# What is Known About Vertex Cover Kernelization?[*]

Michael R. Fellows[1], Lars Jaffke[1], Aliz Izabella Király[1], Frances A. Rosamond[1], and
Mathias Weller[2]

[1]Department of Informatics, University of Bergen, Norway
{michael.fellows, lars.jaffke, aliz.kiraly, frances.rosamond}@uib.no
[2]CNRS, LIGM, Université Paris EST, Marne-la-Vallée, France
mathias.weller@u-pem.fr

May 15, 2019

## Abstract

We are pleased to dedicate this survey on kernelization of the VERTEX COVER problem, to Professor Juraj Hromkovič on the occasion of his 60th birthday. The VERTEX COVER problem is often referred to as the *Drosophila* of parameterized complexity. It enjoys a long history. New and worthy perspectives will always be demonstrated first with concrete results here. This survey discusses several research directions in VERTEX COVER kernelization. The *Barrier Degree* of VERTEX COVER is discussed. We have reduction rules that kernelize vertices of small degree, including in this paper new results that reduce graphs almost to minimum degree five. Can this process go on forever? What is the minimum vertex-degree barrier for polynomial-time kernelization? Assuming the Exponential-Time Hypothesis, there is a minimum degree barrier. The idea of *automated kernelization* is discussed. We here report the first experimental results of an AI-guided branching algorithm for VERTEX COVER whose logic seems amenable for application in finding reduction rules to kernelize small-degree vertices. The survey highlights a central open problem in parameterized complexity. Happy Birthday, Juraj!

## 1 Introduction and Preliminaries

A vertex cover of a graph is a subset of its vertices containing at least one endpoint of each of its edges. The VERTEX COVER problem asks, given a graph $G$ and an integer $k$, whether $G$ contains a vertex cover of size at most $k$.

The study of the VERTEX COVER problem lies at the roots of the theory of NP-completeness: It is one of Karp's 21 NP-complete problems [48] and plays a central role in the monograph of Garey and Johnson [36]. However, interest in the VERTEX COVER problem reaches far beyond pure theory. One reason is that it naturally models *conflict resolution*,[1] a problem occurring in numerous scientific disciplines, with an international workshop devoted to it [2]. Other applications include classification methods (see, e.g., [37]), computational biology (e.g., [14]), and various applications follow from the duality of VERTEX COVER with the CLIQUE

---

[1]In the textbook [17], the problem was entertainingly introduced as 'Bar Fight Prevention'.

problem (see, e.g., [1]). The latter finds numerous applications in fields such as computational biology and bioinformatics [10, 49, 50, 67, 74], computational chemistry [23, 54, 72], and electrical engineering [16, 41].

In *parameterized/multivariate algorithmics* [17, 21, 63], the objects of study are computational problems whose instances are additionally equipped with a integer $k$, the *parameter*, typically expressing some structural measure of the instance of the problem. The goal is to design algorithms for hard problems whose runtime confines the combinatorial explosion to the parameter $k$ rather than the size of the input. A parameterized problem is called *fixed-parameter tractable* if it can be solved in time $f(k) \cdot n^{\mathcal{O}(1)}$ where $f$ is some computable function, $k$ the parameter and $n$ the input size. The second central notion in the field of parameterized algorithms is that of a *kernelization* [20, 22, 28], a polynomial-time algorithm (usually described as a set of *reduction rules*) that takes as input an instance $(I, k)$ of a parameterized problem and outputs an equivalent instance $(I', k')$, where $|I'| + k' \leq g(k)$ for some computable function $g$.[2]

Kernelization (for the first time!) provided a theory of preprocessing with mathematically provable guarantees. On the other end, kernelization has immediate practical implications, as demonstrated by Karsten Weihe's problem [69, 70] (see also [25, 26]) concerning the train systems in Europe. By the means of two simple reduction rules, graphs (instances) on $10,000$ vertices are reduced to equivalent instances whose connected components are of size at most $50$, making the reduced instance solvable exactly even by brute force in reasonable time, after the preprocessing, even though the general problem is NP-hard. Similar reduction rules have been successfully applied in the context of cancer research [4] and spread of virus [24].

The notions of fixed-parameter tractability and kernelization are tightly linked. It has been shown by Cai et al. that a parameterized problem is fixed-parameter tractable if and only if it has a (polynomial-time) kernelization algorithm [11]. Kernelization for the VERTEX COVER problem, which is often referred to as the *Drosophila* of parameterized complexity [21, 33, 39, 63], enjoys a long history. In 1993, the first kernel on $\mathcal{O}(k^2)$ vertices was obtained, and is accredited to Buss [8], with more refined reduction rules given in [3]. Kernels with a linear number of vertices were obtained in various ways. Using classic graph theoretic results, Chor et al. gave a kernel on $3k$ vertices [15] (see also [27]), a kernel on $2k$ vertices was obtained via an LP-relaxation by Chen et al. [12] and another kernel on $2k$ vertices without the use of linear programming was obtained by Dehne et al. [18]. The next series of improvements gave kernels on $2k - c$ vertices [65] and the current champion which is due to Lampis has $2k - c \log k$ vertices [53], where in the latter two $c$ is any fixed constant. Another kernel on $2k - \mathcal{O}(\log k)$ vertices was observed in [61]. An experimental evaluation of several of the earlier kernels was carried out in [1].

There is no known subquadratic bound on the number of *edges* in any kernel for VERTEX COVER, and the question whether such a kernel exists was a long standing open question in multivariate algorithmics. It was finally shown that up to logarithmic factors, VERTEX COVER kernels with a quadratic number of edges are likely to be optimal: Dell and van Melkebeek, building on work of Bodlaender, Downey, Fellows and Hermelin [6, 7], also Fortnow and Santhanam [35], showed that there is no kernel on $\mathcal{O}(n^{2-\varepsilon})$ bits, for any $\varepsilon > 0$, unless NP $\subseteq$ coNP/poly [19]. The latter would imply that the polynomial hierarchy collapses to its third level [73] which is widely considered to be implausible by complexity theorists.[3]

In another line of research, following the *parameter ecology* program [31], the existence

---

[2]As the focus of this text is on the VERTEX COVER problem, we refer to [38, 56] for general surveys on the subject of kernelization and to [59] for a survey on the corresponding lower bound machinery.

[3]We also refer to [46, pages 19f] and [71, Appendix A] for brief accounts of the implausibility of NP $\subseteq$ coNP/poly.

of kernels for VERTEX COVER w.r.t. parameters that take on smaller values than the vertex cover number was studied. Such parameterizations are typically referred to as *structural* parameterizations of VERTEX COVER. The first such result is due to Jansen and Bodlaender who gave a kernel on $\mathcal{O}(\ell^3)$ vertices, where $\ell$ is the size of a feedback vertex set of the graph [47]. Further results include polynomial kernels where the parameter is the size of an odd cycle traversal or a König deletion set [52], the size of vertex deletion sets to maximum degree at most two [58], pseudoforest [34] and $d$-quasi forest [42], or small treedepth [9]. Using the above mentioned lower bound machinery, it was shown that there is no kernel polynomial in the size of a vertex deletion set to chordal or perfect graphs unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$ [5, 31].

As VERTEX COVER is the primary intellectual "lab animal" in parameterized complexity, *new and worthy perspectives will always be demonstrated first with concrete results here.* We discuss several research directions in (VERTEX COVER) kernelization. The first one is based on the observation that several reduction rules are known to kernelize vertices of small degree [8, 32, 66]; a natural question is whether this process can go on 'forever', i.e., whether we can find, for any fixed constant $d \in \mathbb{N}$, a set of reduction rules that kernelize in polynomial time to a reduced graph (the kernel) of minimum degree $d$. On the negative side, we observe that unless the Exponential-Time Hypothesis [44, 45] fails, this is not the case even if the exponent in the polynomial-time kernelization is some arbitrary function of $d$. On the positive side, we give a clear account of reduction rules for VERTEX COVER that were first observed by Fellows and Stege [32] that kernelize instances to minimum degree 'almost five' (see Theorem 5 for the exact statement) and discuss how this question is closely related to finding faster fpt-algorithms for VERTEX COVER, a question that lies at the very heart of parameterized complexity research.

In the light of the ongoing machine-learning and artificial intelligence revolution, one might wonder whether AI could assist in the search for new reduction rules of parameterized problems as well. While this question seems far out, we report first experimental results of an AI-guided branching algorithm for VERTEX COVER whose logic seems amenable for application in finding new reduction rules to kernelize to increasing minimum degree.

The rest of this paper is organized as follows. In the remainder of this section, we give preliminary definitions and introduce the necessary background. In Section 2 we review some classic VERTEX COVER kernels. Section 3 is devoted to the topic of kernelizing small-degree vertices. We there give a description of reduction rules observed by Fellows and Stege [32] (see also [66]). In Section Section 4 we report results on an AI-guided branching algorithm whose ideas might lay the foundations of *automatically generated* reduction rules for VERTEX COVER. We conclude with an open problem in Section 5.

**Technical Preliminaries and Notation.** For two integers $a$ and $b$ with $a < b$, we let $[a..b] := \{a, a+1, \ldots, b\}$ and for a positive integer $a$, we let $[a] := [1..a]$.

Throughout the paper, each graph is finite, undirected and simple. Let $G$ be a graph. We denote the vertex set of $G$ by $V(G)$ and the edge set of $G$ by $E(G) \subseteq \binom{V}{2}$. For a vertex $v \in V(G)$, we denote by $N(v)$ the *(open) neighborhood* of $G$, i.e., $N(v) := \{w \mid \{v, w\} \in E(G)\}$. The *degree of $v$* is the size of the neighborhood of $v$, i.e., $\deg(v) := |N(v)|$. We define the *closed neighborhood* of $v$ as $N[v] := N(v) \cup \{v\}$. For a set of vertices $W \subseteq V(G)$, we let $N(W) := \bigcup_{w \in W} N(w)$ and $N[W] := N(W) \cup W$. For a set of vertices $\{v_1, \ldots, v_r\}$, we use the shorthand $N(v_1, \ldots, v_r) := N(\{v_1, \ldots, v_r\})$. A vertex set $C \subseteq V(G)$ is called a *clique*, if for each pair of distinct vertices $c_1, c_2 \in C$, $\{c_1, c_2\} \in E(G)$. A vertex set $I \subseteq V(G)$ is called *independent*, if for each pair of distinct vertices $v_1, v_2 \in I$, $\{v_1, v_2\} \notin E(G)$. A graph $G$ is called *bipartite*, if there is a partition $(X, Y)$ of its vertex set such that $X$ and $Y$ are independent.

For two graphs $G$ and $H$, we denote by $H \subseteq G$ that $H$ is a *subgraph* of $G$, i.e. that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a vertex set $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of $G$ *induced* by $X$, i.e., $G[X] \coloneqq (X, E(G) \cap \binom{X}{2})$. We let $G - X \coloneqq G[V(G) \setminus X]$ and we use the shorthand $G - v$ for $G - \{v\}$. For two disjoint vertex subsets $X, Y \subseteq V(G)$, we denote by $G[X, Y]$ the bipartite subgraph of $G$ induced by $(X, Y)$, that is $G[X, Y] \coloneqq (X \cup Y, \{\{x, y\} \in E(G) \mid x \in X, y \in Y\})$.

A subgraph $P \subseteq G$ is called a *path* if all its vertices have degree at most two in $P$ and there are precisely two distinct vertices in $V(P)$ that have degree one in $P$, called the *endpoints* of $P$. For $s, t \in V(G)$, a path is called $(s, t)$-*path* if it is a path with endpoints $s$ and $t$.

We call two edges $e, f \in E(G)$ *adjacent* if they share an endpoint, i.e., if there exist vertices $v, w, x \in V(G)$ such that $e = \{v, w\}$ and $f = \{v, x\}$. A *matching* is a set of pairwise non-adjacent edges. We say that a matching $M$ *saturates* a set of vertices $W \subseteq V(G)$, if for all $v \in W$, there is a pair $\{v, w\} \in M$.

Given a set of vertices $X \subseteq V(G)$, we call the operation of adding to $G$ a new vertex $x$ with neighborhood $N(X)$ and deleting all vertices in $X$ the *contraction* of $X$.

**Exponential-Time Hypothesis (ETH).**   In 2001, Impagliazzo and Paturi made a conjecture about the complexity of 3-SAT, the problem of determining whether a given Boolean formula in conjunctive normal form with clauses of size at most 3 has a satisfying assignment. This conjecture is known as the *Exponential-Time Hypothesis (ETH)* and has lead to a plethora of conditional lower bounds, see, e.g., the survey [55] or [17, Chapter 14]. Formally, ETH can be stated as:[4]

**Conjecture 1 (ETH [44, 45]).** *There is an $\varepsilon > 0$ such that* 3-SAT *on $n$ variables cannot be solved in time $\mathcal{O}^*(2^{\varepsilon n})$.*

## 2   Standard Methods

In this section, we review some classic results in VERTEX COVER kernelization. In particular, we discuss the Buss kernel [8] in Subsection 2.1. Subsection 2.2 is devoted to the kernel based on the notion of a crown decomposition [15, 27] (see Definition 1). A linear-programming-based kernel [12] is discussed in Subsection 2.3.

We would like to remark that the technical parts of the expositions given in the remainder of this section are based on [17, Sections 2.2.1, 2.3 and 2.5] and we refer to this text for several details.

### 2.1   Buss Kernelization

The first kernel for VERTEX COVER appeared several years before the notion of kernelization was formally introduced and is attributed to Buss [8]. It relies on two observations. The first one is that by definition, there is no need to include an isolated vertex in a vertex cover, as it does not have any incident edges that need to be covered.

**Reduction R.1.** *If $G$ has an isolated vertex $v$, then reduce $(G, k)$ to $(G - v, k)$.*

The second observation is that, if $G$ has a vertex $v$ of degree more than $k$, then we have no choice but to include $v$ in any size-$k$ vertex cover of $G$: If we did not include $v$, we would have

---

[4]The $\mathcal{O}^*$-notation suppresses polynomial factors in $n$.

to include all of its at least $k + 1$ neighbors, exceeding the budget of $k$ vertices we are given. Hence, $G$ has a vertex cover of size $k$ if and only if $G - v$ has a vertex cover of size $k - 1$, so we have observed that the following reduction rule is *safe*, meaning that the original instance is a YES-instance if and only if the reduced instance is a YES-instance.

**Reduction R.2.** *If $G$ has a vertex $v$ with $\deg(v) > k$, then reduce $(G, k)$ to $(G - v, k - 1)$.*

Now, after exhaustively applying Reduction R.2, $G$ has maximum degree at most $k$, so if $G$ contains more than $k^2$ edges, then we are dealing with a NO-instance: It is not possible to cover more than $k^2$ edges with $k$ vertices of degree at most $k$. On the other hand, if $(G, k)$ is a YES-instance, then $G$ has a vertex cover $X$ of size at most $k$. After exhaustively applying Reduction R.1, $G$ does not contain any isolated vertices so we can assume that every vertex of $V(G) \setminus X$ has a neighbor in $X$. Since the maximum degree of $G$ is at most $k$, we can conclude that $|V(G) \setminus X| \leq k^2$, which implies that $|V(G)| \leq k^2 + k$. Hence, if $G$ has more than $k^2 + k$ vertices, we can again conclude that we are dealing with a NO-instance. Since Reductions R.1 and R.2 clearly run in polynomial time, we have the following theorem.

**Theorem 1 (Buss and Goldsmith [8]).** VERTEX COVER *admits a kernel with at most $k^2 + k$ vertices and $k^2$ edges.*

## 2.2 Crown Reduction

The key insight above was that any vertex of degree at least $k + 1$ has to be contained in any size-$k$ vertex cover of a graph. The kernel we present in this section follows a similar motivation. The goal is to identify a set of vertices that we can always *assume* to be contained in a size-$k$ vertex cover of a graph. In other words, we want to find a set of vertices $S$, such that if $G$ contains a vertex cover of size $k$ then $G$ contains a vertex cover of size $k$ that contains $S$. The process of identifying such a set $S$ is based on a structural decomposition of the input graph, called the *crown decomposition*. Formally, a crown decomposition is defined as follows and we illustrate it in Figure 1.

**Definition 1 (Crown Decomposition).** Let $G$ be a graph. A *crown decomposition* of $G$ is a partition $(C, H, B)$ of $V(G)$, where $C$ is called the *crown*, $H$ the *head* and $B$ the *body*, such that the following hold.

   (i) $C$ is a non-empty independent set in $G$.

  (ii) There are no edges between vertices in $C$ and vertices in $B$.

 (iii) $G[C, H]$ contains a matching that saturates $H$.

The motivation for using the above definition in VERTEX COVER kernelization is as follows. Suppose we are given a crown decomposition $(C, H, B)$ of $G$ and consider the bipartite graph $G[C, H]$. Clearly, any vertex cover of $G$ has to cover the edges in $G[C \cup H]$. However, by (iii) we know that there is a matching in $G[C, H]$ saturating $H$, hence any vertex cover of $G[H, C]$ has size at least $|H|$. On the other hand, $H$ is a vertex cover of $G[C, H]$ and since $C$ is independent by (i), of $G[C \cup H]$. This allows us to conclude that $G$ has a vertex cover of size $k$ if and only if $G - (C \cup H)$ has a vertex cover of size $k - |H|$. Hence, the following reduction rule is safe.

**Reduction R.3.** *If $G$ has a crown decomposition $(C, H, B)$, then reduce $(G, k)$ to $(G - (C \cup H), k - |H|)$.*
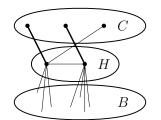
**Figure 1.** Illustration of a crown decomposition (Definition 1). The bold edges in $G[C, H]$ show a matching saturating $H$.

However, two questions remain. Namely whether we can find a crown decomposition of a graph in polynomial time and how to obtain the linear bound on the number of vertices in the resulting kernel. Both questions are answered by the following lemma whose proof is based on classic results in graph theory by König [51] and Hall [40], and polynomial-time algorithms for bipartite matching such as the classic algorithm due to Hopcroft and Karp [43].[5]

**Lemma 1 (Lemma 2.14 in [17] based on [15]).** *Let $G$ be a graph on at least $3k + 1$ vertices. There is a polynomial-time algorithm that either*

*1) finds a matching of size at least $k + 1$ in $G$; or*

*2) finds a crown decomposition of $G$.*

Now, in Case 1) we can immediately conclude that $(G, k)$ is a No-instance and in Case 2) we can apply Reduction R.3. By an exhaustive application of Lemma 1 in combination with Reduction R.3 (and Reduction R.1 to get rid of isolated vertices), we have the following theorem.

**Theorem 2 (Chor et al. [15]).** VERTEX COVER *admits a kernel with at most $3k$ vertices.*

We would like to remark that recently, a kernel on $2k$ vertices that only uses crown decomposition was obtained [57].

## 2.3 LP-Based Kernel

The VERTEX COVER problem is one of many NP-hard problems that can be expressed as an integer linear program [64], a fact which is commonly exploited in the field of approximation algorithms [68]. In this section, we show how to use linear programming to obtain a kernel for VERTEX COVER on at most $2k$ vertices. We first recall how to formulate VERTEX COVER as an integer linear program.

For each vertex $v \in V(G)$, we introduce a variable $x_v \in \{0, 1\}$ with the interpretation that $x_v = 1$ if and only if the vertex $v$ is included in the vertex cover witnessed by a solution to the (integer) linear program. We can then formulate the constraints in a natural way, directly applying the definition of vertex covers: For each edge $uv \in E(G)$, the requirement that at least one of $u$ and $v$ has to be contained in the solution translates to the constraint $x_u + x_v \geq 1$. Since we are looking for a vertex cover of minimum size, the objective function minimizes the sum

---

[5]For a more fine-grained analysis one could apply the faster algorithm [60].

over all $x_v$'s.

$$\min \sum_{v \in V(G)} x_v$$

$$\text{subject to} \quad x_u + x_v \geq 1 \quad \forall uv \in E(G) \tag{1}$$

$$x_v \in \{0,1\} \quad \forall v \in V(G) \tag{2}$$

To make the program feasible to compute, we relax the integrality constraints (2) to $x_v \in \mathbb{R}$, $x_v \geq 0$. (Note that we can drop the constraints $x_v \leq 1$ since the objective function is a minimization.) The resulting linear program is solvable in polynomial time, but may not always return a feasible solution for the original VERTEX COVER instance. However, we are chasing a different goal here, a kernelization algorithm.

Given an optimal solution $(x_v)_{v \in V(G)}$ of the (relaxed) linear program, we define the sets $V_0 := \{v \in V(G) \mid x_v < \frac{1}{2}\}$, $V_1 := \{v \in V(G) \mid x_v > \frac{1}{2}\}$, and $V_{\frac{1}{2}} := \{v \in V(G) \mid x_v = \frac{1}{2}\}$. The key ingredient is the following theorem due to Nemhauser and Trotter [62].

**Theorem 3 (Nemhauser and Trotter [62]).** *There is a minimum vertex cover $X$ of $G$ such that $V_1 \subseteq X \subseteq V_1 \cup V_{\frac{1}{2}}$.*

We derive a reduction rule from Theorem 3. First, we note that in any YES-instance of VERTEX COVER, $\sum_{v \in V(G)} x_v \leq k$. Furthermore, let $X$ be a vertex cover of $G$ of size $k$ with $V_1 \subseteq X$ and $X \cap V_0 = \emptyset$ (whose existence is guaranteed by Theorem 3), then $X \setminus V_1$ is a vertex cover of $G - (V_0 \cup V_1)$ of size $k - |V_1|$. Conversely, if $G - (V_0 \cup V_1)$ has a vertex cover $X'$ of size $k'$, we observe that by the constraints (1), for any edge $vw \in E(G)$ with $v \in V_0$, we have that $w \in V_1$. Hence, $X' \cup V_1$ is a vertex cover of $G$ of size $k' + |V_1|$. We have argued that the following reduction rule is safe.

**Reduction R.4.** *Let $(x_v)_{v \in V(G)}$, $V_0$, $V_{\frac{1}{2}}$ and $V_1$ be as above. If $\sum_{v \in V(G)} x_v > k$, then conclude that we are dealing with a NO-instance. Otherwise, reduce $(G, k)$ to $(G - V_0 \cup V_1, k - |V_1|)$.*

The number of vertices in the reduced instance after applying Reduction R.4 is

$$|V(G) \setminus (V_0 \cup V_1)| = \left| V_{\frac{1}{2}} \right| = \sum_{v \in V_{\frac{1}{2}}} 2x_v \leq 2 \cdot \sum_{v \in V(G)} x_v \leq 2k,$$

so we have obtained the following kernel for VERTEX COVER.[6]

**Theorem 4 (Chen et al. [12]).** VERTEX COVER *admits a kernel with at most $2k$ vertices.*

## 3 Towards the Barrier – What is the Maximum Minimum Vertex Degree of the Kernel that Can be Achieved in Polynomial Time?

In the previous section, we have seen that by Reduction R.2 we can kernelize all vertices whose degree is larger than the target value $k$ of the given vertex cover instance. Hence, after applying

---

[6]We would like to remark that while LINEAR PROGRAMMING can be solved in polynomial time (and hence our reduction runs in polynomial time), the corresponding algorithms are often slow in practice. However, for the case of VERTEX COVER there is good news: One can show that a solution of the above linear program can be found via a reduction to BIPARTITE MATCHING (see, e.g., [17, Section 2.5]) which has fast practical algorithms.

| Chen et al. [13] | $d = 5$ | $d = 6$ | $d = 7$ | $d = 10$ | $d = 25$ | $d = 100$ |
|---|---|---|---|---|---|---|
| $1.2738^k$ | $1.3247^k$ | $1.2852^k$ | $1.2555^k$ | $1.1975^k$ | $1.1005^k$ | $1.0346^k$ |

**Table 1.** (Dependence on $k$ of the) runtime of the resulting simple branching FPT-algorithm when using a kernelization algorithm to minimum degree $d$, for several values of $d$, versus the current fastest known FPT-algorithm for VERTEX COVER [13].

this rule exhaustively there will be no vertex of degree larger than $k$ in the kernelized instance. But what about vertices of small degree? Vertices of degree zero, i.e., isolated vertices, can be removed from a VERTEX COVER instance according to Reduction R.1. Furthermore, we will see below that there are fairly simple reduction rules that kernelize vertices of degree one and two (see Reductions R.5 and R.7). A natural question arises: Can this process go on 'forever', i.e., can we, for any fixed constant $d \in \mathbb{N}$, give a reduction rule that kernelizes all vertices of degree $d$ from a given VERTEX COVER instance?

The answer to this question is *probably not* — even if the degree of the polynomial in the runtime of the kernelization algorithm can depend on $d$: It is well-known (see, e.g., [17, 21, 29]) that unless ETH fails, there is some barrier constant $\zeta_{VC} > 0$ such that the fastest possible algorithm for VERTEX COVER runs in time $(1 + \zeta_{VC})^k \cdot n^{\mathcal{O}(1)}$. If we could kernelize VERTEX COVER in polynomial time to arbitrarily large minimum degree, one could devise a straightforward branching algorithm that runs in time $(1 + \zeta_{VC} - \varepsilon)^k \cdot n^{\mathcal{O}(1)}$, for some $0 < \varepsilon < \zeta_{VC}$, where $\varepsilon$ can be arbitrarily close to the value of $\zeta_{VC}$. We coin the corresponding integer $\delta_{VC} \in \mathbb{N}$ the *barrier degree* of VERTEX COVER kernelization and now prove formally its existence (assuming ETH).

**Proposition 1.** *Unless ETH fails, there is some constant $\delta_{VC} \in \mathbb{N}$, such that VERTEX COVER cannot be kernelized to instances of minimum degree $\delta_{VC}$.*

*Proof.* Using standard arguments about branching algorithms (see, e.g., [17, Chapter 3]) one can show that there is an algorithm solving vertex cover in time $\lambda^k \cdot n^{\mathcal{O}(1)}$, where $\lambda$ satisfies
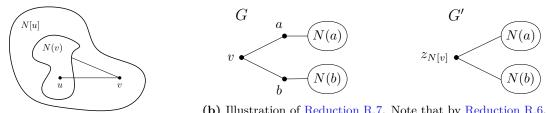
$$\lambda \leq \lambda^d(\lambda - 1), \tag{3}$$

if the input graph always has a vertex of degree at least $d$ to branch on. Now suppose that the statement of the proposition is false, then we can guarantee the existence of such a vertex for constant but arbitrarily large $d$ (with only polynomial time overhead at each stage of the branching). Now let $\varepsilon > 0$ with $\varepsilon < \zeta_{VC}$. (Note that this implies that $\varepsilon < 1$ as $\zeta_{VC} < 0.2738$ [13].) We substitute $\lambda$ with $(1 + \varepsilon)$ in (3) and obtain:

$$1 + \varepsilon \leq (1 + \varepsilon)^d \varepsilon \iff (1 + \varepsilon)^d \geq \frac{1 + \varepsilon}{\varepsilon} \iff d \log(1 + \varepsilon) \geq \log\left(\frac{1 + \varepsilon}{\varepsilon}\right)$$

$$\iff d \geq \frac{\log\left(\frac{1+\varepsilon}{\varepsilon}\right)}{\log(1 + \varepsilon)} = \frac{\log(1 + \varepsilon) - \log(\varepsilon)}{\log(1 + \varepsilon)} = 1 - \frac{\log(\varepsilon)}{\log(1 + \varepsilon)}.$$

This shows that for any such $\varepsilon$, there is a constant $d_\varepsilon \in \mathbb{N}$ such that, if we could kernelize VERTEX COVER to minimum degree $d_\varepsilon$, then we could solve it in $(1 + \varepsilon)^k \cdot n^{\mathcal{O}(1)}$ time, where $\varepsilon < \zeta_{VC}$ by our choice. This contradicts ETH by, e.g., [29, Theorem 1]. $\square$

The proof of Proposition 1 also provides some very natural motivation for the question of kernelizing VERTEX COVER to larger and larger minimum degree; such kernels immediately provide new FPT-algorithms for the problem. In particular, kernelizing to minimum degree seven

**(a)** The situation of Reduction R.6.

**(b)** Illustration of Reduction R.7. Note that by Reduction R.6, we can assume that $a$ and $b$ are not adjacent.

**Figure 2.** Illustrations of Reductions R.6 and R.7, respectively.

would already improve upon the current best known algorithm for VERTEX COVER, yielding first progress in a very attractive research question in over a decade! We illustrate the runtime of such algorithms for several concrete values of $d$ in Table 1.

In the remainder of this section, we present a set of reduction rules that were first observed by Fellows and Stege [32] to kernelize a vertex cover instance to minimum degree 'almost five', in the following sense: We show that a vertex can be kernelized if its degree is at most three or its degree is four and there are more than two edges between the vertices in its neighborhood.

Before we give the reduction rules to kernelize vertices of degree one and two, we would like to remark that later in the text, we introduce two *auxiliary* reduction rules, mostly to deal with structures arising in the kernelization of vertices of degree three and four which as a byproduct also kernelize degree one and two vertices. For explanatory purposes, however, we describe the reduction rules for vertices of degree one and two separately first.

**Reduction R.5.** *If $G$ has a pendant edge $\{u, v\}$ with $\deg(u) = 1$, then reduce $(G, k)$ to $(G - \{u, v\}, k - 1)$.*

**Proposition 2.** *Reduction R.5 is safe, i.e., if $G$ has a pendant edge $\{u, v\}$ with $\deg(u) = 1$, then $G$ has a vertex cover of size $k$ if and only if $G - \{u, v\}$ has a vertex cover of size $k - 1$.*

*Proof.* ($\Rightarrow$) Suppose $G$ has a vertex cover $X^*$ of size $k$. Since $\{u, v\}$ is an edge of $G$, at least one of $u$ and $v$ is contained in $X^*$. If $v \notin X^*$, then we let $X := X^* \setminus \{u\} \cup \{v\}$. Note that $X$ is a vertex cover since $v$ is the only neighbor of $u$. If $v \in X^*$, we simply let $X := X^*$. Since $v \in X$, $X \setminus \{v\}$ is a vertex cover of $G - \{u, v\}$ of size $k - 1$.

($\Leftarrow$) Let $X'$ be a vertex cover of $G - \{u, v\}$ of size $k - 1$. We observe that any edge in $E(G) \setminus E(G - \{u, v\})$ is incident with $v$ and conclude that $X' \cup \{v\}$ is a vertex cover of $G$ of size $k$. $\square$

Before we show how to kernelize degree two vertices, we give the first auxiliary reduction rule.

**Reduction R.6.** *If $G$ has two adjacent vertices $u$ and $v$ such that $N(v) \subseteq N[u]$, then reduce $(G, k)$ to $(G - u, k - 1)$.*

For an illustration of the situation of Reduction R.6, see Figure 2a.

**Proposition 3.** *Reduction R.6 is safe, i.e., if $G$ has two adjacent vertices $u$ and $v$, and $N(v) \subseteq N[u]$, then $G$ contains a vertex cover of size $k$ if and only if $G - u$ has a vertex cover of size $k - 1$.*

*Proof.* ($\Rightarrow$) Suppose $G$ has a vertex cover $X^*$ of size $k$. If $u \notin X^*$ then $N(u)$ must be in $X^*$, so by assumption, it contains $N[v] \setminus \{u\}$. But then, $X := X^* \setminus \{v\} \cup \{u\}$ is also a vertex cover of $G$ of size $k$, so we can assume that $u \in X$. Then, $X \setminus \{u\}$ is a vertex cover of $G - u$ of size $k - 1$.

($\Leftarrow$) is immediate since for any vertex cover $X'$ of $G - u$, $X' \cup \{u\}$ is a vertex cover of $G$. $\square$

The next reduction rule takes care of vertices of degree two and is illustrated in Figure 2b.

**Reduction R.7.** *If Reduction R.6 cannot be applied and $G$ has a vertex $v$ with $\deg(v) = 2$, then reduce $(G, k)$ to $(G', k-1)$, where $G'$ is the graph obtained from $G$ by contracting $N[v]$ to a single vertex.*

**Proposition 4.** *Reduction R.7 is safe, i.e., under its stated conditions, $G$ has a vertex cover of size $k$ if and only if $G'$ has a vertex cover of size $k - 1$.*

*Proof.* Throughout the proof, we denote the neighborhood of $v$ in $G$ by $N(v) = \{a, b\}$ and the vertex in $G'$ that was created due to the contraction of $N[v]$ by $z_{N[v]}$. We can assume that $\{a, b\} \notin E(G)$: If the edge $\{a, b\}$ was present, then $N(v) \subseteq N[a]$ (and $N(v) \subseteq N[b]$), so we could have applied Reduction R.6.

($\Rightarrow$) We observe that each edge in $E(G') \setminus E(G)$ has an endpoint in $\{z_{N[v]}\} \cup N(a, b)$. Let $X$ be a vertex cover of $G$ of size $k$. If $X \cap N(v) = \emptyset$, then $\{v\} \cup N(a, b) \subseteq X$ and we can conclude that $X \setminus \{v\}$ is a vertex cover of $G'$. If $N(v) \subseteq X$, then $X' := X \setminus N(v) \cup \{z_{N[v]}\}$ is a vertex cover of $G'$. (Note that in this case, $X'$ has size at most $k - 1$ as well.) If $X$ contains precisely one vertex from $N(v)$, assume w.l.o.g. that $X \cap N(v) = \{a\}$, then $v \in X$ (otherwise the edge $\{v, b\}$ is not covered), so $X \setminus \{v, a\} \cup \{z_{N[v]}\}$ is a vertex cover of $G'$ of size $k - 1$.

($\Leftarrow$) Let $X'$ be a vertex cover of $G'$ of size $k - 1$. We distinguish the cases when $z_{N[v]} \in X'$ and when $z_{N[v]} \notin X'$. In the former case, $X' \setminus \{z_{N[v]}\} \cup \{a, b\}$ is a vertex cover of $G$, since each edge in $E(G) \setminus E(G')$ is incident with a vertex in $\{a, b\}$. In the latter case, $N(a, b) \subseteq X'$ since $z_{N[v]} \notin X'$, and we have that $X' \cup \{v\}$ is a vertex cover of $G$: Since $\{a, b\} \notin E(G)$, each edge in $E(G) \setminus E(G')$ is incident with a vertex in $\{v\} \cup N(a, b)$. In both cases, the size of the resulting vertex cover is $k$. $\square$

Before we proceed with kernelizing vertices of degree larger than two, we require one more auxiliary reduction rule. This reduction rule will be crucially used to argue that we can exclude certain structures appearing in the subgraphs induced by the neighborhoods of small-degree vertices. It captures [32, Reductions R.4 and R.5] and is illustrated in Figure 3. Note that due to its complexity, it will only be executed for vertices whose degree is bounded by a fixed constant $\alpha$ (independent of $k$). In particular, for our purposes it will be sufficient to make use of the following reduction for $\alpha \leq 4$.

**Reduction R.8.** *Suppose $G$ has a vertex $v$ such that the following hold. There is a partition $(C_1, C_2)$ of $N_G(v)$ where $|C_1| \geq |C_2|$ and the following hold.*

*(i) $C_i$ is a clique for all $i \in [2]$.*

*(ii) Let $M$ be the set of non-edges of $G[C_1, C_2]$. For each $c_1 \in C_1$, there is precisely one $f \in M$ such that $c_1 \in f$.*

*Then, reduce $(G, k)$ to $(G', k - |C_2|)$, where $G'$ is obtained from $G$ by*
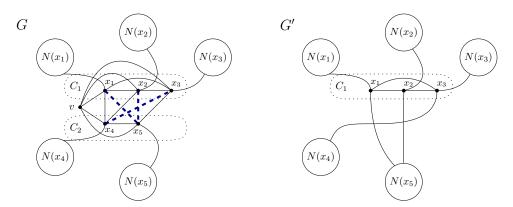
*(1.) deleting $v$ and $C_2$, and*

**Figure 3.** Illustration of Reduction R.8. Note that $|C_1| > |C_2|$, and the bold dotted lines between vertices of $C_1$ and $C_2$ are the set of non-edges $M$ in $G[C_1, C_2]$ satisfying condition (ii): For every vertex $x \in C_1$ there is precisely one element in $M$ containing $x$.

*(2.) for all $\{c_1, c_2\} \in M$ with $c_1 \in C_1$ and $c_2 \in C_2$, adding all edges between $c_1$ and $N_G(c_2)$.*

**Proposition 5.** *Reduction R.8 is safe, i.e. under its stated conditions, $G$ has a vertex cover of size $k$ if and only if $G'$ has a vertex cover of size $k - |C_2|$.*

*Proof.* Since by assumption (i) of Reduction R.8, $C_1$ and $C_2$ are cliques in $G$, and since $C_1$ remains a clique in $G'$, we make the following observation.

**Observation 1.** *Every vertex cover of $G$ contains at least $|C_i| - 1$ vertices from $C_i$ for all $i \in [2]$, and every vertex cover of $G'$ contains at least $|C_1| - 1$ vertices from $C_1$.*

We now prove the proposition by a case analysis on the structure of the intersection of vertex covers of $G$ and $G'$ with $N_G(v) = C_1 \cup C_2$ and $C_1$, respectively. Observation 1 will be used later to argue that we covered all possible cases.

**Claim 1.** *$G$ contains a vertex cover $X$ of size $k$ such that $N_G(v) \subseteq X$ if and only if $G'$ contains a vertex cover $X'$ of size $k - |C_2|$ such that $C_1 \subseteq X'$.*

*Proof.* ($\Rightarrow$) Let $X$ be a vertex cover of $G$ of size $k$ such that $N_G(v) \subseteq X$. (Note that we can assume that $v \notin X$.) We have that $X' := X \setminus C_2$ is a vertex cover of $G^* := G - (\{v\} \cup C_2)$. By construction, any edge in $E(G') \setminus E(G^*)$ is incident with a vertex in $C_1 \subseteq X'$, so $X'$ is a vertex cover of $G'$. Clearly, $|X'| = k - |C_2|$.

($\Leftarrow$) Let $X'$ be a vertex cover of $G'$ of size $k - |C_2|$ such that $C_1 \subseteq X'$. Then, $X := X' \cup C_2$ is a vertex cover of $G$, since every edge in $E(G) \setminus E(G')$ is either incident with a vertex in $C_2$ or with $v$. For the latter case, we observe that $N_G(v) = (C_1 \cup C_2) \subseteq X$. Clearly, $|X| = k$.  ⌟

We observe that Claim 1 also covers the case when a size-$k$ vertex cover of $G$ misses precisely one vertex from $N_G(v)$: Let $X^*$ be such a vertex cover and let $c \in N_G(v) \setminus X$. Since $X^*$ has to contain an endpoint of the edge $\{v, c\}$ and $c \notin X^*$, we can conclude that $v \in X^*$. Now, we simply let $X := X^* \setminus \{v\} \cup \{c\}$ and observe that $X$ is a vertex cover of $G$ of size $k$ such that $N_G(v) \subseteq X$.

**Claim 2.** *$G$ contains a vertex cover $X$ of size $k$ with $|C_i \setminus X| = 1$ for all $i \in [2]$ if and only if $G'$ contains a vertex cover $X'$ of size $k - |C_2|$ with $|C_1 \setminus X'| = 1$.*

*Proof.* ($\Rightarrow$) Let $X$ be a size-$k$ vertex cover of $G$ such that for all $i \in [2]$, $|C_i \setminus X| = 1$ and let $c_i \in C_i \setminus X$ be the unique vertex in $C_i$ that is not contained in $X$. First, since $c_i \notin X$, we have that $v \in X$, otherwise the edge $\{v, c_i\}$ is not covered by $X$. Furthermore, we can conclude that $\{c_1, c_2\} \notin E(G)$, since if $\{c_1, c_2\}$ was an edge of $G$, then this edge was not covered by $X$. Clearly, since $\{c_1, c_2\} \notin E(G)$, we have that $\{c_1, c_2\} \notin E(G[C_1, C_2])$, so $\{c_1, c_2\} \in M$.

We have argued that $\{c_1, c_2\} \in M$, and by condition (ii) of Reduction R.8 we know that $\{c_1, c_2\}$ is the only element in $M$ that contains $c_1$. We now show that

$$X' := X \cap V(G') = X \setminus (\{v\} \cup C_2)$$

is a vertex cover of $G'$. Clearly, $X'$ is a vertex cover of $G^* := G - (\{v\} \cup C_2)$. Now, consider an edge $e' \in E(G') \setminus E(G^*)$. By construction, one of the endpoints of $e'$, say $x$, is from $C_1$. If $x \neq c_1$, then the edge $e'$ is covered by $X'$, since $C_1 \setminus \{c_1\} \subseteq X'$. Now suppose that $x = c_1$ and denote the other endpoint of $e'$ by $y$. Since $e' \in E(G') \setminus E(G^*)$, following the construction of Reduction R.8, we can conclude that there is some $\{c_1, z\} \in M$ such that $y \in N_G(z)$. We can infer that $z = c_2$, since (ii) asserts that there is only one element in $M$ that contains $c_1$ and we know by the above argument that $\{c_1, c_2\} \in M$. As $X$ is a vertex cover of $G$ and $c_2 \notin X$ by assumption, we know that $y \in X$, and so:

$$y \in N_G(c_2) \cap V(G') \subseteq X \cap V(G') = X',$$

hence the edge $\{c_1, y\}$ is covered by $X'$. We can conclude that $X'$ is a vertex cover of $G'$. Since we obtained $X'$ from $X$ by removing from it the vertex $v$ and $|C_2| - 1$ vertices from $C_2$, we have that $|X'| = k - |C_2|$. Clearly, $|C_1 \setminus X'| = 1$.

($\Leftarrow$) Let $X'$ be a vertex cover of $G'$ of size $k - |C_2|$ such that $|C_1 \setminus X'| = 1$ and denote by $c_1 \in C_1 \setminus X'$ the unique vertex of $C_1$ that is not contained in $X'$. Let furthermore $c_2 \in C_2$ be such that $\{c_1, c_2\} \in M$. By condition (ii), such a vertex $c_2$ exists and it is unique. We argue that $X := X' \cup \{v\} \cup (C_2 \setminus \{c_2\})$ is a vertex cover of $G$. Suppose for a contradiction that there is an edge $e \in E(G)$ that is not covered by $X$. Since $X'$ is a vertex cover of $G'$ and $X \supseteq X'$, we have that $e \in E(G) \setminus E(G')$. By construction, each such edge $e$ has (at least) one endpoint in $\{v\} \cup C_2$. Since $\{v\} \cup (C_2 \setminus \{c_2\}) \subseteq (N_G[v] \setminus \{c_1, c_2\}) \subseteq X$, we can conclude that $c_2$ is an endpoint of $e$, and the endpoint of $e$ other than $c_2$, say $y$, is contained in $N_G(c_2) \setminus (N_G[v] \setminus \{c_1\})$. Since $\{c_1, c_2\} \in M$, it is a non-edge, and so $y \in N_G(c_2) \setminus N_G[v]$. From the construction of Reduction R.8, we infer that $y \in N_G(c_2) \cap V(G') \subseteq N_{G'}(c_1) \cap V(G')$.

To conclude, since $X \cap V(G') = X' \cap V(G')$, we have that if $y \notin X$, then $y \notin X'$. Since $y \in N_{G'}(c_1)$ and $c_1 \notin X'$ by assumption, the edge $\{c_1, y\}$ is not covered by $X'$, a contradiction with the assumption that $X'$ was a vertex cover of $G'$. It is clear that $|X| = k$ and that for all $i \in [2]$, $|C_i \setminus X| = 1$. ⌐

We are now ready to finalize the proof of safeness of Reduction R.8. Suppose $G$ has a vertex cover $X$ of size $k$. Then, by Observation 1, we are in one of the following cases: (I) $N_G(v) \subseteq X$, (II) $|N_G(v) \setminus X| = 1$, or (III) for all $i \in [2]$, $|C_i \setminus X| = 1$. In cases (I) and (II), we can conclude that $G'$ has a vertex cover of size $k - |C_2|$ by Claim 1 (and the remark thereafter). In case (III), $G'$ has a vertex cover of size $k - |C_2|$ by Claim 2.

For the other direction, suppose $G'$ has a vertex cover $X'$ of size $k - |C_2|$. Again by Observation 1, we are in one of the following two cases: (IV) $C_1 \subseteq X'$, or (V) $|C_1 \setminus X'| = 1$. In case (IV), we can use Claim 1 to conclude that $G$ has a vertex cover of size $k$ and in case (V) we can use Claim 2. This finishes the proof of Proposition 5. □
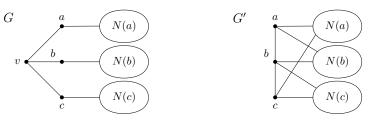
**Figure 4.** Illustration of Reduction R.9.

Before we turn to kernelizing degree-three vertices, we observe that a combination of Reductions R.1, R.6 and R.8 kernelizes vertices of degree one and two as well. Suppose $v$ is a vertex of degree one in $G$ whose only neighbor is $u$. Then, $N(v) = \{u\} \subseteq N[u]$, so following Reduction R.6, we could have removed the vertex $u$ and decreased the parameter value by one. In $G - u$, the vertex $v$ is an isolated vertex, so by Reduction R.1 it can be removed. These two steps together have the same effect as an application of Reduction R.5, the rule for kernelizing vertices of degree one.

Next, suppose that $v$ is a vertex of degree two and let $\{a, b\} := N(v)$. There are two cases we have to consider. If $\{a, b\} \in E(G)$, then $N(v) \subseteq N[a]$ and we could have applied Reduction R.6. In the resulting instance whose graph is $G - a$, the vertex $v$ is of degree one so it would be removed by a combination of Reductions R.1 and R.6, following the same argument as above. If $\{a, b\} \notin E(G)$, then $v$ trivially satisfies the conditions of Reduction R.8 by considering the partition of $\{a, b\}$ into parts $\{a\}$ and $\{b\}$. Applying Reduction R.8, we can remove the vertex $v$.

**Observation 2.** *After an exhaustive application of Reductions R.1, R.6 and R.8 (where for Reduction R.8, $\alpha = 2$), the resulting graph has minimum degree three.*

We are now ready to kernelize degree-three vertices.

**Reduction R.9.** *If neither Reduction R.6 nor Reduction R.8 can be applied and $G$ contains a vertex $v$ of degree three (where $N(v) = \{a, b, c\}$), then reduce $(G, k)$ to $(G', k)$ where $G'$ is the graph on vertex set $V(G) \setminus \{v\}$ and edge set $(E(G) \cap \binom{V(G')}{2}) \cup F$, where*

$$F := \{\{a, b\}, \{b, c\}\} \cup \{\{a, x\} \mid x \in N_G(b)\} \cup \{\{b, y\} \mid y \in N_G(c)\} \tag{4}$$
$$\cup \{\{c, z\} \mid z \in N_G(a)\}.$$

We illustrate the above reduction rule in Figure 4.

**Proposition 6.** *Reduction R.9 is safe, i.e., if its conditions are satisfied, then $G$ contains a vertex cover of size $k$ if and only if $G'$ contains a vertex cover of size $k$.*

*Proof.* We first show that we can assume that there are no edges between the vertices in $N(v)$.

**Claim.** *If neither Reduction R.6 nor Reduction R.8 can be applied, then $N(v) = \{a, b, c\}$ is an independent set in $G$.*

*Proof.* If $G[N(v)]$ contains at least two edges, then these two edges have a common endpoint, say $x \in N(v)$. But then, $N(v) \subseteq N[x]$, so we could have applied Reduction R.6, a contradiction. If $G[N(v)]$ contains precisely one edge, assume w.l.o.g. that $\{a, b\} \in E(G)$, then we could have applied Reduction R.8 with $C_1 = \{a, b\}$, and $C_2 = \{c\}$. Clearly, $\{a, b\}$ and $\{c\}$ are cliques and $M := \{\{a, c\}, \{b, c\}\}$, the set of non-edges of $G[C_1, C_2]$, satisfies the conditions of Reduction R.8(ii). ⌋

Due to the previous claim, we will assume that $N(v) = \{a, b, c\}$ is an independent set throughout the following.

**Claim 3.** *If $G$ has a vertex cover of size $k$, then $G'$ has a vertex cover of size at most $k$.*

*Proof.* We first observe that, for each edge $e' \in E(G')$, either $e' \in E(G - v)$ or $e' \in F$. Hence, any vertex cover $X^*$ of $G - v$ is a vertex cover of $G'$ if each edge in $F$ has an endpoint in $X^*$, since by definition, $X^*$ contains an endpoint of each edge in $E(G - v)$.

Let $X$ be a vertex cover of $G$ of size $k$. If $v \notin X$, then $N(v) = \{a, b, c\} \subseteq X$. By (4), each edge in $F$ has at least one endpoint in $\{a, b, c\}$ and hence in $X$, so we can conclude that $X$ is a vertex cover of $G'$ of size $k$.

Suppose $v \in X$ and note for the remainder of the proof that $X \setminus \{v\}$ is a vertex cover of $G - v$ of size $k - 1$. We argue that we can assume that at most one vertex from $N(v)$ is contained in $X$: For the case that $N(v) \subseteq X$, we can apply the same argument as above to conclude that $X \setminus \{v\}$ is a vertex cover of $G'$ of size $k - 1$. If $X$ contains precisely two vertices from $N(v) = \{a, b, c\}$, assume w.l.o.g. that $\{a, b\} \subseteq X$, then $X' := X \setminus \{v\} \cup \{c\}$ is a vertex cover of $G'$ of size $k$, since again, $X'$ contains $N(v)$.

We assume that $X$ contains at most one vertex from $N(v)$. If $X$ contains no vertex of $N(v)$, then $X$ must contain all of $N_G(a, b, c)$. Hence the only edges in $F$ that are not covered by $X$ – see (4) – are incident with the vertex $b$. Together with the fact that $X \setminus \{v\}$ is a vertex cover of $G - v$, we can conclude that $X \setminus \{v\} \cup \{b\}$ is a vertex cover of $G'$.

From now on, we assume that precisely one vertex of $N(v)$ is contained in the vertex cover $X$ of $G$. If $a \in X$, then $b, c \notin X$ and hence $N_G(b, c) \subseteq X$. Again, $X \setminus \{v\}$ is a vertex cover of $G - v$ and we observe that any edge in $e' \in F$ that does not have an endpoint in $X \setminus \{v\}$ is incident with the vertex $c$. By (4), either $e' = \{b, c\}$ or $e' = \{c, z\}$ for some $z \in N(a)$. We can conclude that $X \setminus \{v\} \cup \{c\}$ is a vertex cover of $G'$. The remaining cases can be argued for similarly: If $b \in X$, then $X \setminus \{v\} \cup \{a\}$ is a vertex cover of $G'$ and if $c \in X$, then $X \setminus \{v\} \cup \{b\}$ is a vertex cover of $G'$.                                                                  ⌟

**Claim 4.** *If $G'$ has a vertex cover of size $k$ then $G$ has a vertex cover of size $k$.*

*Proof.* Throughout the following, let $X'$ be a vertex cover of $G'$ of size $k$. Since $\{a, b, c\}$ is *not* an independent set in $G'$, we know that $X'$ has to contain at least one vertex of $\{a, b, c\}$. If $\{a, b, c\} = N(v) \subseteq X'$, then $X'$ contains an endpoint of each edge in $E(G) \setminus E(G') = \{\{v, x\} \mid x \in \{a, b, c\}\}$, so we can conclude that $X'$ is a vertex cover of $G$.

We now consider the cases when $X'$ contains precisely two vertices from $\{a, b, c\}$. If $\{a, b\} \subseteq X'$ and hence $c \notin X'$, then $X'$ contains $N_G(a)$ as well, to cover the edges between the vertex $c$ and vertices in $N(a)$. It follows that $X' \setminus \{a\}$ is a vertex cover of $G - v$. Since each edge in $E(G) \setminus E(G - v)$ is incident with $v$, we can conclude that $X' \setminus \{a\} \cup \{v\}$ is a vertex cover of $G$. By similar arguments we have that if $X' \cap \{a, b, c\} = \{b, c\}$, then $X' \setminus \{b\} \cup \{v\}$ is a vertex cover of $G$ and if $X' \cap \{a, b, c\} = \{a, c\}$, then $X' \setminus \{c\} \cup \{v\}$ is a vertex cover of $G$.

It remains to argue the case when $X'$ contains precisely one vertex from $\{a, b, c\}$. Note that the only possible such case is when this vertex is $b$. If $X'$ contained only the vertex $a$ (resp., $c$), then the edge $\{b, c\}$ (resp., $\{a, b\}$) would remain uncovered by $X'$. Suppose $X' \cap \{a, b, c\} = \{b\}$, so $a, c \notin X'$, implying that $N(a, b, c) \subseteq X'$. Hence, $X' \setminus \{b\}$ is a vertex cover of $G - v$ and $X' \setminus \{b\} \cup \{v\}$ is a vertex cover of $G$.                                                                  ⌟

In the light of Claims 3 and 4, the proposition is proved.                                       □
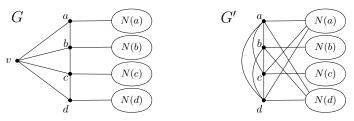
**Figure 5.** Illustration of Reduction R.10.

The next reduction rule kernelizes all vertices that have degree four and whose neighborhood induces a subgraph with more than two edges.

**Reduction R.10.** *If neither Reduction R.6 nor Reduction R.8 can be applied and $G$ contains a vertex $v$ with degree four such that $G[N(v)]$ has at least three edges, then we can assume that (up to renaming the vertices in $N(v) = \{a, b, c, d\}$) $G[N(v)]$ is an $(a, d)$-path. We reduce $(G, k)$ to $(G', k)$, where $G'$ is the graph on vertex set $V(G) \setminus \{v\}$ and edge set $(E(G) \cap \binom{V(G')}{2}) \cup F$, where*

$$F := \binom{N(v)}{2} \cup \{\{x, y\} \mid x \in \{a, b\}, y \in N(d)\} \cup \{\{x, y\} \mid x \in \{c, d\}, y \in N(a)\}. \tag{5}$$

We illustrate Reduction R.10 in Figure 5.

**Proposition 7.** *Reduction R.10 is safe, i.e., if its conditions are satisfied, then $G[N(v)]$ is a path and $G$ has a vertex cover of size $k$ if and only if $G'$ has a vertex cover of size $k$.*

*Proof.* We first justify the assumption that $G[N(v)]$ induces a path.

**Claim 5.** *If neither Reduction R.6 nor Reduction R.8 can be applied and $G$ and $v$ are as above, then $G[N(v)]$ induces a path.*

*Proof.* Suppose not. If $G[N(v)]$ contains at least five edges, then there has to be a vertex $x \in \{a, b, c, d\}$ which is incident with three of these edges. Hence, $N(v) \subseteq N[x]$ and we could have applied Reduction R.6, a contradiction.

Suppose there are four edges in $G[N(v)]$. There are only two non-isomorphic graphs on four vertices and four edges, see the right hand side of Figure 6. If $G[N(v)]$ induces a $C_4$, assume w.l.o.g. that its vertices appear in the order $a - b - c - d - a$, then we can partition $N_G(v)$ into cliques $C_1 = \{a, b\}$ and $C_2 = \{c, d\}$, and we observe that the set $M := \{\{a, c\}, \{b, d\}\}$ of non-edges of $G[C_1, C_2]$ satisfies the condition of Reduction R.8(ii): For $x \in \{a, b\}$ there is precisely one element in $M$ that contains $x$. We could have applied Reduction R.8, a contradiction. If $G[N(v)]$ induces an $H_4$, assume w.l.o.g. that the 3-cycle is $a - b - c - a$ and $d$ is the pendant vertex adjacent to $c$, then we have that $N(v) \subseteq N[c]$ and we could have applied Reduction R.6.

We can assume that $G[N(v)]$ contains precisely three edges. There are three pairwise non-isomorphic graphs on four vertices and three edges, shown in the left-hand side of Figure 6. If $G[N(v)]$ induces a star ($S_4$) with center $a$, then we have that $N(v) \subseteq N[a]$, so we could have applied Reduction R.6, a contradiction. If $G[N(v)]$ induces a $K_3 \cup K_1$, then let $C_1 := \{a, b, c\}$ be the vertices that induce a $K_3$ and $C_2 = \{d\}$, where $d$ is the remaining vertex of $N(v)$. Clearly, $C_1$ and $C_2$ induce cliques and the set of non-edges of $G[C_1, C_2]$ is such that it contains precisely one element incident with each vertex in $C_1$. Hence, we could have applied Reduction R.8, a contradiction. We can conclude that the only case that has not been covered is when $G[N(v)]$ induces a $P_4$, which proves the claim. ⌟
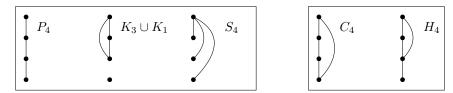
**Figure 6.** In the left box, all pairwise non-isomorphic graphs on four vertices and three edges and in the right box, all pairwise non-isomorphic graphs on four vertices and four edges are shown.

**Claim 6.** *G contains a vertex cover of size $k$ if and only if $G'$ contains a vertex cover of size $k$.*

*Proof.* ($\Rightarrow$) Suppose $G$ has a vertex cover $X$ of size $k$. If $N(v) \subseteq X$, then every edge in $E(G') \setminus E(G)$ has at least one endpoint in $X$ by construction, so $X$ is a vertex cover of $G'$ as well. If $X$ contains precisely three vertices of $N(v)$, suppose w.l.o.g. that $X \cap N(v) = \{a, b, c\}$, then $v$ has to be contained in $X$ as well, otherwise the edge $\{v, d\}$ remains uncovered. Hence, $X \setminus \{v\} \cup \{d\} \supseteq N(v)$ is a vertex cover of $G'$ of size $k$.

From now on suppose that $X$ contains precisely two vertices from $N(v)$ and note that in all of the following cases, $v \in X$. The only vertex covers of $G[N(v)]$ of size two are $\{b, c\}$, $\{a, c\}$ and $\{b, d\}$. First observe that any triple of vertices from $N_G(v)$ covers the edges $\binom{N(v)}{2}$, so we do not have to consider them explicitly in the following discussion.

Suppose $X \cap N(v) = \{b, c\}$. Then, $N(a, d) \subseteq X$, so all edges in $E(G') \setminus E(G) \setminus \binom{N(v)}{2}$ are covered by $X$. We can conclude that $X \setminus \{v\} \cup \{a\}$ (also $X \setminus \{v\} \cup \{d\}$) is a vertex cover of $G'$. If $X \cap N(v) = \{a, c\}$, then $N(b, d) \subseteq X$ and all edges in $E(G') \setminus E(G) \setminus \binom{N(v)}{2}$ that are not covered by $X$ are between $d$ and $N(a)$, so $X \setminus \{v\} \cup \{d\}$ is a vertex cover of $G'$. Similarly, if $X \cap N(v) = \{b, d\}$, then $X \setminus \{v\} \cup \{a\}$ is a vertex cover of $G'$. Since $G[N(v)]$ does not have a vertex cover of size at most 1, this concludes the proof of the first direction.

($\Leftarrow$) Since $N_G(v)$ is a clique in $G'$, any vertex cover of $G'$ contains at least three vertices from $N_G(v)$. Let $X'$ be a vertex cover of $G'$ of size $k$. If $N_G(v) \subseteq X'$, then $X'$ is also a vertex cover of $G$ since each edge in $E(G) \setminus E(G')$ has an endpoint in $N_G(v)$. In the remainder, we can assume that $X'$ contains precisely three vertices from $N_G(v)$. Suppose $X' \cap N_G(v) = \{a, b, c\}$. Since $d \notin X'$, we have that $N(a, d) \subseteq X'$, hence all edges between $a$ and $N(a)$ are covered by $N(a) \subseteq X'$. Together with the observation that $\{b, c\}$ is a vertex cover of $G[N(v)]$, we can conclude that $X' \setminus \{a\} \cup \{v\}$ is a vertex cover of $G$. If $X' \cap N_G(v) = \{a, b, d\}$, then $N(a, c) \subseteq X'$ since $c \notin X'$. By the same reasoning as before, we can observe that $X' \setminus \{a\} \cup \{v\}$ is a vertex cover of $G$. Similarly, if $X' \cap N_G(v) = \{b, c, d\}$ or if $X' \cap N_G(v) = \{a, c, d\}$ then we can argue that $X' \setminus \{d\} \cup \{v\}$ is a vertex cover of $G$.                                             ⌐

Now, by Claim 5, we know that under the conditions stated in Reduction R.10, $G[N(v)]$ induces a path which together with Claim 6 proves the proposition.                                             □

It is easy to see that Reductions R.5 to R.7, R.9 and R.10 can be executed in polynomial time. We observe (naively) that Reduction R.8 can be executed in time $\mathcal{O}\left(n \cdot 2^\alpha \cdot \alpha^{\mathcal{O}(1)}\right)$, where $\alpha$ denotes the degree of the vertex $v$. Since for our purposes, $\alpha \leq 4$ is sufficient, Reduction R.8 runs in polynomial time as well and we have the following theorem. (Note that none of the presented reductions increases the parameter value.)

**Theorem 5 (cf. Fellows and Stege [32]).** *There is a polynomial-time algorithm that given an instance $(G, k)$ of VERTEX COVER outputs an equivalent instance $(G', k')$, where $k' \leq k$,*
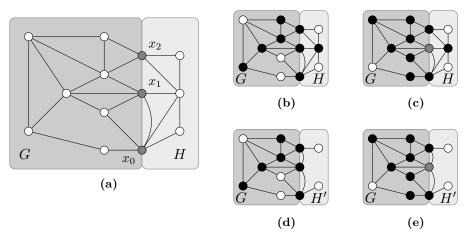
**Figure 7.** Illustration of notation. **(a)** shows a graph $G \oplus H$ resulting from gluing $G$ (darker background) and $H$ (lighter background). **(b)** shows a vertex cover of $G \oplus H$ compatible with $\{2\}$, minimizing the intersection with $V(H)$ and, thus, implying $P_H^G(\{2\}) = 4$. **(c)** shows a vertex cover $S$ of $G \oplus H$ compatible with $\{0, 1, 2\}$, minimizing the intersection with $V(H)$ and, thus, implying $P_H^G(\{0, 1, 2\}) = 3$. Note that $S$ is not necessarily minimum or even minimal for $G \oplus H$. **(d)** and **(e)** show the same as (b) and (c) but with the graph $H'$ resulting from the application of Reduction R.6 to a vertex in $H$, proving $P_{H'}^G(X) = P_H^G(X) - 1$ for $X \in \{\{2\}, \{0, 1, 2\}\}$.

*(I) the minimum degree of $G'$ is at least four and*

*(II) for all vertices $v \in V(G')$ with $\deg_{G'}(v) = 4$, $G'[N(v)]$ contains at most two edges.*

## 4    Automated Vertex Cover Kernelization

Many reduction rules for various problems are instances of the same class that can be described as "find a subgraph $H$ with boundary $X$ and replace it with a graph $H^*$". For example, Reduction R.7 can be formulated as "find a $P_3$ $(a, v, b)$ with boundary $\{a, b\}$ (a kind of "local surgery") and replace it with $z_{N[v]}$". Note however, that boundary connections might change during the replacement. Proposed reduction rules of this type can be checked in roughly $2^{|X|}$ times the time it takes to compute a minimum vertex cover in $H$ and $H^*$.

**Notation.**    See Figure 7 for an illustration. Let $G$ be a graph in which $t$ non-isolated vertices $x_1, \ldots, x_t$ are bijectively labeled with the integers in $[t]$. Then, we call $G$ *t-boundaried*. If $\{x_1, \ldots, x_t\}$ is an independent set in $G$, we call $G$ *strongly t-boundaried*. For a $t$-boundaried graph $H$ and a strongly $t$-boundaried graph $G$, we let $G \oplus H$ denote the result of *gluing $G$ and $H$*, that is, identifying the vertices with the same label in the disjoint union of $G$ and $H$. We call a set $S \subseteq V(G)$ *compatible* with a set $X \subseteq [t]$ in $G$ if $N_G(x_i) \subseteq S \iff i \in X$ for all $i \in [t]$. Let $P_H^G : 2^{[t]} \to \mathbb{N}$ be such that, for all $X \subseteq [t]$, $P_H^G(X)$ is the smallest number of vertices of $H$ contained in any vertex cover $S$ of $G \oplus H$ that is compatible with $X$ in $G$ (and $\infty$ if no such $S$ exists). Then, we call $P_H^G$ the *profile* of $H$ in $G$. It turns out that the profile is indeed independent of $G$, so we drop the superscript.

**Lemma 2.** *Let $G$, $G'$ be strongly $t$-boundaried, let $H$ be $t$-boundaried and let $X \subseteq [t]$. Then, $P_H^G(X) = P_H^{G'}(X)$ for all $X \subseteq [t]$.*

*Proof.* Let $X \subseteq [t]$ and let $S$ and $S'$ be vertex covers of $G \oplus H$ and $G' \oplus H$ that are compatible with $X$ in $G$ and $G'$, respectively, such that $|S \cap V(H)|$ and $|S' \cap V(H)|$ are minimum among

all such vertex covers. To prove the claim, we show that $S^* \coloneqq (S \setminus V(H)) \cup (S' \cap V(H))$ is a vertex cover of $G \oplus H$. By symmetry, the same follows for $S$ and $S'$ inversed, which then implies the lemma. Towards a contradiction, assume that $G \oplus H$ contains an edge $uv$ such that $u, v \notin S^*$. Then, exactly one of $u$ and $v$ is in $H$ as, otherwise, $S$ is not a vertex cover of $G \oplus H$. Without loss of generality, let $u \in V(H)$ and $v \notin V(H)$. Thus, $u$ is a boundary vertex $x_i$ and $v \notin S$. Since $S$ and $S'$ are compatible with $X$ in $G$ and $G'$, respectively, we know that $N_G(x_i) \subseteq S \iff i \in X \iff N_{G'}(x_i) \subseteq S'$ and, as $v \in N_G(u) \setminus S$, we have $N_{G'}(u) \nsubseteq S'$. However, since $S'$ is a vertex cover of $G' \oplus H$, we have $u \in S'$, which contradicts $u \notin S^*$ since $u \in V(H)$. $\square$

We observe that two $t$-boundaried graphs $H$ and $H'$ with the same profile can be swapped for one another in any graph $G$ without changing the size of an optimal vertex cover, that is, $G \oplus H$ and $G \oplus H'$ have the same vertex cover number. More generally, for any $c \in \mathbb{N}$, we say that $H$ and $H'$ are *c-equivalent* if $\forall_{X \subseteq [t]} \, P_H(X) = P_{H'}(X) + c$. In this way, for any fixed size $t$, the profile gives rise to an equivalence relation on the set of $t$-boundaried graphs. This relation allows automated discovery of reduction rules that remove vertices with undesirable properties from the input graph. The idea is, for each induced subgraph $H$ having an undesirable property $\Pi$, to replace $H$ by some $c$-equivalent $H'$ that does not suffer from $\Pi$, while reducing $k$ by $c$.

**Lemma 3.** *Let $G$ be strongly $t$-boundaried, let $H$ and $H'$ be $t$-boundaried and $c$-equivalent for some $c \in \mathbb{N}$, and let $k \in \mathbb{N}$. Then, $G \oplus H$ has a vertex cover of size at most $k$ if and only if $G \oplus H'$ has a vertex cover of size at most $k - c$.*

*Proof.* As "$\Rightarrow$" is completely analogous to "$\Leftarrow$", we only prove the latter. To this end, let $S$ be a smallest vertex cover of $G \oplus H$ that, among all such vertex covers, minimizes $|S \cap V(H)|$. Let $X \coloneqq \{i \mid N_G(x_i) \subseteq S\}$ and note that $S$ is compatible with $X$ in $G$ and $|S \cap V(H)| = P_H(X)$. Let $S'$ be a smallest vertex cover of $G \oplus H'$ that is compatible with $X$ in $G$ and, among all such vertex covers, minimizes $|S' \cap V(H')|$. As $H$ and $H'$ are $c$-equivalent, we know that $|S \cap V(H)| = |S' \cap V(H')| + c$. We show that $S^* \coloneqq (S \setminus V(H)) \cup (S' \cap V(H))$ is a vertex cover of $G \oplus H'$ (clearly, $|S^*| \leq |S \setminus V(H)| + |S' \cap V(H')| = |S \setminus V(H)| + |S \cap V(H)| + c = |S| + c$). Towards a contradiction, assume that there is an edge $uv$ of $G \oplus H'$ with $u, v \notin S^*$. If $u, v \notin V(H')$, then $S$ is not a vertex cover of $G \oplus H$ and, if $u, v \in V(H')$, then $S'$ is not a vertex cover of $G \oplus H'$. Thus, without loss of generality, $u \in V(H')$ and $v \notin V(H')$, implying that $u$ is a boundary vertex $x_i$. Since $u \notin S^*$, we know that $u \notin S'$ and, since $S'$ is a vertex cover of $G \oplus H'$, we have $N_G(u) \subseteq S'$. Since $S'$ is compatible with $X$, we have $i \in X$ and, since $S$ is compatible with $X$, we have $N_G(u) \subseteq S$, implying $v \in S$ which contradicts $v \notin S^*$ since $v \notin V(H)$. $\square$

Given a $t$-boundaried graph $H$ and a property $\Pi$, we can enumerate all $t$-boundaried graphs $H'$ that are $c$-equivalent to $H$ for some $c$ and that do not suffer from $\Pi$.

**Two Examples.** A proof-of-concept implementation[7] was used to attack the remaining cases of degree-four vertices (see Section 3). For a given $t$-boundaried graph $H$ or profile $P_H$ and a given number $n$, the implementation enumerates all strongly $t$-boundaried, $n$-vertex graphs $H'$ and outputs $H'$ if $P_H(X) = P_{H'}(X)$ for all $X \subseteq [t]$. Feeding the graphs displayed in Figures 8a and 9a, the implementation yielded, in 5s and 6s, respectively, reduction rules that remove degree-four vertices whose neighborhood contains exactly two edges.

---

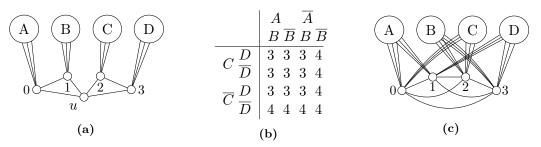[7] https://github.com/igel-kun/VC_min_deg

**Figure 8.** Illustration of Reduction R.11. **(a)** shows the degree-four vertex $u$ with two edges in its neighborhood. $A$–$D$ represent the sets of neighbors of 0–3, respectively, in the rest of the graph ($A$–$D$ may mutually intersect or be empty). **(b)** shows the profile of (a) where the entry 3 at position $(A, \overline{B}, C, D)$ means that three vertices are needed to cover all edges of (a), assuming all vertices in $A$, $C$, and $D$ are already in the cover. Indeed, $\{u, 1, 2\}$ is a size-3 cover in this case. **(c)** shows a subgraph that is 0-equivalent to (a), that is, (b) is also the profile of (c).

**Reduction R.11.** *Let $G$ contain the 4-boundaried graph $H$ depicted in Figure 8a as an induced subgraph. Then, replace $H$ by the 4-boundaried graph $H'$ depicted in Figure 8c.*

**Reduction R.12.** *Let $G$ contain the 4-boundaried graph $H$ depicted in Figure 9a as an induced subgraph. Then, replace $H$ by the 4-boundaried graph $H'$ depicted in Figure 9c.*

By Lemma 3, correctness of Reductions R.11 and R.12 can be verified by convincing oneself that the profiles in Figures 8 and 9 are indeed the profiles of the graphs of Figures 8c and 9c, respectively (implying that the subgraphs are 0-equivalent).

Indeed, Reductions R.11 and R.12 cover all cases of degree-four vertices with two edges in the neighborhood, leaving only the cases of one and no edges in the neighborhood in order to be able to reduce to graphs of minimum degree five.

## 5   Conclusion and Open Problems

In Section 3, we have discussed the *barrier degree* constant $\delta_{VC}$ for VERTEX COVER kernelization: We observed that, for some $\delta_{VC} \in \mathbb{N}$, VERTEX COVER cannot be kernelized to instances of minimum degree $\delta_{VC}$ unless ETH fails. In terms of FPT algorithms, the equivalent concept is that of the existence of the *barrier constant* $\zeta_{VC} > 0$ which is such that there is no algorithm for VERTEX COVER running in time $(1 + \zeta_{VC})^k \cdot n^{\mathcal{O}(1)}$ modulo ETH (e.g., [17, 21, 29]). So far it is only known that $\zeta_{VC} < 0.2738$ [13] and that $\delta_{VC} > 3$ (Section 3, see also [32]). However, observe that the question of determining the concrete value of $\delta_{VC}$ is much more tangible than the one of finding the value of $\zeta_{VC}$: Suppose one can show that a reduction rule that kernelizes degree-$d$ vertices violates ETH, for some $d \in \mathbb{N}$. Then one might be able to adapt the gadgets used in that proof to show an ETH-violation via a reduction rule for degree $d + 1$, $d + 2$, ... vertices as well. We pose: What is the exact value of $\delta_{VC}$?

The main theme of this paper has been to gather (from hitherto unpublished sources), carefully verify, and advance research on the question: to what minimum degree $d$ can the (*formidable naturally parameterized*) VERTEX COVER problem be kernelized to kernels of minimum degree $d$, even if the exponent of the polynomial running time bound grows wildly in $d$? ETH enforces a limit.
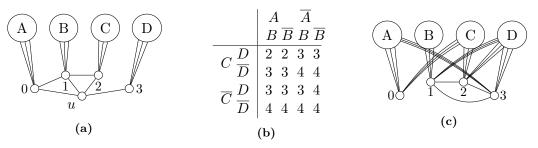
**Figure 9.** Illustration of Reduction R.12. **(a)** shows the degree-four vertex $u$ with two edges in its neighborhood. $A$–$D$ represent the sets of neighbors of 0–3, respectively, in the rest of the graph. **(b)** shows the profile of (a) where the entry 2 at position $(A, \overline{B}, C, D)$ means that two vertices are needed to cover all edges of (a), assuming all vertices in $A$, $C$, and $D$ are already in the cover. Indeed, $\{u, 1\}$ is a size-2 cover in this case. **(c)** shows a subgraph that is 0-equivalent to (a), that is, (b) is also the profile of (c).

# References

[1] Faisal N. Abu-Khzam, Rebecca L. Collins, Michael R. Fellows, Michael A. Langston, W. Henry Suters, and Christopher T. Symons. Kernelization algorithms for the vertex cover problem: Theory and experiments. In Lars Arge, Guiseppe F. Italiano, and Robert Sedgewick, editors, *Proceedings 6th Workshop on Algorithm Engineering and Experiments and 1st Workshop on Analytic Algorithms and Combinatorics (ALENEX/ANALC)*, pages 62–69. SIAM, 2004.

[2] Reyhan Aydoğan, Tim Baarslag, Enrico Gerding, Catholijn M. Jonker, Vicente Julian, and Victor Sanchez-Anguix. *Proceedings 2nd International Workshop on Conflict Resolution in Decision Making (COREDEMA 2016)*, volume 10238 of *Lecture Notes in Artificial Intelligence (LNAI)*. Springer, 2017.

[3] R. Balasubramanian, Michael R. Fellows, and Venkatesh Raman. An improved fixed-parameter algorithm for vertex cover. *Information Processing Letters*, 65(3):163–168, 1998.

[4] Regina Berretta and Pablo Moscato. Cancer biomarker discovery: the entropic hallmark. *PLoS One*, 5(8):e12262, 2010.

[5] Hans L. Bodlaender, Bart M. P. Jansen, and Stefan Kratsch. Kernelization lower bounds by cross-composition. *SIAM Journal on Discrete Mathematics*, 28(1):277–305, 2014.

[6] Hans L. Bodlaender, Rodney G. Downey, Michael R. Fellows, and Danny Hermelin. On problems without polynomial kernels. In *Proceedings of ICALP 2008*, volume 5125 of *Lecture Notes in Computer Science (LNCS)*, pages 563–574. Springer, 2008.

[7] Hans L. Bodlaender, Rodney G. Downey, Michael R. Fellows, and Danny Hermelin. On problems without polynomial kernels. *Journal of Computer and System Sciences*, 75:423–434, 2009.

[8] Jonathan F. Buss and Judy Goldsmith. Nondeterminism within P. *SIAM Journal on Computing*, 22(3):560–572, 1993.

[9] Marin Bougeret and Ignasi Sau. How Much Does a Treedepth Modulator Help to Obtain Polynomial Kernels Beyond Sparse Graphs? In *Proceedings of the 12th International Sympo-*

*sium on Parameterized and Exact Computation (IPEC)*, volume 89 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 10:1–10:13, 2017.

[10] Sergiy Butenko and Wilbert E. Wilhelm. Clique-detection models in computational biochemistry and genomics. *European Journal of Operational Research*, 173(1):1–17, 2006.

[11] Liming Cai, Jianer Chen, Rodney G. Downey, and Michael R. Fellows. Advice classes of parameterized tractability. *Annals of Pure and Applied Logic*, 84(1):119–138, 1997.

[12] Jianer Chen, Iyad A. Kanj, and Weijia Jia. Vertex cover: Further observations and further improvements. *Journal of Algorithms*, 41(2):280–301, 2001.

[13] Jianer Chen, Iyad A. Kanj, and Ge Xia. Improved upper bounds for vertex cover. *Theoretical Computer Science*, 411(40-42):3736–3756, 2010. Previously appeared in MFCS 2006 as 'Improved parameterized upper bounds for vertex cover'.

[14] Tammy M. K. Cheng, Yu-En Lu, Michele Vendruscolo, Tom L. Blundell, et al. Prediction by graph theoretic measures of structural effects in proteins arising from non-synonymous single nucleotide polymorphisms. *PLoS Computational Biology*, 4(7):e1000135, 2008.

[15] Benny Chor, Michael R. Fellows, and David Juedes. Linear kernels in linear time, or how to save $k$ colors in $\mathcal{O}(n^2)$ steps. In Juraj Hromkovič, Manfred Nagl, and Bernhard Westfechtel, editors, *Proceedings 30th International Workshop on Graph-Theoretic Concepts in Computer Science (WG)*, volume 3353 of *Lecture Notes in Computer Science (LNCS)*, pages 257–269. Springer, 2004.

[16] Jason Cong and M'Lissa Smith. A parallel bottom-up clustering algorithm with applications to circuit partitioning in vlsi design. In *Proceedings 30th International Design Automation Conference*, pages 755–760. ACM, 1993.

[17] Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 1st edition, 2015.

[18] Fank K. H. A. Dehne, Michael R. Fellows, Frances A. Rosamond, and Peter Shaw. Greedy localization, iterative compression, and modeled crown reductions: New FPT techniques, an improved algorithm for set splitting, and a novel $2k$ kernelization for vertex cover. In *Proceedings 1st International Workshop on Parameterized and Exact Computation (IWPEC)*, volume 3162 of *Lecture Notes in Computer Science (LNCS)*, pages 271–280. Springer, 2004.

[19] Holger Dell and Dieter van Melkebeek. Satisfiability allows no nontrivial sparsification unless the polynomial-time hierarchy collapses. *Journal of the ACM (JACM)*, 61(4):23, 2014. Previously appeared in STOC 2010.

[20] Rodney G. Downey and Michael R. Fellows. Parameterized computational feasibility. In *Feasible Mathematics II*, pages 219–244. Springer, 1995.

[21] Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013.

[22] Rodney G. Downey, Michael R. Fellows, and Ulrike Stege. Parameterized complexity: A framework for systematically confronting computational intractability. In *Contemporary trends in discrete mathematics: From DIMACS and DIMATIA to the future*, volume 49, pages 49–99, 1999.

[23] Geoffrey M. Downs and Peter Willett. Similarity searching in databases of chemical structures. *Reviews in Computational Chemistry*, 7:1–66, 1996.

[24] Jessica Enright and Kitty Meeks. Deleting edges to restrict the size of an epidemic: A new application for treewidth. *Algorithmica*, 80(6):1857–1889, 2018. previously appeared in COCOA 2015.

[25] Michael R. Fellows. Parameterized complexity: New developments and research frontiers. In Rodney G. Downey and Denis R. Hirschfeldt, editors, *Aspects of Complexity. Minicourses in Algorithmics, Complexity and Computational Algebra*, volume 4 of *De Gruyter Series in Logic and Its Applications*, pages 51–72, Kaikoura, 2000. De Gruyter.

[26] Michael R. Fellows. Parameterized complexity: The main ideas and some research frontiers. In Peter Eades and Tadao Takaoka, editors, *Proceedings 12th International Symposium on Algorithms and Computation (ISAAC)*, volume 2223 of *Lecture Notes in Computer Science (LNCS)*, pages 291–307, Christchurch, New Zealand, 2001. Springer.

[27] Michael R. Fellows. Blow-ups, win/win's, and crown rules: Some new directions in FPT. In Hans L. Bodlaender, editor, *Proceedings 29th International Workshop on Graph-Theoretic Concepts in Computer Science (WG)*, volume 2880 of *Lecture Notes in Computer Science (LNCS)*, pages 1–12. Springer, 2003.

[28] Michael R. Fellows. The lost continent of polynomial time: Preprocessing and kernelization. In *Proceedings 2nd International Workshop on Parameterized and Exact Computation (IWPEC)*, volume 4169 of *Lecture Notes in Computer Science (LNCS)*, pages 276–277. Springer, 2006.

[29] Michael R. Fellows. Some open problems in parameterized complexity related to the work of Jianer Chen. *Tsinghua Science and Technology*, 19(4):325–328, 2014.

[30] Michael R. Fellows, Lars Jaffke, Aliz Izabella Király, Frances A. Rosamond, and Mathias Weller. What is known about vertex cover kernelization? In Hans-Joachim Böckenhauer, Dennis Komm, and Walter Unger, editors, *Adventures Between Lower Bounds and Higher Altitudes. Essays Dedicated to Juraj Hromkovič on the Occasion of His 60th Birthday*, volume 11011 of *Lecture Notes in Computer Science (LNCS)*, pages 330–356, Springer, 2018

[31] Michael R. Fellows, Bart M. P. Jansen, and Frances A. Rosamond. Towards fully multivariate algorithmics: Parameter ecology and the deconstruction of computational complexity. *European Journal of Combinatorics*, 34(3):541–566, 2013.

[32] Michael R. Fellows and Ulrike Stege. An improved fixed-parameter tractable algorithm for vertex cover. 1999.

[33] Fedor V. Fomin, Serge Gaspers, Petr A. Golovach, Dieter Kratsch, and Saket Saurabh. Parameterized algorithm for eternal vertex cover. *Information Processing Letters*, 110(16):702–706, 2010.

[34] Fedor V. Fomin and Torstein J. F. Strømme. Vertex cover structural parameterization revisited. In *Proceedings 42nd International Workshop on Graph-Theoretic Concepts in Computer Science (WG)*, volume 9941 of *Lecture Notes in Computer Science (LNCS)*, pages 171–182. Springer, 2016.

[35] Lance Fortnow and Rahul Santhanam. Infeasibility of instance compression and succinct PCPs for NP. In *Proceedings Fortieth Annual ACM Symposium on Theory of Computing (STOC)*, pages 133–142. ACM, 2008.

[36] Michael R. Garey and David S. Johnson. *Computers and Intractability. A Guide to the Theory of NP-Completeness*. W.H. Freeman and Company, New York, 1979.

[37] Lee-Ad Gottlieb, Aryeh Kontorovich, and Robert Krauthgamer. Efficient classification for metric data. *IEEE Transactions on Information Theory*, 60(9):5750–5759, 2014.

[38] Jiong Guo and Rolf Niedermeier. Invitation to data reduction and problem kernelization. *ACM SIGACT News*, 38(1):31–45, 2007.

[39] Jiong Guo, Rolf Niedermeier, and Sebastian Wernicke. Parameterized complexity of vertex cover variants. *Theory of Computing Systems*, 41(3):501–520, 2007.

[40] Philip Hall. On representatives of subsets. *Journal of the London Mathematical Society*, 10(1):26–30, 1935.

[41] Ilker Hamzaoglu and Janak H. Patel. Test set compaction algorithms for combinational circuits. In *Proceedings IEEE/ACM iInternational Conference on Computer-Aided Design*, pages 283–289. ACM, 1998.

[42] Eva-Maria C. Hols and Stefan Kratsch. Smaller parameters for vertex cover kernelization. In *Proceedings 12th International Symposium on Parameterized and Exact Computation (IPEC)*, volume 89 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 20:1–20:12, Schloss Dagstuhl Publishing, 2017.

[43] John E. Hopcroft and Richard M. Karp. An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. *SIAM Journal on Computing*, 2(4):225–231, 1973.

[44] Russel Impagliazzo and Ramamohan Paturi. On the complexity of $k$-sat. *Journal of Computer and System Sciences*, 62(2):367–375, 2001.

[45] Russel Impagliazzo, Ramamohan Paturi, and Francis Zane. Which problems have strongly exponential complexity? *Journal of Computer and System Sciences*, 63(4):512–530, 2001.

[46] Bart M. P. Jansen. *The Power of Data Reduction. Kernels for Fundamental Graph Problems*. PhD thesis, Utrecht University, The Netherlands, 2013.

[47] Bart M. P. Jansen and Hans L. Bodlaender. Vertex cover kernelization revisited. *Theory of Computing Systems*, 53(2):263–299, 2013.

[48] Richard M. Karp. Reducibility among combinatorial problems. In *Complexity of Computer Computations*, pages 85–103. Springer, 1972.

[49] Richard M. Karp. Heuristic algorithms in computational molecular biology. *Journal of Computer and System Sciences*, 77(1):122–128, 2011.

[50] Ina Koch, Thomas Lengauer, and Egon Wanke. An algorithm for finding maximal common subtopologies in a set of protein structures. *Journal of Computational Biology*, 3(2):289–306, 1996.

[51] Dénes König. Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. *Mathematische Annalen*, 77(4):453–465, 1916.

[52] Stefan Kratsch and Magnus Wahlström. Representative sets and irrelevant vertices: New tools for kernelization. In *Proceedings 53rd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 450–459. IEEE, 2012.

[53] Michael Lampis. A kernel of order $2k - c \log k$ for vertex cover. *Information Processing Letters*, 111(23):1089–1091, 2011.

[54] Andrew R. Leach and Valerie J. Gillet. *An Introduction to Chemoinformatics.* Springer, 2007.

[55] Daniel Lokshtanov, Dániel Marx, and Saket Saurabh. Lower bounds based on the exponential time hypothesis. *Bulletin of EATCS*, 3(105), 2013.

[56] Daniel Lokshtanov, Neeldhara Misra, and Saket Saurabh. Kernelization – Preprocessing with a guarantee. In Hans L. Bodlaender, Rodney G. Downey, Fedor V. Fomin, and Daniel Marx, editors, *The Multivariate Algorithmic Revolution and Beyond. Essays Dedicated to Michael R. Fellows on the Occasion of His 60th Birthday*, volume 7370 of *Lecture Notes in Computer Science (LNCS)*, pages 129–161. Springer, 2012.

[57] Wenjun Li and Binhai Zhu. A $2k$-kernelization algorithm for vertex cover based on crown decomposition. *Theoretical Computer Science*, 739:80–85, 2018.

[58] Diptapriyo Majumdar, Venkatesh Raman, and Saket Saurabh. Kernels for structural parameterizations of vertex cover-case of small degree modulators. In *Proceedings 10th International Symposium on Parameterized and Exact Computation (IPEC)*, volume 43 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 331–342. Schloss Dagstuhl Publishing, 2015.

[59] Neeldhara Misra, Venkatesh Raman, and Saket Saurabh. Lower bounds on kernelization. *Discrete Optimization*, 8(1):110–128, 2011.

[60] Marcin Mucha and Piotr Sankowski. Maximum matchings via gaussian elimination. In *Proceedings 45th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 248–255. IEEE, 2004.

[61] N. S. Narayanaswamy, Venkatesh Raman, M. S. Ramanujan, and Saket Saurabh. LP can be a cure for parameterized problems. In Christoph Dürr and Thomas Wilke, editors, *Proceedings 29th Symposium on Theoretical Aspects of Computer Science (STACS)*, volume 14 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 338–349. Schloss Dagstuhl Publishing, 2012.

[62] George L. Nemhauser and Leslie E. Trotter. Properties of vertex packing and independence system polyhedra. *Mathematical Programming*, 6(1):48–61, 1974.

[63] Rolf Niedermeier. *Invitation to Fixed-Parameter Algorithms*, volume 31 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, 2002.

[64] Alexander Schrijver. *Theory of Linear and Integer Programming*. John Wiley & Sons, 1998.

[65] Arezou Soleimanfallah and Anders Yeo. A kernel of order $2k - c$ for vertex cover. *Discrete Mathematics*, 311(10):892–895, 2011.

[66] Ulrike Stege. *Resolving Conflicts in Problems from Computational Biology*. PhD thesis, ETH Zürich, 2000.

[67] Dawn M. Strickland, Earl Barnes, and Joel S. Sokol. Optimal protein structure alignment using maximum cliques. *Operations Research*, 53(3):389–402, 2005.

[68] Vijay V. Vazirani. *Approximation Algorithms*. Springer, 1st edition, 2003.

[69] Karsten Weihe. Covering trains by stations or the power of data reduction. In *Proceedings 1st Conference on Algorithms and Experiments (ALEX98)*, pages 1–8, Trento, Italy, 1998.

[70] Karsten Weihe. On the difference between "practical" and "applied". In *Proceedings 4th International Workshop on Algorithm Engineering (WAE)*, volume 1982 of *Lecture Notes in Computer Science (LNCS)*, pages 1–10. Springer, 2000.

[71] Mathias Weller. *Aspects of Preprocessing Applied to Combinatorial Graph Problems*. PhD thesis, TU Berlin, 2013.

[72] Peter Willett, John M. Barnard, and Geoffrey M. Downs. Chemical similarity searching. *Journal of Chemical Information and Computer Sciences*, 38(6):983–996, 1998.

[73] Chee K. Yap. Some consequences of non-uniform conditions on uniform classes. *Theoretical Computer Science*, 26(3):287–300, 1983.

[74] Esti Yeger-Lotem, Shmuel Sattath, Nadav Kashtan, Shalev Itzkovitz, Ron Milo, Ron Y. Pinter, Uri Alon, and Hanah Margalit. Network motifs in integrated cellular networks of transcription-regulation and protein-protein interaction. *Proceedings of the National Academy of Sciences of the United States of America*, 101(16):5934–5939, 2004.