

# Finding Detours is Fixed-parameter Tractable\*

Ivona Bezáková<sup>1</sup>, Radu Curticapean<sup>2</sup>, Holger Dell<sup>3</sup>, and Fedor V. Fomin<sup>4</sup>

<sup>1</sup>Department of Computer Science, Rochester Institute of Technology, Rochester, NY, U.S.A., [ib@cs.rit.edu](mailto:ib@cs.rit.edu)

<sup>2</sup>Institute for Computer Science and Control, Hungarian Academy of Sciences (MTA SZTAKI), Budapest, Hungary, [radu.curticapean@gmail.com](mailto:radu.curticapean@gmail.com)

<sup>3</sup>Saarland University and Cluster of Excellence, MMCI, Saarbrücken, Germany, [hdell@mmci.uni-saarland.de](mailto:hdell@mmci.uni-saarland.de)

<sup>4</sup>University of Bergen, Bergen, Norway, [fomin@ii.uib.no](mailto:fomin@ii.uib.no)

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## Abstract

We consider the following natural “above guarantee” parameterization of the classical LONGEST PATH problem: For given vertices  $s$  and  $t$  of a graph  $G$ , and an integer  $k$ , the problem LONGEST DETOUR asks for an  $(s, t)$ -path in  $G$  that is at least  $k$  longer than a shortest  $(s, t)$ -path. Using insights into structural graph theory, we prove that LONGEST DETOUR is fixed-parameter tractable (FPT) on undirected graphs and actually even admits a single-exponential algorithm, that is, one of running time  $\exp(O(k)) \cdot \text{poly}(n)$ . This matches (up to the base of the exponential) the best algorithms for finding a path of length at least  $k$ .

Furthermore, we study the related problem EXACT DETOUR that asks whether a graph  $G$  contains an  $(s, t)$ -path that is exactly  $k$  longer than a shortest  $(s, t)$ -path. For this problem, we obtain a randomized algorithm with running time about  $2.746^k \cdot \text{poly}(n)$ , and a deterministic algorithm with running time about  $6.745^k \cdot \text{poly}(n)$ , showing that this problem is FPT as well. Our algorithms for EXACT DETOUR apply to both undirected and directed graphs.

## 1 Introduction

The LONGEST PATH problem asks, given an undirected  $n$ -vertex graph  $G$  and an integer  $k$ , to decide whether  $G$  contains a path of length at least  $k$ , that is, a self-avoiding walk with at least  $k$  edges. This problem is a natural generalization of the classical NP-complete HAMILTONIAN PATH problem, and the parameterized complexity community has paid exceptional attention to it. For instance, Monien [29] and Bodlaender [4] showed *avant la lettre* that LONGEST PATH is fixed-parameter tractable with parameter  $k$  and admits algorithms with running time  $2^{O(k \log k)} n^{O(1)}$ . This led Papadimitriou and Yannakakis [30] to conjecture that LONGEST PATH is solvable in polynomial time for  $k = \log n$ , and indeed, this conjecture was resolved in a seminal paper of Alon, Yuster, and Zwick [2], who introduced the method of color coding and derived from it the first algorithm with running time  $2^{O(k)} n$ . Since this breakthrough of Alon et al. [2], the problem LONGEST PATH occupied a central place in parameterized algorithmics, and several novel approaches were developed in order to reduce the base of the exponent in the running time [20, 23, 9, 8, 24, 35, 16, 16, 3]. We refer to two review articles in Communications of ACM [15, 25] as well as to the textbook [12, Chapter 10] for an extensive overview of parameterized algorithms for LONGEST PATH. Let us however note that the fastest known randomized algorithm for LONGEST PATH is due to Björklund et al. [3] and runs in time  $1.657^k \cdot n^{O(1)}$ , whereas the fastest known deterministic algorithm is due to Zehavi [36] and runs in time  $2.597^k \cdot n^{O(1)}$ .

In the present paper, we study the problem LONGEST PATH from the perspective of an “above guarantee” parameterization that can attain small values even for long paths: For a pair of vertices  $s, t \in V(G)$ , we use  $d_G(s, t)$  to denote the distance, that is, the length of a shortest path from  $s$  to  $t$ . We then ask for an  $(s, t)$ -path of length at least  $d_G(s, t) + k$ , and we parameterize by this offset  $k$  rather than the actual length of the path to obtain the problem LONGEST DETOUR. In other words, the first  $d_G(s, t)$  steps on a path sought by LONGEST DETOUR are complimentary and will not be counted towards the

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parameter value. This reflects the fact that shortest paths can be found in polynomial time and could (somewhat embarrassingly) be much better solutions for LONGEST PATH than the paths of logarithmic length found by algorithms that parameterize by the path length.

We study two variants of the detour problem, one asking for a detour of length at least  $k$ , and another asking for a detour of length exactly  $k$ .

LONGEST DETOUR **Parameter:**  $k$   
**Input:** Graph  $G$ , vertices  $s, t \in V(G)$ , and integer  $k$ .  
**Task:** Decide whether there is an  $(s, t)$ -path in  $G$  of length at least  $d_G(s, t) + k$ .

EXACT DETOUR **Parameter:**  $k$   
**Input:** Graph  $G$ , vertices  $s, t \in V(G)$ , and integer  $k$ .  
**Task:** Decide whether there is an  $(s, t)$ -path in  $G$  of length exactly  $d_G(s, t) + k$ .

Our parameterization above the length of a shortest path is a new example in the general paradigm of “above guarantee” parameterizations, which was introduced by Mahajan and Raman [27]. Their approach was successfully applied to various problems, such as finding independent sets in planar graphs (where an independent set of size at least  $\frac{n}{4}$  is guaranteed to exist by the Four Color Theorem), or the maximum cut problem, see e.g. [1, 11, 18, 17, 28].

## Our results

We show the following tractability results for LONGEST DETOUR and EXACT DETOUR:

- LONGEST DETOUR is fixed-parameter tractable (FPT) on undirected graphs. The running time of our algorithm is single-exponential, i.e., it is of the type  $2^{O(k)} \cdot n^{O(1)}$  and thus asymptotically matches the running time of algorithms for LONGEST PATH. Our approach requires a non-trivial argument in graph structure theory to obtain the single-exponential algorithm; a mere FPT-algorithm could be achieved with somewhat less effort. It should also be noted that a straightforward reduction rules out a running time of  $2^{o(k)} \cdot n^{O(1)}$  unless the exponential-time hypothesis of Impagliazzo and Paturi [21] fails.
- EXACT DETOUR is FPT on directed and undirected graphs. Actually, we give a polynomial-time Turing reduction from EXACT DETOUR to the standard parameterization of LONGEST PATH, in which we ask on input  $u, v$  and  $k \in \mathbf{N}$  whether there is a  $(u, v)$ -path of length  $k$ . This reduction only makes queries to instances with parameter at most  $2k + 1$ . Pipelined with the fastest known algorithms for LONGEST PATH mentioned above, this implies that EXACT DETOUR admits a bounded-error randomized algorithm with running time  $2.746^k n^{O(1)}$ , and a deterministic algorithm with running time  $6.745^k n^{O(1)}$ .

By a self-reducibility argument, we also show how to construct the required paths rather than just detect their existence. This reduction incurs only polynomial overhead.

## Techniques

The main idea behind the algorithm for LONGEST DETOUR is the following combinatorial theorem, which shows the existence of specific large planar minors in large-treewidth graphs while circumventing the full machinery used in the Excluded Grid Theorem [32]. Although the Excluded Grid Theorem already shows that graphs of sufficiently large treewidth contain arbitrary fixed planar graphs, resorting to more basic techniques allows us to show that linear treewidth suffices for our specific cases. More specifically, we show that there exists a global constant  $c \in \mathbf{N}$  such that every graph of treewidth at least  $c \cdot k$  contains as a subgraph a copy of a graph  $K_4^{\geq k}$ , which is any graph obtained from the complete graph  $K_4$  by replacing every edge by a path with at least  $k$  edges. The proof of this result is based on the structural theorems of Leaf and Seymour [26] and Raymond and Thilikos [31].

With the combinatorial theorem at hand, we implement the following win/win approach: If the treewidth of the input graph is less than  $c \cdot k$ , we use known algorithms [5, 16] to solve the problem in single-exponential time. Otherwise the treewidth of the input graph is at least  $c \cdot k$  and there must be a  $K_4^{\geq k}$ , which we use to argue that any path visiting the same two-connected component as  $K_4^{\geq k}$  can be prolonged by rerouting it through  $K_4^{\geq k}$ . To this end, we set up a fixed system of linear inequalities corresponding to the possible paths in  $K_4^{\geq k}$  such that rerouting is possible if and only if the system is

unsatisfiable. We then verify the unsatisfiability of this fixed system by means of a computer-aided proof (more specifically, a linear programming solver). From LP duality, we also obtain a short certificate for the unsatisfiability, which we include in the appendix.

The algorithm for EXACT DETOUR is based on the following idea. We run breadth-first search (BFS) from vertex  $v$  to vertex  $u$ . Then, for every  $(u, v)$ -path  $P$  of length  $d_G(u, v) + k$ , all but at most  $k$  levels of the BFS-tree contain exactly one vertex of  $P$ . Using this property, we are able to devise a dynamic programming algorithm for EXACT DETOUR, provided it is given access to an oracle for LONGEST PATH.

The remaining part of the paper is organized as follows: §2 contains definitions and preliminary results used in the technical part of the paper. In §3, we give an algorithm for LONGEST DETOUR while §4 is devoted to EXACT DETOUR. We provide a search-to-decision reduction for LONGEST DETOUR and EXACT DETOUR in §5. In §A, we give short certificates for the unsatisfiability of the linear programs from §3.

## 2 Preliminaries

We consider graphs  $G$  to be undirected, and we denote by  $uv$  an undirected edge joining vertices  $u, v \in V(G)$ . A *path* is a self-avoiding walk in  $G$ ; the *length* of the path is its number of edges. An  $(s, t)$ -path for  $s, t \in V(G)$  is a path that starts at  $s$  and ends at  $t$ . We allow paths to have length 0, in which case  $s = t$  holds. For a vertex set  $X \subseteq V(G)$ , denote by  $G[X]$  the subgraph induced by  $X$ .

**Tree decompositions.** A *tree decomposition*  $\mathcal{T}$  of a graph  $G$  is a pair  $(T, \{X_t\}_{t \in V(T)})$ , where  $T$  is a tree in which every node  $t$  is assigned a vertex subset  $X_t \subseteq V(G)$ , called a bag, such that the following three conditions hold:

- (T1) Every vertex of  $G$  is in at least one bag, that is,  $V(G) = \bigcup_{t \in V(T)} X_t$ .
- (T2) For every  $uv \in E(G)$ , there exists a node  $t \in V(T)$  such that  $X_t$  contains both  $u$  and  $v$ .
- (T3) For every  $u \in V(G)$ , the set  $T_u$  of all nodes of  $T$  whose corresponding bags contain  $u$ , induces a connected subtree of  $T$ .

The *width* of the tree decomposition  $\mathcal{T}$  is the integer  $\max_{t \in V(T)} |X_t| - 1$ , that is, the size of its largest bag minus 1. The *treewidth* of a graph  $G$ , denoted by  $\text{tw}(G)$ , is the smallest possible width that a tree decomposition of  $G$  can have.

We will need the following algorithmic results about treewidth.

**Proposition 1 ([6]).** *There is a  $2^{O(k)} \cdot n$  time algorithm that, given a graph  $G$  and an integer  $k$ , either outputs a tree decomposition of width at most  $5k + 4$ , or correctly decides that  $\text{tw}(G) > k$ .*

**Proposition 2 ([5, 16]).** *There is an algorithm with running time  $2^{O(\text{tw}(G))} \cdot n^{O(1)}$  that computes a longest path between two given vertices of a given graph.*

Let us note that the running time of Proposition 2 can be improved to  $2^{O(\text{tw}(G))} \cdot n$  by making use of the matroid-based approach from [16].

Our main theorem is based on graph minors, and we introduce some notation here.

**Definition 3.** A *topological minor model* of  $H$  in  $G$  is a pair of functions  $(f, p)$  with  $f : V(H) \rightarrow V(G)$  and  $p : E(H) \rightarrow 2^{E(G)}$  such that

1.  $f$  is injective, and
2. for every edge  $uv \in E(H)$ , the graph  $G[p(uv)]$  is a path from  $f(u)$  to  $f(v)$  in  $G$ , and
3. for edges  $e, g \in E(H)$  with  $e \neq g$ , the paths  $G[p(e)]$  and  $G[p(g)]$  intersect only in endpoints or not at all.

The graph  $T$  induced by the topological minor model  $(f, p)$  is the subgraph of  $G$  that consists of the union of all paths  $G[p(uv)]$  over all  $uv \in E(H)$ . The vertices in  $f(V(H))$  are the *branch vertices* of  $T$ , and  $G[p(e)]$  realizes the edge  $e$  in  $T$ .

### 3 Win/Win algorithm for Longest Detour

Throughout this section, let  $G$  be an undirected graph with  $n$  vertices and  $m$  edges, let  $s, t \in V(G)$  and  $k \in \mathbb{N}$ . We wish to decide in time  $2^{O(k)} \cdot n^{O(1)}$  whether  $G$  contains an  $(s, t)$ -path of length at least  $d_G(s, t) + k$ . To avoid trivialities, we assume without loss of generality that  $G$  is connected and  $s \neq t$  holds. Moreover, we can safely remove vertices  $v$  that are not part of any  $(s, t)$ -path.

**Definition 4.** Let  $G$  be a graph and let  $s, t \in V(G)$ . The  $(s, t)$ -relevant part of  $G$  is the graph induced by all vertices contained in some  $(s, t)$ -path. We denote it by  $G_{s,t}$ .

The graph  $G_{s,t}$  can be computed efficiently from the block-cut tree of  $G$ . Recall that the *block-cut tree* of a connected graph  $G$  is a tree where each vertex corresponds to a *block*, that is, a maximal biconnected component  $B \subseteq V(G)$ , or to a *cut vertex*, that is, a vertex whose removal disconnects the graph. A block  $B$  and a cut vertex  $v$  are adjacent in the block-cut tree if and only if there is a block  $B'$  such that  $B \cap B' = \{v\}$ .

**Lemma 5.** Let  $B_s$  and  $B_t$  denote the blocks of  $G$  that contain  $s$  and  $t$ , respectively. Furthermore, let  $P$  be the unique  $(B_s, B_t)$ -path in the block-cut tree of  $G$ . Then  $G_{s,t}$  is the graph induced by the union of all blocks visited by  $P$ .

*Proof.* Let  $v \in G_{s,t}$ . Then there is an  $(s, t)$ -path that contains  $v$ ; in particular, there is an  $(s, v)$ -path  $p_1$  and a  $(v, t)$ -path  $p_2$  such that  $p_1$  and  $p_2$  are internally vertex disjoint. If  $v$  was not in one of the blocks visited by  $P$ , it would be hidden behind a cut vertex and  $p_1$  and  $p_2$  would have to intersect in the cut vertex; therefore,  $v$  is contained in one of the blocks visited by  $P$ .

For the other direction, let  $v$  be a vertex contained in a block  $B$  visited by  $P$ . Suppose that  $u$  is the cut vertex preceding  $B$  in  $P$  (or  $u = s$  in case  $B = B_s$ ) and  $w$  is the cut vertex following  $B$  in  $P$  (or  $w = t$  in case  $B = B_t$ ). Then  $u \neq w$  holds, and there is an  $(s, u)$ -path and a  $(w, t)$ -path that are vertex-disjoint. Since  $B$  is biconnected, there are paths from  $u$  to  $v$  and from  $v$  to  $w$  that are internally vertex-disjoint. Combined, these path segments yield an  $(s, t)$ -path that visits  $v$ . ■

We formulate an immediate implication of Lemma 5 that will be useful later.

**Corollary 6.** The block-cut tree of  $G_{s,t}$  is a  $(B_s, B_t)$ -path.

Hopcroft and Tarjan [19] proved that the block-cut tree of a graph can be computed in linear time using DFS. Hence we obtain an algorithm for computing  $G_{s,t}$  from  $G$ .

**Corollary 7.** There is a linear-time algorithm that computes  $G_{s,t}$  from  $G$ .

#### 3.1 The algorithm

By definition, the graph  $G_{s,t}$  contains the same set of  $(s, t)$ -paths as  $G$ . Our algorithm for LONGEST DETOUR establishes a “win/win” situation as follows: We prove that, if the treewidth of  $G_{s,t}$  is “sufficiently large”, then  $(G, s, t, k)$  is a YES-instance of LONGEST DETOUR. Otherwise the treewidth is small, and we use a known treewidth-based dynamic programming algorithm for computing the longest  $(s, t)$ -path. Hence the algorithm builds upon the following subroutines:

1. The algorithm from Corollary 7, computing the relevant part  $G_{s,t}$  of  $G$  in time  $O(n + m)$ .
2. COMPUTE TREewidth( $G, w$ ) from Proposition 1, which is given  $G$  and  $w \in \mathbb{N}$  as input, and either constructs a tree-decomposition  $T$  of  $G$  whose width is bounded by  $5w + 4$ , or outputs LARGE. If the algorithm outputs LARGE, then  $\text{tw}(G) > w$  holds. The running time is  $2^{O(w)} \cdot n$ .
3. LONGEST PATH( $G, T, s, t$ ) from Proposition 2, which is given  $G, s, t$  and additionally a tree-decomposition  $T$  of  $G$ , and outputs a longest  $(s, t)$ -path in  $G$ . The running time is  $2^{O(w)} \cdot n^{O(1)}$ , where  $w$  denotes the width of  $T$ .

We now formalize what we mean by “sufficiently large” treewidth.

**Definition 8.** A function  $f : \mathbb{N} \rightarrow \mathbb{N}$  is *detour-enforcing* if, for all  $k \in \mathbb{N}$  and all graphs  $G$  with vertices  $s$  and  $t$ , the following implication holds: If  $\text{tw}(G_{s,t}) > f(k)$ , then  $G$  contains an  $(s, t)$ -path of length at least  $d_G(s, t) + k$ .

**Theorem 9.** *The function  $f : k \mapsto 32k + 2$  is detour-enforcing.*

We defer the proof of this theorem to the next section, and instead state Algorithm D, which uses  $f$  to solve LONGEST DETOUR. Algorithm D turns out to be an FPT-algorithm already when any detour-enforcing function  $f$  is known (as long as it is polynomial-time computable), and it becomes faster when detour-enforcing  $f$  of slower growth are used.

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**Algorithm D** (LONGEST DETOUR) *Given  $(G, s, t, k)$ , this algorithm decides whether the graph  $G$  contains an  $(s, t)$ -path of length at least  $d_G(s, t) + k$ .*

**D1** (Restrict to relevant part) Compute  $G_{s,t}$  using Corollary 7.

**D2** (Compute shortest path) Compute the distance  $d$  between  $s$  and  $t$  in  $G_{s,t}$ .

**D3** (Compute tree-decomposition) Call COMPUTE TREEWIDTH( $G_{s,t}, f(k)$ ).

**D3a** (Small treewidth) If the subroutine found a tree-decomposition  $T$  of width at most  $f(k)$ , call LONGEST PATH( $G_{s,t}, T, s, t$ ). Output YES if there is an  $(s, t)$ -path of length at least  $d + k$ , otherwise output NO.

**D3b** (Large treewidth) If the subroutine returned LARGE, output YES.

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We prove the running time and correctness of Algorithm D.

**Lemma 10.** *For every polynomial-time computable detour-enforcing function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , Algorithm D solves LONGEST DETOUR in time  $2^{O(f(k))} \cdot n^{O(1)}$ .*

*Proof.* Using COMPUTE TREEWIDTH( $G_{s,t}, f(k)$ ), we first determine in time  $2^{O(f(k))} \cdot n^{O(1)}$  whether  $\text{tw}(G_{s,t}) \leq f(k)$ .

- If  $\text{tw}(G_{s,t}) \leq f(k)$ , then COMPUTE TREEWIDTH yields a tree decomposition  $T$  of  $G_{s,t}$  whose width is bounded by  $5 \cdot f(k) + 4$ . We invoke the algorithm for LONGEST PATH to compute a longest  $(s, t)$ -path in  $G_{s,t}$  and we output YES if and only if its length is at least  $d(s, t) + k$ . Since the  $(s, t)$ -paths in  $G$  are precisely the  $(s, t)$ -paths in  $G_{s,t}$ , this answer is correct. The running time of this step is at most  $2^{O(f(k))} \cdot n^{O(1)}$ .
- If  $\text{tw}(G_{s,t}) > f(k)$ , we output YES. Since  $f$  is detour-enforcing, the graph  $G$  indeed contains an  $(s, t)$ -path of length at least  $d_G(s, t) + k$ .

We conclude that Algorithm D is correct and observe also that its running time is bounded by  $2^{O(f(k))} \cdot n^{O(1)}$ . ■

Theorem 9 and Lemma 10 imply a  $2^{O(k)} \cdot n^{O(1)}$  time algorithm for LONGEST DETOUR.

### 3.2 Overview of the proof of Theorem 9

In our proof of Theorem 9, large subdivisions of  $K_4$  play an important role. Intuitively speaking, a sufficiently large subdivision of  $K_4$  in  $G_{s,t}$  allows us to route some  $(s, t)$ -path through it and then exhibit a long detour within that subdivision.

**Definition 11.** For  $k \in \mathbb{N}$ , a graph  $F$  is a  $K_4^{\geq k}$  if it can be obtained by subdividing each edge of  $K_4$  at least  $k$  times. Please note that the numbers of subdivisions do not need to agree for different edges.

We show in Section 3.3 that graphs  $G$  containing  $K_4^{\geq k}$  subgraphs in  $G_{s,t}$  have  $k$ -detours.

**Lemma 12.** *Let  $G$  be a graph and  $k \in \mathbb{N}$ . If  $G_{s,t}$  contains a  $K_4^{\geq k}$  subgraph, then  $G$  contains an  $(s, t)$ -path of length at least  $d_G(s, t) + k$ .*

Since the graph obtained by subdividing each edge of  $K_4$  exactly  $k$  times is a planar graph on  $O(k)$  vertices, the Excluded Grid Theorem yields a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that every graph of treewidth at least  $f(k)$  contains some  $K_4^{\geq k}$  minor. Furthermore, since every  $K_4^{\geq k}$  has maximum degree 3, this actually shows that  $G$  contains some  $K_4^{\geq k}$  as a *subgraph*. Thus, Lemma 12 implies that  $f$  is detour-enforcing, and a proof of this lemma immediately implies a weak version of Theorem 9.

By recent improvements on the Excluded Grid Theorem [7, 10], the function  $f$  above is at most a polynomial. However, even equipped with this deep result we cannot obtain a single-exponential algorithm for LONGEST DETOUR using the approach of Lemma 10: It would require  $f$  to be linear. In fact, excluding grids is too strong a requirement for us, since every function  $f$  obtained as a corollary of the full Excluded Grid Theorem must be super-linear [33]. We circumvent the use of the Excluded Grid Theorem and prove the following lemma from more basic principles.

**Lemma 13.** *For graphs  $G$  and  $k \in \mathbb{N}$ , if  $\text{tw}(G) \geq 32k + 2$ , then  $G$  contains a  $K_4^{\geq k}$  subgraph.*

Together, Lemmas 13 and 12 imply Theorem 9.

*Proof (of Theorem 9).* Let  $G$  and  $s, t \in V(G)$  and  $k \in \mathbb{N}$  be such that  $\text{tw}(G_{s,t}) > f(k)$ . By Lemma 13, the graph  $G_{s,t}$  contains a  $K_4^{\geq k}$  subgraph, so Lemma 12 implies that  $G$  contains an  $(s, t)$ -path of length  $d_G(s, t) + k$ . This shows that  $f$  is indeed detour-enforcing. ■

### 3.3 Proof of Lemma 12: Rerouting in subdivided tetrahedra

Let  $(G, s, t, k)$  be an instance for LONGEST DETOUR such that  $G_{s,t}$  contains a  $K_4^{\geq k}$  subgraph  $M$ . We want to prove that  $G_{s,t}$  has a path of length at least  $d_G(s, t) + k$ ; in fact, we construct the desired detour entirely in the subgraph  $M$ , for which reason we first need to route some  $(s, t)$ -path through  $M$ .

**Lemma 14.** *There are two distinct vertices  $u, v \in V(M)$  and two vertex-disjoint paths  $P_s$  and  $P_t$  in  $G$  such that  $P_s$  is an  $(s, u)$ -path,  $P_t$  is a  $(v, t)$ -path, and they only intersect with  $V(M)$  at  $u$  and  $v$ .*

The proof of this lemma uses the fact that every block in the block-cut tree is biconnected.

*Proof.* Since  $K_4^{\geq k}$  is biconnected,  $M$  is contained in a single block  $C$  of  $G_{s,t}$ . By Corollary 6, the block-cut tree of  $G_{s,t}$  is a path. Let  $s'$  be the cut vertex preceding  $C$  in this block-cut tree (or  $s' = s$  if  $C$  is the first block) and let  $t'$  be the cut vertex following  $C$  in the tree (or  $t' = t$  if  $C$  is the last block). Then clearly  $s', t' \in C$ .

By the properties of the block-cut tree, there is an  $(s, s')$ -path  $p_s$  and a  $(t', t)$ -path  $p_t$ , the two paths are vertex disjoint, and they intersect  $C$  only in  $s'$  and  $t'$ , respectively. We let  $p_s$  be the first segment of  $P_s$  and  $p_t$  be the last segment of  $P_t$ . It remains to complete  $P_s$  and  $P_t$  within  $C$  using two disjoint paths that lead to  $M$ . Since  $C$  is biconnected, there are two vertex-disjoint paths from  $\{s', t'\}$  to  $V(M)$ . Moreover, both paths can be shortened if they intersect  $V(M)$  more than once. Hence we have an  $(s', u)$ -path  $p_1$  for some  $u \in V(M)$  and a  $(v, t')$ -path  $p_2$  for some  $v \in V(M)$  with the property that  $p_1$  and  $p_2$  are disjoint and their internal vertices avoid  $V(M)$ .

We concatenate the paths  $p_s$  and  $p_1$  to obtain  $P_s$  and the paths  $p_2$  and  $p_t$  to obtain  $P_t$ . ■

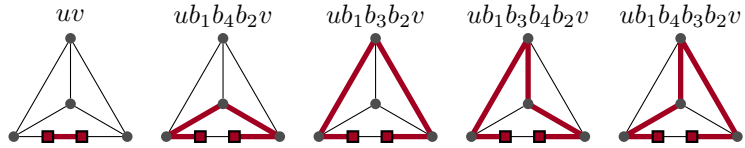
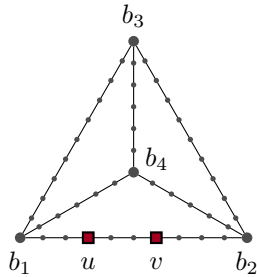
Next we show that every  $K_4^{\geq k}$ -graph  $M$  contains long detours.

**Lemma 15.** *Let  $M$  be a  $K_4^{\geq k}$ -graph. For every two distinct vertices  $u, v \in V(M)$ , there is a  $(u, v)$ -path of length at least  $d_M(u, v) + k$  in  $M$ .*

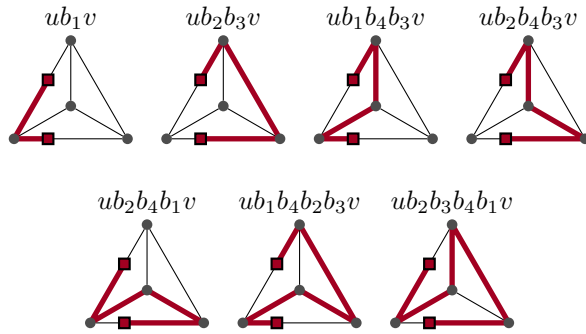
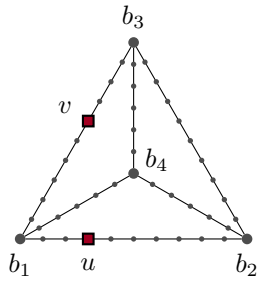
The proof idea is to distinguish cases depending on where  $u, v$  lie in  $M$  relative to each other. For each case, we can exhaustively list all  $(u, v)$ -paths (see Figure 1). We do not quite know the lengths of these paths, but we do know that each has length at least  $d_M(u, v)$ ; moreover, each  $(b_i, b_j)$ -path in  $M$  for two distinct degree-3 vertices  $b_i$  and  $b_j$  has length at least  $k$ , since we subdivided  $K_4$  at least  $k$  times. The claim of Lemma 15 is that one of the  $(u, v)$ -paths must have length at least  $d_M(u, v)$ . To prove this, we set up a linear program where the variables are  $d_M(u, v)$ ,  $k$ , and the various path segment lengths; its infeasibility informs us that indeed a path that is longer by  $k$  must exist.

*Proof (of Lemma 15).* Let  $M$  be a  $K_4^{\geq k}$ -graph, let  $u, v \in V(M)$ , and let  $b_1, \dots, b_4$  denote the four degree-3 vertices of  $M$ . Let  $P_u$  be a path in  $M$  that realizes an edge of  $K_4$  and satisfies  $u \in V(P_u)$ , and let  $P_v$  be such a path with  $v \in V(P_v)$ . We distinguish three cases as depicted in Figure 1:

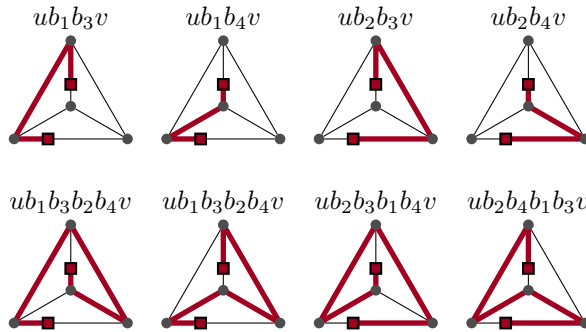
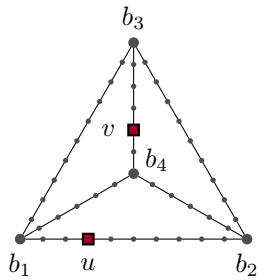
1. The two paths are the same, that is,  $P_u = P_v$ .
2. The two paths share a degree-3 vertex, that is,  $|V(P_u) \cap V(P_v)| = 1$ .
3. The two paths are disjoint, that is,  $V(P_u) \cap V(P_v) = \emptyset$ .



(a)  $u$  and  $v$  lie on the same subdivided edge.



(b)  $u$  and  $v$  lie on two adjacent subdivided edges.



(c)  $u$  and  $v$  lie on two non-adjacent subdivided edges.

Fig. 1. *Left:* Depicted are all three possible cases for the relative positions of vertices  $u$  and  $v$  (red squares) in a subdivided tetrahedron  $K_4^{\geq k}$  with degree-3 vertices  $b_1, \dots, b_4$  (gray dots) and at least  $k = 5$  subdivision vertices (small gray dots). *Right:* An exhaustive list of all  $(u, v)$ -paths (thick red); in each of the three cases, Lemma 15 implies that the longest among them is at least  $k$  longer than the shortest one.

By the symmetries of  $K_4$ , this case distinction is exhaustive. Since  $K_4$  has automorphisms that map any edge to any other edge, we can further assume that  $P_u$  is the path implementing the edge  $b_1b_2$  such that  $P_u$  visits the vertices  $b_1, u, v$ , and  $b_2$  in this order, see Figure 1.

We exhaustively list the set  $\mathcal{P}$  of  $(u, v)$ -paths of  $M$  in Figure 1. Each path is uniquely specified by the sequence of the degree-3 vertices it visits. For example, consider the path  $ub_1b_4b_2v$ : This path consists of the four edge-disjoint segments  $ub_1, b_1b_4, b_4b_2$ , and  $b_2v$ ; in the example figure, these segments have length 3, 6, 6, and 4, respectively. Given a path  $P \in \mathcal{P}$ , let  $S(P)$  be the set of its segments between  $u, v$ , and the degree-3 vertices. For a path or a path segment  $s$ , we denote its length by  $\ell(s)$ .

Since  $M$  is a  $K_4^{\geq k}$ , every edge of  $K_4$  is realized by a path of length at least  $k$  in  $M$ . Hence,  $\ell(b_ib_j) \geq k$  holds for all  $i, j$  with  $i \neq j$ . Moreover, we have  $\ell(b_1b_2) = \ell(b_1u) + \ell(uv) + \ell(vb_2)$  in case 1. Let  $d = d_M(u, v)$ ; clearly  $\ell(P) \geq d$  holds for all  $P \in \mathcal{P}$ . Our goal is to show that  $M$  has a  $(u, v)$ -path  $P$  with  $\ell(P) \geq d + k$ . To this end, we treat  $d, k$ , and all path segment lengths  $\ell(b_ib_j)$  for  $i \neq j$  and  $\ell(b_1u), \ell(uv), \ell(vb_2)$  as variables in a system of linear inequalities and establish that the claim holds if this system is unsatisfiable:

$$\ell(b_ib_j) \geq k, \quad \text{for all } i, j \text{ with } i \neq j, \quad (1)$$

$$\ell(b_1u) + \ell(uv) + \ell(vb_2) = \ell(b_1b_2), \quad (2)$$

$$\sum_{s \in S(P)} \ell(s) \geq d, \quad \text{for all } P \in \mathcal{P}, \quad (3)$$

$$\sum_{s \in S(P)} \ell(s) \leq d + k - 1, \quad \text{for all } P \in \mathcal{P}. \quad (4)$$

This system has eleven variables. Please note that  $d$  and  $k$  are also considered as variables in our formulation. The constraints in (1) express that  $M$  realizes each edge of  $K_4$  by a path of length at least  $k$ . The constraints in (2) express that  $u$  and  $v$  lie on the path  $b_1b_2$  and break it up into segments. The constraints in (3) express that no  $(u, v)$ -path is shorter than  $d$  in length, and the constraints in (4) express that every  $(u, v)$ -path has length strictly less than  $d + k$ . We prove in the appendix that this linear program is infeasible, and so every setting for the variables that satisfies (1)–(3) must violate an inequality from (4); this means that  $M$  must contain a  $(u, v)$ -path of length at least  $d + k$  in case 1.

The proof is analogous when  $u$  and  $v$  are on different subdivided edges of the subdivided tetrahedron; what changes is the set  $\mathcal{P}$  of  $(u, v)$ -paths as well as the constraints (2). In case 2, we may assume by symmetry that  $P_u$  is the  $b_1b_2$ -path and  $P_v$  is the  $b_1b_3$ -path of  $M$ . Then (2) is replaced with the following constraints.

$$\begin{aligned} \ell(b_1u) + \ell(ub_2) &= \ell(b_1b_2), \\ \ell(b_1v) + \ell(vb_3) &= \ell(b_1b_3). \end{aligned}$$

The resulting linear equation system in case 2 has twelve variables and is again infeasible. Similarly, in case 3, the constraints (2) are replaced with the following.

$$\begin{aligned} \ell(b_1u) + \ell(ub_2) &= \ell(b_1b_2), \\ \ell(b_3v) + \ell(vb_4) &= \ell(b_3b_4). \end{aligned}$$

This also leads to an infeasible linear equation system with twelve variables.

We conclude that, no matter how  $u$  and  $v$  lie relative to each other in  $M$ , there is always a  $(u, v)$ -path that is at least  $k$  longer than a shortest one.  $\blacksquare$

This allows us to conclude Lemma 12 rather easily.

*Proof (of Lemma 12).* Let  $d = d_G(s, t)$  be the length of a shortest  $(s, t)$ -path in  $G$ . Let  $M$  be a  $K_4^{\geq k}$  in  $G_{s,t}$ , and let  $P_s, P_t, u$ , and  $v$  be the objects guaranteed by Lemma 14. Let  $P_{uv}$  be a shortest  $(u, v)$ -path that only uses edges of  $M$ ; its length is  $d_M(u, v)$ . Since the combined path  $P_s, P_{uv}, P_t$  is an  $(s, t)$ -path, its length is at least  $d$ .

Finally, Lemma 15 guarantees that there is a  $(u, v)$ -path  $Q_{uv}$  in  $M$  whose length is at least  $d_M(u, v) + k$ . Therefore, the length of the  $(s, t)$ -path  $P_s, Q_{uv}, P_t$  satisfies

$$\begin{aligned} \ell(P_s) + \ell(Q_{uv}) + \ell(P_t) &\geq \ell(P_s) + (d_M(u, v) + k) + \ell(P_t) \\ &= \ell(P_s) + \ell(P_{uv}) + \ell(P_t) + k \geq d + k. \end{aligned}$$

We constructed a path of at least length  $d + k$  as required.  $\blacksquare$



### 3.4 Proof of Lemma 13: Large treewidth entails subdivided tetrahedra

To prove Lemma 13, we require some preliminaries from graph minors theory, among them a term for vertex sets that enjoy very favorable connectivity properties.

**Definition 16 ([14]).** Let  $G$  be a graph and  $A, B \subseteq V(G)$ . The pair  $(A, B)$  is a *separation* in  $G$  if the sets  $A \setminus B$  and  $B \setminus A$  are non-empty and no edge runs between them. The *order* of  $(A, B)$  is the cardinality of  $A \cap B$ .

For  $S \subseteq V(G)$ , we say that  $S$  is *linked* in  $G$  if, for every  $X, Y \subseteq S$  with  $|X| = |Y|$ , there are  $|X|$  vertex-disjoint paths between  $X$  and  $Y$  that intersect  $S$  exactly at its endpoints.

The notion of left-containment conceptually connects separators and minor models.

**Definition 17.** Let  $H$  be a graph on  $k \in \mathbb{N}$  vertices. Recall Definition 3 for the notion of a minor model. We say that  $(A, B)$  *left-contains*  $H$  if  $G[A]$  contains a minor model  $f$  of  $H$  with  $|f(v) \cap (A \cap B)| = 1$  for all  $v \in V(H)$ .

With these definitions at hand, we can adapt a result by Leaf and Seymour [26] to prove the following lemma on topological minor containment in graphs of sufficiently large treewidth. For any forest  $F$  on  $k$  vertices, with maximum degree 3, it asserts that graphs  $G$  of treewidth  $\Omega(k)$  admit a separation such that one side contains  $F$  as a topological minor, with the branch vertices of this topological minor being contained in  $A \cap B$  and linked in  $G$ . We will use this lemma to complete the topological  $F$ -minor in  $G[A]$  to a larger graph by using disjoint paths between vertices in  $A \cap B$ .

**Lemma 18.** *Let  $F$  be a forest on  $k > 0$  vertices with maximum degree 3 and let  $G$  be a graph. If  $\text{tw}(G) \geq \frac{3}{2}k - 1$ , then  $G$  has a separation  $(A, B)$  of order  $|V(F)|$  such that:*

1. *There is a topological minor model  $(f, p)$  of  $F$  in  $G[A]$ .*
2. *For every vertex  $v \in V(F)$  of degree  $\leq 2$ , we have  $f(v) \in A \cap B$ .*
3.  *$A \cap B$  is linked in  $G[B]$ .*

We defer the proof of this lemma to §3.5. Building upon Lemma 18, we prove Lemma 13 by adapting work of Raymond and Thilikos [31], who used a variant of Lemma 18 to prove the existence of  $k$ -wheel minors in graphs of treewidth  $\Omega(k)$ . To this end, let  $T$  and  $P$  be obtained by  $k$ -subdividing the full binary tree with 8 leaves, and the path with 8 vertices, respectively. We invoke Lemma 18 with  $F$  instantiated to the disjoint union  $T \cup P$ . Since  $F$  has  $21k + 2$  vertices, we obtain from Lemma 18 that any graph  $G$  with  $\text{tw}(G) \geq 32k + 2 \geq \frac{3}{2} \cdot (21k + 2)$  has a separation  $(A, B)$  of order  $|V(F)|$  that contains  $F$  in  $G[A]$  and has  $A \cap B$  linked in  $G[B]$ .

Let  $X_F$  denote the eight leaves of  $T$ , and let  $Y_F$  denote the eight non-subdivision vertices of  $P$ . Furthermore, let  $X_G, Y_G \subseteq A \cap B$  denote the images of  $X_F$  and  $Y_F$  in  $G[A]$  under a topological minor model guaranteed by Lemma 18. Since  $A \cap B$  is linked, we can find eight disjoint paths connecting  $X_G$  and  $Y_G$  in  $G[B]$ . We then prove that, regardless of how these paths connect  $X_G$  and  $Y_G$ , they always complete the topological minor model of  $F$  to one of  $K_4^{\geq k}$  in  $G$ . Lemma 13 then follows.

*Proof (of Lemma 13).* Let  $k \in \mathbb{N}$  and let  $G$  be a graph with  $\text{tw}(G) \geq 32k + 2$ . As before, let  $T$  denote the full binary tree with 8 leaves, with root  $r$ , after each edge was subdivided  $k$  times. Let  $P$  denote the path on 8 vertices after subdividing each edge  $k$  times.

We write  $X_F = \{x_1, \dots, x_8\}$  for the leaves of  $T$ , and we write  $Y_F = \{y_1, \dots, y_8\}$  for the vertices in  $P$  that were not obtained as subdivision vertices. Finally, we write  $F$  for the disjoint union  $T \cup P$  and consider  $X_F, Y_F \subseteq V(F)$ . Note that  $|V(F)| = 21k + 2$  and that the degree of all vertices in  $X_F \cup Y_F$  is bounded by 2.

By Lemma 18, there is a separation  $(A, B)$  in  $G$  of order  $|V(F)|$  such that  $A \cap B$  is linked, and there is a topological minor model  $(f, p)$  of  $F$  in  $G[A]$  with  $f(X_F \cup Y_F) \subseteq A \cap B$ . We write  $X_G = \{f(v) \mid v \in X_F\}$  and  $Y_G = \{f(v) \mid v \in Y_F\}$ . In the following, we aim at completing the subgraph induced by  $(f, p)$  in  $G$  to a  $K_4^{\geq k}$  subgraph.

Since  $A \cap B$  is linked in  $G[B]$ , there are vertex-disjoint paths  $L_1, \dots, L_8$  between  $X_G$  and  $Y_G$  in  $G[B]$  that avoid  $A \cap B$  except at their endpoints. For  $i \in [8]$ , denote the endpoints of  $L_i$  in  $X_G$  and  $Y_G$  by  $s_i$  and  $t_i$ , respectively. Assume without limitation of generality (by reordering paths) that  $t_i = f(y_i)$  holds for all  $i \in [8]$ . Furthermore, for  $x \in X_G$ , write  $\sigma(x)$  for the vertex of  $Y_G$  that  $x$  is connected to via its path among  $L_1, \dots, L_8$ .

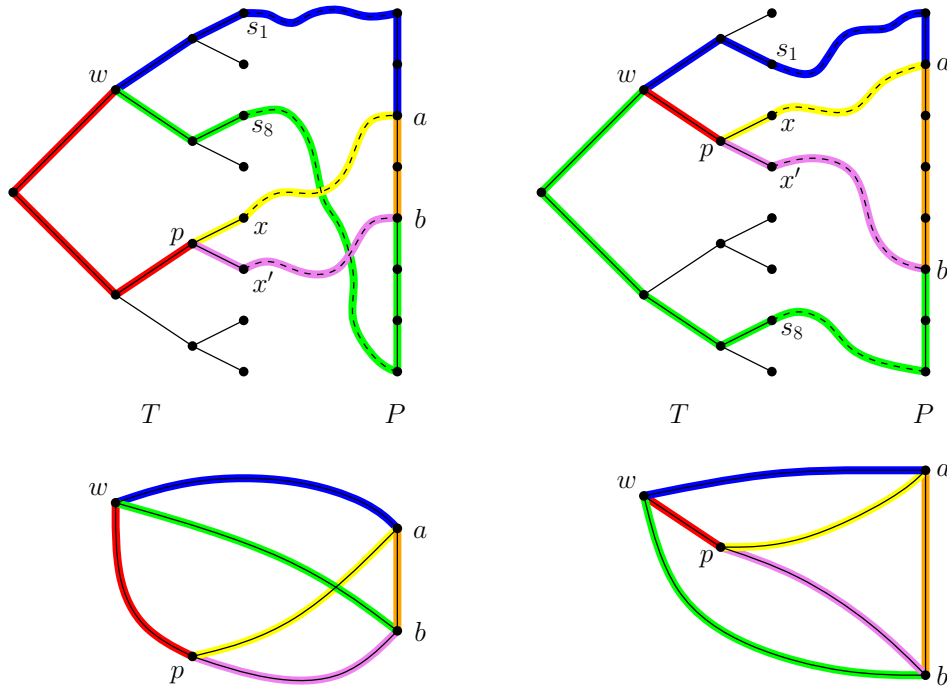


Fig. 2. The two cases relevant in the proof of Lemma 13. The top figures depict the cases and the bottom figures show the corresponding  $K_4^{>k}$  subgraphs. The *bullets* correspond to the main vertices of the tree  $T$  and the path  $P$ . The *lines* represent a path of length  $k$ , created by subdividing the original tree and path. The *dashed curves* correspond to the linkage that is guaranteed to exist between the 8 leaves of  $T$  and the 8 main vertices of  $P$ . The vertices  $w$ ,  $p$ ,  $a$ , and  $b$  are the branch vertices of the  $K_4^{>k}$ -graph that we find. The vertex  $s_1$  is the linkage-partner of the first path vertex, and  $s_8$  is the linkage-partner of the last path vertex. *Top left:* In Case 1,  $s_1$  and  $s_8$  have a least common ancestor  $w$  that is not the root. *Top right:* In Case 2,  $s_1$  and  $s_8$  have the root as their least common ancestor. *Bottom:* A schematic view of the corresponding  $K_4^{>k}$  subgraphs, where each of the six subdivided edges (that is, paths of length greater than  $k$ ) is shown in a different color. The same color is used to highlight the path in the graph above.

Let  $S$  denote the image of  $T$  under  $(f, p)$ , which is a tree; let  $\text{root}(S) = f(r)$ . Write  $S_1, S_2$  for the two subtrees of  $S$  rooted at the children of  $\text{root}(S)$ . Let  $\text{lca}(s_1, s_8)$  denote the lowest common ancestor of  $s_1$  and  $s_8$  in  $S$ . We distinguish two cases (see Figure 2).

**Case 1:** We have  $\text{lca}(s_1, s_8) \neq \text{root}(S)$ . That is,  $s_1$  and  $s_8$  are both in  $S_1$  or both in  $S_2$ . Assume without limitation of generality that  $s_1, s_8 \in V(S_1)$ , as the argument proceeds symmetrically otherwise. Let  $x$  and  $x'$  be two distinct leaves of  $S_2$ . Then we find a  $K_4^{\geq k}$  in  $G$  by defining branch vertices  $w = \text{lca}(s_1, s_8)$ ,  $p = \text{lca}(x, x')$ ,  $a = \sigma(x)$ , and  $b = \sigma(x')$ . Note that  $p \notin \{x, x'\}$  and that the four vertices are distinct.

We realize the edge  $pw$  along the  $(p, w)$ -path present in  $S$ , and  $ab$  along the  $(a, b)$ -path present in  $P$ . We realize  $pa$  by concatenating the  $(p, x)$ -path in  $S$  and the  $(x, a)$ -path in  $G[B]$ , and we realize  $pb$  likewise. To realize  $wa$ , we proceed as follows: If  $a$  precedes  $b$  in the order on  $P$ , then concatenate the  $(w, s_1)$ -path in  $S$  with  $L_1$  and the  $(y_1, a)$ -path in  $P$ . If  $b$  precedes  $a$ , then concatenate the  $(w, s_8)$ -path in  $S$  with  $L_8$  and the  $(y_8, a)$ -path in  $P$ . Realize  $wb$  symmetrically. Then every edge between pairs in  $\{w, p, a, b\}$  is realized, and it is so by a path of length at least  $k$ . This gives a topological minor model of  $K_4^{\geq k}$  in  $G$ .

**Case 2:** We have  $\text{lca}(s_1, s_8) = \text{root}(S)$ . That is,  $s_1$  and  $s_8$  are in different subtrees  $S_1$  and  $S_2$ . Let  $R$  be a subtree of height 2 in  $S$  that is disjoint from the  $(s_1, s_8)$ -path in  $S$ . It is easy to verify that such a subtree indeed exists; denote its root by  $p$ , its leaves by  $x, x'$ , and its parent in  $S$  by  $w$ . Furthermore, define  $a = \sigma(x)$  and  $b = \sigma(x')$ . We declare  $\{w, p, a, b\}$  as branch vertices and connect them as in the previous case.

In both cases, the constructed topological minor model shows that  $G$  contains a  $K_4^{\geq k}$  subgraph. This proves the lemma.  $\blacksquare$

### 3.5 Proof of Lemma 18

For the following part, we need to define the notion of a *minor model*. Note that only *topological* minor models were defined in the main text.

**Definition 19.** Let  $H$  and  $G$  be undirected. A *minor model* of  $H$  in  $G$  is a function  $f : V(H) \rightarrow 2^{V(G)}$  such that

1.  $G[f(v)]$  is connected for all  $v \in V(H)$ , and
2.  $f(u) \cap f(v) = \emptyset$  for all  $u, v \in V(H)$  with  $u \neq v$ , and
3. for all  $uv \in E(H)$ , there is an edge in  $G$  from a vertex in  $f(u)$  to a vertex in  $f(v)$ .

Furthermore, we require the notion of left-containment:

**Definition 20.** If  $G$  is a graph with separation  $(A, B)$ , we say that  $(A, B)$  *left-contains*  $H$  if  $G[A]$  contains a minor model  $f$  of  $H$  with  $|f(v) \cap (A \cap B)| = 1$

We can now state a lemma by Leaf and Seymour [26] that can be easily adapted to obtain Lemma 18.

**Lemma 21 ([26]).** *Let  $F$  be a forest on  $k > 0$  vertices and let  $G$  be a graph. If  $\text{tw}(G) \geq \frac{3}{2}k - 1$ , then there exists a separation  $(A, B)$  of order  $|V(F)|$  in  $G$  such that  $(A, B)$  left-contains  $F$ , and  $A \cap B$  is linked in  $G[B]$ .*

Finally, we prove Lemma 18.

*Proof (of Lemma 18).* Let  $(A, B)$  be the separation from Lemma 21. Then, the third condition holds so we just need to prove the first two conditions. By the same lemma,  $G[A]$  contains a minor model  $f'$  of  $F$  with  $|f'(v) \cap (A \cap B)| = 1$  for all  $v \in V(F)$ . It will be convenient to fix a spanning tree inside each  $G[f'(v)]$  for every  $v \in V(F)$ ; let us denote it by  $T(v)$ . Additionally, for every  $v_1 v_2 \in E(F)$ , we will fix one edge  $u_1 u_2 \in E(G)$  such that  $u_i \in f'(v_i)$ . We define the topological minor model  $(f, p)$  as follows. For every  $v \in V(F)$  of degree  $\leq 2$ , let  $f(v) = u$  where  $u \in f'(v) \cap (A \cap B)$ , the only such vertex. This satisfies the second condition of the corollary.

For  $v \in V(F)$  of degree 3, let  $v_1, v_2, v_3$  be its adjacent vertices in  $F$ . Then, for each edge  $vv_i$  let  $u_i u'_i$  be the corresponding fixed edge where  $u_i \in f'(v)$  (and  $u'_i \in f'(v_i)$ ). If  $u_1, u_2, u_3$  are not distinct, then

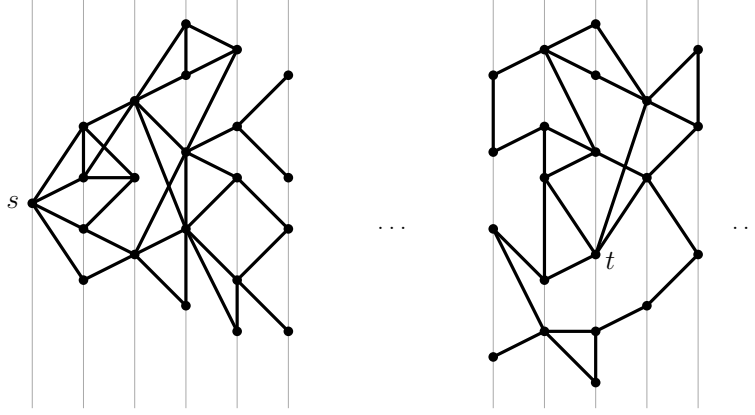


Fig. 3. The *fine vertical lines* in this drawing of an example graph represent distance layers, that is, vertices whose distance  $d(v)$  from  $s$  is equal.

choose one of them to be  $f(v)$ . If they are distinct, then take the spanning tree  $T(v)$ , root it at  $u_3$  and let  $f(v)$  be the vertex that is the lowest common ancestor of  $u_1$  and  $u_2$ .

We defined  $f$ , now we define the paths  $p$ . For edge  $v_1v_2 \in E(F)$ , let  $p(v_1v_2)$  be the path defined by following the spanning tree  $T(v_1)$  from  $f(v_1)$  to the edge  $u_1u_2 \in E(G)$  that we fixed for  $v_1v_2 \in E(F)$ , and then following the spanning tree  $T(v_2)$  to  $f(v_2)$ . It remains to show that the paths are vertex-disjoint, except for their endpoints. Every path uses exactly one edge not in  $G[f'(v)]$  for some  $v \in V(F)$ ; these edges are distinct as they correspond to different edges of  $F$ . Therefore, path intersections could only happen inside one of the  $G[f'(v)]$  graphs. However, the selected vertex  $f(v)$  is connected to the vertices  $u_1, u_2, u_3$  defined in the previous paragraph via disjoint paths in the spanning tree, proving that the paths  $p$  are vertex-disjoint within  $G[f'(v)]$  apart from the endpoints. This proves that  $(f, p)$  is a topological minor model and concludes the proof of the corollary. ■

## 4 Dynamic programming algorithm for exact detour

We devise an algorithm for EXACT DETOUR using a reduction to EXACT PATH, the problem that is given  $(G, s, t, k)$  to determine whether there is an  $(s, t)$ -path of length exactly  $k$ .

**Theorem 22.** EXACT DETOUR is fixed-parameter tractable. In particular, it has a bounded-error randomized algorithm with running time  $2.746^k \text{ poly}(n)$ , and a deterministic algorithm with running time  $6.745^k \text{ poly}(n)$ .

*Proof.* Let  $(G, s, t, k)$  be an instance of EXACT DETOUR. We use Lemma 23 and run the deterministic polynomial-time reduction in algorithm A on this instance, which makes queries to EXACT PATH whose parameter  $k'$  is at most  $2k + 1$ . To answer these queries, we use the best known algorithm as a subroutine. Using the deterministic algorithm by Zehavi [36], we obtain a running time of  $2.597^{k'} \cdot \text{poly}(n) \leq 6.745^k \cdot \text{poly}(n)$  for EXACT DETOUR. Using the randomized algorithm by Björklund et al. [3], we obtain a running time of  $1.657^{k'} \cdot \text{poly}(n) \leq 2.746^k \cdot \text{poly}(n)$ . ■

Before we state the algorithm, let us introduce some notation. Let  $s, t \in V(G)$ . For any  $x \in V(G)$ , we abbreviate  $d_G(s, x)$ , that is, the distance from  $s$  to  $x$  in  $G$ , with  $d(x)$ , and we let the  $i$ -th layer of  $G$  be the set of vertices  $x$  with  $d(x) = i$  (see Figure 3). For  $u, v \in V(G)$  with  $d(u) < d(v)$ , we write  $G_{[u, v]}$  for the graph  $G[X]$  induced by the vertex set  $X$  that contains  $u, v$ , and all vertices  $x$  with  $d(u) < d(x) < d(v)$  (see Figure 4). We also write  $G_{[u, \infty]}$  for the graph  $G[X]$  induced by the vertex set  $X$  that contains  $u$  and all vertices  $x$  with  $d(u) < d(x)$ . These graphs can be computed in linear time using breadth-first search starting at  $s$ . We now describe an algorithm for EXACT DETOUR that makes queries to an oracle for EXACT PATH.

The general idea is as follows. Let  $G$  be an undirected graph, and consider an  $(s, t)$ -path  $P$  of length  $d + k$  where  $d = d_G(s, t)$ , and let  $x$  be a token that travels along this path from  $s$  to  $t$ . As the token advances one step in the path, the number  $d(x)$  can be incremented, decremented, or stay the same. When  $x$  moves from  $s$  to  $t$ , we must increment  $d(x)$  at least  $d$  times, can decrement it at most  $k/2$  times,

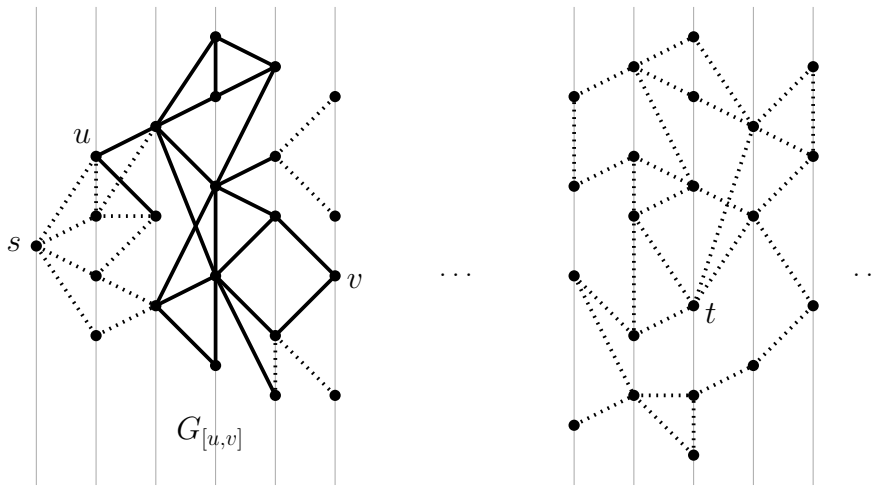


Fig. 4. The *solid* edges are the edges of an example graph  $G_{[u,v]}$ ; the other edges of  $G$  are *dashed*.

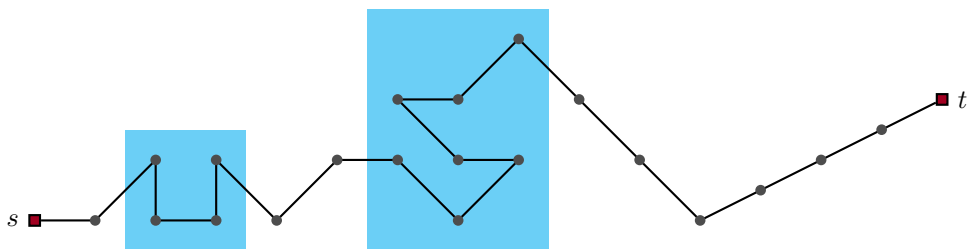


Fig. 5. This is an example of a long  $(s, t)$ -path in a graph with  $d_G(s, t) = 15$ ; the distance from  $s$  increases from left to right as in Figure 3. The path has length 21, so it has  $k = 6$  more edges than a shortest path. Each of the five marked layers (*cyan shading*) contains more than one vertex of the path, and any path of length 21 can have at most 6 such layers.

and keep it unchanged at most  $k$  times; the reason is that the path must reach  $t$  but must use exactly  $k$  edges more than a shortest path. The crucial observation is that there are at most  $k$  different layers whose intersection with the path  $P$  contains more than one vertex (see Figure 5). The idea for the algorithm is to guess the layers with more than one vertex and run an algorithm for EXACT PATH on them.

---

**Algorithm A** (EXACT DETOUR) *Given  $(G, s, t, k)$ , this algorithm decides whether the graph  $G$  contains an  $(s, t)$ -path of length exactly  $d_G(s, t) + k$ .*

**A1** (Initialize table) For each  $x \in V(G)$  with  $d(x) \leq d(t)$ , set  $T[x] = \emptyset$ .

*When the algorithm halts, every entry  $T[x]$  of the table is meant to satisfy the following property  $Q_x$ : For each integer  $\ell$  with  $d(t) - d(x) \leq \ell \leq d(t) - d(x) + k$ , the set  $T[x]$  contains  $\ell$  if and only if  $G_{[x, \infty)}$  contains an  $(x, t)$ -path of length  $\ell$ .*

**A2** (Compute entries for the last  $k + 1$  layers) For each  $x \in V(G)$  with  $d(t) - k \leq d(x) \leq d(t)$ , let  $T[x]$  be the set of all integers  $\ell$  with  $\ell \in \{0, \dots, 2k\}$  such that there is an  $(x, t)$ -path of length  $\ell$  in  $G_{[x, \infty)}$  (that is, call EXACT PATH  $(G_{[x, \infty)}, x, t, \ell)$ ).

*When this step finishes, all vertices  $x$  in the last  $k + 1$  layers satisfy property  $Q_x$ .*

**A3** (Inductively fill in earlier layers) For each  $d$  from  $d(t) - k - 1$  down to 0, for each  $x$  with  $d(x) = d$ , and for each  $y$  with  $d(x) < d(y) \leq d(x) + k + 1$ , we do the following:

**A3a** Compute the set  $L$  of all  $\ell' \in \{0, \dots, 2k + 1\}$  such that there is an  $(x, y)$ -path of length  $\ell'$  in  $G_{[x, y]}$  (that is, call EXACT PATH  $(G_{[x, y]}, x, y, \ell')$ ).

**A3b** Set  $T[x] := T[x] \cup (L + T[y])$ .

*We will show that, when all vertices of a layer  $d$  have been considered, all vertices  $x$  in the layers  $d$  and higher satisfy property  $Q_x$ .*

**A4** Accept if and only if  $(d_G(s, t) + k) \in T[s]$  holds.

---

**Lemma 23.** *Algorithm A is a polynomial-time Turing reduction from EXACT DETOUR to EXACT PATH; on instances with parameter  $k$ , all queries have parameter at most  $2k + 1$ .*

*Proof.* The running time of A is polynomially bounded since breadth-first search can be used to discover all partial graphs  $G_{[x, y]}$  and  $G_{[x, \infty)}$ , and we loop at most over every pair of vertices in A2 and A3. For the parameter bound, note that the queries in A2 and A3 are for paths of length at most  $2k$  and  $2k + 1$ , respectively. It remains to prove the correctness.

We execute algorithm A on an instance  $(G, s, t, k)$ . For the correctness, it suffices to prove that property  $Q_s$  holds at the end of the execution: Note that  $\ell$  with  $\ell = d_G(s, t) + k$  lies in the interval  $[d(t) - d(s), d(t) - d(s) + k]$  since  $d(s) = 0$  and  $d(t) = d_G(s, t)$  holds. Moreover, we have  $G_{[s, \infty)} = G$ . Thus  $Q_s$  guarantees that  $\ell \in T[s]$  holds if and only if  $G$  contains an  $(s, t)$ -path of length  $\ell$ , which by step A4 implies that A accepts if and only if  $(G, s, t, k)$  is a yes-instance of EXACT DETOUR. Therefore it remains to prove that  $Q_s$  holds at the end of the execution of A. We do so using the following claim.

*Claim:* For all  $x$  with  $0 \leq d(x) \leq d(t)$ , property  $Q_x$  holds forever after the entry  $T[x]$  is written to for the last time.

We prove this claim by induction on  $d(x)$ . For the base case, let  $x$  be a vertex with  $d(x) \geq d(t) - k$ . The entry  $T[x]$  is only written to in step A2. To prove that  $Q_x$  holds after A2, let  $\ell$  be an integer with  $d(t) - d(x) \leq \ell \leq d(t) - d(x) + k$ . Note that  $d(t) - d(x) \geq 0$  and  $d(t) - d(x) + k \leq d(t) - (d(t) - k) + k \leq 2k$  holds, and so step A2 adds  $\ell$  to  $T[x]$  if and only if the graph  $G_{[x, \infty)}$  contains an  $(x, t)$ -path of length  $\ell$ . Therefore,  $Q_x$  holds forever after A2 has been executed.

For the induction step, let  $x$  be a vertex with  $d(x) < d(t) - k$ . By the induction hypothesis,  $Q_y$  holds for all  $y$  with  $d(y) > d(x)$ . The entry  $T[x]$  is only written to in step A3b, and when it is first written to, the outer  $d$ -loop in A3 has fully processed all layers larger than  $d(x)$ . Thus already when  $T[x]$  is written to for the first time,  $Q_y$  holds for all  $y$  with  $d(y) > d(x)$ . Let  $T$  be the table right after A3b writes to  $T[x]$  for the last time. It remains to prove that  $T[x]$  satisfies  $Q_x$ . Let  $\ell$  be an integer with  $d(t) - d(x) \leq \ell \leq d(t) - d(x) + k$ .

*Claim:* There is an  $(x, t)$ -path of length  $\ell$  if and only if  $T[x]$  contains  $\ell$ .

For the forward direction, let  $P$  be an  $(x, t)$ -path in  $G_{[x, \infty)}$  of length exactly  $\ell$ . There are exactly  $\ell$  vertices  $u \in V(P) \setminus \{x\}$ . Moreover, since every edge  $uv \in E(P)$  satisfies  $|d(u) - d(v)| \leq 1$ , every  $d \in \{d(x) + 1, \dots, d(t)\}$  must have some vertex  $u \in V(P)$  with  $d(u) = d$ . Since  $\ell \leq d(t) - d(x) + k$ , there

are at most  $k$  distinct  $d$  where more than one vertex  $u \in V(P)$  satisfies  $d(u) = d$ . By the pigeon hole principle, there exists an integer  $d$  in the  $(k + 1)$ -element set  $\{d(x) + 1, \dots, d(x) + k + 1\}$  such that there is exactly one vertex  $y \in V(P)$  with  $d(y) = d$ .

Let  $P_{[x,y]}$  be the subpath of  $P$  between  $x$  and  $y$ , and let  $\ell'$  be its length. By construction,  $P_{[x,y]}$  is an  $(x, y)$ -path in  $G_{[x,y]}$ . Moreover, we have  $\ell' \leq \ell - (d(t) - d(y))$  since  $V(P) \setminus \{x\}$  contains  $\ell$  vertices  $u$ , at least  $d(t) - d(y)$  of which satisfy  $d(u) > d(y)$ . By choice of  $\ell$  and  $y$ , we obtain  $\ell' \leq d(y) - d(x) + k \leq 2k + 1$ . For this setting of  $y$  and  $\ell'$ , step A3a detects the path  $P_{[x,y]}$  and  $\ell'$  is added to the set  $L$ . The second piece  $P_{[y,\infty]}$  of the path  $P$  is a  $(y, t)$ -path in  $G_{[y,\infty]}$  of some length  $\ell''$  between  $d(t) - d(y)$  and  $d(t) - d(y) + k$ ; since  $Q_y$  holds when A3b is executed for  $x$  and  $y$ , the set  $T[y]$  contains  $\ell''$ , and so  $\ell = \ell' + \ell''$  gets added to  $T[x]$ . Since elements never get removed from  $T[x]$ , the forward direction of the claim holds.

For the backward direction of the claim, assume that  $T[x]$  contains  $\ell$ . This means that  $\ell$  is added in step A3b during the execution of the algorithm; in particular, consider the variables  $y \in V(G)$ ,  $\ell' \in L$ , and  $\ell'' \in T[y]$  when  $\ell = \ell' + \ell''$  is added to  $T[x]$ . By the induction hypothesis,  $\ell'' \in T[y]$  implies that there is a  $(y, t)$ -path in  $G_{[y,\infty]}$  of length  $\ell''$ . Moreover,  $\ell'$  was set in A3a in such a way that there is an  $(x, y)$ -path of length  $\ell'$  in the graph  $G_{[x,y]}$ . Combined, these two paths yield a single  $(x, t)$ -path in  $G_{[x,\infty]}$  of length  $\ell$ . The backward direction of the claim follows. ■

The randomized algorithm of Björklund et al. [3] is for a variant of EXACT PATH where the terminal vertices  $s$  and  $t$  are not given, that is, any path of length exactly  $k$  yields a YES-instance. Their algorithm applies to our problem as well, with the same running time. We sketch an argument for this observation here. Recall that the idea is to reduce the problem to checking whether a certain polynomial is identically zero – this polynomial is defined by summing over all possible labelled walks of length  $k$  (see [12, Sec. 10.4.3]). We modify the polynomial by adding two leaf-edges, one incident to  $s$  and one to  $t$ , and restricting our attention only to  $(k + 2)$ -walks that contain these two edges. The required information for such walks can still be computed efficiently as before. The crux of the proof is that walks that are not paths cancel out over a field of characteristic two; this argument works by a local re-orientation of segments of the walk – an operation that does not change the vertices of the walk and must therefore keep  $s$  and  $t$  fixed. The graph  $G$  contains a  $k$ -path if and only if the polynomial is not identically zero; this property remains true in our case. The rest of the argument goes through as before, so the algorithm of Björklund et al. applies to EXACT PATH with no significant loss in the running time.

The deterministic algorithm of Zehavi [36] also does not expect the terminal vertices to be given, but this algorithm works for the weighted version of the problem. In the weighted  $k$ -path problem, we are given a graph  $G$ , weights  $w_e$  on each edge, a number  $k$ , and a number  $W$ , and the question is whether there is a path of length exactly  $k$  such that the sum of all edge weights along the path is at most  $W$ . We observe the following simple reduction from EXACT PATH (with terminal vertices  $s$  and  $t$ ) to the weighted  $k$ -path problem (without terminal vertices): Every edge gets assigned the same edge weight 2, except for the new leaf-edges at  $s$  and  $t$ , which get edge weight 1. Now every path with exactly  $k + 2$  edges has weight at most  $W = 2k + 2$  if and only if the first and the last edges of the path are the leaf-edges we added. Due to this reduction, Zehavi's algorithm applies to EXACT PATH with no significant loss in the running time.

Theorem 22 follows from Algorithm A by using either the algorithm of Björklund et al. [3] or Zehavi [36] as the oracle. We remark that Theorem 22 and Algorithm A apply to directed graphs as well, in which case an algorithm for EXACT PATH in directed graphs needs to be used (color coding yields the fastest randomized algorithm [2], while Zehavi's deterministic algorithm also applies to directed graphs).

## 5 Search-to-decision reduction

Our graph-minor based algorithm for LONGEST DETOUR does not directly construct a good path since the algorithm merely says “yes” when the tree width is large enough. Similarly, our dynamic programming algorithm uses an algorithm for EXACT PATH as a subroutine, and these algorithms also do not typically find a path directly.

In this section, we present a search-to-decision reduction for LONGEST DETOUR and EXACT DETOUR that uses a simple downward self-reducibility argument. In the interest of brevity, we focus on LONGEST DETOUR: Given a decision oracle for this problem, we show how to construct a detour with only polynomial overhead in the running time.

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**Algorithm B** (*Search-to-decision reduction*) *Given  $(G, s, t, k)$  and access to an oracle for LONGEST DETOUR, this algorithm computes an  $(s, t)$ -path of length at least  $d_G(s, t) + k$ .*

- B0** (Trivial case) If  $(G, s, t, k)$  is a no-instance of LONGEST DETOUR, halt and reject.
- B1** (Add a new shortest path) Add  $d := d_G(s, t)$  new edges to  $G$ , forming a new shortest  $(s, t)$ -path  $p_1, \dots, p_d$ .
- B2** (Delete unused edges) For each  $e \in E(G) \setminus \{p_1, \dots, p_d\}$ : If  $(G - e, s, t, k)$  is a yes-instance of LONGEST DETOUR, then set  $G := G - e$ .
- B3** (Delete the added path) Let  $G := G - \{p_1, \dots, p_d\}$ .
- B4** (Output detour) Now  $G$  is an  $(s, t)$ -path of length at least  $d_G(s, t) + k$ .
- 

**Lemma 24.** *Algorithm B is a polynomial-time algorithm when given oracle access to LONGEST DETOUR, and it outputs a path of length at least  $d_G(s, t) + k$ .*

*Proof.* It is clear that B runs in polynomial time; we only need to show correctness. Let  $G_0$  be the graph at the beginning of the algorithm, let  $G_1$  be the remaining graph after B2. If  $(G_0, s, t, k)$  is a yes-instance, then  $(G_1, s, t, k)$  is also a yes-instance. Moreover, deleting any edge from  $E(G) \setminus \{p_1, \dots, p_d\}$  would turn it into a no-instance. Since  $(G_1, s, t, k)$  is a yes-instance, it contains an  $(s, t)$ -path  $q_1, \dots, q_\ell$  for  $\ell \geq d_G(s, t) + k$ . Since the yes-instance is minimal, we have  $E(G_1) = \{p_1, \dots, p_d\} \cup \{q_1, \dots, q_\ell\}$ .

Finally, since  $p_1, \dots, p_d$  got added to  $G$  as a new path, it is edge-disjoint from every other  $(s, t)$ -path in  $G$ . Therefore, by removing  $\{p_1, \dots, p_d\}$  from  $G_1$ , we get the path  $q_1, \dots, q_\ell$ , of length at least  $d_G(s, t) + k$ . ■

## 6 Conclusion

We conclude with the following open problem: what is the complexity of LONGEST DETOUR in directed graphs? So far, our attempts to mimic the algorithm for undirected graphs did not work. By the celebrated work of Kawarabayashi and Kreutzer [22], every directed graph of sufficiently large directed treewidth contains a large directed grid as a butterfly minor. It is tempting to use this theorem in order to obtain a Win/Win algorithm for LONGEST DETOUR on directed graphs; however, there are several obstacles on this path. Actually we do not know if the problem is in the class XP, that is, if there is an algorithm that solves directed LONGEST DETOUR in time  $n^{f(k)}$  for some function  $f$ . Can one even find an  $(s, t)$ -path of length  $\geq d_G(s, t) + 1$  in polynomial time?

For undirected planar graphs, by standard bidimensionality arguments [13], our algorithm can be sped up to run in time  $2^{O(\sqrt{k})}n^{O(1)}$ , but we do not know if LONGEST DETOUR in directed planar graphs is in XP.

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## A Unsatisfiability of the rerouting inequality systems

In this section, we verify manually that the three systems of linear inequalities established in the proof of Lemma 15 are unsatisfiable, as claimed. To this end, we interpret each system as a linear program and verify that its respective dual program is unbounded. From LP duality, it then follows that the primal program is infeasible.

### A.1 Preliminaries from LP theory

Recall that a linear program  $\mathcal{P}$  in standard form is specified by an objective vector  $\mathbf{c} \in \mathbf{Q}^n$ , a matrix  $A \in \mathbf{Q}^{m \times n}$ , and a bound vector  $\mathbf{b} \in \mathbf{Q}^m$ . A vector  $\mathbf{x} \in \mathbf{Q}^n$  is *feasible* for  $\mathcal{P}$  if  $A\mathbf{x} \leq \mathbf{b}$  holds coordinate-wise. The program  $\mathcal{P}$  then asks for a feasible vector  $\mathbf{x} \in \mathbf{Q}^n$  that maximizes the inner product  $\mathbf{c}^T \mathbf{x}$ . Such a vector does not necessarily exist since  $\mathcal{P}$  may fall into one of the following degenerate cases:

- There may be no feasible vector for  $\mathcal{P}$  at all. In this case, we say that  $\mathcal{P}$  is *infeasible*.

- The function  $\mathbf{c}^T \mathbf{x}$  may attain arbitrarily large values for feasible  $\mathbf{x}$ . In this case, we say that  $\mathcal{P}$  is *unbounded*.

These two degenerate cases are intimately linked by the theory of LP duality: The dual of  $\mathcal{P}$ , denoted by  $\mathcal{D}(\mathcal{P})$ , asks for a vector  $\mathbf{y} \in \mathbf{Q}^m$  that minimizes  $\mathbf{b}^T \mathbf{y}$  subject to  $A^T \mathbf{y} = \mathbf{c}$  and  $\mathbf{y} \geq 0$ . It is known that, if  $\mathcal{D}(\mathcal{P})$  is unbounded, then  $\mathcal{P}$  is infeasible [34].

## A.2 Proving unboundedness of the duals

Given a system of linear inequalities  $A\mathbf{x} \leq \mathbf{b}$ , we define a linear program  $\mathcal{P}$  by endowing the system with the objective vector  $\mathbf{c} = \mathbf{0}$ . We show that  $\mathcal{P}$  is infeasible by proving  $\mathcal{D}(\mathcal{P})$  to be unbounded. To do so, it suffices to exhibit a vector  $\mathbf{y}^*$  with  $\mathbf{y}^* \geq \mathbf{0}$  and  $A^T \mathbf{y}^* = \mathbf{0}$  and  $\mathbf{b}^T \mathbf{y}^* < 0$ . The multiples  $\alpha \cdot \mathbf{y}^*$  with  $\alpha > 0$  are then feasible as well, and they attain arbitrarily small objective values; thus,  $\mathbf{y}^*$  is a witness of the fact that  $\mathcal{D}(\mathcal{P})$  is unbounded.

To find such vectors  $\mathbf{y}^*$ , we take a closer look at the systems of inequalities that appear in the proof of Lemma 15. We transform these inequalities into normal form, and observe that  $\mathbf{b}$  has non-zero entries only at rows generated by the inequalities (4) that act as upper bounds on the path lengths. Furthermore, these entries are all equal to  $-1$ . To prove infeasibility of the primal program, it thus suffices to find a vector  $\mathbf{y}^* \geq 0$  that assigns a non-zero value to at least one variable corresponding to a primal constraint from (4).

For each of the three cases in Lemma 15, we exhibit such vectors  $\mathbf{y}^*$ . To improve legibility, we display these vectors as linear combinations of the involved rows and omit rows whose corresponding coefficient is zero. Furthermore, we abbreviate expressions like  $\ell(b_1 u)$  to  $\ell_{1,u}$ . Note that each of the three listed linear combinations indeed

- involves only inequalities from the respective case,
- is a feasible solution to the dual because it evaluates to the zero vector and involves only non-negative coefficients, and
- proves the unboundedness of the dual because it assigns a positive coefficient to some row generated by the set of inequalities (4) in the respective case.

We found these solutions in a bleary-eyed state using the computer algebra system MATLAB, but this is irrelevant, as their correctness can be verified immediately by hand.

### Case 1:

$$\begin{aligned} \mathbf{y}^* \cdot A = & \quad 2 \cdot \left( \begin{array}{cccccccccc} 0 & k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\ell_{3,4} \end{array} \right) \\ & +1 \cdot \left( \begin{array}{cccccccccc} -d & -k & \ell_{1,u} & \ell_{2,v} & 0 & \ell_{1,4} & \ell_{2,3} & 0 & \ell_{3,4} & 0 \end{array} \right) \\ & +1 \cdot \left( \begin{array}{cccccccccc} -d & -k & \ell_{1,u} & \ell_{2,v} & \ell_{1,3} & 0 & 0 & \ell_{2,4} & \ell_{3,4} & 0 \end{array} \right) \\ & +1 \cdot \left( \begin{array}{cccccccccc} d & 0 & -\ell_{1,u} & -\ell_{2,v} & -\ell_{1,3} & 0 & -\ell_{2,3} & 0 & 0 & 0 \end{array} \right) \\ & +1 \cdot \left( \begin{array}{cccccccccc} d & 0 & -\ell_{1,u} & -\ell_{2,v} & 0 & -\ell_{1,4} & 0 & -\ell_{2,4} & 0 & 0 \end{array} \right) \end{aligned}$$

For clarity, we remark that  $\mathbf{y}^*$  has five non-zero entries: one is 2 and four are 1. The dual variable set to 2 in the first line corresponds to the equation  $\ell(b_3 b_4) \geq k$ , which is a constraint from (1). The second and third line correspond to the constraints of type (4) for the paths  $ub_1 b_4 b_3 b_2 v$  and  $ub_1 b_3 b_4 b_2 v$ . The fourth and fifth line correspond to the constraints of type (3) for the paths  $ub_1 b_3 b_2 v$  and  $ub_1 b_4 b_2 v$ .

Finally, note that  $\mathbf{y}^* \cdot A = 0$  and  $\mathbf{b}^T \mathbf{y}^* = -2$  hold. The latter follows since the entries of  $\mathbf{b}$  are equal to  $-1$  at inequalities (4), and zero otherwise.

### Case 2:

$$\begin{aligned} \mathbf{y}^* \cdot A = & \quad 2 \cdot \left( \begin{array}{cccccccccc} 0 & k & 0 & 0 & 0 & 0 & 0 & -\ell_{2,3} & 0 & 0 \end{array} \right) \\ & +1 \cdot \left( \begin{array}{cccccccccc} -d & -k & \ell_{1,u} & 0 & 0 & \ell_{3,v} & \ell_{1,4} & \ell_{2,3} & \ell_{2,4} & 0 \end{array} \right) \\ & +1 \cdot \left( \begin{array}{cccccccccc} -d & -k & 0 & \ell_{2,u} & \ell_{1,v} & 0 & \ell_{1,4} & \ell_{2,3} & 0 & \ell_{3,4} \end{array} \right) \\ & +1 \cdot \left( \begin{array}{cccccccccc} d & 0 & 0 & -\ell_{2,u} & -\ell_{1,v} & 0 & -\ell_{1,4} & 0 & -\ell_{2,4} & 0 \end{array} \right) \\ & +1 \cdot \left( \begin{array}{cccccccccc} d & 0 & -\ell_{1,u} & 0 & 0 & -\ell_{3,v} & -\ell_{1,4} & 0 & 0 & -\ell_{3,4} \end{array} \right) \end{aligned}$$

Case 3:

$$\begin{array}{l}
 \mathbf{y}^* \cdot A = \quad 2 \cdot ( \quad 0 \quad k \quad 0 \quad 0 \quad 0 \quad 0 \quad -\ell_{1,3} \quad 0 \quad 0 \quad ) \\
 \quad +1 \cdot ( \quad -d \quad -k \quad 0 \quad \ell_{2,u} \quad 0 \quad \ell_{4,v} \quad \ell_{1,3} \quad \ell_{1,4} \quad \ell_{2,3} \quad ) \\
 \quad +1 \cdot ( \quad -d \quad -k \quad \ell_{1,u} \quad 0 \quad \ell_{3,v} \quad 0 \quad \ell_{1,3} \quad 0 \quad 0 \quad ) \\
 \quad +1 \cdot ( \quad d \quad 0 \quad -\ell_{1,u} \quad 0 \quad 0 \quad -\ell_{4,v} \quad 0 \quad -\ell_{1,4} \quad 0 \quad ) \\
 \quad +1 \cdot ( \quad d \quad 0 \quad 0 \quad -\ell_{2,u} \quad -\ell_{3,v} \quad 0 \quad 0 \quad 0 \quad -\ell_{2,3} \quad )
 \end{array}$$