \Delta\right], \tag{2.5.41b}$$

where

$$\begin{aligned}\Delta^2 =& 4Y_{22}^2 + 4Y_{33}^2 + 8Y_{23}\left(2Y_{23}^* + v^2 Z_{1123}^*\right) - 4Y_{22}\left[2Y_{33} - v^2\left(Z_{1122} - Z_{1133}\right)\right] \\ &+ v^2\left[8Y_{23}^* Z_{1123} + v^2\left(4|Z_{1123}|^2 + \left(Z_{1122} - Z_{1133}\right)^2\right) - 4Y_{33}\left(Z_{1122} - Z_{1133}\right)\right].\end{aligned} \tag{2.5.42}$$

The neutral mass-squared matrix without the would-be Goldstone boson in the basis

$$\{\tilde{\chi}_2,\, \tilde{\chi}_3,\, \tilde{\eta}_1,\, \tilde{\eta}_2,\, \tilde{\eta}_3\} \tag{2.5.43}$$

is as follows:

$$\mathcal{M}^2_{\text{Neutral}} = \begin{pmatrix} \left(\mathcal{M}^2\right)_{11} & \left(\mathcal{M}^2\right)_{12} & \left(\mathcal{M}^2\right)_{13} & 0 & \left(\mathcal{M}^2\right)_{15} \\ \left(\mathcal{M}^2\right)_{12} & \left(\mathcal{M}^2\right)_{22} & \left(\mathcal{M}^2\right)_{23} & \left(\mathcal{M}^2\right)_{24} & \left(\mathcal{M}^2\right)_{25} \\ \left(\mathcal{M}^2\right)_{13} & \left(\mathcal{M}^2\right)_{23} & \left(\mathcal{M}^2\right)_{33} & \left(\mathcal{M}^2\right)_{34} & \left(\mathcal{M}^2\right)_{35} \\ 0 & \left(\mathcal{M}^2\right)_{24} & \left(\mathcal{M}^2\right)_{34} & \left(\mathcal{M}^2\right)_{44} & \left(\mathcal{M}^2\right)_{45} \\ \left(\mathcal{M}^2\right)_{15} & \left(\mathcal{M}^2\right)_{25} & \left(\mathcal{M}^2\right)_{35} & \left(\mathcal{M}^2\right)_{45} & \left(\mathcal{M}^2\right)_{55} \end{pmatrix}, \tag{2.5.44}$$

where

$$\left(\mathcal{M}^2\right)_{11} = Y_{22} + \frac{1}{2}v^2\left(Z_{1122} - 2Z_{1212} + Z_{1221}\right), \tag{2.5.45a}$$
$$\left(\mathcal{M}^2\right)_{12} = \operatorname{Re}\left(Y_{23}\right) + \frac{1}{2}v^2\left[Z_{1231} + \operatorname{Re}\left(Z_{1123}\right) - \operatorname{Re}\left(Z_{1213}\right)\right], \tag{2.5.45b}$$
$$\left(\mathcal{M}^2\right)_{13} = -v^2 \operatorname{Im}\left(Z_{1112}\right), \tag{2.5.45c}$$
$$\left(\mathcal{M}^2\right)_{15} = \operatorname{Im}\left(Y_{23}\right) + \frac{1}{2}v^2 \operatorname{Im}\left(Z_{1123} - Z_{1213}\right), \tag{2.5.45d}$$
$$\left(\mathcal{M}^2\right)_{22} = Y_{33} + \frac{1}{2}v^2\left[Z_{1133} + Z_{1331} - 2\operatorname{Re}\left(Z_{1313}\right)\right], \tag{2.5.45e}$$
$$\left(\mathcal{M}^2\right)_{23} = -v^2 \operatorname{Im}\left(Z_{1113}\right), \tag{2.5.45f}$$
$$\left(\mathcal{M}^2\right)_{24} = -\operatorname{Im}\left(Y_{23}\right) - \frac{1}{2}v^2 \operatorname{Im}\left(Z_{1123} + Z_{1213}\right), \tag{2.5.45g}$$
$$\left(\mathcal{M}^2\right)_{25} = -v^2 \operatorname{Im}\left(Z_{1313}\right), \tag{2.5.45h}$$
$$\left(\mathcal{M}^2\right)_{33} = 2v^2 Z_{1111}, \tag{2.5.45i}$$

$$\left(\mathcal{M}^2\right)_{34} = v^2\,\mathrm{Re}\left(Z_{1112}\right), \tag{2.5.45j}$$

$$\left(\mathcal{M}^2\right)_{35} = v^2\,\mathrm{Re}\left(Z_{1113}\right), \tag{2.5.45k}$$

$$\left(\mathcal{M}^2\right)_{44} = Y_{22} + \frac{1}{2}v^2\left(Z_{1122} + 2Z_{1212} + Z_{1221}\right), \tag{2.5.45l}$$

$$\left(\mathcal{M}^2\right)_{45} = \mathrm{Re}\left(Y_{23}\right) + \frac{1}{2}v^2\left[Z_{1231} + \mathrm{Re}\left(Z_{1123}\right) + \mathrm{Re}\left(Z_{1213}\right)\right], \tag{2.5.45m}$$

$$\left(\mathcal{M}^2\right)_{55} = Y_{33} + \frac{1}{2}v^2\left[Z_{1133} + Z_{1331} + 2\,\mathrm{Re}\left(Z_{1313}\right)\right]. \tag{2.5.45n}$$

It follows that the neutral mass-squared matrix is block-diagonal in the real basis:

$$\mathcal{M}^2_{\mathrm{Neutral}} = \mathrm{diag}\left(\mathcal{M}^2_{\tilde{\eta}},\, \mathcal{M}^2_{\tilde{\chi}}\right). \tag{2.5.46}$$

The neutral mass-squared matrix for the $\tilde{\eta}$ sector is:

$$\mathcal{M}^2_{\tilde{\eta}} = \begin{pmatrix} 2v^2 Z_{1111} & v^2 Z_{1112} & v^2 Z_{1113} \\ v^2 Z_{1112} & Y_{22} + \frac{1}{2}v^2\left(Z_{1122} + 2Z_{1212} + Z_{1221}\right) & Y_{23} + \frac{1}{2}v^2\left(Z_{1123} + Z_{1213} + Z_{1231}\right) \\ v^2 Z_{1113} & Y_{23} + \frac{1}{2}v^2\left(Z_{1123} + Z_{1213} + Z_{1231}\right) & Y_{33} + \frac{1}{2}v^2\left(Z_{1133} + 2Z_{1313} + Z_{1331}\right) \end{pmatrix}. \tag{2.5.47}$$

The neutral mass-squared matrix for the $\tilde{\chi}$ sector is:

$$\mathcal{M}^2_{\tilde{\chi}} = \begin{pmatrix} Y_{22} + \frac{1}{2}v^2\left(Z_{1122} - 2Z_{1212} + Z_{1221}\right) & Y_{23} + \frac{1}{2}v^2\left(Z_{1123} - Z_{1213} + Z_{1231}\right) \\ Y_{23} + \frac{1}{2}v^2\left(Z_{1123} - Z_{1213} + Z_{1231}\right) & Y_{33} + \frac{1}{2}v^2\left(Z_{1133} - Z_{1313} + Z_{1331}\right) \end{pmatrix}, \tag{2.5.48}$$

In both of these cases the mass-squared parameters are too involved to be analytically expressed.

By inspecting the mass-squared matrices (2.5.40, 2.5.44) an interesting conclusion can be drawn: although the scalar potential can be expressed in terms of 33 parameters, the mass-squared matrices depend only on 12 Z_{ijkl} and 3 Y_{ij} couplings. The VEV of only the first doublet is non-zero (2.5.1) and thus the mass-squared parameters depend on the following combinations:

$$Z_{111i},\quad Z_{11ij},\quad Z_{1i1j},\quad Z_{1ij1},\quad Y_{ij},\quad \text{for } \{i,j\} = \overline{1,3}. \tag{2.5.49}$$

Chapter 3

The C-III-c Model

In this chapter we consider the C-III-c vacuum configuration, given by:

$$\left\{\hat{w}_1 e^{i\sigma_1},\, \hat{w}_2 e^{i\sigma_2},\, 0\right\}. \tag{3.0.1}$$

First of all, we analyse the mass-squared matrices of the exact S_3 symmetry and find that this case is unappealing due to two massless states as presented in Table 2.5. We apply the concept of soft symmetry breaking to promote massless scalars to massive ones. Of particular interest are the possible DM models and thus only properties of such models are taken into account: the mass-squared matrices and if the models are CP violating.

3.1 Model With Exact S_3 Symmetry

We consider that the gauge-eigenstates are given by eq. (2.1.3). The gauge-eigenstates are not uniquely defined and the $SU(2)$ doublets can be expressed in other ways.

The exact C-III-c vacuum configuration is given by eq. (3.0.1) alongside with the constraints:

$$\mu_1^2 = -\left(\lambda_1 - \lambda_2\right) v^2, \tag{3.1.1a}$$

$$\lambda_2 + \lambda_3 = 0, \tag{3.1.1b}$$

$$\lambda_4 = 0, \tag{3.1.1c}$$

where we made use of eq. (2.1.4). From now on we will be using $\hat{w}_1^2 + \hat{w}_2^2 = v^2$ to simplify the look of the VEVs whenever it is possible.

3.1.1 Freedom of the Basis Redefinition

The Higgs basis rotation for the C-III-c vacuum configuration is given by:

$$\begin{aligned}
\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} &= \frac{1}{v}\begin{pmatrix} \hat{w}_1 & \hat{w}_2 & 0 \\ -\hat{w}_2 & \hat{w}_1 & 0 \\ 0 & 0 & v \end{pmatrix}\begin{pmatrix} e^{-i\sigma_1} & 0 & 0 \\ 0 & e^{-i\sigma_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix} \\
&= \frac{1}{v}\begin{pmatrix} e^{-i\sigma_1}\hat{w}_1 & e^{-i\sigma_2}\hat{w}_2 & 0 \\ -e^{-i\sigma_1}\hat{w}_2 & e^{-i\sigma_2}\hat{w}_1 & 0 \\ 0 & 0 & v \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{R}_{\text{HB}-2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix} \equiv \mathcal{R}_{\text{HB}}\begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix}.
\end{aligned} \tag{3.1.2}$$

Let us discuss what was done here. First of all, the phases of the VEVs were rotated away so that VEVs depend only on the absolute value $\hat{w}_i$; note that when going into the Higgs basis, the full

$SU(2)$ doublets are rotated and not only the VEVs. Next, as in the Higgs basis only one VEV acquires a non-zero value, we performed a rotation of the absolute values $\hat{w}_i$ so that $\langle H_1 \rangle = v$ as given by eq. (2.5.1). The rotation of the absolute values can be expressed in terms of the well-known t_β parameter:

$$\mathcal{R}_\beta = \begin{pmatrix} \mathrm{c}_\beta & \mathrm{s}_\beta & 0 \\ -\mathrm{s}_\beta & \mathrm{c}_\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{diag}(\mathcal{R}_{\beta\text{-}2},\, 1), \tag{3.1.3}$$

where the rotation angle β is determined by:

$$\mathrm{t}_\beta = \frac{\hat{w}_2}{\hat{w}_1} = \frac{\mathrm{s}_\beta\, v}{\mathrm{c}_\beta\, v}. \tag{3.1.4}$$

There is a freedom to rotate the VEVs (3.0.1) of the C-III-c model. We can parametrize the fields, for instance, in the following way:

$$\begin{aligned} h_i &= e^{i(\sigma_i - \sigma_i)} \begin{pmatrix} h_i^+ \\ \frac{1}{\sqrt{2}} \left(\hat{w}_i e^{i\sigma_i} + \tilde{\eta}_i + i\tilde{\chi}_i \right) \end{pmatrix} \\ &= e^{i\sigma_i} \begin{pmatrix} e^{-i\sigma_i} h_i^+ \\ \frac{1}{\sqrt{2}} \left(\hat{w}_i + e^{-i\sigma_i}\tilde{\eta}_i + i e^{-i\sigma_i}\tilde{\chi}_i \right) \end{pmatrix} \\ &\equiv e^{i\sigma_i} \begin{pmatrix} h_i^+ \\ \frac{1}{\sqrt{2}} \left(\hat{w}_i + \tilde{\eta}_i + i\tilde{\chi}_i \right) \end{pmatrix}, \end{aligned} \tag{3.1.5}$$

where in the last equality we abused the mathematical notation not to introduce additional abbreviations for the fields. We, however, will make a note and relate both cases so that the fields are expressed in terms of eq. (2.1.3) if a non-trivial re-definition of the fields is performed. After changing the basis, the $SU(2)$ doublets are no longer consistent with eq. (2.1.3).

Due to the $U(1)$ invariance of the scalar potential, it is possible to rotate all of the doublets by the same phase:

$$\begin{pmatrix} h_1' \\ h_2' \\ h_S' \end{pmatrix} = e^{-i\sigma_2} \mathcal{I}_3 \begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix}, \tag{3.1.6}$$

so that VEVs are now:

$$\left\{ \hat{w}_1 e^{i(\sigma_1 - \sigma_2)},\, \hat{w}_2,\, 0 \right\} \equiv \left\{ \hat{w}_1 e^{i\sigma},\, \hat{w}_2,\, 0 \right\}. \tag{3.1.7}$$

In this case, the Higgs basis rotation $\mathcal{R}_{\mathrm{HB}}$ is given by:

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = \begin{pmatrix} \mathrm{c}_\beta & \mathrm{s}_\beta & 0 \\ -\mathrm{s}_\beta & \mathrm{c}_\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\sigma} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_1' \\ h_2' \\ h_S' \end{pmatrix}. \tag{3.1.8}$$

3.1.2 The Mass-Squared Matrices

Of particular interest are mass-eigenstates and therefore we need to find rotation matrices $\mathcal{R}$ of gauge-eigenstates, which would result in diagonal mass-squared matrices:

$$\hat{\mathcal{M}}^2 = \mathcal{R}\mathcal{M}^2\mathcal{R}^\dagger, \tag{3.1.9}$$

where the hatted mass-squared matrix $\hat{\mathcal{M}}^2$ indicates that it is a diagonal matrix.

Before we evaluate the mass-squared matrices we want to show that those are block diagonal for the C-III-c model and thus there is no mixing between the S_3 doublet and S_3 singlet parts as we are interested in the inert S_3 singlet. The only couplings, apart from λ_4 as in C-III-c it is set to zero, which could result in a mixing of the S_3 doublet and singlet are λ_5, or λ_6, or λ_7. In terms of the doublets h_i, the dependence is either $\sim h_1^2 h_S^2$ or $\sim h_2^2 h_S^2$. If we take into consideration the vacuum configuration, especially that $\hat{w}_S = 0$, it becomes obvious that there is no mixing: the only case when the terms are non-zero is when $h_i^2 \to \hat{w}_i^2$, for $i = \overline{1,2}$, and therefore from eq. (2.3.14) it follows that the S_3 singlet fields acquire mass terms $\mathcal{M}^2_{(\xi_S^*)_i(\xi_S)_j}$ from:

$$V_2\Big(\left(\xi_k^S\right)^* \xi_l^S\Big) \text{ and } V_4\Big(\hat{w}_i\hat{w}_j \left(\xi_k^S\right)^* \xi_l^S\Big), \text{ for } i = \overline{1,2}, \tag{3.1.10}$$

where $\xi^S = \{h_S^\pm, \tilde{\eta}_S, \tilde{\chi}_S\}$. Thus we can treat the S_3 singlet mass terms separately. On the other hand, if we did not have the $\lambda_4 = 0$ constraint, this would obviously lead to mixing. This is governed by the fact that there would be terms proportional to $h_1^2 h_2 h_S$ and $h_2^3 h_S$, *i.e.*, even in the limit of $\hat{w}_S = 0$, mixing would arise between the S_3 doublet and singlet gauge fields. Since $\lambda_4 = 0$ we get no mixing.

We consider that the $SU(2)$ doublets are parameterized by (3.1.6) and the overall phase σ is extracted afterwards as in eq. (3.1.5):

$$h_1 \to h_1 e^{i\sigma}, \quad h_2 \to h_2, \quad h_S \to h_S, \tag{3.1.11}$$

and the VEVs are given by (3.1.7). Therefore, only a single overall phase $\sigma = \sigma_1 - \sigma_2$ is present. All further calculations are performed assuming that the $SU(2)$ doublets in terms of the gauge fields are given by:

$$\begin{aligned}
h_1 &= e^{i(\sigma_1-\sigma_2)} \begin{pmatrix} e^{-i\sigma_1} h_1^+ \\ \frac{1}{\sqrt{2}}\left(\hat{w}_1 + e^{-i\sigma_1}\tilde{\eta}_1 + ie^{-i\sigma_1}\tilde{\chi}_1\right) \end{pmatrix} \equiv e^{i\sigma} \begin{pmatrix} h_1^+ \\ \frac{1}{\sqrt{2}}\left(\hat{w}_1 + \tilde{\eta}_1 + i\tilde{\chi}_1\right) \end{pmatrix}, \\
h_2 &= e^{i(\sigma_2-\sigma_2)} \begin{pmatrix} e^{-i\sigma_2} h_2^+ \\ \frac{1}{\sqrt{2}}\left(\hat{w}_2 + e^{-i\sigma_2}\tilde{\eta}_2 + ie^{-i\sigma_2}\tilde{\chi}_2\right) \end{pmatrix} \equiv \begin{pmatrix} h_2^+ \\ \frac{1}{\sqrt{2}}\left(\hat{w}_2 + \tilde{\eta}_2 + i\tilde{\chi}_2\right) \end{pmatrix}, \\
h_S &= e^{i(\sigma_2-\sigma_2)} \begin{pmatrix} e^{-i\sigma_2} h_S^+ \\ \frac{1}{\sqrt{2}}\left(e^{-i\sigma_2}\tilde{\eta}_S + ie^{-i\sigma_2}\tilde{\chi}_S\right) \end{pmatrix} \equiv \begin{pmatrix} h_S^+ \\ \frac{1}{\sqrt{2}}\left(\tilde{\eta}_S + i\tilde{\chi}_S\right). \end{pmatrix}
\end{aligned} \tag{3.1.12}$$

After performing the rotation $\mathcal{R}_\beta$ of eq. (3.1.3), the charged mass-squared matrix is already block-diagonal:

$$\hat{\mathcal{M}}^2_{\text{Charged}} = \mathcal{R}_\beta\, \mathcal{M}^2_{\text{Charged}}\, \mathcal{R}_\beta^{-1}. \tag{3.1.13}$$

We get the following masses of the charged scalars:

$$m^2_{H^\pm} = 2\lambda_2 v^2, \tag{3.1.14a}$$

$$m^2_{S^\pm} = \mu_0^2 + \frac{1}{2}\lambda_5 v^2. \tag{3.1.14b}$$

In order to get the charged physical fields and to identify the would-be Goldstone boson, one needs to apply the rotation matrix to the gauge fields. In terms of the charged sector, the mass-eigenstates are given by:

$$\mathcal{R}_\beta \begin{pmatrix} h_1^+ \\ h_2^+ \\ h_S^+ \end{pmatrix} = \begin{pmatrix} G^+ \\ H^+ \\ S^+ \end{pmatrix}, \quad \begin{pmatrix} G^- \\ H^- \\ S^- \end{pmatrix} = \begin{pmatrix} (G^+)^\dagger \\ (H^+)^\dagger \\ (S^+)^\dagger \end{pmatrix}. \tag{3.1.15}$$

We write down explicitly the charged fields:

$$G^\pm = \mathrm{c}_\beta h_1^\pm + \mathrm{s}_\beta h_2^\pm, \tag{3.1.16a}$$

$$H^\pm = -\mathrm{s}_\beta h_1^\pm + \mathrm{c}_\beta h_2^\pm, \tag{3.1.16b}$$

$$S^\pm = h_S^\pm. \tag{3.1.16c}$$

Next, we consider the neutral scalar sector. As discussed before, there is no mixing between the S_3 doublet and singlet, *i.e.*, in the basis $(h_1,\, h_2)$ - h_S. Consider the gauge fields of the S_3 doublet. The $\mathcal{R}_\beta$ rotation results in the following fields:

$$\begin{pmatrix} H_1' \\ H_2' \\ G^0 \\ H_3' \end{pmatrix} = \mathcal{R}_{\beta\text{-}4} \begin{pmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \\ \tilde{\chi}_1 \\ \tilde{\chi}_2 \end{pmatrix}, \tag{3.1.17}$$

where

$$\mathcal{R}_{\beta\text{-}4} = \mathcal{I}_2 \otimes \mathcal{R}_{\beta\text{-}2}. \tag{3.1.18}$$

We identify the neutral would-be Goldstone boson as $G^0 = \mathrm{c}_\beta \tilde{\chi}_1 + \mathrm{s}_\beta \tilde{\chi}_2$, governed by the same logic as when identifying the charged would-be Goldstone boson $G^\pm$ in eq. (3.1.15).

After identifying the state corresponding to the would-be Goldstone boson, we take a look at the neutral sector mass-squared matrix:

$$\mathcal{M}^2_{\mathrm{Neutral}-12} = \left.\frac{\partial^2 V}{\partial \zeta_i^{12} \partial \zeta_j^{12}}\right|_{\langle v \rangle} = \begin{pmatrix} (\mathcal{M}_a^2)_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.1.19}$$

where

$$\zeta_i^{12} = \{H_1',\ H_2',\ H_3'\}. \tag{3.1.20}$$

An interesting feature of the C-III-c vacuum configuration can be seen. It is obvious that there are two massless scalars as there is only one non-zero element present in the matrix $\mathcal{M}^2_{\mathrm{Neutral}-12}$. This was explained in section 2.3. Due to the structure of $\mathcal{M}^2_{\mathrm{Neutral}-12}$, as it is already diagonal, we identify the physical states:

$$H_1 = \mathrm{c}_\beta \tilde{\eta}_1 + \mathrm{s}_\beta \tilde{\eta}_2, \tag{3.1.21a}$$
$$H_2 = -\,\mathrm{s}_\beta \tilde{\eta}_1 + \mathrm{c}_\beta \tilde{\eta}_2, \tag{3.1.21b}$$
$$H_3 = -\,\mathrm{s}_\beta \tilde{\chi}_1 + \mathrm{c}_\beta \tilde{\chi}_2. \tag{3.1.21c}$$

Out of all these states only one is massive:

$$m_{H_1}^2 = 2\left(\lambda_1 - \lambda_2\right) v^2, \tag{3.1.22a}$$
$$m_{H_2}^2 = 0, \tag{3.1.22b}$$
$$m_{H_3}^2 = 0. \tag{3.1.22c}$$

In order to get massive scalar states H_2 and H_3 one needs to softly break the S_3-symmetric potential. This case is taken into consideration in section 3.2.

The only part we are left with is to determine the mass-squared matrix of the S_3 singlet:

$$\mathcal{M}^2_{\mathrm{Neutral}-\mathrm{S}} = \left.\frac{\partial^2 V}{\partial \zeta_i^{S} \partial \zeta_j^{S}}\right|_{\langle v \rangle} = \begin{pmatrix} (\mathcal{M}_b^2)_{11} & (\mathcal{M}_b^2)_{12} \\ (\mathcal{M}_b^2)_{12} & (\mathcal{M}_b^2)_{22} \end{pmatrix}, \tag{3.1.23}$$

where

$$\zeta_i^S = \{\tilde{\eta}_S,\ \tilde{\chi}_S\}. \tag{3.1.24}$$

The elements of the mass-squared matrix are:

$$(\mathcal{M}_b^2)_{11} = \mu_0^2 + \frac{1}{2}\left(\lambda_5 + \lambda_6\right) v^2 + \lambda_7 \left(\mathrm{c}_{2\sigma} \hat{w}_1^2 + \hat{w}_2^2\right), \tag{3.1.25a}$$
$$(\mathcal{M}_b^2)_{12} = \lambda_7 \mathrm{s}_{2\sigma} \hat{w}_1^2, \tag{3.1.25b}$$

$$(\mathcal{M}_b^2)_{22} = \mu_0^2 + \frac{1}{2}\left(\lambda_5 + \lambda_6\right) v^2 - \lambda_7 \left(\mathrm{c}_{2\sigma}\hat{w}_1^2 + \hat{w}_2^2\right). \tag{3.1.25c}$$

The mass-squared matrix $\mathcal{M}^2_{\text{Neutral}-\text{S}}$ is diagonalizable by performing the following rotation:

$$\hat{\mathcal{M}}^2_{\text{Neutral}-\text{S}} = \mathcal{R}_\gamma \mathcal{M}^2_{\text{Neutral}-\text{S}} \mathcal{R}_\gamma^{-1}, \tag{3.1.26}$$

where the rotation matrix is:

$$\mathcal{R}_\gamma = \begin{pmatrix} \mathrm{c}_\gamma & \mathrm{s}_\gamma \\ -\mathrm{s}_\gamma & \mathrm{c}_\gamma \end{pmatrix}. \tag{3.1.27}$$

In this case the rotation angle γ is not trivial anymore. Calculations yield that the rotation angle γ is determined by[1]:

$$\mathrm{t}_\gamma = \frac{\mathrm{s}_{2\sigma}\hat{w}_1^2}{\mathrm{c}_{2\sigma}\hat{w}_1^2 + \hat{w}_2^2 + \sqrt{\hat{w}_1^4 + 2\mathrm{c}_{2\sigma}\hat{w}_1^2\hat{w}_2^2 + \hat{w}_2^4}}. \tag{3.1.28}$$

We get the following masses of the S_3 singlet scalars:

$$m^2_{S_1} = \mu_0^2 + \frac{1}{2}\left(\lambda_5 + \lambda_6\right) v^2 - \lambda_7 \sqrt{\hat{w}_1^4 + 2\mathrm{c}_{2\sigma}\hat{w}_1^2\hat{w}_2^2 + \hat{w}_2^4}, \tag{3.1.29a}$$

$$m^2_{S_2} = \mu_0^2 + \frac{1}{2}\left(\lambda_5 + \lambda_6\right) v^2 + \lambda_7 \sqrt{\hat{w}_1^4 + 2\mathrm{c}_{2\sigma}\hat{w}_1^2\hat{w}_2^2 + \hat{w}_2^4}. \tag{3.1.29b}$$

The corresponding neutral fields are:

$$S_1 = \mathrm{c}_\gamma \tilde{\eta}_S + \mathrm{s}_\gamma \tilde{\chi}_S, \tag{3.1.30a}$$

$$S_2 = -\mathrm{s}_\gamma \tilde{\eta}_S + \mathrm{c}_\gamma \tilde{\chi}_S. \tag{3.1.30b}$$

The $SU(2)$ doublets in terms of the mass-eigenstates in the basis of eq. (3.1.12) are:

$$h_1 = e^{i\sigma} \begin{pmatrix} \mathrm{c}_\beta G^+ - \mathrm{s}_\beta H^+ \\ \frac{1}{\sqrt{2}}\left(\hat{w}_1 + \mathrm{c}_\beta H_1 - \mathrm{s}_\beta H_2 + i\left(\mathrm{c}_\beta G^0 - \mathrm{s}_\beta H_3\right)\right) \end{pmatrix}, \tag{3.1.31a}$$

$$h_2 = \begin{pmatrix} \mathrm{s}_\beta G^+ + \mathrm{c}_\beta H^+ \\ \frac{1}{\sqrt{2}}\left(\hat{w}_2 + \mathrm{s}_\beta H_1 + \mathrm{c}_\beta H_2 + i\left(\mathrm{s}_\beta G^0 + \mathrm{c}_\beta H_3\right)\right) \end{pmatrix}, \tag{3.1.31b}$$

$$h_S = \begin{pmatrix} S^+ \\ \frac{1}{\sqrt{2}}\left(e^{i\gamma}\left(S_1 + iS_2\right)\right) \end{pmatrix}. \tag{3.1.31c}$$

The $SU(2)$ doublets, denoted as H_i, in the Higgs basis are trivially simplified to:

$$H_1 = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(v + \varphi_1 + iG^0\right) \end{pmatrix}, \tag{3.1.32a}$$

$$H_2 = \begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}}\left(\varphi_2 + i\varphi_3\right) \end{pmatrix}, \tag{3.1.32b}$$

$$H_3 = \begin{pmatrix} S^+ \\ \frac{1}{\sqrt{2}}\left(e^{i\gamma}\left(S_1 + iS_2\right)\right) \end{pmatrix}, \tag{3.1.32c}$$

where we used φ_i for the neutral fields to distinguish those from the $SU(2)$ doublets as it is common to use the H_i notation for the $SU(2)$ doublets in the Higgs basis. The neutral fields φ_i are the same as the H_i fields of eq. (3.1.21) discussed in this section.

[1] It should be noted that this solution is not unique.

3.1.3 The Mass-Squared Matrices in Different Bases

So far we have considered that the C-III-c vacuum is: $\left\{\hat{w}_1 e^{i\sigma},\, \hat{w}_2,\, 0\right\}$. For completeness, we would like to briefly mention the other possible choices and what they result in.

The generic vacuum configuration $\{\hat{w}_1 e^{i\sigma_1},\, \hat{w}_2 e^{i\sigma_2},\, 0\}$ results in a hermitian charged mass-squared matrix:

$$\mathcal{M}^2_{\text{Charged}} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & 0 \\ \mathcal{M}^*_{12} & \mathcal{M}_{22} & 0 \\ 0 & 0 & \mathcal{M}_{33} \end{pmatrix}, \tag{3.1.33}$$

and a symmetric neutral mass-squared matrices:

$$\mathcal{M}^2_{\text{Neutral}} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} & \mathcal{M}_{14} & 0 & 0 \\ \mathcal{M}_{12} & \mathcal{M}_{22} & \mathcal{M}_{23} & \mathcal{M}_{24} & 0 & 0 \\ \mathcal{M}_{13} & \mathcal{M}_{23} & \mathcal{M}_{33} & \mathcal{M}_{34} & 0 & 0 \\ \mathcal{M}_{14} & \mathcal{M}_{24} & \mathcal{M}_{34} & \mathcal{M}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{M}_{55} & \mathcal{M}_{56} \\ 0 & 0 & 0 & 0 & \mathcal{M}_{56} & \mathcal{M}_{66} \end{pmatrix}, \tag{3.1.34}$$

in the basis:

$$\{\tilde{\eta}_1,\, \tilde{\eta}_2,\, \tilde{\chi}_1,\, \tilde{\chi}_2,\, \tilde{\eta}_S,\, \tilde{\chi}_S\}. \tag{3.1.35}$$

The following $\mathcal{M}^2_{\text{Neutral}}$ of this section are evaluated in this basis. Although this is the most trivial choice of the $SU(2)$ doublets, it results in a fact that the charged mass-squared matrix is diagonalizable by a complex matrix while the neutral states are potentially combinations of the gauge-eigenstates $\tilde{\eta}_i$ - $\tilde{\chi}_i$.

If the σ_i phases are extracted from the $SU(2)$ doublets $h_i \to e^{i\sigma_i} h_i$ as in eq. (3.1.5), this results in:

$$\mathcal{M}^2_{\text{Charged}} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & 0 \\ \mathcal{M}_{12} & \mathcal{M}_{22} & 0 \\ 0 & 0 & \mathcal{M}_{33} \end{pmatrix}, \tag{3.1.36a}$$

$$\mathcal{M}^2_{\text{Neutral}} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & 0 & 0 & 0 & 0 \\ \mathcal{M}_{12} & \mathcal{M}_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{M}_{55} & \mathcal{M}_{56} \\ 0 & 0 & 0 & 0 & \mathcal{M}_{56} & \mathcal{M}_{66} \end{pmatrix}. \tag{3.1.36b}$$

The charged mass-eigenstates no longer depend on a phase and the neutral mass-eigenstates do not mix the $\tilde{\eta}_i$ and $\tilde{\chi}_i$ fields.

In Ref. [48] it was shown that the C-III-c vacuum configuration can be written in a basis in which VEVs are given by:

$$\{a e^{i\delta},\, a e^{-i\delta},\, 0\}. \tag{3.1.37}$$

This specific choice of the basis results in $\mathrm{t}_\beta = 1$ of eq. (3.1.4). It can be proven that eq. (3.1.37) is a special basis of the C-III-c model by considering additional symmetries of section 2.3. The $\lambda_4 = 0$ constraint results in a continuous $SO(2)$ symmetry. This fact can be used to rotate the active doublets:

$$\begin{pmatrix} \mathrm{c}_\theta & \mathrm{s}_\theta \\ -\mathrm{s}_\theta & \mathrm{c}_\theta \end{pmatrix} \begin{pmatrix} e^{-i\sigma_2} & 0 \\ 0 & e^{-i\sigma_2} \end{pmatrix} \begin{pmatrix} \hat{w}_1 e^{i\sigma_1} \\ \hat{w}_2 e^{i\sigma_2} \end{pmatrix} = \begin{pmatrix} \hat{w}_1 \mathrm{c}_\theta \mathrm{c}_\sigma + \hat{w}_2 \mathrm{s}_\theta + i \hat{w}_1 \mathrm{c}_\theta \mathrm{s}_\sigma \\ -\hat{w}_1 \mathrm{c}_\sigma \mathrm{s}_\theta + \hat{w}_2 \mathrm{c}_\theta - i \hat{w}_1 \mathrm{s}_\theta \mathrm{s}_\sigma \end{pmatrix} \equiv \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix}. \tag{3.1.38}$$

The complex VEVs $\tilde{w}_i$ can be written as:

$$\tilde{w}_i' = |\tilde{w}_i| e^{i \arg(\tilde{w}_i)}. \tag{3.1.39}$$

The modulus or the phases[2] of the complex values $\tilde{w}_i'$ can be related as there is a free θ parameter. Provided that the absolute values are equal

$$\mathrm{t}_{2\theta} = \frac{\hat{w}_2^2 - \hat{w}_1^2}{2\hat{w}_1\hat{w}_2 \mathrm{c}_\sigma}, \tag{3.1.40}$$

this results in the vacuum configuration: $\{ae^{i\delta_1},\, ae^{i\delta_2},\, 0\}$, with

$$a = \sqrt{\hat{w}_1^2 \mathrm{c}_\theta^2 \mathrm{s}_\sigma^2 + (\hat{w}_1 \mathrm{c}_\theta \mathrm{c}_\sigma + \hat{w}_2 s_\theta)^2}. \tag{3.1.41}$$

Due to the $U(1)$ invariance of the potential, an overall rotation by $e^{-\frac{i}{2}(\delta_1+\delta_2)}$ is possible. This results in the vacuum configuration: $\{ae^{i\delta}, ae^{-i\delta}, 0\}$. The δ phases can be extracted as follows: $h_1 \to e^{i\delta} h_1$ and $h_2 \to e^{-i\delta} h_2$. In this case, the mass-squared matrices are:

$$\mathcal{M}^2_{\mathrm{Charged}} = \begin{pmatrix} \mathcal{M}_{11} & -\mathcal{M}_{11} & 0 \\ -\mathcal{M}_{11} & \mathcal{M}_{11} & 0 \\ 0 & 0 & \mathcal{M}_{33} \end{pmatrix}, \tag{3.1.42a}$$

$$\mathcal{M}^2_{\mathrm{Neutral}} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{11} & 0 & 0 & 0 & 0 \\ \mathcal{M}_{11} & \mathcal{M}_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{M}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{M}_{66} \end{pmatrix}. \tag{3.1.42b}$$

Benefits of this basis are a fixed $\beta = \pi/4$ parameter as well as the diagonalized S_3 singlet states.

It can be seen that in different bases the mass-eigenstates can be expressed differently. The mass squared parameters are physical quantities and therefore are basis independent. Although there is no preferred basis, it should be noted that based on what one is interested in, a re-definition of the $SU(2)$ doublets is valid as long as it is unitary and results in a special direction of the mass-eigenstates.

3.2 Soft Symmetry Breaking

We next consider the possibility to promote massless states of the C-III-c model to massive ones. One of the possible solutions is the soft symmetry breaking concept. The idea behind this method is to introduce additional bilinear terms V_2' and therefore softly break the S_3 symmetry:

$$\tilde{V} = V_2 + V_2' + V_4, \tag{3.2.1}$$

The most general form of these terms is [49]:

$$V_2' = \frac{1}{2}\nu^2 \left(h_1^\dagger h_2 + \mathrm{h.c.}\right) + \mu_2^2 \left(h_1^\dagger h_1 - h_2^\dagger h_2\right) + \frac{1}{2}\mu_3^2 \left(h_S^\dagger h_1 + \mathrm{h.c.}\right) + \frac{1}{2}\mu_4^2 \left(h_S^\dagger h_2 + \mathrm{h.c.}\right). \tag{3.2.2}$$

Due to the fact that there are new terms present in the scalar potential, this leads to altered minimization conditions. We assume that the C-III-c model VEVs are consistent with eq. (3.0.1): $\left\{\hat{w}_1 e^{i\sigma_1},\, \hat{w}_2 e^{i\sigma_2},\, 0\right\}$ if not specified otherwise. The $SU(2)$ doublets are expanded in terms of VEVs

[2] There is no meaningful solutions when trying to solve for θ in terms of $\arg(\tilde{w}_1) = \arg(\tilde{w}_2)$.

before differentiating the potential and regardless can be fixed if those are a subject of the minimization conditions. To distinguish the softly broken models from the exact S_3-symmetric C-III-c model we denote these models as C-III-c-X, where X is a string of the softly broken parameters.

The most general model is C-III-c-ν^2-μ_2^2-μ_3^2-μ_4^4. This model results in the following constraints after minimizing the scalar potential:

$$\nu^2 = -4\left(\lambda_2+\lambda_3\right) c_{\sigma_1-\sigma_2}\hat{w}_1\hat{w}_2, \tag{3.2.3a}$$
$$\mu_1^2 = -\left(\lambda_1-\lambda_2\right) v^2, \tag{3.2.3b}$$
$$\mu_2^2 = -\left(\lambda_2+\lambda_3\right)\left(\hat{w}_1^2-\hat{w}_2^2\right), \tag{3.2.3c}$$
$$\mu_3^2 = -2\lambda_4 c_{2(\sigma_1-\sigma_2)}\hat{w}_1\hat{w}_2, \tag{3.2.3d}$$
$$\mu_4^2 = -\lambda_4\left(\hat{w}_1^2-\hat{w}_2^2\right). \tag{3.2.3e}$$

In this case, we no longer get the $\lambda_4=0$ constraint and thus there arises additional mixing between the S_3 doublet and singlet. It is clear that the soft breaking terms μ_3^2 and μ_4^2 are the ones responsible for the non-zero coupling value of λ_4. To be more precise, the aforementioned soft breaking terms are proportional to $h_S^\dagger h_i$ and are not consistent with the constraint $\lambda_4=0$. From another point of view, it should be noted that both μ_3^2 and μ_4^2 couplings break the $\mathbb{Z}_2$ symmetry.

In principle, one could solve for minimization conditions not in terms of μ_3^2 and μ_4^2, but in terms of λ_4:

$$\lambda_4 = \frac{\mu_3^2 c_{\sigma_1}\hat{w}_1+\mu_4^2 c_{\sigma_2}\hat{w}_2}{-\left(c_{2\sigma_1-\sigma_2}+2c_{\sigma_2}\right)\hat{w}_1^2\hat{w}_2+c_{\sigma_2}\hat{w}_2^3}. \tag{3.2.4}$$

Assume that μ_4^2 depends on μ_3^2 so that:

$$\mu_4^2 = -\mu_3^2\frac{c_{\sigma_1}\hat{w}_1}{c_{\sigma_2}\hat{w}_2}. \tag{3.2.5}$$

Although it might seem that such dependence leads to the constraint $\lambda_4=0$, this is not entirely correct. The λ_4 minimization condition of eq. (3.2.4) is not sufficient to minimize the scalar potential as before two equations (3.2.3d, 3.2.3e) were solved and not a single one. The other derivative results in:

$$\mu_3^2\frac{s_{\sigma_1-\sigma_2}}{c_{\sigma_2}}\hat{w}_1 = 0. \tag{3.2.6}$$

A trivial $\mu_3^2=0$ is consistent with $\lambda_4=0$, but in this case both μ_3^2 and μ_4^2 vanish.

Another possible solution would be to consider $\sigma_2=\sigma_1$. The derivatives with respect to σ_i vanish and therefore the total number of independent minimization conditions is reduced to three:

$$\mu_1^2 = -\frac{\nu^2+4\left(\lambda_1+\lambda_3\right)\hat{w}_1\hat{w}_2}{4\hat{w}_1\hat{w}_2}v, \tag{3.2.7a}$$
$$\mu_2^2 = \frac{1}{4}\nu^2\frac{\hat{w}_1^2-\hat{w}_2^2}{\hat{w}_1\hat{w}_2}, \tag{3.2.7b}$$
$$\mu_3^2 = -\frac{\hat{w}_2}{\hat{w}_1}\left[\mu_4^2+\lambda_4\left(3\hat{w}_1^2-\hat{w}_2^2\right)\right]. \tag{3.2.7c}$$

This case does not result in the $\lambda_4=0$ constraint.

For completeness, we once again consider the minimization condition in terms of eq. (3.2.4), but this time relax the (3.2.5) condition as it was artificially introduced. The scalar potential is minimized provided that an additional constraint is satisfied:

$$\mu_3^2\left(\hat{w}_1^2-\hat{w}_2^2\right)-2\mu_4^2 c_{\sigma_1-\sigma_2}\hat{w}_1\hat{w}_2 = 0. \tag{3.2.8}$$

Assume that we express μ_4^2 in terms of μ_3^2. Full minimization of the potential results in the $\mu_3^2\sim\lambda_4$ dependence. Provided that $\lambda_4=0$, both μ_3^2 and μ_4^2 vanish.

We are interested in a possible DM candidate and therefore the soft breaking terms of interest are reduced:

$$V_2' = \frac{1}{2}\nu^2\left(h_1^\dagger h_2 + h_2^\dagger h_1\right) + \mu_2^2\left(h_1^\dagger h_1 - h_2^\dagger h_2\right). \tag{3.2.9}$$

The quadratic part of the potential with respect to VEVs is therefore:

$$V_2'\big|_{\langle v\rangle} = \frac{1}{2}\left[\nu^2 \mathrm{c}_{\sigma_1-\sigma_2}\hat{w}_1\hat{w}_2 + \mu_2^2\left(\hat{w}_1^2 - \hat{w}_2^2\right)\right]. \tag{3.2.10}$$

For simplicity, we do not identify the fields in terms of the mass-eigenstates and therefore the most general mass-squared matrices are considered, *i.e.*, without defining the Goldstone bosons. All of the calculations are performed in the basis of eq. (2.1.3) unless specified otherwise.

We show that models with soft breaking parameters, which are consistent with the $\lambda_4 = 0$ constraint, result in the same eigenvalues for the charged sector and the S_3 singlet as for the exact C-III-c model by taking appropriate substitutions of the vacuum configuration if needed[3]. The only significant change remains for the S_3 doublet neutral eigenvalues. After all, the main idea behind why we consider soft symmetry breaking is to give masses to otherwise massless states.

3.2.1 The C-III-c-ν^2 Model

In the C-III-c-ν^2 model, the additional bilinear term is:

$$V_2' = \frac{1}{2}\nu^2\left(h_1^\dagger h_2 + h_2^\dagger h_1\right). \tag{3.2.11}$$

In principle, we do not assume that the numerical value of the term ν^2 is bounded by the μ_0^2 and μ_1^2 terms.

Due to the soft symmetry breaking term ν^2 derivatives get modified as follows:

$$\frac{\partial \tilde{V}}{\partial \hat{w}_1}\bigg|_{\langle v\rangle} = \frac{\partial V}{\partial \hat{w}_1}\bigg|_{\langle v\rangle} + \frac{1}{2}\nu^2\hat{w}_2\mathrm{c}_{(\sigma_1-\sigma_2)}, \tag{3.2.12a}$$

$$\frac{\partial \tilde{V}}{\partial \hat{w}_2}\bigg|_{\langle v\rangle} = \frac{\partial V}{\partial \hat{w}_2}\bigg|_{\langle v\rangle} + \frac{1}{2}\nu^2\hat{w}_1\mathrm{c}_{(\sigma_1-\sigma_2)}, \tag{3.2.12b}$$

$$\frac{\partial \tilde{V}}{\partial \sigma_1}\bigg|_{\langle v\rangle} = \frac{\partial V}{\partial \sigma_1}\bigg|_{\langle v\rangle} - \frac{1}{2}\nu^2\hat{w}_1\hat{w}_2\mathrm{s}_{(\sigma_1-\sigma_2)}, \tag{3.2.12c}$$

$$\frac{\partial \tilde{V}}{\partial \sigma_2}\bigg|_{\langle v\rangle} = \frac{\partial V}{\partial \sigma_2}\bigg|_{\langle v\rangle} + \frac{1}{2}\nu^2\hat{w}_1\hat{w}_2\mathrm{s}_{(\sigma_1-\sigma_2)}. \tag{3.2.12d}$$

The soft breaking term ν^2 can be defined by taking a look at the derivatives with respect to phases eq. (3.2.12c) and eq. (3.2.12d):

$$\nu^2 = -4\left(\lambda_2 + \lambda_3\right)\hat{w}_1\hat{w}_2\mathrm{c}_{(\sigma_1-\sigma_2)}. \tag{3.2.13}$$

In case of the exact S_3-symmetric potential we found that one of the minimization condition (3.1.1b) required $\lambda_2 + \lambda_3 = 0$. The soft symmetry breaking term ν^2 "eats" the $\lambda_2 + \lambda_3 = 0$ constraint and this results in a massive state. The constraint $\lambda_4 = 0$ survives as there are no additional terms coming from the derivative with respect to $\hat{w}_S$ (3.2.10). A direct inspection of the derivatives eq. (3.2.12a) and eq. (3.2.12b) leads to the constraint $\hat{w}_1 = \hat{w}_2$.

Taking everything into consideration we end up with the following vacuum configuration:

$$\left\{\hat{w}_1 e^{i\sigma_1},\, \hat{w}_1 e^{i\sigma_2},\, 0\right\}, \tag{3.2.14}$$

[3]We consider two models, *i.e.*, C-III-c-ν^2 and C-III-c-μ_2^2, which after solving for the minimization conditions constrain the vacuum configuration. In the C-III-c-ν^2 model absolute values $\hat{w}_i$ are further limited while in the C-III-c-μ_2^2 model the overall phases σ_i are fixed.

with constraints:

$$\lambda_4 = 0, \tag{3.2.15a}$$
$$\mu_1^2 = -\left(\lambda_1 - \lambda_2\right) v^2, \tag{3.2.15b}$$
$$\nu^2 = -2\left(\lambda_2 + \lambda_3\right) \mathrm{c}_{(\sigma_1 - \sigma_2)} v^2, \tag{3.2.15c}$$

where

$$v^2 = 2\hat{w}_1^2. \tag{3.2.16}$$

This model is analysed more thoroughly in chapter 4.

3.2.2 The C-III-c-μ_2^2 Model

Another possible soft breaking term is μ_2^2:

$$V_2' = \mu_2^2 \left(h_1^\dagger h_1 - h_2^\dagger h_2\right). \tag{3.2.17}$$

In this case derivatives get modified:

$$\left.\frac{\partial \tilde{V}}{\partial \hat{w}_1}\right|_{\langle v \rangle} = \left.\frac{\partial V}{\partial \hat{w}_1}\right|_{\langle v \rangle} + \mu_2^2 \hat{w}_1, \tag{3.2.18a}$$
$$\left.\frac{\partial \tilde{V}}{\partial \hat{w}_2}\right|_{\langle v \rangle} = \left.\frac{\partial V}{\partial \hat{w}_2}\right|_{\langle v \rangle} - \mu_2^2 \hat{w}_2. \tag{3.2.18b}$$

This results in the following minimization conditions:

$$\lambda_4 = 0, \tag{3.2.19a}$$
$$\mu_1^2 = -\frac{1}{2}\left(2\lambda_1 - \lambda_2 + \lambda_3 + \left(\lambda_2 + \lambda_3\right) \mathrm{c}_{2(\sigma_1 - \sigma_2)}\right) v^2, \tag{3.2.19b}$$
$$\mu_2^2 = -\left(\lambda_2 + \lambda_3\right) \mathrm{s}^2_{\sigma_1 - \sigma_2} \left(\hat{w}_1^2 - \hat{w}_2^2\right) \tag{3.2.19c}$$

alongside with

$$\left(\lambda_2 + \lambda_3\right) \mathrm{s}_{2(\sigma_1 - \sigma_2)} \hat{w}_1 \hat{w}_2 = 0. \tag{3.2.20}$$

Provided that $\mu_2^2 \neq 0$, either phases should be constrained or one of the VEVs should vanish. This leads to several possibilities.

As a first case, we consider the limit $\sigma_1 = \frac{1}{4}\pi$ and $\sigma_2 = -\sigma_1$, the minimization conditions are reduced to:

$$\lambda_4 = 0, \tag{3.2.21a}$$
$$\mu_1^2 = -\left(\lambda_1 - \lambda_2\right) v^2, \tag{3.2.21b}$$
$$\mu_2^2 = -\left(\lambda_2 + \lambda_3\right) \left(\hat{w}_1^2 - \hat{w}_2^2\right). \tag{3.2.21c}$$

The charged mass-squared matrix is given by:

$$\mathcal{M}^2_{\text{Charged}} = \begin{pmatrix} 2\lambda_2 \hat{w}_2^2 & 2i\lambda_2 \hat{w}_1 \hat{w}_2 & 0 \\ -2i\lambda_2 \hat{w}_1 \hat{w}_2 & 2\lambda_2 \hat{w}_1^2 & 0 \\ 0 & 0 & \mu_0^2 + \frac{1}{2}\lambda_5 v^2 \end{pmatrix}, \tag{3.2.22}$$

with eigenstates:

$$m^2_{H^\pm} = 2\lambda_2 v^2, \tag{3.2.23a}$$
$$m^2_{S^\pm} = \mu_0^2 + \frac{1}{2}\lambda_5 v^2. \tag{3.2.23b}$$

The charged scalar masses are identical to the ones of the exact C-III-c model (3.1.14). Note that the mass-squared matrix $\mathcal{M}^2_{\text{Charged}}$ is diagonalizable by a complex rotation matrix $\mathcal{R}$.

The S_3 doublet neutral-mass squared matrix in the basis

$$\{\tilde{\eta}_1,\, \tilde{\eta}_2,\, \tilde{\chi}_1,\, \tilde{\chi}_2\} \tag{3.2.24}$$

is of the following form:

$$\mathcal{M}^2_{\text{Neutral}-12} = \begin{pmatrix} \left(\mathcal{M}^2_a\right)_{11} & \left(\mathcal{M}^2_a\right)_{12} & \left(\mathcal{M}^2_a\right)_{13} & \left(\mathcal{M}^2_a\right)_{14} \\ \left(\mathcal{M}^2_a\right)_{12} & \left(\mathcal{M}^2_a\right)_{22} & \left(\mathcal{M}^2_a\right)_{23} & \left(\mathcal{M}^2_a\right)_{24} \\ \left(\mathcal{M}^2_a\right)_{13} & \left(\mathcal{M}^2_a\right)_{23} & \left(\mathcal{M}^2_a\right)_{11} & \left(\mathcal{M}^2_a\right)_{34} \\ \left(\mathcal{M}^2_a\right)_{14} & \left(\mathcal{M}^2_a\right)_{24} & \left(\mathcal{M}^2_a\right)_{34} & \left(\mathcal{M}^2_a\right)_{22} \end{pmatrix}, \tag{3.2.25}$$

where

$$\left(\mathcal{M}^2_a\right)_{11} = (\lambda_1 + \lambda_3)\, \hat{w}_1^2 + (\lambda_2 + \lambda_3)\, \hat{w}_2^2, \tag{3.2.26a}$$
$$\left(\mathcal{M}^2_a\right)_{12} = (\lambda_1 - \lambda_2)\, \hat{w}_1 \hat{w}_2, \tag{3.2.26b}$$
$$\left(\mathcal{M}^2_a\right)_{13} = (\lambda_1 + \lambda_3)\, \hat{w}_1^2 - (\lambda_2 + \lambda_3)\, \hat{w}_2^2, \tag{3.2.26c}$$
$$\left(\mathcal{M}^2_a\right)_{14} = -\,(\lambda_1 - 3\lambda_2 - 2\lambda_3)\, \hat{w}_1 \hat{w}_2, \tag{3.2.26d}$$
$$\left(\mathcal{M}^2_a\right)_{22} = (\lambda_2 + \lambda_3)\, \hat{w}_1^2 + (\lambda_1 + \lambda_3)\, \hat{w}_2^2, \tag{3.2.26e}$$
$$\left(\mathcal{M}^2_a\right)_{23} = (\lambda_1 - 3\lambda_2 - 2\lambda_3)\, \hat{w}_1 \hat{w}_2, \tag{3.2.26f}$$
$$\left(\mathcal{M}^2_a\right)_{24} = (\lambda_2 + \lambda_3)\, \hat{w}_1^2 - (\lambda_1 + \lambda_3)\, \hat{w}_2^2, \tag{3.2.26g}$$
$$\left(\mathcal{M}^2_a\right)_{34} = -\,(\lambda_1 - \lambda_2)\, \hat{w}_1 \hat{w}_2. \tag{3.2.26h}$$

The neutral scalars of the S_3 doublet acquire the following masses:

$$m^2_{H_1} = 2v^2\, (\lambda_2 + \lambda_3)\,, \tag{3.2.27a}$$
$$m^2_{H_2} = v^2\, (\lambda_1 + \lambda_3) - \Delta, \tag{3.2.27b}$$
$$m^2_{H_3} = v^2\, (\lambda_1 + \lambda_3) + \Delta, \tag{3.2.27c}$$

where

$$\Delta^2 = v^4\, (\lambda_1 + \lambda_3)^2 - 16\, (\lambda_1 - \lambda_2)\, (\lambda_2 + \lambda_3)\, \hat{w}_1^2 \hat{w}_2^2. \tag{3.2.28}$$

The mass-eigenstates are a combination of all gauge-eigenstates, which potentially leads to CP-indefinite states.

The mass-squared matrix of the S_3 singlet is:

$$\mathcal{M}^2_{\text{Neutral}-\text{S}} = \begin{pmatrix} \mu_0^2 + \frac{1}{2}\,(\lambda_5 + \lambda_6)\, v^2 & \lambda_7 \left(\hat{w}_1^2 - \hat{w}_2^2\right), \\ \lambda_7 \left(\hat{w}_1^2 - \hat{w}_2^2\right) & \mu_0^2 + \frac{1}{2}\,(\lambda_5 + \lambda_6)\, v^2 \end{pmatrix}, \tag{3.2.29}$$

with eigenvalues:

$$m^2_{S_1} = \mu_0^2 + \frac{1}{2}\,(\lambda_5 + \lambda_6)\, v^2 - \lambda_7 \left(\hat{w}_1^2 - \hat{w}_2^2\right), \tag{3.2.30a}$$
$$m^2_{S_2} = \mu_0^2 + \frac{1}{2}\,(\lambda_5 + \lambda_6)\, v^2 + \lambda_7 \left(\hat{w}_1^2 - \hat{w}_2^2\right). \tag{3.2.30b}$$

This coincides with masses of the exact C-III-c model (3.1.29) by substituting values of the σ_i phases.

Another possibility to satisfy eq. (3.2.20) is to consider that either $\hat{w}_1$ or $\hat{w}_2$ vanishes. If we substitute $\hat{w}_i = 0$ directly into the vacuum configuration, this no longer results in the C-III-c model. To be more precise, such change results in a special case of either R-I-1 or R-II-2. Both of these cases are of no particular interest. Therefore, we conclude that the C-III-c-μ_2^2 vacuum configuration is given by:

$$\{\hat{w}_1 e^{i\frac{\pi}{4}},\, \hat{w}_2 e^{-i\frac{\pi}{4}},\, 0\}. \tag{3.2.31}$$

3.2.3 The C-III-c-ν^2-μ_2^2 Model

The aforementioned soft symmetry breaking terms ν^2 and μ_2^2 can be combined together resulting in the following additional bilinear terms:

$$V_2' = \frac{1}{2}\nu^2 \left(h_1^\dagger h_2 + h_2^\dagger h_1\right) + \mu_2^2 \left(h_1^\dagger h_1 - h_2^\dagger h_2\right). \tag{3.2.32}$$

In this case constraints are:

$$\lambda_4 = 0, \tag{3.2.33a}$$
$$\mu_1^2 = -\left(\lambda_1 - \lambda_2\right) v^2, \tag{3.2.33b}$$
$$\mu_2^2 = -\left(\lambda_2 + \lambda_3\right)\left(\hat{w}_1^2 - \hat{w}_2^2\right), \tag{3.2.33c}$$
$$\nu^2 = -4\left(\lambda_2 + \lambda_3\right) \mathrm{c}_{\sigma_1-\sigma_2} \hat{w}_1 \hat{w}_2. \tag{3.2.33d}$$

The charged mass-squared matrix is given by:

$$\mathcal{M}^2_{\text{Charged}} = \begin{pmatrix} 2\lambda_2\hat{w}_2^2 & -2e^{-i(\sigma_1-\sigma_2)}\lambda_2\hat{w}_1\hat{w}_2 & 0 \\ -2e^{i(\sigma_1-\sigma_2)}\lambda_2\hat{w}_1\hat{w}_2 & 2\lambda_2\hat{w}_1^2 & 0 \\ 0 & 0 & \mu_0^2 + \frac{1}{2}\lambda_5 v^2 \end{pmatrix}, \tag{3.2.34}$$

with eigenvalues:

$$m_{H^\pm}^2 = 2v^2\lambda_2, \tag{3.2.35a}$$
$$m_{S^\pm}^2 = \mu_0^2 + \frac{1}{2}v^2\lambda_5. \tag{3.2.35b}$$

The charged scalar masses are identical to the ones of the exact C-III-c model (3.1.14).

The neutral-mass squared matrix of the S_3 doublet in the basis (3.2.24) is of the following form:

$$\mathcal{M}^2_{\text{Neutral}-12} = \begin{pmatrix} \left(\mathcal{M}_a^2\right)_{11} & \left(\mathcal{M}_a^2\right)_{12} & \left(\mathcal{M}_a^2\right)_{13} & \left(\mathcal{M}_a^2\right)_{14} \\ \left(\mathcal{M}_a^2\right)_{12} & \left(\mathcal{M}_a^2\right)_{22} & \left(\mathcal{M}_a^2\right)_{23} & \left(\mathcal{M}_a^2\right)_{24} \\ \left(\mathcal{M}_a^2\right)_{13} & \left(\mathcal{M}_a^2\right)_{23} & \left(\mathcal{M}_a^2\right)_{33} & \left(\mathcal{M}_a^2\right)_{34} \\ \left(\mathcal{M}_a^2\right)_{14} & \left(\mathcal{M}_a^2\right)_{24} & \left(\mathcal{M}_a^2\right)_{34} & \left(\mathcal{M}_a^2\right)_{44} \end{pmatrix}, \tag{3.2.36}$$

where

$$\left(\mathcal{M}_a^2\right)_{11} = 2\left(\lambda_1 + \lambda_3\right)\hat{w}_1^2 c_{\sigma_1}^2 + 2\left(\lambda_2 + \lambda_3\right)\hat{w}_2^2 c_{\sigma_2}^2, \tag{3.2.37a}$$
$$\left(\mathcal{M}_a^2\right)_{12} = 2\left(\lambda_1 - \lambda_2\right)\hat{w}_1\hat{w}_2 c_{\sigma_1} c_{\sigma_2}, \tag{3.2.37b}$$
$$\left(\mathcal{M}_a^2\right)_{13} = \left(\lambda_1 + \lambda_3\right)\hat{w}_1^2 s_{2\sigma_1} + \left(\lambda_2 + \lambda_3\right)\hat{w}_2^2 s_{2\sigma_2}, \tag{3.2.37c}$$
$$\left(\mathcal{M}_a^2\right)_{14} = 2\hat{w}_1\hat{w}_2\left[\left(\lambda_1 - 2\lambda_2 - \lambda_3\right) c_{\sigma_1} s_{\sigma_2} + \left(\lambda_2 + \lambda_3\right) c_{\sigma_2} s_{\sigma_1}\right], \tag{3.2.37d}$$
$$\left(\mathcal{M}_a^2\right)_{22} = 2\left(\lambda_2 + \lambda_3\right)\hat{w}_1^2 c_{\sigma_1}^2 + 2\left(\lambda_1 + \lambda_3\right)\hat{w}_2^2 c_{\sigma_2}^2, \tag{3.2.37e}$$
$$\left(\mathcal{M}_a^2\right)_{23} = 2\hat{w}_1\hat{w}_2\left[\left(\lambda_1 - 2\lambda_2 - \lambda_3\right) c_{\sigma_2} s_{\sigma_1} + \left(\lambda_2 + \lambda_3\right) c_{\sigma_1} s_{\sigma_2}\right], \tag{3.2.37f}$$
$$\left(\mathcal{M}_a^2\right)_{24} = \left(\lambda_2 + \lambda_3\right)\hat{w}_1^2 s_{2\sigma_1} + \left(\lambda_1 + \lambda_3\right)\hat{w}_2^2 s_{2\sigma_2}, \tag{3.2.37g}$$
$$\left(\mathcal{M}_a^2\right)_{33} = 2\left(\lambda_1 + \lambda_3\right)\hat{w}_1^2 s_{\sigma_1}^2 + 2\left(\lambda_2 + \lambda_3\right)\hat{w}_2^2 s_{\sigma_2}^2, \tag{3.2.37h}$$
$$\left(\mathcal{M}_a^2\right)_{34} = 2\left(\lambda_1 - \lambda_2\right)\hat{w}_1\hat{w}_2 s_{\sigma_1} s_{\sigma_2}, \tag{3.2.37i}$$
$$\left(\mathcal{M}_a^2\right)_{44} = 2\left(\lambda_2 + \lambda_3\right)\hat{w}_1^2 s_{\sigma_1}^2 + 2\left(\lambda_1 + \lambda_3\right)\hat{w}_2^2 s_{\sigma_2}^2. \tag{3.2.37j}$$

This results in the following mass-squared terms:

$$m_{H_1}^2 = 2v^2\left(\lambda_2 + \lambda_3\right), \tag{3.2.38a}$$
$$m_{H_2}^2 = v^2\left(\lambda_1 + \lambda_3\right) - \Delta, \tag{3.2.38b}$$

$$m_{H_3}^2 = v^2\left(\lambda_1+\lambda_3\right)+\Delta \tag{3.2.38c}$$

where

$$\Delta^2 = v^4\left(\lambda_1+\lambda_3\right)^2 - 16\left(\lambda_1-\lambda_2\right)\left(\lambda_2+\lambda_3\right)\mathrm{s}_{\sigma_1-\sigma_2}^2 \hat{w}_1^2\hat{w}_2^2. \tag{3.2.39}$$

The only difference from the C-III-c-μ_2^2 model is how the Δ parameter is defined. The Δ parameter for the C-III-c-μ_2^2 model was defined in (3.2.28). In the C-III-c-ν^2-μ_2^2 model the σ_i phases are no longer fixed and thus this results in additional dependence of the Δ parameter on $\mathrm{s}_{\sigma_1-\sigma_2}^2$.

The S_3 singlet mass-squared matrix is:

$$\mathcal{M}^2_{\mathrm{Neutral-S}} = \begin{pmatrix} \left(\mathcal{M}_b^2\right)_{11} & \left(\mathcal{M}_b^2\right)_{12} \\ \left(\mathcal{M}_b^2\right)_{12} & \left(\mathcal{M}_b^2\right)_{22} \end{pmatrix}, \tag{3.2.40}$$

where

$$\left(\mathcal{M}_b^2\right)_{11} = \mu_0^2 + \frac{1}{2}v^2\left(\lambda_5+\lambda_6\right)+\lambda_7\left(\mathrm{c}_{2\sigma_1}\hat{w}_1^2+\mathrm{c}_{2\sigma_2}\hat{w}_2^2\right), \tag{3.2.41a}$$

$$\left(\mathcal{M}_b^2\right)_{12} = \lambda_7\left(\mathrm{s}_{2\sigma_1}\hat{w}_1^2+\mathrm{s}_{2\sigma_2}\hat{w}_2^2\right), \tag{3.2.41b}$$

$$\left(\mathcal{M}_b^2\right)_{22} = \mu_0^2 + \frac{1}{2}v^2\left(\lambda_5+\lambda_6\right)-\lambda_7\left(\mathrm{c}_{2\sigma_1}\hat{w}_1^2+\mathrm{c}_{2\sigma_2}\hat{w}_2^2\right). \tag{3.2.41c}$$

The mass-eigenstates of the S_3 singlet are:

$$m_{S_1}^2 = \mu_0^2 + \frac{1}{2}v^2\left(\lambda_5+\lambda_6\right)-\lambda_7\sqrt{\hat{w}_1^4+2\mathrm{c}_{2(\sigma_1-\sigma_2)}\hat{w}_1^2\hat{w}_2^2+\hat{w}_2^4}, \tag{3.2.42a}$$

$$m_{S_2}^2 = \mu_0^2 + \frac{1}{2}v^2\left(\lambda_5+\lambda_6\right)+\lambda_7\sqrt{\hat{w}_1^4+2\mathrm{c}_{2(\sigma_1-\sigma_2)}\hat{w}_1^2\hat{w}_2^2+\hat{w}_2^4}, \tag{3.2.42b}$$

which coincide with the terms for the exact C-III-c model (3.1.29).

3.3 CP Violation in the Scalar Sector of the C-III-c Models

Our primarily assumption was that the couplings of the scalar potential are real. Although the scalar potential does not violate CP explicitly, CP can still be violated spontaneously. The concept of spontaneous CP violation makes sense only if the scalar Lagrangian conserves CP explicitly. Spontaneous CP violation in the context of the 2HDM was proposed in Ref. [50]. A method to check if there is spontaneous CP violation in the Multi-Higgs-Doublet Model (NHDM) was presented in Ref. [51]. The main idea behind the method is to check if there exists a symmetry U that leaves the scalar Lagrangian invariant, and the vacuum satisfies:

$$U:\ U_{ij}\left\langle h_j\right\rangle^* = \left\langle h_i\right\rangle. \tag{3.3.1}$$

If this is true, then there is no spontaneous CP violation. This relation follows from the general CP transformation of the $SU(2)$ doublets:

$$h_i \xrightarrow{CP} U'_{ij}h_j^*. \tag{3.3.2}$$

If the aforementioned mapping is possible, the CP is explicitly conserved. In general, the transformation matrix U'_{ij}, in terms of the 3HDM it is $U_{ij}\in U(3)$, is not a symmetry of the scalar Lagrangian. Provided that this symmetry leaves the scalar Lagrangian intact, eq. (3.3.1) is automatically satisfied. Nevertheless, CP violation can still be achieved through the Yukawa Lagrangian.

In the context of the C-III-c vacuum configuration, only the S_3 doublet should be checked. In Ref. [34] it was shown that the C-III-c vacuum configuration does not result in spontaneous CP violation. In Ref. [48] it was proposed that CP violation is easier to analyse in the Higgs basis[4]. We,

[4] After the Higgs basis transformation only a single real VEV is left. The complex couplings in the Higgs basis result in CP violation provided those cannot be rotated into the real ones.

governed by this assumption, however, use a slightly different method and check if the couplings in the Higgs basis are real. Provided that such reverse engineering results in a possible direction in the Higgs basis, the $SU(2)$ doublets in the generic basis can be re-constructed and checked if the result is in agreement with Refs. [34, 48]. The Higgs basis transformations are quite involved. The general approach can be found in section 2.5.

We consider the C-III-c and C-III-c-X (3.2.2) models consistent with the $\lambda_4 = 0$ constraint. The generic $SU(2)$ doublets (2.1.3) are rotated into the Higgs basis so that CP is explicitly conserved in the generic basis. The Higgs basis transformation is given by eq. (3.1.2). The Euler rotation matrices for the C-III-c model in the basis of eq. (2.5.14) are:

$$\mathrm{t}_{\beta_1} = \frac{\hat{w}_2}{\hat{w}_1}, \tag{3.3.3a}$$

$$\beta_2 = 0. \tag{3.3.3b}$$

The coefficients of the potential are:

$$Y_{11} = Y_{22} = \mu_1^2, \tag{3.3.4a}$$

$$Y_{33} = \mu_0^2, \tag{3.3.4b}$$

and

$$Z_{1111} = Z_{2222} = \lambda_1 - \lambda_2, \tag{3.3.5a}$$

$$Z_{1122} = 2\left(\lambda_1 + \lambda_2\right), \tag{3.3.5b}$$

$$Z_{1133} = Z_{2233} = \lambda_5, \tag{3.3.5c}$$

$$Z_{1221} = -4\lambda_2, \tag{3.3.5d}$$

$$Z_{1313} = \lambda_7\left(e^{-2i\sigma_1}\mathrm{c}_{\beta_1}^2 + e^{-2i\sigma_2}\mathrm{s}_{\beta_1}^2\right), \tag{3.3.5e}$$

$$Z_{1323} = \lambda_7 \mathrm{s}_{2\beta_1}\left(-e^{-2i\sigma_1} + e^{-2i\sigma_2}\right), \tag{3.3.5f}$$

$$Z_{1331} = Z_{2332} = \lambda_6, \tag{3.3.5g}$$

$$Z_{2323} = \lambda_7\left(e^{-2i\sigma_1}\mathrm{s}_{\beta_1}^2 + e^{-2i\sigma_2}\mathrm{c}_{\beta_1}^2\right), \tag{3.3.5h}$$

$$Z_{3333} = \lambda_8. \tag{3.3.5i}$$

At first, it might seem that the scalar potential in the Higgs basis is CP violating. In the Higgs basis there is a freedom to rotate the $SU(2)$ doublets with zero VEVs by a $U(2)$ transformation, see eq. (2.5.19). As a consequence, there is a direction in the Higgs basis where all of the couplings become real. The choice of the Higgs basis transformation

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\sigma} & e^{i\sigma} & 0 \\ -ie^{-i\sigma} & ie^{i\sigma} & 0 \\ 0 & 0 & \sqrt{2}i \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix}, \tag{3.3.6}$$

alongside the vacuum configuration $\{\hat{w}e^{i\sigma},\, \hat{w}e^{-i\sigma},\, 0\}$ results in:

$$Z_{1313} = -Z_{2323} = -\mathrm{c}_{2\sigma}\lambda_7, \tag{3.3.7a}$$

$$Z_{1323} = -2\mathrm{s}_{2\sigma}\lambda_7, \tag{3.3.7b}$$

and the other couplings are the same as in eqs. (3.3.4, 3.3.5). Therefore, the vacuum configuration of eq. (3.1.37) is CP conserving. We verify the claim of Refs. [34, 48] that the exact C-III-c vacuum configuration does not result in spontaneous CP violation[5]. It should be noted that the aforementioned Higgs basis rotation is not uniquely defined, *e.g.*, transformation

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{-i\sigma_1} & \frac{1}{\sqrt{2}}e^{i\sigma} & 0 \\ -\frac{i}{2}e^{-i\sigma} & \frac{i}{2}e^{i\sigma} & \frac{i}{\sqrt{2}} \\ \frac{i}{2}e^{-i\sigma} & -\frac{i}{2}e^{i\sigma} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix} \tag{3.3.8}$$

[5] It was shown that this specific vacuum configuration does not lead to explicit CP violation and there exists a unitary matrix which satisfies eq. (3.3.1).

with VEVs $\{\hat{w}e^{i\sigma},\, \hat{w}e^{-i\sigma},\, 0\}$ results in real couplings, but this time there are no vanishing couplings $Z_{ijkl} \neq 0$.

An interesting consequence is that although the exact C-III-c vacuum configuration is CP conserving, the soft symmetry breaking terms may result in CP violation. The corresponding case in the 2HDM was studied in Ref. [52]. When soft symmetry breaking terms are added, for the Higgs basis transformation (3.1.2), the quadratic terms result in:

$$Y_{11} = -\frac{1}{2}\nu^2 \mathrm{s}_{2\beta_1}\mathrm{c}_{\sigma_1-\sigma_2} + \mu_1^2 + \mathrm{c}_{2\beta_1}\mu_2^2, \tag{3.3.9a}$$

$$Y_{12} = \frac{1}{4}\nu^2 \left(\mathrm{c}_{2\beta_1+\sigma_1-\sigma_2} + \mathrm{c}_{2\beta_1-\sigma_1+\sigma_2} - 2i\mathrm{s}_{\sigma_1-\sigma_2}\right) + \mathrm{s}_{2\beta_1}\mu_2^2, \tag{3.3.9b}$$

$$Y_{13} = \frac{1}{2}e^{-i\sigma_1}\mathrm{c}_{\beta_1}\mu_3^2 - \frac{1}{2}e^{-i\sigma_2}\mathrm{s}_{\beta_1}\mu_4^2, \tag{3.3.9c}$$

$$Y_{22} = \frac{1}{2}\nu^2 \mathrm{s}_{2\beta_1}\mathrm{c}_{\sigma_1-\sigma_2} + \mu_1^2 - \mathrm{c}_{2\beta_1}\mu_2^2, \tag{3.3.9d}$$

$$Y_{23} = \frac{1}{2}e^{-i\sigma_1}\mathrm{s}_{\beta_1}\mu_3^2 + \frac{1}{2}e^{-i\sigma_2}\mathrm{c}_{\beta_1}\mu_4^2, \tag{3.3.9e}$$

$$Y_{33} = \mu_0^2. \tag{3.3.9f}$$

The Higgs basis transformation of eq. (3.3.6) results in complex quadratic couplings Y_{ij}:

$$Y_{12} = -\frac{1}{2}\nu^2 s_{2\sigma} + i\mu_2^2, \tag{3.3.10a}$$

$$Y_{13} = -\frac{i}{2\sqrt{2}}\left[e^{-i\sigma}\mu_3^2 + e^{i\sigma}\mu_4^2\right], \tag{3.3.10b}$$

$$Y_{23} = \frac{1}{2\sqrt{2}}\left[-e^{-i\sigma}\mu_3^2 + e^{i\sigma}\mu_4^2\right]. \tag{3.3.10c}$$

Of particular interest is only the Y_{12} coupling as the μ_3^2 and μ_4^2 couplings are inconsistent with $\lambda_4 = 0$. From eq. (3.3.10a) it follows that the C-III-c-ν^2 vacuum configuration does not violate CP spontaneously. The C-III-c-μ_2^2 vacuum configuration $\{\hat{w}e^{i\frac{\pi}{4}},\, \hat{w}e^{-i\frac{\pi}{4}},\, 0\}$ can be rotated into: $\{\hat{w}e^{i\delta},\, \hat{w},\, 0\}$, see eq. (3.1.38). Provided that the overall phase value is fixed at $\sigma = \pi/2$, the couplings in the Higgs basis become real. The C-III-c-ν^2-μ_2^2 model does not result in a real basis due to the dependence $Y \sim \mu_2^2 + i\nu^2\mathrm{s}_\sigma$. In terms of the generic vacuum configuration (3.0.1), the C-III-c-ν^2-μ_2^2 model results in real couplings in the Higgs basis provided that the vacuum configuration is given by $\{\hat{w}_1 e^{i\sigma},\, \hat{w}_2 e^{i\sigma},\, 0\}$. This configuration becomes real and is inconsistent with the C-III-c model.

Chapter 4

The C-III-c-ν^2 Model

In this chapter we revisit the C-III-c-ν^2 model presented in section 3.2.1. This model is of particular interest as it provides the simplest extension of the S_3 scalar potential and results in massive scalars as desired. Moreover, it does not result in spontaneous CP violation as claimed in section 3.3. This model is analysed by taking a look at tree-level couplings involving scalars. Theoretical constraints of the C-III-c-ν^2 model are also considered.

4.1 The Mass-Squared Matrices

If we expanded the scalar potential in terms of the generic $SU(2)$ doublet of eq. (2.1.3), the charged mass-squared matrix would be hermitian and the neutral mass-squared matrices would depend on trigonometric functions of both σ_1 and σ_2. In both cases, eigenvalues of the mass-squared matrices depend on a single combination of angles $\sigma_1 - \sigma_2$. The minimization condition in terms ν^2 is also a function of a single trigonometric function $\mathrm{c}_{\sigma_1-\sigma_2}$, see eq. (3.2.15c). It is possible to rotate one of the phases away. In this case the vacuum configuration of eq. (3.2.14) changes to the following form:

$$\left\{\hat{w}e^{i\sigma},\, \hat{w},\, 0\right\}, \tag{4.1.1}$$

where $\hat{w} \equiv \hat{w}_1$ and $\sigma \equiv \sigma_1 - \sigma_2$. The constraint ν^2 of eq. (3.2.15c) gets modified:

$$\nu^2 = -2\left(\lambda_2 + \lambda_3\right)\mathrm{c}_\sigma v^2. \tag{4.1.2}$$

We expand the scalar potential in terms of the following $SU(2)$ doublets:

$$\begin{aligned}
h_1 &= e^{-i\sigma_2}\begin{pmatrix} h_1^+ \\ \frac{1}{\sqrt{2}}\left(\hat{w}e^{i\sigma_1} + \tilde{\eta}_1 + i\tilde{\chi}_1\right)\end{pmatrix} \equiv e^{i\sigma}\begin{pmatrix} h_1^+ \\ \frac{1}{\sqrt{2}}\left(\hat{\omega} + \tilde{\eta}_1 + i\tilde{\chi}_1\right)\end{pmatrix}, \\
h_2 &= e^{-i\sigma_2}\begin{pmatrix} h_2^+ \\ \frac{1}{\sqrt{2}}\left(\hat{w}e^{i\sigma_2} + \tilde{\eta}_2 + i\tilde{\chi}_2\right)\end{pmatrix} \equiv \begin{pmatrix} h_2^+ \\ \frac{1}{\sqrt{2}}\left(\hat{\omega} + \tilde{\eta}_2 + i\tilde{\chi}_2\right)\end{pmatrix}, \\
h_S &= e^{-i\sigma_2}\begin{pmatrix} h_S^+ \\ \frac{1}{\sqrt{2}}\left(\tilde{\eta}_S + i\tilde{\chi}_S\right)\end{pmatrix} \equiv \begin{pmatrix} h_S^+ \\ \frac{1}{\sqrt{2}}\left(\tilde{\eta}_S + i\tilde{\chi}_S\right)\end{pmatrix},
\end{aligned} \tag{4.1.3}$$

which is equivalent to the transformation (3.1.12). This time, however, the $SU(2)$ doublets were simultaneously rotated by $e^{-i\sigma_2}$ due to the $U(1)$ invariance (3.1.6) without extracting the phases as in eq. (3.1.5).

The modulus of the VEVs is fixed, $\hat{w}_1 = \hat{w}_2$, the rotation angle is also defined, $\beta = \pi/4$. Therefore the rotation matrix $\mathcal{R}_\beta$ in this specific model is given by:

$$\mathcal{R}_\beta = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2}\end{pmatrix}. \tag{4.1.4}$$

Taking everything into consideration we get the following Higgs basis rotation:

$$\mathcal{R}_{\text{HB}} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\sigma} & 1 & 0 \\ -e^{-i\sigma} & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \tag{4.1.5}$$

After we determined the modified vacuum configuration and constraints of the C-III-c-ν^2 vacuum, we are ready to consider the mass-squared matrices. We start by investigating the charged scalar sector. It is diagonalizable by performing the rotation $\mathcal{R}_\beta$. The charged would-be Goldstone boson and the charged physical scalar fields are:

$$G^\pm = \frac{1}{\sqrt{2}} \left(h_1^\pm + h_2^\pm \right), \tag{4.1.6a}$$

$$H^\pm = \frac{1}{\sqrt{2}} \left(-h_1^\pm + h_2^\pm \right), \tag{4.1.6b}$$

$$S^\pm = h_S^\pm. \tag{4.1.6c}$$

The masses of the charged scalar bosons are:

$$m_{H^\pm}^2 = 2\lambda_2 v^2, \tag{4.1.7a}$$

$$m_{S^\pm}^2 = \mu_0^2 + \frac{1}{2}\lambda_5 v^2. \tag{4.1.7b}$$

We note that the masses $m_{H^\pm}^2$ and $m_{S^\pm}^2$ are the same as those of eq. (3.1.14) for the exact C-III-c model.

Next, we consider the neutral scalar sector. After performing the $\mathcal{R}_\beta$ rotation one can identify the would-be Goldstone boson:

$$G^0 = \frac{1}{\sqrt{2}} \left(\tilde{\chi}_1 + \tilde{\chi}_2 \right). \tag{4.1.8}$$

Afterwards we take a look at the neutral sector mass-squared matrix:

$$\mathcal{M}^2_{\text{Neutral}-12} = \left. \frac{\partial^2 \tilde{V}}{\partial \zeta_i^{12} \partial \zeta_j^{12}} \right|_{\langle v \rangle} = \begin{pmatrix} \left(\mathcal{M}_a^2\right)_{11} & 0 & \left(\mathcal{M}_a^2\right)_{13} \\ 0 & \left(\mathcal{M}_a^2\right)_{22} & 0 \\ \left(\mathcal{M}_a^2\right)_{13} & 0 & \left(\mathcal{M}_a^2\right)_{33} \end{pmatrix}, \tag{4.1.9}$$

where ζ_i^{12} is identical to the one of eq. (3.1.20). The non-zero elements of the mass-squared matrix $\mathcal{M}^2_{\text{Neutral}-12}$ are:

$$(\mathcal{M}_a^2)_{11} = 2 \left[\lambda_1 - \lambda_2 \mathrm{s}_\sigma^2 + \lambda_3 \mathrm{c}_\sigma^2 \right] v^2, \tag{4.1.10a}$$

$$(\mathcal{M}_a^2)_{13} = \left(\lambda_2 + \lambda_3 \right) \mathrm{s}_{2\sigma} v^2, \tag{4.1.10b}$$

$$(\mathcal{M}_a^2)_{22} = 2 \left(\lambda_2 + \lambda_3 \right) v^2, \tag{4.1.10c}$$

$$(\mathcal{M}_a^2)_{33} = 2 \left(\lambda_2 + \lambda_3 \right) \mathrm{s}_\sigma^2 v^2. \tag{4.1.10d}$$

Note that elements $(\mathcal{M}_a^2)_{13}$, and $(\mathcal{M}_a^2)_{22}$, and $(\mathcal{M}_a^2)_{33}$ vanish in the limit of $\lambda_2 + \lambda_3 = 0$. An additional rotation in terms of an angle α is needed for the complete diagonalization procedure:

$$\mathcal{R}_\alpha = \begin{pmatrix} \mathrm{c}_\alpha & 0 & \mathrm{s}_\alpha \\ 0 & 1 & 0 \\ -\mathrm{s}_\alpha & 0 & \mathrm{c}_\alpha \end{pmatrix}. \tag{4.1.11}$$

After performing the rotation in terms of α we get the following eigenvalues:

$$m_{H_1}^2 = 2 \left[\lambda_1 \mathrm{c}_\alpha^2 - \lambda_2 \left(\mathrm{c}_\alpha^2 - \mathrm{c}_{\alpha-\sigma}^2 \right) + \lambda_3 \mathrm{c}_{\alpha-\sigma}^2 \right] v^2, \tag{4.1.12a}$$

$$m_{H_2}^2 = 2\left(\lambda_2 + \lambda_3\right) v^2, \tag{4.1.12b}$$
$$m_{H_3}^2 = 2\left[\lambda_1 \mathrm{s}_\alpha^2 + \lambda_2\left(\mathrm{c}_\alpha^2 - \mathrm{c}_{\alpha-\sigma}^2\right) + \lambda_3 \mathrm{s}_{\alpha-\sigma}^2\right] v^2. \tag{4.1.12c}$$

These eigenvalues can further be simplified by identifying the rotation angle α:

$$\mathrm{t}_{2\alpha} = \frac{\left(\lambda_2 + \lambda_3\right) \mathrm{s}_{2\sigma}}{\lambda_1 - \lambda_2 + \left(\lambda_2 + \lambda_3\right) \mathrm{c}_{2\sigma}}\ , \tag{4.1.13}$$

and thus

$$m_{H_1}^2 = \left(\lambda_1 + \lambda_3 - \Delta\right) v^2, \tag{4.1.14a}$$
$$m_{H_2}^2 = 2\left(\lambda_2 + \lambda_3\right) v^2, \tag{4.1.14b}$$
$$m_{H_3}^2 = \left(\lambda_1 + \lambda_3 + \Delta\right) v^2, \tag{4.1.14c}$$

where for simplification we introduced an additional abbreviation

$$\Delta^2 = \left(\lambda_1 - \lambda_2\right)^2 + \left(\lambda_2 + \lambda_3\right)^2 + 2\left(\lambda_1 - \lambda_2\right)\left(\lambda_2 + \lambda_3\right) \mathrm{c}_{2\sigma}. \tag{4.1.15}$$

It follows that we can fix $0 \leq \sigma \leq \pi/2$ and $0 \leq \alpha \leq \pi/2$.

From dependence of masses on λ couplings it follows that masses are strictly ordered as $m_{H_1} < m_{H_2} < m_{H_3}$. Therefore m_{H_1} is the lightest scalar and we associate it with the SM-like Higgs boson.

The neutral S_3 doublet mass-eigenstates are:

$$H_1 = \frac{1}{\sqrt{2}}\left[\mathrm{c}_\alpha\left(\tilde{\eta}_1 + \tilde{\eta}_2\right) + \mathrm{s}_\alpha\left(-\tilde{\chi}_1 + \tilde{\chi}_2\right)\right], \tag{4.1.16a}$$
$$H_2 = \frac{1}{\sqrt{2}}\left(-\tilde{\eta}_1 + \tilde{\eta}_2\right), \tag{4.1.16b}$$
$$H_3 = \frac{1}{\sqrt{2}}\left[-\mathrm{s}_\alpha\left(\tilde{\eta}_1 + \tilde{\eta}_2\right) + \mathrm{c}_\alpha\left(-\tilde{\chi}_1 + \tilde{\chi}_2\right)\right]. \tag{4.1.16c}$$

Based on couplings ZH_iH_j in section 4.2, we note that both H_1 and H_3 are CP-even states and H_2 is the CP-odd state.

Relation between the gauge-eigenstates and the mass-eigenstates is given by the following rotation matrix:

$$\begin{pmatrix} H_1 \\ H_2 \\ G^0 \\ H_3 \end{pmatrix} = \begin{pmatrix} \frac{c_\alpha}{\sqrt{2}} & \frac{c_\alpha}{\sqrt{2}} & -\frac{s_\alpha}{\sqrt{2}} & \frac{s_\alpha}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{s_\alpha}{\sqrt{2}} & -\frac{s_\alpha}{\sqrt{2}} & -\frac{c_\alpha}{\sqrt{2}} & \frac{c_\alpha}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \\ \tilde{\chi}_1 \\ \tilde{\chi}_2 \end{pmatrix}. \tag{4.1.17}$$

We finally consider the S_3 singlet. The mass-squared matrix is given by:

$$\mathcal{M}_{\mathrm{Neutral-S}}^2 = \begin{pmatrix} \mu_0^2 + \frac{1}{2}\left(\lambda_5 + \lambda_6\right) v^2 + \lambda_7 \mathrm{c}_\sigma^2 v^2 & \frac{1}{2}\lambda_7 \mathrm{s}_{2\sigma} v^2 \\ \frac{1}{2}\lambda_7 \mathrm{s}_{2\sigma} v^2 & \mu_0^2 + \frac{1}{2}\left(\lambda_5 + \lambda_6\right) v^2 - \lambda_7 \mathrm{c}_\sigma^2 v^2 \end{pmatrix}. \tag{4.1.18}$$

It is possible to diagonalize the $\mathcal{M}_{\mathrm{Neutral-S}}^2$ mass-squared matrix by the following rotation matrix:

$$\mathcal{R}_\gamma = \begin{pmatrix} \mathrm{c}_\gamma & \mathrm{s}_\gamma \\ -\mathrm{s}_\gamma & \mathrm{c}_\gamma \end{pmatrix}. \tag{4.1.19}$$

The off-diagonal elements are zero provided that:

$$\mathrm{s}_{(2\gamma-\sigma)} = 0, \quad \text{or} \quad \mathrm{s}_{2\sigma-2\gamma} = \mathrm{s}_{2\gamma}. \tag{4.1.20}$$

Therefore we take the value of $\gamma = \sigma/2$.

We find the following neutral scalar masses of the S_3 singlet:

$$m_{S_1}^2 = \mu_0^2 + \frac{1}{2}(\lambda_5 + \lambda_6) v^2 - \lambda_7 c_\sigma v^2, \tag{4.1.21a}$$

$$m_{S_2}^2 = \mu_0^2 + \frac{1}{2}(\lambda_5 + \lambda_6) v^2 + \lambda_7 c_\sigma v^2, \tag{4.1.21b}$$

and as expected those coincide with the exact C-III-c model (3.1.29) by making appropriate changes.

The S_3 singlet neutral physical fields are:

$$S_1 = \quad c_\gamma \tilde{\eta}_S + s_\gamma \tilde{\chi}_S, \tag{4.1.22a}$$

$$S_2 = -s_\gamma \tilde{\eta}_S + c_\gamma \tilde{\chi}_S. \tag{4.1.22b}$$

Although it it known that both states S_1 and S_2 are of opposite CP parities, due to the interaction ZS_1S_2, as presented in section 4.2, it is impossible to assign which of them is CP-even and which is CP-odd.

Taking everything into consideration, the $SU(2)$ doublets in terms of the mass-eigenstates are given by:

$$h_1 = e^{i\sigma}\frac{1}{\sqrt{2}}\begin{pmatrix} G^+ - H^+ \\ \frac{1}{\sqrt{2}}\left(v + H_1 e^{-i\alpha} - H_2 - H_3 i e^{-i\alpha} + iG^0\right)\end{pmatrix}, \tag{4.1.23a}$$

$$h_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} G^+ + H^+ \\ \frac{1}{\sqrt{2}}\left(v + H_1 e^{i\alpha} + H_2 + H_3 i e^{i\alpha} + iG^0\right)\end{pmatrix}, \tag{4.1.23b}$$

$$h_S = \begin{pmatrix} S^+ \\ \frac{1}{\sqrt{2}}\left(S_1 e^{i\gamma} + S_2 i e^{i\gamma}\right)\end{pmatrix}. \tag{4.1.23c}$$

In the Higgs basis, the $SU(2)$ doublets in terms of the mass-eigenstates are as follows:

$$H_1 = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(v + \varphi_1 c_\alpha - \varphi_3 s_\alpha + iG^0\right)\end{pmatrix}, \tag{4.1.24a}$$

$$H_2 = \begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}}\left(i(\varphi_1 s_\alpha + \varphi_3 c_\alpha) + \varphi_2\right)\end{pmatrix}, \tag{4.1.24b}$$

$$H_3 = \begin{pmatrix} S^+ \\ \frac{1}{\sqrt{2}}\left(S_1 e^{i\gamma} + S_2 i e^{i\gamma}\right)\end{pmatrix}, \tag{4.1.24c}$$

where $\varphi_i \equiv H_i$ of eq. (4.1.17).

4.1.1 Quartic Couplings in Terms of Masses

We invert equations eq. (4.1.7a), eq. (4.1.14) and eq. (4.1.21) containing the mass-squared parameters and get equations for couplings as functions of masses. The most straightforward way is to determine relations for λ_2 and λ_5. This can be done by taking a look at $m_{H^\pm}^2$ and $m_{S^\pm}^2$:

$$\lambda_2 = \frac{1}{2v^2} m_{H^\pm}^2, \tag{4.1.25a}$$

$$\lambda_5 = \frac{2}{v^2}\left(m_{S^\pm}^2 - \mu_0^2\right). \tag{4.1.25b}$$

Afterwards, one can determine relation for the coupling λ_3 from $m_{H_2}^2$:

$$\lambda_3 = \frac{1}{2v^2}\left(m_{H_2}^2 - m_{H^\pm}^2\right) \tag{4.1.26}$$

Next, we take a look at the two combinations $m_{H_1}^2 \pm m_{H_3}^2$ and find that:

$$\lambda_1 = \frac{1}{2v^2}\left(m_{H_1}^2 - m_{H_2}^2 + m_{H_3}^2 + m_{H^\pm}^2\right), \tag{4.1.27a}$$

$$\Delta = \frac{1}{2v^2}\left(-m_{H_1}^2 + m_{H_3}^2\right). \tag{4.1.27b}$$

From the Δ parameter we can determine the value of σ:

$$\mathrm{c}_{2\sigma} = 1 - \frac{2m_{H_1}^2 m_{H_3}^2}{m_{H_2}^2\left(m_{H_1}^2 - m_{H_2}^2 + m_{H_3}^2\right)}. \tag{4.1.28}$$

From here, an interesting conclusion can be drawn. Although σ is an overall phase between the two S_3 doublets, it turns out that it can be expressed in terms of physical quantities and thus an arbitrary phase is promoted to a physical parameter.

From the combination $m_{S_1}^2 \pm m_{S_2}^2$ it follows that:

$$\begin{aligned} \lambda_6 &= \frac{1}{v^2}\left(-2m_{S^\pm}^2 + m_{S_1}^2 + m_{S_2}^2\right), \\ \lambda_7 &= \frac{1}{2\mathrm{c}_\sigma v^2}\left(-m_{S_1}^2 + m_{S_2}^2\right). \end{aligned} \tag{4.1.29}$$

The only undetermined couplings are μ_0^2 and λ_8, which are left as free parameters of the model.

For convenience, we collect all of the relations together. In case of the C-III-c-ν^2 model we get the following relations:

$$\mu_1^2 = -\frac{1}{2}\left(m_{H_1}^2 - m_{H_2}^2 + m_{H_3}^2\right), \tag{4.1.30a}$$

$$\nu^2 = -m_{H_2}^2\sqrt{1 - \frac{m_{H_1}^2 m_{H_3}^2}{m_{H_2}^2\left(m_{H_1}^2 - m_{H_2}^2 + m_{H_3}^2\right)}}, \tag{4.1.30b}$$

$$\lambda_1 = \frac{1}{2v^2}\left(m_{H_1}^2 - m_{H_2}^2 + m_{H_3}^2 + m_{H^\pm}^2\right), \tag{4.1.30c}$$

$$\lambda_2 = \frac{1}{2v^2}m_{H^\pm}^2, \tag{4.1.30d}$$

$$\lambda_3 = \frac{1}{2v^2}\left(m_{H_2}^2 - m_{H^\pm}^2\right), \tag{4.1.30e}$$

$$\lambda_4 = 0, \tag{4.1.30f}$$

$$\lambda_5 = \frac{2}{v^2}\left(m_{S^\pm}^2 - \mu_0^2\right), \tag{4.1.30g}$$

$$\lambda_6 = \frac{1}{v^2}\left(-2m_{S^\pm}^2 + m_{S_1}^2 + m_{S_2}^2\right), \tag{4.1.30h}$$

$$\lambda_7 = \frac{1}{2\mathrm{c}_\sigma v^2}\left(-m_{S_1}^2 + m_{S_2}^2\right), \tag{4.1.30i}$$

$$\mathrm{c}_{2\sigma} = 1 - \frac{2m_{H_1}^2 m_{H_3}^2}{m_{H_2}^2\left(m_{H_1}^2 - m_{H_2}^2 + m_{H_3}^2\right)}. \tag{4.1.30j}$$

4.2 Scalar-Gauge Boson Interactions

The kinetic part of the Lagrangian is given by:

$$\mathcal{L}_K = \sum_{i=1,2}\left(D^\mu h_i\right)^\dagger\left(D_\mu h_i\right) + \left(D^\mu h_S\right)^\dagger\left(D_\mu h_S\right), \tag{4.2.1}$$

where D_μ is the covariant derivative:

$$D_\mu = \partial_\mu + ig\tau_i W_\mu^i + ig' Y B_\mu, \tag{4.2.2}$$

where g' and g are the $U(1)$ and $SU(2)$ coupling constants, W_μ^i ($i = \overline{1,3}$) and B_μ are gauge fields, τ_i are the generators of the $SU(2)$ group and Y is the hypercharge. The fields W_μ^i and B_μ are in the gauge basis. These fields are to be diagonalized to correspond to the physical observable states $W_\mu^\pm$, Z_μ and A_μ.

The interaction Lagrangian of the gauge bosons with scalars can be expressed as:

$$\mathcal{L}_K = \mathcal{L}_{V-\text{mass}} + \mathcal{L}_{V-S}, \tag{4.2.3}$$

where the $\mathcal{L}_{V-\text{mass}}$ part is responsible for generation of the mass terms of the gauge bosons and $\mathcal{L}_{V-S}$ is responsible for interactions of the scalar and gauge bosons fields.

Expansion of the covariant derivative with respect to the $SU(2)$ generators yields:

$$\begin{aligned} D_\mu &= \partial_\mu + ig\frac{1}{2}\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} + ig' Y \begin{pmatrix} B_\mu & 0 \\ 0 & B_\mu \end{pmatrix} \\ &= \partial_\mu + \frac{i}{2}\begin{pmatrix} gW_\mu^3 + g'\,2YB_\mu & g\left(W_\mu^1 - iW_\mu^2\right) \\ g\left(W_\mu^1 + iW_\mu^2\right) & -gW_\mu^3 + g'\,2YB_\mu \end{pmatrix}. \end{aligned} \tag{4.2.4}$$

We define the physical fields in the following way:

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right), \tag{4.2.5a}$$

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}}\left(gW_\mu^3 - g'B_\mu\right), \tag{4.2.5b}$$

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}\left(g'W_\mu^3 + gB_\mu\right), \tag{4.2.5c}$$

where we assigned the value $Y = \frac{1}{2}$ to the hypercharge of the scalar doublets. The specific choice of the hypercharge decouples Z_μ from A_μ. This is done as we are interested in massless photon states.

We extract masses of the gauge bosons by evaluating interactions at vacuum:

$$\begin{aligned} \mathcal{L}_{V-\text{mass}} &= \sum_{i=1,2} \frac{1}{8} w_i^2 \left[g^2 (W_\mu^+)^2 + g^2 (W_\mu^-)^2 + \left(g^2 + g'^2\right) Z_\mu^2 + 0 \times A_\mu^2\right] \\ &\equiv m_W^2 W_\mu^\dagger W^\mu + \frac{1}{2} m_Z^2 Z_\mu Z^\mu. \end{aligned} \tag{4.2.6}$$

The mass terms of the gauge bosons can be evaluated using the above mass Lagrangian:

$$m_W = \frac{g}{2} v, \tag{4.2.7a}$$

$$m_Z = \frac{\sqrt{g^2 + g'^2}}{2} v, \tag{4.2.7b}$$

$$m_A = 0. \tag{4.2.7c}$$

We list some of the useful relations:

$$e = g\,\mathrm{s}_w = g'\,\mathrm{c}_w, \tag{4.2.8a}$$

$$\mathrm{c}_w = \frac{g}{\sqrt{g^2 + g'^2}}, \tag{4.2.8b}$$

$$\mathrm{s}_w = \frac{g'}{\sqrt{g^2 + g'^2}}, \tag{4.2.8c}$$

$$\mathrm{t}_w = \frac{g'}{g}, \tag{4.2.8d}$$

$$\frac{m_W}{m_Z} = \mathrm{c}_w. \tag{4.2.8e}$$

It is straightforward to rewrite the covariant derivative D_μ in terms of the physical gauge bosons fields. The only part which requires some calculation is the upper-left component of eq. (4.2.4):

$$\begin{aligned} \frac{i}{2}\left(gW^3_\mu + g'\,2YB_\mu\right) &= \frac{ig}{2\mathrm{c}_w}\left[\mathrm{s}_w B_\mu + \mathrm{c}_w W^3_\mu\right], \\ &= \frac{ig}{2\mathrm{c}_w}\left[\mathrm{s}_w\left(-\mathrm{s}_w Z_\mu + \mathrm{c}_w A_\mu\right) + \mathrm{c}_w\left(\mathrm{c}_w Z_\mu + \mathrm{s}_w A_\mu\right)\right], \\ &= \frac{ig}{2\mathrm{c}_w}\left[\left(\mathrm{c}^2_w - \mathrm{s}^2_w\right) Z_\mu + 2\mathrm{s}_w\mathrm{c}_w A_\mu\right], \\ &= \frac{ig}{2}\frac{\mathrm{c}_{2w}}{\mathrm{c}_w} Z_\mu + ieA_\mu \end{aligned} \tag{4.2.9}$$

Acting with the covariant derivative on the $SU(2)$ scalar doublet yields:

$$D_\mu \begin{pmatrix} h^+_\alpha \\ h^0_\alpha \end{pmatrix} = \begin{pmatrix} \left[\partial_\mu + \frac{ig}{2}\frac{\mathrm{c}_{2w}}{\mathrm{c}_w} Z_\mu + ieA_\mu\right] h^+_\alpha + \frac{ig}{\sqrt{2}} W^+_\mu h^0_\alpha \\ \frac{ig}{\sqrt{2}} W^-_\mu h^+_\alpha + \left[\partial_\mu - \frac{ig}{2\mathrm{c}_w} Z_\mu\right] h^0_\alpha \end{pmatrix}, \qquad \alpha = \{1,\, 2,\, S\}. \tag{4.2.10}$$

Before substituting eq. (4.2.10) into eq. (4.2.1), for convenience the $\mathcal{L}_{V-S}$ of eq. (4.2.3) can be split into several parts based on interaction properties:

$$\mathcal{L}_{V-S} = \mathcal{L}_{VHH} + \mathcal{L}_{VVH} + \mathcal{L}_{VVHH}. \tag{4.2.11}$$

For the general result we present the interaction Lagrangian $\mathcal{L}_{V-S}$ in the gauge basis of the scalars:

$$h_\alpha = \begin{pmatrix} h^+_\alpha \\ \frac{1}{\sqrt{2}} h^0_\alpha \end{pmatrix} = \begin{pmatrix} h^+_\alpha \\ \frac{1}{\sqrt{2}}\left(w_\alpha + \tilde{\eta}_\alpha + \tilde{\chi}_\alpha\right) \end{pmatrix}. \tag{4.2.12}$$

The resulting interaction Lagrangian parts are:

$$\begin{aligned} \mathcal{L}_{VVH} &= \left[\frac{g}{2\mathrm{c}_w} m_Z Z_\mu Z^\mu + g m_W W^+_\mu W^{\mu-}\right]\tilde{\eta}_\alpha \\ &\quad + \left\{\left[e m_W A^\mu W^+_\mu - g m_Z \mathrm{s}^2_w Z^\mu W^+_\mu\right] h^-_\alpha + \mathrm{h.c.}\right\}, \end{aligned} \tag{4.2.13a}$$

$$\mathcal{L}_{VHH} = -\frac{ig}{4\mathrm{c}_w} Z^\mu (h^0_\alpha)^* \overleftrightarrow{\partial}_\mu h^0_\alpha - \frac{g}{2}\left\{ iW^+_\mu h^-_\alpha \overleftrightarrow{\partial}^\mu h^0_\alpha \right\} + \left[ieA^\mu + \frac{ig}{2}\frac{\mathrm{c}_{2w}}{\mathrm{c}_w} Z^\mu\right] h^+_\alpha \overleftrightarrow{\partial}_\mu h^-_\alpha, \tag{4.2.13b}$$

$$\begin{aligned} \mathcal{L}_{VVHH} &= \left[\frac{g^2}{8\mathrm{c}^2_w} Z_\mu Z^\mu + \frac{g^2}{4} W^+_\mu W^{\mu-}\right](h^0_\alpha)^* h^0_\alpha + \left\{\left[\frac{eg}{2} A^\mu W^+_\mu - \frac{g^2}{2}\frac{\mathrm{s}^2_w}{\mathrm{c}_w} Z^\mu W^+_\mu\right] h^-_\alpha h^+_\alpha + \mathrm{h.c.}\right\} \\ &\quad + \left[e^2 A_\mu A^\mu + eg\frac{\mathrm{c}_{2w}}{\mathrm{c}_w} A_\mu Z^\mu + \frac{g^2}{4}\frac{\mathrm{c}^2_{2w}}{\mathrm{c}^2_w} Z_\mu Z^\mu + \frac{g^2}{2} W^-_\mu W^{\mu+}\right] h^-_\alpha h^+_\alpha. \end{aligned} \tag{4.2.13c}$$

In order to extract interactions of physical states from the kinetic Lagrangian one needs to work with the mass-eigenstates of eq. (4.1.23). The resulting parts of the interaction Lagrangian $\mathcal{L}_{V-S}$ are:

$$\begin{aligned} \mathcal{L}_{VVH} &= \left[\frac{g}{2\mathrm{c}_w} m_Z Z_\mu Z^\mu + g m_W W^+_\mu W^{\mu-}\right]\left(\mathrm{c}_\alpha H_1 - \mathrm{s}_\alpha H_3\right) \\ &\quad + \left\{\left[e m_W A^\mu W^+_\mu - g m_Z \mathrm{s}^2_w Z^\mu W^+_\mu\right] G^- + \mathrm{h.c.}\right\}, \end{aligned} \tag{4.2.14a}$$

$$\begin{aligned}
\mathcal{L}_{VHH} =& -\frac{g}{2\mathrm{c}_w} Z^\mu \left(-\mathrm{s}_\alpha H_1 \overset{\leftrightarrow}{\partial}_\mu H_2 + \mathrm{c}_\alpha H_1 \overset{\leftrightarrow}{\partial}_\mu G^0 + \mathrm{c}_\alpha H_2 \overset{\leftrightarrow}{\partial}_\mu H_3 - \mathrm{s}_\alpha H_3 \overset{\leftrightarrow}{\partial}_\mu G^0 + S_1 \overset{\leftrightarrow}{\partial}_\mu S_2 \right) \\
& - \frac{g}{2} \Big\{ i W_\mu^+ \Big(\mathrm{c}_\alpha G^- \overset{\leftrightarrow}{\partial^\mu} H_1 - \mathrm{s}_\alpha G^- \overset{\leftrightarrow}{\partial^\mu} H_3 + i G^- \overset{\leftrightarrow}{\partial^\mu} G^0 + i \mathrm{s}_\alpha H^- \overset{\leftrightarrow}{\partial^\mu} H_1 \\
& \qquad + H^- \overset{\leftrightarrow}{\partial^\mu} H_2 + i \mathrm{c}_\alpha H^- \overset{\leftrightarrow}{\partial^\mu} H_3 + e^{i\gamma} S^- \overset{\leftrightarrow}{\partial^\mu} S_1 + i e^{i\gamma} S^- \overset{\leftrightarrow}{\partial^\mu} S_2 \Big) + \text{h.c.} \Big\} \\
& + \left[i e A^\mu + \frac{ig}{2} \frac{\mathrm{c}_{2w}}{\mathrm{c}_w} Z^\mu \right] \left(G^+ \overset{\leftrightarrow}{\partial}_\mu G^- + H^+ \overset{\leftrightarrow}{\partial}_\mu H^- + S^+ \overset{\leftrightarrow}{\partial}_\mu S^- \right),
\end{aligned} \tag{4.2.14b}$$

$$\begin{aligned}
\mathcal{L}_{VVHH} =& \left[\frac{g^2}{8\mathrm{c}_w^2} Z_\mu Z^\mu + \frac{g^2}{4} W_\mu^+ W^{\mu -} \right] \left(H_1^2 + H_2^2 + H_3^2 + (G^0)^2 + S_1^2 + S_2^2 \right) \\
& + \Big\{ \left[\frac{eg}{2} A^\mu W_\mu^+ - \frac{g^2}{2} \frac{\mathrm{s}_w^2}{\mathrm{c}_w} Z^\mu W_\mu^+ \right] \big(i\mathrm{s}_\alpha H_1 H^- + H_2 H^- + i\mathrm{c}_\alpha H_3 H^- + \mathrm{c}_\alpha H_1 G^- \\
& \qquad - \mathrm{s}_\alpha H_3 G^- + i G^0 G^- + e^{i\gamma} S_1 S^- + i e^{i\gamma} S_2 S^- \big) + \text{h.c.} \Big\} \\
& + \left[e^2 A_\mu A^\mu + e g \frac{\mathrm{c}_{2w}}{\mathrm{c}_w} A_\mu Z^\mu + \frac{g^2}{4} \frac{\mathrm{c}_{2w}^2}{\mathrm{c}_w^2} Z_\mu Z^\mu + \frac{g^2}{2} W_\mu^- W^{\mu +} \right] \left(H^- H^+ + G^- G^+ + S^- S^+ \right).
\end{aligned} \tag{4.2.14c}$$

From the interaction terms ZZH_1 and ZZH_3 it follows that the states H_1 and H_3 act as CP-even. From the terms $Z\varphi_i\varphi_j$ the following information can be extracted: H_2 and G^0 are CP-odd, states S_1 and S_2 have opposite CP numbers. The two photon state $A_\mu A^\mu$ couples to a pair of the same species charged particles $\varphi_i^\pm \varphi_i^\mp$, and thus the state $\varphi_i^\pm \varphi_i^\mp$ is CP-even.

The Feynman rules for the interactions of the scalars φ_i and the gauge bosons V_j are:

$$\begin{aligned}
\varphi_i V_j V_k &= i\, S\, g\,(\varphi_i V_j V_k)\, g^{\mu\nu}, \\
\varphi_i \overset{\leftrightarrow}{\partial} \varphi_j V_k &= S\, g\,(\varphi_i \varphi_j V_k)\,(p_j - p_i)^\mu \,, \text{ for all momenta ingoing}, \\
\varphi_i \varphi_j V_k V_l &= i\, S\, g\,(\varphi_i \varphi_j V_k V_l)\, g^{\mu\nu},
\end{aligned} \tag{4.2.15}$$

where S is the symmetry factor: $S = \Pi_i n_i!$, for the i identical particles of species n, $g\,(\varphi_i V_j V_k)$, $g\,(\varphi_i \overset{\leftrightarrow}{\partial} \varphi_j V_k)$, and $g\,(\varphi_i \varphi_j V_k V_l)$ are the corresponding scalar-gauge bosons coefficients of vertices obtained directly from eqs. (4.2.14a-4.2.14c), and p_i are the incoming four-momenta.

4.3 Scalar-Fermion Interactions

The general Yukawa Lagrangian is as follows:

$$\mathcal{L}_Y = \mathcal{L}_Y^d + \mathcal{L}_Y^u + \mathcal{L}_Y^e + \mathcal{L}_Y^\nu. \tag{4.3.1}$$

We work in the massless neutrino limit and thus the $\mathcal{L}_Y^\nu$ term vanishes. Since the scalar potential is S_3-symmetric, we consider that the Yukawa Lagrangian is also S_3-symmetric. We assume the following S_3 structure $(2+1)_\mathrm{L} \otimes (2+1)_\mathrm{R} \otimes (2+1)_h$:

$$\begin{aligned}
&S_3 \text{ doublets:} \quad \begin{pmatrix} \overline{Q}_1 \\ \overline{Q}_2 \end{pmatrix}_L, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_R, \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}_R, \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}; \\
&S_3 \text{ singlets:} \quad \overline{Q}_{3L},\ u_{3R},\ d_{3R},\ h_S\,;
\end{aligned} \tag{4.3.2}$$

where indices $\overline{1,3}$ label quark families.

The Yukawa Lagrangian must be in the invariant singlet. Singlets of S_3 can be obtained from multiplication of two singlets or two doublets, where one factor could arise from the product of two doublets:

$$\begin{aligned}
\mathbf{1} \otimes \text{any} &= \text{any}, \\
\mathbf{2} \otimes \mathbf{2} &= \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2}.
\end{aligned} \tag{4.3.3}$$

Consider possible Yukawa terms. As an example, if we want to couple the scalar singlet h_S to fermions, we should couple it to a fermion singlet. Yukawa couplings of d-quarks that are singlets under S_3 can thus be constructed in 5 different ways, with independent coefficients $\{y_1^d, \ldots, y_5^d\}$ [35, 53]:

$$y_1^d: \quad \bar{Q}_{1L}^0 h_S d_{1R}^0 + \bar{Q}_{2L}^0 h_S d_{2R}^0, \tag{4.3.4a}$$
$$y_2^d: \quad (\bar{Q}_{1L}^0 h_2 + \bar{Q}_{2L}^0 h_1) d_{1R}^0 + (\bar{Q}_{1L}^0 h_1 - \bar{Q}_{2L}^0 h_2) d_{2R}^0, \tag{4.3.4b}$$
$$y_3^d: \quad \bar{Q}_{3L}^0 h_S d_{3R}^0, \tag{4.3.4c}$$
$$y_4^d: \quad (\bar{Q}_{1L}^0 h_1 + \bar{Q}_{2L}^0 h_2) d_{3R}^0, \tag{4.3.4d}$$
$$y_5^d: \quad \bar{Q}_{3L}^0 (h_1 d_{1R}^0 + h_2 d_{2R}^0), \tag{4.3.4e}$$

where the Yukawa couplings y_i^d are assumed to be real. The quark sector of the Yukawa Lagrangian is:

$$\mathcal{L}_Y^q = \mathcal{L}_Y^d + \mathcal{L}_Y^u + \text{h.c.}, \tag{4.3.5}$$

where

$$\begin{aligned} -\mathcal{L}_Y^d = &\, y_1^d \left[\bar{Q}_{1L}^0 h_S d_{1R}^0 + \bar{Q}_{2L}^0 h_S d_{2R}^0\right] + y_2^d \left[(\bar{Q}_{1L}^0 h_2 + \bar{Q}_{2L}^0 h_1) d_{1R}^0 + (\bar{Q}_{1L}^0 h_1 - \bar{Q}_{2L}^0 h_2) d_{2R}^0\right] \\ &+ y_3^d \bar{Q}_{3L}^0 h_S d_{3R}^0 + y_4^d \left[(\bar{Q}_{1L}^0 h_1 + \bar{Q}_{2L}^0 h_2) d_{3R}^0\right] + y_5^d \bar{Q}_{3L}^0 (h_1 d_{1R}^0 + h_2 d_{2R}^0), \end{aligned} \tag{4.3.6a}$$

$$\begin{aligned} -\mathcal{L}_Y^u = &\, y_1^u \left[\bar{Q}_{1L}^0 \tilde{h}_S u_{1R}^0 + \bar{Q}_{2L}^0 \tilde{h}_S u_{2R}^0\right] + y_2^u \left[(\bar{Q}_{1L}^0 \tilde{h}_2 + \bar{Q}_{2L}^0 \tilde{h}_1) u_{1R}^0 + (\bar{Q}_{1L}^0 \tilde{h}_1 - \bar{Q}_{2L}^0 \tilde{h}_2) u_{2R}^0\right] \\ &+ y_3^u \bar{Q}_{3L}^0 \tilde{h}_S u_{3R}^0 + y_4^u \left[(\bar{Q}_{1L}^0 \tilde{h}_1 + \bar{Q}_{2L}^0 \tilde{h}_2) u_{3R}^0\right] + y_5^u \bar{Q}_{3L}^0 (\tilde{h}_1 u_{1R}^0 + \tilde{h}_2 u_{2R}^0). \end{aligned} \tag{4.3.6b}$$

As a result, the most general fermion mass matrix is of the following form:

$$\mathcal{M}_u = \frac{1}{\sqrt{2}} \begin{pmatrix} y_1^u w_S^* + y_2^u w_2^* & y_2^u w_1^* & y_4^u w_1^* \\ y_2^u w_1^* & y_1^u w_S^* - y_2^u w_2^* & y_4^u w_2^* \\ y_5^u w_1^* & y_5^u w_2^* & y_3^u w_S^* \end{pmatrix}, \tag{4.3.7a}$$

$$\mathcal{M}_d = \frac{1}{\sqrt{2}} \begin{pmatrix} y_1^d w_S + y_2^d w_2 & y_2^d w_1 & y_4^d w_1 \\ y_2^d w_1 & y_1^d w_S - y_2^d w_2 & y_4^d w_2 \\ y_5^d w_1 & y_5^d w_2 & y_3^d w_S \end{pmatrix}. \tag{4.3.7b}$$

Decomposition of the mass matrices into interactions with different scalar doublets yields:

$$\mathcal{M}_1 \approx \begin{pmatrix} 0 & y_2 & y_4 \\ y_2 & 0 & 0 \\ y_5 & 0 & 0 \end{pmatrix}, \tag{4.3.8a}$$

$$\mathcal{M}_2 \approx \begin{pmatrix} y_2 & 0 & 0 \\ 0 & -y_2 & y_4 \\ 0 & y_5 & 0 \end{pmatrix}, \tag{4.3.8b}$$

$$\mathcal{M}_S \approx \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & y_3 \end{pmatrix}. \tag{4.3.8c}$$

It is straightforward to consider which of the vacuum configurations lead to unrealistic cases, *e.g.*, models with a single active $SU(2)$ doublet result in unrealistic eigenvalues. For h_1 (R-I-2a), the first generation becomes massless, and for h_2 (R-II-2), one of the eigenvalues is negative, and for h_S (R-I-1), two states are mass-degenerate.

Of particular interest are mass-eigenstates of fermion states. Rotation from the weak basis into the mass basis is performed by the following unitary transformations:

$$\begin{aligned} u_L^0 &= V_u u_L, \qquad & u_R^0 &= U_u u_R, \\ d_L^0 &= V_d d_L, \qquad & d_R^0 &= U_d d_R. \end{aligned} \tag{4.3.9}$$

In the mass-eigenstates basis we expect to get definite fermion masses. The diagonal entries should therefore correspond to:

$$\begin{aligned} \mathrm{diag}\,(m_u,\, m_c,\, m_t) &\equiv \hat{\mathcal{M}}_u = V_u^\dagger \mathcal{M}_u U_u, \\ \mathrm{diag}\,(m_d,\, m_s,\, m_b) &\equiv \hat{\mathcal{M}}_d = V_d^\dagger \mathcal{M}_d U_d. \end{aligned} \tag{4.3.10}$$

We also define hermitian mass-squared matrices:

$$\begin{aligned} \mathcal{H}_u &= \mathcal{M}_u \left(\mathcal{M}_u\right)^\dagger = V_u \hat{\mathcal{M}}_u^2 V_u^\dagger, \\ \mathcal{H}_d &= \mathcal{M}_d \left(\mathcal{M}_d\right)^\dagger = V_d \hat{\mathcal{M}}_d^2 V_d^\dagger, \end{aligned} \tag{4.3.11}$$

or equivalently:

$$\begin{aligned} \mathcal{H}_u' &= \left(\mathcal{M}_u\right)^\dagger \mathcal{M}_u = U_u \hat{\mathcal{M}}_u^2 U_u^\dagger, \\ \mathcal{H}_d' &= \left(\mathcal{M}_d\right)^\dagger \mathcal{M}_d = U_d \hat{\mathcal{M}}_d^2 U_d^\dagger, \end{aligned} \tag{4.3.12}$$

The left-handed diagonalization matrix of eq. (4.3.9) is defined by solving eq. (4.3.11) and the right-handed diagonalization matrix from:

$$U = \mathcal{M}^{-1} V \hat{\mathcal{M}}. \tag{4.3.13}$$

We define hermitian mass-squared matrix invariants:

$$A_{(u,d)} \equiv \mathrm{tr}(\mathcal{H}_{(u,d)}), \tag{4.3.14a}$$

$$\begin{aligned} B_{(u,d)} \equiv &-(\mathcal{H}_{(u,d)})_{11}(\mathcal{H}_{(u,d)})_{22} - (\mathcal{H}_{(u,d)})_{22}(\mathcal{H}_{(u,d)})_{33} - (\mathcal{H}_{(u,d)})_{33}(\mathcal{H}_{(u,d)})_{11} \\ &+ (\mathcal{H}_{(u,d)})_{12}(\mathcal{H}_{(u,d)})_{21} + (\mathcal{H}_{(u,d)})_{23}(\mathcal{H}_{(u,d)})_{32} + (\mathcal{H}_{(u,d)})_{31}(\mathcal{H}_{(u,d)})_{13}, \end{aligned} \tag{4.3.14b}$$

$$C_{(u,d)} \equiv \det(\mathcal{H}_{(u,d)}). \tag{4.3.14c}$$

CP violation can be inspected by inspecting the determinant [54–56]:

$$J = \mathrm{Det}\left(\mathcal{H}_d \mathcal{H}_u - \mathcal{H}_u \mathcal{H}_d\right). \tag{4.3.15}$$

The C-III-c-ν^2 vacuum configuration is given by $\{e^{i\sigma}\frac{v}{\sqrt{2}},\, \frac{v}{\sqrt{2}},\, 0\}$. Substituting the vacuum configuration into eq. (4.3.7) results in the quark mass matrices:

$$\mathcal{M}_u = \frac{v}{2}\begin{pmatrix} y_2^u & e^{-i\sigma} y_2^u & e^{-i\sigma} y_4^u \\ e^{-i\sigma} y_2^u & -y_2^u & y_4^u \\ e^{-i\sigma} y_5^u & y_5^u & 0 \end{pmatrix}, \tag{4.3.16a}$$

$$\mathcal{M}_d = \frac{v}{2}\begin{pmatrix} y_2^d & e^{i\sigma} y_2^d & e^{i\sigma} y_4^d \\ e^{i\sigma} y_2^d & -y_2^d & y_4^d \\ e^{i\sigma} y_5^d & y_5^d & 0 \end{pmatrix}, \tag{4.3.16b}$$

and hermitian mass-squared matrices are:

$$\mathcal{H}_u = \frac{1}{2}v^2 \begin{pmatrix} \left(y_2^u\right)^2 + \frac{1}{2}\left(y_4^u\right)^2 & i\mathrm{s}_\sigma \left(y_2^u\right)^2 + \frac{1}{2}e^{-i\sigma}\left(y_4^u\right)^2 & \mathrm{c}_\sigma y_2^u y_5^u \\ -i\mathrm{s}_\sigma \left(y_2^u\right)^2 + \frac{1}{2}e^{i\sigma}\left(y_4^u\right)^2 & \left(y_2^u\right)^2 + \frac{1}{2}\left(y_4^u\right)^2 & 0 \\ \mathrm{c}_\sigma y_2^u y_5^u & 0 & \left(y_5^u\right)^2 \end{pmatrix}, \tag{4.3.17a}$$

$$\mathcal{H}_d = \frac{1}{2}v^2 \begin{pmatrix} \left(y_2^d\right)^2 + \frac{1}{2}\left(y_4^d\right)^2 & -i\mathrm{s}_\sigma \left(y_2^d\right)^2 + \frac{1}{2}e^{i\sigma}\left(y_4^d\right)^2 & \mathrm{c}_\sigma y_2^d y_5^d \\ i\mathrm{s}_\sigma \left(y_2^d\right)^2 + \frac{1}{2}e^{-i\sigma}\left(y_4^d\right)^2 & \left(y_2^d\right)^2 + \frac{1}{2}\left(y_4^d\right)^2 & 0 \\ \mathrm{c}_\sigma y_2^d y_5^d & 0 & \left(y_5^d\right)^2 \end{pmatrix}. \tag{4.3.17b}$$

Hermitian mass-squared matrix invariants are:

$$A_u = \frac{1}{2}v^2 \left(y_5^u\right)^2 + \frac{1}{2}v^2 \left[2\left(y_2^u\right)^2 + \left(y_4^u\right)^2\right], \tag{4.3.18a}$$

$$A_d = \frac{1}{2}v^2 \left(y_5^d\right)^2 + \frac{1}{2}v^2 \left[2\left(y_2^d\right)^2 + \left(y_4^d\right)^2\right], \tag{4.3.18b}$$

and

$$B_u = -\frac{1}{8}v^4 \left[\mathrm{c}_{2\sigma}\left(\left(y_2^u\right)^2 - \left(y_4^u\right)^2 - \left(y_5^u\right)^2\right)\left(y_2^u\right)^2 + \left(y_2^u\right)^4 + 3\left(\left(y_4^u\right)^2 + \left(y_5^u\right)^2\right)\left(y_2^u\right)^2 + 2\left(y_4^u\right)^2\left(y_5^u\right)^2\right], \tag{4.3.19a}$$

$$B_d = -\frac{1}{8}v^4 \left[\mathrm{c}_{2\sigma}\left(\left(y_2^d\right)^2 - \left(y_4^d\right)^2 - \left(y_5^d\right)^2\right)\left(y_2^d\right)^2 + \left(y_2^d\right)^4 + 3\left(\left(y_4^d\right)^2 + \left(y_5^d\right)^2\right)\left(y_2^d\right)^2 + 2\left(y_4^d\right)^2\left(y_5^d\right)^2\right], \tag{4.3.19b}$$

and

$$C_u = \frac{1}{32}v^6 \left(5 - 3\mathrm{c}_{2\sigma}\right)\left(y_2^u\right)^2\left(y_4^u\right)^2\left(y_5^u\right)^2, \tag{4.3.20a}$$

$$C_d = \frac{1}{32}v^6 \left(5 - 3\mathrm{c}_{2\sigma}\right)\left(y_2^d\right)^2\left(y_4^d\right)^2\left(y_5^d\right)^2. \tag{4.3.20b}$$

The CP check results in:

$$\begin{aligned} J =& \frac{1}{1024}\left(e^{-4i\sigma} - 1\right)v^{12}\mathrm{c}_\sigma^2 \left[\left(y_4^d\right)^2\left(y_2^u\right)^2 + \left(y_2^d - y_4^d\right)\left(y_2^d + y_4^d\right)\left(y_4^u\right)^2\right] \\ & \Bigg[\left(y_2^u\left(y_2^d y_5^d y_2^u + \left(y_2^d\right)^2 y_5^u - \left(y_4^d\right)^2 y_5^u\right) - e^{2i\sigma} y_2^d\left(y_2^d y_2^u y_5^u + y_5^d\left(y_2^u\right)^2 - y_5^d\left(y_4^u\right)^2\right)\right) \\ & \left(-y_2^d\left(y_2^d y_2^u y_5^u + y_5^d\left(y_2^u\right)^2 - y_5^d\left(y_4^u\right)^2\right) + e^{2i\sigma} y_2^u\left(y_2^d y_5^d y_2^u + \left(y_2^d\right)^2 y_5^u - \left(y_4^d\right)^2 y_5^u\right)\right) \\ & - e^{2i\sigma}\left(2y_2^d y_5^d\left(y_5^u\right)^2 - y_2^d y_5^d\left(2\left(y_2^u\right)^2 + \left(y_4^u\right)^2\right) + \left(2\left(y_2^d\right)^2 + \left(y_4^d\right)^2 - 2\left(y_5^d\right)^2\right) y_2^u y_5^u\right)^2\Bigg]. \end{aligned} \tag{4.3.21}$$

Both $\mathcal{H}_{(u,d)}$ are hermitian. By using the complex number identities in polar coordinates it is possible to rotate away complex phases. Consider $\mathcal{H}_u$ of eq. (4.3.17a). Redefinition yields:

$$\begin{aligned} i\mathrm{s}_\sigma\left(y_2\right)^2 + \frac{1}{2}e^{-i\sigma}\left(y_4\right)^2 &\equiv r_1 e^{i\varphi_1}, \\ i\mathrm{s}_\sigma\left(y_2\right)^2 + \frac{1}{2}e^{-i\sigma}\left(y_5\right)^2 &\equiv r_2 e^{i\varphi_2}, \end{aligned} \tag{4.3.22}$$

and thus hermitian mass-squared matrices become:

$$\mathcal{M}\mathcal{M}^\dagger = \frac{1}{2}v^2 \begin{pmatrix} \left(y_2\right)^2 + \frac{1}{2}\left(y_4\right)^2 & r_1 e^{i\varphi_1} & \mathrm{c}_\sigma y_2 y_5 \\ r_1 e^{-i\varphi_1} & \left(y_2\right)^2 + \frac{1}{2}\left(y_4\right)^2 & 0 \\ \mathrm{c}_\sigma y_2 y_5 & 0 & \left(y_5\right)^2 \end{pmatrix}, \tag{4.3.23a}$$

$$\mathcal{M}^\dagger\mathcal{M} = \frac{1}{2}v^2 \begin{pmatrix} \left(y_2\right)^2 + \frac{1}{2}\left(y_5\right)^2 & r_2 e^{-i\varphi_2} & \mathrm{c}_\sigma y_2 y_4 \\ r_2 e^{i\varphi_2} & \left(y_2\right)^2 + \frac{1}{2}\left(y_5\right)^2 & 0 \\ \mathrm{c}_\sigma y_2 y_4 & 0 & \left(y_4\right)^2 \end{pmatrix}. \tag{4.3.23b}$$

After performing rotations $\mathcal{R}_{\varphi_1}^\dagger \mathcal{M}\mathcal{M}^\dagger \mathcal{R}_{\varphi_1}$ and $\mathcal{R}_{\varphi_2}^\dagger \mathcal{M}^\dagger\mathcal{M}\mathcal{R}_{\varphi_2}$, where

$$\mathcal{R}_{\varphi_{(1,2)}} = \mathrm{diag}\left(1,\, e^{\mp i\varphi_{(1,2)}},\, 1\right), \tag{4.3.24}$$

hermitian mass-squared matrices become symmetric:

$$\mathcal{H} \approx \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{11} & 0 \\ H_{13} & 0 & H_{33} \end{pmatrix}. \tag{4.3.25}$$

Full diagonalization of hermitian matrices is performed in terms of 4 parameters, three angles of the $SO(3)$ rotation and the phase rotation of eq. (4.3.24). Due to complexity of analytical expressions we perform a numerical fit. We note that it is impossible to get a physically meaningful solution when trying to reduce the number of Yukawa couplings, *i.e.*, setting some of the $y_i = 0$.

In total, we need to fit 7 different parameters of which 6 are Yukawa couplings, counting both up and down Yukawa couplings separately, and also the overall phase σ of h_1. In the C-III-c model there are only three non-zero Yukawa couplings and this is the minimal number of couplings needed to fit mass values of the three fermion generations. Since σ is a free parameter, we found that the best fit for masses is achieved by considering the following exponential polynomial:

$$y_i = e^{\sum_{j=0}^{5} A^i_j \sigma^j}, \tag{4.3.26}$$

where A^i_j are real numbers and in some cases are equal to zero. In principle, we need to fit 3 different Yukawa couplings and thus it makes sense to take a look at different orderings of the Yukawa couplings. For simplicity, we use the following notation:

$$Y_{ijk} \equiv y_i > y_j > y_k. \tag{4.3.27}$$

One of the constraints on the Yukawa couplings comes from the CKM matrix. The absolute value of the CKM matrix

$$V_{\text{CKM}} = V_u^\dagger V_d \tag{4.3.28}$$

is a well-known quantity. The PDG presents the following CKM matrix [29]:

$$V_{\text{CKM}} = \begin{pmatrix} 0.9742 & 0.2243 & 0.00394 \\ 0.218 & 0.997 & 0.0422 \\ 0.0081 & 0.0394 & 1.019 \end{pmatrix}. \tag{4.3.29}$$

As stated earlier, we consider that σ is a free parameter and therefore it can be used to fit the value of the CKM matrix. If there was at least one additional free parameter there would arise a possibility to cancel the net effect of the σ value. Therefore the σ phase needs to be fixed so that both experimental and theoretical values of the CKM matrix are in agreement. We found that in most cases diagonal elements of the CKM matrix are close to unity when $\sigma = 0$. It should be noted that based on results of the spectrum generator of section 6.2, the minimum value of σ is close to 0.16π.

Consider different Yukawa couplings orderings (4.3.27). Both orderings Y_{245} and Y_{254} result in unnaturally low second family masses and thus are not considered. Best fit is achieved in the limit

$\sigma = 0.16\pi$:

$$\begin{aligned} Y^u_{425}Y^d_{425}: \quad |V_{\rm CKM}| &= \begin{pmatrix} 0.999 & 0.032 & 0.001 \\ 0.028 & 0.876 & 0.482 \\ 0.016 & 0.481 & 0.876 \end{pmatrix}, \\ Y^u_{452}Y^d_{452}: \quad |V_{\rm CKM}| &= \begin{pmatrix} 0.876 & 0.028 & 0.482 \\ 0.032 & 0.999 & 0.001 \\ 0.481 & 0.016 & 0.876 \end{pmatrix}, \\ Y^u_{524}Y^d_{524}: \quad |V_{\rm CKM}| &= \begin{pmatrix} 0.624 & 0.782 & 0.011 \\ 0.782 & 0.624 & 0.005 \\ 0.011 & 0.005 & 1 \end{pmatrix}, \\ Y^u_{542}Y^d_{542}: \quad |V_{\rm CKM}| &= \begin{pmatrix} 0.877 & 0.481 & 0.001 \\ 0.481 & 0.877 & 0.001 \\ 0.001 & 0.001 & 1 \end{pmatrix}. \end{aligned} \tag{4.3.30}$$

The only meaningful model is $Y^u_{542}Y^d_{542}$. Another interesting case is to consider non-identical orderings of the up-/down-quarks. For $\sigma = 0.36\pi$ we get something even closer to the PDG value:

$$\begin{aligned} Y^u_{524}Y^d_{542}: \quad |V_{\rm CKM}| &= \begin{pmatrix} 0.975 & 0.224 & 0.001 \\ 0.224 & 0.975 & 0.001 \\ 0.001 & 0.001 & 1 \end{pmatrix}, \\ Y^u_{542}Y^d_{524}: \quad |V_{\rm CKM}| &= \begin{pmatrix} 0.975 & 0.223 & 0.005 \\ 0.223 & 0.975 & 0.005 \\ 0.005 & 0.005 & 1 \end{pmatrix}. \end{aligned} \tag{4.3.31}$$

We assume that the only sensible orderings are $Y^u_{524}Y^d_{542}$ and $Y^u_{542}Y^d_{524}$. Since σ is a free parameter and neither result is in perfect agreement with the experimental CKM value, we consider the following acceptable range:

$$\sigma = [0.34\pi,\, 0.38\pi]\,. \tag{4.3.32}$$

In this range we get the following CKM matrix values:

$$\begin{aligned} Y^u_{524}Y^d_{542}: \quad |V_{\rm CKM}| &= \begin{pmatrix} 0.966\text{ - }0.982 & 0.257\text{ - }0.191 & 0.002\text{ - }0.001 \\ 0.257\text{ - }0.191 & 0.966\text{ - }0.982 & 0.001 \\ 0.002\text{ - }0.001 & 0.002\text{ - }0.001 & 1 \end{pmatrix}, \\ Y^u_{542}Y^d_{524}: \quad |V_{\rm CKM}| &= \begin{pmatrix} 0.967\text{ - }0.982 & 0.256\text{ - }0.19 & 0.006\text{ - }0.004 \\ 0.256\text{ - }0.19 & 0.967\text{ - }0.982 & 0.006\text{ - }0.004 \\ 0.006\text{ - }0.004 & 0.005\text{ - }0.004 & 1 \end{pmatrix}. \end{aligned} \tag{4.3.33}$$

The nearly identity form of the CKM matrix $|V_{\rm CKM}| = \mathcal{I} + \mathcal{O}(10^{-2})$ is achieved for both cases when $\sigma \geq 0.493\pi$. This choice of σ results in nearly mass degenerate states $m_{S_1} = m_{S_2}$.

Now, as we have two viable models, we move to the scalar-fermion interactions. The scalar-fermion interactions can be extracted from the Yukawa Lagrangian:

$$g(\xi_i \bar{f}_j f_k) = \left(V_f^\dagger \mathcal{M}_i^{\rm int} U_f P_R + U_f^\dagger \left(\mathcal{M}_i^{\rm int}\right)^\dagger V_f P_L \right)_{jk}, \tag{4.3.34}$$

where the interaction matrices $\mathcal{M}_i^{\rm int}$ are equivalent to the ones of eq. (4.3.7) with substitution of $w_i \to \xi_i$, where ξ_i are fields of interest multiplied by appropriate coefficients. For simplicity, consider the u-quark sector:

$$\mathcal{M}^{\text{int}}_{H_1,u} = \frac{1}{2}\begin{pmatrix} e^{-i\alpha}y^u_2 & \frac{1}{2}e^{i(\alpha-\sigma)}y^u_2 & e^{i(\alpha-\sigma)}y^u_4 \\ e^{i(\alpha-\sigma)}y^u_2 & -e^{-i\alpha}y^u_2 & e^{-i\alpha}y^u_4 \\ e^{i(\alpha-\sigma)}y^u_5 & e^{-i\alpha}y^u_5 & 0 \end{pmatrix}, \tag{4.3.35a}$$

$$\mathcal{M}^{\text{int}}_{H_2,u} = \frac{1}{2}\begin{pmatrix} y^u_2 & -e^{-i\sigma}y^u_2 & -e^{-i\sigma}y^u_4 \\ -e^{-i\sigma}y^u_2 & -y^u_2 & y^u_4 \\ -e^{-i\sigma}y^u_5 & y^u_5 & 0 \end{pmatrix}, \tag{4.3.35b}$$

$$\mathcal{M}^{\text{int}}_{H_3,u} = \frac{1}{2}\begin{pmatrix} -ie^{-i\alpha}y^u_2 & ie^{i(\alpha-\sigma)}y^u_2 & ie^{i(\alpha-\sigma)}y^u_4 \\ ie^{i(\alpha-\sigma)}y^u_2 & iy^u_2e^{-i\alpha} & -ie^{-i\alpha}y^u_4 \\ ie^{i(\alpha-\sigma)}y^u_5 & -ie^{-i\alpha}y^u_5 & 0 \end{pmatrix}, \tag{4.3.35c}$$

where α is the CP-even sector diagonalization angle given by eq. (4.1.13).

We denote

$$\tilde{\mathcal{M}}^{\text{int}}_{(i,f)} \equiv V^\dagger_f \mathcal{M}^{\text{int}}_i U_f \tag{4.3.36}$$

and therefore[1]

$$g(\xi_i \bar{f} f) = \tilde{\mathcal{M}}^{\text{int}}_{(i,f)} P_R + \left(\tilde{\mathcal{M}}^{\text{int}}_{(i,f)}\right)^\dagger P_L. \tag{4.3.37}$$

The resulting diagonal interactions of the form $\xi_i \bar{f}_j f_j$ are:

$$\tilde{\mathcal{M}}_{jj}\gamma_5 + \mathbb{R}e(\tilde{\mathcal{M}}_{jj})\,(1-\gamma_5)\,, \tag{4.3.38}$$

and the FCNC $\xi_i \bar{f}_j f_k$ are of the form:

$$\frac{1}{2}\left[\left(\tilde{\mathcal{M}}_{jk} + (\tilde{\mathcal{M}}_{kj})^*\right) + \gamma_5\left(\tilde{\mathcal{M}}_{jk} - (\tilde{\mathcal{M}}_{kj})^*\right)\right], \tag{4.3.39}$$

where $\tilde{\mathcal{M}}_{jk}$ are the elements of $\tilde{\mathcal{M}}^{\text{int}}_{(i,f)}$. It follows that there are FCNC present if and only if the off-diagonal elements of the $\tilde{\mathcal{M}}^{\text{int}}_{(i,f)}$ matrix are non-zero.

Consider the C-III-c-ν^2 vacuum configuration $\{e^{i\sigma}\frac{v}{\sqrt{2}},\, \frac{v}{\sqrt{2}},\, 0\}$. The left-handed and right-handed matrices V and U diagonalize the fermion mass matrices. Therefore it follows that any sort of combination $A\{e^{i\sigma},\, 1,\, 0\}$, where A is either complex or real, results in a diagonal interaction form. Provided that coefficients next to the fields do not obey the aforementioned condition, the resulting off-diagonal elements of the $\tilde{\mathcal{M}}^{\text{int}}_{(i,f)}$ are non-zero and thus result in FCNC.

Due to the fact that the fermion sector diagonalization is performed numerically, it is not obvious how the elements of interaction matrices look like. We consider Yukawa models $Y^u_{524}Y^d_{542}$ and $Y^u_{542}Y^d_{524}$. Of particular interest are interactions involving the SM-like Higgs boson H_1. In case of the vanishing $\alpha = 0$ we achieve the SM limit:

$$g\left(H_1 \bar{f} f\right) = \frac{m_f}{v}. \tag{4.3.40}$$

For $\alpha \neq 0$, the diagonal elements can be expressed systematically as:

$$g_{ii} = \text{diag}\Big((\mathbb{R}+\mathbb{I}\gamma_5)_{11},\, (\mathbb{R}+\mathbb{I}\gamma_5)_{22},\, \mathbb{R}_{33}\Big), \tag{4.3.41}$$

[1] We do not consider the charged scalar interactions here. The charged flavour currents result in the CKM matrix. The Yukawa Lagrangian should be changed appropriately:

$$(\tilde{\mathcal{M}}^{\text{int}}_{(i,u)})_{\text{Charged}} = \tilde{\mathcal{M}}^{\text{int}}_{(i,u)} V_{\text{CKM}},$$
$$(\tilde{\mathcal{M}}^{\text{int}}_{(i,d)})_{\text{Charged}} = V_{\text{CKM}} \tilde{\mathcal{M}}^{\text{int}}_{(i,d)}.$$

where $\mathbb{R}$ is a real number, $\mathbb{I}$ is an imaginary number. The off-diagonal elements are of the following form:

$$g_{ij} = (\mathbb{R} + \mathbb{I})_{ij} + (\mathbb{R} + \mathbb{I})'_{ij}\,\gamma_5, \tag{4.3.42}$$

and are hermitian, $g_{ij} = g_{ji}^\dagger$. One of the disastrous consequences of the model are FCNC, see eqs. (6.2.1, 6.2.2). We found that in most cases the off-diagonal elements dominate over the diagonal ones. There is no way to control such processes as the model is numerically fixed, there are no free parameters.

In both Yukawa models, with an increasing value of σ, the value of the diagonal third family coupling g_{33} tends to zero. This indicates that there should exist an upper boundary condition for the angle α so that the coupling of the third family to the SM-like Higgs boson is meaningful. We assume that the SM-like limit is achieved by $\mathrm{c}_\alpha \geqslant 0.9$. Another interesting property is that with an increasing α value, values of the diagonal elements g_{ii} decrease while of the off-diagonal elements g_{ij} increase. For $\alpha = 0.14\pi$, the third family quarks couple with a strength $g_{33} \approx 0.9\frac{m_3}{v}$.

Interactions involving the H_2 scalar do not depend on the angle α. Elements of the interaction matrix are of the same form as given by eqs. (4.3.41, 4.3.42). One would assume that each following fermion generation would couple stronger to the scalar then the previous one. This is not the case for the coupling g_{33}. In principle, both g_{11} and g_{22} couple to H_2 as CP-indefinite, while the third family as CP-odd. However, the value of g_{33} is only one magnitude different from g_{11}, *e.g.*, for $Y^u_{524}Y^d_{542}$ and $\sigma = 0.36\pi$ the diagonal elements for the up-quarks are:

$$\begin{aligned}
g^u_{11} &\approx -1.4\times 10^{-6} + 2.2\times 10^{-6}\, i\,\gamma_5,\\
g^u_{22} &\approx -2.4\times 10^{-9} - 2.2\times 10^{-3}\, i\,\gamma_5,\\
g^u_{33} &\approx 1.6\times 10^{-5}\, i\,\gamma_5.
\end{aligned} \tag{4.3.43}$$

For the down-quarks: $g^d_{33} = \mathcal{O}(10^{-8})$. The same is true for the $Y^u_{542}Y^d_{524}$ model: $g^u_{33} = 0$ and $g^d_{33} = \mathcal{O}(10^{-6})$. From the element g^u_{33} it follows that for σ in the range (4.3.32) there is no interaction $H_2\bar{t}t$.

Fermion-scalar interactions involving H_3 once again depend on the angle α. The diagonal elements are of the same form as for H_1 (4.3.41). The off-diagonal elements are of the same form as in eq. (4.3.42). In contrast to $H_1\bar{f}f$, with an increasing α value, values of the diagonal elements g_{ii} increase while the off-diagonal elements g_{ij} decrease. This is a totally expected behavior and arises from the fact that both states H_1 and H_3 were diagonalized by the same angle α.

So far we considered only the quark sector. We assume that there are no right-handed neutrinos ν_R and therefore neglect neutrino masses and the PMNS matrix. The PMNS matrix is not fixed in terms of the left-/right-handed diagonalization matrices and therefore this results in a freedom of the Yukawa couplings ordering. Governed by the quarks sector result, we scanned the range $\sigma = [0.34\pi,\, 0.38\pi]$ and $\alpha = [0,\, 0.14\pi]$. We found no significant deviations from the general conditions for the quark sector. The only interesting observation involves interactions of the H_2 scalar with the third family:

$$\begin{aligned}
Y^e_{425} &= Y^e_{524} = \mathcal{O}\left(10^{-5}\right),\\
Y^e_{452} &= Y^e_{542} = \mathcal{O}\left(10^{-10}\right).
\end{aligned} \tag{4.3.44}$$

It seems unnatural that the scalar H_2 would "not bother" about the third family and thus we consider models with higher g_{33} value. We conclude that models $Y^u_{524}Y^d_{542}Y^e_{425}$ and $Y^u_{524}Y^d_{542}Y^e_{524}$ are of interest and thus those are considered for numerical evaluation in section 6.2.

All in all, the Yukawa sector does not behave as desired: unrealistic CKM matrix, FCNC contribution is too significant, the H_2 field interacts with a really low strength with the third family. The scalar potential is softly broken and therefore additional singlets could be introduced to the Yukawa sector. Additional Yukawa singlets may fix some of the issues. We propose that the following singlets[2] could be added to (4.3.4):

$$y^d_6 : \;\; \bar{Q}^0_{3L} h_1 d^0_{3R}, \tag{4.3.45a}$$

[2] In case of $w_1 = 0$ and $w_2 = 0$, to generate realistic fermion masses the y_8 coupling could be added.

$$y_7^d: \quad \bar{Q}_{3L}^0 h_2 d_{3R}^0, \tag{4.3.45b}$$
$$y_8^d: \quad \bar{Q}_{2L}^0 h_S d_{2R}^0, \tag{4.3.45c}$$

so that the decomposition (4.3.8) results in:

$$\mathcal{M}_1 \approx \begin{pmatrix} 0 & y_2 & y_4 \\ y_2 & 0 & 0 \\ y_5 & 0 & y_6 \end{pmatrix}, \tag{4.3.46a}$$
$$\mathcal{M}_2 \approx \begin{pmatrix} y_2 & 0 & 0 \\ 0 & -y_2 & y_4 \\ 0 & y_5 & y_7 \end{pmatrix}, \tag{4.3.46b}$$
$$\mathcal{M}_S \approx \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_1 + y_8 & 0 \\ 0 & 0 & y_3 \end{pmatrix}. \tag{4.3.46c}$$

Even a single additional Yukawa coupling, $y_7 = y_6$, should result in a more realistic CKM matrix. This is due to the fact that now there is an additional free parameter y_6, but the CKM matrix depends on both y_6^u and y_6^d. Additional parameters may also enable control over the FCNC. A full analysis of the broken Yukawa sector is beyond the scope of the thesis.

4.4 Scalar-Scalar Interactions

The Feynman rules are obtained by expanding the S_3-symmetric scalar potential with respect to the mass-eigenstates and multiplying the relevant terms by $-iS$, where S is the symmetry factor defined in the same way as in section 4.2. We consider that the $SU(2)$ doublets in terms of the mass-eigenstates are given by eq. (4.1.23).

For simplicity, we express the rotation angle of the S_3 singlet in terms of the overall phase $\gamma = \sigma/2$, which we got from the diagonalization procedure of the S_3 singlet neutral sector (4.1.20). We note that the scalar-scalar couplings are written down in a slightly different way. The couplings $g\,(H_i H_j H_k)$ and $g\,(H_i H_j H_k H_l)$ are presented with the symmetry factor S. Therefore, the Feynman rules are defined as:

$$H_i H_j H_k H_l = -ig\,(H_i H_j H_k H_l)\,. \tag{4.4.1}$$

The trilinear couplings involving the same species are:

$$g\,(H_1 H_1 H_1) = 3v\left[(2\lambda_1 - \lambda_2 + \lambda_3)\,\mathrm{c}_\alpha + (\lambda_2 + \lambda_3)\,\mathrm{c}_{3\alpha-2\sigma}\right], \tag{4.4.2a}$$
$$g\,(H_3 H_3 H_3) = -3v\left[(2\lambda_1 - \lambda_2 + \lambda_3)\,\mathrm{s}_\alpha - (\lambda_2 + \lambda_3)\,\mathrm{s}_{3\alpha-2\sigma}\right]. \tag{4.4.2b}$$

The trilinear couplings involving only the neutral fields of the S_3 doublet are:

$$g\,(H_1 H_1 H_3) = -v\left[(2\lambda_1 - \lambda_2 + \lambda_3)\,\mathrm{s}_\alpha + 3\,(\lambda_2 + \lambda_3)\,\mathrm{s}_{3\alpha-2\sigma}\right], \tag{4.4.3a}$$
$$g\,(H_1 H_2 H_2) = v\left[(2\lambda_1 + \lambda_2 + 3\lambda_3)\,\mathrm{c}_\alpha - (\lambda_2 + \lambda_3)\,\mathrm{c}_{\alpha-2\sigma}\right], \tag{4.4.3b}$$
$$g\,(H_1 H_3 H_3) = v\left[(2\lambda_1 - \lambda_2 + \lambda_3)\,\mathrm{c}_\alpha - 3\,(\lambda_2 + \lambda_3)\,\mathrm{c}_{3\alpha-2\sigma}\right], \tag{4.4.3c}$$
$$g\,(H_2 H_2 H_3) = -v\left[(2\lambda_1 + \lambda_2 + 3\lambda_3)\,\mathrm{s}_\alpha - (\lambda_2 + \lambda_3)\,\mathrm{s}_{\alpha-2\sigma}\right]. \tag{4.4.3d}$$

The trilinear couplings involving both neutral fields of the S_3 doublet and singlet are:

$$g\,(H_1 S_1 S_1) = v\left[(\lambda_5 + \lambda_6)\,\mathrm{c}_\alpha + 2\lambda_7 \mathrm{c}_{\alpha-\sigma}\right], \tag{4.4.4a}$$
$$g\,(H_1 S_2 S_2) = v\left[(\lambda_5 + \lambda_6)\,\mathrm{c}_\alpha - 2\lambda_7 \mathrm{c}_{\alpha-\sigma}\right], \tag{4.4.4b}$$
$$g\,(H_2 S_1 S_2) = -2v\lambda_7 \mathrm{s}_\sigma, \tag{4.4.4c}$$
$$g\,(H_3 S_1 S_1) = -v\left[(\lambda_5 + \lambda_6)\,\mathrm{s}_\alpha + 2\lambda_7 \mathrm{s}_{\alpha-\sigma}\right], \tag{4.4.4d}$$

$$g\left(H_3S_2S_2\right) = -v\left[\left(\lambda_5+\lambda_6\right)\mathrm{s}_\alpha - 2\lambda_7\mathrm{s}_{\alpha-\sigma}\right]. \tag{4.4.4e}$$

The trilinear couplings involving the charged fields are:

$$g\left(H_1H^\pm H^\mp\right) = v\left[\left(2\lambda_1+\lambda_2-\lambda_3\right)\mathrm{c}_\alpha - \left(\lambda_2+\lambda_3\right)\mathrm{c}_{\alpha-2\sigma}\right], \tag{4.4.5a}$$
$$g\left(H_1S^\pm S^\mp\right) = v\lambda_5\mathrm{c}_\alpha, \tag{4.4.5b}$$
$$g\left(H_3H^\pm H^\mp\right) = -v\left[\left(2\lambda_1+\lambda_2-\lambda_3\right)\mathrm{s}_\alpha - \left(\lambda_2+\lambda_3\right)\mathrm{s}_{\alpha-2\sigma}\right], \tag{4.4.5c}$$
$$g\left(H_3S^\pm S^\mp\right) = -v\lambda_5\mathrm{s}_\alpha, \tag{4.4.5d}$$
$$g\left(S_1H^\pm S^\mp\right) = \mp ie^{\pm\frac{i\sigma}{2}}v\lambda_7\mathrm{s}_\sigma, \tag{4.4.5e}$$
$$g\left(S_2H^\pm S^\mp\right) = -e^{\pm\frac{i\sigma}{2}}v\lambda_7\mathrm{s}_\sigma. \tag{4.4.5f}$$

The quartic couplings involving the same species are:

$$g\left(H_1H_1H_1H_1\right) = g\left(H_3H_3H_3H_3\right) = 3\left[2\lambda_1-\lambda_2+\lambda_3+\left(\lambda_2+\lambda_3\right)\mathrm{c}_{4\alpha-2\sigma}\right], \tag{4.4.6a}$$
$$g\left(H_2H_2H_2H_2\right) = 6\left(\lambda_1-\lambda_2\mathrm{s}_\sigma^2+\lambda_3\mathrm{c}_\sigma^2\right), \tag{4.4.6b}$$
$$g\left(S_1S_1S_1S_1\right) = g\left(S_2S_2S_2S_2\right) = 6\lambda_8, \tag{4.4.6c}$$
$$g\left(H^\pm H^\mp H^\pm H^\mp\right) = 4\left(\lambda_1-\lambda_2\mathrm{s}_\sigma^2+\lambda_3\mathrm{c}_\sigma^2\right), \tag{4.4.6d}$$
$$g\left(S^\pm S^\mp S^\pm S^\mp\right) = 4\lambda_8. \tag{4.4.6e}$$

The quartic couplings involving only the neutral fields of the S_3 doublet are:

$$g\left(H_1H_1H_1H_3\right) = -3\left(\lambda_2+\lambda_3\right)\mathrm{s}_{4\alpha-2\sigma}, \tag{4.4.7a}$$
$$g\left(H_1H_1H_2H_2\right) = 2\left[\lambda_1+\lambda_3-\left(\lambda_2+\lambda_3\right)\mathrm{s}_{2\alpha-\sigma}\mathrm{s}_\sigma\right], \tag{4.4.7b}$$
$$g\left(H_1H_1H_3H_3\right) = 2\lambda_1-\lambda_2+\lambda_3-3\left(\lambda_2+\lambda_3\right)\mathrm{c}_{4\alpha-2\sigma}, \tag{4.4.7c}$$
$$g\left(H_1H_2H_2H_3\right) = -2\left(\lambda_2+\lambda_3\right)\mathrm{c}_{2\alpha-\sigma}\mathrm{s}_\sigma, \tag{4.4.7d}$$
$$g\left(H_1H_3H_3H_3\right) = 3\left(\lambda_2+\lambda_3\right)\mathrm{s}_{4\alpha-2\sigma}, \tag{4.4.7e}$$
$$g\left(H_2H_2H_3H_3\right) = 2\left[\lambda_1+\lambda_3+\left(\lambda_2+\lambda_3\right)\mathrm{s}_{2\alpha-\sigma}\mathrm{s}_\sigma\right]. \tag{4.4.7f}$$

The quartic couplings involving only the neutral fields of the S_3 singlet are:

$$g\left(S_1S_1S_2S_2\right) = 2\lambda_8. \tag{4.4.8}$$

The quartic couplings involving both neutral fields of the S_3 doublet and singlet are:

$$g\left(H_1H_1S_1S_1\right) = g\left(H_3H_3S_2S_2\right) = \lambda_5+\lambda_6+2\lambda_7\mathrm{c}_{2\alpha-\sigma}, \tag{4.4.9a}$$
$$g\left(H_1H_1S_2S_2\right) = g\left(H_3H_3S_1S_1\right) = \lambda_5+\lambda_6-2\lambda_7\mathrm{c}_{2\alpha-\sigma}, \tag{4.4.9b}$$
$$g\left(H_1H_2S_1S_2\right) = 2\lambda_7\mathrm{s}_{\alpha-\sigma}, \tag{4.4.9c}$$
$$g\left(H_1H_3S_1S_1\right) = -2\lambda_7\mathrm{s}_{2\alpha-\sigma}, \tag{4.4.9d}$$
$$g\left(H_1H_3S_2S_2\right) = 2\lambda_7\mathrm{s}_{2\alpha-\sigma}, \tag{4.4.9e}$$
$$g\left(H_2H_2S_1S_1\right) = \lambda_5+\lambda_6+2\lambda_7\mathrm{c}_\sigma, \tag{4.4.9f}$$
$$g\left(H_2H_2S_2S_2\right) = \lambda_5+\lambda_6-2\lambda_7\mathrm{c}_\sigma, \tag{4.4.9g}$$
$$g\left(H_2H_3S_1S_2\right) = 2\lambda_7\mathrm{c}_{\alpha-\sigma}. \tag{4.4.9h}$$

The quartic couplings involving either the neutral fields of the S_3 doublet or singlet along with either charged fields of the S_3 doublet or singlet are:

$$g\left(H_1H_1H^\pm H^\mp\right) = 2\lambda_1+\left(\lambda_2-\lambda_3\right)\mathrm{c}_{2\alpha}-\left(\lambda_2+\lambda_3\right)\mathrm{c}_{2(\alpha-\sigma)}, \tag{4.4.10a}$$
$$g\left(H_1H_1S^\pm S^\mp\right) = g\left(H_2H_2S^\pm S^\mp\right) = g\left(H_3H_3S^\pm S^\mp\right) = \lambda_5, \tag{4.4.10b}$$
$$g\left(H_1H_3H^\pm H^\mp\right) = -\left(\lambda_2-\lambda_3\right)\mathrm{s}_{2\alpha}+\left(\lambda_2+\lambda_3\right)\mathrm{s}_{2(\alpha-\sigma)}, \tag{4.4.10c}$$
$$g\left(H_2H_2H^\pm H^\mp\right) = 2\left(\lambda_1-\lambda_2\mathrm{s}_\sigma^2+\lambda_3\mathrm{c}_\sigma^2\right), \tag{4.4.10d}$$

$$g\left(H_3H_3H^{\pm}H^{\mp}\right) = 2\lambda_1 - (\lambda_2-\lambda_3)\,\mathrm{c}_{2\alpha} + (\lambda_2+\lambda_3)\,\mathrm{c}_{2(\alpha-\sigma)}, \tag{4.4.10e}$$
$$g\left(S_1S_1H^{\pm}H^{\mp}\right) = g\left(S_2S_2H^{\pm}H^{\mp}\right) = \lambda_5, \tag{4.4.10f}$$
$$g\left(S_1S_1S^{\pm}S^{\mp}\right) = g\left(S_2S_2S^{\pm}S^{\mp}\right) = 2\lambda_8. \tag{4.4.10g}$$

The quartic couplings involving both neutral fields of the S_3 doublet and singlet along with a pair of the charged fields of the S_3 doublet and singlet are:

$$g\left(H_1S_1H^{\pm}S^{\mp}\right) = \mp\frac{i}{2}e^{\pm\frac{i\sigma}{2}}\left(\lambda_6\mathrm{s}_{\alpha} - 2\lambda_7\mathrm{s}_{\alpha-\sigma}\right), \tag{4.4.11a}$$
$$g\left(H_1S_2H^{\pm}S^{\mp}\right) = \frac{1}{2}e^{\pm\frac{i\sigma}{2}}\left(\lambda_6\mathrm{s}_{\alpha} + 2\lambda_7\mathrm{s}_{\alpha-\sigma}\right), \tag{4.4.11b}$$
$$g\left(H_2S_1H^{\pm}S^{\mp}\right) = \frac{1}{2}e^{\pm\frac{i\sigma}{2}}\left(\lambda_6 + 2\lambda_7\mathrm{c}_{\sigma}\right), \tag{4.4.11c}$$
$$g\left(H_2S_2H^{\pm}S^{\mp}\right) = \pm\frac{i}{2}e^{\pm\frac{i\sigma}{2}}\left(\lambda_6 - 2\lambda_7\mathrm{c}_{\sigma}\right), \tag{4.4.11d}$$
$$g\left(H_3S_1H^{\pm}S^{\mp}\right) = \mp\frac{i}{2}e^{\pm\frac{i\sigma}{2}}\left(\lambda_6\mathrm{c}_{\alpha} - 2\lambda_7\mathrm{c}_{\alpha-\sigma}\right), \tag{4.4.11e}$$
$$g\left(H_3S_2H^{\pm}S^{\mp}\right) = \frac{1}{2}e^{\pm\frac{i\sigma}{2}}\left(\lambda_6\mathrm{c}_{\alpha} + 2\lambda_7\mathrm{c}_{\alpha-\sigma}\right). \tag{4.4.11f}$$

The quartic couplings involving only the charged fields are:

$$g\left(H^{\pm}H^{\mp}S^{\pm}S^{\mp}\right) = \lambda_5 + \lambda_6, \tag{4.4.12a}$$
$$g\left(H^{\pm}H^{\pm}S^{\mp}S^{\mp}\right) = 4e^{\pm i\sigma}\lambda_7\mathrm{c}_{\sigma}. \tag{4.4.12b}$$

The trilinear couplings involving only the neutral fields and the would-be Goldstone boson are:

$$g\left(G^0G^0H_1\right) = v\left[(\lambda_2+\lambda_3)\,c_{\alpha-2\sigma} + (2\lambda_1-\lambda_2+\lambda_3)\,c_{\alpha}\right], \tag{4.4.13a}$$
$$g\left(G^0G^0H_3\right) = -v\left[(\lambda_2+\lambda_3)\,\mathrm{s}_{\alpha-2\sigma} + (2\lambda_1-\lambda_2+\lambda_3)\,\mathrm{s}_{\alpha}\right], \tag{4.4.13b}$$
$$g\left(G^0H_1H_2\right) = 2\,(\lambda_2+\lambda_3)\,vc_{\sigma}\mathrm{s}_{\alpha-\sigma}, \tag{4.4.13c}$$
$$g\left(G^0H_2H_3\right) = 2\,(\lambda_2+\lambda_3)\,vc_{\sigma}c_{\alpha-\sigma}, \tag{4.4.13d}$$
$$g\left(G^0S_1S_2\right) = 2\lambda_7vc_{\sigma}. \tag{4.4.13e}$$

The trilinear couplings involving the charged fields and the would-be Goldstone boson are:

$$g\left(H_1G^{\mp}G^{\pm}\right) = v\left[(\lambda_2+\lambda_3)\,c_{\alpha-2\sigma} + (2\lambda_1-\lambda_2+\lambda_3)\,c_{\alpha}\right], \tag{4.4.14a}$$
$$g\left(H_1G^{\mp}H^{\pm}\right) = \pm iv\left[(\lambda_2+\lambda_3)\,\mathrm{s}_{\alpha-2\sigma} + (\lambda_2-\lambda_3)\,\mathrm{s}_{\alpha}\right], \tag{4.4.14b}$$
$$g\left(H_2G^{\mp}H^{\pm}\right) = 2\lambda_3v, \tag{4.4.14c}$$
$$g\left(H_3G^{\mp}G^{\pm}\right) = -v\left[(\lambda_2+\lambda_3)\,\mathrm{s}_{\alpha-2\sigma} + (2\lambda_1-\lambda_2+\lambda_3)\,\mathrm{s}_{\alpha}\right], \tag{4.4.14d}$$
$$g\left(H_3G^{\mp}H^{\pm}\right) = \pm iv\left[(\lambda_2+\lambda_3)\,c_{\alpha-2\sigma} + (\lambda_2-\lambda_3)\,c_{\alpha}\right], \tag{4.4.14e}$$
$$g\left(S_1G^{\mp}S^{\pm}\right) = \frac{1}{2}e^{\mp\frac{i\sigma}{2}}v\left(\lambda_6 + 2\lambda_7\mathrm{c}_{\sigma}\right), \tag{4.4.14f}$$
$$g\left(S_2G^{\mp}S^{\pm}\right) = \mp\frac{1}{2}ie^{\mp\frac{i\sigma}{2}}v\left(\lambda_6 - 2\mathrm{c}_{\sigma}\lambda_7\right). \tag{4.4.14g}$$

The quartic couplings involving neutral states with at least one would-be Goldstone boson are:

$$g\left(G^0G^0G^0G^0\right) = 3\left[2\lambda_1 - \lambda_2 + \lambda_3 + (\lambda_2+\lambda_3)\,c_{2\sigma}\right], \tag{4.4.15a}$$
$$g\left(G^0G^0G^0H_2\right) = -3\,(\lambda_2+\lambda_3)\,\mathrm{s}_{2\sigma}, \tag{4.4.15b}$$
$$g\left(G^0G^0H_1H_1\right) = 2\left[\lambda_1 + \lambda_3 + (\lambda_2+\lambda_3)\,\mathrm{s}_{\sigma}\mathrm{s}_{2\alpha-\sigma}\right], \tag{4.4.15c}$$
$$g\left(G^0G^0H_1H_3\right) = 2\,(\lambda_2+\lambda_3)\,\mathrm{s}_{\sigma}c_{2\alpha-\sigma}, \tag{4.4.15d}$$

$$g\left(G^0G^0H_2H_2\right) = 2\lambda_1 - \lambda_2 + \lambda_3 - 3\left(\lambda_2+\lambda_3\right)c_{2\sigma}, \tag{4.4.15e}$$
$$g\left(G^0G^0H_3H_3\right) = 2\left[\lambda_1 + \lambda_3 - \left(\lambda_2+\lambda_3\right)\mathrm{s}_\sigma \mathrm{s}_{2\alpha-\sigma}\right], \tag{4.4.15f}$$
$$g\left(G^0H_1H_1H_2\right) = 2\left(\lambda_2+\lambda_3\right)c_\sigma \mathrm{s}_{2\alpha-\sigma}, \tag{4.4.15g}$$
$$g\left(G^0H_1H_2H_3\right) = 2\left(\lambda_2+\lambda_3\right)c_\sigma c_{2\alpha-\sigma}, \tag{4.4.15h}$$
$$g\left(G^0H_2H_2H_2\right) = 3\left(\lambda_2+\lambda_3\right)\mathrm{s}_{2\sigma}, \tag{4.4.15i}$$
$$g\left(G^0H_2H_3H_3\right) = -2\left(\lambda_2+\lambda_3\right)c_\sigma \mathrm{s}_{2\alpha-\sigma}, \tag{4.4.15j}$$
$$g\left(G^0G^0S_1S_1\right) = \lambda_5 + \lambda_6 - 2\lambda_7\mathrm{c}_\sigma, \tag{4.4.15k}$$
$$g\left(G^0G^0S_1S_1\right) = \lambda_5 + \lambda_6 + 2\lambda_7\mathrm{c}_\sigma, \tag{4.4.15l}$$
$$g\left(G^0H_1S_1S_2\right) = 2\lambda_7\mathrm{c}_{\alpha-\sigma}, \tag{4.4.15m}$$
$$g\left(G^0H_2S_1S_1\right) = 2\lambda_7\mathrm{s}_\sigma, \tag{4.4.15n}$$
$$g\left(G^0H_2S_2S_2\right) = -2\lambda_7\mathrm{s}_\sigma, \tag{4.4.15o}$$
$$g\left(G^0H_3S_1S_2\right) = -2\lambda_7\mathrm{s}_{\alpha-\sigma}. \tag{4.4.15p}$$

The quartic couplings involving the charged fields and the neutral fields along with the would-be Goldstone boson are:

$$g\left(G^0G^0G^\mp G^\pm\right) = 2\lambda_1 - \lambda_2 + \lambda_3 + \left(\lambda_2+\lambda_3\right)\mathrm{c}_{2\sigma}, \tag{4.4.16a}$$
$$g\left(G^0G^0G^\mp H^\pm\right) = \mp i\left(\lambda_2+\lambda_3\right)\mathrm{s}_{2\sigma}, \tag{4.4.16b}$$
$$g\left(G^0G^0H^\mp H^\pm\right) = 2\lambda_1 + \lambda_2 - \lambda_3 - \left(\lambda_2+\lambda_3\right)\mathrm{c}_{2\sigma}, \tag{4.4.16c}$$
$$g\left(G^0G^0S^\mp S^\pm\right) = \lambda_5, \tag{4.4.16d}$$
$$g\left(G^0H_1G^\mp H^\pm\right) = 2\lambda_3\mathrm{s}_\alpha, \tag{4.4.16e}$$
$$g\left(G^0H_2G^\mp G^\pm\right) = -\left(\lambda_2+\lambda_3\right)\mathrm{s}_{2\sigma}, \tag{4.4.16f}$$
$$g\left(G^0H_2G^\mp H^\pm\right) = \mp i\left[\lambda_2 - \lambda_3 + \left(\lambda_2+\lambda_3\right)\mathrm{c}_{2\sigma}\right], \tag{4.4.16g}$$
$$g\left(G^0H_2H^\mp H^\pm\right) = \left(\lambda_2+\lambda_3\right)\mathrm{s}_{2\sigma}, \tag{4.4.16h}$$
$$g\left(G^0H_3G^\mp H^\pm\right) = 2\lambda_3\mathrm{c}_\alpha, \tag{4.4.16i}$$
$$g\left(H_2H_2G^\mp G^\pm\right) = 2\lambda_1 + \lambda_2 - \lambda_3 - \left(\lambda_2+\lambda_3\right)\mathrm{c}_{2\sigma}, \tag{4.4.16j}$$
$$g\left(H_2H_2G^\mp H^\pm\right) = \pm i\left(\lambda_2+\lambda_3\right)\mathrm{s}_{2\sigma}, \tag{4.4.16k}$$
$$g\left(H_1H_2G^\mp H^\pm\right) = 2\lambda_3\mathrm{c}_\alpha, \tag{4.4.16l}$$
$$g\left(H_2H_3G^\mp H^\pm\right) = -2\lambda_3\mathrm{s}_\alpha, \tag{4.4.16m}$$
$$g\left(H_1H_1G^\mp G^\pm\right) = 2\lambda_1 + \left(\lambda_2+\lambda_3\right)\mathrm{c}_{2(\alpha-\sigma)} - \left(\lambda_2-\lambda_3\right)\mathrm{c}_{2\alpha}, \tag{4.4.16n}$$
$$g\left(H_1H_1G^\mp H^\pm\right) = \pm i\left[\left(\lambda_2+\lambda_3\right)\mathrm{s}_{2(\alpha-\sigma)} + \left(\lambda_2-\lambda_3\right)\mathrm{s}_{2\alpha}\right], \tag{4.4.16o}$$
$$g\left(H_1H_3G^\mp G^\pm\right) = -\left(\lambda_2+\lambda_3\right)\mathrm{s}_{2(\alpha-\sigma)} + \left(\lambda_2-\lambda_3\right)\mathrm{s}_{2\alpha}, \tag{4.4.16p}$$
$$g\left(H_1H_3G^\mp H^\pm\right) = \pm i\left[\left(\lambda_2+\lambda_3\right)\mathrm{c}_{2(\alpha-\sigma)} + \left(\lambda_2-\lambda_3\right)\mathrm{c}_{2\alpha}\right], \tag{4.4.16q}$$
$$g\left(H_3H_3G^\mp G^\pm\right) = 2\lambda_1 - \left(\lambda_2+\lambda_3\right)\mathrm{c}_{2(\alpha-\sigma)} + \left(\lambda_2-\lambda_3\right)\mathrm{c}_{2\alpha}, \tag{4.4.16r}$$
$$g\left(H_3H_3G^\mp H^\pm\right) = \mp i\left[\left(\lambda_2-\lambda_3\right)\mathrm{s}_{2\alpha} + \left(\lambda_2+\lambda_3\right)\mathrm{s}_{2(\alpha-\sigma)}\right], \tag{4.4.16s}$$
$$g\left(S_1S_1G^\mp G^\pm\right) = g\left(S_2S_2G^\mp G^\pm\right) = \lambda_5, \tag{4.4.16t}$$

and

$$g\left(G^0S_1G^\mp S^\pm\right) = \pm\frac{1}{2}ie^{\mp\frac{i\sigma}{2}}\left(\lambda_6 - 2\lambda_7\mathrm{c}_\sigma\right), \tag{4.4.17a}$$
$$g\left(G^0S_2G^\mp S^\pm\right) = \frac{1}{2}e^{\mp\frac{i\sigma}{2}}\left(\lambda_6 + 2\lambda_7\mathrm{c}_\sigma\right), \tag{4.4.17b}$$
$$g\left(G^0S_1H^\mp S^\pm\right) = \lambda_7e^{\mp\frac{i\sigma}{2}}\mathrm{s}_\sigma, \tag{4.4.17c}$$
$$g\left(G^0S_2H^\mp S^\pm\right) = g\left(H_2S_1G^\mp S^\pm\right) = \pm i\lambda_7e^{\mp\frac{i\sigma}{2}}\mathrm{s}_\sigma, \tag{4.4.17d}$$
$$g\left(H_2S_2G^\mp S^\pm\right) = -\lambda_7e^{\mp\frac{i\sigma}{2}}\mathrm{s}_\sigma, \tag{4.4.17e}$$

$$g\left(H_1S_1G^{\mp}S^{\pm}\right) = \frac{1}{2}e^{\mp\frac{i\sigma}{2}}\left(\lambda_6 c_\alpha + 2\lambda_7 c_{\alpha-\sigma}\right), \tag{4.4.17f}$$

$$g\left(H_1S_2G^{\mp}S^{\pm}\right) = \mp\frac{1}{2}ie^{\mp\frac{i\sigma}{2}}\left(\lambda_6 c_\alpha - 2\lambda_7 c_{\alpha-\sigma}\right), \tag{4.4.17g}$$

$$g\left(H_3S_1G^{\mp}S^{\pm}\right) = -\frac{1}{2}e^{\mp\frac{i\sigma}{2}}\left(\lambda_6 s_\alpha + 2\lambda_7 s_{\alpha-\sigma}\right), \tag{4.4.17h}$$

$$g\left(H_3S_2G^{\mp}S^{\pm}\right) = \pm\frac{1}{2}ie^{\mp\frac{i\sigma}{2}}\left(\lambda_6 s_\alpha - 2\lambda_7 s_{\alpha-\sigma}\right). \tag{4.4.17i}$$

The quartic couplings involving only the charged fields and the would-be Goldstone boson are:

$$g\left(G^{\mp}G^{\pm}G^{\mp}G^{\pm}\right) = 2\left[2\lambda_1 - \lambda_2 + \lambda_3 + (\lambda_2+\lambda_3)\,c_{2\sigma}\right], \tag{4.4.18a}$$

$$g\left(G^{\mp}G^{\pm}G^{\mp}H^{\pm}\right) = \mp 2i\left(\lambda_2+\lambda_3\right)s_{2\sigma}, \tag{4.4.18b}$$

$$g\left(G^{\mp}H^{\pm}G^{\mp}H^{\pm}\right) = 4\left(\lambda_2+\lambda_3\right)c_\sigma^2, \tag{4.4.18c}$$

$$g\left(G^{\mp}S^{\pm}G^{\mp}S^{\pm}\right) = 4\lambda_7 e^{\mp i\sigma}c_\sigma, \tag{4.4.18d}$$

$$g\left(G^{\mp}G^{\pm}H^{\mp}G^{\pm}\right) = \pm 2i\left(\lambda_2+\lambda_3\right)s_{2\sigma}, \tag{4.4.18e}$$

$$g\left(G^{\mp}G^{\pm}H^{\mp}H^{\pm}\right) = 2\left[\lambda_1 + \lambda_3 - (\lambda_2+\lambda_3)\,c_{2\sigma}\right], \tag{4.4.18f}$$

$$g\left(G^{\mp}H^{\pm}H^{\mp}H^{\pm}\right) = \pm 2i\left(\lambda_2+\lambda_3\right)s_{2\sigma}, \tag{4.4.18g}$$

$$g\left(G^{\mp}S^{\pm}H^{\mp}S^{\pm}\right) = \pm 4i\lambda_7 e^{\mp i\sigma}s_\sigma, \tag{4.4.18h}$$

$$g\left(G^{\mp}G^{\pm}S^{\mp}S^{\pm}\right) = \lambda_5 + \lambda_6. \tag{4.4.18i}$$

From the trilinear couplings involving the same species states it follows that the states H_1 and H_3 are CP-even. From the neutral trilinear couplings we conclude that the H_2 state is CP-odd and the states S_1 and S_2 have an opposite CP quantum numbers. This is in agreement with the extracted information from the scalar-gauge bosons couplings, see section 4.2.

4.5 Constraints

4.5.1 Constraints From the Scalar Masses

We consider constraints from the scalar masses. First of all, we assume that all of the scalars masses squared are positive and non-zero. Secondly, we assume that the H_1 field corresponds to the recently discovered Higgs boson with $m_H = 125$ GeV.

Taking into consideration the first assumption, from the charged sector we get that:

$$\lambda_2 > 0, \tag{4.5.1a}$$

$$\lambda_5 > -\frac{2\mu_0^2}{v^2}. \tag{4.5.1b}$$

From the neutral sector of the S_3 doublet it follows that:

$$\lambda_3 > -\min\left(\lambda_1,\ \lambda_2\right), \tag{4.5.2a}$$

$$\lambda_1 + \lambda_3 > \Delta. \tag{4.5.2b}$$

The determinant of the mass-squared matrix $\mathcal{M}^2_{\text{Neutral}-12}$ of eq. (4.1.9) should be positive definite:

$$\det\left(\mathcal{M}^2_{\text{Neutral}-12}\right) = 8\left(\lambda_1-\lambda_2\right)\left(\lambda_2+\lambda_3\right)^2 s_\sigma^2 v^6 > 0. \tag{4.5.3}$$

Therefore it follows that:

$$\lambda_1 > \lambda_2. \tag{4.5.4}$$

Taking into consideration the neutral masses of the S_3 singlet we get the following constraints:

$$\begin{aligned} \frac{2\mu_0^2}{v^2} + \lambda_5 + \lambda_6 &> 0, \\ \frac{2\mu_0^2}{v^2} + \lambda_5 + \lambda_6 - \left|\lambda_7 c_{2\sigma}\right| &> 0. \end{aligned} \tag{4.5.5}$$

4.5.2 The Standard Model Limit

In the SM, the Higgs boson and gauge boson Feynman rules are given by [57]:

$$hW_\mu^\pm W_\nu^\mp = igm_W g_{\mu\nu}, \tag{4.5.6a}$$
$$hZ_\mu Z_\nu = i\frac{g}{\mathrm{c}_w} m_Z g_{\mu\nu}, \tag{4.5.6b}$$
$$hhW_\mu^\pm W_\nu^\mp = \frac{i}{2} g^2 g_{\mu\nu}, \tag{4.5.6c}$$
$$hhZ_\mu Z_\nu = \frac{i}{2}\frac{g^2}{\mathrm{c}_w^2} g_{\mu\nu}, \tag{4.5.6d}$$

and the Higgs boson and fermion Feynman rules are:

$$hf\bar{f} = -i\frac{m_f}{v}. \tag{4.5.7}$$

Assuming that the SM-like Higgs boson is H_1, in the SM limit we find that $\mathrm{c}_\alpha = 1$ and therefore $\alpha = 0$. The rotation angle α was defined in eq. (4.1.13). It follows that alongside $\alpha = 0$ another condition should be satisfied:

$$(\lambda_2 + \lambda_3)\,\mathrm{s}_{2\sigma} = 0, \tag{4.5.8}$$

where $\lambda_2 + \lambda_3 = 0$ leads to the exact C-III-c[3] case as in this limit we get $\nu^2 = 0$. The other option leads to $\sigma = 0$ or $\sigma = \frac{1}{2}\pi$. The $\sigma = 0$ constraint results in a real vacuum configuration. Moreover, solving for the minimization conditions we get that $\nu^2 = 0$ and this is exactly the R-II-3 vacuum configuration. In the case of $\sigma = \frac{1}{2}\pi$ we get that $\nu^2 = 0$. Therefore we conclude that there is no exact SM limit for the C-III-c-ν^2 vacuum configuration. Nevertheless, we assume that the SM limit is achieved by $\mathrm{c}_\alpha \geq 0.9$.

In the SM, the Higgs-Higgs boson H trilinear and quartic Feynman rules are:

$$\begin{aligned} HHH &= -3i\frac{m_h^2}{v}, \\ HHHH &= -3i\frac{m_h^2}{v^2}. \end{aligned} \tag{4.5.9}$$

The C-III-c-ν^2 model results in $g\,(H_1H_1H_1)$ (4.4.2a) and $g\,(H_1H_1H_1H_1)$ (4.4.6a). In principle, there is some freedom and we do not compare these terms against the SM couplings. For an insight of the non 3HDM see Refs. [58–60].

4.5.3 Potential Stability

The scalar potential needs to be stable, see Refs. [16, 61]. This implies that the scalar potential of eq. (2.1.7) should be positive in all space directions for asymptotically large values of fields, *i.e.*, for $|h_1|$, and $|h_2|$, and $|h_S|$ approaching infinity. This is the most basic constraint as it forces the existence of a stable minimum.

Necessary potential stability conditions were presented in Ref. [37]:

$$\lambda_1 > 0, \tag{4.5.10a}$$
$$\lambda_8 > 0, \tag{4.5.10b}$$
$$\lambda_1 + \lambda_3 > 0, \tag{4.5.10c}$$
$$2\lambda_1 + (\lambda_3 - \lambda_2) > |\lambda_2 + \lambda_3|, \tag{4.5.10d}$$
$$\lambda_5 + 2\sqrt{\lambda_8\,(\lambda_1 + \lambda_3)} > 0, \tag{4.5.10e}$$
$$\lambda_5 + \lambda_6 + 2\sqrt{\lambda_8\,(\lambda_1 + \lambda_3)} > 2|\lambda_7|, \tag{4.5.10f}$$
$$\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 + 2\lambda_7 + \lambda_8 > 2|\lambda_4|. \tag{4.5.10g}$$

[3] We remind that the exact C-III-c vacuum configurations results in only one massive neutral scalar.

In Ref. [34], following the approach of Refs. [62,63], it was shown that although for the most general scalar potential stability conditions are quite involved, for vacuum configurations with the $\lambda_4 = 0$ constraint there exists an explicit direction in the space of the scalar potential. Another method, in terms of bilinears, was discussed in Ref. [64].

We present the general formulation of Ref. [34]. The $SU(2)$ doublets can be re-expressed as follows:

$$h_i = ||h_i||\hat{h}_i, \quad \text{for } i = \{1,\, 2,\, S\}, \tag{4.5.11}$$

where $||h_i||$ is the norm of the spinor, and $\hat{h}_i$ is a unit spinor. Assuming that the two different systems have the same origin, the norms can be parameterized in terms of relations between the Cartesian and spherical coordinates:

$$||h_1|| = r\mathrm{c}_\gamma \mathrm{s}_\theta, \tag{4.5.12a}$$
$$||h_2|| = r\mathrm{s}_\gamma \mathrm{s}_\theta, \tag{4.5.12b}$$
$$||h_S|| = r\mathrm{c}_\theta, \tag{4.5.12c}$$

where $r \in [0, \infty)$, $\gamma \in [0, \pi/2]$, $\theta \in [0, \pi/2]$. The $SU(2)$ invariant products are:

$$\hat{h}_2^\dagger \hat{h}_1 = \rho_3 e^{i\theta_3}, \tag{4.5.13a}$$
$$\hat{h}_S^\dagger \hat{h}_2 = \rho_1 e^{i\theta_1}, \tag{4.5.13b}$$
$$\hat{h}_1^\dagger \hat{h}_S = \rho_2 e^{i\theta_2}, \tag{4.5.13c}$$
$$\hat{h}_i^\dagger \hat{h}_j = \left(\hat{h}_j^\dagger \hat{h}_i\right)^*, \tag{4.5.13d}$$

where $\theta_i \in [0, 2\pi)$ and $\rho_i \in [0, 1]$ due to the fact that solutions lie within a unit sphere.

For asymptotically large field values the main contribution comes from the quartic terms V_4 of the scalar potential. Thus we consider only the relevant quartic couplings. In this case the potential stability condition is given by:

$$V_4 = r^4 \sum_{i=1}^{8} \lambda_i A_i \geq 0, \quad \forall \, \{\rho_i,\, \theta_i,\, \gamma,\, \theta\}, \tag{4.5.14}$$

where

$$A_1 = \mathrm{s}_\theta^4, \tag{4.5.15a}$$
$$A_2 = -\rho_3^2 \mathrm{s}_{\theta_3}^2 \mathrm{s}_{2\gamma}^2 \mathrm{s}_\theta^4, \tag{4.5.15b}$$
$$A_3 = \left(\mathrm{c}_{2\gamma}^2 + \rho_3^2 \mathrm{c}_{\theta_3}^2 \mathrm{s}_{2\gamma}^2\right) \mathrm{s}_\theta^4, \tag{4.5.15c}$$
$$A_4 = 2\left(\rho_1 \mathrm{c}_{\theta_1} \mathrm{c}_{2\gamma} + 2\rho_2\rho_3 \mathrm{c}_{\theta_2} \mathrm{c}_{\theta_3} \mathrm{c}_\gamma^2\right) \mathrm{s}_\gamma \mathrm{c}_\theta \mathrm{s}_\theta^3, \tag{4.5.15d}$$
$$A_5 = \frac{1}{4}\mathrm{s}_{2\theta}^2, \tag{4.5.15e}$$
$$A_6 = \frac{1}{4}\left(\rho_1^2 \mathrm{s}_\gamma^2 + \rho_2^2 \mathrm{c}_\gamma^2\right) \mathrm{s}_{2\theta}^2, \tag{4.5.15f}$$
$$A_7 = \frac{1}{2}\left(\rho_1^2 \mathrm{c}_{2\theta_1} \mathrm{s}_\gamma^2 + \rho_2^2 \mathrm{c}_{2\theta_2} \mathrm{c}_\gamma^2\right) \mathrm{s}_{2\theta}^2, \tag{4.5.15g}$$
$$A_8 = \mathrm{c}_\theta^4. \tag{4.5.15h}$$

Necessary and sufficient conditions, provided that the $\lambda_4 = 0$ constraint is applied, are [49]:

$$\lambda_1 > 0, \tag{4.5.16a}$$
$$\lambda_8 > 0, \tag{4.5.16b}$$
$$\lambda_1 - \lambda_2 > 0, \tag{4.5.16c}$$
$$\lambda_1 + \lambda_3 > 0, \tag{4.5.16d}$$
$$\lambda_5 + \min\left[0, \lambda_6 - 2|\lambda_7|\right] > -2\sqrt{\lambda_1 \lambda_8}, \tag{4.5.16e}$$

$$\lambda_5 + \min\left[0, \lambda_6 - 2|\lambda_7|\right] > -2\sqrt{(\lambda_1 - \lambda_2)\,\lambda_8}, \tag{4.5.16f}$$
$$\lambda_5 + \min\left[0, \lambda_6 - 2|\lambda_7|\right] > -2\sqrt{(\lambda_1 + \lambda_3)\,\lambda_8}. \tag{4.5.16g}$$

In the C-III-c-ν^2 model eqs. (4.5.16a, 4.5.16c, 4.5.16d) are by default satisfied due to the mass-squared parameters. Eq. (4.5.16b) puts a lower bound on the free coupling λ_8 of the model. The left side of terms (4.5.16e-4.5.16g) is equivalent and thus only the lowest value of the square root should be considered as it results in the most severe constraint. It turns out that only eq. (4.5.16f) can be considered. For simplicity, it can be split into:

$$\lambda_5 > -2\sqrt{(\lambda_1 - \lambda_2)\,\lambda_8}, \tag{4.5.17a}$$
$$\lambda_5 + \lambda_6 - 2|\lambda_7| > -2\sqrt{(\lambda_1 - \lambda_2)\,\lambda_8}. \tag{4.5.17b}$$

4.5.4 Perturbativity

The soft perturbativity limit is given by directly imposing constraints on the quartic couplings:

$$|\lambda_i| \leq \lambda_{\rm max}. \tag{4.5.18}$$

The most conservative choice is to set $\lambda_{\rm max} = 4\pi$. A smaller value $\lambda_{\rm max} = 2\pi$ was adopted in Ref. [65]. We adopt a more widely used convention of $\lambda_{\rm max} = 4\pi$.

For the C-III-c-ν^2, the number of checks of (4.5.18) can be reduced. We discuss relations between the couplings and masses in section 4.1.1. We suppose that the heaviest states are $m_\xi = 1$ TeV. First of all, from $|\lambda_2| \leq 4\pi$ it follows that $m_{H^\pm} \leq 1234.36$ GeV. Next, the $|\lambda_5| \leq 4\pi$ is fixed by μ_0^2:

$$\mu_0^2 \in \left[-2\pi v^2 + m_{S^\pm}^2,\, 2\pi v^2 + m_{S^\pm}^2\right]. \tag{4.5.19}$$

The constraint $|\lambda_3| \leq 4\pi$ can also be neglected as it requires mass splitting of order 1200 GeV, which in our case makes little sense. From $|\lambda_1| \leq 4\pi$ it follows that:

$$m_{H^\pm}^2 < |-m_{H_2}^2 + m_{H_3}^2 + m_{H^\pm}^2| \leq (1228.01\,\text{GeV})^2, \tag{4.5.20}$$

and the lower boundary follows from the fact that $m_{H_2} \leq m_{H_3}$. From $|\lambda_6| \leq 4\pi$ one can derive:

$$m_{S^\pm} \in \left[\text{Max}\Big[\text{Re}\left(\sqrt{\frac{-4\pi v^2 + m_{S_1}^2 + m_{S_2}^2}{2}}\right), 100\Big], \text{Min}\Big[\text{Re}\left(\sqrt{\frac{4\pi v^2 + m_{S_1}^2 + m_{S_2}^2}{2}}\right), 1000\Big]\right], \tag{4.5.21}$$

in GeV units.

The only constraint to be checked is $|\lambda_7| \leq 4\pi$, which turns out to be way too involved to be analytically checked.

We also take into consideration a more severe perturbativity limit in terms of limiting the overall strength of the quartic scalar-scalar interactions[4]:

$$|g\left(\varphi_i \varphi_j \varphi_k \varphi_l\right)| \leq 4\pi, \tag{4.5.22}$$

where the quartic couplings were presented in section 4.4. One of the most obvious limits comes from the coupling $g\left(S_1 S_1 S_1 S_1\right) = 6\pi$ (4.4.6c) by directly considering the perturbativity limit (4.5.22), $\lambda_8 \leq \frac{2}{3}\pi$. Some of the couplings depend on a single λ_5, *e.g.*, $g\left(H_1 H_1 S^\pm S^\mp\right) = \lambda_5$ (4.4.10b), and such checks simplify to (4.5.18). The other simple relation is $g\left(H^\pm H^\mp S^\pm S^\mp\right) = \lambda_5 + \lambda_6$ (4.4.12a). Most of the quartic scalar-scalar couplings depend on trigonometric functions and thus should be numerically checked if those satisfy the perturbativity condition (4.5.22).

[4] Interactions involving the would-be Goldstone bosons are not considered.

4.5.5 Tree-Level Unitarity

The two-body scattering processes involving longitudinal bosons and the Higgs boson in the SM were pioneered by Lee, Quigg and Thacker [66] and further on analysed in Refs. [67, 68]. The unitarity constraints of the 2HDM are well-known and different methods can be found in Refs. [69, 70]. The general idea behind the unitarity bound is that the Born amplitude for elastic longitudinal vector-boson scattering may not result in a higher than unity amplitude.

The process of finding the unitarity limit is straightforward due to the Goldstone equivalence theorem, which relates the longitudinally polarized vector boson and the Goldstone bosons in the high-energy limit [71, 72]. It is sufficient to take a look at the $2 \to 2$ scattering processes of the gauge-eigenstates. The number of 2-body states is given by a binomial:

$$m = \binom{n+1}{2} = \frac{1}{2} n\left(1+n\right), \tag{4.5.23}$$

where n is the number of scalar degrees of freedom. The most general case results in a matrix of dimension $\dim(m)$.

Due to computational complexity of the eigenvalues of the most general scattering matrix it is worth a try to find a basis in which the scattering matrix S is block-diagonal. This approach makes sense as not all of the 2-body scattering processes are possible as those are restricted by the S_3-symmetric potential and by discrete symmetries like CP or $\mathbb{Z}_2$.

The eigenvalues of the block-diagonal matrix S is a list of eigenvalues of each sub-block diagonal matrix:

$$\det\left(S - \lambda\mathcal{I}\right) = \det\left(S_1 - \lambda\mathcal{I}\right) \times \cdots \times \det\left(S_n - \lambda\mathcal{I}\right). \tag{4.5.24}$$

The electric charge should be conserved in the scattering processes and thus it is straightforward to split the scattering matrix S based on the total charge. The neutral scattering matrix is denoted by S^0, and the singly charged scattering matrix is denoted by S^+, and the doubly charged scattering matrix is given by S^{++}. Therefore the form of the scattering matrix including the channels based on the electrical charge is as follows:

$$S = \mathrm{diag}\left(S^0,\, S^+,\, S^{++}\right). \tag{4.5.25}$$

We start with the neutral channel. The neutral scattering matrix can be expressed as:

$$S^0 = \mathrm{diag}\left(S_1^0,\, S_2^0,\, S_3^0,\, S_4^0,\, S_5^0\right). \tag{4.5.26}$$

Each of the matrices S_i^n is obtained by taking a look at the states $\langle\Psi_i^n|\Psi_i^n\rangle$, where the two-particles states Ψ_i^n are given by:

$$\Psi_1^0 = \left\{\left|h_1^+ h_2^-\right\rangle, \left|h_1^- h_2^+\right\rangle, \left|\tilde\eta_1\tilde\eta_2\right\rangle, \left|\tilde\chi_1\tilde\chi_2\right\rangle, \left|\tilde\eta_1\tilde\chi_2\right\rangle, \left|\tilde\eta_2\tilde\chi_1\right\rangle\right\}, \tag{4.5.27a}$$

$$\Psi_2^0 = \left\{\left|h_1^+ h_S^-\right\rangle, \left|h_1^- h_S^+\right\rangle, \left|\tilde\eta_1\tilde\eta_S\right\rangle, \left|\tilde\chi_1\tilde\chi_S\right\rangle, \left|\tilde\eta_1\tilde\chi_S\right\rangle, \left|\tilde\eta_S\tilde\chi_1\right\rangle\right\}, \tag{4.5.27b}$$

$$\Psi_3^0 = \left\{\left|\tilde\eta_1\tilde\chi_1\right\rangle, \left|\tilde\eta_2\tilde\chi_2\right\rangle, \left|\tilde\eta_S\tilde\chi_S\right\rangle\right\}, \tag{4.5.27c}$$

$$\Psi_4^0 = \left\{\left|h_2^+ h_S^-\right\rangle, \left|h_2^- h_S^+\right\rangle, \left|\tilde\eta_2\tilde\eta_S\right\rangle, \left|\tilde\chi_2\tilde\chi_S\right\rangle, \left|\tilde\eta_2\tilde\chi_S\right\rangle, \left|\tilde\eta_S\tilde\chi_2\right\rangle\right\}, \tag{4.5.27d}$$

$$\begin{aligned}\Psi_5^0 = \Bigg\{&\left|h_1^+ h_1^-\right\rangle, \left|h_2^- h_2^+\right\rangle, \left|h_S^- h_S^+\right\rangle, \frac{1}{\sqrt{2}}\left|\tilde\eta_1\tilde\eta_1\right\rangle, \frac{1}{\sqrt{2}}\left|\tilde\eta_2\tilde\eta_2\right\rangle, \\ &\frac{1}{\sqrt{2}}\left|\tilde\eta_S\tilde\eta_S\right\rangle, \frac{1}{\sqrt{2}}\left|\tilde\chi_1\tilde\chi_1\right\rangle, \frac{1}{\sqrt{2}}\left|\tilde\chi_2\tilde\chi_2\right\rangle, \frac{1}{\sqrt{2}}\left|\tilde\chi_S\tilde\chi_S\right\rangle\Bigg\},\end{aligned} \tag{4.5.27e}$$

where the factor of $\frac{1}{\sqrt{2}}$ is due to the Bose-Einstein statistics. States are organized so that each of the separate sets $\langle\Psi_i^n|\Psi_i^n\rangle$ is a block-diagonal component. The elements of the neutral scattering matrix are:

$$S_1^0 = \begin{pmatrix} 2(\lambda_1-\lambda_2) & 4(\lambda_2+\lambda_3) & 2\lambda_3 & 2\lambda_3 & -2i\lambda_2 & 2i\lambda_2 \\ 4(\lambda_2+\lambda_3) & 2(\lambda_1-\lambda_2) & 2\lambda_3 & 2\lambda_3 & 2i\lambda_2 & -2i\lambda_2 \\ 2\lambda_3 & 2\lambda_3 & 2(\lambda_1+\lambda_3) & 2(\lambda_2+\lambda_3) & 0 & 0 \\ 2\lambda_3 & 2\lambda_3 & 2(\lambda_2+\lambda_3) & 2(\lambda_1+\lambda_3) & 0 & 0 \\ 2i\lambda_2 & -2i\lambda_2 & 0 & 0 & 2(\lambda_1-2\lambda_2-\lambda_3) & 2(\lambda_2+\lambda_3) \\ -2i\lambda_2 & 2i\lambda_2 & 0 & 0 & 2(\lambda_2+\lambda_3) & 2(\lambda_1-2\lambda_2-\lambda_3) \end{pmatrix}, \tag{4.5.28a}$$

$$S_2^0 = \begin{pmatrix} \lambda_5+\lambda_6 & 4\lambda_7 & \frac{1}{2}(\lambda_6+2\lambda_7) & \frac{1}{2}(\lambda_6+2\lambda_7) & \frac{i}{2}(\lambda_6-2\lambda_7) & -\frac{i}{2}(\lambda_6-2\lambda_7) \\ 4\lambda_7 & \lambda_5+\lambda_6 & \frac{1}{2}(\lambda_6+2\lambda_7) & \frac{1}{2}(\lambda_6+2\lambda_7) & -\frac{i}{2}(\lambda_6-2\lambda_7) & \frac{i}{2}(\lambda_6-2\lambda_7) \\ \frac{1}{2}(\lambda_6+2\lambda_7) & \frac{1}{2}(\lambda_6+2\lambda_7) & \lambda_5+\lambda_6+2\lambda_7 & 2\lambda_7 & 0 & 0 \\ \frac{1}{2}(\lambda_6+2\lambda_7) & \frac{1}{2}(\lambda_6+2\lambda_7) & 2\lambda_7 & \lambda_5+\lambda_6+2\lambda_7 & 0 & 0 \\ -\frac{i}{2}(\lambda_6-2\lambda_7) & \frac{i}{2}(\lambda_6-2\lambda_7) & 0 & 0 & \lambda_5+\lambda_6-2\lambda_7 & 2\lambda_7 \\ \frac{i}{2}(\lambda_6-2\lambda_7) & -\frac{i}{2}(\lambda_6-2\lambda_7) & 0 & 0 & 2\lambda_7 & \lambda_5+\lambda_6-2\lambda_7 \end{pmatrix}, \tag{4.5.28b}$$

$$S_3^0 = \begin{pmatrix} 2(\lambda_1+\lambda_3) & 2(\lambda_2+\lambda_3) & 2\lambda_7 \\ 2(\lambda_2+\lambda_3) & 2(\lambda_1+\lambda_3) & 2\lambda_7 \\ 2\lambda_7 & 2\lambda_7 & 2\lambda_8 \end{pmatrix}, \tag{4.5.28c}$$

$$S_4^0 = S_2^0, \tag{4.5.28d}$$

$$S_5^0 = \begin{pmatrix} S_{5_{11}}^0 & S_{5_{12}}^0 \\ S_{5_{12}}^{0\ \mathrm{T}} & S_{5_{22}}, \end{pmatrix} \tag{4.5.28e}$$

where

$$S_{5_{11}}^0 = \begin{pmatrix} 4(\lambda_1+\lambda_3) & 2(\lambda_1-\lambda_2) & \lambda_5+\lambda_6 & \sqrt{2}(\lambda_1+\lambda_3) & \sqrt{2}(\lambda_1-\lambda_3) \\ 2(\lambda_1-\lambda_2) & 4(\lambda_1+\lambda_3) & \lambda_5+\lambda_6 & \sqrt{2}(\lambda_1-\lambda_3) & \sqrt{2}(\lambda_1+\lambda_3) \\ \lambda_5+\lambda_6 & \lambda_5+\lambda_6 & 4\lambda_8 & \frac{\lambda_5}{\sqrt{2}} & \frac{\lambda_5}{\sqrt{2}} \\ \sqrt{2}(\lambda_1+\lambda_3) & \sqrt{2}(\lambda_1-\lambda_3) & \frac{\lambda_5}{\sqrt{2}} & 3(\lambda_1+\lambda_3) & \lambda_1+\lambda_3 \\ \sqrt{2}(\lambda_1-\lambda_3) & \sqrt{2}(\lambda_1+\lambda_3) & \frac{\lambda_5}{\sqrt{2}} & \lambda_1+\lambda_3 & 3(\lambda_1+\lambda_3) \end{pmatrix}, \tag{4.5.29a}$$

$$S_{5_{12}}^0 = \begin{pmatrix} \frac{\lambda_5}{\sqrt{2}} & \sqrt{2}(\lambda_1+\lambda_3) & \sqrt{2}(\lambda_1-\lambda_3) & \frac{\lambda_5}{\sqrt{2}} \\ \frac{\lambda_5}{\sqrt{2}} & \sqrt{2}(\lambda_1-\lambda_3) & \sqrt{2}(\lambda_1+\lambda_3) & \frac{\lambda_5}{\sqrt{2}} \\ \sqrt{2}\lambda_8 & \frac{\lambda_5}{\sqrt{2}} & \frac{\lambda_5}{\sqrt{2}} & \sqrt{2}\lambda_8 \\ \frac{1}{2}(\lambda_5+\lambda_6+2\lambda_7) & \lambda_1+\lambda_3 & \lambda_1-2\lambda_2-\lambda_3 & \frac{1}{2}(\lambda_5+\lambda_6-2\lambda_7) \\ \frac{1}{2}(\lambda_5+\lambda_6+2\lambda_7) & \lambda_1-2\lambda_2-\lambda_3 & \lambda_1+\lambda_3 & \frac{1}{2}(\lambda_5+\lambda_6-2\lambda_7) \end{pmatrix}, \tag{4.5.29b}$$

$$S_{5_{22}}^0 = \begin{pmatrix} 3\lambda_8 & \frac{1}{2}(\lambda_5+\lambda_6-2\lambda_7) & \frac{1}{2}(\lambda_5+\lambda_6-2\lambda_7) & \lambda_8 \\ \frac{1}{2}(\lambda_5+\lambda_6-2\lambda_7) & 3(\lambda_1+\lambda_3) & \lambda_1+\lambda_3 & \frac{1}{2}(\lambda_5+\lambda_6+2\lambda_7) \\ \frac{1}{2}(\lambda_5+\lambda_6-2\lambda_7) & \lambda_1+\lambda_3 & 3(\lambda_1+\lambda_3) & \frac{1}{2}(\lambda_5+\lambda_6+2\lambda_7) \\ \lambda_8 & \frac{1}{2}(\lambda_5+\lambda_6+2\lambda_7) & \frac{1}{2}(\lambda_5+\lambda_6+2\lambda_7) & 3\lambda_8 \end{pmatrix}. \tag{4.5.29c}$$

The singly charged scattering matrix is:

$$S^+ = \mathrm{diag}\left(S_1^+,\, S_2^+,\, S_3^+,\, S_4^+\right). \tag{4.5.30}$$

The singly charged two-particles states are:

$$\Psi_1^+ = \left\{\left|h_1^+\tilde{\eta}_1\right\rangle,\, \left|h_2^+\tilde{\eta}_2\right\rangle,\, \left|h_S^+\tilde{\eta}_S\right\rangle,\, \left|h_1^+\tilde{\chi}_1\right\rangle,\, \left|h_2^+\tilde{\chi}_2\right\rangle,\, \left|h_S^+\tilde{\chi}_S\right\rangle\right\}, \tag{4.5.31a}$$

$$\Psi_2^+ = \left\{\left|h_1^+\tilde{\eta}_2\right\rangle,\, \left|h_2^+\tilde{\eta}_1\right\rangle,\, \left|h_1^+\tilde{\chi}_2\right\rangle,\, \left|h_2^+\tilde{\chi}_1\right\rangle\right\}, \tag{4.5.31b}$$

$$\Psi_3^+ = \left\{\left|h_1^+\tilde{\eta}_S\right\rangle,\, \left|h_S^+\tilde{\eta}_1\right\rangle,\, \left|h_1^+\tilde{\chi}_S\right\rangle,\, \left|h_S^+\tilde{\chi}_1\right\rangle\right\}, \tag{4.5.31c}$$

$$\Psi_4^+ = \left\{\left|h_2^+\tilde{\eta}_S\right\rangle,\, \left|h_S^+\tilde{\eta}_2\right\rangle,\, \left|h_2^+\tilde{\chi}_S\right\rangle,\, \left|h_S^+\tilde{\chi}_2\right\rangle\right\}. \tag{4.5.31d}$$

The sub-matrices of the singly charged scattering matrix are:

$$S_1^+ = \begin{pmatrix} 2(\lambda_1+\lambda_3) & 2\lambda_3 & \frac{1}{2}(\lambda_6+2\lambda_7) & 0 & -2i\lambda_2 & \frac{i}{2}(\lambda_6-2\lambda_7) \\ 2\lambda_3 & 2(\lambda_1+\lambda_3) & \frac{1}{2}(\lambda_6+2\lambda_7) & -2i\lambda_2 & 0 & \frac{i}{2}(\lambda_6-2\lambda_7) \\ \frac{1}{2}(\lambda_6+2\lambda_7) & \frac{1}{2}(\lambda_6+2\lambda_7) & 2\lambda_8 & \frac{i}{2}(\lambda_6-2\lambda_7) & \frac{i}{2}(\lambda_6-2\lambda_7) & 0 \\ 0 & 2i\lambda_2 & -\frac{i}{2}(\lambda_6-2\lambda_7) & 2(\lambda_1+\lambda_3) & 2\lambda_3 & \frac{1}{2}(\lambda_6+2\lambda_7) \\ 2i\lambda_2 & 0 & -\frac{i}{2}(\lambda_6-2\lambda_7) & 2\lambda_3 & 2(\lambda_1+\lambda_3) & \frac{1}{2}(\lambda_6+2\lambda_7) \\ -\frac{i}{2}(\lambda_6-2\lambda_7) & -\frac{i}{2}(\lambda_6-2\lambda_7) & 0 & \frac{1}{2}(\lambda_6+2\lambda_7) & \frac{1}{2}(\lambda_6+2\lambda_7) & 2\lambda_8 \end{pmatrix}, \tag{4.5.32a}$$

$$S_2^+ = \begin{pmatrix} 2(\lambda_1-\lambda_3) & 2\lambda_3 & 0 & 2i\lambda_2 \\ 2\lambda_3 & 2(\lambda_1-\lambda_3) & 2i\lambda_2 & 0 \\ 0 & -2i\lambda_2 & 2(\lambda_1-\lambda_3) & 2\lambda_3 \\ -2i\lambda_2 & 0 & 2\lambda_3 & 2(\lambda_1-\lambda_3) \end{pmatrix}, \tag{4.5.32b}$$

$$S_3^+ = \begin{pmatrix} \lambda_5 & \frac{1}{2}(\lambda_6+2\lambda_7) & 0 & -\frac{i}{2}(\lambda_6-2\lambda_7) \\ \frac{1}{2}(\lambda_6+2\lambda_7) & \lambda_5 & -\frac{i}{2}(\lambda_6-2\lambda_7) & 0 \\ 0 & \frac{i}{2}(\lambda_6-2\lambda_7) & \lambda_5 & \frac{1}{2}(\lambda_6+2\lambda_7) \\ \frac{i}{2}(\lambda_6-2\lambda_7) & 0 & \frac{1}{2}(\lambda_6+2\lambda_7) & \lambda_5 \end{pmatrix}, \tag{4.5.32c}$$

$$S_4^+ = S_3^+. \tag{4.5.32d}$$

The doubly-charged two-particle states are given by:

$$\Psi^{++} = \left\{ \frac{1}{\sqrt{2}} \left| h_1^+ h_1^+ \right\rangle, \frac{1}{\sqrt{2}} \left| h_2^+ h_2^+ \right\rangle, \frac{1}{\sqrt{2}} \left| h_S^+ h_S^+ \right\rangle \right\}. \tag{4.5.33}$$

The doubly-charged scattering matrix is:

$$S^{++} = \begin{pmatrix} 2(\lambda_1+\lambda_3) & 2(\lambda_2+\lambda_3) & 2\lambda_7 \\ 2(\lambda_2+\lambda_3) & 2(\lambda_1+\lambda_3) & 2\lambda_7 \\ 2\lambda_7 & 2\lambda_7 & 2\lambda_8 \end{pmatrix}. \tag{4.5.34}$$

After solving for eigenvalues of the scattering matrix S we find that in total there are 18 particular eigenvalues:

$$\Lambda_1 = 2(\lambda_1 \pm \lambda_2), \tag{4.5.35a}$$
$$\Lambda_2 = \lambda_5 \pm \lambda_6, \tag{4.5.35b}$$
$$\Lambda_3 = \lambda_5 \pm 2\lambda_7, \tag{4.5.35c}$$
$$\Lambda_4 = 2(\lambda_1 \pm \lambda_2 - 2\lambda_3), \tag{4.5.35d}$$
$$\Lambda_5 = 2(\lambda_1 + \lambda_2 + 4\lambda_3), \tag{4.5.35e}$$
$$\Lambda_6 = 2(\lambda_1 - 5\lambda_2 - 2\lambda_3), \tag{4.5.35f}$$
$$\Lambda_7 = \lambda_5 + 2\lambda_6 \pm 6\lambda_7, \tag{4.5.35g}$$
$$\Lambda_8 = \lambda_1 - \lambda_2 + 2\lambda_3 + \lambda_8 \pm \sqrt{2\lambda_6^2 + (\lambda_1 - \lambda_2 + 2\lambda_3 - \lambda_8)^2}, \tag{4.5.35h}$$
$$\Lambda_9 = \lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_8 \pm \sqrt{8\lambda_7^2 + (\lambda_1 + \lambda_2 + 2\lambda_3 - \lambda_8)^2}, \tag{4.5.35i}$$
$$\Lambda_{10} = 5\lambda_1 - \lambda_2 + 2\lambda_3 + 3\lambda_8 \pm \sqrt{2(2\lambda_5 + \lambda_6)^2 + (5\lambda_1 - \lambda_2 + 2\lambda_3 - 3\lambda_8)^2}. \tag{4.5.35j}$$

We confirm that the eigenvalues we got are in perfect agreement with those of Ref. [37] in the limit of $\lambda_4 \to 0$.

In the high-energy limit, the partial wave amplitude takes the simple form:

$$|a_j| \leq 1. \tag{4.5.36}$$

In Ref. [67] it was suggested that a stronger constraint may be applied based on the reality of the partial-wave amplitude and the Cauchy–Schwarz inequality:

$$|\,\mathrm{Re}(a_j)| \leq \frac{1}{2}. \tag{4.5.37}$$

In principle, for the NHDM models, the unitarity constraint simplifies to a check:

$$|\Lambda_i| \leq 16\pi, \tag{4.5.38}$$

where the factor of 16π comes from the Jacob-Wick expansion [73]. The $|\Lambda_i| \leq 8\pi$ corresponds to eq. (4.5.37).

The number of checks (4.5.38) can be reduced considering the soft perturbativity limit (4.5.18), $|\lambda_{\max}| = 4\pi$. It follows that if a specific Λ_i of eq. (4.5.35) depends on less than four λ_i, such eigenvalues can be neglected, namely eqs. (4.5.35a - 4.5.35c).

4.5.6 Electroweak Oblique Parameters

The electroweak oblique parameters are parametrised by the self-energy functions S, and T, and U [74, 75]. These parameters are defined in a way that they vanish in the SM. In terms of the extended scalar models these parameters limit how far the electroweak sector can be extended from the SM reference point.

Experimental constraints for the reference values $m_{h_{\mathrm{SM,ref}}} = 125$ GeV and $m_{t,\mathrm{ref}} = 172.5$ GeV were presented by the Gfitter group [76, 77]:

$$S = \quad 0.04 \pm 0.11, \tag{4.5.39a}$$
$$T = \quad 0.09 \pm 0.14, \tag{4.5.39b}$$
$$U = -0.02 \pm 0.11. \tag{4.5.39c}$$

The guideline on how to derive the electroweak-oblique parameters for the NHDM was presented in [78, 79]. The $SU(2)$ doublets in the Higgs basis are given by:

$$H_1 = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(v + \frac{1}{\sqrt{2}}\left(\tilde{\eta}_1 + \tilde{\eta}_2\right) + iG_0\right) \end{pmatrix}, \tag{4.5.40a}$$
$$H_2 = \begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}}\left(\varphi_2 + \frac{i}{\sqrt{2}}\left(-\tilde{\chi}_1 + \tilde{\chi}_2\right)\right) \end{pmatrix}, \tag{4.5.40b}$$
$$H_3 = \begin{pmatrix} S^+ \\ \frac{1}{\sqrt{2}}\left(\tilde{\eta}_S + i\tilde{\chi}_S\right) \end{pmatrix}. \tag{4.5.40c}$$

The V and U rotation matrices of Ref. [78] correspond to[5]:

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\left(\tilde{\eta}_1 + \tilde{\eta}_2\right) + iG_0 \\ \varphi_2 + \frac{i}{\sqrt{2}}\left(-\tilde{\chi}_1 + \tilde{\chi}_2\right) \\ \tilde{\eta}_S + i\tilde{\eta}_S \end{pmatrix} = V \begin{pmatrix} G_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ S_1 \\ S_2 \end{pmatrix}, \tag{4.5.41}$$

and

$$\begin{pmatrix} G^+ \\ H^+ \\ S^+ \end{pmatrix} = U \begin{pmatrix} G^+ \\ H^+ \\ S^+ \end{pmatrix}. \tag{4.5.42}$$

[5]Note that the $\varphi_i = H_i$ in the generic basis, see eq. (4.1.24).

Therefore U is an identity matrix $U \equiv \mathcal{I}_3$ and V is:

$$V = \begin{pmatrix} i & c_\alpha & 0 & -s_\alpha & 0 & 0 \\ 0 & is_\alpha & 1 & ic_\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\gamma} & ie^{i\gamma} \end{pmatrix}, \tag{4.5.43}$$

see eq. (4.1.17).

Calculations result in the following values:

$$\begin{aligned} S = \frac{m_W^2 s_w^2}{24\alpha\pi^2 v^2} \Bigg[& \left(1 - 2s_w^2\right)^2 G\left(m_{H^\pm}^2, m_{H^\pm}^2, m_Z^2\right) + \left(1 - 2s_w^2\right)^2 G\left(m_{S^\pm}^2, m_{S^\pm}^2, m_Z^2\right) \\ & + s_\alpha^2 G\left(m_{H_1}^2, m_{H_2}^2, m_Z^2\right) + c_\alpha^2 G\left(m_{H_2}^2, m_{H_3}^2, m_Z^2\right) + G\left(m_{S_1}^2, m_{S_2}^2, m_Z^2\right) \\ & - 2\ln\left(m_{H^\pm}^2\right) - 2\ln\left(m_{S^\pm}^2\right) + \ln\left(m_{H_2}^2\right) + \ln\left(m_{H_3}^2\right) + \ln\left(m_{S_1}^2\right) + \ln\left(m_{S_2}^2\right) \\ & - s_\alpha^2 \left(\hat{G}\left(m_{H_1}^2, m_Z^2\right) - \hat{G}\left(m_{H_3}^2, m_Z^2\right)\right) \Bigg], \end{aligned} \tag{4.5.44a}$$

$$\begin{aligned} T = \frac{1}{16\alpha\pi^2 v^2} \Bigg[& s_\alpha^2 F\left(m_{H^\pm}^2, m_{H_1}^2\right) + F\left(m_{H^\pm}^2, m_{H_2}^2\right) + c_\alpha^2 F\left(m_{H^\pm}^2, m_{H_3}^2\right) + F\left(m_{S^\pm}^2, m_{S_1}^2\right) \\ & + F\left(m_{S^\pm}^2, m_{S_2}^2\right) - s_\alpha^2 F\left(m_{H_1}^2, m_{H_2}^2\right) - c_\alpha^2 F\left(m_{H_2}^2, m_{H_3}^2\right) - F\left(m_{S_1}^2, m_{S_2}^2\right) \\ & + 3s_\alpha^2 \left(F\left(m_W^2, m_{H_1}^2\right) - F\left(m_Z^2, m_{H_1}^2\right) + F\left(m_Z^2, m_{H_3}^2\right) - F\left(m_W^2, m_{H_3}^2\right)\right) \Bigg], \end{aligned} \tag{4.5.44b}$$

$$\begin{aligned} U = \frac{m_W^2 s_w^2}{24\alpha\pi^2 v^2} \Bigg[& s_\alpha^2 G\left(m_{H^\pm}^2, m_{H_1}^2, m_W^2\right) + G\left(m_{H^\pm}^2, m_{H_2}^2, m_W^2\right) + c_\alpha^2 G\left(m_{H^\pm}^2, m_{H_3}^2, m_W^2\right) \\ & + G\left(m_{S^\pm}^2, m_{S_1}^2, m_W^2\right) + G\left(m_{S^\pm}^2, m_{S_2}^2, m_W^2\right) - \left(1 - 2s_w^2\right)^2 G\left(m_{H^\pm}^2, m_{H^\pm}^2, m_Z^2\right) \\ & - \left(1 - 2s_w^2\right)^2 G\left(m_{S^\pm}^2, m_{S^\pm}^2, m_Z^2\right) - s_\alpha^2 G\left(m_{H_1}^2, m_{H_2}^2, m_Z^2\right) - c_\alpha^2 G\left(m_{H_2}^2, m_{H_3}^2, m_Z^2\right) \\ & - G\left(m_{S_1}^2, m_{S_2}^2, m_Z^2\right) + s_\alpha^2 \left(\hat{G}\left(m_{H_1}^2, m_Z^2\right) - \hat{G}\left(m_{H_1}^2, m_W^2\right)\right) \\ & + s_\alpha^2 \left(\hat{G}\left(m_{H_3}^2, m_W^2\right) - \hat{G}\left(m_{H_3}^2, m_Z^2\right)\right) \Bigg]. \end{aligned} \tag{4.5.44c}$$

The functions, which appear in the expressions above, are given by the well-known function [80]:

$$F(I,J) \equiv \begin{cases} \frac{I+J}{2} - \frac{IJ}{I-J} \ln \frac{I}{J} & , I \neq J \\ 0 & , I = J \end{cases}, \tag{4.5.45}$$

and [78, 79]:

$$G(I,J,Q) \equiv -\frac{16}{3} + \frac{5(I+J)}{Q} - \frac{2(I-J)^2}{Q^2} \tag{4.5.46a}$$

$$+ \frac{3}{Q}\left[\frac{I^2+J^2}{I-J} - \frac{I^2-J^2}{Q} + \frac{(I-J)^3}{3Q^2}\right] \ln \frac{I}{J} + \frac{r}{Q^3} f(t,r), \tag{4.5.46b}$$

$$\tilde{G}(I,J,Q) \equiv -2 + \left(\frac{I-J}{Q} - \frac{I+J}{I-J}\right) \ln \frac{I}{J} + \frac{f(t,r)}{Q}, \tag{4.5.46c}$$

$$\hat{G}(I,Q) \equiv G(I,Q,Q) + 12\tilde{G}(I,Q,Q), \tag{4.5.46d}$$

where

$$f(t,r) \equiv \begin{cases} \sqrt{r} \ln \left| \frac{t-\sqrt{r}}{t+\sqrt{r}} \right| & , r > 0 \\ 0 & , r = 0 \\ 2\sqrt{-r} \arctan\left(\frac{\sqrt{-r}}{t}\right) & , r < 0 \end{cases}, \tag{4.5.47}$$

where

$$t \equiv I + J - Q, \tag{4.5.48a}$$

$$r \equiv Q^2 - 2Q(I+J) + (I-J)^2. \tag{4.5.48b}$$

Chapter 5

The R-II-1a Model

In this chapter we consider the R-II-1a model. The vacuum configuration is given by[1]:

$$\{0,\, w,\, w_S\}, \tag{5.0.1}$$

and the minimization conditions are:

$$\mu_0^2 = \frac{1}{2}\lambda_4 \frac{w_2^3}{w_S} - \frac{1}{2}\left(\lambda_5 + \lambda_6 + 2\lambda_7\right) w_2^2 - \lambda_8 w_S^2, \tag{5.0.2a}$$

$$\mu_1^2 = -\left(\lambda_1 + \lambda_3\right) w_2^2 + \frac{3}{2}\lambda_4 w_2 w_S - \frac{1}{2}\left(\lambda_5 + \lambda_6 + 2\lambda_7\right) w_S^2. \tag{5.0.2b}$$

An interesting property of this model, as mentioned in section 2.4, is that the $\mathbb{Z}_2$ symmetry is preserved for:

$$h_1 \to -h_1, \quad h_2 \to h_2, \quad h_S \to h_S. \tag{5.0.3}$$

Thus the DM candidate resides in the inert $SU(2)$ doublet h_1, $\langle h_1 \rangle = 0$. This model is, by default, CP conserving.

5.1 The Mass-Squared Matrices

The charged mass-squared matrix is given by:

$$\mathcal{M}^2_{\text{Charged}} = \begin{pmatrix} (\mathcal{M}^2_{\text{Ch}})_{11} & 0 & 0 \\ 0 & (\mathcal{M}^2_{\text{Ch}})_{22} & (\mathcal{M}^2_{\text{Ch}})_{23} \\ 0 & (\mathcal{M}^2_{\text{Ch}})_{23} & (\mathcal{M}^2_{\text{Ch}})_{33} \end{pmatrix}, \tag{5.1.1}$$

where

$$(\mathcal{M}^2_{\text{Ch}})_{11} = -2\lambda_3 w_2^2 + \frac{5}{2}\lambda_4 w_2 w_S - \frac{1}{2}(\lambda_6 + 2\lambda_7) w_S^2, \tag{5.1.2a}$$

$$(\mathcal{M}^2_{\text{Ch}})_{22} = \frac{1}{2} w_S \left[\lambda_4 w_2 - (\lambda_6 + 2\lambda_7) w_S\right], \tag{5.1.2b}$$

$$(\mathcal{M}^2_{\text{Ch}})_{23} = -\frac{1}{2} w_2 \left[\lambda_4 w_2 - (\lambda_6 + 2\lambda_7) w_S\right], \tag{5.1.2c}$$

$$(\mathcal{M}^2_{\text{Ch}})_{33} = \frac{1}{2}\frac{w_2^2}{w_S} \left[\lambda_4 w_2 - (\lambda_6 + 2\lambda_7) w_S\right]. \tag{5.1.2d}$$

The lower-right components of the mass-squared matrix are diagonalizable by a rotation matrix:

$$\mathcal{R}_\beta = \begin{pmatrix} \mathrm{c}_\beta & \mathrm{s}_\beta \\ -\mathrm{s}_\beta & \mathrm{c}_\beta \end{pmatrix}, \tag{5.1.3}$$

[1] We note that R-II-1a is a real vacuum configuration and thus VEVs are given by the absolute values. The hatted VEVs $\hat{w}_i$ are no longer used.

where the rotation angle is:

$$t_\beta = \frac{w_S}{w_2} = \frac{s_\beta v}{c_\beta v}. \tag{5.1.4}$$

By convention $0 < \beta < \frac{\pi}{2}$. This rotation can be identified as the Higgs basis rotation. The charged scalar states can be expressed as:

$$h^\pm = h_1^\pm, \tag{5.1.5a}$$
$$G^\pm = c_\beta h_2^\pm + s_\beta h_S^\pm, \tag{5.1.5b}$$
$$H^\pm = -s_\beta h_2^\pm + c_\beta h_S^\pm, \tag{5.1.5c}$$

with masses:

$$m_{h^\pm}^2 = -2\lambda_3 w_2^2 + \frac{5}{2}\lambda_4 w_2 w_S - \frac{1}{2}(\lambda_6 + 2\lambda_7) w_S^2, \tag{5.1.6a}$$
$$m_{H^\pm}^2 = \frac{v^2}{2w_S}\left[\lambda_4 w_2 - (\lambda_6 + 2\lambda_7)\, w_S\right]. \tag{5.1.6b}$$

The neutral components of the inert doublet h_1 are already diagonalized. Masses of the two neutral states are given by:

$$m_{\tilde{\eta}}^2 = \frac{9}{2}\lambda_4 w_2 w_S, \tag{5.1.7a}$$
$$m_{\tilde{\chi}}^2 = -2(\lambda_2 + \lambda_3) w_2^2 + \frac{5}{2}\lambda_4 w_2 w_S - 2\lambda_7 w_S^2. \tag{5.1.7b}$$

The doublets h_2 and h_S acquire a non-zero VEV and thus are active. The neutral mass-squared matrix is block-diagonal in the basis:

$$\{\tilde{\eta}_2,\, \tilde{\eta}_S,\, \tilde{\chi}_2,\, \tilde{\chi}_S\}. \tag{5.1.8}$$

Therefore the mass-squared matrix can be split into:

$$\mathcal{M}^2_{\text{Neutral}-2S} = \text{diag}\left(\mathcal{M}^2_{\eta},\, \mathcal{M}^2_{\chi}\right). \tag{5.1.9}$$

The mass-squared matrix of the CP-odd sector is:

$$\mathcal{M}^2_{\chi} = \begin{pmatrix} (\mathcal{M}^2_{\chi})_{11} & (\mathcal{M}^2_{\chi})_{12} \\ (\mathcal{M}^2_{\chi})_{12} & (\mathcal{M}^2_{\chi})_{22} \end{pmatrix}, \tag{5.1.10}$$

where

$$(\mathcal{M}^2_{\chi})_{11} = \frac{1}{2} w_S\left(\lambda_4 w_2 - 4\lambda_7 w_S\right), \tag{5.1.11a}$$
$$(\mathcal{M}^2_{\chi})_{12} = -\frac{1}{2} w_2\left(\lambda_4 w_2 - 4\lambda_7 w_S\right), \tag{5.1.11b}$$
$$(\mathcal{M}^2_{\chi})_{22} = \frac{w_2^2}{2w_S}\left(\lambda_4 w_2 - 4\lambda_7 w_S\right). \tag{5.1.11c}$$

It is diagonalizable by performing a rotation $\mathcal{R}_\beta$ of eq. (5.1.3). The two CP-odd states are:

$$G^0 = c_\beta \tilde{\chi}_2 + s_\beta \tilde{\chi}_S, \tag{5.1.12a}$$
$$A = -s_\beta \tilde{\chi}_2 + c_\beta \tilde{\chi}_S, \tag{5.1.12b}$$

where

$$m_A^2 = \frac{v^2}{2w_S}\left(\lambda_4 w_2 - 4\lambda_7 w_S\right). \tag{5.1.13}$$

The CP-even mass-squared matrix is:

$$(\mathcal{M}^2_{\eta}) = \begin{pmatrix} (\mathcal{M}^2_{\eta})_{11} & (\mathcal{M}^2_{\eta})_{12} \\ (\mathcal{M}^2_{\eta})_{12} & (\mathcal{M}^2_{\eta})_{22} \end{pmatrix}, \tag{5.1.14}$$

where

$$(\mathcal{M}_\eta^2)_{11} = \frac{1}{2} w_2 \left[4\left(\lambda_1+\lambda_3\right) w_2 - 3\lambda_4 w_S\right], \tag{5.1.15a}$$

$$(\mathcal{M}_\eta^2)_{12} = -\frac{1}{2} w_2 \left[3\lambda_4 w_2 - 2\left(\lambda_5+\lambda_6+2\lambda_7\right) w_S\right], \tag{5.1.15b}$$

$$(\mathcal{M}_\eta^2)_{22} = \frac{1}{2w_S}\left(\lambda_4 w_2^3 + 4\lambda_8 w_S^3\right). \tag{5.1.15c}$$

It is diagonalizable by a rotation matrix:

$$\mathcal{R}_\alpha = \begin{pmatrix} \mathrm{c}_\alpha & \mathrm{s}_\alpha \\ -\mathrm{s}_\alpha & \mathrm{c}_\alpha \end{pmatrix}, \tag{5.1.16}$$

where the rotation angle is:

$$\mathrm{t}_{2\alpha} = \frac{-2w_2 w_S (3\lambda_4 w_2 - 2\left(\lambda_5+\lambda_6+2\lambda_7\right) w_S)}{4(\lambda_1+\lambda_3) w_2^2 w_S - \lambda_4\left(w_2^3 + 3w_2 w_S^2\right) - 4\lambda_8 w_S^3}. \tag{5.1.17}$$

The CP-even states are thus:

$$h = \mathrm{c}_\alpha \tilde{\eta}_2 + \mathrm{s}_\alpha \tilde{\eta}_S, \tag{5.1.18a}$$

$$H = -\mathrm{s}_\alpha \tilde{\eta}_2 + \mathrm{c}_\alpha \tilde{\eta}_S, \tag{5.1.18b}$$

with masses:

$$m_h^2 = \frac{1}{4w_S^2}\left[4\left(\lambda_1+\lambda_3\right) w_2^2 w_S^2 + \lambda_4 w_2 w_S\left(w_2^2 - 3w_S^2\right) + 4\lambda_8 w_S^4 - w_S \Delta\right], \tag{5.1.19a}$$

$$m_H^2 = \frac{1}{4w_S^2}\left[4\left(\lambda_1+\lambda_3\right) w_2^2 w_S^2 + \lambda_4 w_2 w_S\left(w_2^2 - 3w_S^2\right) + 4\lambda_8 w_S^4 + w_S \Delta\right], \tag{5.1.19b}$$

where

$$\begin{aligned}\Delta^2 = & -8\left(\lambda_1+\lambda_3\right)\lambda_4 w_2^5 w_S + 2\left[8\left(\lambda_1+\lambda_3\right)^2 + 21\lambda_4^2\right] w_2^4 w_S^2 \\ & - 8\lambda_4\left[3\left(\lambda_1+\lambda_3+2\left(\lambda_5+\lambda_6+2\lambda_7\right)\right) - \lambda_8\right] w_2^3 w_S^3 \\ & + \left[9\lambda_4^2 + 16\left(\left(\lambda_5+\lambda_6+2\lambda_7\right)^2 - 2\left(\lambda_1+\lambda_3\right)\lambda_8\right)\right] w_2^2 w_S^4 \\ & + \lambda_4 w_2^6 + 24\lambda_4\lambda_8 w_2 w_S^5 + 16\lambda_8^2 w_S^6.\end{aligned} \tag{5.1.20}$$

We identify the lighter h state as the SM-like Higgs boson.

On the other hand, governed by the fact that both the charged $\mathcal{M}_{\text{Charged}}^2$ and the CP-odd $\mathcal{M}_\chi^2$ mass-squared matrices are diagonalizable by going into the Higgs basis[2], *i.e.*, by the $\mathcal{R}_\beta$ rotation, we could assume that the $SU(2)$ doublets are rotated into a new basis:

$$\begin{pmatrix} h_2' \\ h_S' \end{pmatrix} = \mathcal{R}_\beta \begin{pmatrix} h_2 \\ h_S \end{pmatrix}. \tag{5.1.21}$$

Taking this into consideration, the CP-even mass-squared matrix $\mathcal{M}_\eta^2$ gets modified:

$$\mathcal{R}_\beta \mathcal{M}_\eta^2 \mathcal{R}_\beta^{\mathrm{T}} \equiv \tilde{\mathcal{M}}_\eta^2 = \begin{pmatrix} (\tilde{\mathcal{M}}_\eta^2)_{11} & (\tilde{\mathcal{M}}_\eta^2)_{12} \\ (\tilde{\mathcal{M}}_\eta^2)_{12} & (\tilde{\mathcal{M}}_\eta^2)_{22} \end{pmatrix}, \tag{5.1.22}$$

where

$$(\tilde{\mathcal{M}}_\eta^2)_{11} = \frac{2}{v^2}\left[\left(\lambda_1+\lambda_3\right) w_2^4 - 2\lambda_4 w_2^3 w_S + \left(\lambda_5+\lambda_6+2\lambda_7\right) w_2^2 w_S^2 + \lambda_8 w_S^4\right], \tag{5.1.23a}$$

[2] To be more precise, due to the $SU(2)$ doublet h_1 we could determine the rotation matrix as $\mathcal{R}_{\beta\text{-}3} \equiv \mathrm{diag}\left(1, \mathcal{R}_\beta\right)$.

$$(\tilde{\mathcal{M}}^2_\eta)_{12} = \frac{w_2}{v^2}\bigg[\lambda_4 w_2 \left(3w_S^2 - w_2^2\right) - \left(2\lambda_1 + 2\lambda_3 - \lambda_5 - \lambda_6 - 2\lambda_7\right) w_2^2 w_S \\ - \left(\lambda_5 + \lambda_6 + 2\lambda_7 - 2\lambda_8\right) w_S^3\bigg], \tag{5.1.23b}$$

$$(\tilde{\mathcal{M}}^2_\eta)_{22} = \frac{w_2}{2v^2 w_S}\left[\lambda_4 \left(6w_2^2 w_S^2 - 3w_S^4 + w_2^4\right) + 4\left(\lambda_1 + \lambda_3 - \lambda_5 - \lambda_6 - 2\lambda_7 + \lambda_8\right) w_2 w_S^3\right]. \tag{5.1.23c}$$

The mass-squared matrix $\tilde{\mathcal{M}}^2_\eta$ is diagonalizable by $\mathcal{R}_{\alpha'}$ of eq. (5.1.16). Due to the $\mathcal{R}_\beta$ rotation, α' is defined by

$$\mathrm{t}_{2\alpha'} = \frac{4w_2 w_S\left[2\left(\lambda_1+\lambda_3\right)w_2^2 w_S + \lambda_4\left(w_2^3 - 3w_2 w_S^2\right) + \lambda_a\left(w_S^3 - w_2^2 w_S\right) - 2\lambda_8 w_S^3\right]}{4\left(\lambda_1+\lambda_3\right)\left(w_2^2 w_S^3 - w_2^4 w_S\right) + \lambda_4\left(14w_2^3 w_S^2 - 3w_2 w_S^4 + w_2^5\right) - 8w_2^2\lambda_a w_S^3 + 4\lambda_8\left(w_2^2 w_S^3 - w_S^5\right)}, \tag{5.1.24}$$

where

$$\lambda_a = \lambda_5 + \lambda_6 + 2\lambda_7. \tag{5.1.25}$$

The CP-even states are now expressed as:

$$h = \mathrm{c}_{\alpha'+\beta}\,\tilde{\eta}_2 + \mathrm{s}_{\alpha'+\beta}\,\tilde{\eta}_S, \tag{5.1.26a}$$

$$H = -\mathrm{s}_{\alpha'+\beta}\,\tilde{\eta}_2 + \mathrm{c}_{\alpha'+\beta}\,\tilde{\eta}_S. \tag{5.1.26b}$$

Masses of the above states coincide with eq. (5.1.19), as expected. The relation between the angles is trivial:

$$\beta + \alpha' = \alpha. \tag{5.1.27}$$

Doublets in terms of the mass-eigenstates are:

$$h_1 = \begin{pmatrix} h^\pm \\ \frac{1}{\sqrt{2}}\left(\tilde{\eta} + i\tilde{\chi}\right)\end{pmatrix}, \tag{5.1.28a}$$

$$h_2 = \begin{pmatrix} \mathrm{c}_\beta G^\pm - \mathrm{s}_\beta H^\pm \\ \frac{1}{\sqrt{2}}\left(\mathrm{c}_\beta v + \mathrm{c}_\alpha h - \mathrm{s}_\alpha H + i\left(\mathrm{c}_\beta G^0 - \mathrm{s}_\beta A\right)\right)\end{pmatrix}, \tag{5.1.28b}$$

$$h_S = \begin{pmatrix} \mathrm{s}_\beta G^\pm + \mathrm{c}_\beta H^\pm \\ \frac{1}{\sqrt{2}}\left(\mathrm{s}_\beta v + \mathrm{s}_\alpha h + \mathrm{c}_\alpha H + i\left(\mathrm{s}_\beta G^0 + \mathrm{c}_\beta A\right)\right)\end{pmatrix}. \tag{5.1.28c}$$

It is not very appealing to deal with VEVs as input parameters and therefore both w_2 and w_S can be traded for t_β and v. In this case, the mass-squared parameters are as follows:

$$\begin{aligned}
m_{h^\pm}^2 &= \frac{1}{2}v^2\left[-4\lambda_3\mathrm{c}_\beta^2 + \frac{5}{2}\lambda_4\mathrm{s}_{2\beta} - \left(\lambda_6 + 2\lambda_7\right)\mathrm{s}_\beta^2\right],\\
m_{H^\pm}^2 &= \frac{1}{2}v^2\left(\lambda_4\frac{1}{\mathrm{t}_\beta} - \lambda_6 - 2\lambda_7\right),\\
m_{\tilde{\eta}}^2 &= \frac{9}{4}v^2\lambda_4\mathrm{s}_{2\beta},\\
m_{\tilde{\chi}}^2 &= \frac{1}{2}v^2\left[-4\left(\lambda_2+\lambda_3\right)\mathrm{c}_\beta^2 + \frac{5}{2}\lambda_4\mathrm{s}_{2\beta} - 4\lambda_7\mathrm{s}_\beta^2\right],\\
m_A^2 &= \frac{1}{2}v^2\left(\lambda_4\frac{1}{\mathrm{t}_\beta} - 4\lambda_7\right),\\
m_h^2 &= v^2\left[\left(\lambda_1+\lambda_3\right)\mathrm{c}_\beta^2 + \frac{1}{4}\lambda_4\frac{1}{\mathrm{t}_\beta}\left(\mathrm{c}_\beta^2 - 3\mathrm{s}_\beta^2\right) + \lambda_8\mathrm{s}_\beta^2\right] - \frac{v^2}{4\mathrm{s}_\beta}\Delta,\\
m_H^2 &= v^2\left[\left(\lambda_1+\lambda_3\right)\mathrm{c}_\beta^2 + \frac{1}{4}\lambda_4\frac{1}{\mathrm{t}_\beta}\left(\mathrm{c}_\beta^2 - 3\mathrm{s}_\beta^2\right) + \lambda_8\mathrm{s}_\beta^2\right] + \frac{v^2}{4\mathrm{s}_\beta}\Delta,
\end{aligned} \tag{5.1.29}$$

where

$$\begin{aligned}\Delta^2 = & -8\left(\lambda_1+\lambda_3\right)\lambda_4 \mathrm{c}_\beta^5 \mathrm{s}_\beta+2\left[8\left(\lambda_1+\lambda_3\right)^2+21 \lambda_4^2\right] \mathrm{c}_\beta^4 \mathrm{s}_\beta^2 \\ & +\left[9 \lambda_4^2+16\left(\left(\lambda_5+\lambda_6+2 \lambda_7\right)^2-2\left(\lambda_1+\lambda_3\right) \lambda_8\right)\right] \mathrm{c}_\beta^2 \mathrm{s}_\beta^4 \\ & -\lambda_4\left[3\left(\lambda_1+\lambda_3+2\left(\lambda_5+\lambda_6+2 \lambda_7\right)\right)-\lambda_8\right] \mathrm{s}_{2\beta}^3 \\ & +\lambda_4^2 \mathrm{c}_\beta^6+24 \lambda_4 \lambda_8 \mathrm{c}_\beta \mathrm{s}_\beta^5+16 \lambda_8^2 \mathrm{s}_\beta^6 .\end{aligned} \tag{5.1.30}$$

5.1.1 Quartic Couplings in Terms of Masses

The mass-squared parameters cannot be inverted in a simple way, as they were in section 4.1.1, to result in λ_i expressed in terms of the mass-squared parameters. This procedure is not trivial due to the more complicative mass-squared parameters. Therefore, we present the result without specifying how we derived it. The λ couplings in terms of the mass-squared parameters are:

$$\lambda_1 = \frac{v^2\left[9\left(m_{h^\pm}^2+\mathrm{s}_\alpha^2 m_h^2+\mathrm{c}_\alpha^2 m_H^2\right)-m_{\tilde{\eta}}^2\right]-9 m_{h^\pm}^2 w_S^2}{18 v^2 w_2^2}, \tag{5.1.31a}$$

$$\lambda_2 = \frac{\left(m_{h^\pm}^2-m_{\tilde{\chi}}^2\right) v^2+\left(m_A^2-m_{h^\pm}^2\right) w_S^2}{2 v^2 w_2^2}, \tag{5.1.31b}$$

$$\lambda_3 = \frac{\left(4 m_{\tilde{\eta}}^2-9 m_{h^\pm}^2\right) v^2+9 m_{h^\pm}^2 w_S^2}{18 v^2 w_2^2}, \tag{5.1.31c}$$

$$\lambda_4 = \frac{2 m_{\tilde{\eta}}^2}{9 w_2 w_S}, \tag{5.1.31d}$$

$$\lambda_5 = \frac{2 m_{h^\pm}^2}{v^2}+\frac{w_2 m_{\tilde{\eta}}^2-\frac{9}{2} \mathrm{s}_{2\alpha} w_S\left(m_H^2-m_h^2\right)}{9 w_2 w_S^2}, \tag{5.1.31e}$$

$$\lambda_6 = \frac{m_A^2-2 m_{h^\pm}^2}{v^2}+\frac{m_{\tilde{\eta}}^2}{9 w_S^2}, \tag{5.1.31f}$$

$$\lambda_7 = \frac{1}{18}\left(\frac{m_{\tilde{\eta}}^2}{w_S^2}-\frac{9 m_A^2}{v^2}\right), \tag{5.1.31g}$$

$$\lambda_8 = \frac{9 w_S^2\left(\mathrm{c}_\alpha^2 m_h^2+\mathrm{s}_\alpha^2 m_H^2\right)-w_2^2 m_{\tilde{\eta}}^2}{18 w_S^4}, \tag{5.1.31h}$$

or in terms of the β parameter:

$$\lambda_1 = \frac{1}{18 v^2 \mathrm{c}_\beta^2}\left(9 m_{h^\pm}^2-9 \mathrm{s}_\beta^2 m_{H^\pm}^2-m_{\tilde{\eta}}^2+9 \mathrm{s}_\alpha^2 m_h^2+9 \mathrm{c}_\alpha^2 m_H^2\right), \tag{5.1.32a}$$

$$\lambda_2 = \frac{1}{2 v^2 \mathrm{c}_\beta^2}\left[\mathrm{s}_\beta^2\left(m_A^2-m_{H^\pm}^2\right)+m_{h^\pm}^2-m_{\tilde{\chi}}^2\right], \tag{5.1.32b}$$

$$\lambda_3 = \frac{1}{18 v^2 \mathrm{c}_\beta^2}\left(-9 m_{h^\pm}^2+9 \mathrm{s}_\beta^2 m_{H^\pm}^2+4 m_{\tilde{\eta}}^2\right), \tag{5.1.32c}$$

$$\lambda_4 = \frac{m_{\tilde{\eta}}^2}{9 v^2 \mathrm{s}_{2\beta}}, \tag{5.1.32d}$$

$$\lambda_5 = \frac{1}{9 v^2}\left[18 m_{H^\pm}^2+m_{\tilde{\eta}}^2 \frac{1}{\mathrm{s}_\beta^2}+9 \frac{\mathrm{s}_{2\alpha}}{\mathrm{s}_\beta}\left(m_h^2-m_H^2\right)\right], \tag{5.1.32e}$$

$$\lambda_6 = \frac{1}{9 v^2}\left(-18 m_{H^\pm}^2+m_{\tilde{\eta}}^2 \frac{1}{\mathrm{s}_\beta^2}+9 m_A^2\right), \tag{5.1.32f}$$

$$\lambda_7 = \frac{1}{18 v^2}\left(m_{\tilde{\eta}}^2 \frac{1}{\mathrm{s}_\beta^2}-9 m_A^2\right), \tag{5.1.32g}$$

$$\lambda_8 = \frac{1}{18v^2 \mathrm{s}_\beta^2}\left(-m_{\tilde{\eta}}^2 \frac{1}{\mathrm{t}_\beta^2} + 9m_h^2 \mathrm{c}_\alpha^2 + 9m_H^2 \mathrm{s}_\alpha^2\right). \tag{5.1.32h}$$

5.2 R-II-1a in the Higgs Basis

The Higgs basis transformation is given by:

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}_\beta & \mathrm{s}_\beta \\ 0 & -\mathrm{s}_\beta & \mathrm{c}_\beta \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix}. \tag{5.2.1}$$

The $SU(2)$ doublets in terms of the mass-eigenstates (5.1.28) are:

$$H_1 = \begin{pmatrix} h^\pm \\ \frac{1}{\sqrt{2}}\left(\tilde{\eta} + i\tilde{\chi}\right) \end{pmatrix}, \tag{5.2.2a}$$

$$H_2 = \begin{pmatrix} G^\pm \\ \frac{1}{\sqrt{2}}\left(v + \mathrm{c}_{\alpha-\beta} h - \mathrm{s}_{\alpha-\beta} H + iG^0\right) \end{pmatrix}, \tag{5.2.2b}$$

$$H_3 = \begin{pmatrix} H^\pm \\ \frac{1}{\sqrt{2}}\left(\mathrm{s}_{\alpha-\beta} h + \mathrm{c}_{\alpha-\beta} H + iA\right) \end{pmatrix}. \tag{5.2.2c}$$

In the basis of eq. (2.5.14), the Euler angles are:

$$\beta_1 = \frac{\pi}{2}, \tag{5.2.3a}$$

$$\mathrm{t}_{\beta_2} = \frac{w_S}{w_2}. \tag{5.2.3b}$$

Although $\hat{w}_1 = 0$, we rotate the $SU(2)$ doublets in a such way that $\langle H_1 \rangle = v$. This results in the following quadratic couplings:

$$Y_{11} = \mathrm{c}_{\beta_2}^2 \mu_1^2 + \mathrm{s}_{\beta_2}^2 \mu_0^2, \tag{5.2.4a}$$

$$Y_{13} = -\frac{1}{2}\mathrm{s}_{2\beta_2}\mu_1^2 + \frac{1}{2}\mathrm{s}_{2\beta_2}\mu_0^2, \tag{5.2.4b}$$

$$Y_{22} = \mu_1^2, \tag{5.2.4c}$$

$$Y_{33} = \mathrm{c}_{\beta_2}^2 \mu_0^2 + \mathrm{s}_{\beta_2}^2 \mu_1^2, \tag{5.2.4d}$$

and the quartic couplings are:

$$Z_{1111} = \lambda_1 \mathrm{c}_{\beta_2}^4 + \lambda_3 \mathrm{c}_{\beta_2}^4 - 2\lambda_4 \mathrm{c}_{\beta_2}^3 \mathrm{s}_{\beta_2} + \frac{1}{4}\lambda_5 \mathrm{s}_{2\beta_2}^2 + \frac{1}{4}\lambda_6 \mathrm{s}_{2\beta_2}^2 + \frac{1}{2}\lambda_7 \mathrm{s}_{2\beta_2}^2 + \lambda_8 \mathrm{s}_{\beta_2}^4, \tag{5.2.5a}$$

$$Z_{1113} = \lambda_1 \mathrm{c}_{\beta_2}^3 \mathrm{s}_{\beta_2} - 2\lambda_3 \mathrm{c}_{\beta_2}^3 \mathrm{s}_{\beta_2} + \lambda_4\left(1 - 2\mathrm{c}_{2\beta_2}\right)\mathrm{c}_{\beta_2}^2 + \frac{1}{4}\lambda_5 \mathrm{s}_{4\beta_2} + \frac{1}{4}\lambda_6 \mathrm{s}_{4\beta_2} + \frac{1}{2}\lambda_7 \mathrm{s}_{4\beta_2} + 2\lambda_8 \mathrm{c}_{\beta_2} \mathrm{s}_{\beta_2}^3, \tag{5.2.5b}$$

$$Z_{1122} = 2\lambda_1 \mathrm{c}_{\beta_2}^2 - 2\lambda_3 \mathrm{c}_{\beta_2}^2 + \lambda_4 \mathrm{s}_{2\beta_2} + \lambda_5 \mathrm{s}_{\beta_2}^2, \tag{5.2.5c}$$

$$Z_{1133} = \frac{1}{2}\lambda_1 \mathrm{s}_{2\beta_2}^2 + \frac{1}{2}\lambda_3 \mathrm{s}_{2\beta_2}^2 + \frac{1}{2}\lambda_4 \mathrm{s}_{4\beta_2} + \frac{1}{4}\lambda_5\left(\mathrm{c}_{4\beta_2} + 3\right) - \frac{1}{2}\lambda_6 \mathrm{s}_{2\beta_2}^2 - \lambda_7 \mathrm{s}_{2\beta_2}^2 + \frac{1}{2}\lambda_8 \mathrm{s}_{2\beta_2}^2, \tag{5.2.5d}$$

$$Z_{1212} = \lambda_2 \mathrm{c}_{\beta_2}^2 + \lambda_3 \mathrm{c}_{\beta_2}^2 + \frac{1}{2}\lambda_4 \mathrm{s}_{2\beta_2} + \lambda_7 \mathrm{s}_{\beta_2}^2, \tag{5.2.5e}$$

$$Z_{1221} = -2\lambda_2 \mathrm{c}_{\beta_2}^2 + 2\lambda_3 \mathrm{c}_{\beta_2}^2 + \lambda_4 \mathrm{s}_{2\beta_2} + \lambda_6 \mathrm{s}_{\beta_2}^2, \tag{5.2.5f}$$

$$Z_{1223} = \lambda_2 \mathrm{s}_{2\beta_2} - \lambda_3 \mathrm{s}_{2\beta_2} + \lambda_4 \mathrm{c}_{2\beta_2} + \frac{1}{2}\lambda_6 \mathrm{s}_{2\beta_2}, \tag{5.2.5g}$$

$$Z_{1232} = -\lambda_2 \mathrm{s}_{2\beta_2} - \lambda_3 \mathrm{s}_{2\beta_2} + \lambda_4 \mathrm{c}_{2\beta_2} + \lambda_7 \mathrm{s}_{2\beta_2}, \tag{5.2.5h}$$

$$Z_{1313} = \frac{1}{4}\lambda_1 s^2_{2\beta_2} + \frac{1}{4}\lambda_3 s^2_{2\beta_2} + \frac{1}{4}\lambda_4 s_{4\beta_2} - \frac{1}{4}\lambda_5 s^2_{2\beta_2} - \frac{1}{4}\lambda_6 s^2_{2\beta_2} + \lambda_7\left(c^4_{\beta_2} + s^4_{\beta_2}\right) + \frac{1}{4}\lambda_8 s^2_{2\beta_2}, \tag{5.2.5i}$$

$$Z_{1322} = \lambda_1 s_{2\beta_2} + \lambda_3 s_{2\beta_2} + \lambda_4 c_{2\beta_2} + \frac{1}{2}\lambda_5 s_{2\beta_2}, \tag{5.2.5j}$$

$$Z_{1331} = \frac{1}{2}\lambda_1 s^2_{2\beta_2} + \frac{1}{2}\lambda_3 s^2_{2\beta_2} + \frac{1}{2}\lambda_4 s_{4\beta_2} - \frac{1}{2}\lambda_5 s^2_{2\beta_2} + \frac{1}{4}\lambda_6\left(c_{4\beta_2} + 3\right) - \lambda_7 s^2_{2\beta_2} + \frac{1}{2}\lambda_8 s^2_{2\beta_2}, \tag{5.2.5k}$$

$$Z_{1333} = -2\lambda_1 c_{\beta_2} s^3_{\beta_2} - 2\lambda_3 c_{\beta_2} s^3_{\beta_2} - \lambda_4\left(2c_{2\beta_2} + 1\right) s^2_{\beta_2} - \frac{1}{4}\lambda_5 s_{4\beta_2} - \frac{1}{4}\lambda_6 s_{4\beta_2} - \frac{1}{2}\lambda_7 s_{4\beta_2} + 2\lambda_8 c^3_{\beta_2} s_{\beta_2}, \tag{5.2.5l}$$

$$Z_{2222} = \lambda_1 + \lambda_3, \tag{5.2.5m}$$

$$Z_{2233} = 2\lambda_1 s^2_{\beta_2} - 2\lambda_3 s^2_{\beta_2} - \lambda_4 s_{2\beta_2} + \lambda_5 c^2_{\beta_2}, \tag{5.2.5n}$$

$$Z_{2323} = \lambda_2 s^2_{\beta_2} + \lambda_3 s^2_{\beta_2} - \frac{1}{2}\lambda_4 s_{2\beta_2} + \lambda_7 c^2_{\beta_2}, \tag{5.2.5o}$$

$$Z_{2332} = -2\lambda_2 s^2_{\beta_2} + 2\lambda_3 s^2_{\beta_2} - \lambda_4 s_{2\beta_2} + \lambda_6 c^2_{\beta_2}, \tag{5.2.5p}$$

$$Z_{3333} = \lambda_1 s^4_{\beta_2} + \lambda_3 s^4_{\beta_2} + 2\lambda_4 c_{\beta_2} s^3_{\beta_2} + \frac{1}{4}\lambda_5 s^2_{2\beta_2} + \frac{1}{4}\lambda_6 s^2_{2\beta_2} + \frac{1}{2}\lambda_7 s^2_{2\beta_2} + \lambda_8 c^4_{\beta_2}. \tag{5.2.5q}$$

5.3 Scalar-Gauge Boson Interactions

After substituting the $SU(2)$ doublets in terms of the mass-eigenstates (5.1.28) into the kinetic Lagrangian of eq. (4.2.14), the resulting terms are:

$$\begin{aligned}\mathcal{L}_{VVH} = &\left[\frac{g}{2c_w} m_Z Z_\mu Z^\mu + g m_W W^+_\mu W^{\mu-}\right]\left(c_{\alpha-\beta}\, h - s_{\alpha-\beta}\, H\right)\\ &+ \left\{\left[e m_W A^\mu W^+_\mu - g m_Z s^2_w Z^\mu W^+_\mu\right] G^- + \text{h.c.}\right\},\end{aligned} \tag{5.3.1a}$$

$$\begin{aligned}\mathcal{L}_{VHH} = &-\frac{g}{2c_w} Z^\mu\left(\tilde{\eta}\overset{\leftrightarrow}{\partial}_\mu\tilde{\chi} + c_{\alpha-\beta} h\overset{\leftrightarrow}{\partial}_\mu G^0 + s_{\alpha-\beta} h\overset{\leftrightarrow}{\partial}_\mu A - s_{\alpha-\beta} H\overset{\leftrightarrow}{\partial}_\mu G^0 + c_{\alpha-\beta} H\overset{\leftrightarrow}{\partial}_\mu A\right)\\ &-\frac{g}{2}\Big\{iW^+_\mu\Big(ih^-\overset{\leftrightarrow}{\partial^\mu}\tilde{\chi} + h^-\overset{\leftrightarrow}{\partial^\mu}\tilde{\eta} + iG^-\overset{\leftrightarrow}{\partial^\mu}G^0 + c_{\alpha-\beta}G^-\overset{\leftrightarrow}{\partial^\mu}h + s_{\alpha-\beta}H^-\overset{\leftrightarrow}{\partial^\mu}h\\ &\qquad -s_{\alpha-\beta}G^-\overset{\leftrightarrow}{\partial^\mu}H + c_{\alpha-\beta}H^-\overset{\leftrightarrow}{\partial^\mu}H + iH^-\overset{\leftrightarrow}{\partial^\mu}A\Big) + \text{h.c.}\Big\}\\ &+\left[ieA^\mu + \frac{ig}{2}\frac{c_{2w}}{c_w}Z^\mu\right]\left(h^+\overset{\leftrightarrow}{\partial}_\mu h^- + G^+\overset{\leftrightarrow}{\partial}_\mu G^- + H^+\overset{\leftrightarrow}{\partial}_\mu H^-\right),\end{aligned} \tag{5.3.1b}$$

$$\begin{aligned}\mathcal{L}_{VVHH} = &\left[\frac{g^2}{8c^2_w}Z_\mu Z^\mu + \frac{g^2}{4}W^+_\mu W^{\mu-}\right]\left(\tilde{\eta}^2 + \tilde{\chi}^2 + h^2 + H^2 + (G^0)^2 + A^2\right)\\ &+\Big\{\left[\frac{eg}{2}A^\mu W^+_\mu - \frac{g^2}{2}\frac{s^2_w}{c_w}Z^\mu W^+_\mu\right]\big(\tilde{\eta}h^- + i\tilde{\chi}h^- + iG^0G^- + c_{\alpha-\beta}hG^- + s_{\alpha-\beta}hH^-\\ &\qquad -s_{\alpha-\beta}HG^- + c_{\alpha-\beta}HH^- + iAH^-\big) + \text{h.c.}\Big\}\\ &+\left[e^2 A_\mu A^\mu + eg\frac{c_{2w}}{c_w}A_\mu Z^\mu + \frac{g^2}{4}\frac{c^2_{2w}}{c^2_w}Z_\mu Z^\mu + \frac{g^2}{2}W^-_\mu W^{\mu+}\right]\left(h^-h^+ + H^-H^+ + G^-G^+\right).\end{aligned} \tag{5.3.1c}$$

From the interaction terms ZZh and ZZH it follows that the states h and H are CP-even and therefore the state A should be CP-odd.

Provided that the h scalar is associated with the SM-like Higgs boson, from the interactions hZZ and $hW^\pm W^\mp$ it follows that the SM limit is reached for $c_{\alpha-\beta} = 1$.

5.4 Scalar-Fermion Interactions

Formulation of the S_3-symmetric Yukawa sector was presented in section 4.3. The general result is considered. The R-II-1a vacuum configuration is given by $\{0,\, c_\beta v,\, s_\beta v\}$. The inert $SU(2)$ doublet is h_1 and therefore from eq. (4.3.8) it follows that the fermion mass matrices are block-diagonal:

$$\mathcal{M}_u = \frac{v}{\sqrt{2}} \begin{pmatrix} y_1^u \mathrm{s}_\beta + y_2^u \mathrm{c}_\beta & 0 & 0 \\ 0 & y_1^u \mathrm{s}_\beta - y_2^u \mathrm{c}_\beta & y_4^u \mathrm{c}_\beta \\ 0 & y_5^u \mathrm{c}_\beta & y_3^u \mathrm{s}_\beta \end{pmatrix}, \tag{5.4.1a}$$

$$\mathcal{M}_d = \frac{v}{\sqrt{2}} \begin{pmatrix} y_1^d \mathrm{s}_\beta + y_2^d \mathrm{c}_\beta & 0 & 0 \\ 0 & y_1^d \mathrm{s}_\beta - y_2^d \mathrm{c}_\beta & y_4^d \mathrm{c}_\beta \\ 0 & y_5^d \mathrm{c}_\beta & y_3^d \mathrm{s}_\beta \end{pmatrix}, \tag{5.4.1b}$$

and hermitian mass-squared matrices are:

$$\mathcal{H}_u = \frac{v^2}{2} \begin{pmatrix} (y_1^u \mathrm{s}_\beta + y_2^u \mathrm{c}_\beta)^2 & 0 & 0 \\ 0 & (y_1^u \mathrm{s}_\beta - y_2^u \mathrm{c}_\beta)^2 + (y_4^u)^2 \mathrm{c}_\beta^2 & \mathrm{c}_\beta \left[(y_1^u y_5^u + y_3^u y_4^u)\, \mathrm{s}_\beta - y_2^u y_5^u \mathrm{c}_\beta \right] \\ 0 & \mathrm{c}_\beta \left[(y_1^u y_5^u + y_3^u y_4^u)\, \mathrm{s}_\beta - y_2^u y_5^u \mathrm{c}_\beta \right] & (y_3^u)^2 \mathrm{s}_\beta^2 + (y_5^u)^2 \mathrm{c}_\beta^2 \end{pmatrix}, \tag{5.4.2a}$$

$$\mathcal{H}_d = \frac{v^2}{2} \begin{pmatrix} \left(y_1^d \mathrm{s}_\beta + y_2^d \mathrm{c}_\beta\right)^2 & 0 & 0 \\ 0 & \left(y_1^d \mathrm{s}_\beta - y_2^d \mathrm{c}_\beta\right)^2 + (y_4^d)^2 \mathrm{c}_\beta^2 & \mathrm{c}_\beta \left[\left(y_1^d y_5^d + y_3^d y_4^d\right) \mathrm{s}_\beta - y_2^d y_5^d \mathrm{c}_\beta \right] \\ 0 & \mathrm{c}_\beta \left[\left(y_1^d y_5^d + y_3^d y_4^d\right) \mathrm{s}_\beta - y_2^d y_5^d \mathrm{c}_\beta \right] & (y_3^d)^2 \mathrm{s}_\beta^2 + (y_5^d)^2 \mathrm{c}_\beta^2 \end{pmatrix}. \tag{5.4.2b}$$

Hermitian mass-squared matrix invariants are (4.3.14):

$$\begin{aligned} A_u = {} & \frac{1}{2} v^2 \left(\mathrm{c}_\beta y_2^u + \mathrm{s}_\beta y_1^u\right)^2 + \frac{1}{2} v^2 \left[-\mathrm{s}_{2\beta} y_1^u y_2^u + \mathrm{c}_\beta^2 \left((y_2^u)^2 + (y_4^u)^2 \right) + \mathrm{s}_\beta^2 \, (y_1^u)^2 \right] \\ & + \frac{1}{2} v^2 \left[\mathrm{c}_\beta^2 \, (y_5^u)^2 + \mathrm{s}_\beta^2 \, (y_3^u)^2 \right], \end{aligned} \tag{5.4.3a}$$

$$\begin{aligned} A_d = {} & \frac{1}{2} v^2 \left(\mathrm{c}_\beta y_2^d + y_1^d \mathrm{s}_\beta\right)^2 + \frac{1}{2} v^2 \left[-\mathrm{s}_{2\beta} y_1^d y_2^d + \mathrm{c}_\beta^2 \left(\left(y_2^d\right)^2 + \left(y_4^d\right)^2 \right) + \mathrm{s}_\beta^2 \left(y_1^d\right)^2 \right] \\ & + \frac{1}{2} v^2 \left[\mathrm{c}_\beta^2 \left(y_5^d\right)^2 + \mathrm{s}_\beta^2 \left(y_3^d\right)^2 \right], \end{aligned} \tag{5.4.3b}$$

and

$$\begin{aligned} B_u = -\frac{1}{4} v^4 \Big[& 2 \mathrm{c}_\beta^3 \mathrm{s}_\beta \left[\left((y_4^u)^2 + (y_5^u)^2 \right) y_1^u + y_3^u y_4^u y_5^u \right] y_2^u \\ & + \frac{1}{4} \mathrm{s}_{2\beta}^2 \left[-2 y_1^u y_3^u y_4^u y_5^u + 2 \, (y_2^u)^2 \, (y_3^u)^2 - (y_1^u)^2 \left(2 \, (y_2^u)^2 - (y_4^u)^2 - (y_5^u)^2 \right) \right] \\ & + \mathrm{c}_\beta^4 \left((y_2^u)^2 + (y_4^u)^2 \right) \left((y_2^u)^2 + (y_5^u)^2 \right) + s_\beta^4 \, (y_1^u)^2 \left((y_1^u)^2 + 2 \, (y_3^u)^2 \right) \Big], \end{aligned} \tag{5.4.4a}$$

$$\begin{aligned} B_u = -\frac{1}{4} v^4 \Big[& 2 \mathrm{c}_\beta^3 \mathrm{s}_\beta \left[\left(\left(y_4^d\right)^2 + \left(y_5^d\right)^2 \right) y_1^d + y_3^d y_4^d y_5^d \right] y_2^d \\ & + \frac{1}{4} \mathrm{s}_{2\beta}^2 \left[-2 y_1^d y_3^d y_4^d y_5^d + 2 \left(y_2^d\right)^2 \left(y_3^d\right)^2 - \left(y_1^d\right)^2 \left(2 \left(y_2^d\right)^2 - \left(y_4^d\right)^2 - \left(y_5^d\right)^2 \right) \right] \\ & + \mathrm{c}_\beta^4 \left(\left(y_2^d\right)^2 + \left(y_4^d\right)^2 \right) \left(\left(y_2^d\right)^2 + \left(y_5^d\right)^2 \right) + s_\beta^4 \left(y_1^d\right)^2 \left(\left(y_1^d\right)^2 + 2 \left(y_3^d\right)^2 \right) \Big], \end{aligned} \tag{5.4.4b}$$

and

$$C_u = \frac{1}{32} v^6 \left(\mathrm{c}_\beta y_2^u + \mathrm{s}_\beta y_1^u\right)^2 \left[\mathrm{c}_{2\beta} \left(y_1^u y_3^u + y_4^u y_5^u \right) + \mathrm{s}_{2\beta} y_2^u y_3^u - y_1^u y_3^u + y_4^u y_5^u \right]^2, \tag{5.4.5a}$$

$$C_d = \frac{1}{32} v^6 \left(\mathrm{c}_\beta y_2^d + \mathrm{s}_\beta y_1^d\right)^2 \left[\mathrm{c}_{2\beta} \left(y_1^d y_3^d + y_4^d y_5^d \right) + \mathrm{s}_{2\beta} y_2^d y_3^d - y_1^d y_3^d + y_4^d y_5^d \right]^2. \tag{5.4.5b}$$

The CP check of eq. (4.3.15) results in an expected value $J = 0$.

Hermitian mass-squared matrices are diagonalizable by the left-handed rotation matrix:

$$V_L^{(u,d)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}_{\theta_L^{(u,d)}} & \mathrm{s}_{\theta_L^{(u,d)}} \\ 0 & -\mathrm{s}_{\theta_L^{(u,d)}} & \mathrm{c}_{\theta_L^{(u,d)}} \end{pmatrix}, \tag{5.4.6}$$

where

$$\mathrm{t}_{2\theta_L^{(u,d)}} = \frac{2\mathrm{c}_\beta \left[c_\beta y_2^{(u,d)} y_5^{(u,d)} - \mathrm{s}_\beta \left(y_3^{(u,d)} y_4^{(u,d)} + y_1^{(u,d)} y_5^{(u,d)} \right) \right]}{-\mathrm{s}_{2\beta} y_1^{(u,d)} y_2^{(u,d)} + \mathrm{c}_\beta^2 \left[\left(y_2^{(u,d)} \right)^2 + \left(y_4^{(u,d)} \right)^2 - \left(y_5^{(u,d)} \right)^2 \right] + \mathrm{s}_\beta^2 \left[\left(y_1^{(u,d)} \right)^2 - \left(y_3^{(u,d)} \right)^2 \right]}. \tag{5.4.7}$$

By the look of the left-handed diagonalization matrix we determine the CKM matrix (4.3.28):

$$V_{\mathrm{CKM}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}_{\theta_L^d - \theta_L^u} & \mathrm{s}_{\theta_L^d - \theta_L^u} \\ 0 & -\mathrm{s}_{\theta_L^d - \theta_L^u} & \mathrm{c}_{\theta_L^d - \theta_L^u} \end{pmatrix}. \tag{5.4.8}$$

Such parameterization can be fitted up to the order $\mathcal{O}(10^{-1})$ and results in unrealistic V_{CKM}.

From eq. (5.4.1) it follows that there are five different non-zero Yukawa couplings present. In order to determine masses, three of those are sufficient. This leaves us with two numerically unconstrained Yukawa couplings. Due to this, there will be a net effect which will contribute to FCNC. By taking a look at the fermion mass matrices eq. (5.4.1) one can notice that those become purely diagonal if we impose an additional $\mathbb{Z}_2$ symmetry on the Yukawa couplings[3]:

$$\begin{aligned} \left\{ y_1^{(u,d)}, y_2^{(u,d)}, y_3^{(u,d)} \right\} &\to \left\{ y_1^{(u,d)}, y_2^{(u,d)}, y_3^{(u,d)} \right\}, \\ \left\{ y_4^{(u,d)}, y_5^{(u,d)} \right\} &\to - \left\{ y_4^{(u,d)}, y_5^{(u,d)} \right\}. \end{aligned} \tag{5.4.9}$$

In this limit, the Yukawa couplings in terms of masses are given by:

$$\begin{aligned} y_1^{(u,d)} &= \frac{\left(m_1^{(u,d)} + m_2^{(u,d)} \right)}{\sqrt{2} v \mathrm{s}_\beta}, \\ y_2^{(u,d)} &= \frac{\left(m_1^{(u,d)} - m_2^{(u,d)} \right)}{\sqrt{2} v \mathrm{c}_\beta}, \\ y_3^{(u,d)} &= \frac{\sqrt{2} m_3^{(u,d)}}{v \mathrm{s}_\beta}, \end{aligned} \tag{5.4.10}$$

where $m_i^{(u,d)}$ stands for a mass of a specific fermion generation. We consider the following ordering of the fermion masses:

$$m_3^{(u,d)} > m_2^{(u,d)} > m_1^{(u,d)}. \tag{5.4.11}$$

The fermion diagonalization matrices are simplified to $V_f = U_f = \mathcal{I}_3$. The interaction matrix (4.3.36) of the SM-like Higgs boson with fermions is therefore:

$$\tilde{\mathcal{M}}_{h\bar{f}f} = \begin{pmatrix} \frac{\left(m_1^f \mathrm{s}_{\alpha+\beta} + m_2^f \mathrm{s}_{\alpha-\beta} \right)}{v\, \mathrm{s}_{2\beta}} & 0 & 0 \\ 0 & \frac{\left(m_1^f \mathrm{s}_{\alpha-\beta} + m_2^f \mathrm{s}_{\alpha+\beta} \right)}{v\, \mathrm{s}_{2\beta}} & 0 \\ 0 & 0 & \frac{m_3^f \mathrm{s}_\alpha}{v\, \mathrm{s}_\beta} \end{pmatrix}. \tag{5.4.12}$$

[3]We consider this limit due to the fact that the CKM matrix is unrealistic and not to introduce additional variables so that FCNC are controlled.

The scalar-fermion couplings are given by eq. (4.3.37) and can be easily extracted from the Yukawa Lagrangian:

$$g\left((h/H)\bar{f}_i f_i\right) = \text{diag}\left(\frac{\left(m_1^f C^1_{(h/H)} + m_2^f C^2_{(h/H)}\right)}{v\,\mathrm{s}_{2\beta}}, \frac{\left(m_1^f C^2_{(h/H)} + m_2^f C^1_{(h/H)}\right)}{v\,\mathrm{s}_{2\beta}}, \frac{m_3^f C^3_{(h/H)}}{v\,\mathrm{s}_\beta}\right), \tag{5.4.13a}$$

$$g\left(A\bar{f}_i f_i\right) = \text{diag}\left((-1)^{\delta_{u,f}}\frac{i\gamma_5\left(m_1^f \mathrm{c}_{2\beta} + m_2^f\right)}{v\,\mathrm{s}_{2\beta}}, (-1)^{\delta_{u,f}}\frac{i\gamma_5\left(m_1^f + m_2^f \mathrm{c}_{2\beta}\right)}{v\,\mathrm{s}_{2\beta}}, (-1)^{\delta_{u,f}}\frac{i\gamma_5 m_3^f}{v\,\mathrm{t}_\beta}\right), \tag{5.4.13b}$$

where

$$C_h^i = \{\mathrm{s}_{\alpha+\beta},\, \mathrm{s}_{\alpha-\beta},\, \mathrm{s}_\alpha\}, \tag{5.4.14a}$$

$$C_H^i = \{\mathrm{c}_{\alpha+\beta},\, \mathrm{c}_{\alpha-\beta},\, \mathrm{c}_\alpha\}, \tag{5.4.14b}$$

and $\delta_{u,f}$ is the Kronecker delta introduced due to the Yukawa Lagrangian term $\bar{u}_L^0\left(h_i^0\right)^* u_R^0$.

The SM limit for h is in agreement with the one in section 5.3, $\mathrm{c}_{\alpha-\beta} = 1$, however, additional constraints in terms of the β angle should be considered. We assume that the SM-like limit is reachable for $\mathrm{c}_{\alpha-\beta} \geq 0.9$.

5.5 Scalar-Scalar Interactions

We provide general trilinear and quartic scalar-scalar couplings. The Feynman rules are given by eq. (4.4.1). The symmetry factors are accounted for, see section 4.4 for a discussion. For simplicity, we define:

$$\lambda_5 + \lambda_6 + 2\lambda_7 = \lambda_a, \tag{5.5.1a}$$

$$\lambda_5 + \lambda_6 - 2\lambda_7 = \lambda_b. \tag{5.5.1b}$$

The trilinear scalar-scalar couplings involving the same species are:

$$g\left(hhh\right) = 3v\left[\mathrm{c}_\alpha^3\left(2\left(\lambda_1+\lambda_3\right)\mathrm{c}_\beta - \lambda_4\mathrm{s}_\beta\right) + \mathrm{c}_\alpha^2\mathrm{s}_\alpha\left(\lambda_a\mathrm{s}_\beta - 3\lambda_4\mathrm{c}_\beta\right) + \lambda_a\mathrm{c}_\alpha\mathrm{c}_\beta\mathrm{s}_\alpha^2 + 2\lambda_8\mathrm{s}_\alpha^3\mathrm{s}_\beta\right], \tag{5.5.2a}$$

$$g\left(HHH\right) = -3v\left[\mathrm{s}_\alpha^3\left(2\left(\lambda_1+\lambda_3\right)\mathrm{c}_\beta - \lambda_4\mathrm{s}_\beta\right) + \mathrm{c}_\alpha\mathrm{s}_\alpha^2\left(3\lambda_4\mathrm{c}_\beta - \lambda_a\mathrm{s}_\beta\right) + \lambda_a\mathrm{c}_\alpha^2\mathrm{c}_\beta\mathrm{s}_\alpha - 2\lambda_8\mathrm{c}_\alpha^3\mathrm{s}_\beta\right]. \tag{5.5.2b}$$

The trilinear couplings involving the neutral fields are:

$$g\left(\tilde{\eta}\tilde{\eta}h\right) = v\left[\mathrm{s}_\alpha\left(\lambda_a\mathrm{s}_\beta + 3\lambda_4\mathrm{c}_\beta\right) + \mathrm{c}_\alpha\left(2\left(\lambda_1+\lambda_3\right)\mathrm{c}_\beta + 3\lambda_4\mathrm{s}_\beta\right)\right], \tag{5.5.3a}$$

$$g\left(\tilde{\eta}\tilde{\eta}H\right) = v\left[\mathrm{c}_\alpha\left(\lambda_a\mathrm{s}_\beta + 3\lambda_4\mathrm{c}_\beta\right) - \mathrm{s}_\alpha\left(2\left(\lambda_1+\lambda_3\right)\mathrm{c}_\beta + 3\lambda_4\mathrm{s}_\beta\right)\right], \tag{5.5.3b}$$

$$g\left(\tilde{\chi}\tilde{\chi}h\right) = v\left[\mathrm{s}_\alpha\left(\lambda_b\mathrm{s}_\beta + \lambda_4\mathrm{c}_\beta\right) + \mathrm{c}_\alpha\left(2\left(\lambda_1 - 2\lambda_2 - \lambda_3\right)\mathrm{c}_\beta + \lambda_4\mathrm{s}_\beta\right)\right], \tag{5.5.3c}$$

$$g\left(\tilde{\chi}\tilde{\chi}H\right) = v\left[\mathrm{c}_\alpha\left(\lambda_b\mathrm{s}_\beta + \lambda_4\mathrm{c}_\beta\right) - \mathrm{s}_\alpha\left(2\left(\lambda_1 - 2\lambda_2 - \lambda_3\right)\mathrm{c}_\beta + \lambda_4\mathrm{s}_\beta\right)\right], \tag{5.5.3d}$$

$$g\left(\tilde{\eta}\tilde{\chi}A\right) = v\left[\lambda_4\mathrm{c}_{2\beta} - \left(\lambda_2 + \lambda_3 - \lambda_7\right)\mathrm{s}_{2\beta}\right], \tag{5.5.3e}$$

$$\begin{aligned} g\left(hhH\right) = -v\Big[&\mathrm{c}_\alpha^3\left(3\lambda_4\mathrm{c}_\beta - \lambda_a\mathrm{s}_\beta\right) + \mathrm{c}_\alpha^2\mathrm{s}_\alpha\left(\left(-2\lambda_a + 6\lambda_1 + 6\lambda_3\right)\mathrm{c}_\beta - 3\lambda_4\mathrm{s}_\beta\right) \\ &+ \lambda_a\mathrm{c}_\beta\mathrm{s}_\alpha^3 - 2\mathrm{c}_\alpha\mathrm{s}_\alpha^2\left(3\lambda_4\mathrm{c}_\beta + \left(3\lambda_8 - 2\lambda_7\right)\mathrm{s}_\beta\right) + \left(\lambda_5+\lambda_6\right)\mathrm{s}_\alpha\mathrm{s}_{2\alpha}\mathrm{s}_\beta\Big], \end{aligned} \tag{5.5.3f}$$

$$\begin{aligned} g\left(hHH\right) = v\Big[&\mathrm{s}_\alpha^3\left(\lambda_a\mathrm{s}_\beta - 3\lambda_4\mathrm{c}_\beta\right) + \mathrm{c}_\alpha\mathrm{s}_\alpha^2\left(\left(-2\lambda_a + 6\lambda_1 + 6\lambda_3\right)\mathrm{c}_\beta - 3\lambda_4\mathrm{s}_\beta\right) \\ &+ \lambda_a\mathrm{c}_\alpha^3\mathrm{c}_\beta + 2\mathrm{c}_\alpha^2\mathrm{s}_\alpha\left(3\lambda_4\mathrm{c}_\beta - \left(\lambda_a - 3\lambda_8\right)\mathrm{s}_\beta\right)\Big], \end{aligned} \tag{5.5.3g}$$

$$g\left(AAh\right) = v\Big[\mathrm{c}_\alpha\left(\lambda_b\mathrm{c}_\beta^3 + 2\lambda_4\mathrm{c}_\beta^2\mathrm{s}_\beta - \lambda_4\mathrm{s}_\beta^3 + \left(\lambda_1+\lambda_3-2\lambda_7\right)\mathrm{s}_\beta\mathrm{s}_{2\beta}\right) + \mathrm{s}_\alpha\mathrm{s}_\beta\left(\lambda_b\mathrm{s}_\beta^2 + 2\left(\lambda_8-2\lambda_7\right)\mathrm{c}_\beta^2 - \frac{1}{2}\lambda_4\mathrm{s}_{2\beta}\right)\Big], \tag{5.5.3h}$$

$$g\left(AAH\right) = -v\Big[\lambda_b\mathrm{c}_\beta^3\mathrm{s}_\alpha - \mathrm{s}_\beta^3\left(\lambda_b\mathrm{c}_\alpha + \lambda_4\mathrm{s}_\alpha\right) + 2\mathrm{c}_\beta^2\mathrm{s}_\beta\left(\left(2\lambda_7-\lambda_8\right)\mathrm{c}_\alpha + \lambda_4\mathrm{s}_\alpha\right) + \mathrm{c}_\beta\mathrm{s}_\beta^2\left(\lambda_4\mathrm{c}_\alpha + 2\left(\lambda_1+\lambda_3-2\lambda_7\right)\mathrm{s}_\alpha\right)\Big]. \tag{5.5.3i}$$

The trilinear couplings involving the charged fields are:

$$g\left(\tilde{\eta}h^\pm H^\mp\right) = \frac{1}{4}v\left[4\lambda_4\mathrm{c}_{2\beta} + \left(-4\lambda_3+\lambda_6+2\lambda_7\right)\mathrm{s}_{2\beta}\right], \tag{5.5.4a}$$

$$g\left(\tilde{\chi}h^\pm H^\mp\right) = \mp\frac{1}{4}iv\left(4\lambda_2+\lambda_6-2\lambda_7\right)\mathrm{s}_{2\beta}, \tag{5.5.4b}$$

$$g\left(hH^\mp H^\pm\right) = -v\Big[\mathrm{c}_\alpha\left(-\lambda_5\mathrm{c}_\beta^3 - 2\lambda_4\mathrm{c}_\beta^2\mathrm{s}_\beta + \left(-2\lambda_1-2\lambda_3+\lambda_6+2\lambda_7\right)\mathrm{c}_\beta\mathrm{s}_\beta^2 + \lambda_4\mathrm{s}_\beta^3\right) + \mathrm{s}_\alpha\mathrm{s}_\beta\left(\left(\lambda_6+2\lambda_7-2\lambda_8\right)\mathrm{c}_\beta^2 + \frac{1}{2}\lambda_4\mathrm{s}_{2\beta} - \lambda_5\mathrm{s}_\beta^2\right)\Big], \tag{5.5.4c}$$

$$g\left(HH^\mp H^\pm\right) = -v\Big[\lambda_5\mathrm{c}_\beta^3\mathrm{s}_\alpha + \mathrm{c}_\beta^2\mathrm{s}_\beta\left(\left(\lambda_6+2\lambda_7-2\lambda_8\right)\mathrm{c}_\alpha + 2\lambda_4\mathrm{s}_\alpha\right) + \mathrm{c}_\beta\mathrm{s}_\beta^2\left(\lambda_4\mathrm{c}_\alpha + \left(2\lambda_1+2\lambda_3-\lambda_6-2\lambda_7\right)\mathrm{s}_\alpha\right) - \mathrm{s}_\beta^3\left(\lambda_5\mathrm{c}_\alpha + \lambda_4\mathrm{s}_\alpha\right)\Big], \tag{5.5.4d}$$

$$g\left(hh^\pm h^\mp\right) = v\left[\mathrm{c}_\alpha\left(2\left(\lambda_1-\lambda_3\right)\mathrm{c}_\beta + \lambda_4\mathrm{s}_\beta\right) + \mathrm{s}_\alpha\left(\lambda_4\mathrm{c}_\beta + \lambda_5\mathrm{s}_\beta\right)\right], \tag{5.5.4e}$$

$$g\left(Hh^\pm h^\mp\right) = v\left[-\mathrm{s}_\alpha\left(2\left(\lambda_1-\lambda_3\right)\mathrm{c}_\beta + \lambda_4\mathrm{s}_\beta\right) + \mathrm{c}_\alpha\left(\lambda_4\mathrm{c}_\beta + \lambda_5\mathrm{s}_\beta\right)\right]. \tag{5.5.4f}$$

The quartic couplings involving the same species are:

$$g\left(\tilde{\eta}\tilde{\eta}\tilde{\eta}\tilde{\eta}\right) = g\left(\tilde{\chi}\tilde{\chi}\tilde{\chi}\tilde{\chi}\right) = 6\left(\lambda_1+\lambda_3\right), \tag{5.5.5a}$$

$$g\left(hhhh\right) = 6\left[\frac{1}{4}\lambda_a\mathrm{s}_{2\alpha}^2 + \left(\lambda_1+\lambda_3\right)\mathrm{c}_\alpha^4 - 2\lambda_4\mathrm{c}_\alpha^3\mathrm{s}_\alpha + \lambda_8\mathrm{s}_\alpha^4\right], \tag{5.5.5b}$$

$$g\left(HHHH\right) = 6\left[\frac{1}{4}\lambda_a\mathrm{s}_{2\alpha}^2 + \lambda_8\mathrm{c}_\alpha^4 + 2\lambda_4\mathrm{c}_\alpha\mathrm{s}_\alpha^3 + \left(\lambda_1+\lambda_3\right)\mathrm{s}_\alpha^4\right], \tag{5.5.5c}$$

$$g\left(AAAA\right) = 6\left[\frac{1}{4}\lambda_a\mathrm{s}_{2\beta}^2 + \lambda_8\mathrm{c}_\beta^4 + 2\lambda_4\mathrm{c}_\beta\mathrm{s}_\beta^3 + \left(\lambda_1+\lambda_3\right)\mathrm{s}_\beta^4\right]. \tag{5.5.5d}$$

The quartic couplings involving only the neutral fields are:

$$g\left(\tilde{\eta}\tilde{\eta}\tilde{\chi}\tilde{\chi}\right) = 2\left(\lambda_1+\lambda_3\right), \tag{5.5.6a}$$

$$g\left(\tilde{\eta}\tilde{\eta}AA\right) = \lambda_b\mathrm{c}_\beta^2 + 2\left(\lambda_1-2\lambda_2-\lambda_3\right)\mathrm{s}_\beta^2 - \lambda_4\mathrm{s}_{2\beta}, \tag{5.5.6b}$$

$$g\left(\tilde{\chi}\tilde{\chi}AA\right) = \lambda_a\mathrm{c}_\beta^2 + 2\left(\lambda_1+\lambda_3\right)\mathrm{s}_\beta^2 - 3\lambda_4\mathrm{s}_{2\beta}, \tag{5.5.6c}$$

$$g\left(\tilde{\eta}\tilde{\eta}hh\right) = \lambda_a\mathrm{s}_\alpha^2 + 2\left(\lambda_1+\lambda_3\right)\mathrm{c}_\alpha^2 + 3\lambda_4\mathrm{s}_{2\alpha}, \tag{5.5.6d}$$

$$g\left(\tilde{\eta}\tilde{\eta}hH\right) = \frac{1}{2}\left(\lambda_a-2\lambda_1-2\lambda_3\right)\mathrm{s}_{2\alpha} + 3\lambda_4\mathrm{c}_{2\alpha}, \tag{5.5.6e}$$

$$g\left(\tilde{\eta}\tilde{\eta}HH\right) = \lambda_a\mathrm{c}_\alpha^2 + 2\left(\lambda_1+\lambda_3\right)\mathrm{s}_\alpha^2 - 3\lambda_4\mathrm{s}_{2\alpha}, \tag{5.5.6f}$$

$$g\left(\tilde{\chi}\tilde{\chi}hh\right) = \lambda_b\mathrm{s}_\alpha^2 + 2\left(\lambda_1-2\lambda_2-\lambda_3\right)\mathrm{c}_\alpha^2 + \lambda_4\mathrm{s}_{2\alpha}, \tag{5.5.6g}$$

$$g\left(\tilde{\chi}\tilde{\chi}hH\right) = \left(\lambda_b-2\lambda_1+4\lambda_2+2\lambda_3\right)\mathrm{c}_\alpha\mathrm{s}_\alpha + \lambda_4\mathrm{c}_{2\alpha}, \tag{5.5.6h}$$

$$g\left(\tilde{\chi}\tilde{\chi}HH\right) = \lambda_b\mathrm{c}_\alpha^2 + 2\left(\lambda_1-2\lambda_2-\lambda_3\right)\mathrm{s}_\alpha^2 - \lambda_4\mathrm{s}_{2\alpha}, \tag{5.5.6i}$$

$$g\left(\tilde{\eta}\tilde{\chi}hA\right) = -\mathrm{c}_\alpha\left[2\left(\lambda_2+\lambda_3\right)\mathrm{s}_\beta - \lambda_4\mathrm{c}_\beta\right] - \mathrm{s}_\alpha\left(\lambda_4\mathrm{s}_\beta - 2\lambda_7\mathrm{c}_\beta\right), \tag{5.5.6j}$$

$$g\left(\tilde{\eta}\tilde{\chi}HA\right) = \mathrm{s}_\alpha\left[2\left(\lambda_2+\lambda_3\right)\mathrm{s}_\beta - \lambda_4\mathrm{c}_\beta\right] - \mathrm{c}_\alpha\left(\lambda_4\mathrm{s}_\beta - 2\lambda_7\mathrm{c}_\beta\right), \tag{5.5.6k}$$

$$g\left(hhhH\right) = -3\mathrm{c}_\alpha\left[\lambda_4\mathrm{c}_{3\alpha} + \mathrm{s}_\alpha\left(\left(\lambda_1+\lambda_3-\lambda_a+\lambda_8\right)\mathrm{c}_{2\alpha} + \lambda_1+\lambda_3-\lambda_8\right)\right], \tag{5.5.6l}$$

$$g\left(hhHH\right) = \frac{1}{4}\left[\lambda_a + 3\lambda_1 + 3\lambda_3 + 3\lambda_8 + 6\lambda_4 s_{4\alpha} - 3\left(\lambda_1 + \lambda_3 - \lambda_a + \lambda_8\right) c_{4\alpha}\right], \tag{5.5.6m}$$

$$g\left(hHHH\right) = -\frac{3}{2} s_\alpha \left[\left(\lambda_a + \lambda_1 + \lambda_3 - 3\lambda_8\right) c_\alpha - \left(\lambda_1 + \lambda_3 - \lambda_a + \lambda_8\right) c_{3\alpha} + 2\lambda_4 s_{3\alpha}\right], \tag{5.5.6n}$$

$$g\left(AAhh\right) = c_\alpha^2 \left[\lambda_b c_\beta^2 + 2\left(\lambda_1 + \lambda_3\right) s_\beta^2 + \lambda_4 s_{2\beta}\right] + s_\alpha^2 \left(\lambda_b s_\beta^2 + 2\lambda_8 c_\beta^2\right) - s_{2\alpha} s_\beta \left(\lambda_4 s_\beta + 4\lambda_7 c_\beta\right), \tag{5.5.6o}$$

$$\begin{aligned} g\left(AAhH\right) = {} & \frac{1}{2} s_{2\alpha}\left[-\lambda_1 - \lambda_3 + \lambda_8 - \lambda_4 s_{2\beta} + \left(\lambda_1 + \lambda_3 - \lambda_b + \lambda_8\right) c_{2\beta}\right] \\ & + s_\alpha^2 s_\beta \left(4\lambda_7 c_\beta + \lambda_4 s_\beta\right) - c_\alpha^2 s_\beta \left(\lambda_4 s_\beta + 4\lambda_7 c_\beta\right), \end{aligned} \tag{5.5.6p}$$

$$g\left(AAHH\right) = c_\alpha^2 \left(\lambda_b s_\beta^2 + 2\lambda_8 c_\beta^2\right) + s_\alpha^2 \left[\lambda_b c_\beta^2 + 2\left(\lambda_1 + \lambda_3\right) s_\beta^2 + \lambda_4 s_{2\beta}\right] + s_{2\alpha} s_\beta \left(\lambda_4 s_\beta + 4\lambda_7 c_\beta\right). \tag{5.5.6q}$$

The quartic couplings involving both neutral and charged fields are:

$$g\left(\tilde{\eta}\tilde{\eta}h^\pm h^\mp\right) = g\left(\tilde{\chi}\tilde{\chi}h^\pm h^\mp\right) = 2\left(\lambda_1 + \lambda_3\right), \tag{5.5.7a}$$

$$g\left(\tilde{\eta}\tilde{\eta}H^\mp H^\pm\right) = -2\left(\lambda_3 - \lambda_1\right) s_\beta^2 - \lambda_4 s_{2\beta} + \lambda_5 c_\beta^2, \tag{5.5.7b}$$

$$g\left(\tilde{\chi}\tilde{\chi}H^\mp H^\pm\right) = -2\left(\lambda_3 - \lambda_1\right) s_\beta^2 - \lambda_4 s_{2\beta} + \lambda_5 c_\beta^2, \tag{5.5.7c}$$

$$g\left(\tilde{\eta}hh^\mp H^\pm\right) = -c_\alpha\left(2\lambda_3 s_\beta - \lambda_4 c_\beta\right) - \lambda_4 s_\alpha s_\beta + \frac{1}{2}\left(\lambda_6 + 2\lambda_7\right) s_\alpha c_\beta, \tag{5.5.7d}$$

$$g\left(\tilde{\eta}Hh^\mp H^\pm\right) = s_\alpha\left(2\lambda_3 s_\beta - \lambda_4 c_\beta\right) - \lambda_4 c_\alpha s_\beta + \frac{1}{2}\left(\lambda_6 + 2\lambda_7\right) c_\alpha c_\beta, \tag{5.5.7e}$$

$$g\left(\tilde{\chi}hh^\mp H^\pm\right) = \pm i\left[2\lambda_2 c_\alpha s_\beta + \frac{1}{2}\left(\lambda_6 - 2\lambda_7\right) s_\alpha c_\beta\right], \tag{5.5.7f}$$

$$g\left(\tilde{\chi}Hh^\mp H^\pm\right) = \pm i\left[-2\lambda_2 s_\alpha s_\beta + \frac{1}{2}\left(\lambda_6 - 2\lambda_7\right) c_\alpha c_\beta\right], \tag{5.5.7g}$$

$$g\left(hhh^\pm h^\mp\right) = 2\left(\lambda_1 - \lambda_3\right) c_\alpha^2 + \lambda_4 s_{2\alpha} + \lambda_5 s_\alpha^2, \tag{5.5.7h}$$

$$g\left(hHh^\pm h^\mp\right) = -\left(\lambda_1 - \lambda_3 - \frac{1}{2}\lambda_5\right) s_{2\alpha} + \lambda_4 c_{2\alpha}, \tag{5.5.7i}$$

$$g\left(HHh^\pm h^\mp\right) = -2\left(\lambda_3 - \lambda_1\right) s_\alpha^2 - \lambda_4 s_{2\alpha} + \lambda_5 c_\alpha^2, \tag{5.5.7j}$$

$$g\left(AAh^\pm h^\mp\right) = -2\left(\lambda_3 - \lambda_1\right) s_\beta^2 - \lambda_4 s_{2\beta} + \lambda_5 c_\beta^2, \tag{5.5.7k}$$

$$g\left(\tilde{\eta}Ah^\pm H^\mp\right) = \pm i\left[-2\lambda_2 s_\beta^2 + \frac{1}{2}\left(\lambda_6 - 2\lambda_7\right) c_\beta^2\right], \tag{5.5.7l}$$

$$g\left(\tilde{\chi}Ah^\pm H^\mp\right) = 2\lambda_3 s_\beta^2 - \lambda_4 s_{2\beta} + \frac{1}{2}\left(\lambda_6 + 2\lambda_7\right) c_\beta^2, \tag{5.5.7m}$$

$$\begin{aligned} g\left(hhH^\mp H^\pm\right) = {} & c_\alpha^2\left[\lambda_5 c_\beta^2 + 2\left(\lambda_1 + \lambda_3\right) s_\beta^2 + \lambda_4 s_{2\beta}\right] \\ & - \lambda_4 s_{2\alpha} s_\beta^2 + \lambda_5 s_\alpha^2 s_\beta^2 - \frac{1}{2}\lambda_6 s_{2\alpha} s_{2\beta} - \lambda_7 s_{2\alpha} s_{2\beta} + 2\lambda_8 c_\beta^2 s_\alpha^2, \end{aligned} \tag{5.5.7n}$$

$$\begin{aligned} g\left(hHH^\mp H^\pm\right) = {} & \frac{1}{2} s_{2\alpha}\left[\left(\lambda_1 + \lambda_3 - \lambda_5 + \lambda_8\right) c_{2\beta} - \lambda_1 - \lambda_3 + \lambda_8 - \lambda_4 s_{2\beta}\right] \\ & + s_\alpha^2 s_\beta\left[\lambda_4 s_\beta + \left(\lambda_6 + 2\lambda_7\right) c_\beta\right] - c_\alpha^2 s_\beta\left[\lambda_4 s_\beta + \left(\lambda_6 + 2\lambda_7\right) c_\beta\right], \end{aligned} \tag{5.5.7o}$$

$$\begin{aligned} g\left(HHH^\mp H^\pm\right) = {} & c_\alpha^2\left(\lambda_5 s_\beta^2 + 2\lambda_8 c_\beta^2\right) + s_\alpha^2\left[2\left(\lambda_1 + \lambda_3\right) s_\beta^2 + \lambda_4 s_{2\beta} + \lambda_5 c_\beta^2\right] \\ & + s_{2\alpha} s_\beta\left[\lambda_4 s_\beta + \left(\lambda_6 + 2\lambda_7\right) c_\beta\right], \end{aligned} \tag{5.5.7p}$$

$$g\left(AAH^\mp H^\pm\right) = 2\left[\left(\lambda_1 + \lambda_3\right) s_\beta^4 + 2\lambda_4 c_\beta s_\beta^3 + \frac{1}{4}\lambda_a s_{2\beta}^2 + \lambda_8 c_\beta^4\right]. \tag{5.5.7q}$$

The quartic couplings involving only the charged fields are:

$$g\left(h^\pm h^\mp h^\pm h^\mp\right) = 4\left(\lambda_1 + \lambda_3\right), \tag{5.5.8a}$$

$$g\left(H^\mp H^\mp H^\pm H^\pm\right) = 4\left[\left(\lambda_1 + \lambda_3\right) s_\beta^4 + 2\lambda_4 c_\beta s_\beta^3 + \frac{1}{4}\lambda_a s_{2\beta}^2 + \lambda_8 c_\beta^4\right], \tag{5.5.8b}$$

$$g\left(h^\pm h^\pm H^\mp H^\mp\right) = 4\left[\left(\lambda_2 + \lambda_3\right) s_\beta^2 - \frac{1}{2}\lambda_4 s_{2\beta} + \lambda_7 c_\beta^2\right], \tag{5.5.8c}$$

$$g\left(h^{\pm}h^{\mp}H^{\mp}H^{\pm}\right) = 2\left(\lambda_1 - \lambda_2\right)s_\beta^2 - 2\lambda_4 s_{2\beta} + \left(\lambda_5 + \lambda_6\right)c_\beta^2 \tag{5.5.8d}$$

The trilinear couplings involving only the neutral fields and the would-be Goldstone boson are:

$$g\left(\tilde{\eta}\tilde{\chi}G^0\right) = 2v\left[\left(\lambda_2 + \lambda_3\right)c_\beta^2 + \frac{1}{2}\lambda_4 s_{2\beta} + \lambda_7 s_\beta^2\right], \tag{5.5.9a}$$

$$g\left(G^0G^0h\right) = v\left[s_\alpha\left(\lambda_a c_\beta^2 s_\beta - \lambda_4 c_\beta^3 + 2\lambda_8 s_\beta^3\right) + c_\alpha c_\beta\left(\lambda_a s_\beta^2 + 2\left(\lambda_1 + \lambda_3\right)c_\beta^2 - \frac{3}{2}\lambda_4 s_{2\beta}\right)\right], \tag{5.5.9b}$$

$$\begin{aligned} g\left(G^0G^0H\right) = -v\Big[& -c_\alpha\left(\lambda_a c_\beta^2 s_\beta - \lambda_4 c_\beta^3 + 2\lambda_8 s_\beta^3\right) \\ & + s_\alpha\left(2\left(\lambda_1 + \lambda_3\right)c_\beta^3 - 3\lambda_4 c_\beta^2 s_\beta + \left(\lambda_5 + \lambda_6\right)c_\beta s_\beta^2 + \lambda_7 s_\beta s_{2\beta}\right)\Big], \end{aligned} \tag{5.5.9c}$$

$$\begin{aligned} g\left(G^0Ah\right) = -v\Big[& c_\alpha\left(\lambda_4 c_\beta^3 + \left(2\lambda_1 + 2\lambda_3 - \lambda_5 - \lambda_6\right)c_\beta^2 s_\beta - 2\lambda_4 c_\beta s_\beta^2 + 2\lambda_7 s_\beta^3\right) \\ & - s_\alpha c_\beta\left(\frac{1}{2}\lambda_4 s_{2\beta} + 2\lambda_7 c_\beta^2 - \left(\lambda_5 + \lambda_6 - 2\lambda_8\right)s_\beta^2\right)\Big], \end{aligned} \tag{5.5.9d}$$

$$\begin{aligned} g\left(G^0AH\right) = -v\Big[& -s_\alpha\left(\lambda_4 c_\beta^3 + \left(2\lambda_1 + 2\lambda_3 - \lambda_5 - \lambda_6\right)c_\beta^2 s_\beta + \lambda_4 s_\beta s_{2\beta} - 2\lambda_7 s_\beta^3\right) \\ & - c_\alpha c_\beta\left(\frac{1}{2}\lambda_4 s_{2\beta} + 2\lambda_7 c_\beta^2 - \left(\lambda_5 + \lambda_6 - 2\lambda_8\right)s_\beta^2\right)\Big]. \end{aligned} \tag{5.5.9e}$$

The trilinear couplings involving the charged fields and the would-be Goldstone boson are:

$$g\left(\tilde{\eta}h^{\pm}G^{\mp}\right) = \frac{1}{2}v\left[4\lambda_3 c_\beta^2 + 2\lambda_4 s_{2\beta} + \left(\lambda_6 + 2\lambda_7\right)s_\beta^2\right], \tag{5.5.10a}$$

$$g\left(\tilde{\chi}h^{\pm}G^{\mp}\right) = \pm iv\left[2\lambda_2 c_\beta^2 - \frac{1}{2}\left(\lambda_6 - 2\lambda_7\right)s_\beta^2\right], \tag{5.5.10b}$$

$$g\left(hG^{\mp}G^{\pm}\right) = v\left[s_\alpha\left(\lambda_a c_\beta^2 s_\beta - \lambda_4 c_\beta^3 + 2\lambda_8 s_\beta^3\right) + c_\alpha c_\beta\left(\lambda_a s_\beta^2 + 2\left(\lambda_1 + \lambda_3\right)c_\beta^2 - \frac{3}{2}\lambda_4 s_{2\beta}\right)\right], \tag{5.5.10c}$$

$$\begin{aligned} g\left(hG^{\mp}H^{\pm}\right) = -\frac{1}{4}v\Big[& c_\alpha\left(\lambda_4 c_\beta + 3\lambda_4 c_{3\beta} + 2s_\beta\left(\left(2\lambda_1 + 2\lambda_3 - \lambda_a\right)c_{2\beta} + 2\left(\lambda_1 + \lambda_3\right) - \lambda_5\right)\right) \\ & - 2c_\beta s_\alpha\left(\left(\lambda_a - 2\lambda_8\right)c_{2\beta} - \lambda_5 + 2\lambda_8 + \lambda_4 s_{2\beta}\right)\Big], \end{aligned} \tag{5.5.10d}$$

$$\begin{aligned} g\left(HG^{\mp}G^{\pm}\right) = v\Big[& c_\alpha\left(\lambda_a c_\beta^2 s_\beta - \lambda_4 c_\beta^3 + 2\lambda_8 s_\beta^3\right) \\ & - s_\alpha\left(2\left(\lambda_1 + \lambda_3\right)c_\beta^3 - 3\lambda_4 c_\beta^2 s_\beta + \left(\lambda_5 + \lambda_6\right)c_\beta s_\beta^2 + \lambda_7 s_\beta s_{2\beta}\right)\Big], \end{aligned} \tag{5.5.10e}$$

$$\begin{aligned} g\left(HG^{\mp}H^{\pm}\right) = \frac{1}{4}v\Big[& 2c_\alpha c_\beta\left(\left(\lambda_a - 2\lambda_8\right)c_{2\beta} - \lambda_5 + 2\lambda_8 + \lambda_4 s_{2\beta}\right) \\ & + s_\alpha\left(\lambda_4 c_\beta + 3\lambda_4 c_{3\beta} + \left(2\lambda_1 + 2\lambda_3 - \lambda_5 + \lambda_6 + 2\lambda_7\right)s_\beta + \left(2\lambda_1 + 2\lambda_3 - \lambda_a\right)s_{3\beta}\right)\Big], \end{aligned} \tag{5.5.10f}$$

$$g\left(AG^{\mp}H^{\pm}\right) = \mp\frac{1}{2}iv\left(\lambda_6 - 2\lambda_7\right). \tag{5.5.10g}$$

The quartic couplings involving the neutral states with at least one neutral would-be Goldstone boson are:

$$g\left(G^0G^0G^0G^0\right) = 6\left[\left(\lambda_1 + \lambda_3\right)c_\beta^4 - 2\lambda_4 c_\beta^3 s_\beta + \frac{1}{4}\lambda_a s_{2\beta}^2 + \lambda_8 s_\beta^4\right], \tag{5.5.11a}$$

$$g\left(G^0G^0G^0A\right) = -3c_\beta\left[\lambda_4 c_{3\beta} + s_\beta\left(\lambda_1 + \lambda_3 - \lambda_8 + \left(\lambda_1 + \lambda_3 - \lambda_a + \lambda_8\right)c_{2\beta}\right)\right], \tag{5.5.11b}$$

$$g\left(G^0G^0AA\right)=\frac{1}{4}\left[3\lambda_1+3\lambda_3+3\lambda_8+6\lambda_4\mathrm{s}_{4\beta}+\lambda_a-3\left(\lambda_1+\lambda_3-\lambda_a+\lambda_8\right)\mathrm{c}_{4\beta}\right], \tag{5.5.11c}$$

$$g\left(G^0AAA\right)=-\frac{3}{2}\mathrm{s}_\beta\left[2\lambda_4\mathrm{s}_{3\beta}+\left(\lambda_a+\lambda_1+\lambda_3-3\lambda_8\right)\mathrm{c}_\beta-\left(\lambda_1+\lambda_3-\lambda_a+\lambda_8\right)\mathrm{c}_{3\beta}\right], \tag{5.5.11d}$$

$$g\left(G^0G^0hh\right)=\mathrm{c}_\alpha^2\left[2\left(\lambda_1+\lambda_3\right)\mathrm{c}_\beta^2-\lambda_4\mathrm{s}_{2\beta}+\lambda_b\mathrm{s}_\beta^2\right]+\mathrm{s}_\alpha^2\left[\lambda_b\mathrm{c}_\beta^2+2\lambda_8\mathrm{s}_\beta^2\right]-\mathrm{c}_\beta\mathrm{s}_{2\alpha}\left[\lambda_4\mathrm{c}_\beta-4\lambda_7\mathrm{s}_\beta\right], \tag{5.5.11e}$$

$$\begin{aligned}g\left(G^0G^0hH\right)=&-\frac{1}{2}\mathrm{s}_\alpha\left[\lambda_1+\lambda_3-\lambda_4\mathrm{s}_{2\beta}-\lambda_8+\left(\lambda_1+\lambda_3-\lambda_b+\lambda_8\right)\mathrm{c}_{2\beta}\right]\\&+\mathrm{c}_\beta\mathrm{s}_\alpha^2\left(\lambda_4\mathrm{c}_\beta-4\lambda_7\mathrm{s}_\beta\right)-\mathrm{c}_\alpha^2\mathrm{c}_\beta\left(\lambda_4\mathrm{c}_\beta-4\lambda_7\mathrm{s}_\beta\right),\end{aligned} \tag{5.5.11f}$$

$$\begin{aligned}g\left(G^0G^0HH\right)=&-\mathrm{s}_\alpha\left[4\lambda_7\mathrm{c}_\alpha\mathrm{s}_{2\beta}+\mathrm{s}_\alpha\mathrm{s}_\beta\left(2\lambda_4\mathrm{c}_\beta-\lambda_b\mathrm{s}_\beta\right)-2\mathrm{c}_\beta^2\left(\lambda_4\mathrm{c}_\alpha+\left(\lambda_1+\lambda_3\right)\mathrm{s}_\alpha\right)\right]\\&+\mathrm{c}_\alpha^2\left(\lambda_b\mathrm{c}_\beta^2+2\lambda_8\mathrm{s}_\beta^2\right),\end{aligned} \tag{5.5.11g}$$

$$\begin{aligned}g\left(G^0Ahh\right)=&-\frac{1}{2}\Big[\mathrm{s}_{2\beta}\left(\lambda_1+\lambda_3-\lambda_8-\lambda_4\mathrm{s}_{2\alpha}+\left(\lambda_1+\lambda_3-\lambda_b+\lambda_8\right)\mathrm{c}_{2\alpha}\right)\\&+2\mathrm{c}_\alpha\mathrm{c}_{2\beta}\left(\lambda_4\mathrm{c}_\alpha-4\lambda_7\mathrm{s}_\alpha\right)\Big],\end{aligned} \tag{5.5.11h}$$

$$g\left(G^0AhH\right)=\frac{1}{4}\left[2\lambda_4\mathrm{s}_{2(\alpha+\beta)}+8\lambda_7\mathrm{c}_{2\alpha}\mathrm{c}_{2\beta}+2\left(\lambda_1+\lambda_3-\lambda_b+\lambda_8\right)\mathrm{s}_{2\alpha}\mathrm{s}_{2\beta}\right], \tag{5.5.11i}$$

$$g\left(G^0AHH\right)=\frac{1}{2}\Big[-2\lambda_4\mathrm{s}_\alpha\mathrm{s}_{\alpha+2\beta}-4\lambda_7\mathrm{c}_{2\beta}\mathrm{s}_{2\alpha}-\mathrm{s}_{2\beta}\left(\lambda_1+\lambda_3-\lambda_8-\left(\lambda_1+\lambda_3-\lambda_b+\lambda_8\right)\mathrm{c}_{2\alpha}\right)\Big], \tag{5.5.11j}$$

$$g\left(\tilde\eta\tilde\eta G^0G^0\right)=2\left(\lambda_1-2\lambda_2-\lambda_3\right)\mathrm{c}_\beta^2+\lambda_4\mathrm{s}_{2\beta}+\lambda_b\mathrm{s}_\beta^2, \tag{5.5.11k}$$

$$g\left(\tilde\eta\tilde\eta G^0A\right)=\frac{1}{2}\left(-2\lambda_1+4\lambda_2+2\lambda_3+\lambda_b\right)\mathrm{s}_{2\beta}+\lambda_4\mathrm{c}_{2\beta}, \tag{5.5.11l}$$

$$g\left(\tilde\chi\tilde\chi G^0G^0\right)=2\left(\lambda_1+\lambda_3\right)\mathrm{c}_\beta^2+3\lambda_4\mathrm{s}_{2\beta}+\lambda_a\mathrm{s}_\beta^2, \tag{5.5.11m}$$

$$g\left(\tilde\chi\tilde\chi G^0A\right)=-\frac{1}{2}\left(2\lambda_1+2\lambda_3-\lambda_a\right)\mathrm{s}_{2\beta}+3\lambda_4\mathrm{c}_{2\beta}, \tag{5.5.11n}$$

$$g\left(\tilde\eta\tilde\chi hG^0\right)=\mathrm{c}_\alpha\left[2\left(\lambda_2+\lambda_3\right)\mathrm{c}_\beta+\lambda_4\mathrm{s}_\beta\right]+\mathrm{s}_\alpha\left(\lambda_4\mathrm{c}_\beta+2\lambda_7\mathrm{s}_\beta\right), \tag{5.5.11o}$$

$$g\left(\tilde\eta\tilde\chi HG^0\right)=-\mathrm{s}_\alpha\left[2\left(\lambda_2+\lambda_3\right)\mathrm{c}_\beta+\lambda_4\mathrm{s}_\beta\right]+\mathrm{c}_\alpha\left(\lambda_4\mathrm{c}_\beta+2\lambda_7\mathrm{s}_\beta\right). \tag{5.5.11p}$$

The quartic couplings involving the charged fields and the neutral fields along with the would-be Goldstone boson are:

$$g\left(\tilde\eta\tilde\eta G^\mp G^\pm\right)=2\left(\lambda_1-\lambda_3\right)\mathrm{c}_\beta^2+\lambda_4\mathrm{s}_{2\beta}+\lambda_5\mathrm{s}_\beta^2, \tag{5.5.12a}$$

$$g\left(\tilde\eta\tilde\eta G^\mp H^\pm\right)=\lambda_4\mathrm{c}_{2\beta}+\frac{1}{2}\left(-2\lambda_1+2\lambda_3+\lambda_5\right)\mathrm{s}_{2\beta}, \tag{5.5.12b}$$

$$g\left(\tilde\chi\tilde\chi G^\mp G^\pm\right)=2\left(\lambda_1-\lambda_3\right)\mathrm{c}_\beta^2+\lambda_4\mathrm{s}_{2\beta}+\lambda_5\mathrm{s}_\beta^2, \tag{5.5.12c}$$

$$g\left(\tilde\chi\tilde\chi G^\mp H^\pm\right)=\lambda_4\mathrm{c}_{2\beta}+\frac{1}{2}\left(-2\lambda_1+2\lambda_3+\lambda_5\right)\mathrm{s}_{2\beta}, \tag{5.5.12d}$$

$$g\left(\tilde\eta hh^\mp G^\pm\right)=\frac{1}{2}\left[2\mathrm{c}_\alpha\left(2\lambda_3\mathrm{c}_\beta+\lambda_4\mathrm{s}_\beta\right)+\mathrm{s}_\alpha\left(2\lambda_4\mathrm{c}_\beta+\left(\lambda_6+2\lambda_7\right)\mathrm{s}_\beta\right)\right], \tag{5.5.12e}$$

$$g\left(\tilde\eta Hh^\mp G^\pm\right)=\frac{1}{2}\mathrm{c}_\alpha\left[2\lambda_4\mathrm{c}_\beta+\left(\lambda_6+2\lambda_7\right)\mathrm{s}_\beta\right]-\mathrm{s}_\alpha\left(2\lambda_3\mathrm{c}_\beta+\lambda_4\mathrm{s}_\beta\right), \tag{5.5.12f}$$

$$g\left(\tilde\chi hh^\mp G^\pm\right)=\pm i\left[-2\lambda_2\mathrm{c}_\alpha\mathrm{c}_\beta+\frac{1}{2}\left(\lambda_6-2\lambda_7\right)\mathrm{s}_\alpha\mathrm{s}_\beta\right], \tag{5.5.12g}$$

$$g\left(\tilde\chi Hh^\mp G^\pm\right)=\pm i\left[2\lambda_2\mathrm{c}_\beta\mathrm{s}_\alpha+\frac{1}{2}\left(\lambda_6-2\lambda_7\right)\mathrm{c}_\alpha\mathrm{s}_\beta\right], \tag{5.5.12h}$$

$$g\left(G^0G^0h^\pm h^\mp\right)=2\left(\lambda_1-\lambda_3\right)\mathrm{c}_\beta^2+\lambda_4\mathrm{s}_{2\beta}+\lambda_5\mathrm{s}_\beta^2, \tag{5.5.12i}$$

$$g\left(G^0Ah^\pm h^\mp\right)=\lambda_4\mathrm{c}_{2\beta}+\frac{1}{2}\left(-2\lambda_1+2\lambda_3+\lambda_5\right)\mathrm{s}_{2\beta}, \tag{5.5.12j}$$

$$g\left(G^0\tilde\eta h^\pm G^\mp\right)=\pm i\left[-2\lambda_2\mathrm{c}_\beta^2+\frac{1}{2}\left(\lambda_6-2\lambda_7\right)\mathrm{s}_\beta^2\right], \tag{5.5.12k}$$

$$g\left(G^0\tilde{\eta}h^\pm H^\mp\right) = g\left(A\tilde{\eta}h^\pm G^\pm\right) = \pm\frac{1}{4}i\left(4\lambda_2+\lambda_6-2\lambda_7\right)\mathrm{s}_{2\beta}, \tag{5.5.12l}$$

$$g\left(G^0\tilde{\chi}h^\pm G^\pm\right) = \frac{1}{2}\left[4\lambda_3\mathrm{c}_\beta^2+2\lambda_4\mathrm{s}_{2\beta}+\left(\lambda_6+2\lambda_7\right)\mathrm{s}_\beta^2\right], \tag{5.5.12m}$$

$$g\left(G^0\tilde{\chi}h^\pm H^\pm\right) = g\left(A\tilde{\chi}h^\pm G^\pm\right) = \frac{1}{4}\left[4\lambda_4\mathrm{c}_{2\beta}+\left(-4\lambda_3+\lambda_6+2\lambda_7\right)\mathrm{s}_{2\beta}\right], \tag{5.5.12n}$$

$$g\left(G^0hG^\mp H^\pm\right) = g\left(AHG^\mp H^\pm\right) = \pm\frac{1}{2}i\left(\lambda_6-2\lambda_7\right)\mathrm{s}_{\alpha-\beta}, \tag{5.5.12o}$$

$$g\left(G^0HG^\mp H^\pm\right) = g\left(AhG^\pm H^\mp\right) = \pm\frac{1}{2}i\left(\lambda_6-2\lambda_7\right)\mathrm{c}_{\alpha-\beta}, \tag{5.5.12p}$$

$$\begin{aligned} g\left(hhG^\mp G^\pm\right) =& \mathrm{c}_\alpha^2\left[2\left(\lambda_1+\lambda_3\right)\mathrm{c}_\beta^2-\lambda_4\mathrm{s}_{2\beta}+\lambda_5\mathrm{s}_\beta^2\right]-\mathrm{c}_\beta\mathrm{s}_{2\alpha}\left(\lambda_4\mathrm{c}_\beta-\left(\lambda_6+2\lambda_7\right)\mathrm{s}_\beta\right)\\ &+\mathrm{s}_\alpha^2\left(\lambda_5\mathrm{c}_\beta^2+2\lambda_8\mathrm{s}_\beta^2\right), \end{aligned} \tag{5.5.12q}$$

$$\begin{aligned} g\left(hhG^\mp H^\pm\right) =& \frac{1}{2}\Big[-2\mathrm{c}_\alpha\mathrm{c}_{2\beta}\left(\lambda_4\mathrm{c}_\alpha-\left(\lambda_6+2\lambda_7\right)\mathrm{s}_\alpha\right)\\ &-\mathrm{s}_{2\beta}\left(\lambda_1+\lambda_3-\lambda_4\mathrm{s}_{2\alpha}-\lambda_8+\left(\lambda_1+\lambda_3-\lambda_5+\lambda_8\right)\mathrm{c}_{2\alpha}\right)\Big], \end{aligned} \tag{5.5.12r}$$

$$\begin{aligned} g\left(hHG^\mp G^\pm\right) =& -\frac{1}{2}\mathrm{s}_{2\alpha}\left[\lambda_1+\lambda_3-\lambda_4\mathrm{s}_{2\beta}-\lambda_8+\left(\lambda_1+\lambda_3-\lambda_5+\lambda_8\right)\mathrm{c}_{2\beta}\right]\\ &+\mathrm{c}_\beta\mathrm{s}_\alpha^2\left[\lambda_4\mathrm{c}_\beta-\left(\lambda_6+2\lambda_7\right)\mathrm{s}_\beta\right]-\mathrm{c}_\alpha^2\mathrm{c}_\beta\left[\lambda_4\mathrm{c}_\beta-\left(\lambda_6+2\lambda_7\right)\mathrm{s}_\beta\right], \end{aligned} \tag{5.5.12s}$$

$$g\left(hHG^\mp H^\pm\right) = \frac{1}{4}\left[2\lambda_4\mathrm{s}_{2(\alpha+\beta)}+2\left(\lambda_6+2\lambda_7\right)\mathrm{c}_{2\alpha}\mathrm{c}_{2\beta}+2\left(\lambda_1+\lambda_3-\lambda_5+\lambda_8\right)\mathrm{s}_{2\alpha}\mathrm{s}_{2\beta}\right], \tag{5.5.12t}$$

$$\begin{aligned} g\left(HHG^\mp G^\pm\right) =& -\lambda_4\mathrm{s}_\alpha^2\mathrm{s}_{2\beta}+\lambda_5\mathrm{s}_\alpha^2\mathrm{s}_\beta^2-\frac{1}{2}\lambda_6\mathrm{s}_{2\alpha}\mathrm{s}_{2\beta}-\lambda_7\mathrm{s}_{2\alpha}\mathrm{s}_{2\beta}\\ &+\mathrm{c}_\alpha^2\left(\lambda_5\mathrm{c}_\beta^2+2\lambda_8\mathrm{s}_\beta^2\right)+2\mathrm{c}_\beta^2\mathrm{s}_\alpha\left[\left(\lambda_1+\lambda_3\right)\mathrm{s}_\alpha+\lambda_4\mathrm{c}_\alpha\right], \end{aligned} \tag{5.5.12u}$$

$$\begin{aligned} g\left(HHG^\mp H^\pm\right) =& -\frac{1}{2}\left(\lambda_5-2\lambda_8\right)\mathrm{c}_\alpha^2\mathrm{s}_{2\beta}-\frac{1}{2}\mathrm{s}_{2\alpha}\left[\lambda_4\mathrm{s}_{2\beta}+\left(\lambda_6+2\lambda_7\right)\mathrm{c}_{2\beta}\right]\\ &-\mathrm{s}_\alpha^2\left[\lambda_4\mathrm{c}_{2\beta}+\frac{1}{2}\left(2\left(\lambda_1+\lambda_3\right)-\lambda_5\right)\mathrm{s}_{2\beta}\right], \end{aligned} \tag{5.5.12v}$$

and

$$g\left(G^0G^0G^\mp G^\pm\right) = 2\left[\left(\lambda_1+\lambda_3\right)\mathrm{c}_\beta^4-2\lambda_4\mathrm{c}_\beta^3\mathrm{s}_\beta+\lambda_a\mathrm{c}_\beta^2\mathrm{s}_\beta^2+\lambda_8\mathrm{s}_\beta^4\right], \tag{5.5.13a}$$

$$g\left(G^0G^0G^\mp H^\pm\right) = -\mathrm{c}_\beta\left[\lambda_4\mathrm{c}_{3\beta}+\mathrm{s}_\beta\left(\lambda_1+\lambda_3-\lambda_8+\left(\lambda_1+\lambda_3-\lambda_a+\lambda_8\right)\mathrm{c}_{2\beta}\right)\right], \tag{5.5.13b}$$

$$g\left(G^0G^0H^\mp H^\pm\right) = \frac{1}{4}\left[\lambda_1+\lambda_3+3\lambda_5-\lambda_6-2\lambda_7+\lambda_8+2\lambda_4\mathrm{s}_{4\beta}-\left(\lambda_1+\lambda_3-\lambda_a+\lambda_8\right)\mathrm{c}_{4\beta}\right], \tag{5.5.13c}$$

$$g\left(G^0AG^\mp G^\pm\right) = -\mathrm{c}_\beta\left[\lambda_1+\lambda_3-\lambda_8+\lambda_4\mathrm{c}_{3\beta}+\mathrm{s}_\beta\left(\left(\lambda_1+\lambda_3-\lambda_a+\lambda_8\right)\mathrm{c}_{2\beta}\right)\right], \tag{5.5.13d}$$

$$g\left(G^0AG^\mp H^\pm\right) = \frac{1}{4}\left[\lambda_1+\lambda_3-\lambda_5+\lambda_6+2\lambda_7+\lambda_8+2\lambda_4\mathrm{s}_{4\beta}-\left(\lambda_1+\lambda_3-\lambda_a+\lambda_8\right)\mathrm{c}_{4\beta}\right], \tag{5.5.13e}$$

$$g\left(G^0AH^\mp H^\pm\right) = -\frac{1}{2}\mathrm{s}_\beta\left[2\lambda_4\mathrm{s}_{3\beta}+\left(\lambda_a+\lambda_1+\lambda_3-3\lambda_8\right)\mathrm{c}_\beta-\left(\lambda_1+\lambda_3-\lambda_a+\lambda_8\right)\mathrm{c}_{3\beta}\right], \tag{5.5.13f}$$

$$g\left(AAG^\mp G^\pm\right) = \frac{1}{4}\left[\lambda_1+\lambda_3+2\lambda_4\mathrm{s}_{4\beta}+3\lambda_5-\lambda_6-2\lambda_7+\lambda_8-\left(\lambda_1+\lambda_3-\lambda_a+\lambda_8\right)\mathrm{c}_{4\beta}\right], \tag{5.5.13g}$$

$$g\left(AAH^\mp G^\pm\right) = -\frac{1}{2}\mathrm{s}_\beta\left[2\lambda_4\mathrm{s}_{3\beta}+\left(\lambda_a+\lambda_1+\lambda_3-3\lambda_8\right)\mathrm{c}_\beta-\left(\lambda_1+\lambda_3-\lambda_a+\lambda_8\right)\mathrm{c}_{3\beta}\right]. \tag{5.5.13h}$$

The quartic couplings involving only the charged fields and the would-be Goldstone boson are:

$$g\left(G^\mp G^\mp G^\pm G^\pm\right) = 4\left[\left(\lambda_1+\lambda_3\right)\mathrm{c}_\beta^4-2\lambda_4\mathrm{c}_\beta^3\mathrm{s}_\beta+\lambda_a\mathrm{c}_\beta^2\mathrm{s}_\beta^2+\lambda_8\mathrm{s}_\beta^4\right], \tag{5.5.14a}$$

$$g\left(G^\mp G^\mp G^\pm H^\pm\right) = -\mathrm{c}_\beta\left[2\lambda_4\mathrm{c}_{3\beta}+2\mathrm{s}_\beta\left(\lambda_1+\lambda_3-\lambda_8+\left(\lambda_1+\lambda_3-\lambda_a+\lambda_8\right)\mathrm{c}_{2\beta}\right)\right], \tag{5.5.14b}$$

$$g\left(G^\mp G^\mp H^\pm H^\pm\right) = \frac{1}{2}\left[\lambda_1+\lambda_3-\lambda_5-\lambda_6+6\lambda_7+\lambda_8+2\lambda_4\mathrm{s}_{4\beta}-\left(\lambda_1+\lambda_3-\lambda_a+\lambda_8\right)\mathrm{c}_{4\beta}\right], \tag{5.5.14c}$$

$$g\left(G^{\mp}H^{\mp}G^{\pm}H^{\pm}\right)=\frac{1}{2}\left[\lambda_1+\lambda_3+\lambda_8+2\lambda_4\mathrm{s}_{4\beta}+\lambda_b-\left(\lambda_1+\lambda_3-\lambda_a+\lambda_8\right)\mathrm{c}_{4\beta}\right], \tag{5.5.14d}$$

$$g\left(G^{\mp}H^{\mp}H^{\pm}H^{\pm}\right)=-\mathrm{s}_{\beta}\left[2\lambda_4\mathrm{s}_{3\beta}+\left(\lambda_a+\lambda_1+\lambda_3-3\lambda_8\right)\mathrm{c}_{\beta}-\left(\lambda_1+\lambda_3-\lambda_a+\lambda_8\right)\mathrm{c}_{3\beta}\right], \tag{5.5.14e}$$

$$g\left(h^{\pm}h^{\pm}G^{\mp}G^{\mp}\right)=4\left[\left(\lambda_2+\lambda_3\right)\mathrm{c}_{\beta}^2+\frac{1}{2}\lambda_4\mathrm{s}_{2\beta}+\lambda_7\mathrm{s}_{\beta}^2\right], \tag{5.5.14f}$$

$$g\left(h^{\pm}h^{\pm}G^{\mp}H^{\mp}\right)=2\lambda_4\mathrm{c}_{2\beta}-2\left(\lambda_2+\lambda_3-\lambda_7\right)\mathrm{s}_{2\beta}, \tag{5.5.14g}$$

$$g\left(h^{\pm}h^{\mp}G^{\pm}G^{\mp}\right)=2\left(\lambda_1-\lambda_2\right)\mathrm{c}_{\beta}^2+2\lambda_4\mathrm{s}_{2\beta}+\left(\lambda_5+\lambda_6\right)\mathrm{s}_{\beta}^2, \tag{5.5.14h}$$

$$g\left(h^{\pm}h^{\mp}H^{\pm}G^{\mp}\right)=2\lambda_4\mathrm{c}_{2\beta}+\frac{1}{2}\left(-2\lambda_1+2\lambda_2+\lambda_5+\lambda_6\right)\mathrm{s}_{2\beta}. \tag{5.5.14i}$$

From the trilinear couplings involving the same species particles it follows that the states h and H are CP-even. From the trilinear couplings involving only the neutral states we get that A is CP-odd. Although the states $\tilde{\eta}$ and $\tilde{\chi}$ do not mix, it is impossible to determine their CP properties from the couplings.

5.6 Constraints

Necessary[4] stability constraints of Ref. [37] are considered, see eq. (4.5.10). Alongside, if the necessary stability conditions are satisfied, we perform an additional numerical minimization of the potential using the Mathematica function NMinimize.

In section 4.5.5 we verified that the unitarity constraints for $\lambda_4=0$ are in agreement with those of Ref. [37]. We also verify that we get the same eigenvalues of the S matrix when $\lambda_4\neq 0$. For convenience, we present the eigenvalues of Ref. [37]:

$$a_1^{\pm}=\left(\lambda_1-\lambda_2+\frac{\lambda_5+\lambda_6}{2}\right)\pm\sqrt{\left(\lambda_1-\lambda_2+\frac{\lambda_5+\lambda_6}{2}\right)^2-4\left\{\left(\lambda_1-\lambda_2\right)\left(\frac{\lambda_5+\lambda_6}{2}\right)-\lambda_4^2\right\}}, \tag{5.6.1a}$$

$$a_2^{\pm}=\left(\lambda_1+\lambda_2+2\lambda_3+\lambda_8\right)\pm\sqrt{\left(\lambda_1+\lambda_2+2\lambda_3+\lambda_8\right)^2-4\left\{\lambda_8\left(\lambda_1+\lambda_2+2\lambda_3\right)-2\lambda_7^2\right\}}, \tag{5.6.1b}$$

$$a_3^{\pm}=\left(\lambda_1-\lambda_2+2\lambda_3+\lambda_8\right)\pm\sqrt{\left(\lambda_1-\lambda_2+2\lambda_3+\lambda_8\right)^2-4\left\{\lambda_8\left(\lambda_1-\lambda_2+2\lambda_3\right)-\frac{\lambda_6^2}{2}\right\}}, \tag{5.6.1c}$$

$$a_4^{\pm}=\left(\lambda_1+\lambda_2+\frac{\lambda_5}{2}+\lambda_7\right)\pm\sqrt{\left(\lambda_1+\lambda_2+\frac{\lambda_5}{2}+\lambda_7\right)^2-4\left\{\left(\lambda_1+\lambda_2\right)\left(\frac{\lambda_5}{2}+\lambda_7\right)-\lambda_4^2\right\}}, \tag{5.6.1d}$$

$$\begin{aligned}a_5^{\pm}&=\left(5\lambda_1-\lambda_2+2\lambda_3+3\lambda_8\right)\\&\quad\pm\sqrt{\left(5\lambda_1-\lambda_2+2\lambda_3+3\lambda_8\right)^2-4\left\{3\lambda_8\left(5\lambda_1-\lambda_2+2\lambda_3\right)-\frac{1}{2}\left(2\lambda_5+\lambda_6\right)^2\right\}},\end{aligned} \tag{5.6.1e}$$

$$\begin{aligned}a_6^{\pm}&=\left(\lambda_1+\lambda_2+4\lambda_3+\frac{\lambda_5}{2}+\lambda_6+3\lambda_7\right)\\&\quad\pm\sqrt{\left(\lambda_1+\lambda_2+4\lambda_3+\frac{\lambda_5}{2}+\lambda_6+3\lambda_7\right)^2-4\left\{\left(\lambda_1+\lambda_2+4\lambda_3\right)\left(\frac{\lambda_5}{2}+\lambda_6+3\lambda_7\right)-9\lambda_4^2\right\}},\end{aligned} \tag{5.6.1f}$$

$$b_1=\lambda_5+2\lambda_6-6\lambda_7, \tag{5.6.1g}$$

$$b_2=\lambda_5-2\lambda_7, \tag{5.6.1h}$$

$$b_3=2\left(\lambda_1-5\lambda_2-2\lambda_3\right), \tag{5.6.1i}$$

$$b_{4,5}=2\left(\lambda_1\pm\lambda_2-2\lambda_3\right), \tag{5.6.1j}$$

[4] The stability conditions are necessary and not sufficient as $\lambda_4\neq 0$, see Ref. [34].

$$b_6 = \lambda_5 - \lambda_6. \tag{5.6.1k}$$

The unitarity constraint is thus:

$$|a_i^{\pm}| \leq 16\pi, \quad |b_i| \leq 16\pi. \tag{5.6.2}$$

The soft perturbativity condition (4.5.18) along with a more severe perturbativity condition in terms of the quartic couplings eq. (4.5.22) are considered. Most of the quartic couplings depend on trigonometric functions and therefore it is not a trivial task to extract limits in terms of λ_i from such terms. By inspecting the coupling $g\left(\tilde{\eta}\tilde{\eta}\tilde{\eta}\tilde{\eta}\right)$ (5.5.5a) the following relation can be extracted: $|\lambda_1 + \lambda_3| \leq \frac{2}{3}\pi$.

The λ_i couplings can be constrained by the mass-squared parameters. From $m_{\tilde{\eta}}^2 > 0$ we get that $\lambda_4 > 0$. We consider that $m_{\tilde{\eta}}^2 < m_{\tilde{\chi}}^2$ and thus:

$$\lambda_2 + \lambda_3 + \lambda_4 \mathrm{t}_\beta + \lambda_7 \mathrm{t}_\beta^2 < 0. \tag{5.6.3}$$

From the m_A^2 mass-squared parameter it follows that:

$$\lambda_4 - 4\lambda_7 \mathrm{t}_\beta > 0. \tag{5.6.4}$$

The inert scalar sector is bounded by the necessary stability conditions, mainly that the terms $\sqrt{\cdots}$ (4.5.10) should be positive definite:

$$9\mathrm{t}_\beta\left(m_h^2 \mathrm{c}_\alpha^2 + m_H^2 \mathrm{s}_\alpha^2\right) > m_{\tilde{\eta}}^2, \tag{5.6.5a}$$

$$m_h^2 \mathrm{s}_\alpha^2 + m_H^2 \mathrm{c}_\alpha^2 + m_A^2 \mathrm{s}_\beta^2 - 2m_{H^+}^2 \mathrm{s}_\beta^2 > \frac{1}{9} m_{\tilde{\eta}}^2 + m_{\tilde{\chi}}^2. \tag{5.6.5b}$$

We use these constraints to numerically bound the mass terms $m_{\tilde{\eta}}$ and $m_{\tilde{\chi}}$ from above.

5.6.1 Electroweak Oblique Parameters

The $SU(2)$ doublets in the Higgs basis were presented in eq. (5.2.2). The rotation matrix for the charged sector is:

$$\begin{pmatrix} G^{\pm} \\ H^{\pm} \\ h^{\pm} \end{pmatrix} = U \begin{pmatrix} G^{\pm} \\ H^{\pm} \\ h^{\pm} \end{pmatrix}, \tag{5.6.6}$$

and thus U is simply an identity matrix $\mathcal{I}_3$. For the neutral sector we get:

$$\begin{pmatrix} \left(h\,\mathrm{c}_{\beta-\alpha} + H\,\mathrm{s}_{\beta-\alpha} + iG^0\right) \\ \left(-h\,\mathrm{s}_{\beta-\alpha} + H\,\mathrm{c}_{\beta-\alpha} + iA\right) \\ \left(\tilde{\eta} + i\tilde{\chi}\right) \end{pmatrix} = V \begin{pmatrix} G^0 \\ A \\ h \\ H \\ \tilde{\eta} \\ \tilde{\chi} \end{pmatrix}, \tag{5.6.7}$$

where

$$V = \begin{pmatrix} i & 0 & \mathrm{c}_{\beta-\alpha} & \mathrm{s}_{\beta-\alpha} & 0 & 0 \\ 0 & i & -\mathrm{s}_{\beta-\alpha} & \mathrm{c}_{\beta-\alpha} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & i \end{pmatrix}. \tag{5.6.8}$$

The expressions for the S, T, U parameters are as follows:

$$\begin{aligned} T = \frac{g^2}{64 m_W^2 \pi^2 \alpha} \Big[& F\left(m_{h^\pm}^2, m_{\tilde{\eta}}^2\right) + F\left(m_{h^\pm}^2, m_{\tilde{\chi}}^2\right) + \mathrm{s}_{\beta-\alpha}^2 F\left(m_{H^\pm}^2, m_h^2\right) + \mathrm{c}_{\beta-\alpha}^2 F\left(m_{H^\pm}^2, m_H^2\right) \\ & + F\left(m_{H^\pm}^2, m_A^2\right) - F\left(m_{\tilde{\eta}}^2, m_{\tilde{\chi}}^2\right) - \mathrm{s}_{\beta-\alpha}^2 F\left(m_h^2, m_A^2\right) - \mathrm{c}_{\beta-\alpha}^2 F\left(m_H^2, m_A^2\right) \\ & + 3\mathrm{s}_{\beta-\alpha}^2 \left(F\left(m_{W^\pm}^2, m_h^2\right) - F\left(m_Z^2, m_h^2\right) + F\left(m_Z^2, m_H^2\right) - F\left(m_{W^\pm}^2, m_H^2\right)\right)\Big], \end{aligned} \tag{5.6.9a}$$

$$
\begin{aligned}
S = \frac{g^2 \mathrm{s}_w^2}{96\pi^2\alpha} \Big[& \mathrm{c}_{2w}^2 \left(G\left(m_{h^\pm}^2, m_{h^\pm}^2, m_Z^2\right) + G\left(m_{H^\pm}^2, m_{H^\pm}^2, m_Z^2\right)\right) + G\left(m_{\tilde{\eta}}^2, m_{\tilde{\chi}}^2, m_Z^2\right) \\
& + \mathrm{s}_{\beta-\alpha}^2 G\left(m_h^2, m_A^2, m_Z^2\right) + \mathrm{c}_{\beta-\alpha}^2 G\left(m_H^2, m_A^2, m_Z^2\right) - 2\ln\!\left(m_{h^\pm}^2\right) - 2\ln\!\left(m_{H^\pm}^2\right) \\
& + \ln\!\left(m_{\tilde{\eta}}^2\right) + \ln\!\left(m_{\tilde{\chi}}^2\right) + \ln\!\left(m_H^2\right) + \ln\!\left(m_A^2\right) + \mathrm{s}_{\beta-\alpha}^2 \Big(\hat{G}\left(m_H^2, m_Z^2\right) - \hat{G}\left(m_h^2, m_Z^2\right)\Big)\Big] ,
\end{aligned}
\tag{5.6.9b}
$$

$$
\begin{aligned}
U = \frac{g^2 \mathrm{s}_w^2}{96\pi^2\alpha} \Big[& G\left(m_{h^\pm}^2, m_{\tilde{\eta}}^2, m_{W^\pm}^2\right) + G\left(m_{h^\pm}^2, m_{\tilde{\chi}}^2, m_{W^\pm}^2\right) + \mathrm{s}_{\beta-\alpha}^2 G\left(m_{H^\pm}^2, m_h^2, m_{W^\pm}^2\right) \\
& + \mathrm{c}_{\beta-\alpha}^2 G\left(m_{H^\pm}^2, m_H^2, m_{W^\pm}^2\right) + G\left(m_{H^\pm}^2, m_A^2, m_{W^\pm}^2\right) - \mathrm{c}_{2w}^2 G\left(m_{h^\pm}^2, m_{h^\pm}^2, m_Z^2\right) \\
& - \mathrm{c}_{2w}^2 G\left(m_{H^\pm}^2, m_{H^\pm}^2, m_Z^2\right) - G\left(m_{\tilde{\eta}}^2, m_{\tilde{\chi}}^2, m_Z^2\right) - \mathrm{s}_{\beta-\alpha}^2 G\left(m_h^2, m_A^2, m_Z^2\right) \\
& - \mathrm{c}_{\beta-\alpha}^2 G\left(m_H^2, m_A^2, m_Z^2\right) + \mathrm{s}_{\beta-\alpha}^2 \Big(\hat{G}\left(m_h^2, m_Z^2\right) - \hat{G}\left(m_h^2, m_{W^\pm}^2\right)\Big) \\
& + \mathrm{s}_{\beta-\alpha}^2 \Big(\hat{G}\left(m_H^2, m_{W^\pm}^2\right) - \hat{G}\left(m_H^2, m_Z^2\right)\Big)\Big] ,
\end{aligned}
\tag{5.6.9c}
$$

see section 4.5.6 for the functions. The electroweak oblique parameters of the R-II-1a model resemble the case of the 2HDM model [81].

Chapter 6

Numerical Analysis

6.1 General Approach

After expressing interactions via physical scalar states in previous chapters, we proceed with numerical evaluation of the C-III-c-ν^2 and the R-II-1a models. We use Mathematica for the spectrum generator. The main input is specified in terms of the physical scalar masses and additional parameters based on the considered model. For the C-III-c-ν^2 model the following input is used:

$$\left\{ m_{H^\pm},\, m_{S^\pm}, m_{H_2},\, m_{H_3},\, m_{S_1},\, m_{S_2},\, \mu_0^2,\, \lambda_8 \right\}, \tag{6.1.1}$$

and for the R-II-1a model we use:

$$\left\{ m_{H^\pm},\, m_{h^\pm}, m_H,\, m_A,\, m_{\tilde{\eta}},\, m_{\tilde{\chi}},\, \beta,\, \alpha \right\}. \tag{6.1.2}$$

Both cases result in an $\mathbb{R}^8$ surface. We do not treat the SM-like Higgs boson mass as a free parameter, it is fixed at the value $m_{H_1} = m_h = 125.09$ GeV. The mass parameters are assumed to be in the range $m_\xi = [0.1,\, 1]$ TeV.

The spectrum generator outputs data of the scalar and fermion sectors based on several checks:

- Stability of the scalar potential;
- Tree-level unitarity;
- Soft perturbativity and the more severe perturbativity conditions based on the quartic scalar-scalar interactions;
- The electroweak oblique parameters;
- Limitations from the CKM matrix;

The C-III-c-ν^2 model constraints were discussed in section 4.5 and the R-II-1a model constraints in section 5.6. For the electroweak oblique parameters we apply direct constraints from the Gfitter group (4.5.39) without the correlation coefficients. Although the absolute value of $V_{\rm CKM}$ is a well-known quantity, neither the C-III-c-ν^2 nor the R-II-1a models result in realistic cases. Since we are solely interested in a possible DM candidate, and not a truly realistic fermionic sector, the off-diagonal couplings should not play a significant role. For the charged scalar decays we assume that the CKM matrix is approximated by the identity matrix, $V_{\rm CKM} = \mathcal{I}_3$, and for the decays involving $W^\pm$ the standard $V_{\rm CKM}$ [29] is used. The general approach is presented in Figure 6.1.

For the relic density evaluation we use micrOMEGAs [82–84]. The freeze-out scenario is considered, the three-body final states are computed for annihilation processes only, VW/VZdecay = 1, the effective vertices $\varphi_i gg$ and $\varphi_i \gamma\gamma$ are not considered. We do not focus on the details of the decays involving the DM. In order to use micrOMEGAs, all of the vertices should be specified. This is not a trivial task. We use SARAH [85, 86] to produce CalcHEP [87] model files that can be subsequently used by micrOMEGAs. The relic density is compared against the Planck [88] results $\Omega_{\rm CDM} h^2 = 0.120 \pm 0.001$. The PDG [29] provides a value $\Omega_{\rm CDM} h^2 = 0.1186 \pm 0.0020$ based

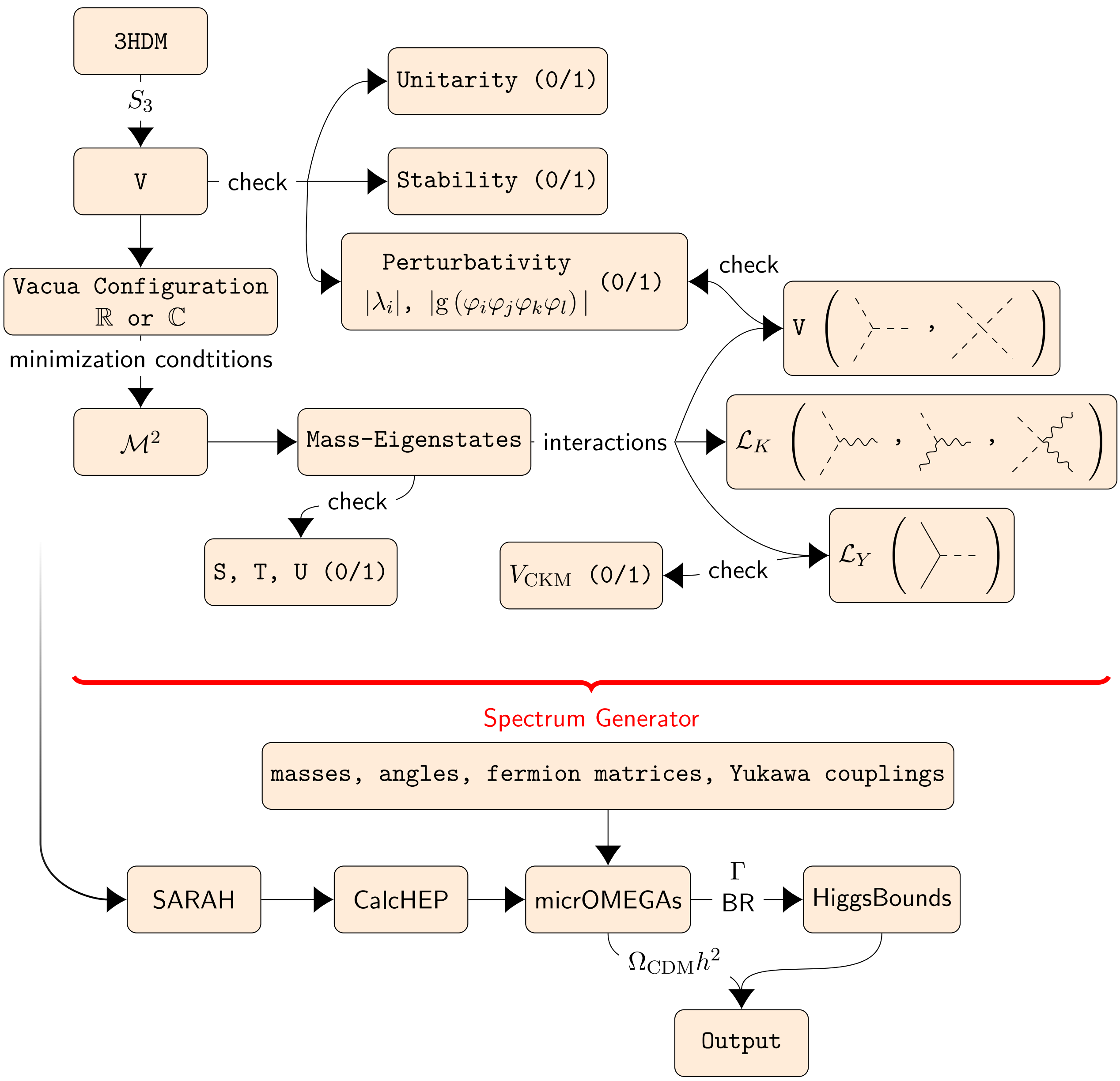

Figure 6.1: The general algorithm of the spectrum generator and additional checks. The "(0/1)" of the spectrum generator indicates the fail/pass switch.

on Planck TT+lowP+lensing and $\Omega_{\rm CDM}h^2 = 0.1184 \pm 0.0012$ based on Planck TT+lowP+lensing+ext. Earlier observations from WMAP [89] resulted in $\Omega_{\rm CDM}h^2 = 0.1147 \pm 0.0051$. We consider a broader acceptable relic density range, $\Omega_{\rm CDM}h^2 = 0.12 \pm 0.01$.

We compare our models against some of the available Higgs boson experimental results. The micrOMEGAs code enables additional comparison against experimental constraints using the HiggsBounds [90–92] code.

The following versions of the codes are used:

- SARAH 4.14.1[1];
- micrOMEGAs 5.0.8;
- HiggsBounds 5.3.2;

[1]We note that as of the current version there is a bug in the function CalcHepVertex of the file *Package/Outputs/calchep.m* when dealing with fields with complex phases and exporting a model using the MakeCHep function.

The full analysis of the results is beyond the scope of the thesis and therefore preliminary results are presented in Figures 6.2 to 6.8. The data should be further constrained.

6.2 The C-III-c-ν^2 Model

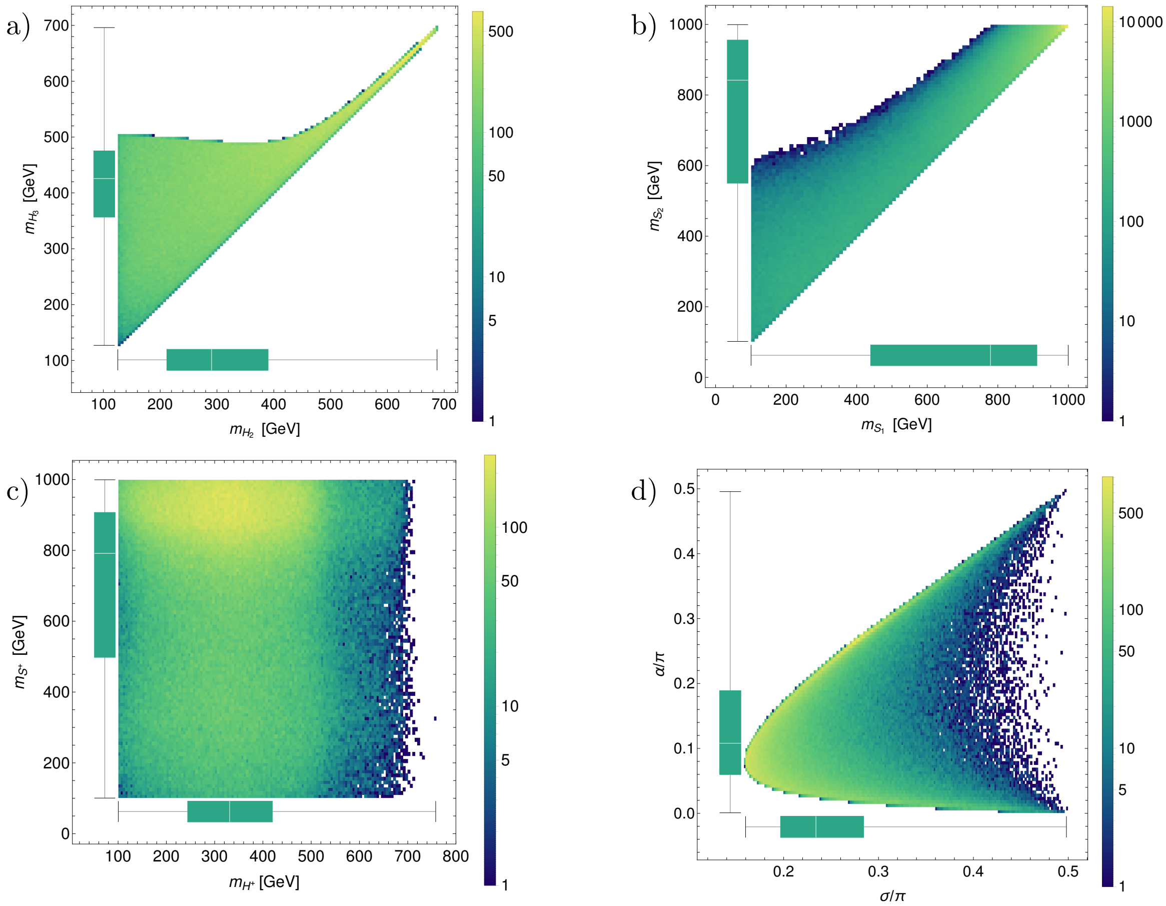

Figure 6.2: General output of the spectrum generator for the C-III-c-ν^2 model. Scatter plots of different parameters are presented: a) masses of the neutral states H_2 - H_3, b) masses of the neutral states S_1 - S_2, c) masses of the charged states $H^\pm$ - $S^\pm$, d) angles σ - α. The boxplot with whiskers indicates where $1/4$ - $3/4$ of the data points are situated along with the medians.

We present the general output of the spectrum generator in Figure 6.2. From the spectrum generator we can extract the following information on masses:

$$\begin{aligned} \max(m_{H_2}) &= 688 \text{ GeV}, \\ \max(m_{H_3}) &= 697 \text{ GeV}, \\ \max(m_{H^\pm}) &= 758 \text{ GeV}, \\ \max(\{m_{S_1}, m_{S_2}, m_{S^\pm}\}) &= 1000 \text{ GeV}, \end{aligned}$$

and the angles in radians lie in:

$$\begin{aligned} 0.159\pi < &\sigma < 0.498\pi, \\ 0 < &\alpha < 0.496\pi. \end{aligned}$$

The quartic couplings lie in the following ranges:

$$\begin{aligned}
0.067 <&\lambda_1/\pi < 1.596,\\
0.026 <&\lambda_2/\pi < 1.510,\\
-0.744 <&\lambda_3/\pi < 1.011,\\
-1.271 <&\lambda_5/\pi < 4,\\
-3.715 <&\lambda_6/\pi < 4,\\
0 <&\lambda_7/\pi < 1.918,\\
0 <&\lambda_8/\pi < \frac{2}{3}.
\end{aligned}$$

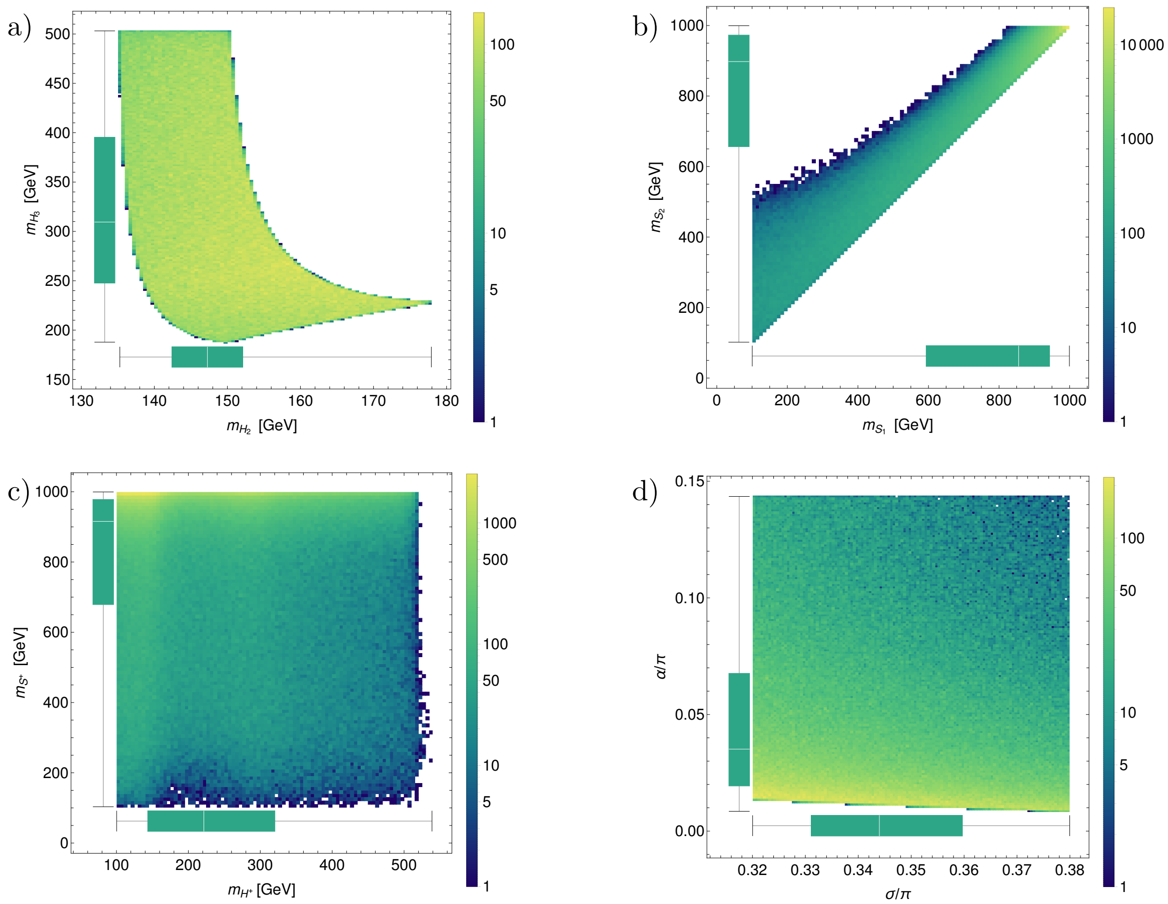

Figure 6.3: Output of the spectrum generator for the C-III-c-ν^2 model based on the SM-like limit and the V_{CKM} absolute value. Scatter plots of different parameters are presented: a) masses of the neutral states H_2 - H_3, b) masses of the neutral states S_1 - S_2, c) masses of the charged states $H^\pm$ - $S^\pm$, d) angles σ - α. The boxplot with whiskers indicates where $^1/_4$ - $^3/_4$ of the data points are situated along with the medians.

Not all of the points of Figure 6.2 are within an acceptable range. The α angle is fixed by the SM-like limit, see section 4.5.2, and the overall phase σ by V_{CKM} (4.3.32). After fixing the angles in the following range: $c_\alpha \geq 0.9$ and $\sigma = [0.34\pi, 0.38\pi]$, the m_{H_2} - m_{H_3} distribution is changed drastically, see Figure 6.3.

We assume that the DM candidate is the scalar S_1. After scanning for an acceptable range of the relic density $\Omega_{\text{CDM}}h^2$ with micrOMEGAs, we found that the annihilation channels are too efficient and neither of the Yukawa models resulted in an acceptable $\Omega_{\text{CDM}}h^2$ value. Moreover, the numerical value of the relic density is several orders of magnitudes lower than the acceptable one, $(\Omega_{\text{CDM}}h^2)_{\text{C-III-c-}\nu^2} < 10^{-3}$. As mentioned in section 4.3, the FCNC are also way too high. The branching ratios of the SM-like Higgs boson H_1 and of the off-diagonal fermion interactions are:

$$\begin{aligned} Y^u_{524}Y^d_{542}:&\ \text{Br}(H_1 \to \bar{f}_i f_j) \approx 10^{-3}, \\ Y^u_{542}Y^d_{524}:&\ \text{Br}(H_1 \to \bar{f}_i f_j) \approx 10^{-2}, \end{aligned} \tag{6.2.1}$$

and the leading decay channels of the other neutral active scalars are[2]:

$$\begin{aligned} &Y^u_{524}Y^d_{542}: && Y^u_{542}Y^d_{524}: \\ &\quad H_2 \to (db,\ cc,\ \mu\tau) && \quad H_2 \to (sb,\ uc,\ \mu\tau) \\ &\quad H_3 \to (H^\pm W^\mp,\ H_2 Z,\ ct) && \quad H_3 \to (H^\pm W^\mp,\ H_2 Z,\ ut) \end{aligned} \tag{6.2.2}$$

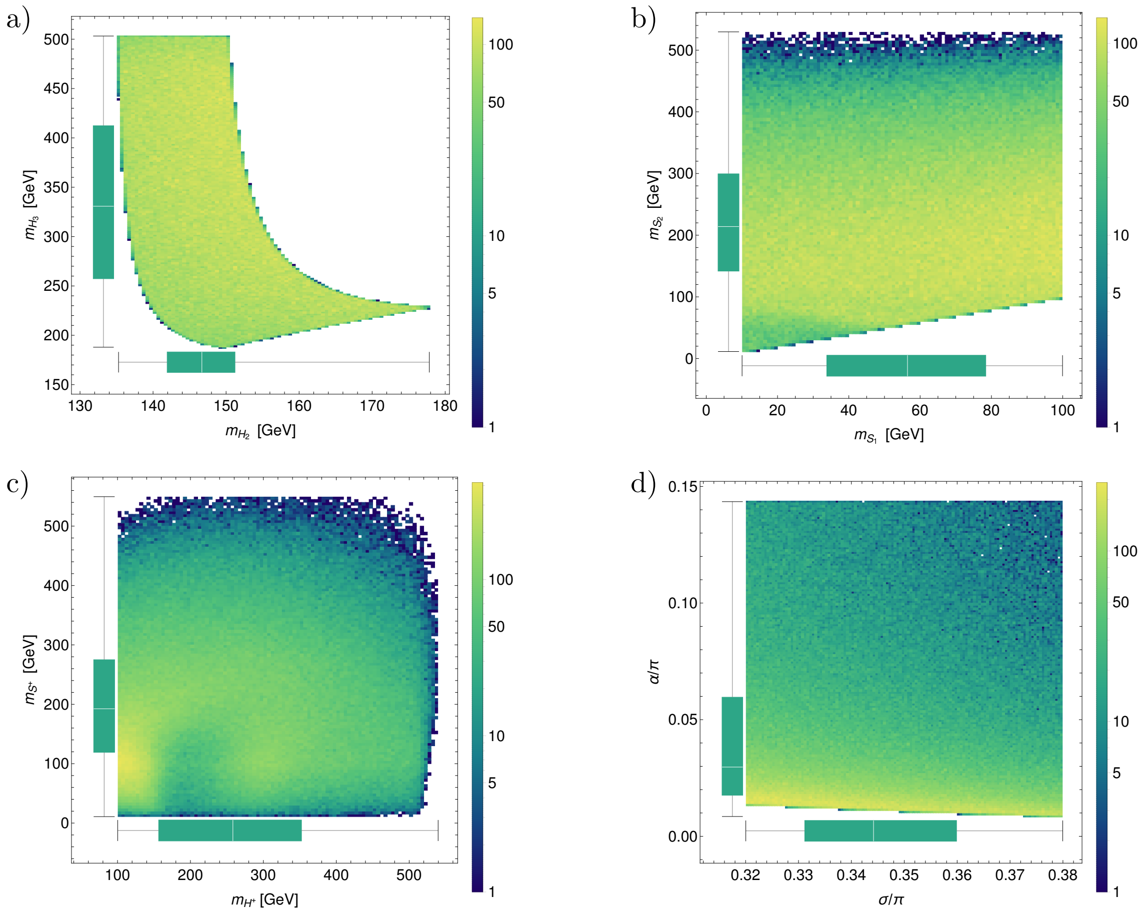

Figure 6.4: Output of the spectrum generator for the C-III-c-ν^2 model based on the SM-like limit and the V_{CKM} absolute value when the masses of the inert doublet h_S are allowed to be lower than 100 GeV. Scatter plots of different parameters are presented: a) masses of the neutral states H_2 - H_3, b) masses of the neutral states S_1 - S_2, c) masses of the charged states $H^\pm$ - $S^\pm$, d) angles σ - α. The boxplot with whiskers indicates where 1/4 - 3/4 of the data points are situated along with the medians.

We performed an additional scan of the data presented in Figure 6.2, when the angles α and β are not fixed. This did not result in a positive result from micrOMEGAs. Therefore, an additional scan of the area $m_{S_1} = [10,\ 100]$ GeV was performed, the corresponding masses of the inert doublet h_S were also allowed in the sub-100 GeV region. The scanned area can be seen in Figure 6.4. This resulted in an acceptable $\Omega_{\text{CDM}}h^2$ value. Nevertheless, this brings another issue: in many cases the primary decay channel of the SM-like Higgs boson H_1 is $\text{Br}(H_1 \to S_1 S_1) \approx (8 - 10) \times 10^{-1}$.

[2] The mass parameters of the scalars should be considered as not all of the decay channels are on-shell at a given scalar mass. The "bars" of the fermions $\bar{f}_i$ are dropped. Branching ratios are within one order of magnitude.

This area corresponds to the region $m_{S_1} = [10, 60)$ of Figure 6.5b. Only the right-side blob of Figure 6.5b may result in more acceptable SM-like Higgs boson decays.

We were not able to discriminate the scanned area based on different Yukawa models, *i.e.*, models $Y^u_{524}Y^d_{542}Y^e_{425}$ and $Y^u_{524}Y^d_{542}Y^e_{524}$, and therefore both models are incorporated in a single scatter plot of Figure 6.5. The HiggsBounds result is not considered as the model did not result in acceptable decay channels. Several possible DM candidates are presented in Table 6.1.

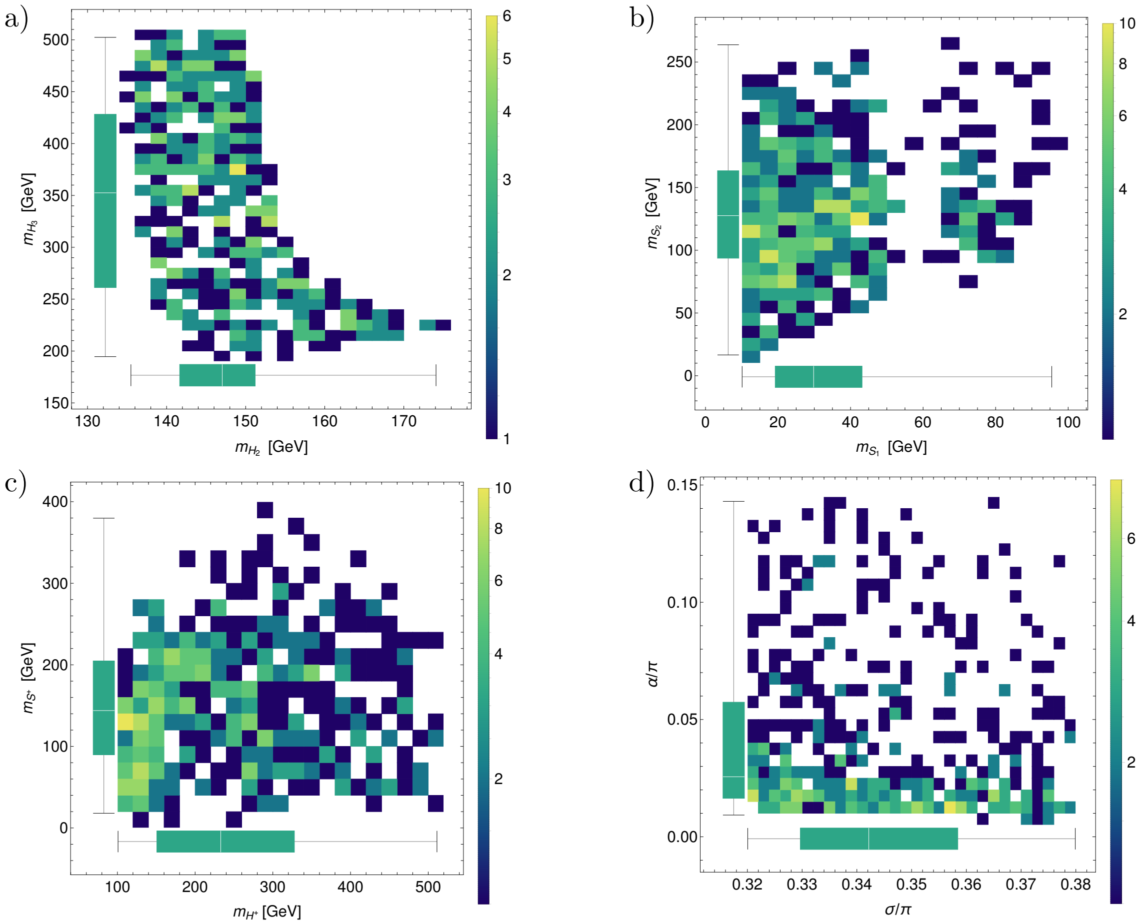

Figure 6.5: Constrained benchmark points of Figure 6.4 based on the acceptable $\Omega_{\mathrm{CDM}}h^2$ parameter range after performing a scan with micrOMEGAs. Yukawa models $Y^u_{524}Y^d_{542}Y^e_{425}$ and $Y^u_{524}Y^d_{542}Y^e_{524}$ are considered. Scatter plots of different parameters are presented: a) masses of the neutral states H_2 - H_3, b) masses of the neutral states S_1 - S_2, c) masses of the charged states $H^{\pm}$ - $S^{\pm}$, d) angles σ - α. The boxplot with whiskers indicates where 1/4 - 3/4 of the data points are situated along with the medians.

Benchmark point	Ωh^2	m_{H_2}	m_{H_3}	$m_{H^\pm}$	m_{S_1}	m_{S_2}	$m_{S^\pm}$	σ	α
A$_1$	0.1207	168.39	218.56	270.10	82.30	121.49	179.26	1.0493	0.4336
A$_2$	0.1202	149.42	427.74	376.22	90.82	180.14	265.60	1.0214	0.0190
A$_3$	0.1199	153.51	320.50	186.42	73.61	132.40	181.51	1.0126	0.1228
B$_1$	0.1208	149.42	427.74	376.22	90.82	180.44	265.60	1.0214	0.0596
B$_2$	0.1198	139.18	493.33	135.22	86.46	230.50	270.44	1.1331	0.0327
B$_3$	0.1999	160.47	213.48	291.10	79.80	131.95	100.43	1.0835	0.3962

Benchmark point	$\Gamma(H_1)$	$\Gamma(H_2)$	$\Gamma(H_3)$	$\Gamma(H^\pm)$	$\Gamma(S_2)$	$\Gamma(S^\pm)$
A$_1$	3.2×10^{-3}	3.23×10^{-3}	1.18×10^{0}	6.43×10^{-1}	6.80×10^{-5}	1.45×10^{-1}
A$_2$	3.35×10^{-3}	3.00×10^{-3}	2.40×10^{1}	8.55×10^{0}	8.22×10^{-3}	2.87×10^{0}
A$_3$	3.36×10^{-3}	3.13×10^{-3}	1.13×10^{1}	1.86×10^{-4}	4.83×10^{-4}	3.00×10^{-1}
B$_1$	3.52×10^{-3}	3.00×10^{-3}	2.40×10^{1}	8.55×10^{0}	8.22×10^{-3}	2.87×10^{0}
B$_2$	3.04×10^{-3}	2.22×10^{-3}	9.54×10^{1}	7.59×10^{-5}	1.07×10^{0}	3.27×10^{0}
B$_3$	3.05×10^{-3}	2.87×10^{-3}	9.70×10^{-1}	1.75×10^{0}	3.33×10^{-4}	3.55×10^{-6}

Table 6.1: Some benchmark points. The mass parameters m_ξ and the total decay width $\Gamma(\xi)$ are given in GeV. The σ and α are given in radians. The benchmark points A_i indicate the $Y^u_{524}Y^d_{542}Y^e_{425}$ Yukawa model and B_i- $Y^u_{524}Y^d_{542}Y^e_{524}$. In the SM, the Higgs boson total width is $\Gamma(m_{h_{\rm SM}}) = 4.2 \times 10^{-3}$ GeV. The SM-like Higgs boson particle is the scalar H_1.

Relative annihilation channel contributions to the relic density $\Omega_{\rm CDM}h^2$ in per cents are:

A$_1$:

- 51% $S_1\, S_1 \to W^+\, W^-$
- 30% $S_1\, S_1 \to \bar{b}\, b$
- 22% $S_1\, S_1 \to \bar{b}\, d$
- 6% $S_1\, S_1 \to Z\, Z$
- 6% $S_1\, S_1 \to \bar{t}\, c$
- 2% $S_1\, S_1 \to \tau\, \tau$

A$_2$:

- 48% $S_1\, S_1 \to W^+\, W^-$
- 29% $S_1\, S_1 \to \bar{b}\, b$
- 15% $S_1\, S_1 \to Z\, Z$
- 4% $S_1\, S_1 \to \bar{t}\, c$
- 3% $S_1\, S_1 \to \bar{c}\, c$
- 2% $S_1\, S_1 \to \tau\, \tau$

A$_3$:

- 80% $S_1\, S_1 \to W^+\, W^-$
- 12% $S_1\, S_1 \to Z\, Z$
- 7% $S_1\, S_1 \to \bar{b}\, b$

B$_1$:

- 48% $S_1\, S_1 \to W^+\, W^-$
- 29% $S_1\, S_1 \to \bar{b}\, b$
- 16% $S_1\, S_1 \to Z\, Z$
- 4% $S_1\, S_1 \to \bar{t}\, c$
- 3% $S_1\, S_1 \to \bar{c}\, c$
- 2% $S_1\, S_1 \to \tau\, \tau$

B$_2$:

- 67% $S_1\, S_1 \to W^+\, W^-$
- 23% $S_1\, S_1 \to \bar{b}\, b$
- 7% $S_1\, S_1 \to Z\, Z$
- 2% $S_1\, S_1 \to \bar{c}\, c$
- 1% $S_1\, S_1 \to \tau\, \tau$

B$_3$:

- 63% $S_1\, S_1 \to W^+\, W^-$
- 21% $S_1\, S_1 \to \bar{b}\, b$
- 10% $S_1\, S_1 \to \bar{b}\, d$
- 2% $S_1\, S_1 \to Z\, Z$
- 1% $S_1\, S_1 \to \tau\, \tau$

6.3 The R-II-1a Model

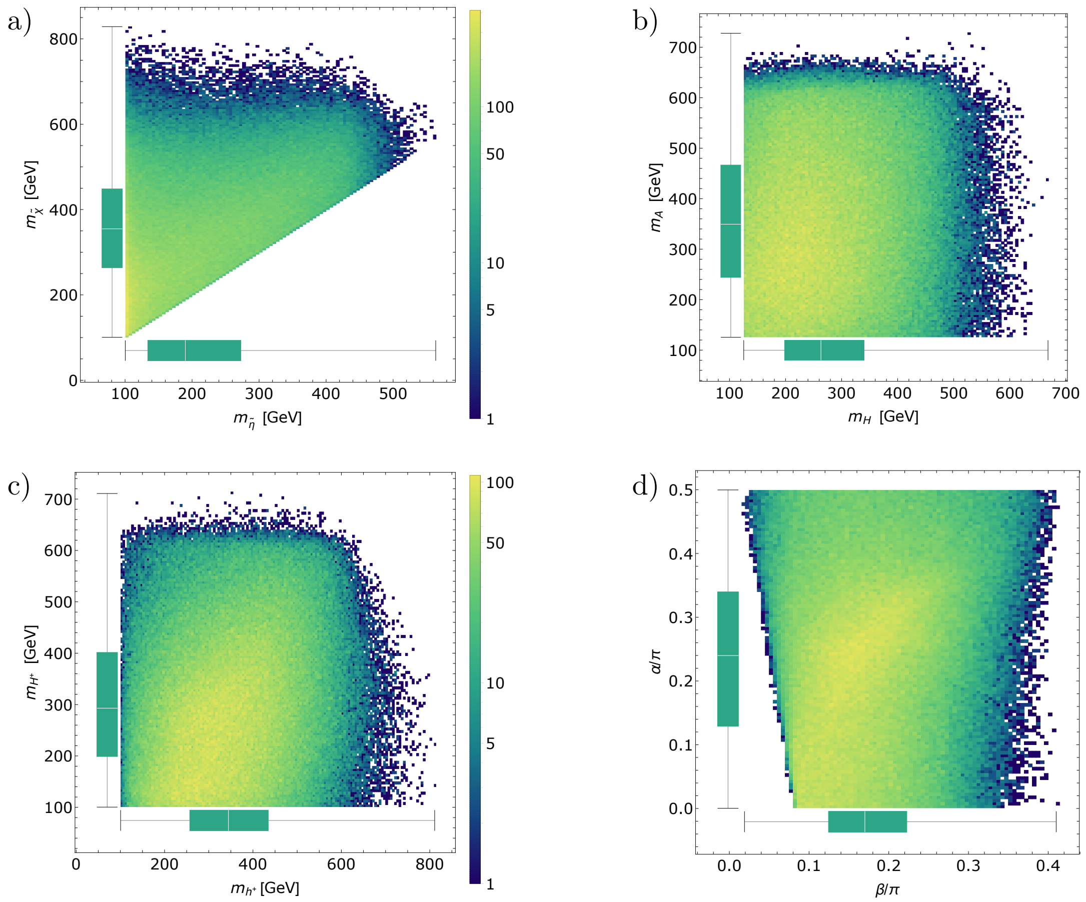

Figure 6.6: General output of the spectrum generator for the R-II-1a model. Scatter plots of different input parameters are presented: a) masses of the neutral states $\tilde{\eta}$ - $\tilde{\chi}$, b) masses of the neutral states H-A, c) masses of the charged states $h^{\pm}$ - $H^{\pm}$, d) angles β - α. The boxplot with whiskers indicates where 1/4 - 3/4 of the data points are situated along with the medians.

We present the general output of the spectrum generator in Figure 6.6. From the spectrum generator we can extract the following information on masses:

$$\begin{aligned}
\max(m_H) &= 652 \text{ GeV}, \\
\max(m_A) &= 687 \text{ GeV}, \\
\max(m_{H^{\pm}}) &= 704 \text{ GeV}, \\
\max(m_{\tilde{\eta}}) &= 526 \text{ GeV}, \\
\max(m_{\tilde{\chi}}) &= 790 \text{ GeV}, \\
\max(m_{h^{\pm}}) &= 792 \text{ GeV},
\end{aligned}$$

and the angles in radians lie in:

$$\begin{aligned}
0.054\pi <&\beta < 0.411\pi, \\
0 <&\alpha < \pi.
\end{aligned}$$

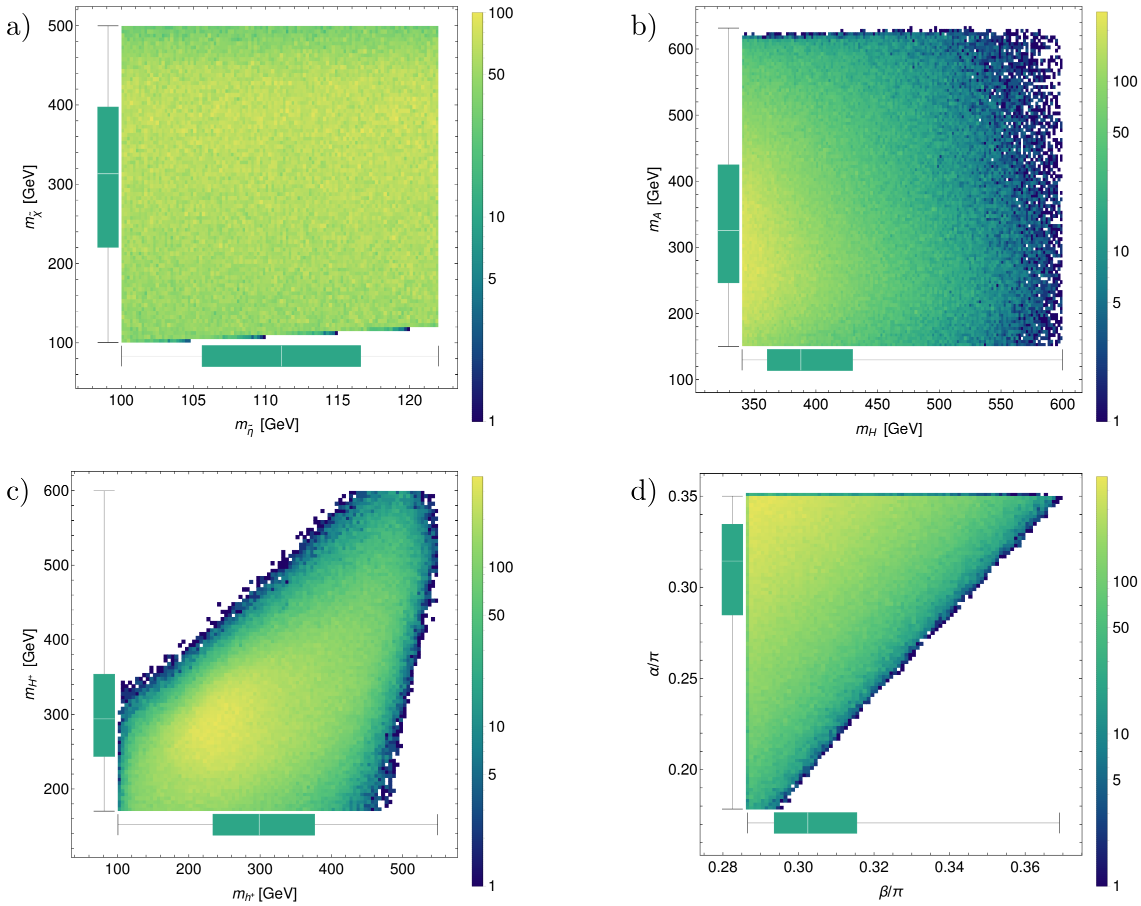

Figure 6.7: Output of the spectrum generator for the R-II-1a model based on additional criteria from the $\Omega_{\rm CDM}h^2$ parameter and the SM-like limit. Scatter plots of different input parameters are presented: a) masses of the neutral states $\tilde{\eta}$ - $\tilde{\chi}$, b) masses of the neutral states H - A, c) masses of the charged states $h^{\pm}$ - $H^{\pm}$, d) angles β - α. The boxplot with whiskers indicates where 1/4 - 3/4 of the data points are situated along with the medians.

The quartic couplings lie in the following ranges:

$$\begin{aligned}
0 <&\lambda_1/\pi < 2.254, \\
-1.082 <&\lambda_2/\pi < 1.875, \\
-1.859 <&\lambda_3/\pi < 0.623, \\
0 <&\lambda_4/\pi < 0.685, \\
-1.651 <&\lambda_5/\pi < 4, \\
-3.850 <&\lambda_6/\pi < 2.326, \\
-1.159 <&\lambda_7/\pi < 0.328, \\
0 <&\lambda_8/\pi < 1.911.
\end{aligned}$$

We assume that the DM candidate is the scalar $\tilde{\eta}$. The relic density $\Omega_{\rm CDM}h^2$ of the benchmark points represented in Figure 6.6 was evaluated using micrOMEGAs. Only a specific range of points resulted in acceptable values of $\Omega_{\rm CDM}h^2$. Based on the result from micrOMEGAs and the SM-limit, the spectrum generator was tweaked appropriately. The new benchmark points are presented in Figure 6.7. Those were further studied using the micrOMEGAs and HiggsBounds codes. As it turned out, the R-II-1a model with an additional $\mathbb{Z}_2$ symmetry of the Yukawa couplings resulted in a higher total width of the SM-like Higgs boson. The SM predicts the total width of the Higgs boson $\Gamma(m_{h_{\rm SM}}) = 4.2 \times 10^{-3}$ GeV, while the typical value of R-II-1a model in the SM-like limit is $\Gamma(m_h) \approx 6 \times 10^{-3}$ GeV. We did not manage to find a single point consistent with the HiggsBounds check. Therefore, we present the result based on just the relic density $\Omega_{\rm CDM}h^2$ scan in Figure 6.8.

Several benchmark points are presented in Table 6.2.

Benchmark point	Ωh^2	m_H	m_A	$m_{H^\pm}$	$m_{\tilde{\eta}}$	$m_{\tilde{\chi}}$	$m_{h^\pm}$	β	α
A_1	0.1200	389.74	344.11	299.77	104.02	348.97	376.15	1.0169	1.0370
A_2	0.1202	427.46	371.65	372.9	106.433	344.80	370.03	0.9862	0.9940
A_3	0.1192	389.91	347.39	196.66	100.97	312.66	469.88	0.9152	1.0350
A_4	0.1205	407.49	515.36	510.42	115.45	363.19	408.44	0.9511	1.0793
A_5	0.1200	445.71	266.09	349.77	115.03	376.74	432.53	0.9312	1.0130

Benchmark point	$\Gamma(h)$	$\Gamma(H)$	$\Gamma(A)$	$\Gamma(H^\pm)$	$\Gamma(\tilde{\chi})$	$\Gamma(h^\pm)$
A_1	6.27×10^{-3}	1.92×10^{1}	9.05×10^{-3}	3.02×10^{0}	8.19×10^{0}	1.19×10^{1}
A_2	6.68×10^{-3}	2.44×10^{1}	3.51×10^{0}	5.94×10^{0}	7.63×10^{0}	1.10×10^{1}
A_3	6.71×10^{-3}	3.04×10^{1}	6.41×10^{0}	3.49×10^{-1}	3.70×10^{1}	7.09×10^{0}
A_4	6.49×10^{-3}	2.33×10^{1}	2.04×10^{1}	1.31×10^{1}	8.97×10^{0}	1.54×10^{1}
A_5	6.46×10^{-3}	3.45×10^{1}	1.46×10^{-2}	6.60×10^{0}	1.06×10^{1}	1.93×10^{1}

Table 6.2: Some benchmark points. The mass parameters m_ξ and the total decay width $\Gamma(\xi)$ are given in GeV. The β and α are given in radians.

Relative annihilation channel contributions to the relic density $\Omega_{\rm CDM}h^2$ in per cents are:

A_1 :

- 51% $\tilde{\eta}\,\tilde{\eta} \to W^+\,W^-$
- 17% $\tilde{\eta}\,\tilde{\eta} \to \bar{c}\,c$
- 15% $\tilde{\eta}\,\tilde{\eta} \to \bar{u}\,u$
- 15% $\tilde{\eta}\,\tilde{\eta} \to Z\,Z$

A_2 :

- 52% $\tilde{\eta}\,\tilde{\eta} \to W^+\,W^-$
- 19% $\tilde{\eta}\,\tilde{\eta} \to Z\,Z$
- 14% $\tilde{\eta}\,\tilde{\eta} \to \bar{c}\,c$
- 14% $\tilde{\eta}\,\tilde{\eta} \to \bar{u}\,u$

A_3 :

- 72% $\tilde{\eta}\,\tilde{\eta} \to W^+\,W^-$
- 10% $\tilde{\eta}\,\tilde{\eta} \to \bar{c}\,c$
- 9% $\tilde{\eta}\,\tilde{\eta} \to Z\,Z$
- 8% $\tilde{\eta}\,\tilde{\eta} \to \bar{u}\,u$
- 1% $\tilde{\eta}\,\tilde{\eta} \to \bar{b}\,b$

A_4 :

- 43% $\tilde{\eta}\,\tilde{\eta} \to W^+\,W^-$
- 25% $\tilde{\eta}\,\tilde{\eta} \to Z\,Z$
- 11% $\tilde{\eta}\,\tilde{\eta} \to \bar{c}\,c$
- 11% $\tilde{\eta}\,\tilde{\eta} \to h\,h$
- 9% $\tilde{\eta}\,\tilde{\eta} \to \bar{u}\,u$

A_5 :

- 43% $\tilde{\eta}\,\tilde{\eta} \to W^+\,W^-$
- 25% $\tilde{\eta}\,\tilde{\eta} \to Z\,Z$
- 12% $\tilde{\eta}\,\tilde{\eta} \to h\,h$
- 10% $\tilde{\eta}\,\tilde{\eta} \to \bar{c}\,c$
- 10% $\tilde{\eta}\,\tilde{\eta} \to \bar{u}\,u$

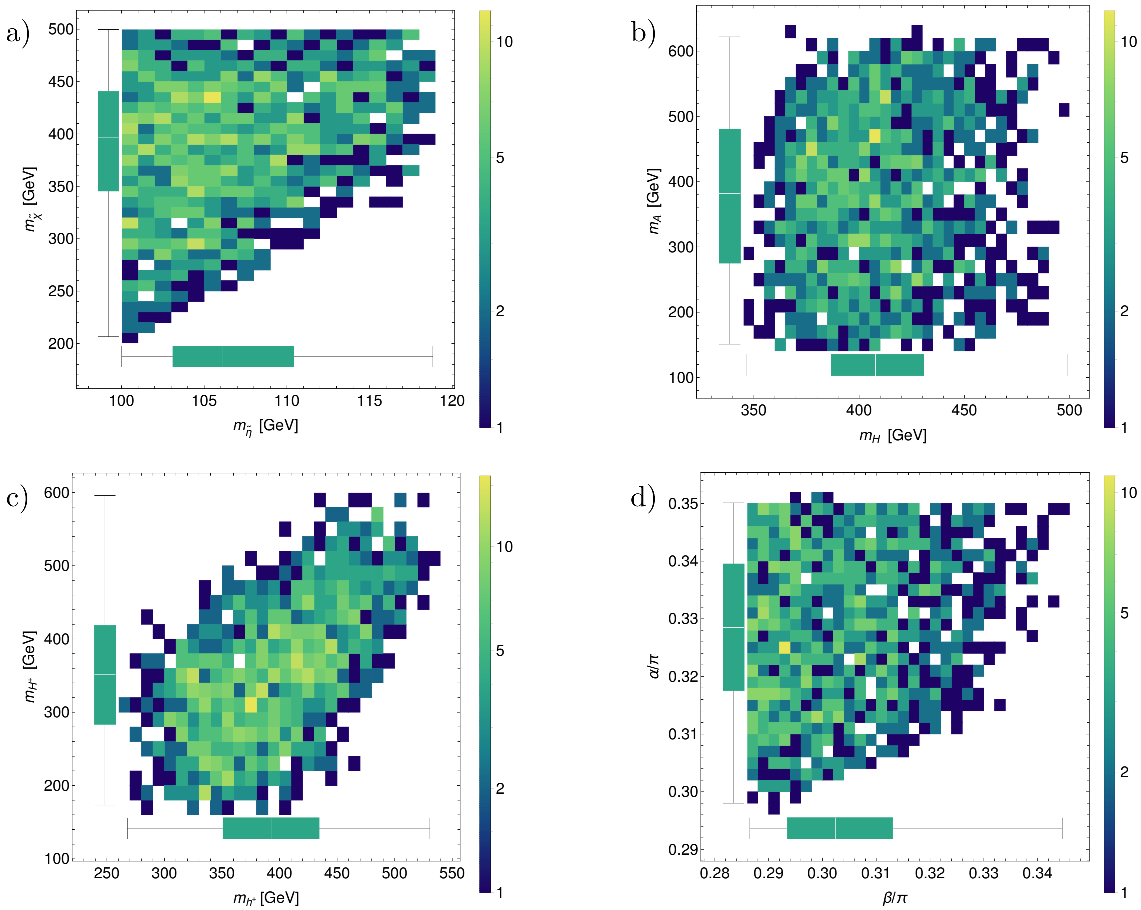

Figure 6.8: Constrained benchmark points of Figure 6.7 based on the acceptable $\Omega_{\mathrm{CDM}}h^2$ parameter range after performing a scan with micrOMEGAs. Scatter plots of different input parameters are presented: a) masses of the neutral states $\tilde{\eta}$ - $\tilde{\chi}$, b) masses of the neutral states H - A, c) masses of the charged states $h^{\pm}$ - $H^{\pm}$, d) angles β - α. The boxplot with whiskers indicates where $^1/_4$ - $^3/_4$ of the data points are situated along with the medians.

Chapter 7

Summary and Conclusions

7.1 Results

The motivation for this work was to gain a better understanding of the S_3-symmetric 3HDM and test if a scalar DM is possible within the framework of the model. The results obtained are summarized as follows:

- The hidden symmetries of the S_3-symmetric scalar potential were analysed. We were able to identify the massless states using the Goldstone theorem. Also, based on the $\mathbb{Z}_2$ symmetry and the mass-squared matrices mixing, the vacuum configurations, which could accommodate DM, were identified.
- One of the models, C-III-c, with massless states was studied. The principle of soft symmetry breaking was applied to promote the massless states to massive ones. The softly broken model C-III-c-ν^2 was further analysed; tree-level interactions and constraints considered. It was shown that the C-III-c-ν^2 model is CP conserving. The S_3-symmetric Yukawa sector resulted in unrealistic V_{CKM} values, the FCNC are way too high. Therefore, the S_3-symmetric Yukawa sector should be further broken. Numerical evaluation of the model resulted in realistic relic density $\Omega_{\text{CDM}}h^2$ values, but the experimental constraints of the SM-like Higgs boson are violated.
- The real vacuum configuration R-II-1a has been analysed. This is the only real vacuum configuration with the $\mathbb{Z}_2$ symmetry preserved by default. The S_3-symmetric Yukawa sector resulted in unrealistic V_{CKM} and thus the Yukawa couplings were further constrained by an additional $\mathbb{Z}_2$ symmetry. Such model resulted in no FCNC, but the decays of the SM-like Higgs boson are violated. Nevertheless, the model resulted in possible relic density $\Omega_{\text{CDM}}h^2$ values.

7.2 Future Research

As this work has shown, the S_3-symmetric 3HDMs have several interesting properties, which have to be addressed further. A number of interesting problems were considered and some possible solutions provided. Nevertheless, only a tiny amount was covered and many questions are still open. Governed by this fact, we mention proposal for future research:

- It should be checked what happens with the additional massless states after renormalization. Of particular interest is how they act in the high-energy limit.
- Some of the models with the zero VEV components do not result in a possible DM candidate. The R-I-2b, and R-I-2c, and C-I-a models involve zero VEV components but there is mixing present between the states. It should be checked more thoroughly if there is a possible direction of the potential, which would result in inert $SU(2)$ doublets. There are two models R-I-1 and C-I-a, which result in mass-degenerate states. The degenerate states should be further analysed to see what that implies.

- The general conditions for CP violation in the Higgs basis should be derived. This might result in several possible solutions based on the $\mathcal{R}_{\beta_i}\mathcal{R}_{\beta_j}$ orderings and if an additional $U(2)$ transformation is considered.

- The C-III-c-ν^2 and R-II-1a models resulted in unrealistic $\mathrm{V}_{\mathrm{CKM}}$. Minimal conditions for realistic values should be further considered. Also, it was assumed that neutrinos are massless. Those should be promoted to massive particles. Moreover, the other vacuum configurations should be checked in terms of the Yukawa sector.

- Although, as shown, both the C-III-c-ν^2 and R-II-1a models are capable of producing realistic relic density $\Omega_{\mathrm{CDM}}h^2$ values and thus are viable candidates, the decay rates of the scalars were not realistic. The total width of the SM-like scalars could be enhanced when loops are considered and the effective vertices of scalars-gluons $\varphi_i gg$ and scalars-photons $\varphi_i \gamma\gamma$ are introduced. The DM decays should be analysed at the further leading orders.

Appendix A

Different Forms of the Scalar Potential

A.1 The Derman Potential

The scalar potential in terms of the S_3-reducible-triplet fields was derived by Derman [33, 38]. He showed that the most general $SU(2) \otimes U(1) \otimes S_3$ Higgs triplet scalar potential can be written as:

$$\begin{aligned} V = &\sum_i \left[-\lambda \left(\phi_i^\dagger \phi_i \right) + A \left(\phi_i^\dagger \phi_i \right)^2 \right] \\ &+ \sum_{i<j} \left\{ \frac{1}{2}\gamma \left(\phi_i^\dagger \phi_j + \text{h.c.} \right) + C \left(\phi_i^\dagger \phi_i \right) \left(\phi_j^\dagger \phi_j \right) + \bar{C} \left(\phi_i^\dagger \phi_j \right) \left(\phi_j^\dagger \phi_i \right) + \frac{1}{2} D \left[\left(\phi_i^\dagger \phi_j \right)^2 + \text{h.c.} \right] \right\} \\ &+ \frac{1}{2} E_1 \sum_{i \neq j} \left[\left(\phi_i^\dagger \phi_i \right) \left(\phi_i^\dagger \phi_j \right) + \text{h.c.} \right] \\ &+ \sum_{i \neq j \neq k \neq i, j<k} \left\{ \frac{1}{2} E_2 \left[\left(\phi_i^\dagger \phi_j \right) \left(\phi_k^\dagger \phi_i \right) + \text{h.c.} \right] + \frac{1}{2} E_3 \left[\left(\phi_i^\dagger \phi_i \right) \left(\phi_k^\dagger \phi_j + \text{h.c.} \right) \right] \right. \\ &\qquad \left. + \frac{1}{2} E_4 \left[\left(\phi_i^\dagger \phi_j \right) \left(\phi_i^\dagger \phi_k \right) + \text{h.c.} \right] \right\}, \end{aligned} \tag{A.1.1}$$

where all of the couplings are assumed to be real. As noted, spontaneous symmetry breaking happens provided that $\lambda > 0$.

It is also worth mentioning that Derman analysed vacua of the form

$$\langle \phi_i \rangle_{\text{min}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_i e^{i\alpha_i} \\ \rho_i e^{i\theta_i} \end{pmatrix}, \; i = \overline{1,3}, \tag{A.1.2}$$

where all of the parameters are real. It was noted that the scalar potential has a local minimum at $\theta_1 - \theta_2 = \theta_2 - \theta_3 = 0$, provided that the couplings $\gamma,\ \bar{C} + D,\ D,\ E_i$ are all negative and thus T-invariance is obtained. Charge conservation is also ensured by choosing couplings in this way.

The Derman potential is related to the potential of eq. (2.1.7) by the following transformations:

$$\begin{pmatrix} \mu_0^2 \\ \mu_1^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -2 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} \lambda \\ \gamma \end{pmatrix}, \tag{A.1.3a}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 4 & 4 & 1 & 1 & -4 & 1 & -2 & 1 \\ 0 & 0 & -3 & 3 & 0 & 3 & 0 & -3 \\ 2 & -1 & 2 & 2 & -2 & -1 & 2 & -1 \\ 4\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 8 & 8 & -4 & -4 & 4 & -4 & 2 & -4 \\ 8 & -4 & 8 & -4 & 4 & 2 & -4 & -4 \\ 4 & -2 & -2 & 4 & 2 & -2 & -2 & 1 \\ 4 & 4 & 4 & 4 & 8 & 4 & 4 & 4 \end{pmatrix} \begin{pmatrix} A \\ C \\ \bar{C} \\ D \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix}. \tag{A.1.3b}$$

Taking into consideration the previously mentioned conditions by Derman, one can get the following constraints in terms of μ_i^2 and λ_i:

$$\left(\mu_0^2 - \mu_1^2\right) < 0, \tag{A.1.4a}$$

$$\left(2\lambda_1 + 8\lambda_3 - 4\sqrt{2}\lambda_4 - 2\lambda_5 + \lambda_6 + 2\lambda_7 + 2\lambda_8\right) < 0, \tag{A.1.4b}$$

$$\left(\lambda_1 + 3\lambda_2 + 4\lambda_3 - 2\sqrt{2}\lambda_4 - \lambda_5 - \lambda_6 + 4\lambda_7 + \lambda_8\right) < 0, \tag{A.1.4c}$$

$$-\left(4\lambda_1 + 4\lambda_3 + \sqrt{2}\lambda_4 - \lambda_5 - \lambda_6 - 2\lambda_7 - 2\lambda_8\right) < 0, \tag{A.1.4d}$$

$$\left(2\lambda_1 + 6\lambda_2 - 4\lambda_3 + 2\sqrt{2}\lambda_4 - 2\lambda_5 + \lambda_6 - 4\lambda_7 + 2\lambda_8\right) < 0, \tag{A.1.4e}$$

$$\left(-4\lambda_1 + 8\lambda_3 + 2\sqrt{2}\lambda_4 + \lambda_5 - 2\lambda_6 - 4\lambda_7 + 2\lambda_8\right) < 0, \tag{A.1.4f}$$

$$\left(\lambda_1 - 3\lambda_2 - 2\lambda_3 + \sqrt{2}\lambda_4 - \lambda_5 - \lambda_6 + \lambda_7 + \lambda_8\right) < 0. \tag{A.1.4g}$$

A.2 Matrix Form

One of the possibilities is to write down the scalar potential in a matrix form, from which elements for the $SU(2)$-covariant form of the potential (2.5.7) can be easily extracted. By directly inspecting different combinations of the $SU(2)$ singlets, $h_{ij} \equiv h_i^\dagger h_j$, we can write down the S_3-symmetric scalar potential as:

$$V = \mathcal{H}_2 M + \mathcal{H}_4 \Lambda \mathcal{H}_4^{\mathrm{T}}, \tag{A.2.1}$$

where the basis vectors are given by:

$$\mathcal{H}_4 = \left(h_{11} \;\; h_{22} \;\; h_{SS} \;\; h_{12} \;\; h_{1S} \;\; h_{21} \;\; h_{2S} \;\; h_{S1} \;\; h_{S2}\right), \tag{A.2.2a}$$

$$\mathcal{H}_2 = \left\{(\mathcal{H}_4)_i : i \in [1,3]\right\}. \tag{A.2.2b}$$

The matrices of eq. (A.2.1) can then be expressed in terms of the couplings as:

$$M = \mathrm{diag}\left(\mu_1^2,\, \mu_1^2,\, \mu_0^2\right),$$

$$\Lambda = \begin{pmatrix} \lambda_1+\lambda_3 & \lambda_1-\lambda_3 & \frac{1}{2}\lambda_5 & 0 & 0 & 0 & \frac{1}{2}\lambda_4 & 0 & \frac{1}{2}\lambda_4 \\ \lambda_1-\lambda_3 & \lambda_1+\lambda_3 & \frac{1}{2}\lambda_5 & 0 & 0 & 0 & -\frac{1}{2}\lambda_4 & 0 & -\frac{1}{2}\lambda_4 \\ \frac{1}{2}\lambda_5 & \frac{1}{2}\lambda_5 & \lambda_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2+\lambda_3 & \frac{1}{2}\lambda_4 & -\lambda_2+\lambda_3 & 0 & \frac{1}{2}\lambda_4 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\lambda_4 & \lambda_7 & \frac{1}{2}\lambda_4 & 0 & \frac{1}{2}\lambda_6 & 0 \\ 0 & 0 & 0 & -\lambda_2+\lambda_3 & \frac{1}{2}\lambda_4 & \lambda_2+\lambda_3 & 0 & \frac{1}{2}\lambda_4 & 0 \\ \frac{1}{2}\lambda_4 & -\frac{1}{2}\lambda_4 & 0 & 0 & 0 & 0 & \lambda_7 & 0 & \frac{1}{2}\lambda_6 \\ 0 & 0 & 0 & \frac{1}{2}\lambda_4 & \frac{1}{2}\lambda_6 & \frac{1}{2}\lambda_4 & 0 & \lambda_7 & 0 \\ \frac{1}{2}\lambda_4 & -\frac{1}{2}\lambda_4 & 0 & 0 & 0 & 0 & \frac{1}{2}\lambda_6 & 0 & \lambda_7 \end{pmatrix}. \tag{A.2.3}$$

Appendix B

Derivatives of the Potential With Respect to the Fields

B.1 The First Derivatives

In order to identify minimization conditions we consider first derivatives of the potential with respect to different fields at vacuum. One should always be careful as the set of all derivatives is not independent, but all of the derivatives must vanish simultaneously after applying the minimization conditions. We consider the following vacuum configuration:

$$\{w_1,\, w_2,\, w_S\}, \tag{B.1.1}$$

where, in general, w_i are complex values. The derivatives are as follows:

$$\left.\frac{\partial V}{\partial h_i^+}\right|_{\langle v\rangle} = \left.\frac{\partial V}{\partial h_S^+}\right|_{\langle v\rangle} = 0, \tag{B.1.2a}$$

$$\left.\frac{\partial V}{\partial h_i^-}\right|_{\langle v\rangle} = \left.\frac{\partial V}{\partial h_S^-}\right|_{\langle v\rangle} = 0, \tag{B.1.2b}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial w_1}\right|_{\langle v\rangle} &= \frac{1}{2}\mu_1^2 w_1^* + \frac{1}{2}\lambda_1 w_1^*\left(|w_1|^2+|w_2|^2\right) + \frac{1}{2}\lambda_2\left(w_1 w_2^{*2} - w_1^*|w_2|^2\right)\\ &+ \frac{1}{2}\lambda_3\left(w_1^*|w_1|^2 + w_1 w_2^{*2}\right) + \frac{1}{2}\lambda_4\left(w_1 w_2^* w_S^* + w_1^* w_2 w_S^* + w_1^* w_2^* w_S\right)\\ &+ \frac{1}{4}\left(\lambda_5+\lambda_6\right) w_1^*|w_S|^2 + \frac{1}{2}\lambda_7 w_1 w_S^{*2} = 0,\end{aligned} \tag{B.1.3a}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial w_2}\right|_{\langle v\rangle} &= \frac{1}{2}\mu_1^2 w_2^* + \frac{1}{2}\lambda_1 w_2^*\left(|w_1|^2+|w_2|^2\right) + \frac{1}{2}\lambda_2\left(w_1^{*2} w_2 - |w_1|^2 w_2\right)\\ &+ \frac{1}{2}\lambda_3\left(w_1^{*2} w_2 + w_2^*|w_2|^2\right) + \frac{1}{4}\lambda_4\left[2 w_S^*\left(|w_1|^2 - |w_2|^2\right) + w_S\left(w_1^{*2} - w_2^{*2}\right)\right]\\ &+ \frac{1}{4}\left(\lambda_5+\lambda_6\right) w_2^*|w_S|^2 + \frac{1}{2}\lambda_7 w_2 w_S^{*2} = 0,\end{aligned} \tag{B.1.3b}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial w_S}\right|_{\langle v\rangle} &= \frac{1}{2}\mu_0^2 w_S^* + \frac{1}{4}\lambda_4\left(2|w_1|^2 w_2^* - w_2^*|w_2|^2 + w_1^{*2} w_2\right) + \frac{1}{4}\left(\lambda_5+\lambda_6\right)\left(|w_1|^2+|w_2|^2\right) w_S^*\\ &+ \frac{1}{2}\lambda_7\left(w_1^{*2} + w_2^{*2}\right) w_S + \frac{1}{2}\lambda_8 w_S^*|w_S|^2 = 0,\end{aligned} \tag{B.1.3c}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial w_1^*}\right|_{\langle v\rangle} &= \frac{1}{2}\mu_1^2 w_1 + \frac{1}{2}\lambda_1 w_1\left(|w_1|^2+|w_2|^2\right) + \frac{1}{2}\lambda_2\left(w_1^* w_2^2 - w_1|w_2|^2\right)\\ &+ \frac{1}{2}\lambda_3 w_1^*\left(w_1^2 + w_2^2\right) + \frac{1}{2}\lambda_4\left(w_1^* w_2 w_S + w_1 w_2^* w_S + w_1 w_2 w_S^*\right)\\ &+ \frac{1}{4}\left(\lambda_5+\lambda_6\right) w_1|w_S|^2 + \frac{1}{2}\lambda_7 w_1^* w_S^2 = 0,\end{aligned} \tag{B.1.3d}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial w_2^*}\right|_{\langle v\rangle} &= \frac{1}{2}\mu_1^2 w_2 + \frac{1}{2}\lambda_1 w_2\left(|w_1|^2+|w_2|^2\right) + \frac{1}{2}\lambda_2\left(w_1^2 w_2^* - |w_1|^2 w_2\right)\\ &+ \frac{1}{2}\lambda_3 w_2^*\left(w_1^2+w_2^2\right) + \frac{1}{4}\lambda_4\left[2\left(|w_1|^2-|w_2|^2\right)w_S + \left(w_1^2-w_2^2\right)w_S^*\right]\\ &+ \frac{1}{4}\left(\lambda_5+\lambda_6\right)w_2|w_S|^2 + \frac{1}{2}\lambda_7 w_2^* w_S^2 = 0,\end{aligned} \tag{B.1.3e}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial w_S^*}\right|_{\langle v\rangle} &= \frac{1}{2}\mu_0^2 w_S + \frac{1}{4}\lambda_4\left[2|w_1|^2 w_2 + w_2^*\left(w_1^2-w_2^2\right)\right] + \frac{1}{4}\left(\lambda_5+\lambda_6\right)w_S\left(|w_1|^2+|w_2|^2\right)\\ &+ \frac{1}{2}\lambda_7 w_S^*\left(w_1^2+w_2^2\right) + \frac{1}{2}\lambda_8|w_S|^2 w_S = 0,\end{aligned} \tag{B.1.3f}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial \tilde\eta_1}\right|_{\langle v\rangle} &= \frac{1}{2}\mu_1^2(w_1^*+w_1) + \frac{1}{2}\lambda_1\left[\left(|w_1|^2+|w_2|^2\right)\left(w_1^*+w_1\right)\right]\\ &+ \frac{1}{2}\lambda_2\left[w_1 w_2^{*2} + w_1^* w_2^2 - |w_2|^2\left(w_1^*+w_1\right)\right]\\ &+ \frac{1}{2}\lambda_3\left[w_1^*|w_1|^2 + w_1 w_2^{*2} + w_1^*(w_1^2+w_2^2)\right]\\ &+ \frac{1}{2}\lambda_4\left(w_1 w_2^* w_S^* + w_1^* w_2 w_S^* + w_1^* w_2^* w_S + w_1^* w_2 w_S + w_1 w_2^* w_S + w_1 w_2 w_S^*\right)\\ &+ \frac{1}{4}\left(\lambda_5+\lambda_6\right)|w_S|^2\left(w_1^*+w_1\right) + \frac{1}{2}\lambda_7\left(w_1 w_S^{*2} + w_1^* w_S^2\right) = 0,\end{aligned} \tag{B.1.4a}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial \tilde\eta_2}\right|_{\langle v\rangle} &= \frac{1}{2}\mu_1^2(w_2^*+w_2) + \frac{1}{2}\lambda_1\left[\left(|w_1|^2+|w_2|^2\right)\left(w_2^*+w_2\right)\right]\\ &+ \frac{1}{2}\lambda_2\left[w_1^2 w_2^* + w_1^{*2} w_2 - |w_1|^2\left(w_2^*+w_2\right)\right]\\ &+ \frac{1}{2}\lambda_3\left[w_2^*|w_2|^2 + w_1^{*2} w_2 + w_2^*(w_1^2+w_2^2)\right]\\ &+ \frac{1}{4}\lambda_4\left[2|w_1|^2\left(w_S^*+w_S\right) + w_S\left(w_1^{*2}-w_2^{*2}-2|w_2|^2\right) + w_S^*\left(w_1^2-w_2^2-2|w_2|^2\right)\right]\\ &+ \frac{1}{4}\left(\lambda_5+\lambda_6\right)|w_S|^2\left(w_2^*+w_2\right) + \frac{1}{2}\lambda_7\left(w_2 w_S^{*2} + w_2^* w_S^2\right) = 0,\end{aligned} \tag{B.1.4b}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial \tilde\eta_S}\right|_{\langle v\rangle} &= \frac{1}{2}\mu_0^2(w_S^*+w_S) + \frac{1}{4}\lambda_4\left(w_1^2 w_2^* + w_1^{*2} w_2 + \left(2|w_1|^2-|w_2|^2\right)\left(w_2^*+w_2\right)\right)\\ &+ \frac{1}{4}\left(\lambda_5+\lambda_6\right)\left(|w_1|^2+|w_2|^2\right)\left(w_S^*+w_S\right) + \frac{1}{2}\lambda_7\left[w_S^*\left(w_1^2+w_2^2\right) + w_S\left(w_1^{*2}+w_2^{*2}\right)\right]\\ &+ \frac{1}{2}\lambda_8|w_S|^2\left(w_S^*+w_S\right) = 0,\end{aligned} \tag{B.1.4c}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial \tilde\chi_1}\right|_{\langle v\rangle} &= \frac{i}{2}\mu_1^2(w_1^*-w_1) + \frac{i}{2}\lambda_1\left[\left(|w_1|^2+|w_2|^2\right)\left(w_1^*-w_1\right)\right]\\ &+ \frac{i}{2}\lambda_2\left[w_1 w_2^{*2} - w_1^* w_2^2 - |w_2|^2\left(w_1^*-w_1\right)\right]\\ &+ \frac{i}{2}\lambda_3\left[w_1^*|w_1|^2 + w_1 w_2^{*2} - w_1^*(w_1^2+w_2^2)\right]\\ &+ \frac{i}{2}\lambda_4\left(w_1 w_2^* w_S^* + w_1^* w_2 w_S^* + w_1^* w_2^* w_S - w_1^* w_2 w_S - w_1 w_2^* w_S - w_1 w_2 w_S^*\right)\\ &+ \frac{i}{4}\left(\lambda_5+\lambda_6\right)|w_S|^2\left(w_1^*-w_1\right) + \frac{i}{2}\lambda_7\left(w_1 w_S^{*2} - w_1^* w_S^2\right) = 0,\end{aligned} \tag{B.1.5a}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial \tilde{\chi}_2}\right|_{\langle v\rangle} &= \frac{i}{2}\mu_1^2(w_2^* - w_2) + \frac{i}{2}\lambda_1\left[\left(|w_1|^2+|w_2|^2\right)(w_2^*-w_2)\right] \\ &+ \frac{i}{2}\lambda_2\left[-w_1^2 w_2^* + w_1^{*2}w_2 - |w_1|^2(w_2^*-w_2)\right] \\ &+ \frac{i}{2}\lambda_3\left[w_2^*|w_2|^2 + w_1^{*2}w_2 - w_2^*(w_1^2+w_2^2)\right] \\ &- \frac{i}{4}\lambda_4\left[2|w_1|^2(w_S^*-w_S) + w_S\left(w_1^{*2}-w_2^{*2}+2|w_2|^2\right) + w_S^*\left(w_1^2-w_2^2+2|w_2|^2\right)\right] \\ &+ \frac{i}{4}(\lambda_5+\lambda_6)|w_S|^2(w_2^*-w_2) + \frac{1}{2}\lambda_7\left(w_2 w_S^{*2} - w_2^* w_S^2\right) = 0,\end{aligned} \tag{B.1.5b}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial \tilde{\chi}_S}\right|_{\langle v\rangle} &= \frac{i}{2}\mu_0^2(w_S^* - w_S) + \frac{i}{4}\lambda_4\left[-w_1^2w_2^* + w_1^{*2}w_2 + \left(2|w_1|^2 - |w_2|^2\right)(w_2^*-w_2)\right] \\ &+ \frac{i}{4}(\lambda_5+\lambda_6)\left(|w_1|^2+|w_2|^2\right)(w_S^*-w_S) + \frac{i}{2}\lambda_7\left[-w_S^*\left(w_1^2+w_2^2\right) + w_S\left(w_1^{*2}+w_2^{*2}\right)\right] \\ &+ \frac{i}{2}\lambda_8|w_S|^2(w_S^*-w_S) = 0.\end{aligned} \tag{B.1.5c}$$

In case of the real vacuum configuration the derivatives are of a different form. Nevertheless, one should realize that it is just a matter of the prefactor, which is a constant. Thus derivatives can be divided by it without leading to another solution.

Next, we consider another possible description of the vacuum configuration by explicitly splitting VEVs into a real part $\hat{w}_i$ and a complex phase σ_i:

$$\{\hat{w}_1 e^{i\sigma_1},\ \hat{w}_2 e^{i\sigma_2},\ \hat{w}_S\}. \tag{B.1.6}$$

The only significant change comes from eq. (B.1.3):

$$\begin{aligned}\left.\frac{\partial V}{\partial \hat{w}_1}\right|_{\langle v\rangle} &= \mu_1^2\hat{w}_1 + \lambda_1\hat{w}_1\left(\hat{w}_1^2+\hat{w}_2^2\right) - 2\lambda_2\hat{w}_1\hat{w}_2^2\mathrm{s}^2_{\sigma_1-\sigma_2} + \lambda_3\hat{w}_1\left(\hat{w}_1^2+\hat{w}_2^2\mathrm{c}_{2(\sigma_1-\sigma_2)}\right) \\ &+ \lambda_4\hat{w}_1\hat{w}_2\hat{w}_S\left(\mathrm{c}_{(2\sigma_1-\sigma_2)} + 2\mathrm{c}_{\sigma_2}\right) + \frac{1}{2}(\lambda_5+\lambda_6)\hat{w}_1\hat{w}_S^2 + \lambda_7\hat{w}_1\hat{w}_S^2\mathrm{c}_{2\sigma_1} = 0,\end{aligned} \tag{B.1.7a}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial \hat{w}_2}\right|_{\langle v\rangle} &= \mu_1^2\hat{w}_2 + \lambda_1\hat{w}_2\left(\hat{w}_1^2+\hat{w}_2^2\right) - 2\lambda_2\hat{w}_1^2\hat{w}_2\mathrm{s}^2_{\sigma_1-\sigma_2} + \lambda_3\hat{w}_2\left(\hat{w}_1^2\mathrm{c}_{2(\sigma_1-\sigma_2)}+\hat{w}_2^2\right) \\ &+ \frac{1}{2}\lambda_4\hat{w}_S\left[\hat{w}_1^2\mathrm{c}_{(2\sigma_1-\sigma_2)} + \left(2\hat{w}_1^2-3\hat{w}_2^2\right)\mathrm{c}_{\sigma_2}\right] + \frac{1}{2}(\lambda_5+\lambda_6)\hat{w}_2\hat{w}_S^2 + \lambda_7\hat{w}_2\hat{w}_S^2\mathrm{c}_{2\sigma_2} = 0,\end{aligned} \tag{B.1.7b}$$

$$\begin{aligned}\left.\frac{\partial V}{\partial \hat{w}_S}\right|_{\langle v\rangle} &= \mu_0^2\hat{w}_S + \frac{1}{2}\lambda_4\hat{w}_2\left[\hat{w}_1^2\mathrm{c}_{(2\sigma_1-\sigma_2)} + \left(2\hat{w}_1^2-\hat{w}_2^2\right)\mathrm{c}_{\sigma_2}\right] \\ &+ \frac{1}{2}(\lambda_5+\lambda_6)\left(\hat{w}_1^2+\hat{w}_2^2\right)\hat{w}_S + \lambda_7\hat{w}_S\left[\hat{w}_1^2\mathrm{c}_{2\sigma_1}+\hat{w}_2^2\mathrm{c}_{2\sigma_2}\right] + \lambda_8\hat{w}_S^3 = 0,\end{aligned} \tag{B.1.7c}$$

$$\left.\frac{\partial V}{\partial \sigma_1}\right|_{\langle v\rangle} = -(\lambda_2+\lambda_3)\hat{w}_1^2\hat{w}_2^2\mathrm{s}_{2(\sigma_1-\sigma_2)} - \lambda_4\hat{w}_1^2\hat{w}_2\hat{w}_S\mathrm{s}_{(2\sigma_1-\sigma_2)} - \lambda_7\hat{w}_1^2\hat{w}_S^2\mathrm{s}_{\sigma_1} = 0, \tag{B.1.7d}$$

$$\left.\frac{\partial V}{\partial \sigma_2}\right|_{\langle v\rangle} = (\lambda_2+\lambda_3)\hat{w}_1^2\hat{w}_2^2\mathrm{s}_{2(\sigma_1-\sigma_2)} + \frac{1}{2}\lambda_4\hat{w}_2\hat{w}_S\left[\hat{w}_1^2\mathrm{s}_{(2\sigma_1-\sigma_2)} - \left(2\hat{w}_1^2-\hat{w}_2^2\right)\mathrm{s}_{\sigma_2}\right] - \lambda_7\hat{w}_2^2\hat{w}_S^2\mathrm{s}_{2\sigma_2} = 0. \tag{B.1.7e}$$

All the other derivatives are of the same form except for the fact that they are not expanded in terms of the absolute value $\hat{w}_i$ and the overall phase $e^{i\sigma_i}$.

Appendix C

Potential in the Higgs Basis

C.1 $\mathcal{R}_{\beta_z}\mathcal{R}_{\beta_x}$ Higgs Basis Rotation

We present relations between the couplings in the Higgs basis and the generic basis of eqs. (2.5.9, 2.5.10) using transformations givens by eq. (2.5.14). The quadratic couplings are:

$$Y_{11} = \mu_0^2 s_{\beta_1}^2 s_{\beta_2}^2 + \mu_1^2 \left(c_{\beta_1}^2 + s_{\beta_1}^2 c_{\beta_2}^2\right), \tag{C.1.1a}$$
$$Y_{12} = \left(\mu_0^2 - \mu_1^2\right) s_{\beta_1} c_{\beta_1} s_{\beta_2}^2, \tag{C.1.1b}$$
$$Y_{13} = \left(\mu_0^2 - \mu_1^2\right) s_{\beta_1} c_{\beta_2} s_{\beta_2}, \tag{C.1.1c}$$
$$Y_{22} = \mu_0^2 c_{\beta_1}^2 s_{\beta_2}^2 + \mu_1^2 \left(s_{\beta_1}^2 + c_{\beta_1}^2 c_{\beta_2}^2\right), \tag{C.1.1d}$$
$$Y_{23} = \left(\mu_0^2 - \mu_1^2\right) c_{\beta_1} c_{\beta_2} s_{\beta_2}, \tag{C.1.1e}$$
$$Y_{33} = \mu_0^2 c_{\beta_2}^2 + \mu_1^2 s_{\beta_2}^2. \tag{C.1.1f}$$

The soft symmetry breaking terms of eq. (3.2.2) result in a change of parameters in the Higgs basis:

$$Y'_{11} = Y_{11} + \frac{1}{2}\nu^2 c_{\beta_2} c_{\sigma_1-\sigma_2} s_{2\beta_1} + \mu_2^2 \left(c_{\beta_1}^2 - c_{\beta_2}^2 s_{\beta_1}^2\right) + \mu_3^2 c_{\beta_1} c_{\sigma_1} s_{\beta_1} s_{\beta_2} + \mu_4^2 c_{\beta_2} c_{\sigma_2} s_{\beta_1}^2 s_{\beta_2}, \tag{C.1.2a}$$
$$\begin{aligned} Y'_{12} = Y_{12} &+ \frac{1}{4}\nu^2 c_{\beta_2} \left(c_{2\beta_1+\sigma_1-\sigma_2} + c_{2\beta_1-\sigma_1+\sigma_2} - 2i s_{\sigma_1-\sigma_2}\right) - \frac{1}{4}\mu_2^2 \left(c_{2\beta_2} + 3\right) s_{2\beta_1} \\ &+ \frac{1}{2}\mu_3^2 s_{\beta_2} \left(c_{2\beta_1} c_{\sigma_1} - i s_{\sigma_1}\right) + \mu_4^2 c_{\beta_1} c_{\beta_2} c_{\sigma_2} s_{\beta_1} s_{\beta_2}, \end{aligned} \tag{C.1.2b}$$
$$\begin{aligned} Y'_{13} = Y_{13} &- \frac{1}{2}\nu^2 e^{-i(\sigma_1-\sigma_2)} c_{\beta_1} s_{\beta_2} + \mu_2^2 c_{\beta_2} s_{\beta_1} s_{\beta_2} + \frac{1}{2}\mu_3^2 e^{-i\sigma_1} c_{\beta_1} c_{\beta_2} \\ &+ \frac{1}{2}\mu_4^2 s_{\beta_1} \left(c_{2\beta_2} c_{\sigma_2} - i s_{\sigma_2}\right), \end{aligned} \tag{C.1.2c}$$
$$Y'_{22} = Y_{22} - \frac{1}{2}\nu^2 c_{\beta_2} c_{\sigma_1-\sigma_2} s_{2\beta_1} + \mu_2^2 \left(s_{\beta_1}^2 - c_{\beta_1}^2 c_{\beta_2}^2\right) - \mu_3^2 c_{\beta_1} c_{\sigma_1} s_{\beta_1} s_{\beta_2} + \mu_4^2 c_{\beta_1}^2 c_{\beta_2} c_{\sigma_2} s_{\beta_2}, \tag{C.1.2d}$$
$$\begin{aligned} Y'_{23} = Y_{23} &+ \frac{1}{2}\nu^2 e^{-i(\sigma_1-\sigma_2)} s_{\beta_1} s_{\beta_2} + \mu_2^2 c_{\beta_1} c_{\beta_2} s_{\beta_2} - \frac{1}{2}\mu_3^2 e^{-i\sigma_1} c_{\beta_2} s_{\beta_1} \\ &+ \frac{1}{2}\mu_4^2 c_{\beta_1} \left(c_{2\beta_2} c_{\sigma_2} - i s_{\sigma_2}\right), \end{aligned} \tag{C.1.2e}$$
$$Y'_{33} = Y_{33} - \mu_2^2 s_{\beta_2}^2 - \mu_4^2 c_{\beta_2} c_{\sigma_2} s_{\beta_2}. \tag{C.1.2f}$$

The real quartic couplings are:

$$\begin{aligned} Z_{1111} = &\lambda_1 \left(c_{\beta_1}^2 + c_{\beta_2}^2 s_{\beta_1}^2\right)^2 - 4\lambda_2 c_{\beta_1}^2 c_{\beta_2}^2 s_{\beta_1}^2 s_{\sigma_1-\sigma_2}^2 + \lambda_3 \left[c_{\beta_1}^4 + 2 c_{\beta_2}^2 c_{\beta_1}^2 c_{2(\sigma_1-\sigma_2)} s_{\beta_1}^2 + c_{\beta_2}^4 s_{\beta_1}^4\right] \\ &+ \lambda_4 \left[c_{\beta_1}^2 \left(c_{2\sigma_1-\sigma_2} + 2 c_{\sigma_2}\right) s_{\beta_1}^2 s_{2\beta_2} - 2 c_{\beta_2}^3 c_{\sigma_2} s_{\beta_1}^4 s_{\beta_2}\right] + \lambda_5 s_{\beta_1}^2 s_{\beta_2}^2 \left(c_{\beta_1}^2 + c_{\beta_2}^2 s_{\beta_1}^2\right) \\ &+ \lambda_6 s_{\beta_1}^2 s_{\beta_2}^2 \left(c_{\beta_1}^2 + c_{\beta_2}^2 s_{\beta_1}^2\right) + 2\lambda_7 s_{\beta_1}^2 s_{\beta_2}^2 \left(c_{\beta_1}^2 c_{2\sigma_1} + c_{\beta_2}^2 c_{2\sigma_2} s_{\beta_1}^2\right) \\ &+ \lambda_8 s_{\beta_1}^4 s_{\beta_2}^4, \end{aligned} \tag{C.1.3a}$$

$$\begin{aligned}
Z_{1122} = & 2\lambda_1 \left(c_{\beta_1}^2 c_{\beta_2}^2 + s_{\beta_1}^2\right) \left(c_{\beta_1}^2 + c_{\beta_2}^2 s_{\beta_1}^2\right) + 8\lambda_2 c_{\beta_1}^2 c_{\beta_2}^2 s_{\beta_1}^2 s_{\sigma_1-\sigma_2}^2 \\
& - 2\lambda_3 \left[c_{\beta_2}^2 c_{\beta_1}^4 - c_{\beta_1}^2 s_{\beta_1}^2 \left(-4 c_{\beta_2}^2 c_{\sigma_1-\sigma_2}^2 + c_{\beta_2}^4 + 1\right) + c_{\beta_2}^2 s_{\beta_1}^4\right] \\
& + \lambda_4 \left[\frac{1}{4} s_{2\beta_2} \left(\left(c_{4\beta_1} + 3\right) c_{\sigma_2} - 4 c_{\sigma_1} c_{\sigma_1-\sigma_2} s_{2\beta_1}^2\right) - c_{\beta_2}^3 c_{\sigma_2} s_{2\beta_1}^2 s_{\beta_2}\right] \\
& + \frac{1}{8}\lambda_5 \left[2\left(c_{4\beta_1} + 3\right) s_{\beta_2}^2 + s_{2\beta_1}^2 s_{2\beta_2}^2\right] - 2\lambda_6 c_{\beta_1}^2 s_{\beta_1}^2 s_{\beta_2}^4 \\
& - 4\lambda_7 c_{\beta_1}^2 s_{\beta_1}^2 s_{\beta_2}^2 \left(c_{2\sigma_1} - c_{\beta_2}^2 c_{2\sigma_2}\right) + 2\lambda_8 c_{\beta_1}^2 s_{\beta_1}^2 s_{\beta_2}^4,
\end{aligned} \tag{C.1.3b}$$

$$\begin{aligned}
Z_{1133} = & 2\lambda_1 s_{\beta_2}^2 \left(c_{\beta_1}^2 + c_{\beta_2}^2 s_{\beta_1}^2\right) - 2\lambda_3 s_{\beta_2}^2 \left(c_{\beta_1}^2 - c_{\beta_2}^2 s_{\beta_1}^2\right) - 2\lambda_4 c_{\beta_2} c_{\sigma_2} s_{\beta_2} \left(c_{\beta_1}^2 - c_{2\beta_2} s_{\beta_1}^2\right) \\
& + \lambda_5 \left[c_{\beta_1}^2 c_{\beta_2}^2 + \frac{1}{4}\left(c_{4\beta_2} + 3\right) s_{\beta_1}^2\right] - 2\lambda_6 c_{\beta_2}^2 s_{\beta_1}^2 s_{\beta_2}^2 - 4\lambda_7 c_{\beta_2}^2 c_{2\sigma_2} s_{\beta_1}^2 s_{\beta_2}^2 + 2\lambda_8 c_{\beta_2}^2 s_{\beta_1}^2 s_{\beta_2}^2,
\end{aligned} \tag{C.1.3c}$$

$$\begin{aligned}
Z_{1221} = & 2\lambda_1 c_{\beta_1}^2 s_{\beta_1}^2 s_{\beta_2}^4 - \frac{1}{2}\lambda_2 c_{\beta_2}^2 \left(c_{4\beta_1} + 2 c_{2(\sigma_1-\sigma_2)} s_{2\beta_1}^2 + 3\right) \\
& + 2\lambda_3 \left[c_{\beta_2}^2 c_{\beta_1}^4 + c_{\beta_1}^2 s_{\beta_1}^2 \left(c_{\beta_2}^4 + 4 c_{\beta_2}^2 s_{\sigma_1-\sigma_2}^2 + 1\right) + c_{\beta_2}^2 s_{\beta_1}^4\right] \\
& + \frac{1}{4}\lambda_4 c_{\beta_2} s_{\beta_2} \left[\left(5 c_{4\beta_1} + 3\right) c_{\sigma_2} - 2 s_{2\beta_1}^2 \left(c_{2\beta_2} c_{\sigma_2} + 2 c_{2\sigma_1-\sigma_2}\right)\right] \\
& - 2\lambda_5 c_{\beta_1}^2 s_{\beta_1}^2 s_{\beta_2}^4 + \lambda_6 s_{\beta_2}^2 \left(c_{\beta_1}^4 + 2 c_{\beta_2}^2 c_{\beta_1}^2 s_{\beta_1}^2 + s_{\beta_1}^4\right) \\
& - 4\lambda_7 c_{\beta_1}^2 s_{\beta_1}^2 s_{\beta_2}^2 \left(c_{2\sigma_1} - c_{\beta_2}^2 c_{2\sigma_2}\right) + 2\lambda_8 c_{\beta_1}^2 s_{\beta_1}^2 s_{\beta_2}^4,
\end{aligned} \tag{C.1.3d}$$

$$\begin{aligned}
Z_{1331} = & 2\lambda_1 c_{\beta_2}^2 s_{\beta_1}^2 s_{\beta_2}^2 - 2\lambda_2 c_{\beta_1}^2 s_{\beta_2}^2 + 2\lambda_3 s_{\beta_2}^2 \left(c_{\beta_1}^2 + c_{\beta_2}^2 s_{\beta_1}^2\right) - 2\lambda_4 c_{\beta_2} c_{\sigma_2} s_{\beta_2} \left(c_{\beta_1}^2 - c_{2\beta_2} s_{\beta_1}^2\right) \\
& + 2\lambda_5 c_{\beta_2}^2 s_{\beta_1}^2 s_{\beta_2}^2 + \lambda_6 \left[c_{\beta_1}^2 c_{\beta_2}^2 + \frac{1}{4}\left(c_{4\beta_2} + 3\right) s_{\beta_1}^2\right] - 4\lambda_7 c_{\beta_2}^2 c_{2\sigma_2} s_{\beta_1}^2 s_{\beta_2}^2 + 2\lambda_8 c_{\beta_2}^2 s_{\beta_1}^2 s_{\beta_2}^2,
\end{aligned} \tag{C.1.3e}$$

$$\begin{aligned}
Z_{2222} = & \lambda_1 \left(c_{\beta_1}^2 c_{\beta_2}^2 + s_{\beta_1}^2\right)^2 - 4\lambda_2 c_{\beta_1}^2 c_{\beta_2}^2 s_{\beta_1}^2 s_{\sigma_1-\sigma_2}^2 + \lambda_3 \left[c_{\beta_1}^4 c_{\beta_2}^4 + 2 c_{\beta_1}^2 c_{\beta_2}^2 c_{2(\sigma_1-\sigma_2)} s_{\beta_1}^2 + s_{\beta_1}^4\right] \\
& + \lambda_4 \left[c_{\beta_1}^2 \left(c_{2\sigma_1-\sigma_2} + 2 c_{\sigma_2}\right) s_{\beta_1}^2 s_{2\beta_2} - 2 c_{\beta_1}^4 c_{\beta_2}^3 c_{\sigma_2} s_{\beta_2}\right] + \lambda_5 c_{\beta_1}^2 s_{\beta_2}^2 \left(c_{\beta_1}^2 c_{\beta_2}^2 + s_{\beta_1}^2\right) \\
& + \lambda_6 c_{\beta_1}^2 s_{\beta_2}^2 \left(c_{\beta_1}^2 c_{\beta_2}^2 + s_{\beta_1}^2\right) + 2\lambda_7 c_{\beta_1}^2 s_{\beta_2}^2 \left(c_{\beta_1}^2 c_{\beta_2}^2 c_{2\sigma_2} + c_{2\sigma_1} s_{\beta_1}^2\right) + \lambda_8 c_{\beta_1}^4 s_{\beta_2}^4,
\end{aligned} \tag{C.1.3f}$$

$$\begin{aligned}
Z_{2233} = & 2\lambda_1 s_{\beta_2}^2 \left(c_{\beta_1}^2 c_{\beta_2}^2 + s_{\beta_1}^2\right) + 2\lambda_3 s_{\beta_2}^2 \left(c_{\beta_1}^2 c_{\beta_2}^2 - s_{\beta_1}^2\right) + 2\lambda_4 c_{\beta_2} c_{\sigma_2} s_{\beta_2} \left(c_{\beta_1}^2 c_{2\beta_2} - s_{\beta_1}^2\right) \\
& + \lambda_5 \left[\frac{1}{4}\left(c_{4\beta_2} + 3\right) c_{\beta_1}^2 + c_{\beta_2}^2 s_{\beta_1}^2\right] - 2\lambda_6 c_{\beta_1}^2 c_{\beta_2}^2 s_{\beta_2}^2 - 4\lambda_7 c_{\beta_1}^2 c_{\beta_2}^2 c_{2\sigma_2} s_{\beta_2}^2 + 2\lambda_8 c_{\beta_1}^2 c_{\beta_2}^2 s_{\beta_2}^2,
\end{aligned} \tag{C.1.3g}$$

$$\begin{aligned}
Z_{2332} = & 2\lambda_1 c_{\beta_1}^2 c_{\beta_2}^2 s_{\beta_2}^2 - 2\lambda_2 s_{\beta_1}^2 s_{\beta_2}^2 + 2\lambda_3 s_{\beta_2}^2 \left(c_{\beta_1}^2 c_{\beta_2}^2 + s_{\beta_1}^2\right) + 2\lambda_4 c_{\beta_2} c_{\sigma_2} s_{\beta_2} \left(c_{\beta_1}^2 c_{2\beta_2} - s_{\beta_1}^2\right) \\
& - 2\lambda_5 c_{\beta_1}^2 c_{\beta_2}^2 s_{\beta_2}^2 + \lambda_6 \left[\frac{1}{4}\left(c_{4\beta_2} + 3\right) c_{\beta_1}^2 + c_{\beta_2}^2 s_{\beta_1}^2\right] - 4\lambda_7 c_{\beta_1}^2 c_{\beta_2}^2 c_{2\sigma_2} s_{\beta_2}^2 + 2\lambda_8 c_{\beta_1}^2 c_{\beta_2}^2 s_{\beta_2}^2,
\end{aligned} \tag{C.1.3h}$$

$$Z_{3333} = \lambda_1 s_{\beta_2}^4 + \lambda_3 s_{\beta_2}^4 + 2\lambda_4 c_{\beta_2} c_{\sigma_2} s_{\beta_2}^3 + \lambda_5 c_{\beta_2}^2 s_{\beta_2}^2 + \lambda_6 c_{\beta_2}^2 s_{\beta_2}^2 + 2\lambda_7 c_{\beta_2}^2 c_{2\sigma_2} s_{\beta_2}^2 + \lambda_8 c_{\beta_2}^4, \tag{C.1.3i}$$

$$\begin{aligned}
Z_{1233} = & -2\lambda_1 c_{\beta_1} s_{\beta_1} s_{\beta_2}^4 + 2\lambda_3 c_{\beta_1} \left(c_{\beta_2}^2 + 1\right) s_{\beta_1} s_{\beta_2}^2 + 2\lambda_4 c_{\beta_2}^3 c_{\sigma_2} s_{2\beta_1} s_{\beta_2} \\
& - \frac{1}{2}\lambda_5 c_{2\beta_2} s_{2\beta_1} s_{\beta_2}^2 - 2\lambda_6 c_{\beta_1} c_{\beta_2}^2 s_{\beta_1} s_{\beta_2}^2 - 2\lambda_7 c_{\beta_2}^2 c_{2\sigma_2} s_{2\beta_1} s_{\beta_2}^2 + \lambda_8 c_{\beta_2}^2 s_{2\beta_1} s_{\beta_2}^2,
\end{aligned} \tag{C.1.3j}$$

$$\begin{aligned}
Z_{1332} = & \lambda_1 c_{\beta_2}^2 s_{2\beta_1} s_{\beta_2}^2 + \lambda_2 s_{2\beta_1} s_{\beta_2}^2 - 2\lambda_3 c_{\beta_1} s_{\beta_1} s_{\beta_2}^4 + 2\lambda_4 c_{\beta_2}^3 c_{\sigma_2} s_{2\beta_1} s_{\beta_2} \\
& - 2\lambda_5 c_{\beta_1} c_{\beta_2}^2 s_{\beta_1} s_{\beta_2}^2 - \frac{1}{2}\lambda_6 c_{2\beta_2} s_{2\beta_1} s_{\beta_2}^2 - 2\lambda_7 c_{\beta_2}^2 c_{2\sigma_2} s_{2\beta_1} s_{\beta_2}^2 + \lambda_8 c_{\beta_2}^2 s_{2\beta_1} s_{\beta_2}^2,
\end{aligned} \tag{C.1.3k}$$

and the complex quartic couplings are:

$$
\begin{aligned}
Z_{1112} = & -2\lambda_1 c_{\beta_1} s_{\beta_1} s_{\beta_2}^2 \left(c_{\beta_1}^2 + c_{\beta_2}^2 s_{\beta_1}^2\right) - 2\lambda_2 c_{\beta_2}^2 s_{2\beta_1} s_{\sigma_1-\sigma_2} \left(c_{2\beta_1} s_{\sigma_1-\sigma_2} + i c_{\sigma_1-\sigma_2}\right) \\
& - 2\lambda_3 e^{-2i(\sigma_1+\sigma_2)} c_{\beta_1} s_{\beta_1} \left(e^{2i\sigma_1} - e^{2i\sigma_2} c_{\beta_2}^2\right) \left(e^{2i\sigma_2} c_{\beta_1}^2 + e^{2i\sigma_1} c_{\beta_2}^2 s_{\beta_1}^2\right) \\
& + \frac{1}{2}\lambda_4 e^{-i(2\sigma_1+\sigma_2)} c_{\beta_1} s_{\beta_1} s_{2\beta_2} \Big[2\left(e^{2i\sigma_1} + e^{2i\sigma_2} + e^{2i(\sigma_1+\sigma_2)}\right) c_{\beta_1}^2 \\
& \qquad - e^{2i\sigma_1} s_{\beta_1}^2 \left(e^{2i\sigma_2}\left(c_{2\beta_2}+3\right) + c_{2\beta_2} + 2e^{2i\sigma_1} + 3\right) \Big] \\
& + \lambda_5 c_{\beta_1} s_{\beta_1} s_{\beta_2}^2 \left(c_{\beta_1}^2 + c_{2\beta_2} s_{\beta_1}^2\right) + \lambda_6 c_{\beta_1} s_{\beta_1} s_{\beta_2}^2 \left(c_{\beta_1}^2 + c_{2\beta_2} s_{\beta_1}^2\right) \\
& + 2\lambda_7 c_{\beta_1} s_{\beta_1} s_{\beta_2}^2 \left[e^{-2i\sigma_1} c_{\beta_1}^2 - s_{\beta_1}^2 \left(-2c_{\beta_2}^2 c_{2\sigma_2} + c_{2\sigma_1} + i s_{2\sigma_1}\right)\right] + 2\lambda_8 c_{\beta_1} s_{\beta_1}^3 s_{\beta_2}^4,
\end{aligned} \tag{C.1.4a}
$$

$$
\begin{aligned}
Z_{1113} = & -\frac{1}{4}\lambda_1 \left[\left(c_{2\beta_1}+3\right) s_{\beta_1} s_{2\beta_2} + s_{\beta_1}^3 s_{4\beta_2}\right] + \lambda_2 e^{-2i\sigma_1}\left(e^{2i\sigma_1} - e^{2i\sigma_2}\right) c_{\beta_1}^2 s_{\beta_1} s_{2\beta_2} \\
& - \lambda_3 \left(2c_{\beta_2}^3 s_{\beta_2} s_{\beta_1}^3 + 2e^{-2i(\sigma_1-\sigma_2)} c_{\beta_1}^2 c_{\beta_2} s_{\beta_2} s_{\beta_1}\right) \\
& + \frac{1}{2}\lambda_4 e^{-i(2\sigma_1+\sigma_2)} s_{\beta_1} \Big[e^{2i\sigma_2}\left(2c_{2\beta_2} c_{\beta_1}^2 + e^{2i\sigma_1}\left(s_{\beta_1}^2 s_{2\beta_2}^2 - 4c_{\beta_1}^2 s_{\beta_2}^2\right)\right) \\
& \qquad - e^{2i\sigma_1} c_{\beta_2}\left(\left(c_{\beta_2} + c_{3\beta_2}\right) s_{\beta_1}^2 - 4c_{\beta_1}^2 c_{\beta_2}\right) \Big] \\
& + \lambda_5 \left(c_{\beta_1}^2 c_{\beta_2} s_{\beta_2} s_{\beta_1} + \frac{1}{4} s_{4\beta_2} s_{\beta_1}^3\right) + \lambda_6 c_{\beta_2} s_{\beta_1} s_{\beta_2} \left(c_{\beta_1}^2 + c_{2\beta_2} s_{\beta_1}^2\right) \\
& + \lambda_7 \left[2e^{-2i\sigma_2} c_{\beta_2} s_{\beta_2} s_{\beta_1}^3 \left(c_{\beta_2}^2 - e^{4i\sigma_2} s_{\beta_2}^2\right) + e^{-2i\sigma_1} c_{\beta_1}^2 s_{2\beta_2} s_{\beta_1}\right] + 2\lambda_8 c_{\beta_2} s_{\beta_1}^3 s_{\beta_2}^3,
\end{aligned} \tag{C.1.4b}
$$

$$
\begin{aligned}
Z_{1123} = & -\lambda_1 c_{\beta_1} s_{2\beta_2}\left(c_{\beta_1}^2 + c_{\beta_2}^2 s_{\beta_1}^2\right) - \lambda_2 \left(1 - e^{-2i(\sigma_1-\sigma_2)}\right) c_{\beta_1} s_{\beta_1}^2 s_{2\beta_2} \\
& + \lambda_3 c_{\beta_1} s_{2\beta_2}\left[c_{\beta_1}^2 + s_{\beta_1}^2\left(s_{\beta_2}^2 + e^{-2i(\sigma_1-\sigma_2)}\right)\right] \\
& + \lambda_4 c_{\beta_1} \Big[-2e^{-i\sigma_2} c_{\beta_2}^4 s_{\beta_1}^2 + e^{i\sigma_2 - 2i\sigma_1} s_{\beta_1}^2 \left(-c_{2\beta_2} + e^{2i\sigma_1}\left(c_{2\beta_2}+2\right) s_{\beta_2}^2\right) \\
& \qquad + c_{\beta_1}^2 \left(c_{2\beta_2} c_{\sigma_2} - i s_{\sigma_2}\right) \Big] \\
& + \frac{1}{2}\lambda_5 c_{\beta_1} s_{2\beta_2}\left(c_{\beta_1}^2 + c_{2\beta_2} s_{\beta_1}^2\right) - 2\lambda_6 c_{\beta_1} c_{\beta_2} s_{\beta_1}^2 s_{\beta_2}^3 \\
& + \lambda_7 c_{\beta_1} s_{\beta_1}^2 \left[2e^{-2i\sigma_2} c_{\beta_2}^3 s_{\beta_2} - 2e^{2i\sigma_2} c_{\beta_2} s_{\beta_2}^3 - e^{-2i\sigma_1} s_{2\beta_2}\right] + 2\lambda_8 c_{\beta_1} c_{\beta_2} s_{\beta_1}^2 s_{\beta_2}^3,
\end{aligned} \tag{C.1.4c}
$$

$$
\begin{aligned}
Z_{1212} = & \lambda_1 c_{\beta_1}^2 s_{\beta_1}^2 s_{\beta_2}^4 + \lambda_2 e^{-2i(\sigma_1+\sigma_2)} c_{\beta_2}^2 \left(e^{2i\sigma_2} c_{\beta_1}^2 + e^{2i\sigma_1} s_{\beta_1}^2\right)^2 \\
& + \lambda_3 \left[e^{-2i(\sigma_1-\sigma_2)} c_{\beta_2}^2 c_{\beta_1}^4 + e^{2i(\sigma_1-\sigma_2)} c_{\beta_2}^2 s_{\beta_1}^4 + \left(c_{\beta_2}^4 + 1\right) c_{\beta_1}^2 s_{\beta_1}^2\right] \\
& + \lambda_4 e^{-i(2\sigma_1+\sigma_2)} c_{\beta_2} s_{\beta_2} \left[e^{2i\sigma_2} c_{\beta_1}^4 - e^{2i\sigma_1}\left(1 + e^{2i\sigma_2}\right)\left(c_{\beta_2}^2 + 2\right) c_{\beta_1}^2 s_{\beta_1}^2 + e^{4i\sigma_1} s_{\beta_1}^4\right] \\
& - \lambda_5 c_{\beta_1}^2 s_{\beta_1}^2 s_{\beta_2}^4 - \lambda_6 c_{\beta_1}^2 s_{\beta_1}^2 s_{\beta_2}^4 \\
& + \lambda_7 s_{\beta_2}^2 \left[\frac{1}{4}\left(c_{4\beta_1}+3\right) c_{2\sigma_1} + 2c_{\beta_1}^2 c_{\beta_2}^2 c_{2\sigma_2} s_{\beta_1}^2 - i c_{2\beta_1} s_{2\sigma_1}\right] + \lambda_8 c_{\beta_1}^2 s_{\beta_1}^2 s_{\beta_2}^4,
\end{aligned} \tag{C.1.4d}
$$

$$
\begin{aligned}
Z_{1213} = & 2\lambda_1 c_{\beta_1} c_{\beta_2} s_{\beta_1}^2 s_{\beta_2}^3 - 2\lambda_2 c_{\beta_1} c_{\beta_2} s_{\beta_2}\left(s_{\beta_1}^2 + e^{-2i(\sigma_1-\sigma_2)} c_{\beta_1}^2\right) \\
& - \lambda_3 \left(2c_{\beta_2}^3 c_{\beta_1} s_{\beta_1}^2 s_{\beta_2} + 2e^{-2i(\sigma_1-\sigma_2)} c_{\beta_2} c_{\beta_1}^3 s_{\beta_2}\right) \\
& + \frac{1}{4}\lambda_4 e^{-i(2\sigma_1+\sigma_2)} \Big[4e^{2i\sigma_2} c_{\beta_1}^3 c_{2\beta_2} \\
& \qquad - e^{2i\sigma_1} c_{\beta_1} s_{\beta_1}^2 \left(6c_{2\beta_2} + c_{4\beta_2} - 4e^{2i\sigma_2}\left(c_{2\beta_2}+3\right) s_{\beta_2}^2 + 5\right) \Big] \\
& - 2\lambda_5 c_{\beta_1} c_{\beta_2} s_{\beta_1}^2 s_{\beta_2}^3 - 2\lambda_6 c_{\beta_1} c_{\beta_2} s_{\beta_1}^2 s_{\beta_2}^3 \\
& + 2\lambda_7 c_{\beta_1} c_{\beta_2} s_{\beta_2} \left[e^{-2i\sigma_1} c_{\beta_1}^2 + e^{-2i\sigma_2} s_{\beta_1}^2 \left(c_{\beta_2}^2 - e^{4i\sigma_2} s_{\beta_2}^2\right)\right] + 2\lambda_8 c_{\beta_1} c_{\beta_2} s_{\beta_1}^2 s_{\beta_2}^3,
\end{aligned} \tag{C.1.4e}
$$

$$
\begin{aligned}
Z_{1222} =& -2\lambda_1 \mathrm{c}_{\beta_1}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}^2\left(\mathrm{c}_{\beta_1}^2\mathrm{c}_{\beta_2}^2+\mathrm{s}_{\beta_1}^2\right)+2\lambda_2\mathrm{c}_{\beta_2}^2\mathrm{s}_{2\beta_1}\mathrm{s}_{\sigma_1-\sigma_2}\left(\mathrm{c}_{2\beta_1}\mathrm{s}_{\sigma_1-\sigma_2}+i\mathrm{c}_{\sigma_1-\sigma_2}\right)\\
&-2\lambda_3 e^{-2i(\sigma_1+\sigma_2)}\mathrm{c}_{\beta_1}\mathrm{s}_{\beta_1}\left(e^{2i\sigma_2}-e^{2i\sigma_1}\mathrm{c}_{\beta_2}^2\right)\left(e^{2i\sigma_2}\mathrm{c}_{\beta_1}^2\mathrm{c}_{\beta_2}^2+e^{2i\sigma_1}\mathrm{s}_{\beta_1}^2\right)\\
&+2\lambda_4 e^{-i(2\sigma_1+\sigma_2)}\mathrm{c}_{\beta_1}\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}\Big[-e^{2i\sigma_2}\mathrm{c}_{\beta_1}^2\\
&\qquad -e^{2i\sigma_1}\left(1+e^{2i\sigma_2}\right)\left(\mathrm{c}_{\beta_1}^2\left(\mathrm{c}_{\beta_2}^2+1\right)-\mathrm{s}_{\beta_1}^2\right)+e^{4i\sigma_1}\mathrm{s}_{\beta_1}^2\Big]\\
&+\lambda_5\mathrm{c}_{\beta_1}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}^2\left(\mathrm{c}_{2\beta_2}\mathrm{c}_{\beta_1}^2+\mathrm{s}_{\beta_1}^2\right)+\lambda_6\mathrm{c}_{\beta_1}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}^2\left(\mathrm{c}_{2\beta_2}\mathrm{c}_{\beta_1}^2+\mathrm{s}_{\beta_1}^2\right)\\
&+\frac{1}{2}\lambda_7\mathrm{s}_{\beta_2}^2\left[8\mathrm{c}_{\beta_2}^2\mathrm{c}_{\beta_1}^3\mathrm{c}_{2\sigma_2}\mathrm{s}_{\beta_1}-\mathrm{c}_{2\sigma_1}\mathrm{s}_{4\beta_1}+2i\mathrm{s}_{2\beta_1}\mathrm{s}_{2\sigma_1}\right]+2\lambda_8\mathrm{c}_{\beta_1}^3\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}^4,
\end{aligned}
\tag{C.1.4f}
$$

$$
\begin{aligned}
Z_{1223} =& 2\lambda_1\mathrm{c}_{\beta_1}^2\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}^3+2\lambda_2\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}\left(\mathrm{s}_{\beta_1}^2+e^{-2i(\sigma_1-\sigma_2)}\mathrm{c}_{\beta_1}^2\right)\\
&+2\lambda_3\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}\left[-\mathrm{s}_{\beta_1}^2+\mathrm{c}_{\beta_1}^2\left(-\mathrm{c}_{\beta_2}^2+e^{-2i(\sigma_1-\sigma_2)}-1\right)\right]\\
&+\frac{1}{8}\lambda_4 e^{-i(2\sigma_1+\sigma_2)}\mathrm{s}_{\beta_1}\Big[-2e^{2i\sigma_1}\left(1+e^{2i\sigma_2}\right)\mathrm{c}_{4\beta_2}\mathrm{c}_{\beta_1}^2+e^{2i\sigma_1}\left(-1+e^{2i\sigma_2}\right)\left(5\mathrm{c}_{2\beta_1}+1\right)\\
&\qquad +2\mathrm{c}_{2\beta_2}\left(-e^{2i\sigma_1}\left(3\mathrm{c}_{2\beta_1}+1\right)-2e^{2i\sigma_2}\left(1+\left(1+e^{2i\sigma_1}\right)\mathrm{c}_{2\beta_1}\right)\right)\Big]\\
&-2\lambda_5\mathrm{c}_{\beta_1}^2\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}^3+\lambda_6\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}\left(\mathrm{c}_{2\beta_2}\mathrm{c}_{\beta_1}^2+\mathrm{s}_{\beta_1}^2\right)\\
&-2\lambda_7 e^{-2i(\sigma_1+\sigma_2)}\mathrm{c}_{\beta_1}^2\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}\left[e^{2i\sigma_1}\left(-\mathrm{c}_{\beta_2}^2+e^{4i\sigma_2}\mathrm{s}_{\beta_2}^2\right)+e^{2i\sigma_2}\right]+2\lambda_8\mathrm{c}_{\beta_1}^2\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}^3,
\end{aligned}
\tag{C.1.4g}
$$

$$
\begin{aligned}
Z_{1231} =& 2\lambda_1\mathrm{c}_{\beta_1}\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}^2\mathrm{s}_{\beta_2}^3+2\lambda_2\mathrm{c}_{\beta_1}\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_2}\left(\mathrm{c}_{\beta_1}^2+e^{2i(\sigma_1-\sigma_2)}\mathrm{s}_{\beta_1}^2\right)\\
&+2\lambda_3\mathrm{c}_{\beta_1}\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_2}\left[-\mathrm{c}_{\beta_1}^2+\mathrm{s}_{\beta_1}^2\left(-\mathrm{c}_{\beta_2}^2+e^{2i(\sigma_1-\sigma_2)}-1\right)\right]\\
&+\lambda_4 e^{-i\sigma_2}\Big[\mathrm{c}_{\beta_1}^3\left(-\mathrm{s}_{\beta_2}^2+e^{2i\sigma_2}\mathrm{c}_{\beta_2}^2\right)\\
&\qquad -\frac{1}{4}\mathrm{c}_{\beta_1}\mathrm{s}_{\beta_1}^2\left(\left(2+4\left(e^{2i\sigma_1}+e^{2i\sigma_2}\right)\right)\mathrm{c}_{2\beta_2}+e^{2i\sigma_2}\left(\mathrm{c}_{4\beta_2}+3\right)+\mathrm{c}_{4\beta_2}-3\right)\Big]\\
&-2\lambda_5\mathrm{c}_{\beta_1}\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}^2\mathrm{s}_{\beta_2}^3+\lambda_6\mathrm{c}_{\beta_1}\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_2}\left(\mathrm{c}_{\beta_1}^2+\mathrm{c}_{2\beta_2}\mathrm{s}_{\beta_1}^2\right)\\
&+2\lambda_7 e^{-2i\sigma_2}\mathrm{c}_{\beta_1}\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}^2\mathrm{s}_{\beta_2}\left(e^{4i\sigma_2}\mathrm{c}_{\beta_2}^2-\mathrm{s}_{\beta_2}^2-e^{2i(\sigma_1+\sigma_2)}\right)+2\lambda_8\mathrm{c}_{\beta_1}\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}^2\mathrm{s}_{\beta_2}^3,
\end{aligned}
\tag{C.1.4h}
$$

$$
\begin{aligned}
Z_{1232} =& 2\lambda_1\mathrm{c}_{\beta_1}^2\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}^3+2\lambda_2\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}\left(-\mathrm{c}_{\beta_1}^2-e^{2i(\sigma_1-\sigma_2)}\mathrm{s}_{\beta_1}^2\right)\\
&-\lambda_3\left(2\mathrm{c}_{\beta_1}^2\mathrm{c}_{\beta_2}^3\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}+e^{2i(\sigma_1-\sigma_2)}\mathrm{s}_{\beta_1}^3\mathrm{s}_{2\beta_2}\right)\\
&+\lambda_4 e^{-i\sigma_2}\left[e^{2i\sigma_1}\mathrm{c}_{2\beta_2}\mathrm{s}_{\beta_1}^3+\mathrm{c}_{\beta_1}^2\mathrm{s}_{\beta_1}\left(\left(\mathrm{c}_{2\beta_2}+3\right)\mathrm{s}_{\beta_2}^2-e^{2i\sigma_2}\mathrm{c}_{\beta_2}^2\left(\mathrm{c}_{2\beta_2}+2\right)\right)\right]\\
&-2\lambda_5\mathrm{c}_{\beta_1}^2\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}^3-2\lambda_6\mathrm{c}_{\beta_1}^2\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}^3\\
&+2\lambda_7 e^{-2i\sigma_2}\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}\left[\mathrm{c}_{\beta_1}^2\left(-\mathrm{s}_{\beta_2}^2+e^{4i\sigma_2}\mathrm{c}_{\beta_2}^2\right)+e^{2i(\sigma_1+\sigma_2)}\mathrm{s}_{\beta_1}^2\right]+2\lambda_8\mathrm{c}_{\beta_1}^2\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_1}\mathrm{s}_{\beta_2}^3,
\end{aligned}
\tag{C.1.4i}
$$

$$
\begin{aligned}
Z_{1313} =& \lambda_1\mathrm{c}_{\beta_2}^2\mathrm{s}_{\beta_1}^2\mathrm{s}_{\beta_2}^2+\lambda_2 e^{-2i(\sigma_1-\sigma_2)}\mathrm{c}_{\beta_1}^2\mathrm{s}_{\beta_2}^2+\lambda_3\mathrm{s}_{\beta_2}^2\left(\mathrm{c}_{\beta_2}^2\mathrm{s}_{\beta_1}^2+e^{-2i(\sigma_1-\sigma_2)}\mathrm{c}_{\beta_1}^2\right)\\
&+\lambda_4 e^{-i\sigma_2}\mathrm{c}_{\beta_2}\mathrm{s}_{\beta_2}\left[\mathrm{s}_{\beta_1}^2\left(\mathrm{c}_{\beta_2}^2-e^{2i\sigma_2}\mathrm{s}_{\beta_2}^2\right)-e^{-2i(\sigma_1-\sigma_2)}\mathrm{c}_{\beta_1}^2\right]+\lambda_5\mathrm{c}_{\beta_2}^2\mathrm{s}_{\beta_1}^2\mathrm{s}_{\beta_2}^2\\
&-\lambda_6\mathrm{c}_{\beta_2}^2\mathrm{s}_{\beta_1}^2\mathrm{s}_{\beta_2}^2+\lambda_7\left[e^{-2i\sigma_1}\mathrm{c}_{\beta_1}^2\mathrm{c}_{\beta_2}^2+e^{-2i\sigma_2}\mathrm{s}_{\beta_1}^2\left(\mathrm{c}_{\beta_2}^4+e^{4i\sigma_2}\mathrm{s}_{\beta_2}^4\right)\right]+\lambda_8\mathrm{c}_{\beta_2}^2\mathrm{s}_{\beta_1}^2\mathrm{s}_{\beta_2}^2,
\end{aligned}
\tag{C.1.4j}
$$

$$\begin{aligned}
Z_{1322} = & -2\lambda_1 c_{\beta_2} s_{\beta_1} s_{\beta_2}\left(c_{\beta_1}^2 c_{\beta_2}^2 + s_{\beta_1}^2\right) + 2\lambda_2\left(-1 + e^{-2i(\sigma_1-\sigma_2)}\right) c_{\beta_1}^2 c_{\beta_2} s_{\beta_1} s_{\beta_2} \\
& + 2\lambda_3 c_{\beta_2} s_{\beta_1} s_{\beta_2}\left[s_{\beta_1}^2 + c_{\beta_1}^2\left(s_{\beta_2}^2 + e^{-2i(\sigma_1-\sigma_2)}\right)\right] \\
& + \frac{1}{8}\lambda_4 e^{-i(2\sigma_1+\sigma_2)} s_{\beta_1}\Big[-2e^{2i\sigma_1}\left(1 + e^{2i\sigma_2}\right) c_{4\beta_2} c_{\beta_1}^2 \\
& \qquad + e^{2i\sigma_1}\left(-1 + e^{2i\sigma_2}\right)\left(5c_{2\beta_1} + 1\right) \\
& \qquad + 2c_{2\beta_2}\left(-e^{2i\sigma_1}\left(3c_{2\beta_1} + 1\right) - 2e^{2i\sigma_2}\left(1 + \left(1 + e^{2i\sigma_1}\right) c_{2\beta_1}\right)\right)\Big] \\
& + \lambda_5 c_{\beta_2} s_{\beta_1} s_{\beta_2}\left(c_{2\beta_2} c_{\beta_1}^2 + s_{\beta_1}^2\right) - 2\lambda_6 c_{\beta_1}^2 c_{\beta_2} s_{\beta_1} s_{\beta_2}^3 \\
& + 2\lambda_7 e^{-2i(\sigma_1+\sigma_2)} c_{\beta_1}^2 c_{\beta_2} s_{\beta_1} s_{\beta_2}\left[e^{2i\sigma_1}\left(-c_{\beta_2}^2 + e^{4i\sigma_2} s_{\beta_2}^2\right) + e^{2i\sigma_2}\right] + 2\lambda_8 c_{\beta_1}^2 c_{\beta_2} s_{\beta_1} s_{\beta_2}^3,
\end{aligned} \tag{C.1.4k}$$

$$\begin{aligned}
Z_{1323} = & \lambda_1 c_{\beta_2}^2 s_{2\beta_1} s_{\beta_2}^2 - 2\lambda_2 e^{-2i(\sigma_1-\sigma_2)} c_{\beta_1} s_{\beta_1} s_{\beta_2}^2 + 2\lambda_3 c_{\beta_1} s_{\beta_1} s_{\beta_2}^2\left(c_{\beta_2}^2 - e^{-2i(\sigma_1-\sigma_2)}\right) \\
& + 2\lambda_4 e^{-i(2\sigma_1+\sigma_2)} c_{\beta_1} c_{\beta_2} s_{\beta_1} s_{\beta_2}\left[e^{2i\sigma_1}\left(c_{\beta_2}^2 - e^{2i\sigma_2} s_{\beta_2}^2\right) + e^{2i\sigma_2}\right] - 2\lambda_5 c_{\beta_1} c_{\beta_2}^2 s_{\beta_1} s_{\beta_2}^2 \\
& + 2\lambda_6 c_{\beta_1} c_{\beta_2}^2 s_{\beta_1} s_{\beta_2}^2 + 2\lambda_7 c_{\beta_1} s_{\beta_1}\left(e^{-2i\sigma_2} c_{\beta_2}^4 - e^{-2i\sigma_1} c_{\beta_2}^2 + e^{2i\sigma_2} s_{\beta_2}^4\right) + \lambda_8 c_{\beta_2}^2 s_{2\beta_1} s_{\beta_2}^2,
\end{aligned} \tag{C.1.4l}$$

$$\begin{aligned}
Z_{1333} = & -2\lambda_1 c_{\beta_2} s_{\beta_1} s_{\beta_2}^3 - 2\lambda_3 c_{\beta_2} s_{\beta_1} s_{\beta_2}^3 - \lambda_4 e^{-i\sigma_2} s_{\beta_1} s_{\beta_2}^2\left[1 + \left(1 + e^{2i\sigma_2}\right) c_{2\beta_2}\right] \\
& + \frac{1}{4}\lambda_5 s_{\beta_1} s_{4\beta_2} - \frac{1}{4}\lambda_6 s_{\beta_1} s_{4\beta_2} + 2\lambda_7 e^{-2i\sigma_2} c_{\beta_2} s_{\beta_1} s_{\beta_2}\left(-c_{\beta_2}^2 + e^{4i\sigma_2} s_{\beta_2}^2\right) + 2\lambda_8 c_{\beta_2}^3 s_{\beta_1} s_{\beta_2},
\end{aligned} \tag{C.1.4m}$$

$$\begin{aligned}
Z_{2223} = & -2\lambda_1 c_{\beta_1} c_{\beta_2} s_{\beta_2}\left(c_{\beta_1}^2 c_{\beta_2}^2 + s_{\beta_1}^2\right) + \lambda_2 e^{-2i\sigma_1}\left(e^{2i\sigma_1} - e^{2i\sigma_2}\right) c_{\beta_1} s_{\beta_1}^2 s_{2\beta_2} \\
& - \lambda_3\left(2c_{\beta_1}^3 c_{\beta_2}^3 s_{\beta_2} + 2e^{-2i(\sigma_1-\sigma_2)} c_{\beta_1} c_{\beta_2} s_{\beta_1}^2 s_{\beta_2}\right) \\
& + \lambda_4 e^{-i(2\sigma_1+\sigma_2)} c_{\beta_1}\Big[e^{2i\sigma_2}\left(c_{2\beta_2} s_{\beta_1}^2 + 2e^{2i\sigma_1} s_{\beta_2}^2\left(c_{\beta_1}^2 c_{\beta_2}^2 - s_{\beta_1}^2\right)\right) \\
& \qquad - e^{2i\sigma_1} c_{\beta_2}^2\left(c_{\beta_1}^2 c_{2\beta_2} - 2s_{\beta_1}^2\right)\Big] \\
& + \lambda_5 c_{\beta_1} c_{\beta_2} s_{\beta_2}\left(c_{2\beta_2} c_{\beta_1}^2 + s_{\beta_1}^2\right) + \lambda_6 c_{\beta_1} c_{\beta_2} s_{\beta_2}\left(c_{2\beta_2} c_{\beta_1}^2 + s_{\beta_1}^2\right) \\
& + 2\lambda_7 c_{\beta_1} c_{\beta_2} s_{\beta_2}\left[e^{-2i\sigma_2} c_{\beta_1}^2\left(c_{\beta_2}^2 - e^{4i\sigma_2} s_{\beta_2}^2\right) + e^{-2i\sigma_1} s_{\beta_1}^2\right] + 2\lambda_8 c_{\beta_1}^3 c_{\beta_2} s_{\beta_2}^3,
\end{aligned} \tag{C.1.4n}$$

$$\begin{aligned}
Z_{2323} = & \lambda_1 c_{\beta_1}^2 c_{\beta_2}^2 s_{\beta_2}^2 + \lambda_2 e^{-2i(\sigma_1-\sigma_2)} s_{\beta_1}^2 s_{\beta_2}^2 + \lambda_3 s_{\beta_2}^2\left(c_{\beta_1}^2 c_{\beta_2}^2 + e^{-2i(\sigma_1-\sigma_2)} s_{\beta_1}^2\right) \\
& + \lambda_4 e^{-i\sigma_2} c_{\beta_2} s_{\beta_2}\left[c_{\beta_1}^2\left(c_{\beta_2}^2 - e^{2i\sigma_2} s_{\beta_2}^2\right) - e^{-2i(\sigma_1-\sigma_2)} s_{\beta_1}^2\right] + \lambda_5 c_{\beta_1}^2 c_{\beta_2}^2 s_{\beta_2}^2 \\
& - \lambda_6 c_{\beta_1}^2 c_{\beta_2}^2 s_{\beta_2}^2 + \lambda_7\left[e^{-2i\sigma_2} c_{\beta_1}^2\left(c_{\beta_2}^4 + e^{4i\sigma_2} s_{\beta_2}^4\right) + e^{-2i\sigma_1} c_{\beta_2}^2 s_{\beta_1}^2\right] + \lambda_8 c_{\beta_1}^2 c_{\beta_2}^2 s_{\beta_2}^2,
\end{aligned} \tag{C.1.4o}$$

$$\begin{aligned}
Z_{2333} = & -2\lambda_1 c_{\beta_1} c_{\beta_2} s_{\beta_2}^3 - 2\lambda_3 c_{\beta_1} c_{\beta_2} s_{\beta_2}^3 - \lambda_4 e^{-i\sigma_2} c_{\beta_1} s_{\beta_2}^2\left[1 + \left(1 + e^{2i\sigma_2}\right) c_{2\beta_2}\right] \\
& + \frac{1}{4}\lambda_5 c_{\beta_1} s_{4\beta_2} - \frac{1}{4}\lambda_6 c_{\beta_1} s_{4\beta_2} + 2\lambda_7 e^{-2i\sigma_2} c_{\beta_1} c_{\beta_2} s_{\beta_2}\left(-c_{\beta_2}^2 + e^{4i\sigma_2} s_{\beta_2}^2\right) + 2\lambda_8 c_{\beta_1} c_{\beta_2}^3 s_{\beta_2}.
\end{aligned} \tag{C.1.4p}$$

Couplings which potentially lead to CP violation are presented in eq. (C.1.4). In terms of the generic basis, only terms multiplied by λ_2, or λ_3, or λ_4, or λ_7 may result in CP violation in the Higgs basis[1]. If the soft symmetry breaking terms of the generic basis (3.2.2) are added, the bilinear terms in the Higgs basis may also lead to CP violation. In this case the couplings Y'_{12}, or Y'_{13}, or Y'_{23} may result in CP violation. Not all of the Z couplings should be checked as:

$$\mathrm{Im}\left(Z_{1112}\right) = \mathrm{Im}\left(Z_{1222}^*\right), \tag{C.1.5a}$$

$$\mathrm{Im}\left(Z_{1123}\right) = \mathrm{Im}\left(Z_{1231}^*\right), \tag{C.1.5b}$$

$$\mathrm{Im}\left(Z_{1223}\right) = \mathrm{Im}\left(Z_{1322}\right). \tag{C.1.5c}$$

It should be noted that this is only true for transformation given by eq. (2.5.14).

[1] This is only true when an additional $U(2)$ transformation is not considered.

C.2 $\mathcal{R}_{\beta_y}\mathcal{R}_{\beta_z}$ Higgs Basis Rotation

We present relations between the couplings in the Higgs basis and the generic basis of eqs. (2.5.9, 2.5.10) using transformations given by eq. (2.5.26).

Quadratic couplings in the Higgs basis are:

$$Y_{11} = \frac{1}{v^2}\left(\mu_0^2 \hat{w}_S^2 + \mu_1^2 w^2\right), \tag{C.2.1a}$$
$$Y_{22} = \mu_1^2, \tag{C.2.1b}$$
$$Y_{13} = \frac{w^2}{N_3 v}\left(-\mu_0^2 + \mu_1^2\right), \tag{C.2.1c}$$
$$Y_{33} = \frac{1}{v^2}\left(\mu_0^2 w^2 + \mu_1^2 \hat{w}_S^2\right), \tag{C.2.1d}$$
$$\tag{C.2.1e}$$

and the soft symmetry breaking terms of eq. (3.2.2) result in:

$$Y'_{11} = Y_{11} + \frac{1}{v^2}\left[\nu^2 \hat{w}_1 \hat{w}_2 c_{\sigma_1-\sigma_2} + \left(\hat{w}_1^2 - \hat{w}_2^2\right)\mu_2^2 + \hat{w}_1 \hat{w}_S c_{\sigma_1}\mu_3^2 + \hat{w}_2 \hat{w}_S c_{\sigma_2}\mu_4^2\right], \tag{C.2.2a}$$
$$Y'_{12} = \frac{1}{N_2 v^2}\left[\frac{1}{2}\nu^2 e^{-i(\sigma_1+\sigma_2)} v\left(e^{2i\sigma_1}\hat{w}_2^2 - e^{2i\sigma_2}\hat{w}_1^2\right) + 2v\hat{w}_1\hat{w}_2\mu_2^2 + \frac{1}{2}e^{i\sigma_1} v\hat{w}_2\hat{w}_S\mu_3^2 - \frac{1}{2}e^{i\sigma_2} v\hat{w}_1\hat{w}_S\mu_4^2\right], \tag{C.2.2b}$$
$$\begin{aligned} Y'_{13} = Y_{13} + \frac{1}{N_3 v}\Bigg[&\nu^2 \hat{w}_1\hat{w}_2 c_{\sigma_1-\sigma_2} + \left(\hat{w}_1^2 - \hat{w}_2^2\right)\mu_2^2 + \frac{1}{2}e^{-i\sigma_1}\hat{w}_1\mu_3^2\left(X + e^{2i\sigma_1}\hat{w}_S\right) \\ &+ \frac{1}{2}e^{-i\sigma_2}\hat{w}_2\mu_4^2\left(X + e^{2i\sigma_2}\hat{w}_S\right)\Bigg], \end{aligned} \tag{C.2.2c}$$
$$Y'_{22} = Y_{22} + \frac{1}{N_2^2}\left[-\nu^2\hat{w}_1\hat{w}_2 c_{\sigma_1-\sigma_2} + \left(\hat{w}_2^2 - \hat{w}_1^2\right)\mu_2^2\right], \tag{C.2.2d}$$
$$Y'_{23} = \frac{1}{N_2 N_3}\left[\frac{1}{2}\nu^2 e^{-i(\sigma_1+\sigma_2)}\left(e^{2i\sigma_2}\hat{w}_2^2 - e^{2i\sigma_1}\hat{w}_1^2\right) + 2\hat{w}_1\hat{w}_2\mu_2^2 + \frac{1}{2}e^{-i\sigma_1}\hat{w}_2 X\mu_3^2 - \frac{1}{2}e^{-i\sigma_2}\hat{w}_1 X\mu_4^2\right], \tag{C.2.2e}$$
$$Y'_{33} = Y_{33} + \frac{1}{N_2 N_3^2}\left[\nu^2\hat{w}_1\hat{w}_2 c_{\sigma_1-\sigma_2} + \left(\hat{w}_1^2 - \hat{w}_2^2\right)\mu_2^2 + \hat{w}_1 X c_{\sigma_1}\mu_3^2 + \hat{w}_2 X c_{\sigma_2}\mu_4^2\right]. \tag{C.2.2f}$$

The real quartic couplings are:

$$\begin{aligned} Z_{1111} = \frac{1}{v^4}\Bigg\{&\lambda_1 w^4 - 4\lambda_2\hat{w}_1^2\hat{w}_2^2 s_{\sigma_1-\sigma_2}^2 + \lambda_3\left[2\hat{w}_2^2\hat{w}_1^2 c_{2(\sigma_1-\sigma_2)} + \hat{w}_1^4 + \hat{w}_2^4\right] \\ &+ 2\lambda_4\hat{w}_2\hat{w}_S\left[\hat{w}_1^2\left(c_{2\sigma_1-\sigma_2} + 2c_{\sigma_2}\right) - \hat{w}_2^2 c_{\sigma_2}\right] + \lambda_5 w^2\hat{w}_S^2 \\ &+ \lambda_6 w^2\hat{w}_S^2 + 2\lambda_7\hat{w}_S^2\left(\hat{w}_1^2 c_{2\sigma_1} + \hat{w}_2^2 c_{2\sigma_2}\right) + \lambda_8\hat{w}_S^4\Bigg\}, \end{aligned} \tag{C.2.3a}$$

$$\begin{aligned} Z_{1122} = \frac{1}{N_2^2 v^2}\Bigg\{&2\lambda_1 w^4 + 8\lambda_2\hat{w}_1^2\hat{w}_2^2 s_{\sigma_1-\sigma_2}^2 - 2\lambda_3\left[2\hat{w}_2^2\hat{w}_1^2 c_{2(\sigma_1-\sigma_2)} + \hat{w}_1^4 + \hat{w}_2^4\right] \\ &+ 2\lambda_4\hat{w}_2\hat{w}_S\left[\hat{w}_2^2 c_{\sigma_2} - \hat{w}_1^2\left(c_{2\sigma_1-\sigma_2} + 2c_{\sigma_2}\right)\right] + \lambda_5 w^2\hat{w}_S^2\Bigg\}, \end{aligned} \tag{C.2.3b}$$

$$\begin{aligned} Z_{1133} = \frac{1}{N_3^2 v^2}\Bigg\{&2\lambda_1 w^4 - 8\lambda_2\hat{w}_1^2\hat{w}_2^2 s_{\sigma_1-\sigma_2}^2 + 2\lambda_3\left[2\hat{w}_2^2\hat{w}_1^2 c_{2(\sigma_1-\sigma_2)} + \hat{w}_1^4 + \hat{w}_2^4\right] \\ &+ 2\lambda_4\hat{w}_2\left[\hat{w}_1^2\left(c_{2\sigma_1-\sigma_2} + 2c_{\sigma_2}\right) - \hat{w}_2^2 c_{\sigma_2}\right]\left(\hat{w}_S + X\right) + \lambda_5 w^2\left(\hat{w}_S^2 + X^2\right) \\ &- 2\lambda_6 w^4 - 4\lambda_7 w^2\left(\hat{w}_1^2 c_{2\sigma_1} + \hat{w}_2^2 c_{2\sigma_2}\right) + 2\lambda_8 w^4\Bigg\}, \end{aligned} \tag{C.2.3c}$$

$$Z_{1221} = \frac{1}{N_2^2 v^2}\Bigg\{ -2\lambda_2\left[2\hat{w}_2^2\hat{w}_1^2 c_{2(\sigma_1-\sigma_2)} + \hat{w}_1^4 + \hat{w}_2^4\right] + 2\lambda_3\left[-2\hat{w}_2^2\hat{w}_1^2\left(c_{2(\sigma_1-\sigma_2)} - 2\right) + \hat{w}_1^4 + \hat{w}_2^4\right] + 2\lambda_4\hat{w}_2\hat{w}_S\left[\hat{w}_2^2 c_{\sigma_2} - \hat{w}_1^2\left(c_{2\sigma_1-\sigma_2} + 2c_{\sigma_2}\right)\right] + \lambda_6 w^2\hat{w}_S^2\Bigg\}, \tag{C.2.3d}$$

$$Z_{1331} = \frac{1}{N_3^2 v^2}\Bigg\{2\lambda_1 w^4 - 8\lambda_2\hat{w}_1^2\hat{w}_2^2 s_{\sigma_1-\sigma_2}^2 + 2\lambda_3\left[2\hat{w}_2^2\hat{w}_1^2 c_{2(\sigma_1-\sigma_2)} + \hat{w}_1^4 + \hat{w}_2^4\right] + 2\lambda_4\hat{w}_2\left[\hat{w}_1^2\left(c_{2\sigma_1-\sigma_2} + 2c_{\sigma_2}\right) - \hat{w}_2^2 c_{\sigma_2}\right]\left(\hat{w}_S + X\right) - 2\lambda_5 w^4 + \lambda_6 w^2\left(\hat{w}_S^2 + X^2\right) - 4\lambda_7 w^2\left(\hat{w}_1^2 c_{2\sigma_1} + \hat{w}_2^2 c_{2\sigma_2}\right) + 2\lambda_8 w^4\Bigg\}, \tag{C.2.3e}$$

$$Z_{2222} = \frac{1}{N_2^4}\Bigg\{\lambda_1 w^4 - 4\lambda_2\hat{w}_1^2\hat{w}_2^2 s_{\sigma_1-\sigma_2}^2 + \lambda_3\left[2\hat{w}_2^2\hat{w}_1^2 c_{2(\sigma_1-\sigma_2)} + \hat{w}_1^4 + \hat{w}_2^4\right]\Bigg\}, \tag{C.2.3f}$$

$$Z_{2233} = \frac{1}{N_2^2 N_3^2}\Bigg\{2\lambda_1 w^4 + 8\lambda_2\hat{w}_1^2\hat{w}_2^2 s_{\sigma_1-\sigma_2}^2 - 2\lambda_3\left[2\hat{w}_2^2\hat{w}_1^2 c_{2(\sigma_1-\sigma_2)} + \hat{w}_1^4 + \hat{w}_2^4\right] + \lambda_4\left[2\hat{w}_2^3 X c_{\sigma_2} - 2\hat{w}_1^2\hat{w}_2 X\left(c_{2\sigma_1-\sigma_2} + 2c_{\sigma_2}\right)\right] + \lambda_5 w^2 X^2\Bigg\}, \tag{C.2.3g}$$

$$Z_{2332} = \frac{1}{N_2^2 N_3^2}\Bigg\{ -2\lambda_2\left[2\hat{w}_2^2\hat{w}_1^2 c_{2(\sigma_1-\sigma_2)} + \hat{w}_1^4 + \hat{w}_2^4\right] + 2\lambda_3\left[-2\hat{w}_2^2\hat{w}_1^2\left(c_{2(\sigma_1-\sigma_2)} - 2\right) + \hat{w}_1^4 + \hat{w}_2^4\right] + \lambda_4\left(2\hat{w}_2^3 X c_{\sigma_2} - 2\hat{w}_1^2\hat{w}_2 X\left(c_{2\sigma_1-\sigma_2} + 2c_{\sigma_2}\right)\right] + \lambda_6 w^2 X^2\Bigg\}, \tag{C.2.3h}$$

$$Z_{3333} = \frac{1}{N_3^4}\Bigg\{\lambda_1 w^4 - 4\lambda_2\hat{w}_1^2\hat{w}_2^2 s_{\sigma_1-\sigma_2}^2 + \lambda_3\left[2\hat{w}_2^2\hat{w}_1^2 c_{2(\sigma_1-\sigma_2)} + \hat{w}_1^4 + \hat{w}_2^4\right] + \lambda_4\left[2\hat{w}_1^2\hat{w}_2 X\left(c_{2\sigma_1-\sigma_2} + 2c_{\sigma_2}\right) - 2\hat{w}_2^3 X c_{\sigma_2}\right] + \lambda_5 w^2 X^2 + \lambda_6 w^2 X^2 + 2\lambda_7 X^2\left(\hat{w}_1^2 c_{2\sigma_1} + \hat{w}_2^2 c_{2\sigma_2}\right) + \lambda_8 X^4\Bigg\}, \tag{C.2.3i}$$

and the complex ones are:

$$Z_{1112} = \frac{1}{N_2 v^3}\Bigg\{4\lambda_2\hat{w}_1\hat{w}_2 s_{\sigma_1-\sigma_2}\left[\left(\hat{w}_1^2 - \hat{w}_2^2\right)s_{\sigma_1-\sigma_2} + i w^2 c_{\sigma_1-\sigma_2}\right] + 4\lambda_3\hat{w}_1\hat{w}_2 s_{\sigma_1-\sigma_2}\left[\left(\hat{w}_1^2 - \hat{w}_2^2\right)s_{\sigma_1-\sigma_2} + i w^2 c_{\sigma_1-\sigma_2}\right] + \lambda_4 e^{-i\sigma_2}\hat{w}_1\hat{w}_S\left[\left(3 + 2e^{2i\sigma_1}\right)\hat{w}_2^2 + e^{2i\sigma_2}\left(4\hat{w}_2^2 - \left(2 + e^{-2i\sigma_1}\right)\hat{w}_1^2\right)\right] + 2\lambda_7\left(e^{2i\sigma_1} - e^{2i\sigma_2}\right)\hat{w}_1\hat{w}_2\hat{w}_S^2\Bigg\}, \tag{C.2.4a}$$

$$
\begin{aligned}
Z_{1113} = \frac{1}{N_3 v^3} \Bigg\{ & 2\lambda_1 w^4 - 8\lambda_2 w_1^2 \hat{w}_2^2 s^2_{\sigma_1-\sigma_2} + 2\lambda_3 \left[2\hat{w}_2^2 \hat{w}_1^2 c_{2(\sigma_1-\sigma_2)} + \hat{w}_1^4 + \hat{w}_2^4\right] \\
& + \lambda_4 e^{-i\sigma_2} \hat{w}_2 \Big[2e^{2i\sigma_1} \hat{w}_1^2 \hat{w}_S + e^{-2i(\sigma_1-\sigma_2)} \hat{w}_1^2 \left(\hat{w}_S + X\right) \\
& \qquad + \left(2\hat{w}_1^2 - \hat{w}_2^2\right)\left(2e^{2i\sigma_2}\hat{w}_S + \hat{w}_S + X\right) \Big] \\
& + \lambda_5 w^2 \hat{w}_S \left(\hat{w}_S + X\right) + \lambda_6 w^2 \hat{w}_S \left(\hat{w}_S + X\right) \\
& + 2\lambda_7 \hat{w}_S \left[e^{2i\sigma_2} \hat{w}_2^2 \hat{w}_S + e^{-2i\sigma_1} \hat{w}_1^2 \left(X + e^{4i\sigma_1}\hat{w}_S\right) + e^{-2i\sigma_2}\hat{w}_2^2 X\right] - 2\lambda_8 w^2 \hat{w}_S^2 \Bigg\},
\end{aligned}
\tag{C.2.4b}
$$

$$
\begin{aligned}
Z_{1123} = \frac{1}{N_2 N_3 v^2} \Bigg\{ & - 4i\lambda_2 e^{-i(\sigma_1+\sigma_2)} \hat{w}_1 \hat{w}_2 s_{\sigma_1-\sigma_2} \left(e^{2i\sigma_1}\hat{w}_1^2 + e^{2i\sigma_2}\hat{w}_2^2\right) \\
& - 4i\lambda_3 e^{-i(\sigma_1+\sigma_2)} \hat{w}_1 \hat{w}_2 s_{\sigma_1-\sigma_2} \left(e^{2i\sigma_1}\hat{w}_1^2 + e^{2i\sigma_2}\hat{w}_2^2\right) \\
& - \lambda_4 e^{-i\sigma_2} \hat{w}_1 \Big[e^{2i\sigma_1} \hat{w}_1^2 \hat{w}_S - e^{-2i(\sigma_1-\sigma_2)} \hat{w}_2^2 \left(3e^{2i\sigma_1}\hat{w}_S + \hat{w}_S + X\right) \\
& \qquad + \left(\hat{w}_1^2 - 2\hat{w}_2^2\right)\left(\hat{w}_S + X\right) \Big] \\
& - 2\lambda_7 \left(e^{-2i\sigma_1} - e^{-2i\sigma_2}\right) w^2 \hat{w}_1 \hat{w}_2 \Bigg\},
\end{aligned}
\tag{C.2.4c}
$$

$$
\begin{aligned}
Z_{1212} = \frac{1}{N_2^2 v^2} \Bigg\{ & \lambda_2 e^{-2i(\sigma_1+\sigma_2)} \left(e^{2i\sigma_2}\hat{w}_1^2 + e^{2i\sigma_1}\hat{w}_2^2\right)^2 + \lambda_3 e^{-2i(\sigma_1+\sigma_2)} \left(e^{2i\sigma_2}\hat{w}_1^2 + e^{2i\sigma_1}\hat{w}_2^2\right)^2 \\
& + \lambda_4 e^{-i\sigma_2} \hat{w}_2 \hat{w}_S \left(e^{2i\sigma_1}\hat{w}_2^2 - 3e^{2i\sigma_2}\hat{w}_1^2\right) + \lambda_7 \hat{w}_S^2 \left(e^{2i\sigma_2}\hat{w}_1^2 + e^{2i\sigma_1}\hat{w}_2^2\right) \Bigg\},
\end{aligned}
\tag{C.2.4d}
$$

$$
\begin{aligned}
Z_{1213} = \frac{1}{N_2 N_3 v^2} \Bigg\{ & 4\lambda_2 \hat{w}_1 \hat{w}_2 s_{\sigma_1-\sigma_2} \left[\left(\hat{w}_1^2 - \hat{w}_2^2\right) s_{\sigma_1-\sigma_2} + i w^2 c_{\sigma_1-\sigma_2}\right] \\
& + 4\lambda_3 \hat{w}_1 \hat{w}_2 s_{\sigma_1-\sigma_2} \left[\left(\hat{w}_1^2 - \hat{w}_2^2\right) s_{\sigma_1-\sigma_2} + i w^2 c_{\sigma_1-\sigma_2}\right] \\
& + \lambda_4 e^{-i\sigma_2} \hat{w}_1 \left[2e^{2i\sigma_1}\hat{w}_2^2 \hat{w}_S - 2e^{2i\sigma_2}\left(\hat{w}_1^2 - 2\hat{w}_2^2\right)\hat{w}_S - e^{-2i(\sigma_1-\sigma_2)}\hat{w}_1^2 X + 3\hat{w}_2^2 X\right] \\
& + 2\lambda_7 \left(e^{2i\sigma_1} - e^{2i\sigma_2}\right) \hat{w}_1 \hat{w}_2 \hat{w}_S^2 \Bigg\},
\end{aligned}
\tag{C.2.4e}
$$

$$
\begin{aligned}
Z_{1222} = \frac{1}{N_2^3 v} \Bigg\{ & - 4\lambda_2 \hat{w}_1 \hat{w}_2 s_{\sigma_1-\sigma_2} \left[\left(\hat{w}_1^2 - \hat{w}_2^2\right) s_{\sigma_1-\sigma_2} + i w^2 c_{\sigma_1-\sigma_2}\right] \\
& - 4\lambda_3 \hat{w}_1 \hat{w}_2 s_{\sigma_1-\sigma_2} \left[\left(\hat{w}_1^2 - \hat{w}_2^2\right) s_{\sigma_1-\sigma_2} + i w^2 c_{\sigma_1-\sigma_2}\right] \\
& + \lambda_4 e^{-i\sigma_2} \hat{w}_1 \hat{w}_S \left[e^{2i\sigma_2}\left(\hat{w}_1^2 - 2\hat{w}_2^2\right) - e^{2i\sigma_1}\hat{w}_2^2\right] \Bigg\},
\end{aligned}
\tag{C.2.4f}
$$

$$
\begin{aligned}
Z_{1223} = \frac{1}{N_2^2 N_3 v} \Bigg\{ & - 2\lambda_2 \left[2\hat{w}_2^2 \hat{w}_1^2 c_{2(\sigma_1-\sigma_2)} + \hat{w}_1^4 + \hat{w}_2^4\right] + 2\lambda_3 \left[-2\hat{w}_2^2 \hat{w}_1^2 \left(c_{2(\sigma_1-\sigma_2)} - 2\right) + \hat{w}_1^4 + \hat{w}_2^4\right] \\
& + \lambda_4 e^{-i\sigma_2} \hat{w}_2 \left[-e^{2i\sigma_1}\hat{w}_1^2 \hat{w}_S + \left(\hat{w}_2^2 - 2\hat{w}_1^2\right)\left(X + e^{2i\sigma_2}\hat{w}_S\right) - e^{-2i(\sigma_1-\sigma_2)}\hat{w}_1^2 X\right] - \lambda_6 w^4 \Bigg\},
\end{aligned}
\tag{C.2.4g}
$$

$$\begin{aligned}Z_{1231} = \frac{1}{N_2 N_3 v^2}\Bigg\{ & 4\lambda_2 \hat{w}_1 \hat{w}_2 s_{\sigma_1-\sigma_2}\left[\left(\hat{w}_1^2-\hat{w}_2^2\right)s_{\sigma_1-\sigma_2}+iw^2 c_{\sigma_1-\sigma_2}\right]\\ & + 4\lambda_3 \hat{w}_1 \hat{w}_2 s_{\sigma_1-\sigma_2}\left[\left(\hat{w}_1^2-\hat{w}_2^2\right)s_{\sigma_1-\sigma_2}+iw^2 c_{\sigma_1-\sigma_2}\right]\\ & + \lambda_4 e^{-i\sigma_2}\hat{w}_1\Bigg[-e^{-2i(\sigma_1-\sigma_2)}\hat{w}_1^2\hat{w}_S + e^{2i\sigma_1}\hat{w}_2^2\left(\hat{w}_S+X\right)\\ & \qquad - e^{2i\sigma_2}\left(\hat{w}_1^2-2\hat{w}_2^2\right)\left(\hat{w}_S+X\right)+3\hat{w}_2^2\hat{w}_S\Bigg]\\ & - 2\lambda_7\left(e^{2i\sigma_1}-e^{2i\sigma_2}\right)w^2\hat{w}_1\hat{w}_2\Bigg\},\end{aligned} \tag{C.2.4h}$$

$$\begin{aligned}Z_{1232} = \frac{1}{N_2^2 N_3 v}\Bigg\{ & 2\lambda_2 e^{-2i(\sigma_1+\sigma_2)}\left(e^{2i\sigma_2}\hat{w}_1^2+e^{2i\sigma_1}\hat{w}_2^2\right)^2+2\lambda_3 e^{-2i(\sigma_1+\sigma_2)}\left(e^{2i\sigma_2}\hat{w}_1^2+e^{2i\sigma_1}\hat{w}_2^2\right)^2\\ & + \lambda_4\hat{w}_2\left(e^{2i\sigma_1-i\sigma_2}\hat{w}_2^2-3e^{i\sigma_2}\hat{w}_1^2\right)\left(\hat{w}_S+X\right)-2\lambda_7 w^2\left(e^{2i\sigma_2}\hat{w}_1^2+e^{2i\sigma_1}\hat{w}_2^2\right)\Bigg\},\end{aligned} \tag{C.2.4i}$$

$$\begin{aligned}Z_{1233} = Z_{1332} = \frac{1}{N_2 N_3^2 v}\Bigg\{ & 4\lambda_2 \hat{w}_1 \hat{w}_2 s_{\sigma_1-\sigma_2}\left[\left(\hat{w}_1^2-\hat{w}_2^2\right)s_{\sigma_1-\sigma_2}+iw^2 c_{\sigma_1-\sigma_2}\right]\\ & + 4\lambda_3 \hat{w}_1 \hat{w}_2 s_{\sigma_1-\sigma_2}\left[\left(\hat{w}_1^2-\hat{w}_2^2\right)s_{\sigma_1-\sigma_2}+iw^2 c_{\sigma_1-\sigma_2}\right]\\ & + \lambda_4 e^{-i\sigma_2}\hat{w}_1\Big[e^{2i\sigma_1}\hat{w}_2^2\left(\hat{w}_S+X\right)-e^{2i\sigma_2}\left(\hat{w}_1^2-2\hat{w}_2^2\right)\left(\hat{w}_S+X\right)\\ & \qquad - e^{-2i(\sigma_1-\sigma_2)}\hat{w}_1^2 X+3\hat{w}_2^2 X\Big]\\ & - 2\lambda_7\left(e^{2i\sigma_1}-e^{2i\sigma_2}\right)w^2\hat{w}_1\hat{w}_2\Bigg\},\end{aligned} \tag{C.2.4j}$$

$$\begin{aligned}Z_{1313} = \frac{1}{N_3^2 v^2}\Bigg\{ & \lambda_1 w^4 - 4\lambda_2\hat{w}_1^2\hat{w}_2^2 s_{\sigma_1-\sigma_2}^2+\lambda_3\left(2\hat{w}_2^2\hat{w}_1^2 c_{2(\sigma_1-\sigma_2)}+\hat{w}_1^4+\hat{w}_2^4\right)\\ & + \lambda_4 e^{-i\sigma_2}\hat{w}_2\left[e^{2i\sigma_1}\hat{w}_1^2\hat{w}_S+\left(2\hat{w}_1^2-\hat{w}_2^2\right)\left(X+e^{2i\sigma_2}\hat{w}_S\right)+e^{-2i(\sigma_1-\sigma_2)}\hat{w}_1^2 X\right]-\lambda_5 w^4\\ & - \lambda_6 w^4\hat{w}_S+\lambda_7\left[e^{2i\sigma_2}\hat{w}_2^2\hat{w}_S^2+e^{-2i\sigma_1}\hat{w}_1^2\left(X^2+e^{4i\sigma_1}\hat{w}_S^2\right)+e^{-2i\sigma_2}\hat{w}_2^2X^2\right]+\lambda_8 w^4\Bigg\},\end{aligned} \tag{C.2.4k}$$

$$\begin{aligned}Z_{1322} = \frac{1}{N_2^2 N_3 v}\Bigg\{ & 2\lambda_1 w^4 + 8\lambda_2\hat{w}_1^2\hat{w}_2^2 s_{\sigma_1-\sigma_2}^2-2\lambda_3\left[2\hat{w}_2^2\hat{w}_1^2 c_{2(\sigma_1-\sigma_2)}+\hat{w}_1^4+\hat{w}_2^4\right]\\ & + \lambda_4 e^{-i\sigma_2}\hat{w}_2\left[-e^{2i\sigma_1}\hat{w}_1^2\hat{w}_S+\left(\hat{w}_2^2-2\hat{w}_1^2\right)\left(X+e^{2i\sigma_2}\hat{w}_S\right)-e^{-2i(\sigma_1-\sigma_2)}\hat{w}_1^2 X\right]-\lambda_5 w^4\Bigg\},\end{aligned} \tag{C.2.4l}$$

$$\begin{aligned}Z_{1323} = \frac{1}{N_2 N_3^2 v}\Bigg\{ & -4i\lambda_2 e^{-i(\sigma_1+\sigma_2)}\hat{w}_1\hat{w}_2 s_{\sigma_1-\sigma_2}\left(e^{2i\sigma_1}\hat{w}_1^2+e^{2i\sigma_2}\hat{w}_2^2\right)\\ & - 4i\lambda_3 e^{-i(\sigma_1+\sigma_2)}\hat{w}_1\hat{w}_2 s_{\sigma_1-\sigma_2}\left(e^{2i\sigma_1}\hat{w}_1^2+e^{2i\sigma_2}\hat{w}_2^2\right)\\ & + \lambda_4 e^{-i\sigma_2}\hat{w}_1\left[\hat{w}_S\left(3e^{2i\sigma_2}\hat{w}_2^2-e^{2i\sigma_1}\hat{w}_1^2\right)+2\left(2+e^{-2i(\sigma_1-\sigma_2)}\right)\hat{w}_2^2 X-2\hat{w}_1^2 X\right]\\ & + 2\lambda_7\left(e^{-2i\sigma_1}-e^{-2i\sigma_2}\right)\hat{w}_1\hat{w}_2 X^2\Bigg\},\end{aligned} \tag{C.2.4m}$$

$$Z_{1333} = \frac{1}{N_3^3 v}\Bigg\{2\lambda_1 w^4 - 8\lambda_2 \hat{w}_1^2 \hat{w}_2^2 s_{\sigma_1-\sigma_2}^2 + 2\lambda_3 \left(2\hat{w}_2^2 \hat{w}_1^2 c_{2(\sigma_1-\sigma_2)} + \hat{w}_1^4 + \hat{w}_2^4\right)$$
$$+ \lambda_4 e^{-i\sigma_2} \hat{w}_2 \Bigg[e^{2i\sigma_1} \hat{w}_1^2 \hat{w}_S + \left(2\hat{w}_1^2 - \hat{w}_2^2\right)\left(2X + e^{2i\sigma_2}\left(\hat{w}_S + X\right)\right)$$
$$+ e^{2i\sigma_1} \hat{w}_1^2 X + 2e^{-2i(\sigma_1-\sigma_2)} \hat{w}_1^2 X\Bigg]$$
$$+ \lambda_5 w^2 X \left(\hat{w}_S + X\right) + \lambda_6 w^2 X \left(\hat{w}_S + X\right)$$
$$+ 2\lambda_7 X \left[e^{2i\sigma_2} \hat{w}_2^2 \hat{w}_S + e^{-2i\sigma_1} \hat{w}_1^2 \left(X + e^{4i\sigma_1} \hat{w}_S\right) + e^{-2i\sigma_2} \hat{w}_2^2 X\right] - 2\lambda_8 X^2 w^2\Bigg\}, \tag{C.2.4n}$$

$$Z_{2223} = \frac{1}{N_2^3 N_3}\Bigg\{4i\lambda_2 e^{-i(\sigma_1+\sigma_2)} \hat{w}_1 \hat{w}_2 s_{\sigma_1-\sigma_2} \left(e^{2i\sigma_1} \hat{w}_1^2 + e^{2i\sigma_2} \hat{w}_2^2\right)$$
$$+ 4i\lambda_3 e^{-i(\sigma_1+\sigma_2)} \hat{w}_1 \hat{w}_2 s_{\sigma_1-\sigma_2} \left(e^{2i\sigma_1} \hat{w}_1^2 + e^{2i\sigma_2} \hat{w}_2^2\right) \tag{C.2.4o}$$
$$+ \lambda_4 e^{-i\sigma_2} \hat{w}_1 X \left[\hat{w}_1^2 + \left(-2 - e^{-2i(\sigma_1-\sigma_2)}\right) \hat{w}_2^2\right]\Bigg\},$$

$$Z_{2323} = \frac{1}{N_2^2 N_3^2}\Bigg\{\lambda_2 e^{-2i(\sigma_1+\sigma_2)} \left(e^{2i\sigma_1} \hat{w}_1^2 + e^{2i\sigma_2} \hat{w}_2^2\right)^2 + \lambda_3 e^{-2i(\sigma_1+\sigma_2)} \left(e^{2i\sigma_1} \hat{w}_1^2 + e^{2i\sigma_2} \hat{w}_2^2\right)^2$$
$$+ \lambda_4 \left(e^{i\sigma_2-2i\sigma_1} \hat{w}_2^3 X - 3e^{-i\sigma_2} \hat{w}_1^2 \hat{w}_2 X\right) + \lambda_7 X^2 \left(e^{-2i\sigma_2} \hat{w}_1^2 + e^{-2i\sigma_1} \hat{w}_2^2\right)\Bigg\}, \tag{C.2.4p}$$

$$Z_{2333} = \frac{1}{N_2 N_3^3}\Bigg\{-4i\lambda_2 e^{-i(\sigma_1+\sigma_2)} \hat{w}_1 \hat{w}_2 s_{\sigma_1-\sigma_2} \left(e^{2i\sigma_1} \hat{w}_1^2 + e^{2i\sigma_2} \hat{w}_2^2\right)$$
$$- 4i\lambda_3 e^{-i(\sigma_1+\sigma_2)} \hat{w}_1 \hat{w}_2 s_{\sigma_1-\sigma_2} \left(e^{2i\sigma_1} \hat{w}_1^2 + e^{2i\sigma_2} \hat{w}_2^2\right)$$
$$+ \lambda_4 \hat{w}_1 X \Bigg[\left(\hat{w}_1^2 - 2\hat{w}_2^2\right) c_{2\sigma_1-\sigma_2} + \left(2\hat{w}_1^2 - 7\hat{w}_2^2\right) c_{\sigma_2} \tag{C.2.4q}$$
$$+ i\left(\left(\hat{w}_2^2 - 2\hat{w}_1^2\right) s_{\sigma_2} + \left(\hat{w}_1^2 + 2\hat{w}_2^2\right) s_{2\sigma_1-\sigma_2}\right)\Bigg]$$
$$+ 2\lambda_7 \left(e^{-2i\sigma_1} - e^{-2i\sigma_2}\right) \hat{w}_1 \hat{w}_2 X^2\Bigg\}.$$

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