# Two phase flow including interfacial area as a variable

Hans Fredrik Nordhaug, Helge K. Dahle and Magne S. Espedal
*Department of Mathematics, University of Bergen, Norway*

William G. Gray
*Department of Civil Engineering and Geological Sciences, University of Notre Dame, USA*

Michael A. Celia
*Department of Civil and Environmental Engineering, Princeton University, USA*

ABSTRACT: Based on the procedure of (Gray & Hassanizadeh 1998) we state macroscale conservation equations for multi phase flow in porous media including interfacial area as a variable. The phases we consider are a solid phase, a wetting phase and a non-wetting phase. Some modeling work is done to decide the functionality of the coefficients in the equations. In particular we construct constitutive relationships for the interfacial area as a function of saturation and capillary pressure, and the resistance terms. The final set of equations is reduced to a $2 \times 2$ system with the saturation and the interfacial area as independent variables. Some preliminary numerical experiments are reported.

## 1 INTRODUCTION

Multi phase flow has been intensively studied in fields like reservoir mechanics, ground water research and so on. Models in these fields are normally based on generalisations from one phase flow, e.g. Darcy's law is generalised through the introduction of relative permeability. In this work we take another approach. Conservation equations for a complete two phase model including all physical effects are used. These equations are derived through a general and systematic procedure that employs conservations equations at the microscale and thermodynamics analysis, see (Gray & Hassanizadeh 1998; Gray 1999; Gray 2000). This model is very complex and contains a large number of coefficients that must be specified for numerical computation. In this preliminary study, assumptions are made to reduce the number of variables and facilitate the study of the behaviour of the system of equations. In fact, the general set of equations is reduced to a $2 \times 2$ system that has saturation and interfacial area as the primary dependent variables.

There are three major reasons for this approach: First, when the complete system is simplified, there are no hidden assumptions, and hence it is easier to see what the limitations of the model are. Secondly, there may be practical reasons for calculating amount of interfacial area. Third, the inclusion of interfacial area may help in reducing apparent hysteresis in the model. Thus, the development of models determining the amount of interfacial are important. Both rely on the reasoning that it is better to simplify a general model than to generalise a simple model. In fact, hysteresis effects, e.g. in $P_c - s^w$ curves, may be a re-

sult of taking one phase flow models and generalising them directly to multi phase flow. On the other hand, these effects can be explained and eliminated through a more general approach.

In Section 2 the model and the conservation equations are stated. The basic assumptions are listed and simplifications made. The model is rewritten into a fractional flow formulation in Section 3. This is done for computational and analytical purposes. The resistances introduced in Section 2 is parameterised in Section 4, and capillary pressure is parameterised in Section 5. A numerical experiment is given in Section 6, and some conclusions and final remarks are given in Section 7.

## 2 CONCEPTUAL MODEL

The following terminology will be used:

| | |
|---|---|
| $a^s$ | specific area of the solid phase surface. |
| $a^{wn}$ | specific interfacial area (fluid-fluid). |
| $a^{\alpha s}$ | specific area of the $\alpha s$-interface (fluid-solid). |
| $\mathbf{G}^{wn}$ | geometric tensor. |
| $J^w_{wn}$ | average curvature. |
| $\mathbf{K}$ | permeability for the medium. |
| $L_p, L_x$ | coefficients in linearised equations. |
| $p^\alpha$ | pressure of $\alpha$-phase. |
| $\mathbf{R}^\alpha_{wn}$ | resistance for the $\alpha$-phase due to the $wn$-interface (fluid-fluid). |
| $\mathbf{R}^\alpha_{\alpha s}$ | resistance for the $\alpha$-phase due to the $\alpha s$-interface (fluid-solid). |
| $s^\alpha$ | saturation of the $\alpha$-phase. |
| $\mathbf{u}$ | total Darcy velocity. |
| $\mathbf{u}^\alpha$ | Darcy velocity for the $\alpha$-phase. |

| | |
|---|---|
| $\mathbf{v}^{\alpha}$ | velocity of $\alpha$-phase. |
| $\mathbf{v}^{wn}$ | velocity of the $wn$-interface. |
| $x_s^{ws}$ | fraction of the solid surface covered by the wetting phase. |
| $\gamma^{\alpha\beta}$ | surface tension of the $\alpha\beta$-interface. |
| $\varepsilon$ | porosity of the medium. |
| $\varepsilon^{\alpha} = \varepsilon s^{\alpha}$ | volume fraction of the $\alpha$-phase. |
| $\mu^{\alpha}$ | viscosity for the $\alpha$-phase. |
| $\rho^{\alpha}$ | density of $\alpha$-phase. |
| $\Phi^w$ | average contact angle. |

### 2.1 Conservation equations

Based on the general theory in (Gray & Hassanizadeh 1998; Gray 2000) and the assumptions below we state a simplified set of equations modeling two phase flow in porous media that include interfacial area as a variable. The basic assumptions are:

- No phase change occurs,
- isothermal system,
- immobile, non-deforming solid,
- common lines are neglected,
- the solid-fluid interface dynamics are negligible (i.e no film flow),
- massless fluid-fluid interface,
- inertial terms are negligible in the momentum equations and
- interfacial tensions are all constant and specified.

Because the solid phase is immobile and non-deforming we have a two phase immiscible flow system where the wetting phase is denoted $w$ and the non-wetting phase $n$.

In addition the solid is non-deforming so that $\mathbf{v}^s = 0$ and we get

$$\mathbf{v}^{\alpha,s} = \mathbf{v}^{\alpha} - \mathbf{v}^s = \mathbf{v}^{\alpha}, \quad \alpha = w, n, wn,$$

and

$$\frac{D^s}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}^s \cdot \nabla = \frac{\partial}{\partial t}.$$

Note that bold capital and lowercase letters denote matrices and vectors, respectively.

From these assumptions, the following equations can be stated:

Mass conservation for the $w$-phase:

$$\frac{D^w(\varepsilon^w \rho^w)}{Dt} + \varepsilon^w \rho^w \nabla \cdot \mathbf{v}^w = 0. \quad (1)$$

Momentum conservation for the $w$-phase:

$$-\varepsilon^w \nabla p^w + \varepsilon^w \rho^w \mathbf{g} = \tag{2}$$
$$(\mathbf{R}_{wn}^w + \mathbf{R}_{ws}^w) \cdot \mathbf{v}^w - \mathbf{R}_{wn}^w \cdot \mathbf{v}^{wn}.$$

Mass conservation for the $n$-phase:

$$\frac{D^n(\varepsilon^n \rho^n)}{Dt} + \varepsilon^n \rho^n \nabla \cdot \mathbf{v}^n = 0. \quad (3)$$

Momentum conservation for the $n$-phase:

$$-\varepsilon^n \nabla p^n + \varepsilon^n \rho^n \mathbf{g} = \tag{4}$$
$$(\mathbf{R}_{wn}^n + \mathbf{R}_{ns}^n) \cdot \mathbf{v}^n - \mathbf{R}_{wn}^n \cdot \mathbf{v}^{wn}.$$

The momentum conservation equation for a massless interface indicates that the interfacial velocity is a weighted sum of the velocities of the adjacent phases. Here, for a massless $wn$-interface, the following equality is assumed:

$$(\mathbf{R}_{wn}^w + \mathbf{R}_{wn}^n) \cdot \mathbf{v}^{wn} = \mathbf{R}_{wn}^w \cdot \mathbf{v}^w + \mathbf{R}_{wn}^n \cdot \mathbf{v}^n. \quad (5)$$

Geometry equation for interfacial area:

$$\frac{\partial a^{wn}}{\partial t} - \varepsilon J_{wn}^w \frac{\partial s^w}{\partial t} - a^s \cos \Phi^w \frac{\partial x_s^{ws}}{\partial t} = \tag{6}$$
$$-\nabla \cdot [\mathbf{G}^{wn} a^{wn} \cdot \mathbf{v}^{wn}].$$

Dynamic capillary pressure equation:

$$\varepsilon \frac{\partial s^w}{\partial t} = -L_p[p^n - p^w + \gamma^{wn} J_{wn}^w]. \quad (7)$$

Linearised constitutive equation for surface area fraction:

$$a^s \frac{\partial x_s^{ws}}{\partial t} = -L_x[\gamma^{ws} - \gamma^{ns} + \gamma^{wn} \cos \Phi^w]. \quad (8)$$

The Darcy velocity for phase $\alpha$ is given by $\mathbf{u}^{\alpha} = \varepsilon^{\alpha} \mathbf{v}^{\alpha} = \varepsilon s^{\alpha} \mathbf{v}^{\alpha}$. We substitute these relationships into Equation (1)-(6). The velocity for the interface between the fluids, $\mathbf{v}^{wn}$, will be kept as it is.

To further simplify the set of equations we additionally assume incompressible flow. Equation (1) and (3) then reduces to

$$\varepsilon \frac{\partial s^w}{\partial t} + \nabla \cdot \mathbf{u}^w = 0, \quad (9)$$

$$\varepsilon \frac{\partial s^n}{\partial t} + \nabla \cdot \mathbf{u}^n = 0. \quad (10)$$

In some simple models, e.g. bundle of capillary tubes, it is reasonable to assume that there will be no production of $wn$-interfacial area as saturation changes. Therefore, the second and third terms of Equation (6) will cancel. Application of this condition, in general, is a significant assumption that limits the general applicability of the system of equations. However, it will be applied here consistent with the objective of performing a preliminary analysis of a simplified set of equations. This assumption eliminates the need to model the dynamics of the $ws$-interfacial area, Equation (8), and reduces Equation (6) to:

$$\frac{\partial a^{wn}}{\partial t} + \nabla \cdot [\mathbf{G}^{wn} a^{wn} \cdot \mathbf{v}^{wn}] = 0. \quad (11)$$

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The transcriber coefficient, $L_p$, in Equation (7) is also considered large so that Equation (7) reduces to:

$$\gamma^{wn} J_{wn}^w = p^w - p^n = -P_c. \qquad (12)$$

Adding Equation (9) and (10) and observing that $s^w + s^n = 1$ gives

$$\nabla \cdot [\mathbf{u}^w + \mathbf{u}^n] = \nabla \cdot \mathbf{u} = 0$$

where

$$\mathbf{u}^w + \mathbf{u}^n = \mathbf{u} \qquad (13)$$

is the total Darcy velocity. In 1D $\mathbf{u}$ is a constant.

We are now in the position to rewrite Equation (2), (4), (5), (9)-(12) to a $2 \times 2$ system of equations including fractional flow and capillary diffusion terms. This will be done in the following sections.

## 3 FRACTIONAL FLOW FORMULATION

From Equation (5) we get

$$\mathbf{v}^{wn} = \mathbf{A}^w \cdot \mathbf{u}^w + \mathbf{A}^n \cdot \mathbf{u}^n \qquad (14)$$

where

$$\mathbf{A}^\alpha \overset{\text{def}}{=} (\mathbf{R}_{wn}^w + \mathbf{R}_{wn}^n)^{-1} \cdot \mathbf{R}_{wn}^\alpha / \varepsilon^\alpha, \quad \alpha = w, n.$$

Using this expression for $\mathbf{v}^{wn}$ in Equation (2) and (4) we get

$$-\varepsilon^w \rho^w \nabla \psi^w = \mathbf{B}^w \cdot \mathbf{u}^w - \mathbf{R}_{wn}^w \cdot \mathbf{A}^n \cdot \mathbf{u}^n, \qquad (15)$$

$$-\varepsilon^n \rho^n \nabla \psi^n = \mathbf{B}^n \cdot \mathbf{u}^n - \mathbf{R}_{wn}^n \cdot \mathbf{A}^w \cdot \mathbf{u}^w, \qquad (16)$$

where $\psi^\alpha = p^\alpha / \rho^\alpha - gz$ and

$$\mathbf{B}^\alpha \overset{\text{def}}{=} (\mathbf{R}_{wn}^\alpha \cdot (\mathbf{I} - \mathbf{A}^\alpha) + \mathbf{R}_{\alpha s}^\alpha)/\varepsilon^\alpha, \quad \alpha = w, n.$$

Define

$$\mathbf{C}^n \overset{\text{def}}{=} \mathbf{R}_{wn}^w \cdot \mathbf{A}^n \cdot (\mathbf{B}^n)^{-1},$$

$$\mathbf{C}^w \overset{\text{def}}{=} \mathbf{R}_{wn}^n \cdot \mathbf{A}^w \cdot (\mathbf{B}^w)^{-1}.$$

Add $\mathbf{C}^n$ times Equation (16) to (15), and $\mathbf{C}^w$ times Equation (15) to (16) to get $\mathbf{u}^\alpha$ expressed as a sum of the phase pressures:

$$\mathbf{u}^w = -(\mathbf{M}^w)^{-1} \cdot [\varepsilon^w \rho^w \nabla \psi^w +$$

$$\varepsilon^n \rho^n \mathbf{C}^n \cdot \nabla \psi^n], \qquad (17)$$

$$\mathbf{u}^n = -(\mathbf{M}^n)^{-1} \cdot [\varepsilon^n \rho^n \nabla \psi^n +$$

$$\varepsilon^w \rho^w \mathbf{C}^w \cdot \nabla \psi^w], \qquad (18)$$

where

$$\mathbf{M}^w \overset{\text{def}}{=} \mathbf{B}^w - \mathbf{C}^n \cdot \mathbf{R}_{wn}^n \cdot \mathbf{A}^w,$$

$$\mathbf{M}^n \overset{\text{def}}{=} \mathbf{B}^n - \mathbf{C}^w \cdot \mathbf{R}_{wn}^w \cdot \mathbf{A}^n.$$

$\mathbf{M}^\alpha$ is the total mobility for the $\alpha$-phase and Equation (17) and (18) are generalised Darcy's laws. Note that the superficial velocity of each phase now depends on the potential of the other phase, and that resistances also depends on the amount of interfacial area.

Equation (12) gives the following relationship between $\psi^n$ and $\psi^w$

$$\rho^w \psi^w - \rho^n \psi^n = p^w - p^n + (\rho^n - \rho^w)gz$$

$$= (\gamma^{wn} J_{wn}^w) + (\rho^n - \rho^w)gz$$

$$= -P_c + (\rho^n - \rho^w)gz \qquad (19)$$

We then have 6 unknowns

$$\mathbf{u}^w, \mathbf{u}^n, \mathbf{u}, \psi^n, \psi^w \text{ and } P_c.$$

Using Equation (13), (17)-(19) we are able to express $\mathbf{u}^w$ and $\mathbf{u}^n$ as a function of the total Darcy velocity $\mathbf{u}$ and the capillary pressure $P_c$.

$$\mathbf{u}^w = \mathbf{F}_w \cdot \mathbf{u} + \mathbf{D}_w \cdot \rho^{n,w} \mathbf{g} - \mathbf{D}_w \cdot \nabla P_c, \qquad (20)$$

$$\mathbf{u}^n = \mathbf{F}_n \cdot \mathbf{u} + \mathbf{D}_n \cdot \rho^{n,w} \mathbf{g} - \mathbf{D}_n \cdot \nabla P_c, \qquad (21)$$

where

$$\rho^{n,w} \mathbf{g} = (\rho^n - \rho^w)\mathbf{g},$$

$$\mathbf{F}_w = (\varepsilon^w + \varepsilon^n \mathbf{C}^n) \cdot \mathbf{M}^n \cdot \mathbf{N}^{-1},$$

$$\mathbf{F}_n = (\varepsilon^w \mathbf{C}^w + \varepsilon^n) \cdot \mathbf{M}^w \cdot \mathbf{N}^{-1},$$

$$\mathbf{D}_w = \varepsilon^n \varepsilon^w (1 - \mathbf{C}^w \cdot \mathbf{C}^n) \cdot \mathbf{N}^{-1} = -\mathbf{D}_n,$$

and

$$\mathbf{N} = \varepsilon^n \mathbf{M}^w + \varepsilon^n \mathbf{C}^n \cdot \mathbf{M}^n + \varepsilon^w \mathbf{C}^w \cdot \mathbf{M}^w + \varepsilon^w \mathbf{M}^n.$$

From Equation (14) we get the following expression for the speed of the interface between the fluid phases

$$\mathbf{v}^{wn} = (\mathbf{A}^w \cdot \mathbf{F}_w + \mathbf{A}^n \cdot \mathbf{F}_n) \cdot \mathbf{u} +$$

$$(\mathbf{A}^w \cdot \mathbf{D}_w + \mathbf{A}^n \cdot \mathbf{D}_n) \cdot (\rho^{n,w} \mathbf{g} - \nabla P_c)$$

$$= \mathbf{F}_{wn} \cdot \mathbf{u} + \mathbf{D}_{wn} \cdot (\rho^{n,w} \mathbf{g} - \nabla P_c)$$

where $\mathbf{F}_{wn}$ and $\mathbf{D}_{wn}$ is defined implicitly.

### 3.1 *Resulting $2 \times 2$ system*

Substituting the above expression for $\mathbf{u}^w$ and $\mathbf{v}^{wn}$ into Equation (9) and (11) we get the following equations:

$$\varepsilon \frac{\partial s^w}{\partial t} + \mathbf{u} \cdot \nabla \cdot \mathbf{F}_w + \mathbf{g} \cdot \nabla \cdot \mathbf{D}_w$$

$$= \nabla \cdot (\mathbf{D}_w \cdot \nabla P_c)$$

$$\frac{\partial a^{wn}}{\partial t} + \frac{1}{3}\left(\mathbf{u}\cdot\nabla\cdot(\mathbf{F}_{wn}a^{wn}) + \rho^{n,w}\mathbf{g}\cdot\nabla\cdot(\mathbf{D}_{wn}a^{wn})\right)$$

$$= \frac{1}{3}\nabla\cdot(a^{wn}\mathbf{D}_{wn}\cdot\nabla P_c))$$

This constitutes a $2\times 2$ coupled system for the saturation and the interfacial area.

## 4 PARAMETERISATION OF RESISTANCES

To close the above system we need to specify the resistances. We assume the functional form of $\mathbf{R}^\alpha_{\alpha\beta}$ to be

$$\mathbf{R}^w_{wn} = \mu^w g^w_{wn}(a^{wn})h^w_{wn}(s^w)\mathbf{K}^{-1}\varepsilon^{w^2},$$

$$\mathbf{R}^w_{ws} = \mu^w g^w_{ws}(a^{ws})h_w(s^w)\mathbf{K}^{-1}\varepsilon^{w^2},$$

$$\mathbf{R}^n_{wn} = \mu^n g^n_{wn}(a^{wn})h^n_{wn}(s^n)\mathbf{K}^{-1}\varepsilon^{n^2},$$

$$\mathbf{R}^n_{ns} = \mu^n g^n_{ns}(a^{ns})h_n(s^n)\mathbf{K}^{-1}\varepsilon^{n^2}.$$

The motivation for these forms is as follows: Equation (2) and (4) are generalised forms of Darcy's law. We therefore expect $\mu^\alpha$ and $\mathbf{K}$ to be terms in the resistances. In addition, to make the dimensions on the left and right side of the equations to match, $\varepsilon^{\alpha^2}$ must be a factor too. The $g$'s and $h$'s must be non-dimensional (and positive).

To get expressions for the $g$'s and $h$'s we look at the limit cases where we have only one phase. Equation (2) and (4) should then reduce to the usual Darcy's law.

Case 1: No non-wetting phase

$$s^w \to 1, \quad k_{rw} \to 1, \quad a^{wn}/a^s \to 0,$$

$$a^{ws}/a^s \to 1 \quad \text{and} \quad a^{ns}/a^s \to 0.$$

Consider Equation (2). Since the interfacial area $a^{wn}$ is zero, the resistance due to this interface, $\mathbf{R}^w_{wn}$, should also be zero and Equation (2) reduces to

$$-\varepsilon^w\nabla p^w + \varepsilon^w\rho^w\mathbf{g} = \mathbf{R}^w_{ws}\cdot\mathbf{v}^w.$$

Hence

$$\mathbf{R}^w_{ws} = \mu^w\mathbf{K}^{-1}\varepsilon^{w^2},$$

and we get

$$g^w_{wn}(0)h^w_{wn}(1) = 0 \text{ and } g^w_{ws}(0)h_w(1) = 1. \quad (22)$$

Case 2: No wetting phase

$$s^n \to 1, \quad k_{rn} \to 1, \quad a^{wn}/a^s \to 0,$$

$$a^{ns/a^s} \to 1 \quad \text{and} \quad a^{ws}/a^s \to 0$$

Consider Equation (4). Since there is no (fluid-fluid) interfacial area, the resistance due to this interface, $\mathbf{R}^n_{wn}$, should be zero and Equation (4) reduces to

$$-\varepsilon^n\nabla p^n + \varepsilon^n\rho^n\mathbf{g} = \mathbf{R}^n_{ns}\cdot\mathbf{v}^n.$$

Hence

$$\mathbf{R}^n_{ns} = \mu^n\mathbf{K}^{-1}\varepsilon^{n^2},$$

and we get

$$g^n_{wn}(0)h^n_{wn}(1) = 0 \text{ and } g^n_{ws}(0)h_n(1) = 1. \quad (23)$$

One set of functions that satisfies the constraints imposed by Equation (22) and (23) are:

$$g^w_{wn}(a^{wn}) = \frac{a^{wn}}{a^s}, \qquad h^w_{wn}(s^w) = 1 - s^w$$

$$g^n_{wn}(a^{wn}) = \frac{a^{wn}}{a^s}, \qquad h^n_{wn}(s^n) = 1 - s^n$$

$$g^w_{ws}(a^{ws}) = \frac{a^{ws}}{a^s}, \qquad h_w(s^w) = k^{-1}_{rw}(s^w) = (s^{w^2})^{-1}$$

$$g^n_{ns}(a^{ns}) = \frac{a^{ns}}{a^s}, \qquad h_n(s^n) = k^{-1}_{rn}(s^n) = (s^{n^2})^{-1}$$

Other constraints arise if we consider the case when the wetting fluid covers the medium ($a^{wn}/a^s \to 1$), and the case when the interfacial area is negligible ($a^{wn}/a^s \to 0$). One possible set of functions $g^\alpha_{\alpha s}$ is:

$$g^w_{ws} = 1 + \frac{a^{ns}}{a^s}\frac{a^{wn}}{a^s}$$

$$g^n_{ns} = 1 + \frac{a^{ws}}{a^s}\frac{a^{wn}}{a^s}$$

However, this choice appears to be unphysical since the resistance due to the solid-fluid interface depend on the area of the fluid-fluid interface. We will not use this choice in our numerical experiments.

### 4.1 Resulting fractional flow and diffusion functions

In 1D we can write the fractional flow functions $\mathbf{F}_w$, $\mathbf{F}_n$ and $\mathbf{F}_{wn}$ and the diffusion functions $\mathbf{D}_w$, and $\mathbf{D}_{wn}$ as rational polynomials in saturation and interfacial area. The expression are generated using Maple, but omitted because of their length. Using the parameters given in Section 6 we get the surfaces in Figure 1 and 2. Notice that these functions are only weakly dependent on interfacial area.

## 5 CAPILLARY PRESSURE

In this section a capillary pressure function depending on the (wetting) saturation $s$ and on the interfacial area $a^{wn}$ is constructed. To our knowledge there are almost no available data for $P_c - s^w - a^{wn}$ relationships. We therefore use results from network models, see e.g. (Reeves & Celia 1996), and try to capture the qualitative behaviour of surfaces obtained therein.

In (van Genuchten 1980) the following relationship between capillary pressure $P_c$ and saturation $s$ was proposed:

$$P_c = \frac{((s^w)^{-\frac{n}{n-1}} - 1)^{\frac{1}{n}}}{\alpha_G}. \quad (24)$$
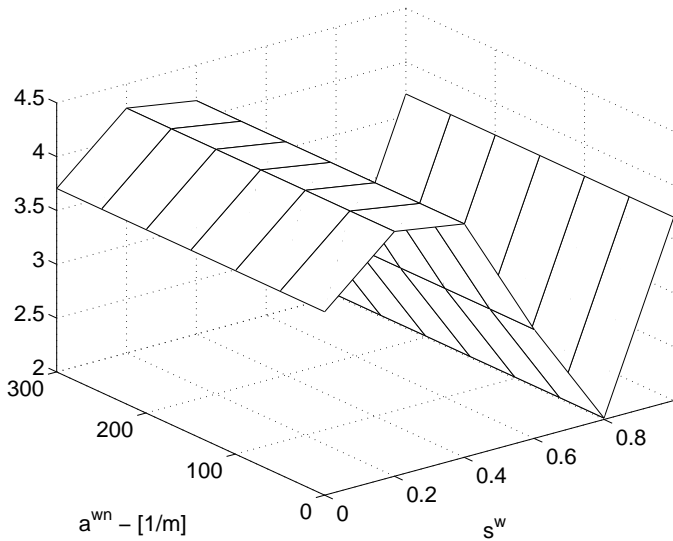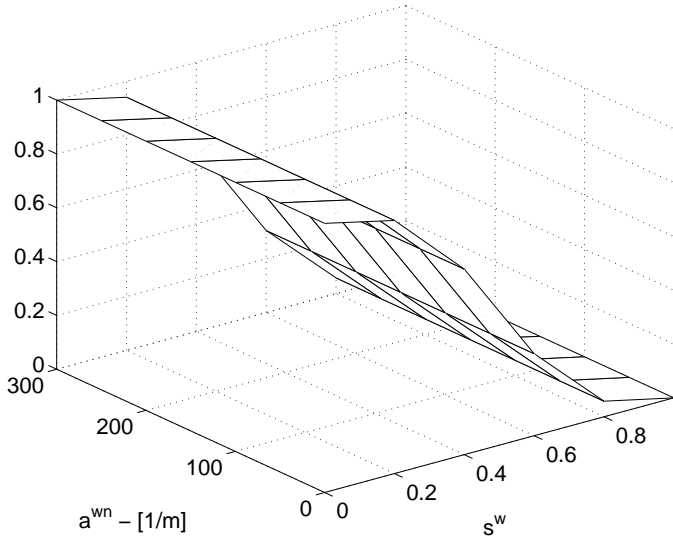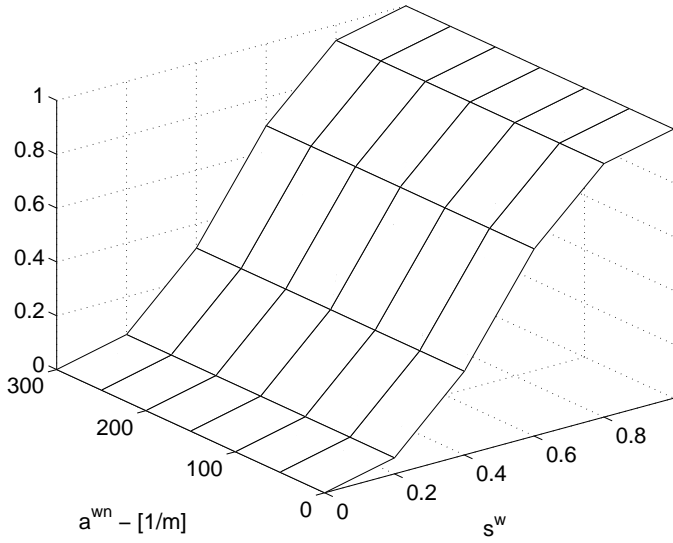
Figure 2: Diffusion function for the $w$-phase (upper) and the $wn$-interface (lower).



Figure 1: Fractional flow function for the $w$-phase (upper), the $n$-phase (middle) and the $wn$-interface (lower).

Starting out with this relationship we fix $n$ and vary $\alpha_G$ to get three curves satisfying

$$P_c^1 \leq P_c^2 \leq P_c^3.$$

Here $P_c^1$ and $P_c^3$ are approximations to the inner and outer envelopes for the $P_c - s^w$ curves. $P_c^2$ is chosen midway between the two boundary curves for interpolation purposes. Along these three curves we specify the interfacial area as

$$a_i^{wn}(s^w) = \alpha_i s^w(1 - s^w) + \beta s^w, \quad i = 1, 2, 3, \ (25)$$

where $\alpha_i$ is chosen such that $a_1^{wn} \leq a_2^{wn} \leq a_3^{wn}$. In Figure 3 and 4 plots of capillary pressure and interfacial area are shown for a set of parameters.

Interpolating (quadratically) from these three curves defines the function. For a given $P_c$ we specify $a^{wn}(s, P_c)$ as

$$a^{wn}(s, P_c) = c_1 s^{w\,2} + c_2 s^w + c_3. \tag{26}$$

Let $s_i^w(P_c) = (P_c^i)^{-1}(P_c)$, where $(P_c^i)^{-1}$ is given by Equation (24) with $\alpha_G = \alpha_G^i$. It follows that
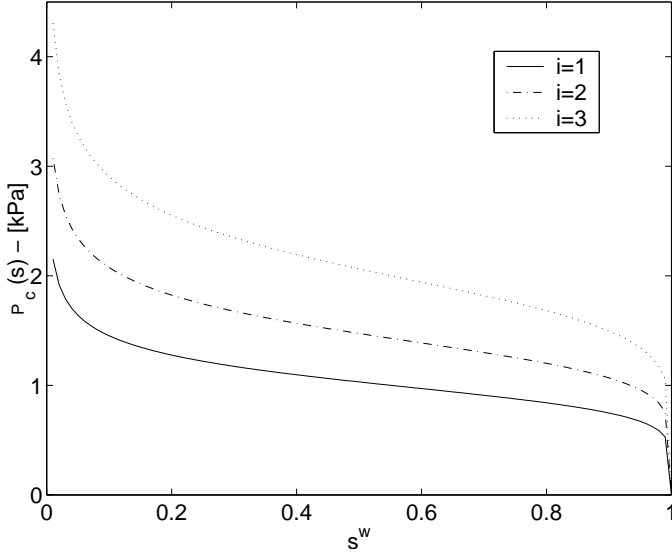
Figure 3: Capillary pressure $P_c^i$ (as a function of saturation) for $n = 7$ and $\alpha = 0.10, 0.07, 0.05$.
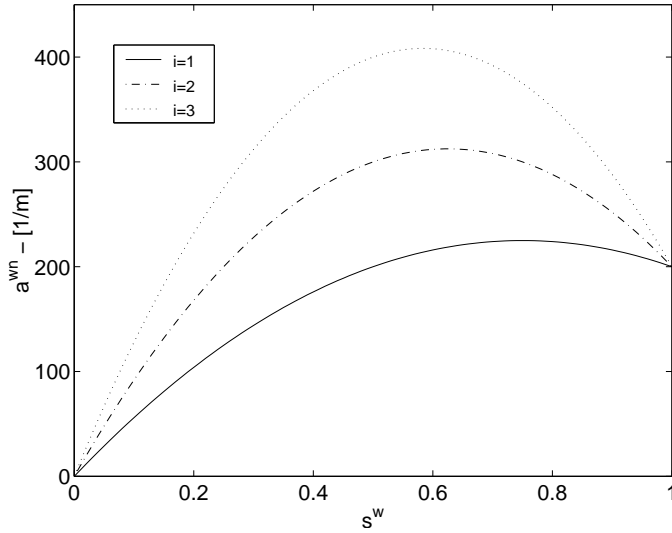


Figure 4: Interfacial area $a_i^{wn}$ on $P_c^i$, $i = 1, 2, 3$.

$a_i^{wn}(s_i^w) = a_i^{wn}(s_i^w(P_c)) = a_i^{wn}(P_c)$, $i = 1, 2, 3$. Using Lagragian interpolating we get the following expression for the interfacial area as a function of capillary pressure and saturation:

$$a^{wn}(P_c, s^w) =$$

$$a_1^{wn}(P_c) \frac{(s^w - s_2^w(P_c))(s^w - s_3^w(P_c))}{(s_1^w(P_c) - s_2^w(P_c))(s_1^w(P_c) - s_3^w(P_c))}$$

$$+ a_2^{wn}(P_c) \frac{(s^w - s_1^w(P_c))(s^w - s_3^w(P_c))}{(s_2^w(P_c) - s_1^w(P_c))(s_2^w(P_c) - s_3^w(P_c))}$$

$$+ a_3^{wn}(P_c) \frac{(s^w - s_1^w(P_c))(s^w - s_2^w(P_c))}{(s_3^w(P_c) - s_1^w(P_c))(s_3^w(P_c) - s_2^w(P_c))}$$

Since the interfacial area is non-monotone in capillary pressure, see Figure 5, it is generally impossible to find a unique expression for $P_c$ as a function
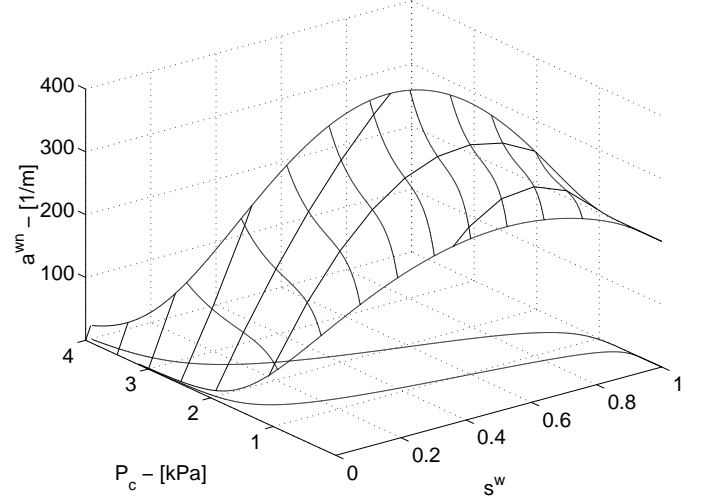


Figure 5: Interfacial area as a function of wetting saturation and capillary pressure given by the three curves above.

of saturation and interfacial area. In the equations we will need the derivative of capillary pressure with respect to saturation and interfacial area. If we assume uniqueness of $P_c$ locally around a point $(s^w, a^{wn})$ we can employ implicit differentiation of function to evaluate

$$\left(\frac{\partial P_c}{\partial s^w}\right)_{a^{wn}} \quad \text{and} \quad \left(\frac{\partial P_c}{\partial a^{wn}}\right)_{s^w}.$$

## 6 EXAMPLES

We have done a simple test case in 1D. Data for parameters are taken from (Dahle & Celia 1999) and from a table over typical values of permeability and porosity for sand stones. The specific area of the solid phase surface is calculated using the Carman-Kozeny equation. The following values are used:

$$
\begin{aligned}
\mu^w &= 10^{-3} \, [Ns/m^2] \\
\mu^n &= 10^{-2} \, [Ns/m^2] \\
\varepsilon &= 0.27 \\
K &= 1.3 \times 10^{-9} \, [m^2] \\
u &= 1.3 \times 10^{-3} \, [m/s] \\
a^s &= 7.6 \times 10^4 \, [1/m] \\
a^{ns} &= 0.7 \times 10^4 \, [1/m] \\
a^{ws} &= 6.9 \times 10^4 \, [1/m] \\
n &= 7 \\
\alpha_G^i &= 10^{-3}, 0.7 \times 10^{-3}, 0.5 \times 10^{-3} \\
\alpha_i &= 400, 800, 1200 \\
\beta &= 0
\end{aligned}
$$

Since we are working in 1D, gravity terms are neglected. To avoid difficulties with boundaries and discontinuities, a smooth ramp is initially specified for the saturation. An increased amount of interfacial area is evenly distributed around this ramp. The equations are solved by sequential time stepping using a straight forward upstream/central difference scheme.

The initial data and solutions for interfacial area and saturation are shown in Figure 6. This problem appears to be advection dominated. It follows that

since the advective flux is nearly insensitive to interfacial area, the dynamics of the saturation is nearly independent of the changes in interfacial area.
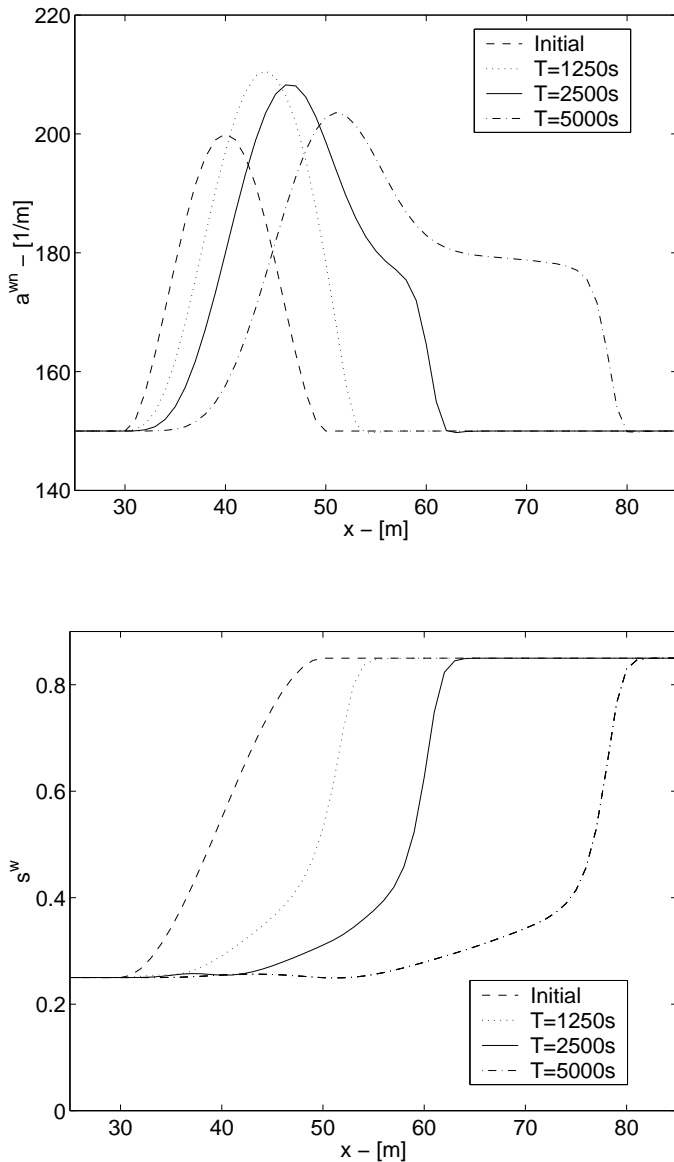


Figure 6: Profiles for interfacial area (upper) and wetting saturation (lower) at $t = 0, 1300, 2600$ and $5200s$.

## 7 CONCLUSIONS

In this work, we have presented initial numerical solutions for an enhanced model of two phase flow which includes fluid-fluid interfacial area as a primary variable. The solutions are for a simplified test problem under a specific set of assumptions, and with a particular choice of functional forms for the nonlinear coefficients. For this particular case, the saturation profiles show little dependence on interfacial area. This result is consistent with the forms of the fractional flow functions, which show minimal sensitivity to interfacial area. This in turn is a reflection of the functional forms chosen to parameterise the various nonlinear resistance terms that arise in the governing equations. Because the correct functional forms for these resistance terms are not known, more general sensitivity studies are required to assess the practical importance of the interfacial area equations to flow modeling. In addition, numerical solutions need to be obtained for cases where the simplifying assumptions applied to the governing equations are systematically examined, so that the sensitivity of the solutions to these assumptions can be assessed. The results presented herein represent a starting point for this more general analysis.

While we cannot reach any general conclusions based on our initial calculations, we can make a few general observations about the overall system of equations. First, inclusion of the interfacial area equations allows for direct calculation of the amount of interfacial area in the system, as a function of space and time. Equations written for the individual phases also incorporate effects of the interfaces through functional dependences in the appropriate nonlinear coefficients. In addition, inclusion of a unique functional relationship between $P_c$, $s^w$, and $a^{wn}$, and the associated transport equation for interfacial area, allow hysteresis within the $P_c - s^w$ plane to be incorporated into the algorithm with no additional effort. This means that arbitrary drainage and imbibition cycles can be simulated without regard for redefinition of the constitutive curves. This appears to be a major advantage of this approach to multi phase flow modeling. The cost of this additional generality and flexibility is the requirement to identify and quantify additional parameters, as well as their functional dependences on interfacial area. Additional equations also need to be solved, as compared to the traditional equations for multi phase flow; and these additional equations remain to be analysed mathematically, as do the associated numerical algorithms implemented for their solution.

The overall significance of interfacial areas in the mathematical description of multi phase flow in porous media remains to be determined. To make this determination, numerical simulators are required to allow a wide range of possible functional forms for the nonlinear coefficients to be explored. The degree to which the governing equations can be simplified must also be explored numerically. This requires a systematic and comprehensive numerical approach. The results presented herein represent a first contribution to the overall numerical study.

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