SUBSTITUTION IN RELEVANT LOGICS

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Abstract. This essay discusses rules and semantic clauses relating to Substitution—Leibniz's law in the conjunctive-implicational form $s \doteq t \wedge A(s) \rightarrow A(t)$—as these are put forward in Priest's books *In Contradiction* and *An Introduction to Non-Classical Logic*: *From If to Is*. The stated rules and clauses are shown to be too weak in some cases and too strong in others. New ones are presented and shown to be correct. Justification for the various rules is probed and it is argued that Substitution ought to fail.

§1. Introduction. Finding the correct clause, be it proof-theoretic or semantic, sufficient for validating some logical principle can be hard work. Even after having specified the proof-theoretic machinery and the semantic interpretation thereof, it may happen that even though the setup is sound and complete and does allow one to draw the intended inferences, it also forces unintended ones; the clauses have simply overshot their target. The result is that the logic in question has become stronger than intended, and possibly stronger than what is desired. This is the case with clauses set forth to validate *Substitution*—Leibniz's law in the form $s \doteq t \wedge A(s) \rightarrow A(t)$—in relevant logics in Priest's most excellent textbook *An Introduction to Non-Classical Logic*: *From If to Is* (Priest, 2008). Let's use 'INCL' to refer to this book from now on. I show in §2 that identity contracts in relevant logics with Substitution provided Ackermann's δ-rule—$A \rightarrow (B \rightarrow C), B \vdash A \rightarrow C$—is available, but that it doesn't in many logics in which this rule is not derivable. §3 then shows that Priest's clauses designed to validate Substitution entail not only it, but also that identity contracts. They are also shown to involve a kind of cross-world reasoning which seems unwarranted given the motivation Priest gives for relevant logics as logics which allows for reasoning with impossible worlds where the laws of logic are different. I correct the mistake and display the correct clauses, but show that these make the semantics quite intractable.

§5 considers the logic of identity as this is presented in Priest's first and second edition of *In Contradiction* (Priest, 2006). Let's use 'IC' to refer to this book from now on. The error of Priest (2008) has its roots in both these editions, although the problem with the logic of identity in IC is different and more intricate than the problem with identity for Substitution in relevant logics in Priest (2008). Although in the same ballpark, the proof theory and semantics of Priest (2006) are slightly different and it is shown that the clauses for identity there too are stronger than what is needed if the goal is simply to validate Substitution for the primitive and non-contraposable conditional $\Rightarrow$. They are, however, shown to be too weak to validate Substitution for the contraposable conditional $\rightarrow$ which is the version Priest states that the semantic clauses do validate. It is then shown how to

Received: December 18, 2018.
2010 *Mathematics Subject Classification*: 03A05, 03B47, 03B53, 03C07, 03C90, 03E70.
Key words and phrases: identity, Leibniz's law, substitution, relevant logics, tableaux system, possible world semantics.

doi:10.1017/S1755020319000467

validate Substitution for the contraposable conditional, clauses which then provide further impetus for thinking that Substitution should not be valid. Before summing up, the short §6 gives three further reasons for why a relevantist should reject Substitution. The third reason connects most closely to one way of motivating relevant logics, namely logics which allow for impossible worlds—worlds in which logic itself goes astray.

But first, a quick note on notation: I follow Priest (Priest, 2008, sec. 12.2.4) in writing $A_z(t)$ for the formula obtained from A by replacing every free occurrence of the variable z by the term t. When there is no danger of confusion, I'll simply write $A(t)$. This term-replacing function is obviously a surjective function from the Cartesian product of the set of formulas, the set of variables and the set of terms onto the set of formulas. Thus there will typically be many formulas A and A' such that $A_z(t)$ and $A'_z(t)$ are the same formulas. For instance $x \doteq x_x(a)$, $x \doteq a_x(a)$, $a \doteq x_x(a)$, and $a \doteq a_x(a)$ are all identical to the formula $a \doteq a$. s and t will be arbitrary terms, whereas a, b and c will be constants or elements in a domain of quantification or both.

§2. The logics B, TW and EW - Hilbert style. The relevant logic $\forall\mathbf{TW}\overset{s}{\doteq}$—quantified **TW** with Substitution—has the following axioms and rules:

(Ax1)	$A \to A$	
(Ax2)	$A \to A \vee B$ and $B \to A \vee B$	
(Ax3)	$A \wedge B \to A$ and $A \wedge B \to B$	
(Ax4)	$\neg\neg A \to A$	
(Ax5)	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$	
(Ax6)	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$	
(Ax7)	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$	
(Ax8)	$(A \to B) \to ((B \to C) \to (A \to C))$	
(Ax9)	$(A \to B) \to ((C \to A) \to (C \to B))$	
(Ax10)	$(A \to \neg B) \to (B \to \neg A)$	
(Q1)	$\forall xA \to A_x(t)$	t free for x
(Q2)	$\forall x(A \vee B) \to A \vee \forall xB$	$x \notin FV\{A\}$
(Q3)	$\forall x(A \to B) \to (A \to \forall xB)$	$x \notin FV\{A\}$
(Q4)	$A_x(t) \to \exists xA$	t free for x
(Q5)	$A \wedge \exists xB \to \exists x(A \wedge B)$	$x \notin FV\{A\}$
(Q6)	$\forall x(B \to A) \to (\exists xB \to A)$	$x \notin FV\{A\}$
(I1)	$\forall x(x \doteq x)$	
(I2)	$\forall x\forall y(x \doteq y \wedge A_z(x) \to A_z(y))$	x & y free for z
(R1)	$A, B \vdash A \wedge B$	
(R2)	$A, A \to B \vdash B$	
(RQ)	$\dfrac{\Gamma \vdash A_x(y)}{\Gamma \vdash \forall xA}$	$y \notin FV(\Gamma \cup \{\forall xA\})$

The logic $\forall\mathbf{B}\overset{s}{\doteq}$ is got simply by weakening Ax8–Ax10 to rule form, whereas the logic $\forall\mathbf{EW}\overset{s}{\doteq}$ is got from $\forall\mathbf{TW}\overset{s}{\doteq}$ by adding Ackermann's δ-rule—the weak permutation rule

$$\text{(R3)} \quad A \to (B \to C), B \vdash A \to C.$$

Neither of these three logics validate the contraction rule—the rule $A \to (A \to B) \vdash A \to B$—but they might, of course, validate it for some restricted part of the language. The current question is whether I2 entails that contraction holds for identity statements. I will now show that this is so only provided that R3 is available.

$\to$	$\bot$	**f**	**t**	$\top$	$\neg$
$\bot$	**t**	**t**	**t**	**t**	$\top$
f	**f**	**t**	**t**	**t**	**t**
t	**f**	**f**	**t**	**t**	**f**
$\top$	**f**	**f**	**f**	**t**	$\bot$

$\doteq$	a	b
a	**t**	**f**
b	**f**	**t**

Fig. 1. The $\forall\mathbf{TW}\doteq$-model $\mathfrak{A}$.

THEOREM 2.1. *Identity contracts in* $\forall\mathbf{EW}\overset{s}{\doteq}$*;*

$$s \doteq t \to (s \doteq t \to C) \vdash_{\forall\mathbf{EW}\overset{s}{\doteq}} s \doteq t \to C.$$

Proof.

(1)	$s \doteq t \to (s \doteq t \to C)$	*assumption*
(2)	$s \doteq t \to (s \doteq t \wedge (s \doteq t \to C))$	*1, fiddling*
(3)	$(s \doteq t \wedge (s \doteq t \to C)) \to (t \doteq t \to C)$	*I2*
(4)	$s \doteq t \to (t \doteq t \to C)$	*2, 3, transitivity of* $\to$
(5)	$t \doteq t$	*I1*
(6)	$s \doteq t \to C$	*4, 5, R3* □

So avoiding that identity contracts is hopeless in stronger logics such as $\forall\mathbf{EW}\overset{s}{\doteq}$. To show that R3 really is needed for the inference to go through I will now present a model for $\forall\mathbf{TW}\overset{s}{\doteq}$ over the empty language in which the contraction inference above does not hold. $\{a, b\}$ will be the domain of quantification. $\neg$ and $\to$ are to be interpreted according to the displayed matrix in Figure 1, $\vee$ and $\exists$ are interpreted as supremum and $\wedge$ and $\forall$ as infimum over the displayed ordering of the truth-values $\{\bot, \mathbf{f}, \mathbf{t}, \top\}$. For ease of reference, let's call the model $\mathfrak{A}$.[1] The propositional part of $\mathfrak{A}$ was found by the matrix generator `MaGIC`.[2] To see that $\mathfrak{A}$ validates I2 one should note that if $x \doteq y$ is evaluated to **t**, then obviously $A_z(x)$ and $A_z(y)$ are evaluated to the same truth-value, and so $x \doteq y \wedge A_z(x) \to A_z(y)$ is evaluated to the designated element **t**. If $x \doteq y$ is evaluated to **f**, then $x \doteq y \wedge A_z(x) \to A_z(y)$ may be evaluated to **f** if it can happen that $A_z(y)$ is evaluated to $\bot$ while $A_z(x)$ is evaluated to some element different from $\bot$. The following lemma will therefore complete the demonstration that the model in fact validates I2:

LEMMA 2.2. *If* $\mathfrak{A}(x \doteq y) = \mathbf{f}$*, then both*

$$\mathfrak{A}(A_z(y)) = \top \Rightarrow \mathfrak{A}(A_z(x)) = \top$$
$$\mathfrak{A}(A_z(y)) = \bot \Rightarrow \mathfrak{A}(A_z(x)) = \bot.$$

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1 It is worth mentioning that the model also validates the following logical principles:

$$A \vee \neg A$$
$$(A \to B) \wedge (B \to C) \to (A \to C)$$
$$A \vdash \neg(A \to \neg A)$$

$$\frac{\Gamma, B \vdash A \qquad \Gamma, C \vdash A}{\Gamma, B \vee C \vdash A} \text{ (Reasoning by Cases)}$$

$$\frac{\Gamma, B_x(y) \vdash A}{\Gamma, \exists x B \vdash A} \; y \notin FV(\Gamma \cup \{\exists x B, A\}) \text{ (Existential Instant.)}$$

Thus it validates the logic $\forall\mathbf{TR}^{d}\overset{s}{\doteq}$. For a better overview over the various relevant logics and how they can be pieced together, see Øgaard (2016, sec. 2). §7 of Øgaard (2016) also has a classification of various versions of Leibniz's law according to strength and relevance.

2 I leave it to the distrusting reader to verify that it is in fact a model for $\forall\mathbf{TW}$. `MaGIC`—an acronym for *Matrix Generator for Implication Connectives*—is an open source computer program created by John K. Slaney (Slaney, 1995).

Proof. *The proof is by induction on the complexity of A. Assume that the variable assignment function is chosen so that* $\mathfrak{A}(x \doteq y) = \mathbf{f}$. *Now if A is a propositional variable then since* $A_z(y)$ *will then simply be A, the consequent follows trivially. If A is an atomic formula* $s \doteq t$ *for terms s and t, then the consequent again follows trivially since identity statements only can be evaluated to* **t** *or* **f**. *For induction hypothesis (IH), assume that the lemma holds for B and C.*

1. $A_z(y) := (B \wedge C)_z(y)$.
   (*1.1*). *If* $\mathfrak{A}(A_z(y)) = \bot$, *then either* $\mathfrak{A}(B_z(y)) = \bot$ *or* $\mathfrak{A}(C_z(y)) = \bot$, *and so, by (IH) it follows that either* $\mathfrak{A}(B_z(x)) = \bot$ *or* $\mathfrak{A}(C_z(x)) = \bot$ *and therefore that* $\mathfrak{A}((B \wedge C)_z(x)) = \bot$.
   (*1.2*). *The case when* $\mathfrak{A}(A_z(y)) = \top$ *is similar to (1.1).*
2. $A_z(y) := \neg B_z(y)$.
   (*2.1*). *If* $\mathfrak{A}(A_z(y)) = \bot$, *then it follows that* $\mathfrak{A}(B_z(y)) = \top$ *and so, by (IH), that* $\mathfrak{A}(B_z(x)) = \top$ *and therefore that* $\mathfrak{A}(\neg B_z(x)) = \bot$.
   (*2.2*). *The case when* $\mathfrak{A}(A_z(y)) = \top$ *is similar to (2.1).*
3. $A_z(y) := (\forall x B)_z(y)$. *Left for the reader.*
4. $A_z(y) := (\exists x B)_z(y)$. *Left for the reader.*
5. $A_z(y) := (B \to C)_z(y)$. *The conclusion follows trivially since it cannot happen that a conditional gets evaluated to other elements than* **t** *or* **f**. □

THEOREM 2.3. *Identity does not contract in* $\forall\mathbf{TW}\overset{s}{\doteq}$.

Proof. *Let the propositional variable p be evaluated to* $\bot$ *by* $\mathfrak{A}$, *and let the individual variables x and y be evaluated to the elements a and b, respectively. Then* $\mathfrak{A}(x \doteq y \to (x \doteq y \to p)) = \mathbf{t}$, *but* $\mathfrak{A}(x \doteq y \to p) = \mathbf{f}$. $\mathfrak{A}$ *is a model for* $\forall\mathbf{TW}\overset{s}{\doteq}$ *by the above considerations, and so the theorem follows.* □

§3. Priest's tableaux system and possible world semantics for B. Unlike the previous section, INCL does not use Hilbert systems for deductions, nor algebraic models. Instead it uses tableaux systems and possible world semantics. In many cases this is quite pleasing as the relation between the semantic and proof-theoretic clauses are most often rather transparent. What can be challenging, however, is figuring out which Hilbertian rule a given change in a tableaux system begets.

I will now give a brief presentation of Priest's tableaux system for **B** and its possible world semantics as this is presented in Priest (2008). I will only specify the $\{\to, \wedge\}$-fragment of $\forall\mathbf{B}$ together with the identity principles. Afterward I'll show that Priest's set-up forces identity to contract in the way the previous section showed was not the case with $\forall\mathbf{B}\overset{s}{\doteq}$. I'll then go on to pin-point what is wrong with the set-up and how to fix it.

A proof is defined to be a tree the nodes of which are of the form *Rijk*, $A, +i$ or $A, -i$ for formulas A and natural numbers i, j and k. The proof system mirrors the possible world semantics which has the semantic counterpart of R, $\mathcal{R}$, as its ternary accessibility relation. $A, +i$ means that A holds, or is true at i, and $A, -i$ means that A does not hold, or is untrue at i. A branch therefore closes if $A, +i$ and $A, -i$ are both on it. '0' stands for the actual world.[3] A tree is a proof of B from assumptions $A_{m \le n}$ if it has all of $A_m, +0$ and $B, -0$

[3] One only needs to assume that there is one possible, or normal world, and so every world different from 0 will throughout this essay be impossible.

as initial nodes and all branches of the tree closes. The consequence relation is therefore to be thought of as truth-preservation over the base world. Further axioms and rules are as specified in Table 1.

	i	ii	iii
I	$\cdot$ $\downarrow$ $R0ii$	$A \to B, +i$ $Rijk$ $\swarrow \searrow$ $A, -j \quad B, +k$	$A \to B, -i$ $\downarrow$ $Rijk$ $A, +j$ $B, -k$
II	$A, +i$ $A, -i$ $\downarrow$ $\times$	$A \wedge B, +i$ $\downarrow$ $A, +i$ $B, +i$	$A \wedge B, -i$ $\swarrow \searrow$ $A, -i \quad B, -i$
III	$\cdot$ $\downarrow$ $t \doteq t, +0$	$s \doteq t, +0$ $A_x(s), +i$ $\downarrow$ $A_x(t), +i$	$s \doteq t, +i$ $\downarrow$ $s \doteq t, +0$

Table 1. *Fragment of Priest's tableaux system for* $\forall\mathbf{B}\doteq$

The rule I.iii requires j & k to be new, and if i is 0, then $j = k$. Rule III.ii carries the usual requirement that s & t are substitutable for x in A. The rules are best read as rules for expanding a branch downward; for instance, rule I.ii allows the branch to bifurcate anywhere below $A \to B, +i$, and $Rijk$, whereas rule I.iii allows one to expand a branch downward with the nodes $Rijk$, $A, +j$, and $B, -k$ provided $A \to B, -i$ is on it and j, k meet the above restrictions.[4]

The following proof shows that identity contracts given the tableaux rules stated above:

[4] These tableaux rules can be found on pages 165 (II.ii–II.iii), 190 (I.i–I.iii), and 549–550 (III.i–III.iii) in Priest (2008).
Priest states the identity rules slightly differently, so let me make a quick note on what this amounts to (this is a bit pedantic, so feel free to skip ahead!): the calculi in Priest (2008) are all sentential calculi; the relations of proof-theoretic and semantic consequences, $\vdash$ and $\vDash$, for the various logics all take a set of closed formulas as their left argument, and a single such closed formula as their right argument (see for instance Priest (2008, sec. 12.3.3)). This contrasts to *In Contradiction* in which open formulas are allowed to be related by the consequence relations (see for instance Priest (2006, p. 78 & sec. 19.7)). Some do make this out to be more than a difference of ideology; in terms of semantic consequence one might argue that this is a relation of truth-preservation, and truth only applies to sentences, whereas satisfaction applies to both open and closed formulas. As long as the sententialist makes sure that there is always sufficiently many individual constants available, however, one could translate between these ideologies without loss of logical strength (see Priest (2008, sec. 12.2) for the INCL way of setting up the syntax of a logic). I have, for unity of presentation and since I myself am not a sententialist, opted for formulating all consequence relations in this essay to allow for open formulas, and so III.i–III.iii are stated for terms t and s and $A_x(s)$ may be an open formula.
The other difference is that Priest states rule III.ii with two restrictions: (1) A in the line $A_z(s), +i$ should be *atomic* and (2) $A_z(s), +i$ must not be $s \doteq t, +0$. The latter here is inconsequential as allowing it would simply allow one to extend a branch with $s \doteq t, +0$ and $t \doteq t, +0$. For the purposes of this article, and so and not to cause confusion when III.ii is to be generalized, I've opted for dropping this restriction. Some generalizations of III.ii do need that A be an arbitrary formula and not just atomic. This is not the case with III.ii, but I have for consistency of presentation opted for stating also it for arbitrary formulas. I will get back to this issue later when the difference between having an atomic rule vs. a non-atomic one makes a difference to the consequence relation.

(1)	$s \doteq t \to (s \doteq t \to B), +0$	assumption
(2)	$s \doteq t \to B, -0$	assumption
(3)	$R011$	I.iii
(4)	$s \doteq t, +1$	2, I.iii
(5)	$B, -1$	2, I.iii
(6)	$s \doteq t, +0$	4, III.iii
(7)	$R000$	I.i
	$\swarrow \searrow$	
(8)	$s \doteq t, -0 \quad s \doteq t \to B, +0$	1, 7, I.ii
(9)	$\times \quad \times$	left: 6 & 8, II.i; right: 2 & 8, II.i

The crucial assumption is undoubtedly rule III.iii. Its corresponding semantic clause is termed the *Subset Constraint* (SC) and allows one to infer that any identity statement true at some world is true at the base world. Before we go on, let's look closer at the possible world semantics for **B**:

DEFINITION 3.1 (Interpretations). *An interpretation for the positive fragment of the relevant predicate logic* **B** *is a structure*

$$\mathcal{S} = \langle \mathcal{D}, \mathcal{W}, \mathcal{N}, \mathcal{R}, *, \mathfrak{s}, \mathfrak{v} \rangle$$

where

- $\mathcal{D}$ *is the non-empty domain of quantification*
- $\mathcal{W}$ *is a non-empty set of worlds*
- $\mathcal{N} \subseteq \mathcal{W}$ *is a non-empty set of* normal *worlds*
- $\mathcal{R}$ *is the ternary accessibility relation on* $\mathcal{W}$—$\mathcal{R} \subseteq \mathcal{W}^3$— *such that for all normal worlds* $n \in \mathcal{N}$, $\mathcal{R}nxy$ *iff* $x = y$
- $*$, *the* Routley star, *is a function from* $\mathcal{W}$ *to* $\mathcal{W}$ *such that* $w^{**} = w$ *for all* $w \in \mathcal{W}$
- $\mathfrak{s}$ *is a variable assignment function:* $\mathfrak{s} : VAR \mapsto \mathcal{D}$
- $\mathfrak{v}$ *is an interpretation function such that*
  - $\mathfrak{v}(a) \in \mathcal{D}$ *for individual constants* a
  - $\mathfrak{v}_w(p) \in \{1, 0\}$ *for propositional constants* p *and* $w \in \mathcal{W}$
  - $\mathfrak{v}_w(P) \subseteq \mathcal{D}^n$ *for n-ary predicate* P *and* $w \in \mathcal{W}$
  - $\mathfrak{v}_w(\doteq) = \{\langle a, a\rangle \mid a \in \mathcal{D}\}$ *for* $w \in \mathcal{N}$
  - $\mathfrak{v}_w(\doteq) \subseteq \mathcal{D}^2$ *for* $w \in \mathcal{W} \setminus \mathcal{N}$.

DEFINITION 3.2 (Satisfaction).

- $\mathfrak{v}_w^{\mathfrak{s}} \vDash p$ *iff* $\mathfrak{v}_w(p) = 1$
- $\mathfrak{v}_w^{\mathfrak{s}} \vDash P(s_1, \ldots, s_n)$ *iff* $\langle \mathfrak{v}^{\mathfrak{s}}(s_1), \ldots, \mathfrak{v}^{\mathfrak{s}}(s_n)\rangle \in \mathfrak{v}_w(P)$
- $\mathfrak{v}_w^{\mathfrak{s}} \vDash \neg A$ *iff* $\mathfrak{v}_{w^*}^{\mathfrak{s}} \nvDash A$
- $\mathfrak{v}_w^{\mathfrak{s}} \vDash A \wedge B$ *iff* $\mathfrak{v}_w^{\mathfrak{s}} \vDash A$ *and* $\mathfrak{v}_w^{\mathfrak{s}} \vDash B$
- $\mathfrak{v}_w^{\mathfrak{s}} \vDash A \vee B$ *iff* $\mathfrak{v}_w^{\mathfrak{s}} \vDash A$ *or* $\mathfrak{v}_w^{\mathfrak{s}} \vDash B$
- $\mathfrak{v}_w^{\mathfrak{s}} \vDash A \to B$ *iff for every world* $\mathcal{R}wxy$, *if* $\mathfrak{v}_x^{\mathfrak{s}} \vDash A$, *then* $\mathfrak{v}_y^{\mathfrak{s}} \vDash B$
- $\mathfrak{v}_w^{\mathfrak{s}} \vDash \forall xA$ *iff* $\mathfrak{v}_w^{\mathfrak{s}(x/a)} \vDash A$ *for all* $a \in \mathcal{D}$
- $\mathfrak{v}_w^{\mathfrak{s}} \vDash \exists xA$ *iff* $\mathfrak{v}_w^{\mathfrak{s}(x/a)} \vDash A$ *for some* $a \in \mathcal{D}$.

DEFINITION 3.3 (Semantic consequence). *Semantic consequence in a structure* $\mathcal{S}$ *is defined as preservation of satisfaction over all* normal *worlds*: $\Theta \vDash_{\mathcal{S}} A$ *iff for all* $n \in \mathcal{N}$, *if* $\mathfrak{v}_n^{\mathfrak{s}} \vDash \theta_i$ *for all* $\theta_i \in \Theta$, *then* $\mathfrak{v}_n^{\mathfrak{s}} \vDash A$. *Semantic consequence* simpliciter *for* **B**, $\Theta \vDash_{\mathbf{B}} A$, *is then semantic consequence over all* **B***-structures.*

In the case of **B** one may assume that there is only one normal world. This is reflected in the tableaux system where '0' stands for this one normal world. The interpretation of the rule III.i is then that $s \doteq s$ is true at the normal world, whereas III.ii says that if $s \doteq t$ is true at the normal world, then A is true of t at any world i if it is true of s there. The model theory validates these two principles since $\doteq$ is interpreted as the real identity predicate at normal worlds and the semantics is compositional so that coreferring terms can't make out a semantic difference. The semantics does not validate III.iii, however, since it allows non-normal worlds to interpret $\doteq$ arbitrarily. Thus distinct object a and b can be identified at non-normal world. It is this possibility the Subset Constraint does away with. Formally, SC is the demand that $\doteq$ be, at every non-normal world in the model, interpreted to be a subset of the set of all identity-pairs:

$$\text{(SC)} \quad v_w(\doteq) \subseteq \{\langle a, a\rangle \mid a \in \mathcal{D}\} \text{ for } w \in \mathcal{W} \setminus \mathcal{N}.$$

Priest adds both III.i and III.ii as default rules for identity for relevant logics, and states that in order to validate Substitution one needs in addition to add III.iii:

> The Subset Constraint nonetheless has an effect on the validity of inferences concerning identity. Without it, $(a = b \wedge Pa) \rightarrow Pb$ is not logically valid in B. (Details are left as an exercise.) With it, it is[. . .]. (Priest, 2008, p. 551)

This is plainly wrong as the counter-model above shows; Substitution does not entail contraction for identity formulas, whereas Priest's clauses for it do. On the same page just quoted from, Priest does prove that Substitution is a theorem of his set-up, and so the only problem with the statement is the claim that Substitution is not a theorem unless III.iii/SC holds.

Without III.iii, the logic only validates the rule $s \doteq t \vdash A(s) \rightarrow A(t)$ which the clause III.ii is designed to validate. III.ii seems itself to be unwarranted as it forces substitution to hold at a world different from where the identity holds; it licenses one to infer that t is A at a world if $A(s)$ is true there and the base world identifies s with t. The reason why this is not unwarranted, however, is that the base world always identifies truly, whereas other worlds can falsely identify objects—provided that the Subset Constraint does not hold of course—or fail to report true identities. It is also instructive to note, as Priest also does, that III.ii can equivalently be replaced by the *Substitutivity of Identity*, the rule

$$\text{(SI)} \quad s \doteq t, A(s) \vdash A(t),$$

the tableaux version of which is most naturally stated as the rule

$$(LL_{00_0}) \qquad \begin{array}{c} s \doteq t, +0 \\ A_x(s), +0 \\ \downarrow \\ A_x(t), +0. \end{array}$$

Thus the only work that III.ii does—let's for consistent nomenclature introduce LL_{0i_i} as just another name for the rule III.ii— is to ensure that the actual world is closed under SI; the cross-world rumpus of III.ii is simply due to the logic of $\rightarrow$ and the restriction on every world to respect true identities even though it is not laid upon every world to confirm these. The legitimacy of these rules, then, are based on the insistence that normal worlds interpret the identity predicate in accordance with identity, and that the semantics must be compositional.

Clause III.iii is supposed to validate Substitution, and without any further stated purposes, one would guess that it is not designed to validate any further logical principle than what follows from Substitution (given the logic at hand). Substitution surely entails SI. This is not reflected in Priest's setup, however—III.ii is surely not derivable using III.i and III.iii alone—and so it is somewhat disappointing to end up with a logic which has to assume SI (in the guise of III.ii) as a primitive rule. To remedy this, one could equivalently replace both III.ii and III.iii by the rule:

$$(LL_{ij_j}) \qquad \begin{array}{c} s \doteq t, +i \\ A_x(s), +j \\ \downarrow \\ A_x(t), +j. \end{array}$$

Here is a proof of this equivalence:

Proof. (I) That LL_{ij_j} entails III.ii is obvious. That it entails III.iii is seen by the following derivation:

(1)	$s \doteq t, +i$	assumption
(2)	$(s \doteq x)_x(s), +0$	III.i
(3)	$s \doteq t, +0$	1, 2, LL_{ij_j}.

That III.ii together with III.iii entail LL_{ij_j} can be seen from the following:

(1)	$s \doteq t, +i$	assumption
(2)	$A(s), +j$	assumption
(3)	$s \doteq t, +0$	1, III.iii
(4)	$A(t), +j$	2, 3, III.ii.

□

Whereas III.iii informs one that only true identities hold at any possible world, this variant of it brings to the light the fact that it entails that $A(t)$ must be true at a world j if $A(s)$ is true there and s and t are identified at some world or other. This is so even though the constraint does not entail the general *Identity Invariance Rule*

$$(\mathrm{IIR}) \qquad \begin{array}{c} s \doteq t, +i \\ \downarrow \\ s \doteq t, +j, \end{array}$$

and so does not entail the only reasonable explanation—apart from assuming that s and t really are identical which would be begging the question—for why $A(t)$ should be true at j, namely that it also identifies s and t. In this case, $s \doteq t$ can fail to obtain at j, but $A(t)$ has to be true regardless. This kind of cross-world substitution seems just plainly unwarranted! It is true that the rule III.ii, aka. LL_{0i_i}, does involve cross-world substitution, but that was shown to follow simply by insisting on a compositional semantics and an identity predicate which at *normal* worlds respects true identities. Why think that this would extend to *impossible* worlds; why think that it is impossible for impossible worlds to—wrongly from the point of view of possible worlds—identify separate objects?

III.iii, then, not only overshoots its target of validating Substitution, but seems also to be objectionable from a philosophical perspective. What rule, then, must we add in order to validate Substitution? In the possible world semantics which Priest makes use of, a conditional $A \rightarrow B$ is true at the base world just in case every A-world is also a B-world, and since conjunction is treated standardly at every world, it is plain to see that Substitution will hold at the base world just in case it is not only closed under SI as required by III.ii, but that *every* world is so closed. Put negatively: in order for Substitution to fail, there would have to be a world i at which both $s \doteq t$, and $A(s)$ hold, but at which $A(t)$ fails. The proper clause for Substitution in a tableaux system for relevant logics is therefore not LL_{ij_j}, but

$$(LL_{ii_i}) \qquad \begin{array}{c} s \doteq t, +i \\ A_x(s), +i \\ \downarrow \\ A_x(t), +i. \end{array}$$

With only III.i and LL_{ii_i} available, it is easy to see that the contraction-proof above would not go through.[5]

I have in this section shown that the Subset Constraint entails the contraction rule for identity formulas, whereas the previous section showed that this is not the case with Substitution. That the Subset Constraint therefore overshoots its stated target shows that it is possible to reject it, while at the same time insisting that every world respects SI and therefore insisting that Substitution be true. I have also given strong reasons for rejecting the principles due to its cross-world substitutional nature. Despite this, one might think there are other reasons why one ought to go for the stronger rule LL_{ij_j} instead of the weaker LL_{ii_i}. The following section gives two such reasons, which, though definitely far from conclusive, do raise important questions in their own right. It also shows which semantic clause is needed to validate Substitution and provides some reflections on its complexity.

§4. Two reasons for keeping the Subset Constraint. *First reason - Symmetry.* An identity predicate worth its salt should not only validate Leibniz's law over non-opaque contexts, but also satisfy reflexivity, symmetry and transitivity. Reflexivity is obviously not derivable using Substitution, and so it is to be expected that one would need a separate axiom for this. One might, however, think that transitivity and symmetry would be. The symmetry rule $s \doteq t \vdash t \doteq s$ is obviously derivable using LL_{00_0}, and so is the transitivity rule $s \doteq t, t \doteq u \vdash s \doteq u$. Neither $\rightarrow$-symmetry, $s \doteq t \rightarrow t \doteq s$, nor transitivity in either the $\rightarrow$-nested form $s \doteq t \rightarrow (t \doteq u \rightarrow s \doteq u)$ or the conjunctive form $s \doteq t \wedge t \doteq u \rightarrow s \doteq u$ is derivable without adding identity principles, however. By adding Substitution one does get the conjunctive form of transitivity.[6] $\rightarrow$-symmetry is not, however, derivable from Substitution for the simple reason that relevant logics fail to validate various forms of

[5] To see this, it might help to look at the proof using LL_{ij_j} instead of III.iii:

(1)	$x \doteq t \rightarrow (s \doteq t \rightarrow B)_x(s), +0$		assumption
(2)	$s \doteq t \rightarrow B, -0$		assumption
(3)	$R011$		I.iii
(4)	$s \doteq t, +1$		2, I.iii
(5)	$B, -1$		2, I.iii
(6)	$t \doteq t \rightarrow (s \doteq t \rightarrow B), +0$		4, 1, LL_{ij_j}
(7)	$R000$		I.i
	$\swarrow$	$\searrow$	
(8)	$t \doteq t, -0$	$s \doteq t \rightarrow B, +0$	1, 7, I.ii
(9)	$t \doteq t, +0$	$\times$	2, 8, II.i
(10)	$\times$		left: 8, 9, II.i

Here it is easy to see that the proof needs the cross-world principle encoded by LL_{ij_j} which allows one to infer that $A(t)$ holds at the base world (6th line) on the bases of $A(s)$ holding there (1st line) together with the fact that some other world identifies s and t (4th line).

[6] Edwin Mares' logical identity predicate in Mares (2004, sec. 6.13) obeys transitivity in this form, and similarly for the arithmetical identity predicate which Richard Routley uses in Routley (1980,

suppression of assumptions; in particular, the rule $A \wedge B \rightarrow C, A \vdash B \rightarrow C$ is not derivable in any relevant logic.[7] Because of this it will not be possible to derive $s \doteq t \rightarrow t \doteq s$ from the Substitution-instance $s \doteq t \wedge s \doteq s \rightarrow t \doteq s$. This is also brought out by trying to derive the symmetry formula:

(1)	$s \doteq t \rightarrow t \doteq s, -0$	assumption
(2)	$R011$	1, I.iii
(3)	$s \doteq t, +1$	1, I.iii
(4)	$t \doteq s, -1$	1, I.iii
(5)	$x \doteq t_x(s), +1$	3, rewrite
(6)	$t \doteq t, +1$	3, 5, LL_{ii_i}.

Using LL_{ij_j}, however, one does get the desired closure: from (3) together with $x \doteq s_x(s), +0$ one then gets $t \doteq s, +0$ using LL_{ij_j}, which applied to (6) using LL_{0i_i} then suffices for deriving $t \doteq s, +1$. The Substitution-instance $s \doteq t \wedge s \doteq s \rightarrow t \doteq s$ is easily seen to translate into the condition that $t \doteq s$ be true at each world where both $s \doteq t$ and $s \doteq s$ are true, thus says nothing about what needs to be true at $s \doteq t$-worlds where $s \doteq s$ fails. Piecing the facts from the derivation together with this it is clear that to build a counterexample to the symmetry axiom it would suffice to have an interpretation $\mathcal{S}$ with a two-world setup $\mathcal{W} = \{0, 0^*\}$, $\mathcal{N} = \{0\}$, $\mathcal{R} = \{\langle 0, 0, 0\rangle, \langle 0, 0^*, 0^*\rangle\}$ and have these two worlds interpret the identity predicate over the domain $\mathcal{D} = \{a, b\}$ as follows:

$$\mathfrak{v}_0(\doteq) = \{\langle a, a\rangle, \langle b, b\rangle\}$$
$$\mathfrak{v}_{0^*}(\doteq) = \{\langle a, b\rangle, \langle b, b\rangle\}.$$

Here both $s \doteq s$ and $t \doteq s$ fail to be true at the non-normal world 0^* when a is chosen as the denotation of s and b as the denotation of t. Both worlds are, however, closed under SI, so the set-up validates Substitution.[8]

The correct attitude here would surely not be to rush to some stronger version of Leibniz's law, but to simply either accept and try to make due with the weaker and derivable version of symmetry, or simply add $\forall x \forall y(x \doteq y \rightarrow y \doteq x)$ as a separate axiom.[9]

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sec. A.9). The more common way to add transitivity, however, at least in relevant arithmetics, is to add some $\rightarrow$-nested version of transitivity (see fn. 9 for references).

[7] It is easily seen to be interderivable with the rule version of weakening, $A \vdash B \rightarrow A$.

[8] I leave it to the reader to verify that the model in fact validates Substitution over the empty language. This model can, incidentally, easily be made into a countermodel to identity contraction: let p be propositional variable such that $\mathfrak{v}_{0^*} \nvDash p$. Since 0^* is not $\mathcal{R}$-related to any world, every conditional is true at 0^*, and since 0^* is the only world in which $a \doteq b$ is true, it follows that $\mathfrak{v}_0 \vDash a \doteq b \rightarrow (a \doteq b \rightarrow p)$. However, since p is not true at 0^*, we have that $\mathfrak{v}_0 \nvDash a \doteq b \rightarrow p$. By adding $\langle 0^*, 0^*, 0^*\rangle$ to $\mathcal{R}$, the interpretation still validates Substitution, but also the contraction rule: $A \rightarrow (A \rightarrow B) \vDash_{\mathcal{S}} A \rightarrow B$.

[9] Symmetry is added as an extra axiom in for instance Edwin Mares' book *Relevant Logic*: *A Philosophical Interpretation* (Mares, 2004) in which identity is treated as a logical predicate. I think it is fair to say, however, that identity is most often taken as a non-logical predicate. This is the case with the literature on relevant arithmetics in which symmetry is always (as far as I know) taken as a separate axiom, or derivable using Ackermann's δ-rule (R3) from the transitivity axiom $\forall x \forall y \forall z(x \doteq y \rightarrow (x \doteq z \rightarrow y \doteq z))$ (see, for instance, Dunn (1979), Friedman & Meyer (1992), Meyer & Restall (1999), Restall (2010), Routley (1980, sec. A.9), and Meyer & Restall (1996)). It is only in Restall (2010, p. 98) that I've found it suggested that one might do arithmetics without symmetry in $\rightarrow$-form (the $\vdash$-form is derivable). Note also that the symmetry axiom can come out false in Belnap's test-model of relevance; see Øgaard (2017, fn. 14), which shows that $\forall x \forall y(x \doteq y \wedge \mathbf{t} \rightarrow y \doteq x)$, however, does come out true on that model, where $\mathbf{t}$ is the Ackermann constant axiomatized by the two-way rule $A \dashv\vdash \mathbf{t} \rightarrow A$.

The same reason why symmetry is not derivable from Substitution applies if one considers function symbols; the "functionality" axiom for unary functions f, $\forall x \forall y(x \doteq y \to f(x) \doteq f(y))$, fails to be derivable, and one needs to take a stand on whether to add this as a logical axiom, or stick with the derivable $\vdash$-version $s \doteq t \vdash f(s) \doteq f(t)$. At least the literature on relevant arithmetics seems to prefer the intuitionistic-style approach where such axioms are added as non-logical axioms, and not necessarily for all function symbols.[10] *End first reason.*

Second reason - Inductiveness. It is easy to show that the rules LL_{00_0} and LL_{0i_i} are in fact interderivable. There is an important difference between them, however, which the following definition brings out:

DEFINITION 4.1. *A rule is* inductive *in a logic* **L** *if the rule restricted to atomic formulas suffices for deriving the unrestricted version of it. A rule is* non-inductive *in that logic if not.*

The difference between LL_{00_0} and LL_{0i_i} is then that the latter is inductive in the logic **B**, whereas the first is non-inductive. That LL_{00_0} is non-inductive in the logic **B** is easily realized by contemplating how an inductive proof would go; since LL_{00_0} is assumed for atomic formulas, the base case is trivial. Assume that LL_{00_0} is derivable for A and B. We should then have to show that it is OK for $A \to B$ as well which would have to get beyond (9) in the following derivation without using LL_{0i_i}:

(1)	$s \doteq t, +0$		assumption
(2)	$(A \to B)_x(s), +0$		assumption
(3)	$(A \to B)_x(t), -0$		assumption
(4)	$R011$		3, I.iii
(5)	$A_x(t), +1$		3, I.iii
(6)	$B_x(t), -1$		3, I.iii
	$\swarrow$	$\searrow$	
(7)	$A_x(s), -1$	$B_x(s), +1$	2, 4, I.ii
(8)	$x \doteq s_x(s), +0$		left: III.i
(9)	$t \doteq s, +0$		left: 1, 8, LL_{00_0}
(10)	$[A_x(s), +1]$	$[B_x(t), +1]$	left: 9, 5, LL_{0i_i}; right: 1, 7, LL_{0i_i}
(11)	$[\times]$	$[\times]$	left: 7r, 10r, II.i; right: 6, 10r, II.i

[10] As far as I know, only Routley has proposed a weaker functionality axiom in the context of relevant arithmetics—the functionality of the successor function is in Routley (1980, sec. A.9) stated as $\forall x \forall y(x \doteq y \wedge 1 \doteq 1 \to x' \doteq y')$ which is slightly stronger than the strongest version of the functionality axiom validated by Belnap's test model for relevance (see Belnap (1960)). That model validates Leibniz's law in any (permuted) version of the following two formulas:

$$\forall \bar{x} \forall \bar{y}(x_n \doteq y_n \wedge \mathbf{t} \to (\ldots(x_1 \doteq y_1 \wedge \mathbf{t} \to (A(\bar{x}) \to A(\bar{y})))\ldots))$$
$$\forall \bar{x} \forall \bar{y}(\textstyle\bigwedge_{i \leq n} x_i \doteq y_i \wedge \mathbf{t} \to (A(\bar{x}) \to A(\bar{y}))).$$

Thus the strongest versions of the functionality axioms that it validates are any permuted version of the following axioms:

$$\forall \bar{x} \forall \bar{y}(x_n \doteq y_n \wedge \mathbf{t} \to (\ldots(x_1 \doteq y_1 \wedge \mathbf{t} \to f(\bar{x}) \doteq f(\bar{y}))\ldots))$$
$$\forall \bar{x} \forall \bar{y}(\textstyle\bigwedge_{i \leq n} x_i \doteq y_i \wedge \mathbf{t} \to f(\bar{x}) \doteq f(\bar{y})).$$

(See Øgaard (2017, sec. 8) for more details on Belnap's test model and the versions of Leibniz's law it validates).

It is here easy to see that LL_{00_0} for A and B will not bridge the world-gap needed to get below line (9).[11]

When it comes to semantics it is vastly more difficult to check that a non-inductive rule holds in a model than checking that an inductive one does. In the case of LL_{00_0} one needs to verify that

$$\text{if } \langle a, b\rangle \in \mathfrak{v}_0(\doteq), \text{ then } \mathfrak{v}_0^{\mathfrak{s}(x/a)} \models A \text{ iff } \mathfrak{v}_0^{\mathfrak{s}(x/b)} \models A.$$

Since there are infinitely many such A-formulas to check, one will in general need a proof by induction in order to verify that LL_{00_0} does indeed hold in a model. The situation with LL_{0i_i} is quite different; in order to validate this rule, one only needs to verify that a given model satisfies

$$\text{if } \langle a, b\rangle \in \mathfrak{v}_0(\doteq), \text{ then } \langle c_1, \ldots, a, \ldots, c_n\rangle \in \mathfrak{v}_i(P) \text{ iff } \langle c_1, \ldots, b, \ldots, c_n\rangle \in \mathfrak{v}_i(P),$$

where i ranges over every world in the model, and P is any *atomic* n-ary predicate.[12] For models for finite languages with finite set of worlds, this can be a far easier task.

As I mentioned in fn. 4, Priest's rule III.ii is stated for only atomic A's and even with this limitation it, together with III.iii, suffices for deriving LL_{ij_j} which also can be shown to be inductive in any logic extending **B**. The rule LL_{ii_i}, however, is, like LL_{00_0}, non-inductive in relevant logics, and so restricting it to atomic formulas will not suffice for deriving unrestricted Substitution. This is maybe easier seen by trying to actually derive for instance $a \doteq b \wedge (B \rightarrow C(a)) \rightarrow (B \rightarrow C(b))$. By inductive hypothesis one may then assume that Substitution for $C(x)$ is safe. We may therefore use LL_{ii_i} on $C(x)$. The best derivation that is possible is this:

(1)	$a \doteq b \wedge (B \rightarrow C(a)) \rightarrow (B \rightarrow C(b)), -0$	assumption
(2)	$R011$	1, I.iii
(3)	$a \doteq b \wedge (B \rightarrow C(a)), +1$	1, I.iii
(4)	$B \rightarrow C(b), -1$	1, I.iii
(5)	$a \doteq b, +1$	3, II.i
(6)	$B \rightarrow C(a), +1$	1, II.i
(7)	$R123$	4, I.iii
(8)	$B, +2$	4, I.iii
(9)	$C(b), -3$	4, I.iii
	$\swarrow \searrow$	
(10)	$B, -2 \quad C(a), +3$	3, I.ii
(11)	$\times$	left: 8 & 10, II.i

[11] A countermodel here would involve a completeness proof for **B** with atomic LL_{00_0}. The models here would have to allow for "non-logical" interpretations of identity; for "logical" interpretations of the identity predicate we demand that normal worlds interpret the identity predicate as the real identity predicate: $\mathfrak{v}_n^{\mathfrak{s}} \models s \doteq t$ iff $\mathfrak{s}(s) = \mathfrak{s}(t)$. Since in general we have that for any assignment function $\mathfrak{s}$, term t and world w, $\mathfrak{v}_w^{\mathfrak{s}(x/\mathfrak{s}(t))} \models A$ iff $\mathfrak{v}_w^{s} \models A_x(t)$, one automatically gets that if $\mathfrak{v}_n^{\mathfrak{s}} \models s \doteq t$, then $\mathfrak{v}_w^{\mathfrak{s}} \models A_x(s)$ iff $\mathfrak{v}_w^{\mathfrak{s}} \models A_x(t)$, thus validating unrestricted LL_{0i_i}. **B** with atomic LL_{00_0} is not complete with regard to such models. To get the appropriate models one would have to relax the identity clause so as to permit that $\{\langle a, a\rangle \mid a \in \mathcal{D}\} \subseteq \mathfrak{v}_n(\doteq)$ even for normal worlds.

[12] Here $\doteq$ needs to be included among the P's. However, by adding transitivity, symmetry and functionality as primitive axioms/rules, which is the more common way to deal with identity in classical mathematical logic, this could then be dispensed with.

The only way to close the right-most branch would be to utilize the fact that $a \doteq b, +1$ together with the fact that $C(a), +3$ to infer $C(b), +3$. The options here seem to be two in number: either somehow derive $a \doteq b, +0$ and then to infer $C(b), +3$ using LL_{0i_i}.[13] Without III.iii, this will not happen, however. The other option is to somehow derive $a \doteq b, +3$. However, if not by magic, then the only general rule which would make this possible would be the rule

$$\begin{array}{c} A, +i \\ Rijk \\ \downarrow \\ A, +k \end{array}$$

which is the rule needed to validate the weakening axiom $A \rightarrow (B \rightarrow A)$. Thus without either weakening or III.iii, LL_{ii_i} is non-inductive. To show how badly it is non-inductive, consider the following model for (the positive fragment of) **B** which validates atomic Substitution over the empty language ($\doteq$ is here regarded as a logical predicate), but in which

$$(a \doteq b \wedge (a \doteq a \rightarrow a \doteq x)_x(a)) \rightarrow (a \doteq a \rightarrow a \doteq x)_x(b)$$

fails:

$$\begin{array}{ll} \mathcal{D} = \{a, b\} & \mathfrak{v}_0(\doteq) = \{\langle a, a\rangle, \langle b, b\rangle\} \\ \mathcal{W} = \{0, 0^*\} & \mathfrak{v}_1(\doteq) = \{\langle a, b\rangle, \langle b, b\rangle\} \\ \mathcal{N} = \{0\} & \mathcal{R} = \{\langle 0, 0, 0\rangle, \langle 0, 0^*, 0^*\rangle, \langle 0^*, 0, 0\rangle\}. \end{array}$$

It is then easy to check that $\mathfrak{v}_{0^*} \vDash a \doteq b \wedge (a \doteq a \rightarrow a \doteq x)_x(a)$, but that $\mathfrak{v}_{0^*} \nvDash a \doteq a \rightarrow a \doteq x_x(b)$ so that $\mathfrak{v}_0$ fails to validate the instance of Substitution despite the fact that both 0 and 0^* validate atomic Substitution.[14]

Setting out looking for a model which validates Substitution without either weakening or the Subset Constraint can therefore be quite an onerous task in that one has to chance upon a set of worlds which for every formula A satisfies that

DEFINITION 4.2 (Semantics of Substitution).

$$\textit{if } \langle a, b\rangle \in \mathfrak{v}_w(\doteq), \textit{ then } \mathfrak{v}_w^{\mathfrak{s}(x/a)} \vDash A \textit{ iff } \mathfrak{v}_w^{\mathfrak{s}(x/b)} \vDash A.$$

As we saw, the A's here might be formulas which involve the identity predicate itself, and so the stated semantical clause for substitution is essentially a closure condition. However, looking at the above model one also realizes that that model can't be extended so as to validate unrestricted Substitution, and so it will not always be possible to generate models from initially permissible models which validate atomic Substitution.[15] *End second reason.*

Problem 9 in Priest (2008, p. 562) asks the reader to consider if one ought to accept the Subset Constraint. I have in the previous section shown that that principle entails the contraction rule for identity formulas which the previous section showed is not the case with Substitution. I then displayed a different, but equivalent, tableaux rule for the constraint, which makes it clearer in what regard it is too strong, and even though it entails

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[13] LL_{0i_i} would here have to be added as an primitive rule since it is not derivable from atomic LL_{ii_i} as the reader should be able to convince themselves.

[14] It is this feature which Edwin Mares exploited in Mares (2004, sec. 6.13) where he argues that unrestricted Substitution should not be regarded as a valid principle, but suggests that we ought to accept $s \doteq t \wedge A(s) \rightarrow A(t)$ when A is a formula over the $\rightarrow$-free fragment of the language.

[15] If this translates into a complexity issue is another matter which I have not tried to decide.

Substitution, and therefore that SI holds at every world, it also entails a cross-world version of SI which seems quite unwarranted.

We have so far seen that Priest's Subset Constraint, as this is formulated in INCL, overshoots the stated goal of validating Substitution; we have seen that the Subset Constraint forces the symmetry of identity to hold in $\rightarrow$-form, as well as identity to contract. The question, therefore, is how to add axioms/rules to the Hilbert calculus for $\forall\mathbf{B}\overset{s}{\doteq}$ to get a logic which is complete with regard to $\forall\mathbf{B}\doteq$-models with the Subset Constraint. A further reason for, temporarily at least, not accepting the Subset Constraint is that this question is as of yet an open one:

OPEN PROBLEM. *What axioms and/or rules need to be added to* $\forall\mathbf{B}\overset{s}{\doteq}$ (*and relatives*) *to get a logic which is sound and complete with regard to* $\forall\mathbf{B}\overset{s}{\doteq}$*-models with the Subset Constraint*?

Added in Press: In vernacular English, the Subset Constrain says that only true identity statements can hold true at other possible worlds. This is captured by Priest's tablaux rule which allows one to infer that if an identity is true at some world, then it is true at the base world. Put differently, then, either an identity statement $s \doteq t$ is true at the base world, or it is true nowhere. Now the semantics for relevant logics is such that a conditional $A \rightarrow B$ is evaluated as true at the base world if B is true at every world at which A is true. Thus, if $s \doteq t$ is true at no world, $s \doteq t \rightarrow A$ will be true at the base world. A completeness proof is beyond the scope of this paper, but I dare guess that is possible to prove that $\forall\mathbf{B}\overset{s}{\doteq}$ augmented with

$$(RM_3^{\doteq}) \quad \forall x \forall y(x \doteq y \vee (x \doteq y \rightarrow A))$$

is sound and complete with regards to $\forall\mathbf{B}\overset{s}{\doteq}$-models with the Subset Constraint.

This, then, yields a new argument against the Subset Constraint. That identity contracts will not be a counterargument to any relevant logician who already accepts contraction. However, that $\forall x \forall y(x \doteq y \vee (x \doteq y \rightarrow A))$ is forced upon one, even though $A \vee (A \rightarrow B)$ is not a theorem of any relevant logic,[a] speaks against accepting the Subset Constraint. Why should "if s is identical to t, then the earth is flat" hold true just because it is not true that s is identical to t?[b]

[a] The three-valued logic $\mathbf{RM}_3$ is got by adding the formula $A \vee (A \rightarrow B)$ to the quasi-relevant logic $\mathbf{RM}$ which again is got by adding $A \rightarrow (A \rightarrow A)$ to the relevant logic $\mathbf{R}$.

[b] Many thanks to one of the referees who pressed the issue of the missing axiom/rule from the Hilbert axiomatization. Alas, the solution presented here dawned on me too late to incorporate it further into the paper.

Having differentiated between the Subset Constraint and the validity of Substitution, Problem 9* arises: is it viable to reject the former, but accept the latter? I have shown that without the Subset Constraint, one will not be able to derive the symmetry of identity in the usual form and that Substitution in tableaux systems is essentially a non-inductive rule which entails that the semantics will not be, in a sense, recursively generatable by specifying the truth-conditions of the atomic formulas. The first point should not move one either way, but the latter is worrying in that the construction of models will be needlessly hard. One might here, however, console oneself with the fact that it will often be fine to try to first construct a model which validates the Subset Constraint, and only if this fails look for models where the Subset Constraint fails, but Substitution holds. I will in §6 give three further reasons why one should not accept Substitution, but first I will examine the roots of the Subset Constraint, namely the second edition of Priest's provocative book *In Contradiction*.

§5. Identity in *In Contradiction*. This section discusses how identity is treated in *In Contraction*—IC for short. Priest states in Priest (2008, sec. 24.10) that the Subset Constraint had its origin in §19.8 of the second edition of IC. A version of the Subset Constraint is there presented in Footnote 30 which reads:[16]

> Note that this condition does not deliver the validity of $(x = y \wedge \beta) \rightarrow \beta(x/y)$. If $x = y$ holds at an impossible world, it is not guaranteed that x and y have the same denotation. The validity of this principle can be obtained by adding the further constraint that at impossible worlds, w, $d^{+}_{w}(=) \subseteq \{\langle a, a\rangle : a \in D\}$. (Priest, 2006, p. 273)

The quote does not, and Priest does, as far as I have found, nowhere else in IC either, state the stronger, and as the previous section of this essay shows, false statement which *An Introduction to Non-Classical Logic* (INCL) does, namely that Substitution fails without the Subset Constraint. The Subset Constraint of IC, however, is different from that of INCL in that it is not only too strong, but also too weak. The essential difference between the two is that the Subset Constraint of INCL appears in the context of an intensional semantics for negation in which the Routley star is used, whereas the semantics of IC is many-valued or relational. The consequence of this is that the semantically basic conditional of INCL is the contraposable conditional $\rightarrow_{INCL}$ which we have been dealing with up until now, whereas the semantically basic conditional of IC is a non-contraposable $\Rightarrow$ from which Priest's "official" entailment conditional $\rightarrow_{IC}$ may be defined as $(A \Rightarrow B) \wedge (\neg B \Rightarrow \neg A)$. The identity rules and principles, including the SC, of INCL relate to $\rightarrow_{INCL}$, whereas the identity rules and principles of IC relate to $\Rightarrow$. It may, of course, be the case that the rules and principles also relate to $\rightarrow_{IC}$, but then again, they might not. As it turns out, not all of them do, and it is the purpose of this section to explain this more thoroughly.

IC was first published in 1987. The second edition came out in 2006 and contained four additional chapters (Chapter 15–18) with new material, one chapter (Chapter 19) with autocommentary on the chapters of the first edition, and finally one chapter containing comments on critics. Priest touches fleetingly on the logic of identity in both Chapters 5 and 6 belonging to the first edition, in Chapter 18 of the second edition, and in the autocommentary on Chapter 6 found in Chapter 19.8 of the second edition.[17] During the 19 years between the first and the second edition, Priest has, as detailed in Chapter 19, changed his views on some things. For starters, the logic of the first edition was a non-relevant logic, whereas Priest of the second edition prefers a relevant logic. With it, his view on identity seems also to have evolved. The two most important differences for our quest is that the first edition thought that all worlds must report all true identities and that all worlds must respect excluded middle. The second edition does not.

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16 'This condition' refers to $d^{+}_{w}(=) = \{\langle a, a\rangle : a \in D\}$ for normal worlds w.

17 Chapter 18 is also reprinted with minor differences in Priest (2011) which came out in 2011. There are no differences between Chapter 18 of Priest (2006) and Priest (2011) in terms of the logic of identity.

The following two subsections go through, respectively, the semantic material and the syntactic material relating to the logic of identity, explain how the logic of identity of the first edition is different from the logic of the second, and what is wrong with the latter.

5.1.* In Contradiction *and the semantics of identity. The semantics laid out for relevant logics in INCL uses the *Routley star* semantics.[18] That semantics interprets identity by adding a $*$-operator on worlds and demands that $\neg A$ be true at a world w if and only if A fails to be true at w's star-mate w^*—$w \vDash \neg A$ iff $w^* \nvDash A$. Because of this one only needs to specify the extension of a predicate at every world in order for the semantics to get going. This allows for the following four options:

($*1$) a is in the extension of the predicate P at w ($a \in \mathfrak{v}_w(P)$)
($*2$) b is not in the extension of the predicate P at w ($b \notin \mathfrak{v}_w(P)$)
($*3$) c is in the extension of the predicate P at w^* ($c \in \mathfrak{v}_{w^*}(P)$)
($*4$) d is not in the extension of the predicate P at w^* ($d \notin \mathfrak{v}_{w^*}(P)$).

These conditions then entail the following four w-facts:

($w1$) P is true of a at w ($\mathfrak{v}_w^{\mathfrak{s}} \vDash P(a)$)
($w2$) P is not true of b at w ($\mathfrak{v}_w^{\mathfrak{s}} \nvDash P(b)$)
($w3$) $\neg P$ is not true of c at w ($\mathfrak{v}_w^{\mathfrak{s}} \nvDash \neg P(c)$)
($w4$) $\neg P$ is true of d at w ($\mathfrak{v}_w^{\mathfrak{s}} \vDash \neg P(d)$).

The semantics of IC, however, is relational; a formula is evaluated by relating it to the truth-values 1 and 0.[19] In the classical case it will relate to precisely one of them. It may, however, be related to neither 1 nor 0—in that case it is called a *gap*—or it may be related to *both* 0 and 1, in which case it is called a *glut*. In such a setting one needs to not only specify the extension of a predicate—the things the predicate is true of (1-related to), but also its *anti-extension*—the things the predicate is false of (0-related to). For any predicate P, $\mathfrak{v}_w^+(P)$ is the *extension* of P at the world w, whereas $\mathfrak{v}_w^-(P)$ is the *anti-extension* of P at the world w. The latter is the set of all things the predicate is false of, whereas the first is the set of things the predicate is true of. The four w-facts above are in this semantics generated by the following four conditions:

(R1) a is in the extension of the predicate P at w ($a \in \mathfrak{v}_w^+(P)$)
(R2) b is not in the extension of the predicate P at w ($b \notin \mathfrak{v}_w^+(P)$)
(R3) c is not in the anti-extension of the predicate P at w ($c \notin \mathfrak{v}_w^-(P)$)
(R4) d is in the anti-extension of the predicate P at w ($d \in \mathfrak{v}_w^-(P)$).

With this cleared up, we can state the recursive model-theoretic clauses for the relational semantics of the logic **FDE**.[20]

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18 The exception here is the logic $\mathbf{N}_4$ discussed in Chapters 9 and 23. However, only the weak rule III.ii, is considered for this logic.

19 The first edition of IC used a many-valued semantics; formulas were there evaluated to the truth-values $\{0\}$, $\{1\}$, and $\{0, 1\}$. Priest notes in the autocommentary (Priest, 2006, sec. 19.7) that he now favors a relational semantics. The semantics are, however, equivalent, but whereas the many-valued semantics suggests that there are three truth-values—False, True and Glut—the relational semantics relates formulas to only the two classical truth-values True and False.

20 Disjunction and the existential quantifier can be defined in the usual way.

DEFINITION 5.1 (Enough relational semantics to be dangerous).

$(A1)$	$\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 P(s_1,\ldots,s_n)$	*iff*	$\langle \mathfrak{v}^{\mathfrak{s}}(s_1),\ldots,\mathfrak{v}^{\mathfrak{s}}(s_n)\rangle \in \mathfrak{v}^{+}_{w}(P)$
$(A0)$	$\mathfrak{v}^{\mathfrak{s}}_{w} \models_0 P(s_1,\ldots,s_n)$	*iff*	$\langle \mathfrak{v}^{\mathfrak{s}}(s_1),\ldots,\mathfrak{v}^{\mathfrak{s}}(s_n)\rangle \in \mathfrak{v}^{-}_{w}(P)$
$(N1)$	$\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 \neg A$	*iff*	$\mathfrak{v}^{\mathfrak{s}}_{w} \models_0 A$
$(N0)$	$\mathfrak{v}^{\mathfrak{s}}_{w} \models_0 \neg A$	*iff*	$\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 A$
$(C1)$	$\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 A\wedge B$	*iff*	$\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 A$ *and* $\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 B$
$(C0)$	$\mathfrak{v}^{\mathfrak{s}}_{w} \models_0 A\wedge B$	*iff*	$\mathfrak{v}^{\mathfrak{s}}_{w} \models_0 A$ *or* $\mathfrak{v}^{\mathfrak{s}}_{w} \models_0 B$
$(U1)$	$\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 \forall xA$	*iff*	$\mathfrak{v}^{\mathfrak{s}(x/a)}_{w} \models_1 A$ *for all* $a \in \mathcal{D}$
$(U0)$	$\mathfrak{v}^{\mathfrak{s}}_{w} \models_0 \forall xA$	*iff*	$\mathfrak{v}^{\mathfrak{s}(x/a)}_{w} \models_0 A$ *for some* $a \in \mathcal{D}$.

FDE itself needs no world parameter w, but these will come in handy in a little while. All worlds in both the first and second edition of IC respects the **FDE**-clauses, but both add to these in different ways. There are significant differences between the first and the second edition of IC, and this section now splits into two subsections to deal with these editions in turn.

5.1.1. The semantics of identity in the first edition of In Contradiction. The first edition of IC made use of a possible world semantics with a base world G which is related to every other world by the binary accessibility relation R. No other R-facts are required to hold. Priest demands that excluded middle should hold for all atomic formulas at all worlds—which then propagates to all formulas—and introduces a non-contraposable conditional $\Rightarrow$ from which the contraposable $\rightarrow$ is defined as $A \rightarrow B =_{df} (A \Rightarrow B) \wedge (\neg B \Rightarrow \neg A)$. The new conditional, $\Rightarrow$, is simply a strict conditional preserving truth over all worlds. The missing semantic clauses are as follows:

(R)	GRw for all $w \in \mathcal{W}$
(BI)	$\mathfrak{v}^{+}_{w}(P) \cup \mathfrak{v}^{-}_{w}(P) = \mathcal{D}^n$ for all $w \in \mathcal{W}$
$(I1)$	$\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 A \Rightarrow B$ iff for all Rwx, $\mathfrak{v}^{\mathfrak{s}}_{x} \models_1 B$ if $\mathfrak{v}^{\mathfrak{s}}_{x} \models_1 A$
$(I0)$	$\mathfrak{v}^{\mathfrak{s}}_{w} \models_0 A \Rightarrow B$ iff for some Rwx, $\mathfrak{v}^{\mathfrak{s}}_{x} \models_1 A$ and $\mathfrak{v}^{\mathfrak{s}}_{x} \models_0 B$
$(BI\doteq)$	$\mathfrak{v}^{+}_{w}(\doteq) \cup \mathfrak{v}^{-}_{w}(\doteq) = \mathcal{D}^2$ for all $w \in \mathcal{W}$
$(\doteq^{+})$	$\mathfrak{v}^{+}_{w}(\doteq) = \{\langle a,a\rangle \mid a \in \mathcal{D}\}$ for all $w \in \mathcal{W}$
$(\doteq^{-})$	$\mathfrak{v}^{-}_{G}(\doteq) = \mathfrak{v}^{-}_{w}(\doteq)$ for all $w \in \mathcal{W}$.

The anti-extension of $\doteq$ can, in other words, be arbitrary, but should be world-invariant.[21] Semantical consequence is defined as $\models_1$-preservation over the base world G. These semantic clauses are referred to as the Δ*-semantics* in the first edition of IC.[22]

THEOREM 5.2. *$s \doteq t \wedge A_z(s) \rightarrow A_z(t)$ is valid on the Δ-semantics without the invariance clause $(\doteq^{-})$.*

Proof. The goal, then, is to show that
$\mathfrak{v}^{\mathfrak{s}}_{G} \models_1 s \doteq t \wedge A_z(s) \rightarrow A_z(t)$. *Since $\rightarrow$ is defined, this reduces to showing*

[21] This, I believe, is the most reasonable interpretation of Priest. We are first told in §5.3 of IC that the anti-extension of $\doteq$ for the extensional **FDE**-fragment can be arbitrary (Priest, 2006, p. 78), and in the next chapter dealing with conditional which involves many worlds, we are then told that "'=' is a two-place predicate whose interpretation is the world-invariant set specified in §5.3." (Priest, 2006, p. 93).

[22] As Priest notes in Priest (2006, pp. 86f) that Δ validates the depth-relevant logic **DR** (see Brady (1989)).

$$(1) \quad \mathfrak{v}^{\mathfrak{s}}_{G} \models_1 s \doteq t \wedge A_z(s) \Rightarrow A_z(t)$$
$$(2) \quad \mathfrak{v}^{\mathfrak{s}}_{G} \models_1 \neg A_z(t) \Rightarrow \neg(s \doteq t \wedge A_z(s)).$$

As for (1), assume that there is a world w such that $\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 s \doteq t \wedge A_z(s)$. Then $\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 s \doteq t$ which then per clause ($\doteq^+$) entails that the two terms really denote one single object. The compositionality of the semantics then entails that $\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 A_z(t)$ since $\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 A_z(s)$. Since w was arbitrary, (1) follows by (I1) and (R).

As for (2), assume that there is a world w such that $\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 \neg A_z(t)$. Now either (i) $\mathfrak{v}^{\mathfrak{s}}_{w} \models_0 s \doteq t$ or (ii) $\mathfrak{v}^{\mathfrak{s}}_{w} \not\models_0 s \doteq t$. If (i), then using (N1) and (C0), we get that $\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 \neg(s \doteq t \wedge A_z(s))$. If (ii), then $\langle \mathfrak{v}^{\mathfrak{s}}(s), \mathfrak{v}^{\mathfrak{s}}(t)\rangle \notin \mathfrak{v}^{-}_{w}(\doteq)$, and so by (BI$\doteq$), $\langle \mathfrak{v}^{\mathfrak{s}}(s), \mathfrak{v}^{\mathfrak{s}}(t)\rangle \in \mathfrak{v}^{+}_{w}(\doteq)$. By ($\doteq^+$) we have then that s and t really do denote the same object. The compositional semantics then entails that $\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 \neg A_z(s)$ which, using (N1) and (C0) again, suffices for $\mathfrak{v}^{\mathfrak{s}}_{w} \models_1 \neg(s \doteq t \wedge A_z(s))$. w was arbitrary and so (2) follows by clauses (I1) and (R). □

Note that the proof of (1) above would have gone through as long as every world was closed under the weak SI-rule $s \doteq t, A_z(s) \vdash A_z(t)$. The appeal to ($\doteq^+$) in (2) could also be dispensed with in favor of having every world closed under this rule. However, the proof of (2.ii) does need (*BI*$\doteq$) to infer $\langle \mathfrak{v}^{\mathfrak{s}}(s), \mathfrak{v}^{\mathfrak{s}}(t)\rangle \in \mathfrak{v}^{+}_{w}(\doteq)$. This amounts to validating excluded middle for identity statements at every world. One could here rather appeal to ($\doteq^-$) to infer $\langle \mathfrak{v}^{\mathfrak{s}}(s), \mathfrak{v}^{\mathfrak{s}}(t)\rangle \notin \mathfrak{v}^{-}_{G}(\doteq)$ and then use (*BI*$\doteq$) to infer that $\langle \mathfrak{v}^{\mathfrak{s}}(s), \mathfrak{v}^{\mathfrak{s}}(t)\rangle \in \mathfrak{v}^{+}_{G}(\doteq)$. That way one would only need ($\doteq^+$) and excluded middle to hold at the base world provided all worlds are closed under the SI-rule. The following theorem states this with a slight twist so as to avoid having to appeal to excluded middle at G:

THEOREM 5.3. *$s \doteq t \wedge A_z(s) \rightarrow A_z(t)$ is valid on the Δ-semantics with clauses*

$$(BI{\doteq}) \quad \mathfrak{v}^{+}_{w}(\doteq) \cup \mathfrak{v}^{-}_{w}(\doteq) = \mathcal{D}^2$$
$$(\doteq^+) \quad \mathfrak{v}^{+}_{w}(\doteq) = \{\langle a, a\rangle \mid a \in \mathcal{D}\}$$
$$(\doteq^-) \quad \mathfrak{v}^{-}_{G}(\doteq) = \mathfrak{v}^{-}_{w}(\doteq) \quad \textit{for all } w \in \mathcal{W}$$

dropped in favor of

$$(\bullet_1) \quad \mathfrak{v}^{+}_{G}(\doteq) = \{\langle a, a\rangle \mid a \in \mathcal{D}\}$$
$$(\bullet_2) \quad \mathcal{D}^2 \setminus \mathfrak{v}^{+}_{G}(\doteq) \subseteq \mathfrak{v}^{-}_{w}(\doteq)$$

provided all worlds are closed under the SI-rule.

($\bullet_2$) here licenses the inference from $s \doteq t$ failing to being false at w to it in fact being true which is what the proof above needs.

Neither of these options for validating $\rightarrow$-substitution seem rather attractive, however. (1) Having excluded middle valid on every world is a recipe for irrelevancies, and the invariance clause seems simply ad-hoc. ($\bullet_2$) also seems ad-hoc; why think that all worlds must report all identity falsehoods?

The first edition of IC simply notes without a proof that $\rightarrow$-Substitution is valid on the Δ-semantics. That it does so with a vengeance, is of course not something to fault it over as it is never claimed that $\rightarrow$-Substitution would fail in a logically skimpier environment.

I will get back to these two ways of validating $\rightarrow$-substitution in the next section where Priest's tableaux system from the second edition of IC is shown forth. Before that, however, let's look at the semantics of identity in the second edition of IC.

5.1.2. The semantics of identity in the second edition of In Contradiction. The conditional $\Rightarrow$ of the first edition of IC was, as I mentioned, simply a strict conditional which preserves truth over all possible worlds. A relevantist will regard this as suspect, however, since strict conditionals bring with them paradoxes. If we restrict our attention to the $\Rightarrow$-free fragment of the language we get that every Δ-world is a **LP**-evaluation and every such **LP**-evaluation validates B if and only if B is a two-valued tautology (Priest, 1979, Theorem III.8). But then for any formula A and two-valued tautology B such that A and B share no propositional parameters, we get that $\mathfrak{v}^{\mathfrak{s}}_G \vDash_1 A \Rightarrow B$. Furthermore, for every formula A and B we get that $\mathfrak{v}^{\mathfrak{s}}_G \vDash_1 A \Rightarrow (B \Rightarrow B)$. There is, as Priest also notes, a tradition within relevant logic, going back to the Routleys' essay Routley & Routley (1972) which claims that total absence of suppression is the deep challenge to entailment.[23] On that note, notice that Δ validates the suppression rules $\Box A, A \wedge B \Rightarrow C \models B \Rightarrow C$, where $\Box A =_{df} \neg A \Rightarrow A$.[24] These irrelevancies do not hold when $\Rightarrow$ is replaced by $\rightarrow$, however. Entailment proper, according to Priest, does not only preserve truth by necessity as $\Rightarrow$ does, but also preserves falsity from the consequent back to the antecedent. Hence $\Rightarrow$ is not, according to Priest, an entailment connective. Of course, not every conditional need to be relevant for the relevantist to accept it, but it does seem that a relevantist should reject a logical theory which has, like Priest's Δ, a necessary truth-preserving conditional littered with irrelevancies. Anyways, there are also pure $\rightarrow$-irrelevancies in Δ such as the Kleene axiom $A \wedge \neg A \rightarrow B \vee \neg B$ for any A's and B's. This generalizes: let A be the negation of a classical tautology[25] and B be a classical tautology which shares no propositional variables with A. Then for every world w, we have that $\mathfrak{v}^{\mathfrak{s}}_w \vDash_1 B$ and $\mathfrak{v}^{\mathfrak{s}}_w \vDash_1 \neg A$. Thus both $\mathfrak{v}^{\mathfrak{s}}_G \vDash_1 A \Rightarrow B$ and $\mathfrak{v}^{\mathfrak{s}}_G \vDash_1 \neg B \Rightarrow \neg A$, and therefore $\mathfrak{v}^{\mathfrak{s}}_G \vDash_1 A \rightarrow B$.

Priest notes that Δ validates some irrelevancies—he mentions the Kleene axiom explicitly—but shurgs this off due to Δ's "simplicity and philosophical perspicuity" (Priest, 2006, p. 92). The semantics for relevant logics at the time of the first edition of IC was complicated and difficult to give good philosophical sense of. That changed, however, with Priest and Sylvan's (formerly known as *Routley*) essay *Simplified Semantics for Basic Relevant Logics* (Priest & Sylvan, 1992) which came out in 1992 and made the ternary semantics of relevant logics much easier to stomach. This was also the year *What is A Non-Normal World?* (Priest, 1992) came out in which Priest develops a relevant semantics (albeit not a ternary one) in which impossible/non-normal worlds figure and motivates the idea that even logical laws can fail.

When the second edition of IC then came out, Priest had dropped the adherence to a strict conditional in favor of a relevant one with a new emphasis on the dichotomy of possible/impossible worlds and the mischiefs of the latter. Both excluded middle and the law of self-identity, $s \doteq s$, are now counted as such laws which hold on all *possible/normal* worlds, but may fail at *impossible/non-normal* ones. The new semantics laid out in §19.8—the autocommentary on Chapter 6 of the first edition— holds on to the relational semantics, and so the **FDE**-clauses above are still enforced, but the binary relation R of the first edition is dropped in favor of the ternary $\mathcal{R}$-relation of relevant logics. Since the new relevant conditional reduces to the strict conditional of Δ in normal worlds, I will not specify the new semantic clauses here.[26] What is new, however, is the identity clauses. The second edition of IC has only two such (Priest, 2006, p. 273):

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23 See Øgaard (2019, sec. 6) where it is shown that absence of suppression is in fact weaker than the variable-sharing property.

24 This is Priest's definition of a necessity operator (Priest, 2006, p. 90). $\Box A$ is true at any world if and only if A is true at all worlds accessible from it.

25 Formulas $C \Rightarrow D$ can be regarded as propositional variable so as to be classically evaluable.

26 They can be gleaned off the proof theory presented in §5.2.

$$(BI_n\doteq) \quad \mathfrak{v}_n^+(\doteq) \cup \mathfrak{v}_n^-(\doteq) = \mathcal{D}^2 \qquad \text{for normal worlds; } n \in \mathcal{N}$$
$$(\doteq_n^+) \quad \mathfrak{v}_n^+(\doteq) = \{\langle a, a\rangle \mid a \in \mathcal{D}\} \qquad \text{for normal worlds; } n \in \mathcal{N}.$$

Priest then notes (Priest, 2006, fn. 30, p. 273) that this is insufficient for validating $\rightarrow$-Substitution and then specifies for the first time a version of the Subset Constraint as sufficient for validating $\rightarrow$-Substitution. To avoid confusing the two versions of the Subset Constraint, let's agree to use 'SC' for the version of INCL and SC_{IC} for the version of IC:

$$(\text{SC}) \quad v_w(\doteq) \subseteq \{\langle a, a\rangle \mid a \in \mathcal{D}\}$$
$$(\text{SC}_{\text{IC}}) \quad v_w^+(\doteq) \subseteq \{\langle a, a\rangle \mid a \in \mathcal{D}\}.$$

SC_{IC} does validate $\Rightarrow$-Substitution, and as in INCL, is stronger than what is needed in this regard. Unlike INCL, however, there is no claim here that SC_{IC} is necessary, only that it is sufficient. But it *is* claimed to be sufficient for $\rightarrow$-Substitution, not only $\Rightarrow$-Substitution, however, and this seems at best doubtful. Note that there are in the second edition of IC no new clauses for how to interpret the anti-extension of $\doteq$ and so one is left wondering if it now can be interpreted willy-nilly as long as excluded middle holds for all identity statements at normal worlds, or if in fact the world invariant clause of the first edition is still to be enforced. If the first is the case, then the semantics doesn't validate $\rightarrow$-Substitution. If the latter, then $\rightarrow$-Substitution is valid as I showed in the comments to Theorem 5.2 in the previous section. However, it would then seem simply misleading to claim, as the quotation at the beginning of this section does, that one needs to add SC_{IC} in addition to $(\doteq^+)$ to validate $\rightarrow$-Substitution. A better interpretation, I think, is rather that Priest is simply wrong here. This is also the most plausible interpretation when looking at the proof theory set forth in Chapter 18 new to the second edition. This is the task of the next section. Here, again, we will find that the proof theory suffices for deriving $\Rightarrow$-Substitution, but not $\rightarrow$-Substitution. I will also consider various rules which are sufficient for deriving $\rightarrow$-Substitution, and end with evaluating the plausibility of $\rightarrow$-Substitution on the relational semantics for relevant logics.

5.2. In Contradiction and the proof theory of identity. The first edition of IC did not specify a proof system. This is rectified with the new 18th chapter of the second edition. Note, however, that the title of that chapter is "Paraconsistent Set Theory", and so stating a proof system isn't the main agenda of the chapter. The proof system is claimed to be sound with regard to the semantics of Priest (2006, sec. 19.8) (Priest, 2006, p. 270, fn. 20). No claim with regard to completeness is made.

The rules of IC's proof system, let's call it *IC2*, are akin to those presented for the INCL-system above, so as not to confuse the two, Table 2 contains the entire proof system for the proposition fragment together with the rules of identity.[27]

First note that the extra closing rule, IC2.i.b, ensures that excluded middle holds, but only at the base world. Secondly, note also the '$\pm$' in the IC5.i-rule which therefore should be read as the conjunctive of the two clauses

- *if $s \doteq t$ is true at world i and A is true of s at world j, then A is also true of t at j.*
- *if $s \doteq t$ is true at world i and A is untrue of s at world j, then A is also untrue of t at j.*

[27] The Leibniz's law rule carries the usual restriction on substitutivity. The same restriction as above also applies here with regards to the world parameters; for the rules involving the conditional, j and k need to be identical if i is 0, and the non-branching rules requires that j and k be new to the branch.

	i	ii	iii
IC1	$\cdot$ $\downarrow$ $R0ii$	$A \Rightarrow B, +i$ $Rijk$ $\swarrow \searrow$ $A, -j \quad B, +k$	$A \Rightarrow B, -i$ $\downarrow$ $Rijk$ $A, +j$ $B, -k$
IC2	**(IC2.i.a)** $A, +i$ $A, -i$ $\downarrow$ $\times$ **(IC2.i.b)** $A, -0$ $\neg A, -0$ $\downarrow$ $\times$	$\neg(A \Rightarrow B), +i$ $\downarrow$ $Rijk$ $A, +j$ $\neg B, +k$	$\neg(A \Rightarrow B), -i$ $Rijk$ $\swarrow \searrow$ $A, -j \quad \neg B, -k$
IC3	$\neg\neg A, \pm i$ $\downarrow$ $A, \pm i$	$\neg(A \wedge B), \pm i$ $\downarrow$ $\neg A \vee \neg B, \pm i$	$\neg(A \vee B), \pm i$ $\downarrow$ $\neg A \wedge \neg B, \pm i$
IC4	$\cdot$ $\downarrow$ $t \doteq t, +0$	$A \wedge B, +i$ $\downarrow$ $A, +i$ $B, +i$	$A \wedge B, -i$ $\swarrow \searrow$ $A, -i \quad B, -i$
IC5	$s \doteq t, +i$ $A_x(s), \pm j$ $\downarrow$ $A_x(t), \pm j$	$A \vee B, +i$ $\swarrow \searrow$ $A, +i \quad B, +i$	$A \vee B, -i$ $\downarrow$ $A, -i$ $B, -i$

Table 2. *The tableaux system IC2 from* In Contradiction

'Untrue' here simply means the same as 'not true', whereas A is false at a world if $\neg A$ is true there. Untruth and falsity will not always be the same thing; indeed the proof system does not have rules which links these concepts except for the closure rule which bars a formula from, at the base world, being both untrue ($A, -0$) and unfalse ($\neg A, -0$). This is the excluded middle rule which could have equivalently been replaced by the rule allowing one to infer that A is false at 0 from the assumption that it is untrue.

Before I process, note that both the positive version of IC5.i, +IC5.i, as well as the negative version -IC5.i, both individually suffice for deriving $s \doteq t \Rightarrow t \doteq s$, which otherwise only holds as a mere rule.

THEOREM 5.4. *$s \doteq t \Rightarrow t \doteq s$ is derivable in IC2 using either +IC5.i or -IC5.i.*

Proof. The left derivation uses the positive rule, whereas the right uses the negative rule:

(1)	$s \doteq t \Rightarrow t \doteq s$	*assumption*
(2)	$R011$	*IC1.iii*
(3)	$s \doteq t, +1$	*IC1.iii*
(4)	$t \doteq s, -1$	*IC1.iii*
(5)	$x \doteq s_x(s), +0$	*IC4.i*
(6)	$t \doteq s, +0$	*3, 5, +IC5.i*
(7)	$x \doteq t_x(s), +1$	*3, rewrite*
(8)	$t \doteq t, +1$	*3, 7, +IC5.i*
(9)	$t \doteq x_x(t), +1$	*8, rewrite*
(10)	$t \doteq s, +1$	*6, 9, +IC5.i*
(11)	$\times$	*4, 10, IC2.i.a*

(1)	$s \doteq t \Rightarrow t \doteq s, -0$	*assumption*
(2)	$R011$	*IC1.iii*
(3)	$s \doteq t, +1$	*IC1.iii*
(4)	$t \doteq s, -1$	*IC1.iii*
(5)	$t \doteq t \Rightarrow t \doteq t, -0$	*3, 1, −IC5.i*
(6)	$R022$	*IC1.iii*
(7)	$t \doteq t, +2$	*IC1.iii*
(8)	$t \doteq t, -2$	*IC1.iii*
(9)	$\times$	*7, 8, IC2.i.a*

□

As I mentioned in §4, Substitution will not suffice for deriving this in relevant logics since one cannot suppress true premises so as to get rid of $s \doteq s$ in the Substitution instance $s \doteq t \wedge x \doteq s_x(s) \Rightarrow x \doteq s_x(t)$.[28]

THEOREM 5.5. *The semantic validity of the positive and negative parts of IC5.i separately coentail SC_{IC}.*

Proof. *That +IC5.i coentails SC_{IC} is obvious from the discussion of SC_{INCL}.*

Now assume that -IC5.i is valid, but that SC_{IC} fails and let (*i*) $\langle a, b\rangle \in v_w^+(\doteq)$ *and* (*ii*) $\langle a, b\rangle \notin v_0^+(\doteq)$ *for some world w. Using -IC5.i on* $\langle a, a\rangle \in v_0^+(\doteq)$ *and* (*ii*) *yields* $\langle b, b\rangle \notin v_0^+(\doteq)$ *which contradicts the clause* ($\bullet_1$) *in the above section.*

Now assume that SC_{IC} is valid, and that $s \doteq t$ is true at w, but that $A(s)$ fails to be true at z. Since $s \doteq t$ is true at w, we can use SC_{IC} to infer that it is true at the base world and that s and t therefore have the same denotation. The compositional semantics then ensures that $A(s)$ and $A(t)$ have to be evaluated the same at every world, and therefore that $A(t)$ is not true at z. □

Just as in INCL, then, the proof-theoretic and semantic clauses for identity are fit for each other. From the discussion of Theorems 5.2 and 5.3 is easy to see that SC_{IC} does not suffice for validating the contraposed version of ⇒-Substitution, but it is instructive to see where the derivation terminates:

(1)	$\neg A(t) \Rightarrow \neg(s \doteq t \wedge A(s)), -0$	assumption
(3)	$R011$	IC1.i
(4)	$\neg A(t), +1$	1, IC1.iii
(5)	$\neg(s \doteq t \wedge A(s)), -1$	1, IC1.iii
(6)	$s \dot{\neq} t \vee \neg A(s), -1$	5, IC3.ii
(7)	$s \dot{\neq} t, -1$	6, IC5.iii
(8)	$\neg A(s), -1$	6, IC5.iii

No further expansion of the tree is possible and the tree will therefore not close. The trouble here is that there is no rule which allows one to eliminate single negations; $s \doteq t$ ends up being not false at 1 (Step 7), but without excluded middle holding there ((BI$_n \doteq$) only guarantees $\doteq$-excluded middle for normal worlds) one may not infer that $s \doteq t$ is true at 1 which would, together with the positive part of IC5.i, suffice for a proof. Without excluded middle holding at every world it seems hard to find a plausible principle which would suffice; one option would be to handle negation via the so-called *Routley Star*, $*$, as the semantics of INCL does. That would indeed be sufficient as the reader can easily check by consulting Priest (2008, sec. 8.5) for the rules regulating it. However, those rules also entail that the conditional will contrapose, and so go against Priest's wish to have a non-contraposable conditional. One might amend this by adding either of the following three rules:

$$\begin{array}{c} s \dot{\neq} t, -i \\ \downarrow \\ s \doteq t, +i \end{array} \qquad \begin{array}{c} (\bar{\neg} LL_{ii_i}) \\ s \dot{\neq} t, -i \\ A(s), -i \\ \downarrow \\ A(t), -i \end{array} \qquad \begin{array}{c} (\bar{\neg} III.iii) \\ s \dot{\neq} t, -i \\ \downarrow \\ s \doteq t, +0. \end{array}$$

[28] Note, furthermore, that IC5.i will not yield symmetry in → form. I leave it to the reader to construct a model which verifies this.

The first, however, seems rather ad hoc; it entails that excluded middle holds at every world for identity statements, which seems strange to claim if one allows excluded middle to fail in general. The second is more interesting; looking at the above derivation it is evident that $\bar{\neg}\mathrm{LL}_{ii_i}$ is just what is needed. The rule preserves the untruth/unfalsity of $A(s)$ unto $A(t)$ given that $s \doteq t$ is unfalse. This is in fact in line with the rest of Priest's tableaux system which do also preserve untruth/unfalsity. Those rules, however, are designed to break up complex formula into their composites and the rules simply ensure that untruth/unfalsity is preserved in the process. $\bar{\neg}\mathrm{LL}_{ii_i}$ has a very different ring to it, and I find it hard to give anything but a technical justification for it: the proof system is set up to display that a formula can be evaluated in four different ways at every world; it could be true at i $(A, +i)$, false at i $(\neg A, +i)$, untrue at i $(A, -i)$ and unfalse at i $(\neg A, -i)$. The inference now encoded by $\bar{\neg}\mathrm{LL}_{ii_i}$ is that it not be possible for $s \doteq t$ to be unfalse, $A(s)$, depending on it is a positive or negated formula, untrue/unfalse, while $A(t)$ be anything other than $A(s)$. Now, per the closure rule IC2.i.a, no formula can be both true/false and untrue/unfalse at the same world. Any of the other four combinations of different valuations are generally possible, however. These five combinations are as follows:

True and untrue	$(A, +i)\&(A, -i)$	ruled out by Logic (IC2.i.a)
True and false	$(A, +i)\&(\neg A, +i)$	ruled out by ex falso quodlibet at i
True and unfalse	$(A, +i)\&(\neg A, -i)$	classical case 1
Untrue and false	$(A, -i)\&(\neg A, +i)$	classical case 2
Untrue and unfalse	$(A, -i)\&(\neg A, -i)$	ruled out by excluded middle at i.

Now if $s \doteq t$ is unfalse yet not true at a world—unless excluded middle holds for identity statements at every world, this is possible—why can't $A(s)$ be unfalse (and maybe true as well), yet $A(t)$ be simply false? The rule seems simply to deliver what is needed for the derivation to go through, without allowing for a justification for why it *should* be included as a valid inference rule in the first place. A slightly better rule, then, would be to add $\bar{\neg}$III.iii, which, as the reader can easily verify, entails $\bar{\neg}\mathrm{LL}_{ii_i}$. $\bar{\neg}$III.iii is the proof-theoretic equivalent of the semantic clause $(\bullet_2)$ in Theorem 5.3. This rule itself, however, seems out of tune with the ideology of impossible worlds: if even the law of self-identity can fail at impossible world, then such worlds need not report all true identities. But then it seems rather far fetched to demand that every such world need to report all false identities.

I have in this rather long section shown that the semantics of the first edition of IC did manage to validate $\rightarrow$-Substitution. The proof-theoretic and semantic clauses for identity in the second edition were shown to match each other and to suffice—with an oomph—for validating $\Rightarrow$-Substitution. However, they fall short of validating $\rightarrow$-Substitution contra what is claimed by Priest. Different principles sufficient for validating $\rightarrow$-Substitution were considered, but were all found to lack justification either by being simply ad hoc, or by being unsound within the ideology of impossible worlds. The cost of insisting that $\rightarrow$-Substitution in the framework of IC seems therefore quite higher than that under INCL where one only need to insist that all worlds be closed under the weak SI-rule $s \doteq t, A_z(s) \vdash A_z(t)$. The problem of validating Substitution, then, dips into the battle between the *The Australian Plan* and the *The American Plan* for a semantics for negation.[29] The trouble with the entailment connective of IC is that it is not semantically basic, but rather definable

[29] *The Australian Plan* is sometimes used to refer to possible world semantics in which negation is treated as a modal or shift-operator on worlds such as as the Routley star does. *The American Plan*

from a non-contraposable conditional. To get the expected laws to hold for the entailment connective, one often has to add to the system in order for it to become sufficiently strong so as to make the requisite laws hold for the entailment connective. Such additions need not always be justifiable, as I have argued is the case with the principles required to validate $\rightarrow$-Substitution.

I think that this shows that one ought not to accept $\rightarrow$-Substitution within the framework of the second edition of *In Contradiction*. Should one, however, accept $\Rightarrow$-Substitution? The next section gives further reasons for not accepting $\Rightarrow$-Substitution in the framework of IC, and not to accept $\rightarrow$-Substitution in the framework of INCL.

§6. Further reasons for dropping Substitution. The rule IC5.i was first set forth in Chapter 18.3 of Priest's book *In Contradiction*. That chapter discusses the prospects of developing a naïve theory of sets using a relevant logic. Priest there claims that such a theory is non-trivial in the logic he sets up. However, I showed in Øgaard (2017, fn. 25) that any relevant logic with Substitution—$\forall\mathbf{B}\doteq$ will do—trivializes naïve set theory. The naïve set theorist should therefore definitely shun $\rightarrow$-Substitution.[30]

I furthermore suggested in Øgaard (2017) that Substitution should be regarded as too strong for the relevantist since it is not valid in Belnap's test-model for relevance. That might be regarded as a barren and technical point, so let's take a step back and look at one possible motivation for relevant logics, namely the need to recognize impossible worlds—worlds at which logic itself is different—in order to make sense of conditionals such as "if intuitionistic logic were correct, the law of double negation would fail".[31] I think it is fair to say that Priest approaches the various logics presented in Priest (2008) by and large by way of reasoning with possible and impossible worlds, and relevant logics are therein commended for allowing such impossible worlds to exist. The following quote is a succinct example: "Since $a = a$ is a logical truth, there may be (impossible) worlds where it fails" (Priest, 2008, p. 550). Priest sets up the laws of identity in such a way that SI must hold at the base world. As a logical law on par with the logical truth of $a \doteq a$, however, it seems reasonable that there should be worlds in which also SI fails to hold. In light of this, it is easy to see the true effect of Substitution—be it formulated with either $\rightarrow$ or $\Rightarrow$—namely the blocking of such impossible worlds from existence. If impossible worlds can fail to report true identities, then why not accept worlds which simply get mixed up about how to interpret identity altogether and therefore fail to report that identity is reflexive, symmetric, transitive and congruent?[32]

is used to refer to the same kind of possible world semantic but where negation is world-bound such as is the case in IC. For more on this, see Berto & Restall (2019).

30 To be fair, it is worth mentioning that Priest states naïve set theory using the naïve comprehension axiom $\exists y \forall x(x \in y \leftrightarrow A)$, where x does not occur in A, whereas the proof in Øgaard (2017) uses the term-rich abstraction version of it, namely $\forall x(x \in \{x \mid A\} \leftrightarrow A)$. The latter obviously entails the first, but it is as of yet an open question if naïve abstraction is derivable from naïve comprehension. Furthermore, the triviality proof does not consider non-contraposable conditionals, and so it might be possible to have $\Rightarrow$-Substitution without trivializing the theory.

31 For more on this way of motivating relevant logics, see Priest (1992) and Priest (2008, secs. 9,7).

32 I should also mention another, albeit quite different, line of attack on Substitution, namely that of Kremer (1999). Kremer there argues that Substitution is at odds with the theory of relevant predication together with the relevant indiscernibility interpretation of identity.

§7. Summary. I have in this essay shown that the proper semantical clause for Leibniz's law of the form $s \doteq t \wedge A(s) \rightarrow A(t)$, *Substitution*, in relevant logics with a contraposable conditional $\rightarrow$, is not Priest's Subset Constraint—that an identity claim true at some world is true at all normal worlds—but instead that every world be closed under the substitutivity of identity, the rule $s \doteq t, A(s) \vdash A(t)$. It was shown that the Subset Constraint entails that identity formulas contract, but that Substitution does not entail this in relevant logics such as **TW**. The Subset Constraint was also shown to entail an unwarranted version of cross-world substitution stronger than Substitution which lacked justification.

I then showed that great care has to be taken when setting up a system with a non-contraposable conditional as the primitive one. This is Priest's approach in *In Contradiction* from whence the Subset Constraint originates. In this case, however, it was shown that the constraint is insufficient to validate $\rightarrow$-Substitution, and that it, like before, is stronger than what is needed for validating Substitution for the primitive non-contraposable conditional $\Rightarrow$. The rules needed to validate $\rightarrow$-Substitution, where $\rightarrow$ is the defined contraposable conditional, were shown to lack justification. Finally I tried fleetingly to motivate why Substitution *ought* to fail: i) it trivializes naïve set theory; ii) it is invalid in Belnap's test-model for relevance; iii) it is in conflict with one of the motivations behind relevant logics as such, namely the need to recognize impossible worlds—worlds at which logic itself is different.

Acknowledgments. An earlier version of this article was presented before the Bergen Logic Group. I am very grateful for helpful comments from the members of the group, as well as the anonymous referee.

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