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A NOTE ON THE CUT-ELIMINATION PROOF IN "TRUTH WITHOUT CONTRA(DI)CTION"

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Abstract. This note shows that the permutation instructions presented by Zardini (2011) for eliminating cuts on universally quantified formulas in the sequent calculus for the noncontractive theory of truth IKT$^{\omega}$ are inadequate. To that purpose the note presents a derivation in the sequent calculus for IKT$^{\omega}$ ending with an application of cut on a universally quantified formula which the permutation instructions cannot deal with. The counterexample is of the kind that leaves open the question whether cut can be shown to be eliminable in the sequent calculus for IKT$^{\omega}$ with an alternative strategy.

§1. Introduction. Following the results by Da Ré & Rosenblatt (2018) and Fjellstad (2018), it is argued by Fjellstad (2018) that we have good reasons to assume that there is an error in the cut-elimination proof for the sequent calculus defining the noncontractive theory of truth IKT$^{\omega}$ presented by Zardini (2011). This note confirms that conjecture by presenting a derivation in the sequent calculus for IKT$^{\omega}$ ending with an application of cut on a universally quantified formula which cannot be eliminated through the permutation instructions for that case proposed by Zardini (2011). An important feature of the counterexample is that the application of cut therein is clearly eliminable *in practice*, though not through the permutation instructions presented by Zardini (2011), and it doesn't show that IKT$^{\omega}$ is inconsistent. Instead, it leaves open the question whether cut can be shown to be eliminable in the sequent calculus for IKT$^{\omega}$ with an alternative strategy.

§2. The sequent calculus for IKT$^{\omega}$. The theory of truth IKT$^{\omega}$ is presented by Zardini (2011) for a first-order language $\mathcal{L}$ without a designated equality-symbol, but with a designated predicate T, and with the connectives $\rightarrow$, $\neg$, $\exists$ and $\forall$ in addition to $\otimes$ as symbol for conjunction and $\oplus$ as symbol for disjunction. We shall in this note restrict $\mathcal{L}$ to $\forall$, $\rightarrow$ and $\neg$. The other connectives are anyway definable in terms of our selection.

Regarding names for formulas, we "pick designated individual constants to serve as canonical names of all sentences in the language" such that "if an individual constant is the canonical name of a sentence A" then $\ulcorner A \urcorner$ or a lower-case letter "will refer to that individual constant" (Zardini, 2011, p. 506).

To show that IKT$^{\omega}$ is consistent, Zardini (2011, p. 524) defines a sequent calculus for IKT$^{\omega}$ based on sequents as multisets of $\mathcal{L}$-formulas, where both the antecedent and succedent multiset of a sequent can contain ω many formulas. For simplicity we shall

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depart from the nomenclature in Zardini (2011) by referring to the sequent calculus in question as $\mathcal{C}$. Moreover, we use upper-case Latin letters such as A and B as meta-linguistic variables for formulas of $\mathcal{L}$, and upper-case Greek letters such as Γ and Δ as meta-linguistic variables for multisets of formulas.

We thus let $\mathcal{C}$ be the sequent calculus with sequents as multisets of $\mathcal{L}$-formulas obtained with the initial sequents

$$A, \Gamma \Rightarrow \Delta, A$$

and the rules

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \text{cut}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ \neg\text{L} \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\ \neg\text{R}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{T\ulcorner A \urcorner, \Gamma \Rightarrow \Delta}\ \text{TL} \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, T\ulcorner A \urcorner}\ \text{TR}$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma' \Rightarrow \Delta'}{A \rightarrow B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \rightarrow\text{L} \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\ \rightarrow\text{R}$$

$$\frac{A(t_0/x), A(t_1/x), A(t_2/x), \ldots \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ \forall\text{L}^{\omega}$$

$$\frac{\Gamma_0 \Rightarrow \Delta_0, A(t_0/x) \qquad \Gamma_1 \Rightarrow \Delta_1, A(t_1/x) \qquad \Gamma_2 \Rightarrow \Delta_2, A(t_2/x) \qquad \ldots}{\Gamma_0, \Gamma_1, \Gamma_2, \ldots \Rightarrow \Delta_0, \Delta_1, \Delta_2, \ldots, \forall x A}\ \forall\text{R}^{\omega}$$

where $t_0, t_1, t_2, \ldots$ represent a complete enumeration of the closed terms of $\mathcal{L}$. Our extensive use of the $\ldots$-notation in the presentation of the rules for the quantifiers is hopefully excusable, and the dissatisfied reader may refer to Zardini (2011) for a less perspicuous but more precise presentation which still involves some use of "$\ldots$".

Based on how the theory of truth IKT$^{\omega}$ is defined by Zardini (2011), it follows that Γ implies Δ in IKT$^{\omega}$ if and only if $\Gamma \Rightarrow \Delta$ is derivable in $\mathcal{C}$. To show that IKT$^{\omega}$ is consistent, then it suffices to show that cut is eliminable for $\mathcal{C}$.

§3. The cut-elimination proof. The main ingredient for a Gentzen-style cut-elimination proof are the permutation instructions on subderivations containing an application of cut only at its final step through which (one of the) the measure(s) for induction is reduced. For the case in which the cut-formula is of the form $T\ulcorner A \urcorner$ and is principal in both premise-sequents, the to-be-permuted subderivation $\mathcal{D}$ takes the form

$$\dfrac{\dfrac{\overset{\vdots}{\Gamma \Rightarrow \Delta, A}}{\Gamma \Rightarrow \Delta, T\ulcorner A \urcorner}\ \text{TR} \qquad \dfrac{\overset{\vdots}{A, \Gamma' \Rightarrow \Delta'}}{T\ulcorner A \urcorner, \Gamma' \Rightarrow \Delta'}\ \text{TL}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \text{cut}$$

and is to be permuted into a derivation $\mathcal{D}'$ of the form:

$$\frac{\overset{\vdots}{\Gamma \Rightarrow \Delta, A} \qquad \overset{\vdots}{A, \Gamma' \Rightarrow \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \text{cut}$$

The issue now is that a cut-elimination proof based on the strategy going back to Gentzen (1934) employs as main measure the complexity of the cut-formula, but that measure might not be reduced through these permutations. A cut-elimination proof for a sequent calculus containing the rules TL and TR requires thus an alternative measure.

The proof by Zardini (2011) follows here the strategy of Petersen (2000) by performing the induction on the number of nodes in a derivation, however in a way that ensures that this number remains finite despite the infinitary rules. The finer details of the measure do not matter for our purposes here. Instead, we will focus on the proposal by Zardini (2011) for how to deal with the case in which the cut-formula is of the form $\forall xA$ and is principal in both premise-sequents for the application of cut, i.e., subderivations of the following form:

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\ \Gamma^0_0 \Rightarrow \Delta^0_0, At_0\end{array} \quad \begin{array}{c}\vdots\\ \Gamma^1_0 \Rightarrow \Delta^1_0, At_1\end{array} \quad \dots}{\Gamma^0_0, \Gamma^1_0, \dots \Rightarrow \Delta^0_0, \Delta^1_0, \dots, \forall xA}\,\forall\mathrm{R}^\omega \quad \cfrac{\begin{array}{c}\vdots\\ At_0, At_1, \dots \Gamma_1 \Rightarrow \Delta_1\end{array}}{\forall xA, \Gamma_1 \Rightarrow \Delta_1}\,\forall\mathrm{L}^\omega}{\Gamma^0_0, \Gamma^1_0, \dots, \Gamma_1 \Rightarrow \Delta^0_0, \Delta^1_0, \dots, \Delta_1}\,\text{cut}$$

To deal with this case, we are according to Zardini (2011) supposed to distinguish between whether the denumerable submultiset of formulas required for $\forall\mathrm{L}^\omega$ has been introduced through an initial sequent or through an application of the rule $\forall\mathrm{R}^\omega$. For the case where a denumerable submultiset of the formulas required for the application of $\forall\mathrm{L}^\omega$ has been introduced through $\forall\mathrm{R}^\omega$, we are according to Zardini (2011, p. 529) supposed to permute a derivation of the following form

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}^0_0\\ \Gamma^0_0 \Rightarrow \Delta^0_0, At_0\end{array} \quad \begin{array}{c}\mathcal{D}^1_0\\ \Gamma^1_0 \Rightarrow \Delta^1_0, At_1\end{array} \quad \dots}{\Gamma^0_0, \Gamma^1_0, \dots \Rightarrow \Delta^0_0, \Delta^1_0, \dots, \forall xAx}\,\forall\mathrm{R}^\omega \quad \cfrac{\begin{array}{c}\cfrac{\begin{array}{c}\mathcal{D}^0_1\\ At_i, \Gamma^0_1 \Rightarrow \Delta^0_1\end{array} \quad \begin{array}{c}\mathcal{D}^1_1\\ At_{i+1}, \Gamma^1_1 \Rightarrow \Delta^1_1\end{array} \quad \dots}{At_i, At_{i+1}, \dots, \Gamma^*_1 \Rightarrow \Delta^*_1}\\ \mathcal{D}^*_1\\ At_0, At_1, \dots, At_i, At_{i+1}, \dots, \Gamma_1 \Rightarrow \Delta_1\end{array}}{\forall xAx, \Gamma_1 \Rightarrow \Delta_1}\,\forall\mathrm{L}^\omega}{\Gamma^0_0, \Gamma^1_0, \dots, \Gamma_1 \Rightarrow \Delta^0_0, \Delta^1_0, \dots, \Delta_1}\,\text{cut}$$

into a derivation of the following form where $\mathcal{D}^{i\$}_0$ and $\mathcal{D}^{i\$}_1$ are obtained from $\mathcal{D}^i_0$ and $\mathcal{D}^i_1$ by "fiddling with" (Zardini, 2011, p. 529) the initial sequents, $\mathcal{D}^{*£}_1$ is obtained from $\mathcal{D}^*_1$ by substituting At_i for $At_i, At_{i+1}, \dots$, and the choice of i is arbitrary:

$$\cfrac{\begin{array}{c}\mathcal{D}^{i-1}_0\\ \Gamma^{i-1}_0 \Rightarrow \Delta^{i-1}_0, At_{i-1}\end{array} \quad \cfrac{\begin{array}{c}\mathcal{D}^{i\$}_0\\ \Gamma^i_0, \Gamma^{i+1}_0, \dots \Rightarrow \Delta^i_0, \Delta^{i+1}_0, \dots, At_i\end{array} \quad \begin{array}{c}\mathcal{D}^{i\$}_1\\ At_i, \Gamma^*_1 \Rightarrow \Delta^*_1\\ \mathcal{D}^{*£}_1\\ At_0, At_1, \dots, At_i, \Gamma_1 \Rightarrow \Delta_1\end{array}}{At_0, At_1, \dots, At_{i-1}, \Gamma^i_0, \Gamma^{i+1}_0, \dots, \Gamma_1 \Rightarrow \Delta^i_0, \Delta^{i+1}_0, \dots, \Delta_1}\,\text{cut}}{At_0, At_1, \dots, At_{i-2}, \Gamma^{i-1}_0, \Gamma^i_0, \Gamma^{i+1}_0, \dots, \Gamma_1 \Rightarrow \Delta^{i-1}_0, \Delta^i_0, \Delta^{i+1}_0, \dots, \Delta_1}\,\text{cut}$$

$$\cfrac{\begin{array}{c}\mathcal{D}^0_0\\ \Gamma^0_0 \Rightarrow \Delta^0_0, At_0\end{array} \qquad \begin{array}{c}\vdots\\ At_0, \Gamma^1_0, \Gamma^2_0, \dots, \Gamma_1 \Rightarrow \Delta^1_0, \Delta^2_0, \dots, \Delta_1\end{array}}{\Gamma^0_0, \Gamma^1_0, \dots, \Gamma_1 \Rightarrow \Delta^0_0, \Delta^1_0, \dots, \Delta_1}\,\text{cut}$$

However, these permutation instructions do not work.

§4. A counterexample. Consider the following application of cut which we assume is the topmost application of cut:

$$\dfrac{\dfrac{\Gamma_0 \Rightarrow \Delta_0, At_0 \quad \Gamma_1 \Rightarrow \Delta_1, At_1 \quad \ldots}{\Gamma \Rightarrow \Delta, \forall xAx}\,\forall\mathrm{R}^\omega \qquad \dfrac{\dfrac{\dfrac{At_0 \Rightarrow At_0 \qquad At_1 \Rightarrow At_1 \qquad \ldots}{At_0, At_1, \ldots \Rightarrow \forall xAx}\,\forall\mathrm{R}^\omega}{At_0, At_1, \ldots, \neg\forall xAx \Rightarrow}\,\neg\mathrm{L}}{\forall xAx, \neg\forall xAx \Rightarrow}\,\forall\mathrm{L}^\omega}{\neg\forall xAx, \Gamma \Rightarrow \Delta}\,\mathrm{cut}$$

On the face of it, this application of cut is straightforwardly eliminable by applying ¬L on $\Gamma \Rightarrow \Delta, \forall xAx$. For the permutation instructions provided by Zardini (2011), however, things are not that straightforward.

In our case, a denumerable subset of $At_0, At_1, \ldots$ is introduced through an application of $\forall\mathrm{R}^\omega$, and it is fairly evident that our case is of the above form: $\mathcal{D}_1^*$ is our application of ¬L and $\mathcal{D}_1^0$ is our initial sequent $At_0 \Rightarrow At_0$. In fact, the denumerable subset of $At_0, At_1, \ldots$ introduced through an application of $\forall\mathrm{R}^\omega$ is $At_0, At_1, \ldots$ itself. Following the instructions, we pick the derivation of the sequent containing the first element of that subset, i.e., the derivation ending with $At_0 \Rightarrow At_0$. By fiddling with the initial sequent(s) of the derivation of that sequent, namely, $At_0 \Rightarrow At_0$ itself, we should obtain a sequent from which we obtain with one application of ¬L the sequent $\neg\forall xAx, At_0 \Rightarrow$. We are thus supposed to permute the above derivation into something along the following lines:

$$\dfrac{\Gamma_0, \Gamma_1, \ldots \Rightarrow \Delta_0, \Delta_1, \ldots, At_0 \qquad \dfrac{At_0 \Rightarrow At_0, \forall xAx}{\neg\forall xAx, At_0 \Rightarrow}\,\neg\mathrm{L}}{\neg\forall xAx, \Gamma \Rightarrow \Delta}\,\mathrm{cut}$$

However, the copy of At_0 in succedent position of $At_0 \Rightarrow At_0, \forall xAx$ has now mysteriously disappeared. As far as the instructions provided by Zardini (2011) go, then, the above cut is not eliminable.

It follows that the permutation instructions provided by Zardini (2011) cannot be employed to show that cut is eliminable in $\mathcal{C}$, and thus cannot be employed to show that IKT^ω is consistent. Of course, the counterexample doesn't show that IKT^ω is inconsistent, and it doesn't exclude cut from being eliminable through some other strategy with alternative permutation instructions for cuts on universally quantified formulas. For all we know cut might still be eliminable in $\mathcal{C}$, but we do not have a proof thereof.

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