Thickness and weak integrability
&
approximation and u-ideals

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Contents

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Chapter 1

Introduction and perspectives

This introduction provides background material for the articles [3], [1], and [2] which respectively constitute Chapters 2 to 4 in this thesis. Section 1.1 contains notation and terminology used in the introduction. In Section 1.2 we present background material for the article [3]. This article contains new characterizations of thick and weak*-thick sets. Subsection 1.2.2 contains a generalization of the notion of thick sets. New results and open problems are also presented here. In Section 1.3 background material for the articles [1] and [2] is presented. These articles contain new results about approximation properties and u-ideals. In subsection 1.3.2 we discuss some open problems related to the notion of u-ideals.

1.1 Notation and terminology

The notation and terminology used throughout this introduction is standard (see e.g. [57]). We will write $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ for the sets of natural numbers, real numbers, and complex numbers, respectively. $\mathbb{K}$ will denote a set that can be either $\mathbb{R}$ or $\mathbb{C}$. The letters X, Y, and Z will denote Banach spaces unless otherwise stated. The letters E, F, and G will typically denote finite dimensional Banach spaces. The closed unit ball of a Banach space X is denoted by B_X and the unit sphere of X is denoted by S_X. We will write X^* for the dual space of X. The sets of extreme points, exposed points, and strongly exposed points of B_X are respectively denoted by $\operatorname{ext} B_X$, $\exp B_X$, and $\text{str-exp}\, B_X$. Similarly $\omega^*\text{-exp}\, B_{X^*}$ and $\omega^*\text{-str-exp}\, B_{X^*}$ denote the sets of weak*-exposed and weak*-strongly exposed points of B_{X^*}.

Let A be a subset of a Banach space X. Then its norm closure, convex hull, absolutely convex hull, and linear span will be denoted by $\overline{A}$, $\text{conv}A$, $\text{absconv}A$, and $\text{span}A$, respectively. We will write $\overline{A}^w$ for the weak closure of A. Similarly, if A is a subset of a dual space X^*, $\overline{A}^{w^*}$ denotes the weak*-closure of A.

Let X and Y be Banach spaces. We will write $\mathcal{L}(Y, X)$ for the Banach space of bounded linear operators from Y to X, and $\mathcal{F}(Y, X)$, $\mathcal{K}(Y, X)$, and $\mathcal{W}(Y, X)$ for its subspaces of finite rank operators, compact operators, and weakly compact operators, respectively. If Z is a subspace of X, then we will write $i_Z : Z \to X$ for the canonical embedding of Z into X. I_X will denote the identity operator on X. If no confusion is possible, we will sometimes also write

I for the identity operator on a Banach space. The natural embedding of X into its bidual will be denoted by $k_X : X \to X^{**}$. $\ker T$ will denote the kernel of a bounded linear operator T.

1.2 Background on thick and weak*-thick sets in Banach spaces

The Banach-Steinhaus Uniform Boundedness Principle (see e.g. [69, p. 43]) is one of the cornerstones in the theory of Banach spaces. Special cases of the theorem dates back to those of Lebesgue [43] in 1909 for the function spaces $L_2[a,b]$, $L_1[a,b]$, $L_\infty[a,b]$, Helly [36] in 1912 for the function space $C[a,b]$, and Toeplitz [77] and Schur [71] in 1913 and 1920 for the sequence space c. The abstract version of the Banach-Steinhaus Uniform Boundedness Principle was published independently by Hahn [34], Banach [6], and Hildebrant [38] in the years 1922 – 1923. Banach and Steinhaus [8] proved a more general version of the principle for second category sets in 1927. The proof of this theorem was modern because it used Baire's Category Theorem [5] (cf. [58, p. 37]) instead of the *gliding hump technique* (cf. [17, pp. 138-142]) used before. This general version of the Banach-Steinhaus Uniform Boundedness Principle essentially tells us that whenever (T_α) is a family of bounded linear operators on some Banach space X, which is pointwise bounded on a set A of the second category in X, then the family is bounded. However, in some cases, boundedness can be obtained from pointwise boundedness on a "smaller set" than the second category. Indeed, the Nikodým-Grothendieck Boundedness Theorem (see e.g. [16, p. 14] or [15, p. 80]) says that if a family (T_α) of bounded linear operators is pointwise bounded on the set of characteristic functions in the unit sphere of the space $B(\Sigma)$ (see text above Theorem 1.2.5), then this family is bounded. This set of characteristic functions is certainly not of the second category, it is even nowhere dense. Thus it is natural to ask: How can we sharpen the Banach-Steinhaus Uniform Boundedness Principle in the sense of weakening the restrictions on the set A on which to test pointwise boundedness?

Building on a result of Kadets and Fonf [26, Proposition 1], Nygaard proposed a property, that he called thickness, which is weaker than the second category, so that the conclusion of the Banach-Steinhaus Uniform Boundedness Principle still holds [59]. Further, Nygaard showed that thickness is the ultimate property in the sense that if a subset B of a Banach space X is not thick, then it is always possible to find an unbounded family of bounded linear operators on X which is pointwise bounded on B. Nygaard noticed also that the thickness property is equivalent to another fundamental property in the theory of linear operators. The property is the one that guarantees that if a bounded linear operator $T : Y \to X$ is onto a subset B of X, then it is onto X.

The paper [3], which is presented in Chapter 2 in this thesis, contains new characterizations of the thickness property, and a weaker dual companion called the weak*-thickness property, in terms of integrability of vector-valued functions.

1.2.1 Basic results on thick and weak*-thick sets

Suppose X and Y are topological vector spaces. The following two problems are of fundamental importance in the theory of linear operators:

Problem 1.2.1. *Assume A is a subset of Y. Find a property on A such that every continuous linear operator $T : X \to Y$ is onto Y if and only if the range of the operator contains A.*

Problem 1.2.2. *Assume A is a subset of Y and that $\mathcal{A}$ is a subset of the space of all continuous linear operators from Y into X. Find a property on A such that $\mathcal{A}$ is bounded if and only if the set $\{Ty : T \in \mathcal{A}\}$ is bounded for each $y \in A$ ($\mathcal{A}$ is pointwise bounded on Y).*

If Y is of finite dimension, the answer to both problems is of course that A has to contain as many independent vectors as the dimension of Y. When Y is of infinite dimension there is, on the contrary, no simple answer to any of the problems.

However, from a classical theorem that appeared already in Théorie des Opérations Linéaires [7], the following result is known.

Theorem 1.2.3 (Banach, 1932). *If T is a bounded linear operator from a Banach space into a normed linear space, then the range of T is either of first category or equal to the range space itself.*

Another classical theorem, the famous category version of the Banach-Steinhaus Uniform Boundedness Principle [8], which appeared in a joint paper of Banach and Steinhaus as early as 1927, reads:

Theorem 1.2.4 (Banach and Steinhaus, 1927). *Let (T_n) be a sequence of bounded linear operators from a Banach space Y into a Banach space X. Suppose $\sup_n \|T_n y\| < \infty$ for every $y \in A$ where A is a set of the second category in Y. Then $\sup_n \sup_{y \in B_Y} \|T_n y\| < \infty$.*

Thus Theorem 1.2.3 and Theorem 1.2.4 tell us that the property *second category* is sufficiently strong to obtain implication in one direction in both Problems 1.2.1 and 1.2.2 when X and Y are Banach spaces. However, there are examples which show that this property is indeed too strong for the reverse implications to hold. In the case of Problem 1.2.1, the spectacular theorem of Seever shows this [72] (see also [16, p. 17]). ($B(\Sigma)$ denotes here the Banach space of uniform limits of simple functions modeled on the σ-algebra Σ.)

Theorem 1.2.5 (Seever, 1968). *Let Σ be a σ-algebra of subsets of a set Ω and let X be a Banach space. Let $T : X \to \mathcal{B}(\Sigma)$ be a bounded linear operator whose range includes the set $\{\chi_E : E \in \Sigma\}$. Then $TX = \mathcal{B}(\Sigma)$.*

In particular Seever's theorem says that if an operator is onto the set of 0-1 sequences in ℓ_∞, then it is onto ℓ_∞.

In the case of Problem 1.2.2, the Nikodým-Grothendieck Boundedness Theorem (see below) shows that second category is a too strong property. Indeed, this is easily seen from Corollary 1.2.7 below which is an immediate consequence of the Nikodým-Grothendieck Boundedness Theorem. (Use the fact that for each bounded linear operator $T : B(\Sigma) \to X$ there corresponds a vector measure $F : \Sigma \to X$ defined by $F(E) = T(\chi_E)$ and then apply Theorem 1.2.6.)

Theorem 1.2.6 (Nikodým and Grothendieck)**.** *Let Σ be a σ-algebra of subsets of a set Ω, let X be a Banach space, and let $\{F_\tau : \tau \in T\}$ be a family of X-valued bounded vector measures defined on Σ. If $\sup_\tau \|F_\tau(E)\| < \infty$ for each $E \in \Sigma$, then the family $\{F_\tau : \tau \in T\}$ is uniformly bounded, i.e. $\sup_{\tau \in T} \|F_\tau\|(\Omega) < \infty$.*

Corollary 1.2.7. *Let Σ be a σ-algebra of subsets of a set Ω. Suppose $\{T_\alpha : \alpha \in A\}$ is a collection of bounded linear operators from $B(\Sigma)$ to a Banach space X such that $\sup_{\alpha \in A} \|T_\alpha \chi_E\| < \infty$ for each $E \in \Sigma$. Then $\sup_\alpha \|T_\alpha\| < \infty$.*

It is clear from the theorems above, that if X and Y are Banach spaces and if A is a subset of Y, then the property on A that solves both Problem 1.2.1 and Problem 1.2.2, is strictly between A being span dense in Y and A being of the second category in Y. But still, what characterizes such a property?

In [26], Kadets and Fonf encovered a property which in fact solves Problem 1.2.1 in the case Y is a Banach space and A is a *bounded* subset of Y.

Theorem 1.2.8 (Kadets and Fonf, 1983)**.** *Let Y be a Banach space and suppose $A \subset S_Y$. The following are equivalent statements:*

(a) *For any Banach space X and any bounded linear operator $T : X \to Y$ such that $T(X) \supset A$, one has $T(X) = Y$.*

(b) *For every representation of A as the union of an increasing sequence of sets, $A = \cup_{i=1}^\infty A_i$, $(A_i \uparrow)$, there is an index j such that*

$$\inf_{y^* \in S_{Y^*}} \sup_{y \in A_j} |y^*(y)| > 0.$$

Theorem 1.2.8 suggests the following definition (cf. [26], [25], and [59]).

Definition 1.2.9. Let Y be a normed linear space. A subset $A \subset Y$ is said to have the *surjectivity property* if for every Banach space X, every $T \in \mathcal{L}(X, Y)$, such that $T(X) \supset A$, we have that T is onto Y. If the same conclusion holds for a subset $\mathcal{A} \subset \mathcal{L}(X, Y)$, we say that A has the *$\mathcal{A}$-restricted surjectivity property.* For the special case when $A \subset Y^*$ and $\mathcal{A}$ is the space of adjoints in $\mathcal{L}(X^*, Y^*)$, we say that A has the *weak*-surjectivity property.*

Note that Theorem 1.2.3 of Banach, says that every second category set in a Banach space has the surjectivity property.

Before we go into a further discussion of Problems 1.2.1 and 1.2.2, we need to agree on some more definitions (cf. [26], [25], and [59]).

Definition 1.2.10. A subset A of a Banach space Y (resp. a dual Banach space Y^*) is said to be *norming (resp. weak*-norming)* if $\inf_{y^* \in S_{Y^*}} \sup_{y \in A} |y^*(y)| > 0$ (resp. $\inf_{y \in S_Y} \sup_{y^* \in A} |y^*(y)| > 0$). The subset A is called *thin (resp. weak*-thin)* if it can be written as a countable increasing union of non-norming (resp. non-weak*-norming) sets. If A is not thin (resp. weak*-thin) it is called *thick (resp. weak*-thick).*

The following geometrical lemmas [59, Lemmas 2.2 and 2.3] are easy consequences of the Hahn-Banach separation Theorem.

Lemma 1.2.11. *Let Y be a real normed space and A a subset of Y. The following statements are equivalent.*

(a) *A is norming.*

(b) *$\overline{conv}(\pm A)$ is norming.*

(c) *There exists $\delta > 0$ such that $\overline{conv}(\pm A) \supset \delta B_Y$.*

Lemma 1.2.12. *Let Y be a real normed space and A a subset of Y^*. The following statements are equivalent.*

(a) *A is weak*-norming.*

(b) *$\overline{conv}^{w^*}(\pm A)$ is weak*-norming.*

(c) *There exists $\delta > 0$ such that $\overline{conv}^{w^*}(\pm A) \supset \delta B_{Y^*}$.*

We remark that if the space Y is complex, Lemma 1.2.11 and Lemma 1.2.12 hold if we replace $\overline{\text{co}}(\pm A)$ with $\overline{\text{co}}(\cup_{|r|=1} rA)$ where r is a complex number.

Of course a norming set in a dual space is weak*-norming. However, it does not need to be weak*-thick. The set of extreme points of the unit ball of ℓ_1 is such an example since it is countable. (Indeed, it is clear that every countable set is thin, or weak*-thin if it is in a dual space). There are also weak*-thick sets which are not norming. The unit ball of every non-reflexive Banach space, considered as a subset of the bidual, is such an example. Next we give an example of a set which is both norming and weak*-thick.

Example 1.2.13. Let $H^\infty(D)$ denote the space of bounded analytic functions on the open unit disk. The Blaschke products in $H^\infty(D)$ is a weak*-thick and norming set [59, Corollary 3.7]. See [70, p. 310] for a definition of Blaschke products. It is unknown whether the Blaschke products forms a thick set.

It is immediate from the definitions that every thick set in a dual space is weak*-thick. From the definitions it is also straightforward to verify that sets of the second category are thick [59, Lemma 3.4]. General examples of thick and weak*-thick sets are given by the results [27, Theorem 4.3], [60, Corollary 2.2], [24, Theorem 1], and [25, Theorem 3*].

Theorem 1.2.14 (Fonf and Lindenstrauss, 2003). *Let X be a separable non-reflexive Banach space. Then the set of functionals in X^* which do not attain their maximum on B_X is a thick set.*

Theorem 1.2.15 (Nygaard, 2006). *Let X be a Banach space. If $x^{**} \in X^{**} \setminus X$, then $\ker x^{**}$ is a weak*-thick subset of X^*.*

Recall that a subset B of the unit sphere S_{X^*} of the dual of a Banach space X is called a James boundary of X, if for every $x \in X$, there exists $x^* \in B$ such that $x^*(x) = \|x\|$.

Theorem 1.2.16 (Fonf, 1989). *Let X be a Banach space. If X does not contain a copy of c_0, then every James boundary of X is weak*-thick.*

Theorem 1.2.17 (Fonf, 1996). *Let X be a separable Banach space. If X does not contain a copy of c_0, then ω^*-$\exp B_{X^*}$ is weak*-thick.*

Definition 1.2.18. Let Y be a normed linear space. A subset $A \subset Y$ is said to have the *boundedness property* if for every normed linear space X, every family $(T_\alpha) \subset \mathcal{L}(Y, X)$, which is pointwise bounded on A, is bounded. If the same conclusion holds for a subset $\mathcal{A} \subset \mathcal{L}(Y, X)$, we say that A has the *$\mathcal{A}$-restricted boundedness property.* For the special case when $A \subset Y^*$ and $\mathcal{A}$ is the space of adjoints in $\mathcal{L}(Y^*, X^*)$, we say that A has the *weak*-boundedness property.*

From the Banach-Steinhaus Uniform Boundedness Principle [8] (see also [69, p. 43]) we have that sets of the second category in Banach spaces have the boundedness property. Note also that Theorem 1.2.5 of Seever and Corollary 1.2.7 of Nikodým and Grothendieck say that the characteristic functions in the unit sphere of $B(\Sigma)$ both have the surjectivity property and the boundedness property.

Nygaard proved in [59] the following general result.

Theorem 1.2.19 (Nygaard, 2002). *Suppose A is a subset of a Banach space Y. The following statements are equivalent.*

(a) *A has the surjectivity property.*

(b) *For every Banach space X, every injection $T : X \to Y$ which is onto A is an isomorphism.*

(c) *A has the boundedness property.*

(d) *Every sequence $(y_n^*) \subset Y^*$ which is pointwise bounded on A is a bounded sequence in Y^*.*

(e) *A is thick.*

Note that from Theorem 1.2.19 it follows that Seever's theorem and the Nikodým-Grothendieck Boundedness Theorem are the same.

In [59] another special case of Problem 1.2.1 was considered, that is the case when A is a subset of the dual of a Banach space X and the operators are adjoints into X^*.

Theorem 1.2.20 (Nygaard, 2002). *Suppose A is a subset of the dual of a Banach space Y. The following statements are equivalent.*

(a) *A has the weak*-surjectivity property.*

(b) *For every Banach space X, every dual injection $T : X^* \to Y^*$ which is onto A is an isomorphism.*

(c) *A has the weak*-boundedness property.*

(d) *Every sequence $(y_n) \subset Y$ which is pointwise bounded on A is a bounded sequence in Y.*

(e) *A is weak*-thick.*

The notion of weak*-thick sets also turns up in the theory of vector measures. Let us recall the basic definitions from this theory (cf. e.g. [16]).

Let X be a Banach space and let $\mathscr{F}$ be an algebra of subsets of a set Ω. A set function $F : \mathscr{F} \to X$ is called a *vector measure* if whenever E_1 and E_2

are disjoint members of $\mathscr{F}$, then $F(E_1 \cup E_2) = F(E_1) + F(E_2)$. If, in addition $F(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} F(E_n)$, with convergence in the norm-topology of X, for all sequences (E_n) of pairwise disjoint members of $\mathscr{F}$ such that $\cup_{n=1}^{\infty} E_n \in \mathscr{F}$, then F is said to be a *countably additive vector measure*. Moreover, a vector measure $F : \mathscr{F} \to X$ is said to be bounded if $\sup_{E \in \mathscr{F}} \|F(E)\| < \infty$.

If Σ is a σ-algebra of subsets of a set Ω, and μ a measure on Σ, then a function $f : \Omega \to X$ is called *weakly μ-measurable* if for every $x^* \in X^*$ the scalar valued function $x^* f$ is μ-measurable.

The following theorem was proved by Dunford already in 1937 (cf. [16, p. 52]).

Theorem 1.2.21 (Dunford, 1937). *Let X be a Banach space, Σ a σ-algebra of subsets of a set Ω, and μ a measure. If $f : \Omega \to X$ is a function such that $x^* f \in L_1(\mu)$ for every $x^* \in X^*$, then for each $E \in \Sigma$ there exists $x_E^{**} \in X^{**}$ satisfying*

$$x_E^{**}(x^*) = \int_E x^*(f) d\mu \tag{1.2.1}$$

for all $x^* \in X^*$.

Based on this result, we can define the Dunford integral.

Definition 1.2.22. A weakly μ-measurable function $f : \Omega \to X$ is called *Dunford integrable* if $x^* f \in L_1(\mu)$ for every $x^* \in X^*$. The Dunford integral of f over $E \in \Sigma$ is defined by the element x_E^{**} of X^{**} in (1.2.1). We denote this integral by $(D) - \int_\Omega f \mathrm{d}\mu$.

Moreover, if $(D) - \int_\Omega f \mathrm{d}\mu \in X$, then f is called Pettis integrable.

In [18] and [14] Dimitrov and Diestel independently proved the following result.

Theorem 1.2.23 (Dimitrov, 1971 and Diestel, 1973). *Let X be a separable Banach space which does not contain isomorphic copies of c_0 and let (Ω, Σ, μ) be a finite measure space. Then every Dunford integrable function $f : \Omega \to X$ is Pettis integrable.*

Using this theorem of Dimitrov and Diestel, in combination with the fact that when a Banach space X is c_0 free, the set $\operatorname{ext} B_{X^*}$ is weak*-thick [24, Theorem 1] (cf. Theorem 1.2.16), Fonf obtained the following theorem.

Theorem 1.2.24 (Fonf, 1989). *Let X be a separable Banach space which does not contain isomorphic copies of c_0. Then, whenever (Ω, Σ, μ) is a finite measure space and a function $f : \Omega \to X$ is such that $x^* f \in L_1(\mu)$ for every $x^* \in \operatorname{ext} B_{X^*}$, we have $x^* f \in L_1(\mu)$ for every $x^* \in X^*$ and f is Pettis integrable.*

The main objective of the article [3] (cf. Chapter 2) is to generalize the above result of Fonf. We do this by giving the following characterization of weak*-thick sets (cf. Chapter 2, Main theorem).

Theorem 1.2.25 (Abrahamsen, Nygaard, and Põldvere, 2006). *Let X be a Banach space. A subset $A \subset X^*$ is weak*-thick if and only if whenever (Ω, Σ, μ) is a measure space and $f : \Omega \to X$ is an essentially separable valued function such that $x^* f \in L_1(\mu)$ for all $x^* \in A$, then $x^* f \in L_1(\mu)$ for all $x^* \in X^*$.*

Let (x_n) be a sequence in a Banach space X. Observe that, for any $x^* \in X^*$, we have $\sum_{n=1}^{\infty} |x^*(x_n)| = \int_{\mathbb{N}} |x^* f| \mathrm{d}c$, where c is the counting measure on the σ-algebra $\mathcal{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$ and $f : \mathbb{N} \to X$ is the function defined by $f = \sum_{n=1}^{\infty} \chi_{\{n\}} x_n$. Now, using Theorem 1.2.25, (b) $\Rightarrow$ (a) in the following characterization of weak*-thin sets, is immediate (cf. Corollary 2.2.4). The reverse implication is proved by using a "gliding hump" argument.

Corollary 1.2.26 (Abrahamsen, Nygaard, and Põldvere, 2006)**.** *Let X be a Banach space and $A \subset X^*$. The following statements are equivalent.*

(a) *A is weak*-thin.*

(b) *There exists a sequence $(x_n) \subset X$ and $x^* \in X^* \setminus A$ such that $\sum_{n=1}^{\infty} |x^*(x_n)|$ diverges, but $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$ for all $x^* \in A$.*

In [19] Elton proved the theorem stated below.

Theorem 1.2.27 (Elton, 1981)**.** *Let X be a Banach space. The following statements are equivalent.*

(a) *X contains a copy of c_0.*

(b) *There exists a divergent series $\sum_{n=1}^{\infty} x_n$ in X such that $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$ for all $x^* \in \operatorname{ext} B_{X^*}$.*

Fonf proved in [25, Theorem 3*] that a separable Banach space X contains c_0 whenever the set ω^*-$\exp B_{X^*}$ is weak*-thin. He then combined this result with the well known Bessaga-Pełczyński Theorem [10] and deduced that the set $\operatorname{ext} B_{X^*}$ can be replaced by the set ω^*-$\exp B_{X^*}$ in the above theorem of Elton [25, Theorem 6].

Using the Nikodým-Grothendieck Boundedness Theorem one can prove the following important result of Dieudonné and Grothendieck (cf. [16, p. 16]).

Theorem 1.2.28 (Dieudonné and Grothendieck)**.** *Let X be a Banach space and let F be an X-valued set function defined on a σ-algebra Σ. Suppose that x^*F is bounded and finitely additive for each x^* belonging to some total subset A of X^*. Then F is a bounded vector measure.*

Note that the additivity of F is immediate from the totality of Γ.

Theorem 1.2.28 may fail for algebras which are not σ-algebras. A stronger property is needed in this case. Indeed, if "total" is replaced by "weak*-thick" in this Theorem 1.2.28, then we get a test for boundedness of vector measures defined merely on algebras. In fact, we also get a new characterization of weak*-thick sets (cf. Propositions 2.3.2 and 2.3.3)

Theorem 1.2.29 (Abrahamsen, Nygaard, and Põldvere, 2006)**.** *Let X be a Banach space and A a subset of X^*. The following statements are equivalent.*

(a) *For every algebra $\mathscr{F}$ and every set function $F : \mathscr{F} \to X$, the function F is a bounded vector measure whenever the function x^*F is bounded and finitely additive for each $x^* \in A$.*

(b) *A is weak*-thick.*

1.2.2 Further results and a generalized thickness notion

Let $|\cdot|$ denote the distance function on $\mathbb{K}$. Recall that a function f from a topological linear space X into the real numbers is said to be *lower semi-countinuous* if $f(x) \leq \liminf_\alpha f(x_\alpha)$ whenever (x_α) is a net in X converging to some element $x \in X$. A function f is called *convex* if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for every $x, y \in X$ and $0 \leq t \leq 1$.

It is not difficult to see that Theorem 1.2.19 can be continued by

(f) *Whenever a sequence of functions $\{f_n : Y \to \mathbb{K}\}$, with the properties that for every natural number n, $|\cdot| \circ f_n$ is lower semi-continuous and convex, is pointwise bounded on A, then this sequence is uniformly bounded on B_Y.*

Evidently every linear functional in a dual Banach space is lower semi-continuous and convex when left composed with $|\cdot|$, so (f) implies (d) in Theorem 1.2.19 above. The fact that (e) in Theorem 1.2.19 implies (f), follows from the same argument as in (e) implies (d) in Theorem 1.2.19. Indeed, assume that A is thick and put $A_n = \{y \in Y \cap A : \sup_k |f_k(y)| \leq n\}$. By the pointwise boundedness, (A_n) form an increasing, countable covering of A. Since A is thick, there exists a natural number m such that A_m is norming. By Lemma 1.2.11, there exists a real number $\delta > 0$ such that $\overline{\text{absconv}}(A_m) \supset \delta B_Y$. Finally, observe that we only need $|\cdot| \circ f_n$ to be convex and lower semi-continuous, to conclude that $\sup_k \sup_{y \in B_Y} |f_k(y)| \leq \frac{m}{\delta}$.

A similar argument as in the preceding paragraph proves that Theorem 1.2.20 can be continued by

(f) *Whenever a sequence of functions $\{f_n : Y^* \to \mathbb{K}\}$, with the properties that for every natural number n, $|\cdot| \circ f_n$ is weak*-lower semi-continuous and convex, is pointwise bounded on A, then this sequence is uniformly bounded on B_{Y^*}.*

As already mentioned, second category sets in a Banach space are thick. The converse is not true. A standard counterexample is the set of 0-1 sequences in ℓ_∞ which is thick by Nikodým-Grothendieck Boundedness Theorem and Theorem 1.2.19. The set is nowhere dense, so it is trivially of the first category. Based on this on can ask: Which Banach spaces contain thick sets of the first category? The interesting and surprising answer is that indeed *every* Banach space does. This follows from the fact that every Banach space contains a Hamelbasis of the first category [9, Proposition 3.2] and Theorem 1.2.19. In other words we can conclude from this that every Banach space contains a set on which (the category version of) the Banach-Steinhaus Uniform Boundedness Principle does not apply, but Theorem 1.2.19 does.

Let X be a Banach space and assume $\mathcal{F}$ is a subset of X^*. Suppose we want to determine whether $\mathcal{F}$ is bounded or not. From Theorem 1.2.19, we know that $\mathcal{F}$ is bounded if and only if it is pointwise bounded on a thick set A in X. But suppose we know in addition that $\mathcal{F}$ belongs to some (weak*-dense linear) subset Γ of X^*. Can we then weaken the restrictions on A and still have an equivalence as in Theorem 1.2.19? We can state the following problem.

Problem 1.2.30. *Let A be a subset of a Banach space X and let $\mathcal{F} \subset \Gamma$ where Γ is a weak*-dense linear subset of X^*. Which condition (P_Γ) must A fulfill so* *that boundedness of $\mathcal{F}$ can be deduced from testing pointwise boundedness of $\mathcal{F}$ on A?*

Note that for $A \subset X$ in case $\Gamma = X \subset X^{**}$, (P_Γ) is exactly the weak*-boundedness property for A.

Let A be a subset of a Banach space X. The following list of examples are special cases of the problem above:

(a) If Y and Z are Banach spaces, $X = \mathcal{L}(Y, Z)$, and $\Gamma = Y \otimes Z^*$.

(b) If Γ is the (norm closed) linear span of the extreme points of B_{X^*} (or of any James boundary).

(c) If X has a Schauder basis and Γ is the (norm closed) linear span of the biorthogonal functionals in X^* associated with the basis.

(d) If Y is a Banach space and $T : X \to Y$ is a bounded linear injection and $\Gamma = T^*(Y^*)$.

(e) If X is a dual Y^* and Γ is the Baire functionals, $\mathrm{Ba}(Y)$, in Y^{**}.

Motivated by Problem 1.2.30 and the definitions of norming and weak*-norming and thin and weak*-thin sets, we make the following definition.

Definition 1.2.31. Let X be a Banach space and Γ a weak*-dense linear subset of X^*. A subset A of X is called Γ-*norming* if $\inf\{\sup_{x\in A} |x^*(x)| : x^* \in S_{X^*} \cap \Gamma\} > 0$. If the set A is not Γ-norming, then it is called *non-Γ-norming.* Moreover, A is said to be Γ-*thin* if it can be written as a countable increasing union of non-Γ-norming sets. If it is not Γ-thin, then it is called Γ-*thick.*

Note that a bounded set is Γ-norming if and only if it is $\overline{\Gamma}$-norming (norm closure in X^*). Thus a set Γ and its norm closure share the same thick sets. However, is the converse true, i.e. is it so that two sets $\Gamma_1, \Gamma_2 \subset X^*$ which share the same thick sets have the same norm closures? Indeed, the following result answers this question in the affirmative, and hence provides a good reason to study the special cases of Problem 1.2.30 listed above.

Theorem 1.2.32. *Let X be a Banach space and let $\Gamma_1 \subset \Gamma_2$ be weak*-dense linear subspaces of X^*. Then Γ_1 and Γ_2 share the same thick sets if and only if Γ_1 and Γ_2 have the same norm-closure.*

We sketch a proof of this result.

Proof. As noted in the paragraph above Γ_1 and Γ_2 share the same thick sets if they have the same norm closures. For the converse one can assume that Γ_1 is not norm-dense in $\overline{\Gamma_2}$, then choose $x^* \in \overline{\Gamma_2} \setminus \overline{\Gamma_1}$ and put $A = \ker x^*$. It is evident that A now is Γ_2-thin and not to hard to show using [16, Lemma 2] that A is Γ_1-norming. This latter fact in combination with Banach's lemma (see e.g. [33, Lemma 82]), is then used to prove that A is Γ_1-thick. □

From the proof of Theorem 1.2.32, the next corollary follows.

Corollary 1.2.33. *Let X be a Banach space. Suppose Γ is a weak*-dense linear subspace of X^* and $\overline{\Gamma} \neq X^*$. If $x^* \in X^* \setminus \Gamma$, then $\ker x^*$ is a thin, but Γ-thick, set.*

Note that this result generalizes [60, Corollary 2.2] of Nygaard presented in Theorem 1.2.15.

1.3 Background on approximation properties and u-ideals

A fundamental question in functional analysis is whether compact operators, from a Banach space Y into a Banach space X, can be approximated in norm by sequences of finite rank operators. (This has been called the approximation problem for obvious reasons.) A Banach space X for which this is true for every Banach space Y, is said to have the approximation property. The first formal treatment of the approximation property was done by Grothendieck [32] in his doctoral thesis from 1955. In his thesis he produced equivalent formulations of the approximation property. It is, however, clear from [67] that Banach and his collaborators, knew many of these equivalences.

In [32], Grothendieck defined stronger forms of the approximation property, e.g. the bounded approximation property and the metric approximation property. A powerful and important result concerning the latter of these two properties, says that for separable dual spaces, the approximation property implies the metric approximation property. This result has never been generalized to non-separable Banach spaces. However, in some unpublished lecture notes (see [16, p. 256]), Rosenthal has shown that for a Banach space with the Radon-Nikodým property which is 1-complemented in its bidual, the approximation property implies the metric approximation property. Thus for a dual Banach space with the Radon-Nikodým property, the approximation property implies the metric approximation property. The result of Rosenthal is actually also implicit in Grothendieck's thesis [32].

Lima and Oja [55] have recently made a new approach to answer the problem of whether Grothendieck's result holds for non-separable spaces. They did so by introducing the weak metric approximation property. The weak metric approximation property is weaker than the metric approximation property and strictly stronger than the approximation property [55, Proposition 2.2]. For dual spaces, however, Lima and Oja has proved that the approximation property implies the weak metric approximation property [55, Corollary 3.4]. So the problem of determining whether Grothendieck's result generalizes to non-separable spaces still remains, but now we are left with the question of whether the weak metric approximation property implies the metric approximation property for non-separable dual spaces.

Most recently [44] the weak metric approximation property has been characterized in terms of ideals of finite rank operators and Hahn-Banach extension operators. The article [1], which constitutes Chapter 3 in this thesis, contains generalized forms of characterizations of the weak metric approximation property obtained in [55] and [44].

The study of u-ideals and the unconditional metric approximation property, emerged from the article [11] by Casazza and Kalton. Casazza and Kalton proved that for a separable reflexive Banach space X with the approximation property, $\mathcal{K}(X,X)$ is a u-ideal in $\mathcal{L}(X,X)$ if and only if X has the unconditional metric approximation property. Lima [50] generalized this result by showing that it holds when the unconditional metric approximation property is replaced by the unconditional metric compact approximation property and when only assuming X to have the Radon-Nikodým property. However, removing the Radon-Nikodým property from the assumption, Lima and Lima [45] showed that the above result is equivalent to $\mathcal{K}(Y,X)$ being a u-ideal in $\mathcal{L}(Y,X)$ for every Banach space Y which in turn is equivalent to $\mathcal{K}(\hat{X},X)$ being a u-ideal in $\mathcal{L}(\hat{X},X)$ for every equivalent renorming $\hat{X}$ of X. A similar result for dual spaces having the unconditional metric compact approximation property with conjugate operators, was also obtained in [45].

In the article [2], which constitutes Chapter 4 in this thesis, we look at the finite rank operators and obtain characterizations for when they are u-ideals in the space of weakly compact operators.

1.3.1 Basic results on approximation properties and u-ideals

A sequence (x_n) in a Banach space X is called a *Schauder basis* for X if for each $x \in X$ there is a unique sequence (α_n) of scalars such that

$$x = \lim_n \sum_{k=1}^{n} \alpha_k x_k.$$

On page 111 in the famous book *Théorie des Opération Linéaires* [7] from 1932, the following problem appears: "Does every separable Banach space have a Schauder basis?" This problem, known as the basis problem, remained open for a long time and was solved in the negative by Enflo [20] in 1973. Enflo constructed a separable, reflexive Banach space without the approximation property, and by doing so he also solved the approximation problem.

Definition 1.3.1 (Grothendieck, 1955)**.** A Banach space X has the *approximation property* (AP) if for every compact set K in X and every $\varepsilon > 0$, there is an operator $T : X \to X$ of finite rank such that $\|Tx - x\| < \varepsilon$, for every $x \in K$. If these approximating finite rank operators can be chosen with $\|T\| \leq \lambda$, for some $\lambda \geq 1$, then X is said to have the λ*-bounded approximation property* (λ-BAP). A Banach space is said to have the *bounded approximation property* (BAP) if it has λ-BAP for some λ. We say that X has the *metric approximation property* if it has 1-BAP.

A Banach space with a Schauder basis has the BAP and hence the AP. So Enflo's space is, in particular, an example of a separable Banach space without a Schauder basis. Right after Enflo's construction was published, Davie [12] simplified it and showed that c_0 and ℓ_p, for $p > 2$, have subspaces without the AP. Later the same decade, Szankowski [74] proved that also ℓ_p, for $1 \leq p < 2$, have subspaces without the AP. Szankowski [75] has also proved that the space of bounded linear operators on an infinite dimensional Hilbert space fails the AP.

In 1973, using Enflo's example, Figiel and Johnson [23] showed that there is a Banach space with the AP which fails the BAP. In 1987 Szarek [76] showed that there exists a reflexive Banach space without a basis which has the BAP. It has also been proved that there are Banach spaces with the BAP which fail the MAP (cf. e.g. [57, p. 42]).

In many cases, however, the AP implies the MAP. A powerful and surprising result of Grothendieck [32] (see e.g. [57, p. 39] for a nice proof of this) reads.

Theorem 1.3.2 (Grothendieck, 1955)**.** *Let X be a separable Banach space which is isometric to a dual space and which has the AP. Then X has the MAP.*

It is, however, still an open problem whether this result holds for non-separable spaces.

Problem 1.3.3. *Does the AP of the dual space X^* of a Banach space X imply the MAP?*

The obvious reason why it is still unknown whether Theorem 1.3.2 holds for non-separable spaces, is that the proof does not generalize to such spaces. The fact that in a separable dual Banach space X^*, the sets B_{X^*} and $B_{X^{**}}$ are compact metric in their corresponding weak* topologies, are crucial parts of the proof.

In 1974 Davis, Figiel, Johnson, and Pełczyński [13, Corollary 1] proved that every weakly compact operator factors through a reflexive Banach space. Lima, Nygaard, and Oja later improved this result in [51, Theorems 2.3 and 2.4] by showing that the factorization can by done isometrically and even uniformly with respect to finite dimensional subspaces. Their proof is based on the Davis-Figiel-Johnson-Pełczyński construction. However, in the Lima-Nygaard-Oja version of the Davis-Figiel-Johnson-Pelczyński construction, the number 2 is replaced by $\sqrt{a}$ for $a > 1$. This seemingly minor change, turns out to be important.

Let $a > 1$ and let K be a closed absolutely convex subset of the unit ball B_X of a Banach space X. For each positive integer n, put $B_n = a^{\frac{n}{2}}K + a^{-\frac{n}{2}}B_X$ and denote by $\|\cdot\|_n$ the equivalent norm on X defined by the gauge on B_n. Let $\|x\|_K = (\sum_{n=1}^{\infty}\|x\|_n)^{\frac{1}{2}}$, $X_K = \{x \in X : \|x\|_K < \infty\}$, $C_K = \{x \in X : \|x\|_K \leq 1\}$, and let J_K denote the identity embedding of X_K into X. Finally, define $f : (1,\infty) \to \mathbb{R}$ by

$$f(a) = \Big(\sum_{n=1}^{\infty}\frac{a^n}{(a^n+1)^2}\Big)^{1/2}.$$

It can be shown that there is a unique $\tilde{a} \in (1,\infty)$ such that $f(\tilde{a}) = 1$. For this fixed number $\tilde{a}$, Lima, Nygaard, and Oja proved in Lemmas 1.1 and 2.1 in [51], the following isometric version of Lemma 1 in [13].

Lemma 1.3.4 (Lima, Nygaard, and Oja, 2000)**.** *Let K be a closed absolutely convex subset of the unit ball B_X of a Banach space X. If $a \in (1,\infty)$ is such that $f(a) = 1$, then*

(a) $K \subset C_K \subset B_X$

(b) $(X_K, \|\cdot\|_K)$ *is a Banach space with closed unit ball C_K, and $J_K \in \mathcal{L}(X_K, X)$ with $\|J_K\| \leq 1$.*

(c) J_K^{**} *is injective.*

(d) X_K *is reflexive if and only if K is weakly compact.*

(e) *The X-norm and the X_K-norm topologies coincide on K.*

(f) *The weak topologies defined by X^* and X_K^* coincide on C_K.*

(g) *C_K as a subset of X is compact, weakly compact, or separable if and only if K has the same property.*

Davis, Figiel, Johnson, and Pełczyński used their version of the preceding result to prove that every weakly compact operator factors through a reflexive space. Similarly Lima, Nygaard, and Oja applied their quantitative modified version to prove that the factorization can be done isometrically and uniformly in the following way.

Theorem 1.3.5 (Lima, Nygaard, and Oja, 2000)**.** *Let F be a finite dimensional subspace of $\mathcal{W}(Y,X)$. Then there exist a reflexive space Z, a norm one operator $J : Z \to X$, and a linear isometry $\Phi : F \to \mathcal{W}(Y,Z)$ such that $T = J \circ \Phi(T)$ for all $T \in F$. Moreover,*

(a) *$Z = X_K$ and $J = J_K$ for the weakly compact absolutely convex set $K = \overline{\mathrm{conv}}\{Ty : T \in B_F \text{ and } y \in B_Y\}$ whenever the number a is fixed so that $f(a) = 1$.*

(b) *T is compact if and only if $\Phi(T)$ is compact.*

(c) *T has finite rank if and only if $\Phi(T)$ has finite rank.*

Corollary 1.3.6 (Lima, Nygaard, and Oja, 2000)**.** *Let F be a finite dimensional subspace of $\mathcal{W}(X,Y)$. Then there exist a reflexive space Z, a norm one operator $J : X \to Z$, and a linear isometry $\Phi : F \to \mathcal{W}(Z,Y)$ such that $T = \Phi(T) \circ J$ for all $T \in F$. Moreover,*

(a) *T is compact if and only if $\Phi(T)$ is compact.*

(b) *T has finite rank if and only if $\Phi(T)$ has finite rank.*

Using their version of the Davis-Figiel-Johnson-Pełczyński construction, Lemma 1.3.4, Lima, Nygaard, and Oja proved in [51, Corollary 1.5] that the approximation property has a "metric" equivalent.

Theorem 1.3.7 (Lima, Nygaard, and Oja, 2000)**.** *Let X be a Banach space. The following statements are equivalent.*

(a) *X has the approximation property.*

(b) *For every Banach space Y and every $T \in \mathcal{W}(Y,X)$, there is a net (T_α) in $\mathcal{F}(Y,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha \to T$ in the strong operator topology.*

(c) *For every separable reflexive Banach space Y and every $T \in \mathcal{K}(Y,X)$, there is a net (T_α) in $\mathcal{F}(Y,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha \to T$ in the strong operator topology.*

One can show that Theorem 1.3.7 can be continued by

(d) *For every Banach space Y and every $T \in \mathcal{W}(Y,X)$, there is a net (S_α) in $\mathcal{F}(X,X)$ with $\sup_\alpha \|S_\alpha T\| \le \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X.*

(e) *For every separable reflexive Banach space Y and every $T \in \mathcal{K}(Y,X)$, there is a net (S_α) in $\mathcal{F}(X,X)$ with $\sup_\alpha \|S_\alpha T\| \leq \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X.*

Proof. We only need to show that (b) $\Rightarrow$ (d). To this end, first note that the net (T_α) in (b) may be assumed to converge uniformly on compact sets in Y. Now, let $\varepsilon > 0$ and $T \in \mathcal{W}(Y,X)$ of norm one. Let $u_k = \sum_{n=1}^{\infty} x_{k,n}^* \otimes x_{k,n} \in X^* \hat{\otimes}_\pi X = (\mathcal{L}(X,X),\tau)^*$ for $k = 1,...,m$ where τ is the topology of uniform convergence on compact sets in X (see e.g. [57, Proposition 1.e.3]). Assume $\sum_{n=1}^{\infty} \|x_{k,n}^*\| < \infty$ and $1 \geq \|x_{k,n}\| \to 0$ for each $k = 1,...,m$. Put $K = \overline{\text{conv}}\{\pm T(B_Y) \cup \{x_{k,n}\} : k = 1,...,m; n = 1,2,...\} \subset B_X$. Let Z be the Banach space constructed from K in Lemma 1.3.4, and let $J : Z \to X$ be the identity embedding of Z into X. Now Z is reflexive and $J \in \mathcal{W}(Z,X)$ is of norm one. From (b) in Theorem 1.3.7 and the two first lines in this paragraph, there is a net $(J_\alpha) \subset \mathcal{F}(Z,X)$ with $\sup_\alpha \|J_\alpha\| \leq \|J\| = 1$ such that $J_\alpha \to J$ uniformly on compact sets in Z. By Lemma 1.3.4 J^*X^* is norm-dense in Z^* and thus we can write $J_\alpha = S_\alpha J$ where S_α is in $\mathcal{F}(X,X)$. For each $x_{k,n}$ and $k = 1,...,m, n = 1,...$ choose $z_{k,n} \in B_Z$ and S in (S_α) such that $Jz_{k,n} = x_{k,n}$ and

$$\begin{aligned}\varepsilon &> \max_{1\leq k\leq m} \Big| \sum_{n=1}^{\infty} \langle SJz_{k,n}, x_{k,n}^* \rangle - \sum_{n=1}^{\infty} \langle Jz_{k,n}, x_{k,n}^* \rangle \Big| \\ &= \max_{1\leq k\leq m} \Big| \sum_{n=1}^{\infty} \langle Sx_{k,n}, x_{k,n}^* \rangle - \sum_{n=1}^{\infty} \langle x_{k,n}, x_{k,n}^* \rangle \Big|.\end{aligned}$$

Thus (d) follows from (b). □

In [55] Lima and Oja introduced the weak metric approximation property.

Definition 1.3.8. A Banach space X has the *weak metric approximation property (weak MAP)* if for every Banach space Y and for every $T \in \mathcal{W}(X,Y)$, there is a net (S_α) in $\mathcal{F}(X,X)$ with $\sup_\alpha \|TS_\alpha\| \leq \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X.

Note that the only difference between Definition 1.3.8 and statement (d) in Theorem 1.3.7 is that the roles of X and Y are interchanged. Comparing definitions it is immediate that MAP $\Rightarrow$ weak MAP $\Rightarrow$ AP. The fact that the weak MAP is strictly stronger than the AP follows from [55, Proposition 2.1]. Recently, Oja [66, Corollary 1] showed that if a Banach space has the weak MAP, then it has the MAP if either its dual or its bidual have the Radon-Nikodým property. It is still unknown if the weak MAP implies the MAP in general. However, in [55, Corollary 3.4] it is shown that for dual spaces the AP implies the weak MAP. Hence Problem 1.3.3 can be restated as follows.

Problem 1.3.9. *Does the weak MAP of the dual space X^* of a Banach space X imply the MAP?*

In [55, Theorem 2.4] Lima and Oja proved the following characterization of the weak MAP.

Theorem 1.3.10 (Lima and Oja, 2005)**.** *Let X be a Banach space. The following statements are equivalent.*

(a) *X has the weak MAP.*

(b) *For every separable reflexive Banach space Y and for every operator $T \in \mathcal{K}(X,Y)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\sup_\alpha \|TS_\alpha\| \le \|T\|$ such that $S_\alpha \to I_X$ in the strong operator topology.*

(c) *For every separable reflexive Banach space Y and for every operator $T \in \mathcal{K}(X,Y)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\sup_\alpha \|TS_\alpha\| \le \|T\|$ such that $TS_\alpha \to T$ in the strong operator topology.*

(d) *For every Banach space Y, for every operator $T \in \mathcal{W}(X,Y)$ with $\|T\| = 1$, and for all sequences $(x_n) \subset X$, and $(y_n^*) \subset Y^*$ with $\sum_{n=1}^\infty \|x_n\|\|y_n^*\| < \infty$, one has the inequality*

$$|\sum_{n=1}^\infty y_n^*(Tx_n)| \le \sup_{\|TS\|\le 1, S\in\mathcal{F}(X,X)} |\sum_{n=1}^\infty y_n^*(TSx_n)|.$$

In [29] Godefroy, Kalton, and Saphar introduced the notion of an ideal.

Definition 1.3.11. A closed subspace X of a Banach space Y is an *ideal* in Y if the annihilator $X^\perp$ is the kernel of a linear norm one projection on Y^*. Such a projection is called an *ideal projection*.

It is straightforward to show that ideals can be expressed in terms of Hahn-Banach extension operators.

Definition 1.3.12. Let X be a subspace of a Banach space Y. A linear operator $\phi : X^* \to Y^*$ is called a *Hahn-Banach extension operator* if $\phi(x^*)(x) = x^*(x)$ and $\|\phi(x^*)\| = \|x^*\|$ for every $x \in X$ and $x^* \in X^*$. We write $\mathrm{HB}(X,Y)$ for the set of all Hahn-Banach extension operators from X^* into Y^*.

The justification for this terminology comes from the Hahn-Banach Theorem, which tells us that every element $x^* \in X^*$ has a norm-preserving extension to Y. A Hahn-Banach extension operator extends all elements in X^* linearly.

The connection between ideals and Hahn-Banach extension operators was announced above. Indeed, if $i_X : X \to Y$ is the natural inclusion and $\phi \in \mathrm{HB}(X,Y)$, then the operator $P = \phi \circ i_X^*$ is an ideal projection on Y^* with $\ker P = X^\perp$ (P is usually called the corresponding ideal projection to ϕ). Conversely, if X is an ideal in Y with an ideal projection P, then $\phi : X^* \to Y^*$ defined by $\phi x^* = Py^*$, where $y^* \in \mathrm{HB}(x^*)$, the set of norm-preserving extensions of x^* to Y, is a Hahn-Banach extension operator (ϕ is called the corresponding Hahn-Banach extension operator to P). Thus $\mathrm{HB}(X,Y) \ne \emptyset$ if and only if X is an ideal in Y.

Lima [44, Theorem 2.6 and Proposition 3.1] has showed that the weak MAP can be characterized in terms of ideals of finite rank operators and Hahn-Banach extension operators.

Theorem 1.3.13 (Lima)**.** *Let X be a Banach space. The following statements are equivalent.*

(a) *X has the weak MAP.*

(b) *For every Banach space Y, $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X^{**})$.*

(c) *For every separable reflexive Banach space Y, $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{K}(Y,X^{**})$.*

(d) *There exists a Hahn-Banach extension operator $\phi \in H\!B(X,X^{**})$ such that for every choice of sequences $(x_n^*)_{n=1}^\infty \subset X^*$ and $(x_n^{**})_{n=1}^\infty \subset X^{**}$ with $\sum_{n=1}^\infty \|x_n^*\|\|x_n^{**}\| < \infty$ and $\sum_{n=1}^\infty x_n^*(x)x_n^{**} = 0$, for all $x \in X$ we have*

$$\sum_{n=1}^\infty \phi(x_n^*)(x_n^{**}) = 0.$$

(e) *There exists a Hahn-Banach extension operator $\phi \in H\!B(X,X^{**})$ such that for every reflexive Banach space Y and operator $T \in \mathcal{W}(Y,X^{**})$ we have $\phi^*|_{X^{**}}T \in \mathcal{F}(Y,X)^{**}$.*

(f) *There exists a Hahn-Banach extension operator $\phi \in H\!B(X,X^{**})$ such that for every reflexive Banach space Y and operator $T \in \mathcal{K}(Y,X^{**})$ we have $\phi^*|_{X^{**}}T \in \mathcal{F}(Y,X)^{**}$.*

In [1] (cf. Theorem 3.2.4) we generalize Theorem 1.3.13 by proving that the extension operator $\phi \in H\!B(X,X^{**})$, can be replaced by an extension operator $\phi_P \in H\!B(X,X^{**})$ such that $P = \phi_P^*|_{X^{**}}$ is a projection on X^{**}. The fact that this can be done, follows from the result below (cf. Theorem 3.2.1). We state Theorem 3.2.1 in a slightly different manner here.

Theorem 1.3.14 (Abrahamsen, 2007)**.** *Let X be a Banach space.*

(a) *If P is a norm one projection on X^{**} with $X \subset P(X^{**})$, then $\varphi_P = P^*k_{X^*} \in H\!B(X,X^{**})$.*

(b) *If there exists a Hahn-Banach extension operator $\varphi \in H\!B(X,X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$, then there exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$.*

Using Theorem 1.3.14 in combination with Lemma 1.3.4 and a result of Godefroy and Saphar [30, Theorem 1.5], one can prove that the following holds (cf. Proposition 3.2.2).

Proposition 1.3.15 (Abrahamsen, 2007)**.** *Let X be a Banach space with the weak MAP. Then there exists a projection P on X^{**} with $X \subset P(X^{**})$ such that for every reflexive Banach space Y and for every $T \in \mathcal{W}(X,Y)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup_\alpha \|TS_\alpha\| \le \|T\|$ such that $S_\alpha \to P$ weak* in $\mathcal{L}(X^{**},X^{**})$.*

Of course Proposition 1.3.15 holds for every Banach space Y and not just for reflexive Y. Indeed, this is immediate from Corollary 1.3.6 by putting $F = \operatorname{span}\{T\}$ for $T \in \mathcal{W}(X,Y)$. On the basis of this, Proposition 1.3.15 should be compared with Definition 1.3.8.

Prior to the notion an ideal, Alfsen and Effors had introduced the notion of an M-ideal in a Banach space in their fundamental article [4] from 1972. Part of their aim was to generalize structure theories for C^*-algebras and L_1-preduals. This becomes transparent from the definition below and the fact that in C^*-algebras M-ideals are exactly the closed two-sided algebraic ideals.

Definition 1.3.16. Let Y be a Banach space. A linear projection P on Y is called an L-*projection* if

$$\|y\| = \|Py\| + \|y - Py\| \text{ for all } y \in Y.$$

A closed subspace $X \subset Y$ is called an L-summand in Y if it is the range of an L-projection. If the annihilator $X^\perp \subset Y^*$ of X is an L-summand, then X is called an M-ideal in Y.

Vaguely spoken, if X is an M-ideal in Y, then the norm of Y^* resembles the ℓ_1-norm and the norm of Y thus ought to resemble the max-norm. M-ideals have been thoroughly studied in many articles. The reader should confer the book [35] for a nice and exhaustive presentation of M-ideal theory.

From the definitions it is immediate that M-ideals are stronger forms of ideals. Also properties intermediate that of being an M-ideal and that of being an ideal, have been studied in the literature (see e.g. [37], [62]). An unconditional ideal is one such property. The notion of an unconditional ideal was introduced by Kalton and Casazza in [11].

Definition 1.3.17. A closed subspace X of a Banach space Y is an unconditional ideal (u-*ideal*) in Y if there exists a linear projection P on Y^* with $\ker P = X^\perp$ such that $\|I - 2P\| = 1$.

It is straightforward to show that this definition is equivalent to $\|v + x^\perp\| = \|v - x^\perp\|$ for every $v \in P(Y^*)$ and $x^\perp \in X^\perp$. Thus, if X is a u-ideal in Y, the norm on Y^* fulfills a symmetry condition.

In [56] Lindenstrauss and Rosenthal showed that finite dimensional subspaces of the bidual of a Banach space X, are more or less the same as those of X. This fact is commonly referred to as the *Principle of Local Reflexivity.* The version of this principle listed below was proved in [41] and is a slightly stronger form of that of Lindenstrauss and Rosenthal.

Theorem 1.3.18 (Principle of Local Reflexivity, 1969)**.** *Let X be a Banach space, and let E and F be finite dimensional subspaces of X^{**} and X^*, respectively. Then, for each $\varepsilon > 0$ there is an injective operator $L : E \to X$ with the following properties:*

(a) *$L(x) = x$ for all $x \in E \cap X$,*

(b) *$\|L\| \cdot \|L^{-1}\| \leq 1 + \varepsilon$,*

(c) *$\langle Lx^{**}, x^{**} \rangle = \langle x^{**}, x^* \rangle$ for all $x^{**} \in E$ and $x^* \in F$.*

Every Banach space X is an ideal in its bidual, since the natural embedding $k_{X^*} : X^* \to X^{***}$ is a Hahn-Banach extension operator. In fact, every ideal in a Banach space can be characterized in terms of local structure similarly to the Principle of Local Reflexivity. This follows from results of Fakhoury [21] and Kalton [42]. Of course Fakhoury and Kalton did not use the term "ideal" which was introduced later, as mentioned above.

Theorem 1.3.19 (Fakhoury, 1972 and Kalton, 1984)**.** *Let X be a subspace of a Banach space Y. Then the following statements are equivalent.*

(a) *X is an ideal in Y.*

(b) *For every finite dimensional subspace E of Y and every $\varepsilon > 0$, there exists a linear operator $L : E \to X$ such that*

(i) $L(x) = x$ *for all* $x \in E \cap X$,

(ii) $\|L\| \leq 1 + \varepsilon$.

Godefroy, Kalton, and Saphar showed that also u-ideals have a local characterization. From [29, Lemma 2.2 and Proposition 3.6] and we have the following result.

Theorem 1.3.20 (Godefroy, Kalton, and Saphar, 1993). *Let Y be a Banach space and let X be a subspace of Y. The following statements are equivalent.*

(a) *X is a u-ideal in Y.*

(b) *There exists a Hahn Banach extension operator $\phi \in I\!H\!B(X, Y)$ such that for every $y \in Y$ there is a net (x_α) in X such that $\phi^*(y) = \lim_\alpha x_\alpha$ in the weak*-topology and* $\limsup_\alpha \|y - 2x_\alpha\| \leq \|y\|$.

(c) *For every finite dimensional subspace E of Y and every $\varepsilon > 0$, there is a linear map $L : E \to X$ such that*

(1) $L(y) = y$ *for every* $y \in E \cap X$, *and*

(2) $\|y - 2L(y)\| \leq (1 + \varepsilon)\|y\|$ *for every* $y \in E$.

There are approximation properties linked to the notion of u-ideals.

Definition 1.3.21. A Banach space X has the *unconditional metric approximation property (UMAP)* if there is a net (T_α) in $\mathcal{F}(X, X)$ with $\limsup_\alpha \|I - 2T_\alpha\| \leq 1$ such that $T_\alpha x \to x$ for every $x \in X$. If the net (T_α) is in $\mathcal{K}(X, X)$ instead of $\mathcal{F}(X, X)$ we say that X has the *unconditional metric compact approximation property (UMKAP).*

The obvious reason for this terminology is given by the following result of Casazza and Kalton [11, Theorem 3.8].

Theorem 1.3.22 (Casazza and Kalton, 1990). *A separable Banach space X has the UMAP if and only if for every $\varepsilon > 0$ there exists a sequence $(T_n) \in \mathcal{F}(X, X)$ with $\sup_n \|T_n\| < \infty$ and $T_n x \to x$ for all $x \in X$, so that if $A_n = T_n - T_{n-1}$ for $n \in \mathbb{N}$ (with $T_0 = 0$) then for every $N \in \mathbb{N}$ and all $\eta_i = \pm 1$, $i = 1, 2, ..., N$ we have*

$$\|\sum_{n=1}^{N} \eta_i A_i\| \leq 1 + \epsilon.$$

In [29, Theorem 8.1] Godefroy, Kalton, and Saphar showed that Theorem 1.3.22 holds when UMAP and $\mathcal{F}$ is replaced by UMKAP and $\mathcal{K}$ respectively.

Casazza and Kalton also proved in [11, Theorem 3.9] that UMAP is related to u-ideals of compact operators in the following way.

Theorem 1.3.23 (Casazza and Kalton, 1990). *Let X be a separable reflexive Banach space with the approximation property. Then the following statements are equivalent.*

(a) *X has UMAP.*

(b) *$\mathcal{K}(X,X)$ is a u-ideal in $\mathcal{L}(X,X)$.*

In [29, Theorem 8.3] Godefroy, Kalton, and Saphar showed that Theorem 1.3.23 holds when UMAP is replaced by UMKAP without assuming X to have the AP. Lima soon generalized this result by showing that the assumptions can be reduced to X having the RNP or $B_{X^*} = \overline{\text{conv}}(\omega^*\text{-str-exp}\, B_{X^*})$ [50, Theorem 4.3]. Note that if X has the AP, then $\mathcal{K}(X,X)$ is the norm closure of $\mathcal{F}(X,X)$. Thus [50, Theorem 4.3] of Lima also generalize Theorem 1.3.23. Theorems 5.2 and 6.1 in [45] show that the following holds without any assumptions on the Banach space X.

Theorem 1.3.24 (Lima and Lima, 2004)**.** *Let X be a Banach space. The following statements are equivalent.*

(a) *X has UMKAP.*

(b) *$\mathcal{K}(Y,X)$ is a u-ideal in $\mathcal{L}(Y,X)$ for every Banach space Y.*

(c) *$\mathcal{K}(\hat{X},X)$ is a u-ideal in $\mathcal{L}(\hat{X},X)$ for every equivalent renorming $\hat{X}$ for X.*

Theorem 1.3.25 (Lima and Lima, 2004)**.** *Let X be a Banach space. The following statements are equivalent.*

(a) *X^* has UMKAP with conjugate operators.*

(b) *$\mathcal{K}(X,Y)$ is a u-ideal in $\mathcal{L}(X,Y)$ for every Banach space Y.*

(c) *$\mathcal{K}(X,\hat{X})$ is a u-ideal in $\mathcal{L}(X,\hat{X})$ for every equivalent renorming $\hat{X}$ for X.*

The results also hold when compact operators and UMKAP are replaced by finite rank operators and UMAP respectively.

The next result was proved is Theorem 3.3 in [51].

Theorem 1.3.26 (Lima, Nygaard, and Oja, 2000)**.** *Let X be a Banach space. The following statements are equivalent.*

(a) *X has the AP.*

(b) *$\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for every Banach space Y.*

(c) *$\mathcal{F}(Y,X)$ is an ideal in $\mathcal{K}(Y,X)$ for every separable reflexive Banach space Y.*

Next we prove that Theorem 1.3.26 can be continued by the following statements:

(d) *$\mathcal{F}(Y,X)$ is an ideal in span $(\mathcal{F}(Y,X),\{T\})$ for every Banach space Y and every $T \in \mathcal{W}(Y,X)$.*

(e) *$\mathcal{F}(Y,X)$ is an ideal in span $(\mathcal{F}(Y,X),\{T\})$ for every separable reflexive Banach space Y and every $T \in \mathcal{K}(Y,X)$.*

Proof. We only have to prove (e) $\Rightarrow$ (c). To do this, we use the ideas from the proofs of [51, Lemma 1.4] and [2, Proposition 2.5].

Let Y be a separable reflexive Banach space and let $T \in \mathcal{K}(Y,X)$. We want to show that $\mathrm{HB}(\mathcal{F}(Y,X),\mathcal{K}(Y,X)) \neq \emptyset$. Since $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{B} =$span $(\mathcal{F}(Y,X),\{T\})$ we can, by using Goldstine's theorem, find a net $(T_\alpha) \subset \mathcal{F}(Y,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha \to \Phi_T^*(T)$ weak*, where $\Phi_T \in \mathrm{HB}(\mathcal{F}(Y,X),\mathcal{B})$ is the extension operator. Now, assume that $y \in B_Y$ is a strongly exposed point. Then by Lemma 3.4 in [50] $x^* \otimes y$ has a unique norm-preserving extension from $\mathcal{F}(Y,X)$ to $\mathcal{L}(Y,X)$ and hence $\Phi_T(x^* \otimes y) = x^* \otimes y$. Since Y has the RNP we get $\Phi_T(x^* \otimes y)$ for every $x^* \in X^*$ and $y \in Y$ by linearity and continuity. By a theorem of Feder and Saphar [22, Theorem 1] $\mathcal{F}(Y,X)^*$ is a quotient of $X^*\hat{\otimes}_\pi Y$ and it follows that Φ_T is just the identity and hence unique. A straightforward calculation shows that $\Phi_T^*(T) = T$. Thus the operator $\Psi = I_{X^*} \otimes I_Y \in \mathrm{HB}(\mathcal{F}(Y,X),\mathcal{K}(Y,X))$ and (c) follows. ☐

If the roles of X and Y are interchanged in Theorem 1.3.26, we get a characterization of the dual of X having the AP [51, Theorem 3.4].

The metric approximation property has also been characterized in terms of ideals of operators similarly to the approximation property.

Theorem 1.3.27 (Lima and Lima, 2004)**.** *Let X be a Banach space. The following statements are equivalent.*

(a) *X has the MAP.*

(b) *$\mathcal{F}(Y,X)$ is an ideal in $\mathcal{L}(Y,X)$ for every Banach space Y.*

(c) *$\mathcal{F}(Y,X)$ is an ideal in $\mathcal{L}(Y,X)$ for every separable Banach space Y.*

(d) *$\mathcal{F}(\hat{X},X)$ is an ideal in $\mathcal{L}(\hat{X},X)$ for every equivalent renorming $\hat{X}$ of X.*

If the roles of X and Y are interchanged in Theorem 1.3.27, we get a characterization of the dual of X having the MAP with conjugate operators [45, Theorem 1.2].

From [52, Theorem 5.1] and [53, Theorem 4.4] (resp. [53, Theorem 4.3]) we have the following result when the space of compact operators is considered as a subspace of the space of weakly compact operators.

Theorem 1.3.28 (Lima and Oja, 1999 and 2004)**.** *Let X be a closed subspace of a Banach space Z. Then $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,Z)$ (resp. $\mathcal{K}(Y,Z)$) for all Banach spaces Y if and only if $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,Z)$ (resp. $\mathcal{K}(Y,Z)$) for all (resp. separable) reflexive Banach spaces Y.*

In [2], which is presented in Chapter 4 in this thesis, we study when the space of finite rank operators is a u-ideal in the space of compact and weakly compact operators as in Theorems, 1.3.29, 1.3.30, 1.3.31, and 1.3.32 below (cf. Theorems 4.3.2, 4.3.8, 4.4.4, and 4.4.6 respectively).

Theorem 1.3.29 (Abrahamsen, Lima, and Lima)**.** *Let X be a Banach space. The following statements are equivalent.*

(a) *$\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,X)$ for every Banach space Y.*

(b) *$\mathcal{F}(Y,X)$ is a u-ideal in span $(\mathcal{F}(Y,X),\{T\})$ for every $T \in \mathcal{W}(Y,X)$ and for every reflexive Banach space Y.*

(c) *For every reflexive Banach space* Y *there exists a Hahn-Banach extension operator* $\Psi \in H\!B(\mathcal{F}(Y,X),\mathcal{W}(Y,X))$ *such that for every* $T \in \mathcal{W}(Y,X)$ *there is a net* $(T_\alpha) \subset \mathcal{F}(Y,X)$ *with* $\limsup_\alpha \|T - 2T_\alpha\| \le \|T\|$ *such that* $T_\alpha \to \Psi^*(T) = T$ *weak** *in* $\mathcal{F}(Y,X)^{**}$.

(d) *For every weakly compact set* $K \subset X$ *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *with* $\lim_\alpha \sup_{x\in K} \|x - 2S_\alpha x\| \le \sup_{x \in K}\|x\|$ *such that* $S_\alpha \to I_X$ *uniformly on compact subsets of* K.

(e) *For every Banach space* Y *and* $T \in \mathcal{W}(Y,X)$ *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *with* $\limsup_\alpha \|T - 2S_\alpha T\| \le \|T\|$ *such that* $S_\alpha \to I_X$ *uniformly on compact sets in* X.

(f) *For every Banach space* Y *and* $T \in \mathcal{W}(Y,X)$ *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *with* $\limsup_\alpha \|T - 2S_\alpha T\| \le \|T\|$ *such that* $S_\alpha \to I_X$ *in the strong operator topology.*

(g) *For every reflexive Banach space* Y *and* $T \in \mathcal{W}(Y,X)$ *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *with* $\limsup_\alpha \|T - 2S_\alpha T\| \le \|T\|$ *such that* $S_\alpha T \to T$ *in the strong operator topology.*

Theorem 1.3.30 (Abrahamsen, Lima, and Lima)**.** *Let* X *be a Banach space. The following statements are equivalent.*

(a) $\mathcal{F}(X,Y)$ *is a* u*-ideal in* $\mathcal{W}(X,Y)$ *for every Banach space* Y.

(b) $\mathcal{F}(X,Y)$ *is a* u*-ideal in* $\mathcal{W}(X,Y)$ *for every reflexive Banach space* Y.

(c) $\mathcal{F}(X,Y)$ *is a* u*-ideal in span* $(\mathcal{F}(X,Y),\{T\})$ *for every* $T \in \mathcal{W}(X,Y)$ *and for every reflexive Banach space* Y.

(d) *For every reflexive Banach space* Y *there exists a Hahn-Banach extension operator* $\Psi \in H\!B(\mathcal{F}(X,Y),\mathcal{W}(X,Y))$ *such that for every* $T \in \mathcal{W}(X,Y)$ *there is a net* $(T_\alpha) \subset \mathcal{F}(X,Y)$ *with* $\limsup_\alpha \|T - 2T_\alpha\| \le \|T\|$ *such that* $T_\alpha \to \Psi^*(T) = T$ *weak** *in* $\mathcal{F}(X,Y)^{**}$.

(e) *For every weakly compact compact set* $K \subset X^*$ *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *with* $\lim_\alpha \sup_{x^*\in K} \|x^* - 2S_\alpha^* x^*\| \le \sup_{x^* \in K}\|x^*\|$ *such that* $S_\alpha^* \to I_{X^*}$ *uniformly on compact subsets of* K.

(f) *For every Banach space* Y *and* $T \in \mathcal{W}(X,Y)$ *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *such that* $\limsup_\alpha \|T - 2TS_\alpha\| \le \|T\|$ *and* $S_\alpha^* \to I_{X^*}$ *uniformly on compact sets in* X^*.

(g) *For every Banach space* Y *and* $T \in \mathcal{W}(X,Y)$ *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *such that* $\limsup_\alpha \|T - 2TS_\alpha\| \le \|T\|$ *and* $S_\alpha^* \to I_{X^*}$ *in the strong operator topology.*

(h) *For every reflexive Banach space* Y *and* $T \in \mathcal{W}(X,Y)$ *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *such that* $\limsup_\alpha \|T - 2TS_\alpha\| \le \|T\|$ *and* $S_\alpha^* T^* \to T^*$ *in the strong operator topology.*

Theorem 1.3.31 (Abrahamsen, Lima, and Lima)**.** *Let* X *be a Banach space. The following statements are equivalent.*

(a) $\mathcal{F}(Y,X)$ *is a u-ideal in* $\mathcal{W}(Y,X^{**})$ *for every Banach space* Y.

(b) X *is a u-ideal in its bidual with unconditional Hahn-Banach extension operator* $\psi \in H\!B(X,X^{**})$ *such that for every Banach space* Y *and* $T \in \mathcal{W}(Y,X^{**})$ *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *with* $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leq \|T\|$ *such that* $S_\alpha^{**}T \to \psi^*T$ *weak* in* $\mathcal{L}(Y,X^{**})$.

(c) *There exists a Hahn-Banach extension operator* $\psi \in H\!B(X,X^{**})$ *such that for every Banach space* Y *and* $T \in \mathcal{W}(Y,X^{**})$ *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *with* $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leq \|T\|$ *such that* $S_\alpha^{**}T \to \psi^*T$ *weak* in* $\mathcal{L}(Y,X^{**})$.

(d) *For every weakly compact compact set* $K \subset X^{**}$ *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *with* $\lim_\alpha \sup_{x^{**}\in K} \|x^{**} - 2S_\alpha^{**}x^{**}\| \leq \sup_{x^{**}\in K} \|x^{**}\|$ *such that* $S_\alpha \to I_X$ *uniformly on compact subsets of* $K \cap X$.

(e) *For every Banach space* Y *and* $T \in \mathcal{W}(Y,X^{**})$, *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *with* $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leq \|T\|$ *such that* $S_\alpha \to I_X$ *uniformly on compact sets in* X.

(f) *For every reflexive Banach space* Y *and* $T \in \mathcal{W}(Y,X^{**})$, *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *with* $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leq \|T\|$ *such that* $S_\alpha \to I_X$ *uniformly on compact sets in* X.

Theorem 1.3.32 (Abrahamsen, Lima, and Lima). *Let* X *be a Banach space. The following statements are equivalent.*

(a) $\mathcal{F}(Y,X)$ *is a u-ideal in* $\mathcal{K}(Y,X^{**})$ *for every Banach space* Y.

(b) X *is a u-ideal in* X^{**} *with unconditional Hahn-Banach extension* ψ *such that* $\psi^*|_{X^{**}}$ *is in the weak*-closure of the* $\mathcal{F}(X,X)$ *in* $\mathcal{L}(X^{**},X^{**})$.

(c) X *is a u-ideal in its bidual with unconditional Hahn-Banach extension operator* $\psi \in H\!B(X,X^{**})$ *such that for every Banach space* Y *and* $T \in \mathcal{K}(Y,X^{**})$ *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *with* $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leq \|T\|$ *such that* $S_\alpha^{**}T \to \psi^*T$ *weak* in* $\mathcal{L}(Y,X^{**})$.

(d) *For every Banach space* Y *and* $T \in \mathcal{K}(Y,X^{**})$ *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *with* $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leq \|T\|$ *such that* $S_\alpha \to I_X$ *uniformly on compact sets in* X.

(e) *For every separable reflexive Banach space* Y *and* $T \in \mathcal{K}(Y,X^{**})$ *there is a net* $(S_\alpha) \subset \mathcal{F}(X,X)$ *with* $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leq \|T\|$ *such that* $S_\alpha \to I_X$ *uniformly on compact sets in* X.

Note that when "u-ideal" is replaced by "ideal" in statement (a) in Theorem 1.3.31 and in (a) in Theorem 1.3.32, these statements are equivalent. This is part of Theorem 1.3.13. On the basis of this, it is interesting to note that the statements in Theorem 1.3.31 are in fact strictly stronger than those in Theorem 1.3.32. Indeed, as remarked in [2] (see Chapter 4) the equivalently renormed version $\hat{\ell_2}$ of ℓ_2 obtained by Oja in [62, Example 3], fulfills the statements in Theorem 1.3.32, but fails to satisfy those of Theorem 1.3.31 (or equivalently Theorems 1.3.29, 1.3.30 since $\hat{\ell_2}$ is reflexive (see the next subsection)). In the next subsection, this renorming is discussed in more detail.

From Theorem 1.3.26, [73], and [49, Corollary 2] (see also [42, Theorem 5.1], [35, p. 138], and [65, Proposition 2.1]) we get the following proposition.

Proposition 1.3.33 (Lima, 1993; Lima, Nygaard, and Oja, 2000)**.** *Let X be a Banach space. The following statements are equivalent.*

(a) *$\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for every Banach space Y.*

(b) *X has the AP.*

(c) *Every separable ideal Z in X has the AP.*

(d) *$\mathcal{F}(Y,Z)$ is an ideal in $\mathcal{W}(Y,Z)$ for every Banach space Y and separable ideal Z in X.*

In [2] (cf. Proposition 4.3.6) we were able to show that the following analogue to Theorem 1.3.33 holds for u-ideals.

Proposition 1.3.34 (Abrahamsen, Lima, and Lima)**.** *Let X be a Banach space and assume $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,X)$ for every Banach space Y. Then a closed subspace Z of X has the AP if and only if $\mathcal{F}(Y,Z)$ is a u-ideal in $\mathcal{W}(Y,Z)$ for every Banach space Y.*

By using Theorem 1.3.33 the next result is immediate.

Corollary 1.3.35. *Let X be a Banach space. The following are equivalent.*

(a) *$\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,X)$ for every Banach space Y.*

(b) *$\mathcal{F}(Y,Z)$ is a u-ideal in $\mathcal{W}(Y,Z)$ for every Banach space Y and ideal Z in X.*

1.3.2 U-ideals and open problems

Before we start to discuss u-ideals, we will take a detour into some related properties. It turns out that known results about these properties are important also in the setting of u-ideals.

The Hahn-Banach theorem asserts that a linear functional defined on a subspace of a normed linear space has at least one norm-preserving extension to the whole space. In some cases, however, this extension is unique (e.g. reflexive spaces). Following Phelps [68] we define.

Definition 1.3.36. Let X be a closed subspace of a normed linear space Y. Then X has property U in Y if every element $x^* \in X^*$ has a unique norm-preserving extension y^* to Y.

Generalizing the concept of an M-ideal, Hennefeld [37] introduced and investigated the concept of IHB-subspaces.

Definition 1.3.37. A closed subspace X of a normed linear space Y is said to be an *IHB-subspace* in Y (or X has property IHB in Y) if its annihilator $X^\perp$ is complemented in Y^* by a subspace X_* such that whenever $x_* \in X_*$ and $x^\perp \in X^\perp \setminus \{0\}$, then $\|x_* + x^\perp\| \geq \|x^\perp\|$ and $\|x_* + x^\perp\| > \|x_*\|$.

It is straightforward to verify that an HB-subspace has property U. Indeed, let X be an HB-subspace in Y and let P be the induced projection on Y^* defined by $P(x^\perp + x_*) = x_*$. Then, for $x^* \in X^*$ and $y^* \in HB(x^*)$, we get $Py^* = y^*$ since $Py^* \in HB(x^*)$. Now, if y_1^* and y_2^* are in $HB(x^*)$, then $y_1^* - y_2^* \in \ker P$. Thus $y_1^* = Py_1^* = Py_2^* = y_2^*$ which shows that X has property U in Y.

There are, however, subspaces with property U which fail to be HB-subspaces. Producing such an example took some years, but finally Oja succeeded in [61] (see also Example 1 in [62]). In fact, Oja showed that there is a subspace of ℓ_∞^3 with property U which fails the property SU [61] (see also [62, Example 1]). The property SU is stronger than the property U. This follows by the same argument as for HB-subspaces.

Definition 1.3.38. Let X be a closed subspace of a normed linear space Y. Then X has the property *SU* in Y if its annihilator $X^\perp$ is complemented in Y^* by a subspace X_* such that whenever $x_* \in X_*$ and $x^\perp \in X^\perp \setminus \{0\}$, then $\|x_* + x^\perp\| > \|x_*\|$.

It is clear from the definitions that HB-subspaces must have property SU, so Oja's example shows in particular that the property HB is strictly stronger than property U. For a subspace with the property SU failing the property HB see Example 2 in [62]. Thus the property SU is strictly between the properties U and HB.

The property U is locally determined in the sense that a subspace X of a Banach space Y has this property in Y if and only X has this property in every subspace Z of Y in which X has codimension 1. Similar results also holds for the properties SU and HB.

Theorem 1.3.39. *Let X be a closed subspace of a Banach space Y. The following statements are equivalent.*

(a) *X has property U (resp. SU, HB) in Y.*

(b) *X has property U (resp. SU, HB) in $Z =$span$(X, \{y\})$ for every $y \in Y$.*

To prove this we will use results which require the following definition [48].

Definition 1.3.40. Let X be a subspace of a Banach space Y and let $n \geq 3$ be a natural number. Then X is said to have the *n.Y. intersection property (n.Y.I.P)* if for every family $(B(x_i, r_i))_{i=1}^n$ of n closed balls with centers $(x_i)_{i=1}^n$ in X and $Y \cap \bigcap_{i=1}^n B(x_i, r_i) \neq \emptyset$, then $X \cap \bigcap_{i=1}^n B(x_i, r_i + \varepsilon) \neq \emptyset$ for every $\varepsilon > 0$.

From [48, Theorem 3.1], [52, Proposition 2.1], and [63, Theorem 1.2] we have the following results.

Theorem 1.3.41 (Lima, 1983; Lima and Oja, 1999)**.** *Let X be a closed subspace of a Banach space Y. The following are equivalent.*

(a) *X is an ideal in $Z =$span$(X, \{y\})$ for every $y \in Y$.*

(b) *X has the n.Y.I.P for all n.*

(c) *If $n \in \mathbb{N}$ $x_1^*, ..., x_n^* \in X^*$ are such that $x_1^* + x_2^* + ... + x_n^* = 0$, then for $i = 1, ..., n$ there exist $y_i^* \in HB(x_i^*)$ such that $y_1^* + ... + y_n^* = 0$.*

Theorem 1.3.42 (Oja, 1991)**.** *Let X be a closed subspace of a Banach space Y. Then the following statements are equivalent.*

(a) *X is an HB subspace of Y.*

(b) *X has property U in Y and there exists an ideal projection P on Y^* satisfying $\|I - P\| = 1$.*

Proof of the U-case of Theorem 1.3.39 . (a) $\Rightarrow$ (b). Let $y \in Y \setminus X$ and put Z =span $(X, \{y\})$. Let $x^* \in X^*$ and $z_1^*, z_2^* \in \mathrm{HB}(x^*) \subset Z^*$. Choose $y_i^* \in \mathrm{HB}(z_i^*) \subset Y^*$ for $i = 1, 2$. Then $y_1^* = y_2^* \in \mathrm{HB}(x^*)$, so $z_1^* = z_2^*$.

(b) $\Rightarrow$ (a). Let $x^* \in X^*$. Suppose that $y_1^*, y_2^* \in \mathrm{HB}(x^*) \subset Y^*$ and that $y_1^* \neq y_2^*$. Choose $y \in Y \setminus X$ such that $y_1^*(y) \neq y_2^*(y)$ and let Z =span $(X, \{y\})$. Since $y_1^*|_Z$ and $y_2^*|_Z$ are extensions of x^* to Z, they have to be equal on Z by assumption, and we get a contradiction. □

Proof of the SU-case of Theorem 1.3.39. (a) $\Rightarrow$ (b). Let $y \in Y \setminus X$ and put Z =span $(X, \{y\})$. Since X has property U in Z and is an ideal in Z the result follows from [62, Theorem].

(b) $\Rightarrow$ (a). By [62, Theorem] it suffices to show that X possesses properties 3.$Y.I.P$ and U in Y. But this follows from Proposition 1.3.41 and Theorem 1.3.39 (U-case). □

Proof of the HB-case of Theorem 1.3.39. (a) $\Rightarrow$ (b). Let $y \in Y \setminus X$, and define Z =span $(X, \{y\})$. Let $z^* \in Z^*$, and let $y^* \in \mathrm{HB}(z^*)$. Since HB-subspaces have property SU it follows from Theorem 1.3.39 (SU-case) that X has property SU in Z. Denote by $i_{X,Z} : X \to Z$, $i_{Z,Y} : Z \to Y$, and $i_{X,Y} : X \to Y$ the natural embeddings. Then $i_{X,Y} = i_{Z,Y} \circ i_{X,Z}$, so $(i_{X,Y})^* = (i_{X,Z})^* \circ (i_{Z,Y})^*$. Let P_{Y^*} and P_{Z^*} denote the unique ideal projections on Y^* and Z^* respectively. Write $P_{Y^*} = \phi \circ (i_{X,Y})^*$ and $P_{Z^*} = \psi \circ (i_{X,Z})^*$ where $\phi \in \mathrm{HB}(X, Y)$ and $\psi \in \mathrm{HB}(X, Z)$. We get

$$\begin{aligned}\|z^* - P_{Z^*}z^*\| &= \|z^* - \psi(i_{X,Z})^*z^*\| = \|z^* - \phi(i_{X,Z})^*z^*|_Z\| \\ &= \|z^* - \phi(i_{X,Z})^*(i_{Z,Y})^*y^*|_Z\| \leq \|y^* - \phi(i_{X,Y})^*y^*\| \\ &= \|y^* - P_{Y^*}y^*\| \leq 1.\end{aligned}$$

Thus the result follows from Theorem 1.3.42.

(b) $\Rightarrow$ (a). From Theorem 1.3.39 (SU-case) we get that X has property SU in Y. Let $y^* \in Y^*$ and $y \in B_Y$ and put Z =span $(X, \{y\})$. Then

$$\langle y^* - \phi(i_{X,Y})^*y^*, y\rangle = \langle y^* - \phi_Z(i_{X,Z})^*(i_{Z,Y})^*y^*, y\rangle = \langle (I_{Z^*} - P_{Z^*})(i_{Z,Y})^*y^*, y\rangle,$$

and the result follows from Theorem 1.3.42. □

The article [48] of Lima, left open the following two questions: Do there exist Banach spaces X and Y, such that X or Y^* has the metric approximation property, for which $\mathcal{K}(Y, X)$ has property U in $\mathcal{L}(Y, X)$, but is not an HB-subspace in $\mathcal{L}(Y, X)$? Could a Banach space have property U in its bidual without being a HB-subspace in its bidual?

A few years after the article of Lima, both of these questions was answered in the negative by Oja in Examples 3 and 4 in [62]. In [62, Example 3], Oja defined a renorming $\hat{\ell}_2$ of ℓ_2 for which $\mathcal{K}(Y, \hat{\ell}_2)$ has property SU in $\mathcal{W}(Y, \hat{\ell}_2)$ for every normed linear space Y, but such that $\mathcal{K}(\ell_1, \hat{\ell_2})$ fails to be an HB-subspace in $\mathcal{W}(\ell_1, \hat{\ell_2})$. This renorming of ℓ_2 is done in the following manner:

$$\|(\xi_1, \xi_2, ...)\| = (\frac{1}{3}\sum_{i=1}^{\infty} \xi_i^2 + \frac{2}{3}\sup_{n\geq 2}(\xi_1^2, (\frac{\xi_1}{\sqrt{2}} + \xi_n)^2))^{1/2},$$

where $(\xi_1, \xi_2, ...) \in \ell_2$.

In Example 4 in [62], Oja showed that for $0 < r < 1$, the equivalently renormed versions c_{0r} of c_0, due to Johnson and Wolfe [40], have property SU in their biduals, but in fact fail to be HB-subspaces in their biduals. For $0 < r \leq 1$, the Johnson-Wolfe renorming of c_0 is done in the following manner:

$$\|(\xi_1, \xi_2, ...)\| = \sup\{|\xi_1|/r, |\xi_1 - \xi_2|, ...\},$$

where $(\xi_1, \xi_2, ...) \in c_0$.

Later, in [64, p. 127], Oja also showed that for $0 < r < 1$ the spaces $\mathcal{K}(c_{0r}, c_{0r})$, $\mathcal{K}(\ell_1, c_{0r})$, and $\mathcal{K}(\hat{\ell_2}, \hat{\ell_2})$ all have property U, in fact SU, in $\mathcal{L}(c_{0r}, c_{0r})$, $\mathcal{L}(\ell_1, c_{0r})$, and $\mathcal{L}(\hat{\ell_2}, \hat{\ell_2})$ respectively. However, all of them fail to be HB-subspaces. Observe that $\mathcal{L}(\hat{\ell_2}, \hat{\ell_2}) = \mathcal{K}(\hat{\ell_2}, \hat{\ell_2})^{**}$, so $\hat{\ell_2}$ is also an example of a Banach space for which $\mathcal{K}(\hat{\ell_2}, \hat{\ell_2})$ has property U, actually SU, in its bidual, without being an HB-subspace in its bidual. If we combine this fact with Theorem 1.3.22, we get that $\hat{\ell_2}$ does not have the UMAP. Thus the UMAP is not preserved under equivalent renormings since ℓ_2 has the UMAP.

Note that, if X is a u-ideal in a Banach space Y, and X has property U in Y, then X is an HB-subspace of Y. Indeed, let P be the unconditional projection on Y^* satisfying $\|I - 2P\| = 1$. Then writing $I - P = \frac{I}{2} + \frac{I-2P}{2}$ and using the triangle inequality, this follows. Since $\mathcal{K}(\ell_1, \hat{\ell_2})$ is not an HB-subspace of $\mathcal{W}(\ell_1, \hat{\ell_2})$, it now follows that $\hat{\ell_2}$ does not fulfill Theorem 1.3.31 as claimed in the last paragraph of subsection 1.3.1.

It now also follows from the examples in the preceding paragraphs that for $0 < r < 1$, c_{0r} and $\mathcal{K}(\hat{\ell_2}, \hat{\ell_2})$ are not u-ideals in their biduals. These two examples leave us with the problem of determining when Banach spaces are u-ideals in their biduals. Some results in this direction are known. If a Banach space X is a u-ideal in its bidual, then from [29, Corollary 4.1] we know that every Banach space being $(1+\varepsilon)$-isomorphic to a $(1+\varepsilon)$-complemented subspace of X, is a u-ideal in its bidual. In particular 1-complemented subspaces of X possess this property. However, it is not known if ideals in X also possess this property. Based on this, one can ask:

Problem 1.3.43. *Suppose a Banach space is a u-ideal in its bidual. Which subspaces of this Banach space inherits the property of being u-ideals in their biduals? In particular, do we have that every ideal in a Banach space is a u-ideal in its bidual whenever the space itself is?*

Godefroy, Kalton, and Saphar proved a result related to this problem, but for h-ideals instead of u-ideals [29, Theorem 6-7]. H-ideals are complex analogues to u-ideals.

Definition 1.3.44. A closed subspace X of a complex Banach space Y is called an *h-ideal* in Y if there exists a projection P on Y^* with $\ker P = X^{\perp}$ such that $\|I - (1 + \lambda)P\| = 1$ for all λ with $|\lambda| = 1$.

Theorem 1.3.45 (Godefroy, Kalton, and Saphar, 1993). *Suppose X is a separable Banach space and X is an h-ideal in its bidual. Let $\phi \in I\!B(X, X^{**})$ be the corresponding Hahn-Banach extension operator. Then every closed subspace Z of X such that $\phi^*(Z^{\perp\perp}) \subset Z^{\perp\perp}$, inherits the property of being an h-ideal in its bidual.*

Actually the proof can be modified so that the result holds for arbitrary Banach spaces being u-ideals in their biduals (see [2, Theorem 2.4])

Theorem 1.3.13, Theorem 1.3.26, and its dual counterpart [51, Theorem 3.4], gives reason to study the following statements. This is done in [2], which constitutes Chapter 4 in this thesis.

(A) *$\mathcal{F}(X, Y)$ is a u-ideal in $\mathcal{W}(X, Y)$ for every Banach space Y.*

(B) *$\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X^{**})$ for every Banach space Y.*

(C) *$\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X)$ for every Banach space Y.*

If X is a reflexive Banach space, then (A), (B), and (C) are equivalent. Indeed, this follows from [2, Theorems 3.2 and 3.5] and [50, Theorem 4.3] using the isometries $\mathcal{F}(X, X) = \mathcal{F}(X^*, X^*)$ and $\mathcal{W}(X, X) = \mathcal{W}(X^*, X^*)$.

For a general Banach space X it is evident that (B) implies (C) by using the local characterization of u-ideals Theorem 1.3.19.

Note that if (A) holds, then X^* has the AP [51, Theorem 3.4]. From [44, Proposition 3.3] we have that $\mathcal{F}(Y, X)^{**} = \mathcal{W}(Y, X^{**})$ for every reflexive Banach space Y if and only if X^* has the AP. Thus, if (A) implies (B) and X is a space satisfying (A), then $\mathcal{F}(Y, X)$ becomes a u-ideal in its bidual for every reflexive Banach space Y.

From [35, Example 4.1], it follows that ℓ_p for $1 < p < \infty$ fulfills (C) and thus (B) and (A) by the paragraph above. In [2] it is remarked that ℓ_1 fulfills (C), but fails (A). Note that this shows that the statement (A) is strictly stronger than the similar statements in [51, Theorem 3.4] for ideals.

We will now prove that also ℓ_1 fulfills statement (B). To do this we will use the recently established fact that $\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X^{**})$ for every Banach spaces Y if and only if $\mathcal{F}(Y, X)$ is a u-ideal in span $(\mathcal{F}(Y, X), \{T\})$ for every Banach space Y and $T \in \mathcal{W}(Y, X^{**})$ [47].

Proof. Let Y be a Banach space and let $T \in \mathcal{W}(Y, \ell_1^{**})$. By the above paragraph, it suffices to prove that $\mathcal{F}(Y, \ell_1)$ is a u-ideal in $\mathcal{B}$ =span $(\mathcal{F}(Y, \ell_1), \{T\})$ for every Banach space Y and $T \in \mathcal{W}(Y, {\ell_1}^{**})$. Let $(S_i)_{i=1}^3 \subset \mathcal{F}(Y, \ell_1)$. Since c_0 is an M-ideal in its bidual [35, p. 105], there exists an L-projection, P, from ℓ_1^{**} onto ℓ_1. Denote by $P_n : \ell_1 \to \ell_1$ the canonical projection onto the first n coordinates. We may assume that $S_i = P_n S_i$ for $i = 1, 2, 3$ for some large n. For $y \in B_Y$, using the fact that P and P_n are L-projections, we get that

$$\begin{aligned}
\|(T + S_i - 2P_n PT)y\| &= \|Ty - PTy\| + \|PTy + S_i y - 2P_n PTy\| \\
&= \|Ty - PTy\| + \|PTy - P_n PTy\| + \|P_n PTy - S_i y\| \\
&= \|Ty - PTy\| + \|PTy - S_i y\| \\
&= \|Ty - S_i y\| \le \|T - S_i\|.
\end{aligned}$$

This means that $2P_nPT \in \mathcal{F}(Y, \ell_1) \cap \bigcap_{i=1}^3 B_{\mathcal{B}}(T + S_i, \|T - S_i\|)$, and the result now follows from [46, Theorem 1.3]. If we have $\|S_i - P_nS_i\| < \varepsilon$ for $i = 1, 2, 3$, then we get $2P_nPT \in \mathcal{F}(Y, \ell_1) \cap \bigcap_{i=1}^3 B_{\mathcal{B}}(T + S_i, \|T - S_i\| + 2\varepsilon)$. □

In [2] it is also remarked that c_0 fulfills (A), (B), and (C), but that ℓ_∞ fails (C) and hence also (B). Note that this shows that the statements in Theorems 1.3.26 and 1.3.13 are strictly weaker than statements (C) and (B) respectively. As far as the author knows, it is open whether ℓ_∞ fails (A). Also, from [75] it follows that $X = \ell_2 \hat{\otimes}_\pi \ell_2$ does not fulfill (A), but does this X fulfill (C) or (B)?

The discussion in this paragraph leaves open the following problem, which seems to be of some importance.

Problem 1.3.46. *Do we have that (A) ⇒ (B)? Moreover, are the implications (A) ⇒ (B) ⇒ (C) strict?*

Another interesting question is whether Corollary 1.3.35 holds when "ideals" Z in X are replaced by "separable ideals". As far as the author knows this question is not answered even with "M-ideal" in place of "u-ideal".

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Chapter 2

On weak integrability and boundedness in Banach spaces

2.1 Introduction

Let X be a Banach space. A subset $B \subset X^*$ is said to be *weak*-norming* if $\inf_{x \in S_X} \sup_{x^* \in B} |x^*(x)| > 0$. Equivalently, the set B is weak*-norming if and only if its weak*-closed absolutely convex hull contains some ball. The set B is said to be *weak*-non-norming* if it is not weak*-norming. In [9] Fonf defined a set $A \subset X^*$ to be *weak*-thin* if it can be represented as a non-decreasing countable union of weak*-non-norming sets. (Remark that Fonf used the term "thin" instead of "weak*-thin".) As in [14] and [15], let us say that the set A is *weak*-thick* if it is not weak*-thin. For characterizations of weak*-thick sets in terms of uniform boundedness of families of functionals in X, and surjectivity of conjugate operators, we refer to [10, Proposition 1] and [15, Theorems 3.4 and 4.6] (see also Theorem 2.4.4 of the present paper for the summary of these characterizations).

In [9, Theorem 1], Fonf proved that if X does not contain any closed subspaces isomorphic to c_0, then $\text{ext} B_{X^*}$, the set of extreme points of the dual unit ball, is weak*-thick. From this he deduced (see [9, Theorem 4]) that *if X is separable and does not contain any isomorphic copies of c_0, then whenever (Ω, Σ, μ) is a finite measure space and a function $f\colon \Omega \to X$ is such that $x^* f \in L^1(\mu)$ for all $x^* \in \text{ext}\, B_{X^*}$, one has $x^* f \in L^1(\mu)$ for all $x^* \in X^*$, i.e., f is weakly integrable* (and thus Pettis integrable by a well known result of Dimitrov and Diestel (see [4] or [3, Theorem 7, p. 54])). The main objective of this paper is to generalize this result by giving the following new characterization of weak*-thick sets.

Main theorem *A subset $A \subset X^*$ is weak*-thick if and only if whenever (Ω, Σ, μ) is a measure space and $f\colon \Omega \to X$ is an essentially separable valued function such that $x^* f \in L^1(\mu)$ for all $x^* \in A$, then $x^* f \in L^1(\mu)$ for all $x^* \in X^*$.*

In Section 2.2, we prove the Main Theorem. As a corollary, it specializes to give a characterization of weak*-thick sets in X^* in terms of weakly unconditionally Cauchy series. In Section 2.3, we prove a characterization of weak*-thick sets in terms of boundedness of vector measures. In Section 2.4, we explain how "thickness", a notion dual to "weak*-thickness", is related to the theory of barrelled spaces.

Throughout this paper, X will be a Banach space. Our notation is standard. The unit ball and the unit sphere of X are denoted, respectively, by B_X and S_X. For a set $A \subset X$, we denote by $\text{ext}A$ the set of extreme points of A, and by $\text{absconv}(A)$ its absolutely convex hull. If some subsets $A_j \subset X$, $j \in \mathbb{N}$, are such that $A_1 \subset A_2 \subset A_3 \subset \ldots$, then, for their union, we sometimes write $\bigcup_{j=1}^{\infty} A_j \uparrow$.

2.2 Thickness and weak integrability

The "if" part of the Main Theorem is an immediate consequence of the following lemma which will be used also in Section 2.3.

Lemma 2.2.1. *Let a subset $A \subset X^*$ be weak*-thin, and let $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$, $j \in \mathbb{N}$. Then there are $x_j \in X$, $j \in \mathbb{N}$, $z^* \in X^* \setminus A$, an increasing sequence of indices $(\nu_j)_{j=1}^{\infty}$, and a real number $\delta > 0$ such that*

$$\sum_{j=1}^{\infty} \alpha_j \, |x^*(x_j)| < \infty \quad \textit{for all } x^* \in A,$$

but $\alpha_{\nu_j} \, |z^*(x_{\nu_j})| > \delta$ for all $j \in \mathbb{N}$.

Corollary 2.2.2. *Let a subset $A \subset X^*$ be weak*-thin, and let (Ω, Σ, μ) be a measure space such that there are pairwise disjoint sets $A_j \in \Sigma$ with $0 < \mu(A_j) < \infty$, $j \in \mathbb{N}$. Then there is a strongly measurable function $f \colon \Omega \to X$ such that $\int_\Omega |x^* f| \, d\mu < \infty$ for all $x^* \in A$, but $\int_\Omega |z^* f| \, d\mu = \infty$ for some $z^* \in X^* \setminus A$.*

Proof. The assertion follows by applying Lemma 2.2.1 for $\alpha_j = \mu(A_j)$, $j \in \mathbb{N}$, and putting $f = \sum_{j=1}^{\infty} \chi_{A_j} x_j$. □

Proof of Lemma 2.2.1. Since A is weak*-thin, it has a representation $A = \bigcup_{j=1}^{\infty} A_j \uparrow$ where all the A_j are weak*-non-norming, i.e., $\inf\limits_{x \in S_X} \sup\limits_{x^* \in A_j} |x^*(x)| = 0$, $j \in \mathbb{N}$. Thus we can pick a sequence $(x_j) \subset X$ with $\alpha_j \, \|x_j\| = 2^j$, $j \in \mathbb{N}$, such that

$$\sup_{x^* \in A_j} \alpha_j \, |x^*(x_j)| \leq \frac{1}{2^j} \quad \text{for all } j \in \mathbb{N}.$$

Note that whenever $x^* \in A$, then there is some $m \in \mathbb{N}$ such that $x^* \in A_j$ for all $j \geq m$, and thus

$$\begin{aligned} \sum_{j=1}^{\infty} \alpha_j \, |x^*(x_j)| &= \sum_{j=1}^{m-1} \alpha_j \, |x^*(x_j)| + \sum_{j=m}^{\infty} \alpha_j \, |x^*(x_j)| \\ &\leq \sum_{j=1}^{m-1} \alpha_j \, |x^*(x_j)| + \sum_{j=m}^{\infty} \frac{1}{2^j} < \infty. \end{aligned}$$

Next pick a sequence $(x_j^*) \subset X^*$ with $\|x_j^*\| \le \frac{1}{2^j}$, $j \in \mathbb{N}$, such that

$$\alpha_j\,|x_j^*(x_j)| > 1 - \frac{1}{4}, \quad j \in \mathbb{N}.$$

Now there are two alternatives:

1) $\lim_{j\to\infty} \alpha_j\,|x_{i_0}^*(x_j)| \ne 0$ for some $i_0 \in \mathbb{N}$;

2) $\lim_{j\to\infty} \alpha_j\,|x_i^*(x_j)| = 0$ for all $i \in \mathbb{N}$.

In the case 1), choose an increasing sequence of indices (ν_j) such that, for some $\delta > 0$, one has $\alpha_{\nu_j}\,|x_{i_0}^*(x_{\nu_j})| > \delta$ for all $j \in \mathbb{N}$, and put $z^* = x_{i_0}^*$.

In the case 2), put $\nu_1 = 1$ and proceed as follows. Given indices $\nu_1 < \nu_2 < \ldots < \nu_{j-1}$ ($j \in \mathbb{N}$, $j \ge 2$), pick an index $\nu_j > \nu_{j-1}$ such that

$$\sum_{i=1}^{j-1} \alpha_{\nu_j}\,|x_{\nu_i}^*(x_{\nu_j})| < \frac{1}{4} \quad \text{and} \quad \frac{2^{\nu_{j-1}}}{2^{\nu_j}} \le \frac{1}{2^{j+1}}.$$

Denoting $z^* = \sum_{i=1}^{\infty} x_{\nu_i}^*$ (this series converges because it converges absolutely), it remains to observe that, whenever $j \in \mathbb{N}$ and $i > j$, one has

$$\alpha_{\nu_j}\,|x_{\nu_i}^*(x_{\nu_j})| \le \alpha_{\nu_j}\,\|x_{\nu_j}\|\,\|x_{\nu_i}^*\| \le \frac{2^{\nu_j}}{2^{\nu_i}} \le \frac{2^{\nu_{i-1}}}{2^{\nu_i}} \le \frac{1}{2^{i+1}},$$

and thus, for all $j \in \mathbb{N}$,

$$\begin{aligned}\alpha_{\nu_j}\,|z^*(x_{\nu_j})| &\ge \alpha_{\nu_j}\,|x_{\nu_j}^*(x_{\nu_j})| - \sum_{i=1}^{j-1} \alpha_{\nu_j}\,|x_{\nu_i}^*(x_{\nu_j})| - \sum_{i=j+1}^{\infty} \alpha_{\nu_j}\,|x_{\nu_i}^*(x_{\nu_j})| \\ &\ge 1 - \frac{1}{4} - \frac{1}{4} - \sum_{i=j+1}^{\infty} \frac{1}{2^{i+1}} \ge \frac{1}{4}.\end{aligned}$$

□

Proof of the Main Theorem. Sufficiency has been proven in Corollary 2.2.2.

Necessity has been essentially proven in [9, Theorem 4]. For the sake of completeness, we shall give the details also here.

Let $A \subset X^*$ be weak*-thick, let (Ω, Σ, μ) be a measure space, and let an essentially separable valued function $f\colon \Omega \to X$ be such that $x^* f \in L^1(\mu)$ for all $x^* \in A$. Denote $A_j = \{x^* \in A\colon\ \int_\Omega |x^* f|\,d\mu \le j\}$, $j \in \mathbb{N}$. Then $A = \bigcup_{j=1}^{\infty} A_j \uparrow$, and the thickness of A implies the existence of some $m \in \mathbb{N}$ and $\delta > 0$ such that $\overline{\operatorname{absconv}}^{w^*}(A_m) \supset \delta B_{X^*}$. Thus it clearly suffices to show that $x^* f \in L^1(\mu)$ for all $x^* \in \overline{\operatorname{absconv}}^{w^*}(A_m)$. Fix an arbitrary $x^* \in \overline{\operatorname{absconv}}^{w^*}(A_m)$. Since f is essentially separable valued, there is a sequence $(y_n^*) \subset \operatorname{absconv}(A_m)$ such that $y_n^* f \to x^* f$ μ-almost everywhere on Ω; hence $x^* f$ is measurable. Since, for any $y^* \in \operatorname{absconv}(A_m)$, one has $\int_\Omega |y^* f|\,d\mu \le m$, by courtesy of Fatou's lemma, also $\int_\Omega |x^* f|\,d\mu \le m$; thus $x^* f \in L^1(\mu)$. □

By the Banach-Steinhaus theorem, from [15, Theorem 3.4] (see also Theorem 2.4.4 of the present paper) it follows that any Banach space is a weak*-thick subset of its bidual. Thus the Main Theorem yields the following corollary (which is probably known although the authors do not know any reference for it).

Corollary 2.2.3. *Let (Ω, Σ, μ) be a measure space, and let $f\colon \Omega \to X^*$ be an essentially separable-valued function. If $xf \in L^1(\mu)$ for all $x \in X$, then $x^{**}f \in L^1(\mu)$ for all $x^{**} \in X^{**}$.*

Recall that a series $\sum_{j=1}^{\infty} x_j$ in X is said to be *weakly unconditionally Cauchy* if $\sum_{j=1}^{\infty} |x^*(x_j)| < \infty$ for all $x^* \in X^*$. Observing that, for any $x^* \in X^*$, $\sum_{j=1}^{\infty} |x^*(x_j)| = \int_{\mathbb{N}} |x^* f|\, dc$, where c is the counting measure on $\mathcal{P}(\mathbb{N})$ and the function $f\colon \mathbb{N} \to X$ is defined by $f = \sum_{j=1}^{\infty} \chi_{\{j\}} x_j$, then from the Main Theorem and the proof of Corollary 2.2.2 we immediately get

Corollary 2.2.4. *A set $A \subset X^*$ is weak*-thick if and only if every series $\sum_{j=1}^{\infty} x_j$ in X satisfying $\sum_{j=1}^{\infty} |x^*(x_j)| < \infty$ for all $x^* \in A$ is weakly unconditionally Cauchy.*

The "only if" part of Corollary 2.2.4 gives the known link between Fonf's theorem stating that if X does not contain any isomorphic copies of c_0, then $\mathrm{ext} B_{X^*}$ is weak*-thick (see [9, Theorem 1]), and a theorem of Elton (see [5, Corollary] or [2, Theorem 15, p. 169]).

2.3 Thickness and bounded vector measures

Let $\mathcal{F}$ be an algebra of subsets of a set Ω, and let $F\colon \mathcal{F} \to X$ be a *vector measure* (i.e., let F be a finitely additive set function). It is standard (see [3, Proposition 11, p. 4]) that F has bounded range if and only if it is of bounded semi-variation, i.e., $\|F\|(\Omega) = \sup_{x^* \in B_{X^*}} |x^*F|(\Omega) < \infty$ (see [3, p. 2] for the definitions of the variation and the semivariation of a vector measure).

An important consequence of the Nikodým boundedness theorem is the following result of Dieudonné and Grothendieck.

Proposition 2.3.1 (see [3, p. 16])**.** *Let F be an X-valued set function defined on a σ-algebra Σ of subsets of a set Ω, and suppose that, for each x^* belonging to some total subset $\Gamma \subset X^*$, the function x^*F is bounded and finitely additive. Then F is a bounded vector measure.*

The interesting part of the theorem is of course the test for boundedness: if Σ is a σ-algebra, then it is enough to test on a total subset $\Gamma \subset X^*$. In general, Proposition 2.3.1 may fail for algebras that are not σ-algebras. We now show that there is a general test for boundedness also if the vector measure is defined merely on an algebra.

Proposition 2.3.2. *Let F be an X-valued set function defined on an algebra $\mathcal{F}$ of subsets of a set Ω, and suppose that, for each x^* belonging to some weak*-thick subset $\Gamma \subset X^*$, the function x^*F is bounded and finitely additive. Then F is a bounded vector measure.*

Proof. By the Hahn-Banach theorem, the additivity of F follows easily from the weak*-denseness of $\operatorname{span}\Gamma$ in X^*, and it remains to show that F is bounded. Put $A_j = \{x^* \in \Gamma\colon |x^*F|(\Omega) \leq j\}$, $j \in \mathbb{N}$. Then $\Gamma = \bigcup_{j=1}^{\infty} A_j \uparrow$, and the weak*-thickness of Γ implies that there are some $m \in \mathbb{N}$ and $\delta > 0$ such that $\overline{\mathrm{absconv}}^{w^*}(A_m) \supset \delta B_{X^*}$. Thus it clearly suffices to show that, for all $x^* \in \overline{\mathrm{absconv}}^{w^*}(A_m)$, one has $|x^*F|(\Omega) \leq m$. Observing that the last inequality holds for all $x^* \in \operatorname{absconv}(A_m)$, it can be easily seen to hold also for all $x^* \in \overline{\operatorname{absconv}}^{w^*}(A_m)$. □

It is natural to ask whether Proposition 2.3.2 characterizes the weak*-thick sets in X^*. More precisely, if a subset $A \subset X^*$ is weak*-thin, then can one always find an algebra $\mathcal{F}$ and an unbounded X-valued vector measure F on $\mathcal{F}$ such that, for all $x^* \in A$, the scalar valued vector measure x^*F is bounded? The following proposition answers this question in the affirmative.

Proposition 2.3.3. *Let a subset $A \subset X^*$ be weak*-thin. Then there is an unbounded X-valued vector measure F on the algebra $\mathcal{F}_{\mathbb{N}}$ of finite and cofinite subsets of $\mathbb{N}$ such that $|x^*F|(\mathbb{N}) < \infty$ for every $x^* \in A$.*

Proof. Applying Lemma 2.2.1 for $\alpha_j = 1$, $j \in \mathbb{N}$, produces some $z_j \in X$, $j \in \mathbb{N}$, $z^* \in X^*$, and $\delta > 0$ such that $\sum_{j=1}^{\infty} |x^*(z_j)| < \infty$ for all $x^* \in A$, but $\operatorname{Re} z^*(z_j) > \delta$ for all $j \in \mathbb{N}$ (just take $z_j = \frac{\overline{z^*(x_{\nu_j})}}{|z^*(x_{\nu_j})|} x_{\nu_j}$ in Lemma 2.2.1). It remains to define the vector measure $F\colon \mathcal{F}_{\mathbb{N}} \to X$ by

$$F(E) = \begin{cases} 0, & \text{if } E = \emptyset \text{ or } E = \mathbb{N}, \\ \sum\limits_{j \in E} z_j, & \text{if } 0 < |E| < \infty, \\ -\sum\limits_{j \in E^c} z_j, & \text{if } 0 < |E^c| < \infty. \end{cases}$$

□

2.4 Notes and remarks

There is a notion dual to "weak*-thickness", namely, "thickness". A subset $B \subset X$ is said to be *norming* if $\inf_{x^* \in S_{X^*}} \sup_{x \in B} |x^*(x)| > 0$. Equivalently, the set B is norming if and only if its closed absolutely convex hull contains some ball. The set B is said to be *non-norming* if it is not norming. In [11], Kadets and Fonf defined a set $A \subset X$ to be *thin* if it can be represented as a non-decreasing countable union of non-norming sets. As in [14] and [15], let us say that the set A is *thick* if it is not thin.

From [11, Proposition 1] and [15, Theorems 3.2 and 4.2], one has the following characterization of thick sets.

Theorem 2.4.1. *Let $A \subset X$. The following assertions are equivalent.*

(i) *The set A is thick.*

(ii) *Whenever Y is a Banach space and $T\colon Y \to X$ is a continuous linear operator such that $TY \supset A$, then $TY = X$.*

(iii) *Whenever a family of continuous linear operators from the space X to some Banach space is pointwise bounded on A, then this family is norm bounded.*

(iv) *Whenever a family of functionals in the dual space X^* is pointwise bounded on A, then this family is norm bounded.*

It is almost verbatim to the proof of the Main Theorem to show that Theorem 2.4.1 can be continued by

(v) *Whenever* (Ω, Σ, μ) *is a measure space and a function* $g\colon \Omega \to X^*$ *is such that* $xg \in L^1(\mu)$ *for all* $x \in A$*, then* $xg \in L^1(\mu)$ *for all* $x \in X$.

The perhaps most famous thick set is the set A of characteristic functions in $B(\Sigma)$, the space of bounded measurable functions on a measurable space (Ω, Σ): Nikodym's boundedness theorem states that A satisfies the condition (iv) in $B(\Sigma)$, Seever's theorem states that A satisfies the condition (iii). Remark that both these theorems were proved before Theorem 2.4.1 was commonly known.

It is well known that every pointwise bounded family of continuous linear operators from a locally convex space (LCS) E to some other LCS is equicontinuous if and only if the space E is *barrelled*, i.e., every absolutely convex closed absorbing set (every *barrell*) in E is a neighbourhood of zero. The theory of barrelled LCS is by now well documented through many books, among them [17] and more recently [8] and [13]. If an LCS is metrizable, then it is barrelled if and only if it is *Baire-like*, i.e., it can not be represented as a countable non-decreasing union of absolutely convex, nowhere dense sets. In this defininition, one may of course assume the sets to be closed. Observing that whenever a subset of a Banach space is thin, then its linear span is thin as well, just comparing the definitions gives

Proposition 2.4.2. *A subset* $A \subset X$ *is thick if and only if its linear span is dense and barrelled.*

Thus the equivalences (i)⇔(iii) and (i)⇔(ii) in Theorem 2.4.1 are, respectively, just a restatement for Banach spaces of the above-mentioned barrelledness criterion, and the following well-known result of Bennett and Kalton.

Theorem 2.4.3 (see [1, Proposition 1])**.** *Let* $Z \subset X$ *be a dense subspace. Then* Z *is barrelled if and only if whenever* Y *is a Banach space and* $T\colon Y \to X$ *is a continuous linear operator such that* $TY \supset Z$*, then* $TY = X$.

From [10, Proposition 1] and [15, Theorems 3.4 and 4.6] one has the following characterization of weak*-thick sets.

Theorem 2.4.4. *Let* $A \subset X^*$. *The following assertions are equivalent.*

(i) *The set* A *is weak***-thick in* X^*.

(ii) *Whenever* Y *is a Banach space and* $T\colon X \to Y$ *is a continuous linear operator such that* $T^*Y^* \supset A$*, then* $T^*Y^* = X^*$.

(iii) *Whenever a family of elements of the space* X *is pointwise bounded on* A*, then this family is norm bounded.*

On the contrary to Theorem 2.4.1, Theorem 2.4.4 has nothing to do with results from the theory of barrelled spaces: it does not say anything about the equicontinuity of weak*-continuous linear functionals, but it gives a test for the equicontinuity of norm continuous linear functionals.

The already mentioned theorem due to Fonf (see [9, Theorem 1]) states that if $\text{ext}B_{X^*}$ is weak*-thin in X^*, then X contains a copy of c_0. If X is separable, the same is true for w^*-expB_{X^*}, the set of weak*-exposed points of B_{X^*}, as is shown in [10, Theorem 3*].

Using results of Fonf, Nygaard showed in [15] that if both X^* and Y are c_0-free, then the set $E = \text{ext}B_{X^{**}} \otimes \text{ext}B_{Y^*}$ is weak*-thick in $\mathcal{L}(X,Y)^*$. From this it follows that if both X^* and Y are c_0-free, then ext$B_{\mathcal{K}(X,Y)^*}$ is weak*-thick in $\mathcal{K}(X,Y)^*$. Note that even $\mathcal{K}(\ell_2)$ contains a copy of c_0.

In the theory of analytic functions, a set A satisfying the condition (iv) of Theorem 2.4.1 is called a *uniform boundedness deciding set (UBD-set)* (see [7]). It has been shown by Fernandez ([6]) that the set of inner functions is a UBD-set in (H^∞, w^*). Later it has been shown by H. Shapiro ([16]) that also the set of the Blaschke-products has this property. Whether the inner functions form a UBD-set in $(H^\infty, \|\cdot\|)$ is still unknown. In other words, it is unknown whether the linear span of the inner functions in H^∞ is barrelled. What is known from [16] is that this linear span is not a Baire space, but the inner functions form a norming set in H^∞. In fact, the closed, convex hull of the Blaschke-products is exactly the unit ball in H^∞ (see [12, Cor 2.6, p. 196]).

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Chapter 3

Weak metric approximation properties and nice projections

3.1 Introduction

Let X and Y be Banach spaces. We denote by $\mathcal{L}(Y, X)$ the Banach space of bounded linear operators from Y to X, and by $\mathcal{F}(Y, X)$, $\mathcal{K}(Y, X)$, $\mathcal{W}(Y, X)$ its subspaces of finite rank operators, compact operators, and weakly compact operators, respectively.

We denote by $X\hat{\otimes}_\pi Y$ the (completed) projective tensor product of X and Y. Recall that we may identify the dual of $X\hat{\otimes}_\pi Y$ with $\mathcal{L}(Y, X^*)$ and that the action of an operator $T : Y \to X^*$, as a linear functional on $X\hat{\otimes}_\pi Y$, is given by

$$\left\langle T, \sum_{n=1}^{\infty} x_n \otimes y_n \right\rangle = \sum_{n=1}^{\infty} (Ty_n)(x_n).$$

Let I_X denote the identity operator on X. Recall that X is said to have the *approximation property (AP)* if there exists a net $(S_\alpha) \subset \mathcal{F}(X, X)$ such that $S_\alpha \to I_X$ uniformly on compact sets in X. If the net (S_α) can be chosen such that $\sup_\alpha \|S_\alpha\| \le 1$, then X is said to have the *metric approximation property (MAP)*.

In [8] Lima and Oja introduced and studied the weak metric approximation property. Following Lima and Oja a Banach space X is said to have the *weak metric approximation property (weak MAP)* if, for every Banach space Y and every operator $T \in \mathcal{W}(X, Y)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\sup_\alpha \|TS_\alpha\| \le \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X.

It is immediate from the definitions that MAP $\Rightarrow$ weak MAP $\Rightarrow$ AP. However, the AP does not imply the weak MAP in general as was shown in [8, Proposition 2.2]. Recently it was also shown [10, Corollary 1] that if a Banach space has the weak MAP then it has the MAP if either its dual or its bidual has the Radon-Nikodým property (RNP). It is, however, not known whether the weak MAP and the MAP in general are equivalent properties.

Let X be a subspace of a Banach space Y. A linear operator $\varphi : X^* \to Y^*$ is called a *Hahn-Banach extension operator* if $(\varphi x^*)(x) = x^*(x)$ and $\|\varphi x^*\| = \|x^*\|$ for every $x \in X$ and $x^* \in X^*$. We denote the set of Hahn-Banach extension operators $\varphi : X^* \to Y^*$ by $\mathrm{HB}(X, Y)$. It is easy to show that $\mathrm{HB}(X, Y)$ is non-void if and only if X is an ideal in Y (in the sense of Godefroy, Kalton and Saphar [2]).

The following result [5, Propositions 2.1 and 2.5] of Lima establishes a connection between the weak MAP and the existence of a Hahn-Banach extension operator.

Theorem 3.1.1 (Lima)**.** *Let X be a Banach space. Then X has the weak MAP if and only if there exists a Hahn-Banach extension operator $\varphi \in \mathrm{HB}(X, X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{F}(X, X)$ in $\mathcal{L}(X^{**}, X^{**})$.*

Note that we can consider $\mathcal{F}(X, X)$ as a subspace of $\mathcal{L}(X^{**}, X^{**})$ through the embedding operator which maps an operator $T \in \mathcal{F}(X, X)$ to its second adjoint $T^{**} \in \mathcal{L}(X^{**}, X^{**})$.

In Section 3.2 we improve Theorem 3.1.1 by showing that we can replace the Hahn-Banach extension operator $\varphi : X^* \to X^{***}$ by a Hahn-Banach extension operator $\varphi_P : X^* \to X^{***}$ such that $P = \varphi_P^*|_{X^{**}}$ is a projection on X^{**}. This result is then thereafter used to improve other characterizations of the weak MAP.

In Section 3.3 we establish similar characterizations to those in Section 3.2 for two, recently introduced [6], natural compact companions of the weak MAP.

We will consider Banach spaces over the real scalar field only. We use standard Banach space notation, as can be found e.g. in [9]. The closed unit ball of a Banach space X is denoted by B_X and the unit sphere of X by S_X. The closure of a set $A \subset X$ is denoted by $\overline{A}$, its linear span by $\mathrm{span} A$, and its convex hull by $\mathrm{conv} A$. We will write X^* for the dual of X.

3.2 The weak MAP

We might ask what more can we say about the Hahn-Banach extension operator in Theorem 3.1.1. In fact, by using a technique of Godefroy and Kalton from [1], we will prove that we can replace the Hahn-Banach extension operator $\varphi \in \mathrm{HB}(X, X^{**})$ in Theorem 3.1.1 by a Hahn-Banach extension operator $\varphi_P \in \mathrm{HB}(X, X^{**})$ such that $P = \varphi_P^*|_{X^{**}}$ is a projection on X^{**}. More explicitly we have the following theorem.

Theorem 3.2.1. *Let X be a Banach space.*

(a) *If P is a norm one projection on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{F}(X, X)$ in $\mathcal{L}(X^{**}, X^{**})$, then there exists a Hahn-Banach extension operator $\varphi \in \mathrm{HB}(X, X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{F}(X, X)$ in $\mathcal{L}(X^{**}, X^{**})$.*

(b) *If there exists a Hahn-Banach extension operator $\varphi \in \mathrm{HB}(X, X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{F}(X, X)$ in $\mathcal{L}(X^{**}, X^{**})$, then there exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{F}(X, X)$ in $\mathcal{L}(X^{**}, X^{**})$.*

Proof. (a) Assume that there exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$. Then put $\varphi_P = P^* i_{X^*}$ where $i_{X^*} : X^* \to X^{***}$ is the natural embedding of X^* into X^{***}. Finally observe that $\varphi_P : X^* \to X^{***}$ is a Hahn-Banach extension operator such that $\varphi_P^*|_{X^{**}} = P$.

(b) We use an argument from the proof of [1, Theorem III.1]. Assume that there exists a Hahn-Banach extension operator $\varphi \in \mathbb{HB}(X,X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$. Now, pick a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $S_\alpha^{**} \to \varphi^*|_{X^{**}}$ weak* in $\mathcal{L}(X^{**},X^{**})$. Let $\mathfrak{S}$ be the convex semi-group generated by the net (S_α^{**}), i.e. the smallest convex semi-group that contains (S_α^{**}). Let $\mathfrak{S}^*$ denote the weak*-closure of $\mathfrak{S}$. Now $\mathfrak{S}^*$ is a convex semi-group. To see this let U and V be in $\mathfrak{S}^*$ and write

$$U = \omega^*\text{-}\lim U_\alpha^{**}$$
$$V = \omega^*\text{-}\lim V_\beta^{**},$$

where U_α^{**} and V_β^{**} are in $\mathfrak{S}$. Choose $u = \sum_{n=1}^\infty x_n^* \otimes x_n^{**} \in X^* \hat{\otimes}_\pi X^{**}$ arbitrarily. Then it follows that

$$\begin{aligned} UV(u) &= \lim_\alpha \sum_{n=1}^\infty \langle x_n^*, U_\alpha^{**} V x_n^{**}\rangle = \lim_\alpha \sum_{n=1}^\infty \langle U_\alpha^* x_n^*, V x_n^{**}\rangle \qquad (3.2.1)\\ &= \lim_\alpha \lim_\beta \sum_{n=1}^\infty \left\langle U_\alpha^* x_n^*, V_\beta^{**} x_n^{**}\right\rangle = \lim_\alpha \lim_\beta (U_\alpha V_\beta)^{**}(u). \end{aligned}$$

Hence UV is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$. It is obvious that $\mathfrak{S}^*$ is convex.

Now put $\mathfrak{S}_0{}^* = \{T \in \mathfrak{S}^* : T|_X = I_X, \|T\| = 1\}$. Note that $\mathfrak{S}_0{}^* \neq \emptyset$ since $\varphi^*|_{X^{**}} \in \mathfrak{S}_0{}^*$. Since $\mathfrak{S}_0{}^*$ is closed under composition, it is a semi-group. It is straightforward to show that it is convex and weak*-closed.

Equip $\mathfrak{S}_0{}^*$ with the order-relation $\leq$ defined by $S \leq T$ if $\|Sx^{**}\| \leq \|Tx^{**}\|$ for every $x^{**} \in X^{**}$. Now let N be any maximal chain in $(\mathfrak{S}_0{}^*, \leq)$ and for $S \in N$ let $N_S = \{T \in N : T \leq S\}$. We can write $N = \bigcup_{S\in N} N_S$. Note that each N_S is weak*-closed. Indeed, choose a net (V_α) in N_S and assume $V_\alpha \underset{\alpha}{\to} V'$ weak*, where $V' \in \mathfrak{S}_0^*$. Then for every $x^{**} \in X^{**}$ we get

$$\|V' x^{**}\| \leq \liminf_\alpha \|V_\alpha x^{**}\| \leq \|S x^{**}\|.$$

By the maximality of N it follows that $V' \in N$ so N_S is weak*-closed. Now choose $(S_i)_{i=1}^n \subset N$ arbitrarily. Then $(N_{S_i})_{i=1}^n$ is a finite family of weak*-closed sets and

$$\bigcap_{i=1}^n N_{S_i} = \{T \in N : T \leq \min_{1\leq i\leq n} S_i\} \neq \emptyset.$$

Since $\mathfrak{S}_0{}^*$ is weak*-compact, every family of closed sets having the finite intersection property has a non-void intersection. Hence $\bigcap_{S\in N} N_S \neq \emptyset$. By the Hausdorff maximality theorem every chain is contained in a maximal chain. Hence, by the above argument, every chain in $\mathfrak{S}_0{}^*$ has a lower bound. It now follows by Zorn's lemma that $\mathfrak{S}_0{}^*$ has a minimal element. Denote such a minimal element by P.

We now show that P is a projection of norm one. Since P is minimal and $\|S\| = 1$ for all $S \in \mathfrak{S}_0{}^*$ we have $\|SPx^{**}\| = \|Px^{**}\|$ for all $S \in \mathfrak{S}_0{}^*$ and all $x^{**} \in X^{**}$. Applying this observation to

$$S_n = \frac{1}{n}(\sum_{i=1}^{n} P^i),$$

which by convexity is in $\mathfrak{S}_0{}^*$, gives

$$\begin{aligned}\|(S_nP^2 - S_nP)x^{**}\| &= \|S_nP(Px^{**} - x^{**})\| \\ &= \|P(Px^{**} - x^{**})\| \\ &= \|P^2x^{**} - Px^{**}\|.\end{aligned}$$

Since we have that

$$S_nP^2 - S_nP = \frac{1}{n}(P^{n+2} - P^2),$$

we get that $\|P^2x^{**} - Px^{**}\| \leq \frac{2}{n}$ for all $n \geq 1$. It follows that P is a projection on X^{**} such that P is in the weak*-closure of $\mathcal{F}(X, X)$ in $\mathcal{L}(X^{**}, X^{**})$. By the definition of $\mathfrak{S}_0{}^*$, P is of norm one and $X \subset P(X^{**})$. □

In fact we can do slightly better than Theorem 3.2.1. The result below tells us that we may assume that the net converging weak* to the projection, satisfies some boundedness property.

Proposition 3.2.2. *Let X be a Banach space with the weak MAP. Then there exist a projection P on X^{**} with $X \subset P(X^{**})$ such that for every reflexive Banach space Y and for every $T \in \mathcal{W}(X, Y)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\limsup_\alpha \|TS_\alpha\| \leq \|T\|$ such that $S_\alpha \to P$ weak* in $\mathcal{L}(X^{**}, X^{**})$.*

Proof. Let $\epsilon > 0$, let Y be a reflexive Banach space, and let $T \in \mathcal{W}(X, Y)$ of norm one. Let $u_k = \sum_{n=1}^{\infty} x^*_{k,n} \otimes x^{**}_{k,n} \in X^*\hat{\otimes}_\pi X^{**}$ for $k = 1, ..., m$. Assume $\sum_{n=1}^{\infty} \|x^{**}_{k,n}\| < \infty$ and $1 \geq \|x^*_{k,n}\| \to 0$ for each $k = 1, ..., m$. Put $K = \overline{\text{conv}}\{\pm x^*_{k,n} : k = 1, ..., m; n = 1, 2, ...\} \subset B_{X^*}$. Let Z be the Banach space constructed from K in the factorization lemma [7, Lemma 1.1], and let $J : Z \to X^*$ be the identity embedding of Z into X^*. Now Z is separable, reflexive and $J \in \mathcal{K}(Z, X^*)$ is of norm one. Define a map $V : X \to Z^* \oplus_\infty Y$ by $Vx = (J^*x, Tx)$. Note that $V \in \mathcal{W}(X, Z^* \oplus_\infty Y)$. By Theorem 3.1.1 and Theorem 3.2.1 there exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{F}(X, X)$ in $\mathcal{L}(X^{**}, X^{**})$. Note that $V^{**}P$ is in the weak*-closure of the convex set $\{V^{**}S^{**} : S \in \mathcal{F}(X, X)\}$ in $\mathcal{W}(X^{**}, Z^* \oplus_\infty Y)$. Since $Z^* \oplus_\infty Y$ is reflexive we have, by [3, Theorem 1.5], that $V^{**}P$ is in the weak*-closure of

$$\{V^{**}S^{**} : S \in \mathcal{F}(X, X), \|V^{**}S^{**}\| < \|V^{**}P\| + \epsilon\}$$

in $\mathcal{W}(X^{**}, Z^* \oplus_\infty Y)$, which again is a subset of the weak*-closure of

$$\{V^{**}S^{**} : S \in \mathcal{F}(X, X), \|VS\| < 1 + \epsilon\}$$

in $\mathcal{W}(X^{**}, Z^* \oplus_\infty Y)$. Now choose $z_{k,n} \in B_Z$ such that $Jz_{k,n} = x^*_{k,n}$ for all k and n. Find S in the above set such that

$$\begin{aligned}
\epsilon &> \max_{1\le k\le m} |V^{**}S^{**}(\sum_{n=1}^{\infty}(z_{k,n},0)\otimes x^{**}_{k,n}) - V^{**}P(\sum_{n=1}^{\infty}(z_{k,n},0)\otimes x^{**}_{k,n})| \\
&= \max_{1\le k\le m} |\sum_{n=1}^{\infty}\langle z_{k,n}, J^*S^{**}x^{**}_{k,n}\rangle - \sum_{n=1}^{\infty}\langle z_{k,n}, J^*Px^{**}_{k,n}\rangle| \\
&= \max_{1\le k\le m} |\sum_{n=1}^{\infty}\langle x^*_{k,n}, S^{**}x^{**}_{k,n}\rangle - \sum_{n=1}^{\infty}\langle x^*_{k,n}, Px^{**}_{k,n}\rangle|.
\end{aligned}$$

Since $\|TS\| \le \|VS\| \le 1+\epsilon$, the result follows. □

When the space X is separable and does not contain a copy of ℓ_1, we know even more about the projection.

Corollary 3.2.3. *Let X be a separable Banach space not containing ℓ_1. Then there exists a Hahn-Banach extension operator $\varphi \in I\!H\!B(X, X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$ if and only if there exist a norm one projection P on X^{**} with weak*-closed kernel and with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$.*

Proof. This follows directly from Theorem 3.2.1 and [1, Claim III.2]. □

Building on Theorem 3.2.1, we arrive at the result below. This improves [5, Propositions 2.5 and 3.1] in the way that the Hahn-Banach extension operator $\varphi : X^* \to X^{***}$, in each of these results, is replaced by Hahn-Banach extension operator $\varphi_P : X^* \to X^{***}$ such that $P = \varphi_P^*|_{X^{**}}$ is a projection on X^{**}.

Theorem 3.2.4. *Let X be a Banach space. The following statements are equivalent.*

(a) *X has the weak-MAP.*

(b) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$.*

(c) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for every reflexive Banach space Y and every operator $T \in \mathcal{W}(Y, X^{**})$, one has $PT \in \mathcal{F}(Y,X)^{**}$.*

(d) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for every separable reflexive Banach space Y and every operator $T \in \mathcal{K}(Y, X^{**})$, one has $PT \in \mathcal{F}(Y,X)^{**}$.*

Proof. (a)⇔(b) follows from Theorem 3.1.1 and Theorem 3.2.1.
(b)⇒(c) is obtained by the same reasoning as in [5, Proposition 3.1 (a)⇒(b)].
(c)⇒(d) is trivial.
(d)⇒(a) is obtained by the same reasoning as in [5, Proposition 3.1 (c)⇒(a)]. □

3.3 The weak MCAP and the very weak MCAP

Recently Lima and Lima [6] introduced and investigated two approximation properties that are natural compact companions of the weak MAP. Following [6], a Banach space X has the *weak metric compact approximation property (weak MCAP)* if, for every Banach space Y and every operator $T \in \mathcal{W}(X,Y)$, there exists a net $(S_\alpha) \subset \mathcal{K}(X,X)$ with $\sup_\alpha \|TS_\alpha\| \leq \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X. Moreover, X is said to have the *very weak metric compact approximation property (very weak MCAP)* if for every Banach space Y and every operator $T \in \mathcal{W}(X,Y)$ there exists a net $(S_\alpha) \subset \mathcal{K}(X,X^{**})$ with $\sup_\alpha \|T^{**}S_\alpha\| \leq \|T\|$ such that $\lim_\alpha \operatorname{tr}(S_\alpha u) = \operatorname{tr}(I_X u)$ for every $u \in X^* \hat{\otimes}_\pi X$. By comparing the definitions, it is immediate that the following implications hold: weak MAP $\Rightarrow$ weak MCAP $\Rightarrow$ very weak MCAP. As pointed out in [6, Remark 5.2], there is a space with the very weak MCAP, but without the weak MCAP. Moreover, the space of Willis [11, Proposition 4] has the weak MCAP, but not the weak MAP.

It should be noted that similar results to Theorem 3.2.1 also hold for the weak MCAP and the very weak MCAP. The results differ from Theorem 3.2.1 only in the way that $\mathcal{F}(X,X)$ is replaced by $\mathcal{K}(X,X)$ in the weak MCAP case, and $\mathcal{K}(X,X^{**})$ in the very weak MCAP case. The proofs of these results are verbatim to that of Theorem 3.2.1, using $\mathcal{K}(X,X)$ and $\mathcal{K}(X,X^{**})$ instead of $\mathcal{F}(X,X)$ respectively. The reason why the arguments work, is that the image of the second adjoint of a compact operator is a subspace of the range space of the operator itself. Hence the calculation in (3.2.1) holds.

Proposition 3.3.1. *Let X be a Banach space.*

(a) *If P is a norm one projection on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $K(X,X)$ [$\mathcal{K}(X,X^{**})$] in $\mathcal{L}(X^{**},X^{**})$, then there exists a Hahn-Banach extension operator $\varphi \in I\!H\!B(X,X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{K}(X,X)$ [$\mathcal{K}(X,X^{**})$] in $\mathcal{L}(X^{**},X^{**})$.*

(b) *If there exists a Hahn-Banach extension operator $\varphi \in I\!H\!B(X,X^{**})$ such that $\varphi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{K}(X,X)$ [$\mathcal{K}(X,X^{**})$] in $\mathcal{L}(X^{**},X^{**})$, then there exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $K(X,X)$ [$\mathcal{K}(X,X^{**})$] in $\mathcal{L}(X^{**},X^{**})$.*

By applying these results in companion with the proof of [5, Proposition 3.1] and the proofs of [6, Theorem 4.3] and [6, Theorem 5.3], we obtain the following strengthenings of [6, Theorem 4.3] for the weak MCAP case, and [6, Theorem 5.3] for the very weak MCAP case. The results improve [6, Theorem 4.3] and [6, Theorem 5.3] in the way that the Hahn-Banach extension operator $\varphi : X^* \to X^{***}$ in each of these theorems is replaced by a Hahn-Banach extension operator $\varphi_P : X^* \to X^{***}$ such that $P = \varphi_P^*|_{X^{**}}$ is a projection on X^{**}.

Theorem 3.3.2. *Let X be a Banach space. The following statements are equivalent.*

(a) *X has the weak MCAP.*

(b) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{K}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$.*

(c) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for every reflexive Banach space Y and every $T \in \mathcal{W}(Y, X^{**})$, one has $PT \in \mathfrak{E}^{**}$ where $\mathfrak{E} = \{S^{**}T : S \in \mathcal{K}(X, X)\} \subset \mathcal{K}(Y, X)$.*

(d) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for every separable reflexive Banach space Y and every $T \in \mathcal{K}(Y, X^{**})$, one has $PT \in \mathfrak{E}^{**}$ where $\mathfrak{E}$ is as in (c).*

(e) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for all sequences $(x_n^*) \subset X^*$ and $(x_n^{**}) \subset X^{**}$ with $\sum_{n=1}^{\infty} \|x_n^*\| \|x_n^{**}\| < \infty$ and $\sum_{n=1}^{\infty} x_n^{**}(S^* x_n^*) = 0$ for all $S \in \mathcal{K}(X, X)$, one has $\sum_{n=1}^{\infty} x_n^{**}(P^* x_n^*) = 0$.*

Proof. (a)⇔(b) follows from [6, Theorem 4.3 (a)⇔(b)] and Proposition 3.3.1.
(b)⇒(c) is similar to the proof of [5, Proposition 3.1 (a)⇒(b)].
(c)⇒(d) is trivial.
(d)⇒(e) is similar to the proof of [6, Theorem 4.3 (f)⇒(g)].
(e)⇒(b) is trivial. □

Theorem 3.3.3. *Let X be a Banach space. The following statements are equivalent.*

(a) *X has the very weak MCAP.*

(b) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that P is in the weak*-closure of $\mathcal{K}(X, X^{**})$ in $\mathcal{L}(X^{**}, X^{**})$.*

(c) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for every reflexive Banach space Y and every $T \in \mathcal{W}(X, Y)$, one has $T^{**}P \in \mathfrak{E}^{**}$, where $\mathfrak{E} = \{T^{**}S : S \in \mathcal{K}(X, X^{**})\} \subset K(X, Y)$.*

(d) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for every reflexive Banach space Y and every $T \in \mathcal{K}(X, Y)$, there exists a net $(S_\alpha) \subset \mathcal{K}(X, X^{**})$, with $\sup_\alpha \|T^{**}S_\alpha\| \leq \|T\|$, such that $\omega^*\text{-}\lim_\alpha S_\alpha^* T^* y = P^* T^* y^*$ in X^{***} for all $y^* \in Y^*$.*

(e) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for every reflexive Banach space Y and every $T \in \mathcal{K}(X, Y)$, there exists a net $(S_\alpha) \subset \mathcal{K}(X, X^{**})$, with $\sup_\alpha \|T^{**}S_\alpha\| \leq \|T\|$, such that $T^{**}S_\alpha^{**} \to T^{**}P$ in the strong operator topology.*

(f) *There exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ such that, for all sequences $(x_n^*) \subset X^*$ and $(x_n^{**}) \subset X^{**}$ with $\sum_{n=1}^{\infty} \|x_n^*\| \|x_n^{**}\| < \infty$ and $\sum_{n=1}^{\infty} x_n^{**}(S^* x_n^*) = 0$ for all $S \in \mathcal{K}(X, X^{**})$, one has $\sum_{n=1}^{\infty} x_n^{**}(P^* x_n^*) = 0$.*

Proof. (a)⇔(b) follows from [6, Theorem 5.3 (a)⇔(b)] and Proposition 3.3.1.
(b)⇒(c)⇒(d)⇒(e) are similar to the proofs of (b)⇒(c)⇒(d)⇒(e) in [6, Theorem 5.3] respectively.
(e)⇒(f). Let $\epsilon > 0$, let $u = \sum_{n=1}^{\infty} x_n^* \otimes x_n^{**} \in X^* \hat{\otimes}_\pi X^{**}$, and assume $\sum_{n=1}^{\infty} \|x_n^{**}\| < \infty$ and $1 \geq \|x_n^*\| \to 0$. Put $K = \overline{\text{conv}}\{\pm x_n^* : n = 1, 2, ...\} \subset B_{X^*}$. Let Z be the Banach space constructed from K in the factorization lemma [7, Lemma 1.1], and $J : Z \to X^*$ the identity embedding of Z into X^*. Now Z is separable, reflexive and $J \in \mathcal{K}(Z, X^*)$ is of norm one. Choose $z_n \in B_Z$ such

that $Jz_n = x_n^*$ for every $n \in \mathbb{N}$. From the assumption it follows that there exists a norm one projection P on X^{**} with $X \subset P(X^{**})$ and a net $(S_\alpha) \in \mathcal{K}(X, X^{**})$ with $\sup_\alpha \|(J^*|_X)^{**} S_\alpha\| \leq 1$ such that $(J^*|_X)^{**} S_\alpha^{**} \to (J^*|_X)^{**} P$ in the strong operator topology. Since $((J^*|_X)^{**} S_\alpha)$ is bounded, we may assume that the net converges to $(J^*|_X)^{**} P$ in the topology τ of uniform convergence on compact sets in X^{**}. By the description of $(\mathcal{L}(X^{**}, Z^*), \tau)^*$, due to Grothendieck [4] (see i.e. [9, Proposition 1.e.3]), it now follows that there exists an $S \in \mathcal{K}(X, X^{**})$ such that

$$\begin{aligned}
\epsilon &> |\sum_{n=1}^{\infty} \langle (J^*|_X)^{**} S^{**} x_n^{**}, z_n \rangle - \sum_{n=1}^{\infty} \langle (J^*|_X)^{**} P x_n^{**}, z_n \rangle | \\
&= |\sum_{n=1}^{\infty} \langle S^{**} x_n^{**}, J z_n \rangle - \sum_{n=1}^{\infty} \langle P x_n^{**}, J z_n \rangle | \\
&= |\sum_{n=1}^{\infty} \langle S^{**} x_n^{**}, x_n^* \rangle - \sum_{n=1}^{\infty} \langle P x_n^{**}, x_n^* \rangle |,
\end{aligned}$$

and we are done.

(f)⇒(b) is clear by using the Hahn-Banach theorem. □

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Chapter 4

Unconditional ideals of finite rank operators

4.1 Introduction

A closed subspace Z of a Banach space X is an *ideal* in X if the annihilator $Z^{\perp}$ is the kernel of a norm one projection on X^*. A linear operator $\varphi : Z^* \to X^*$ is called a *Hahn-Banach extension operator* if $\varphi(z^*)(z) = z^*(z)$ and $\|\varphi(z^*)\| = \|z^*\|$ for every $z \in Z$ and $z^* \in Z^*$. We write $\mathrm{HB}(Z, X)$ for the set of all Hahn-Banach extension operators from Z^* into X^*. It is not difficult to see that $\mathrm{HB}(Z, X) \neq \emptyset$ if and only if Z is an ideal in X. If Z is a subspace of a normed space X, we say that Z is an ideal in X if $\overline{Z}$ is an ideal in $\overline{X}$. The notion of an ideal was introduced and studied by Godefroy, Kalton, and Saphar in [5].

The stronger notion of an *unconditional ideal* (*u-ideal* for short) was introduced and studied by Casazza and Kalton in [2]. If Z is an ideal in X such that the corresponding projection P on X^* satisfies $\|I - 2P\| = 1$, then Z is called a *u-ideal* in X. The projection is called a *u-projection* and the corresponding $\varphi \in \mathrm{HB}(Z, X)$ is called an *unconditional* Hahn-Banach extension operator. From Lemma 2.2 and Proposition 3.6 in [5], we can state the following result.

Theorem 4.1.1 (Godefroy, Kalton, and Saphar). *Let X be a Banach space and let Z be a subspace of X. The following statements are equivalent.*

(a) *Z is a u-ideal in X.*

(b) *There exists a Hahn Banach extension operator $\varphi \in \mathrm{HB}(Z, X)$ such that whenever $\varepsilon > 0$, $x \in X$ and A is a convex subset of Z such that $\varphi^*(x)$ is in the weak*-closure of A then there exists $z \in A$ with $\|x - 2z\| < \|x\| + \varepsilon$.*

(c) *There exists a Hahn Banach extension operator $\varphi \in \mathrm{HB}(Z, X)$ such that for every $x \in X$ there is a net (z_α) in Z such that $\varphi^*(x) = \lim_\alpha z_\alpha$ in the weak*-topology and* $\limsup_\alpha \|x - 2z_\alpha\| \leq \|x\|$.

(d) *For every finite dimensional subspace F of X and every $\varepsilon > 0$, there is a linear map $L : F \to Z$ such that*

   (1) *$L(x) = x$ for every $x \in F \cap Z$, and*

(2) $\|x - 2L(x)\| \leq (1+\varepsilon)\|x\|$ *for every* $x \in F$.

Note that (1) can be substituted by the inequality $\|L(x) - x\| \leq \varepsilon\|x\|$ for every $x \in F \cap Z$. We will sometimes use this fact.

Let X and Y be Banach spaces. We denote by $\mathcal{L}(Y, X)$ the Banach space of bounded linear operators from Y to X, and by $\mathcal{F}(Y, X)$, $\mathcal{K}(Y, X)$, and $\mathcal{W}(Y, X)$ its subspaces of finite rank operators, compact operators, and weakly compact operators, respectively.

In Section 4.2 we show that the set of Hahn-Banach extension operators $\mathrm{HB}(X, Y)$ is a face in the unit ball of $\mathcal{L}(X^*, Y^*)$. We show in Proposition 4.2.2 that an unconditional Hahn-Banach extension operator has to be a center of symmetry in $\mathrm{HB}(X, Y)$. If X contains a copy of ℓ_1 and is a u-ideal in its bidual, then we show that the diam $\mathrm{HB}(X, X^{**}) = 2$. We also show that in some important cases the set $\mathrm{HB}(X, Y)$ consists of a single element. The subspaces Z of X such that $\varphi^*|_{X^{**}}(Z^{\perp\perp}) \subset Z^{\perp\perp}$ where $\varphi \in \mathrm{HB}(X, X^{**})$ is unconditional are characterized.

In Section 4.3 we establish in Theorem 4.3.2 characterizations of the case when $\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X)$ for every Banach space Y. The characterizations include a statement similar to Theorem 4.1.1 (b) involving a Hahn-Banach extension operator, a statement which is an approximation property for X and statements about approximating weakly compact operators by finite rank operators. In Theorem 4.3.8 we give similar characterizations of the case when $\mathcal{F}(X, Y)$ is a u-ideal in $\mathcal{W}(X, Y)$ for every Banach space Y.

In Section 4.4 we characterize the property that $\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X^{**})$ for every Banach space Y in Theorem 4.4.4, and the property that $\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{K}(Y, X^{**})$ for every Banach space Y in Theorem 4.4.6 by statements similar to those in Theorems 4.3.2 and 4.3.8. An example due to Oja [25, Example 3] shows that the latter property is strictly weaker (see Remark 4.4.7 below). We define an unconditional version of the weak metric approximation property. We show by giving an example that this property is strictly weaker than $\mathcal{F}(Y, X)$ being a u-ideal in $\mathcal{K}(Y, X^{**})$ for every Banach space Y.

We will frequently use the isometric version of the Davis-Figiel-Johnson-Pełczyński factorization lemma [3] due to Lima, Nygaard, and Oja [16]. Let X be a Banach space and let K be a closed absolutely convex subset of the unit ball B_X of X. If Z is the Banach space constructed from K in the factorization lemma and J is the norm one identity embedding of Z into X (see [16, Lemma 1.1]), we will write

$$[Z, J] = \mathrm{DFJP}(K).$$

From the factorization lemma we know that Z is reflexive if and only if K is weakly compact. The factorization lemma also says that if K is compact, then Z is separable and J is compact.

From the isometric version of the factorization lemma proved by Lima, Nygaard, and Oja [16, Theorem 2.3] we get that if $G \subset \mathcal{W}(Y, X)$ is a finite dimensional subspace, then there exist a reflexive Banach space Z, a norm one operator $J : Z \to X$, and a linear isometry $\Phi : G \to \mathcal{W}(Y, Z)$ such that $T = J \circ \Phi(T)$ for every $T \in G$. We will write

$$[Z, J, \Phi] = \mathrm{DFJP}(G), \tag{4.1.1}$$

for this construction. Similarly, using [16, Corollary 2.4], we get that if $G \subset \mathcal{W}(X,Y)$ is a finite dimensional subspace, then there exists a reflexive Banach space Z, a norm one operator $J : X \to Z$, and a linear isometry $\Phi : G \to \mathcal{W}(Z,Y)$ such that $T = \Phi(T) \circ J$ for every $T \in G$. We will write

$$[Z, \Phi, J] = \mathrm{DFJP}(G), \tag{4.1.2}$$

for this construction.

We use standard Banach space notation as used by Lindenstrauss and Tzafriri in [23]. Only real Banach spaces are considered unless otherwise stated. The closed unit ball of a Banach space X is denoted by B_X and the identity operator on X is denoted by I_X. We will write X^* for the dual space of X. If $Z \subset X$ is a subspace of X, then we will write $i_Z : Z \to X$ for the canonical embedding. We will write $k_X : X \to X^{**}$ for the natural embedding of X into its bidual. $\operatorname{ext} B_X$ denotes the set of extreme points in B_X. If $T : X \to Y$ is an operator and $x \in X$, then we will write Tx instead of $T(x)$ when there is no danger of confusion.

4.2 Unconditional Hahn-Banach extension operators

Let us start with a general result about the location and the size of the set of Hahn-Banach extension operators.

Proposition 4.2.1. *Let Y be a Banach space. If X is an ideal in Y, then $\mathrm{HB}(X,Y)$ is a face in $B_{\mathcal{L}(X^*,Y^*)}$.*

Proof. Let ϕ_1, $\phi_2 \in B_{\mathcal{L}(X^*,Y^*)}$ and suppose $\varphi = \frac{\phi_1+\phi_2}{2} \in \mathrm{HB}(X,Y)$. We then get that

$$\frac{i_X^*\phi_1 + i_X^*\phi_2}{2} = i_X^*\varphi = I_{X^*} \in \operatorname{ext} B_{\mathcal{L}(X^*,X^*)}.$$

Thus $i_X^*\phi_i = I_{X^*}$ and $\phi_i \in \mathrm{HB}(X,Y)$ for $i = 1, 2$. ☐

In Lemma 3.1 in [5] there is an algebraic proof of the fact that an unconditional Hahn-Banach extension operator is unique. Next we have a geometric proof. (Recall that x is a *center of symmetry* in a subset A of a linear space X if $2x - y \in A$ for every $y \in A$.)

Proposition 4.2.2. *Let X be a u-ideal in Y with unconditional $\varphi \in \mathrm{HB}(X,Y)$. For $x^* \in X^*$, let $\mathrm{HB}(x^*) \subset Y^*$ be the set of norm preserving extensions of x^* to Y. Then $\varphi(x^*)$ is the center of symmetry in $\mathrm{HB}(x^*)$ for every $x^* \in X^*$.*
In particular, the unconditional Hahn-Banach extension operator φ is unique, and φ is a center of symmetry in $\mathrm{HB}(X,Y)$.

Proof. Let $y^* \in \mathrm{HB}(x^*)$ and let $P_\varphi = \varphi i_X^*$ be the u-projection. Then $\|x^*\| = \|y^*\| = \|(I - 2P_\varphi)y^*\| = \|y^* - 2\varphi(x^*)\|$, so that $2\varphi(x^*) - y^* \in \mathrm{HB}(x^*)$. Hence $\varphi(x^*)$ is a center of symmetry in $\mathrm{HB}(x^*)$. Since a center of symmetry in a convex bounded set is unique, it follows that there is at most one unconditional extension operator in $\mathrm{HB}(X,Y)$.

If $\psi \in \mathrm{HB}(X,Y)$ and $x^* \in X^*$, then $\psi(x^*) \in \mathrm{HB}(x^*)$. Using that $\varphi(x^*)$ is a center of symmetry in $\mathrm{HB}(x^*)$ we get $2\varphi(x^*) - \psi(x^*) \in \mathrm{HB}(x^*)$. Hence we get $2\varphi - \psi \in \mathrm{HB}(X,Y)$ and φ is a center of symmetry in $\mathrm{HB}(X,Y)$. □

The following result shows that if a Banach space X contains a subspace isomorphic to ℓ_1 and is a u-ideal in its bidual, then the diameter of $\mathrm{HB}(X,X^{**})$ is as large as possible.

Proposition 4.2.3. *Let X be a Banach space which contains a subspace isomorphic to ℓ_1. If X is a u-ideal in its bidual, then* $\operatorname{diam} \mathrm{HB}(X,X^{**}) = 2$.

Proof. Let $\pi = k_{X^*}k_X^*$ and $P_\varphi = \varphi k_X^*$ respectively be the canonical projection and the u-projection on X^{***}. By Proposition 4.2.2 the unconditional Hahn-Banach extension operator φ is a center of symmetry in $\mathrm{HB}(X,X^{**})$, i.e. $\psi = 2\varphi - k_{X^*} \in \mathrm{HB}(X,X^{**})$. Let $P_\psi = \psi k_X^*$ and note that P_ψ is an ideal projection on X^{***}. By Proposition 2.6 in [5] we have $\|I - 2\pi\| = 3$, so

$$2 \geq \|P_\psi - \pi\| = \|2P_\varphi - 2\pi\| \geq \|I - 2\pi\| - \|I - 2P\varphi\| = 2.$$

Hence $\|\psi - k_{X^*}\| = \|P_\psi - \pi\| = 2$, so $\operatorname{diam} \mathrm{HB}(X,X^{**}) = 2$. □

Note that the proof of Proposition 1 in [1] shows that if a non-reflexive Banach space X is 1-complemented in its bidual by a projection P, then $\mathrm{HB}(X,X^{**})$ consists of at least two elements.

One direction of the following theorem was proved for separable h-ideals in [5, Theorem 6.7]. Our argument, as the proof of Theorem 6.7 in [5], is based on an application of Theorem 4.1.1 (b).

Theorem 4.2.4. *Let X be a Banach space. Assume that X is a u-ideal in X^{**} with unconditional $\varphi \in \mathrm{HB}(X,X^{**})$. Let Z be a closed subspace of X. Then $\varphi^*(Z^{\perp\perp}) \subset Z^{\perp\perp}$ if and only if Z is a u-ideal in Z^{**} with unconditional Hahn-Banach extension operator $\psi \in \mathrm{HB}(Z,Z^{**})$ such that $i_Z^{**}\psi^*|_{Z^{**}} = \varphi^* i_Z^{**}$.*

Proof. Suppose $\varphi^*(Z^{\perp\perp}) \subset Z^{\perp\perp}$. $i_Z : Z \to X$ is the natural embedding, so i_Z^* is the restriction and i_Z^{**} is weak*-weak* continuous, isometric, and onto $Z^{\perp\perp}$.

Define $\psi : Z^* \to Z^{***}$ by

$$\psi(z^*) = \psi(x^* + Z^\perp) = i_Z^{***}\varphi(x^*)$$

for $z^* = x^* + Z^\perp \in Z^*$. Since $i_Z^{**}(Z^{**}) \subset Z^{\perp\perp}$ we get that ψ is well-defined:

$$\langle \psi(z^*), z^{**}\rangle = \langle x^* + Z^\perp, \varphi^*(i_Z^{**}(z^{**}))\rangle = \langle x^*, \varphi^*(i_Z^{**}(z^{**}))\rangle = \langle i_Z^{***}\varphi(x^*), z^{**}\rangle$$

for $z^{**} \in Z^{**}$. Thus we have $\psi(i_Z^*(x^*) = i_Z^{***}\varphi(x^*)$ for all $x^* \in X^*$. Taking adjoints we get $i_Z^{**}\psi^*|_{Z^{**}} = \varphi^* i_Z^{**}$.

Let us show that ψ is an unconditional Hahn-Banach extension operator. Clearly ψ is linear with norm at most one. For $z \in Z$ and $z^* = x^* + Z^\perp \in Z^*$ we have

$$\psi(z^*)(z) = \langle \varphi(x^*), i_Z(z)\rangle = \langle x^*, i_Z(z)\rangle = z^*(z).$$

Let $z^{**} \in B_{Z^{**}}$ and $\varepsilon > 0$. Since X is a u-ideal in X^{**} and $\varphi^*(i_Z^{**}(z^{**}))$ is in the w*-closure of B_Z in X^{**} there exists a $z \in B_Z$ such that

$$\|z^{**} - 2z\| = \|i_Z^{**}(z^{**}) - 2i_Z(z)\| < \|z^{**}\| + \varepsilon$$

by Theorem 4.1.1 (b). Thus there is a net $(z_\alpha) \subset B_Z$ with $\limsup_\alpha \|z^{**} - 2z_\alpha\| \leq \|z^{**}\|$ such that $z_\alpha \to \psi^*(z^{**})$ weak* in Z^{**} (here we used $i_Z^{**}\psi^*|_{Z^{**}} = \varphi^* i_Z^{**}$). Hence $\|z^{**} - 2k_Z^{**}(\psi(z^{**}))\| \leq \|z^{**}\|$ and ψ is unconditional.

For the converse assume that Z is a u-ideal in Z^{**} with unconditional $\psi \in \mathrm{HB}(Z, Z^{**})$ such that $i_Z^{**}\psi^*|_{Z^{**}} = \varphi^* i_Z^{**}$. Let $x^{**} \in Z^{\perp\perp}$ in X^{**} and choose $z^{**} \in Z^{**}$ such that $i_Z^{**}(z^{**}) = x^{**}$, then $\varphi^*(x^{**}) = i_Z^{**}(\psi^* z^{**}) \in Z^{\perp\perp}$. □

Recall that a Banach space X is said to have the *approximation property (AP)* if there exists a net $(S_\alpha) \subset \mathcal{F}(X, X)$ such that $S_\alpha \to I_X$ uniformly on compact sets in X. Lima, Nygaard, and Oja have proved [16, Theorem 3.3] that a Banach space X has the AP if and only if the set $\mathrm{HB}(\mathcal{F}(Y,X), \mathcal{W}(Y,X))$ of Hahn-Banach extension operators is non-empty for every Banach space Y.

In some cases the set of Hahn-Banach extension operators consists of a single element. For example if X is an M-ideal in a Banach space Y, then $\mathrm{HB}(X, Y)$ contains a single element (see [7, Proposition 1.2]. Cf. [7, p. 1] for definition of an M-ideal). A Banach space X such that $\mathrm{HB}(X, X^{**})$ consists of a single element is said to have the *unique extension property (UEP)*. This notion was introduced and studied by Godefroy and Saphar in [6]. They proved in [6, Corollary 5.4] that if X and Y are Banach spaces such that X is reflexive and Y^* has the Radon-Nikodým property (RNP) and contains no proper norming subspace, then $X \otimes_\varepsilon Y$ and $\mathcal{K}(X, Y)$ have the UEP. (Recall that a subspace Z of Y^* is *norming* if $\|y\| = \sup\{y^*(y) : y^* \in Z, \|y^*\| \leq 1\}$ for $y \in Y$.)

From [24] we also know that $\mathrm{HB}(\mathcal{F}(Y,X), \mathcal{L}(Y,X))$ contains a single element for every Banach space Y whenever X is either ℓ_p or the Lorentz sequence space $d(\omega, p)$ for $1 < p < \infty$ (see also [7, Example 4.1] for the case $X = \ell_p$ and $Y = \ell_q$ where $1 < q \leq p < \infty$). Dually we also have that $\mathrm{HB}(\mathcal{F}(X,Y), \mathcal{L}(X,Y))$ contains a single element for every Y whenever X is either ℓ_p or $d(\omega,p)^*$ for $1 < p < \infty$. From [26, Theorem 3] we have in addition that the above holds if X is a closed subspace of either ℓ_p, $d(\omega, p)$ or $d(\omega, p)^*$ with the AP. Also the set $\mathrm{HB}(\mathcal{F}(Y, c_0), \mathcal{L}(Y, c_0))$ consists of a single element for every Banach space Y ($\mathcal{F}(Y, c_0)$ is an M-ideal in $\mathcal{L}(Y, c_0)$, see [7, Example 4.1]). The next results tell us that in many more cases the set of Hahn-Banach extension operators consists of a single element.

Proposition 4.2.5. *Let X and Y be Banach spaces. If X has the AP and Y is reflexive, then $\mathrm{HB}(\mathcal{F}(Y,X), \mathcal{W}(Y,X))$ consists of one element only.*

Proof. Let $\Phi \in \mathrm{HB}(\mathcal{F}(Y,X), \mathcal{W}(Y,X))$, let $x^* \in X^*$ and $y \in B_Y$. Assume that y is a strongly exposed point. Then by Lemma 3.4 in [15] $x^* \otimes y$ has a unique norm-preserving extension from $\mathcal{F}(Y,X)$ to $\mathcal{W}(Y,X)$ and hence $\Phi(x^* \otimes y) = x^* \otimes y$. Since Y has the RNP we get $\Phi(x^* \otimes y)$ for every $x^* \in X^*$ and $y \in Y$ by linearity and continuity. By a theorem of Feder and Saphar [4, Theorem 1] $\mathcal{F}(Y,X)^*$ is a quotient of $X^* \otimes Y$ and it follows that Φ is just the identity and hence unique. □

A Banach space X has the AP if and only if $\mathcal{F}(Y,X)$ is dense in $\mathcal{K}(Y,X)$ for every Banach space Y (cf. e.g. [23, Theorem 1.e.4]). By [17, Theorem 5.1] X has the AP if and only if $\mathcal{F}(Y,X)$ is a (trivially unconditional) ideal in $\mathcal{K}(Y,X)$ for every Banach space Y.

For Y reflexive, we can combine Proposition 4.2.5 with the isometries $\mathcal{F}(X,Y) = \mathcal{F}(Y^*, X^*)$ and $\mathcal{W}(X,Y) = \mathcal{W}(Y^*, X^*)$ and we get the following corollary.

Corollary 4.2.6. *Let X and Y be Banach spaces. If X^* has the AP and Y is reflexive, then $\mathbb{H}\mathbb{B}(\mathcal{F}(X,Y), \mathcal{W}(X,Y))$ consists of one element only.*

The dual of a Banach space X has the AP if and only if $\mathcal{F}(X,Y)$ is dense in $\mathcal{K}(X,Y)$ for every Banach space Y (cf. e.g. [23, Theorem 1.e.5]). By [17, Theorem 5.2] X^* has the AP if and only if $\mathcal{F}(X,Y)$ is a (trivially unconditional) ideal in $\mathcal{K}(X,Y)$ for every Banach space Y.

4.3 $\mathcal{F}(Y,X)$ as a u-ideal in $\mathcal{W}(Y,X)$

From [17, Theorem 5.1] and [19, Theorem 4.4] (resp. [19, Theorem 4.3]) we have the following result.

Theorem 4.3.1 (Lima and Oja). *Let X be a closed subspace of a Banach space Z. Then $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,Z)$ (resp. $\mathcal{K}(Y,Z)$) for all Banach spaces Y if and only if $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,Z)$ (resp. $\mathcal{K}(Y,Z)$) for all (resp. separable) reflexive Banach spaces Y.*

The next theorem characterizes the property that $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,X)$ for every Banach space Y in terms of convergence of nets of finite rank operators. The statements should be compared with their prototypes in similar results on ideals (see [11, Theorem 5.2] and [20, Theorem 2.3]).

Theorem 4.3.2. *Let X be a Banach space. The following statements are equivalent.*

(a) *$\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,X)$ for every Banach space Y.*

(b) *$\mathcal{F}(Y,X)$ is a u-ideal in span$(\mathcal{F}(Y,X), \{T\})$ for every $T \in \mathcal{W}(Y,X)$ and for every reflexive Banach space Y.*

(c) *For every reflexive Banach space Y there exists a Hahn-Banach extension operator $\Psi \in \mathbb{H}\mathbb{B}(\mathcal{F}(Y,X), \mathcal{W}(Y,X))$ such that for every $T \in \mathcal{W}(Y,X)$ there is a net $(T_\alpha) \subset \mathcal{F}(Y,X)$ with $\limsup_\alpha \|T - 2T_\alpha\| \le \|T\|$ such that $T_\alpha \to \Psi^*(T) = T$ weak* in $\mathcal{F}(Y,X)^{**}$.*

(d) *For every weakly compact set $K \subset X$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\lim_\alpha \sup_{x\in K} \|x - 2S_\alpha x\| \le \sup_{x\in K} \|x\|$ such that $S_\alpha \to I_X$ uniformly on compact subsets of K.*

(e) *For every Banach space Y and $T \in \mathcal{W}(Y,X)$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup_\alpha \|T - 2S_\alpha T\| \le \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X.*

(f) *For every Banach space Y and $T \in \mathcal{W}(Y,X)$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup_\alpha \|T - 2S_\alpha T\| \le \|T\|$ such that $S_\alpha \to I_X$ in the strong operator topology.*

(g) *For every reflexive Banach space Y and $T \in \mathcal{W}(Y,X)$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup_\alpha \|T - 2S_\alpha T\| \le \|T\|$ such that $S_\alpha T \to T$ in the strong operator topology.*

Proof. (a) $\Rightarrow$ (b) is immediate from the local characterization of u-ideals, Theorem 4.1.1.

(b) $\Rightarrow$ (c). Assume that Y is reflexive and let $T \in \mathcal{W}(Y,X)$. Since $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{B} = \operatorname{span}(\mathcal{F}(Y,X), \{T\})$ we can, using the local characterization of u-ideals Theorem 4.1.1, find a net $(T_\alpha) \subset \mathcal{F}(Y,X)$ with $\limsup_\alpha \|T - 2T_\alpha\| \leq \|T\|$ such that $T_\alpha \to \Phi_T^*(T)$ weak*, where $\Phi_T \in \mathbb{H}\mathbb{B}(\mathcal{F}(Y,X), \mathcal{B})$ is the unconditional extension operator. From the argument in the proof of Proposition 4.2.5 Φ_T is unique and of the form $\Phi_T = I_{X^*} \otimes I_Y$. A straightforward calculation shows that $\Phi_T^*(T) = T$. Thus the operator $\Psi = I_{X^*} \otimes I_Y \in \mathbb{H}\mathbb{B}(\mathcal{F}(Y,X), \mathcal{W}(Y,X))$ satisfies (c) in Theorem 4.1.1.

(c) $\Rightarrow$ (d). Let $K \subset X$ be weakly compact, $\varepsilon > 0$, and $u = \sum_{n=1}^{\infty} x_n^* \otimes x_n \in X^* \hat{\otimes}_\pi X$. Assume that K is a symmetric subset of B_X. Assume also that $1 \geq \|x_n\| \to 0$ and that $\sum_{n=}^{\infty} \|x_n^*\| < \infty$. Put $[Z, J] = \mathrm{DFJP}(\overline{\operatorname{conv}}\{\pm K \cup x_n : n = 1, ..., \infty\})$. Now Z is reflexive, $J \in \mathcal{W}(Z,X)$, and $\|J\| \leq 1$. Find $z_n \in B_Z$ such that $x_n = Jz_n$. Choose a net $(J_\alpha) \subset \mathcal{F}(Z,X)$ with $\limsup_\alpha \|J - 2J_\alpha\| \leq \|J\|$ such that $J_\alpha \to J$ weak* in $\mathcal{F}(Z,X)^{**}$. Since $J^* X^*$ is norm-dense in Z^* [16, Lemma 1.1] we can write $J_\alpha = S_\alpha J$ where $(S_\alpha) \subset \mathcal{F}(X,X)$ (see the proof of [21, Theorem 3.2]). Now we can find an S among the S_α's such that

$$\varepsilon > |\sum_{n=1}^{\infty} \langle SJz_n, x_n^* \rangle - \sum_{n=1}^{\infty} \langle Jz_n, x_n^* \rangle | = |\sum_{n=1}^{\infty} \langle Sx_n, x_n^* \rangle - \sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle |$$

and $\sup_{x \in K} \|x - 2Sx\| \leq \sup_{z \in B_Z} \|Jz - 2SJz\| \leq \|J - 2SJ\| < 1 + \varepsilon$.

(d) $\Rightarrow$ (e). Let Y be a Banach space and let $T \in \mathcal{W}(Y,X)$ of norm one. Let $C \subset B_X$ be compact and let $\varepsilon > 0$. Define $K = \overline{\operatorname{conv}}(\pm(C \cup T(B_Y)))$ and note that $K \subset B_X$ and weakly compact. By assumption there is $S \in \mathcal{F}(X,X)$ with $\sup_{x \in K} \|x - 2Sx\| < 1 + \varepsilon$ and $\sup_{x \in C} \|x - Sx\| < \varepsilon$. From this (e) follows.

(e) $\Rightarrow$ (f) and (f) $\Rightarrow$ (g) are trivial.

(g) $\Rightarrow$ (a). Let Y be a Banach space, let $\varepsilon > 0$, and choose a finite dimensional subspace $F \subset \mathcal{W}(Y,X)$. Put $[Z, J, \Phi] = \mathrm{DFJP}(F)$ (see (4.1.1)) and let $G = F \cap \mathcal{F}(Y,X)$. Then

$$K = \overline{\bigcup_{T \in B_G} T(B_Y)}$$

is a compact subset of X and of Z. It follows from the assumptions that we can find an $S \in \mathcal{F}(X,X)$ with $\|J - 2SJ\| \leq 1 + \varepsilon$ such that $\|z - Sz\| \leq \varepsilon$ for every $z \in K$. Define $L : F \to \mathcal{F}(Y,X)$ by $L(T) = ST$. Then $\|T - L(T)\| \leq \|\Phi(T)\| \|J - SJ\| \leq \varepsilon \|T\|$ for every $T \in G$ and $\|T - 2L(T)\| = \|T - 2ST\| \leq \|\Phi(T)\| \|J - 2SJ\| \leq (1 + \varepsilon)\|T\|$ for $T \in F$. The result now follows from local characterization of u-ideals in Theorem 4.1.1. □

Remark 4.3.3. Let $\hat{\ell}_2$ be the equivalently renormed version of ℓ_2 defined by Oja and denoted F in Example 3 in [25]. The space $\mathcal{F}(\ell_1, \hat{\ell}_2)$ is not a u-ideal in $\mathcal{W}(\ell_1, \hat{\ell}_2)$ (by [25, Example 3] and [27, Theorem 1.2] or [28, Proposition 1]). Since $\hat{\ell}_2$ has the AP, $\mathcal{F}(Y, \hat{\ell}_2)$ is an ideal in $\mathcal{W}(Y, \hat{\ell}_2)$ for all Banach spaces Y (see [25, Example 3] or [16, Theorem 3.3]). Thus statement (a) in Theorem 4.3.2 is strictly stronger than statement (a) in Proposition 4.3.5 below. Note that this implies that the bound $\limsup_\alpha \|T - 2S_\alpha T\| \leq \|T\|$ in statement (f) in 4.3.2 is strictly stronger than the bound $\limsup_\alpha \|T_\alpha\| \leq \|T\|$ in (iii) in Corollary 1.5

in [16].

Since $\hat{\ell}_2$ is reflexive, we also get that $\mathcal{F}(\hat{\ell}_2^*, \ell_\infty)$ is not a u-ideal in $\mathcal{W}(\hat{\ell}_2^*, \ell_\infty)$. Hence, also ℓ_∞ is an example of a Banach space X such that $\mathcal{F}(Y, X)$ is an ideal in $\mathcal{W}(Y, X)$ for all Banach spaces Y, without being a u-ideal for all Y. Also, if for $0 < r < 1$, Y_r are the equivalently renormed versions of c_0 defined in [8], then $\mathcal{F}(\ell_1, Y_r)$ is not a u-ideal in $\mathcal{W}(\ell_1, Y_r)$ for any $0 < r < 1$, even though $\mathcal{F}(Y, Y_r)$ is an ideal in $\mathcal{W}(Y, Y_r)$ for all Banach spaces Y and $0 < r < 1$ (see last paragraph in [25]).

Remark 4.3.4. Let X be a Banach space and let $K \subset B_X$ be a weakly compact subset. If X has the AP, then there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\sup_{x \in K} \|S_\alpha x\| \leq 1$ such that $S_\alpha \to I_X$ uniformly on compact sets in X. Indeed, put $[Z, J] = \mathrm{DFJP}(\overline{\mathrm{conv}}(\pm K))$. Using [4, Theorem 1] we get that $B_{\mathcal{F}(Z,X)}$ cannot be strongly separated from $\mathrm{conv}(S_\alpha J)$. This should be compared with statement (d) in Theorem 4.3.2.

A Banach space X is said to have the *unconditional metric approximation property* (UMAP) if there is a net $(T_\alpha) \subset \mathcal{F}(X, X)$ with $\limsup_\alpha \|I_X - 2T_\alpha\| \leq 1$ such that $T_\alpha(x) \to x$ for all $x \in X$. Like u-ideals, also the notion of the UMAP (for separable spaces using sequences) was introduced by Casazza and Kalton in [2].

In Theorem 5.2 in [11] it was proved that X has the UMAP if and only if $\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{L}(Y, X)$ for every Banach space Y.

If X is reflexive, then (d) in Theorem 4.3.2 says that X has the UMAP. By [2, Theorem 3.9], it follows that in this case $\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X)$ for all Banach spaces Y if and only if $\mathcal{F}(X, X)$ is a u-ideal in $\mathcal{W}(X, X)$.

From [16, Theorem 3.3] and [14, Corollary 2] (see also [9, Theorem 5.1], [30, Proposition 2.1]) we get the following proposition.

Proposition 4.3.5. *Let X be a Banach space. The following are equivalent.*

(a) *$\mathcal{F}(Y, X)$ is an ideal in $\mathcal{W}(Y, X)$ for every Banach space Y.*

(b) *X has the AP.*

(c) *Every separable ideal Z in X has the AP.*

(d) *$\mathcal{F}(Y, Z)$ is an ideal in $\mathcal{W}(Y, Z)$ for every Banach space Y and separable ideal Z in X.*

For u-ideals we have the following result.

Proposition 4.3.6. *Let X be a Banach space and assume $\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X)$ for every Banach space Y. Then a closed subspace Z of X has the AP if and only if $\mathcal{F}(Y, Z)$ is a u-ideal in $\mathcal{W}(Y, Z)$ for every Banach space Y.*

Proof. One direction is immediate from Proposition 4.3.5.

For the reverse direction let Y be a reflexive Banach space, Z a subspace of X with the AP, and $T \in \mathcal{W}(Y, Z)$. Put $\hat{T} = i_Z \circ T$, choose a compact subset K of Z, and let $\varepsilon > 0$. By Theorem 4.3.2 there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\limsup_\alpha \|\hat{T} - 2S_\alpha \hat{T}\| \leq \|\hat{T}\| = \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets. Since Z has the AP, there is a net $(U_\beta) \subset \mathcal{F}(Z, Z)$ such that $U_\beta \to I_Z$ uniformly on compact sets. After switching to the product index set we may suppose that (U_β) is indexed by the same set as (S_α). Hence we shall write

(U_α) from now on.

Now let $u \in \mathcal{F}(Y,X)^*$. Since Y is reflexive and X has the AP $\mathcal{F}(Y,X)^*$ is isometrically isomorphic to a quotient of $X^*\hat{\otimes}_\pi Y$ by a theorem of Feder and Saphar [4, Theorem 1]. Choose a representation $\sum_{n=1}^\infty x_n^* \otimes y_n$ for u. For the net $T_\alpha = S_\alpha i_Z T - i_Z U_\alpha T$, we have

$$\begin{aligned}\langle u, T_\alpha\rangle &= \sum_{n=1}^\infty \langle x_n^*, (S_\alpha i_Z T - i_Z U_\alpha T)(y_n)\rangle \\ &\to \sum_{n=1}^\infty \langle i_Z^* x_n^*, T y_n\rangle - \sum_{n=1}^\infty \langle i_Z^* x_n^*, T y_n\rangle = 0.\end{aligned}$$

Hence $T_\alpha \to 0$ weakly in $\mathcal{F}(Y,X)$. Consequently a suitable net of convex combinations of T_α converges in norm to 0. Thus there exist α_0, $\hat{S}_{\alpha_0} \in \text{co}\{S_\alpha : \alpha > \alpha_0\}$, and $\hat{U}_{\alpha_0} \in \text{co}\{U_\alpha : \alpha > \alpha_0\}$ such that $\|\hat{S}_{\alpha_0} i_Z T - i_Z \hat{U}_{\alpha_0} T\| \le \varepsilon$, $\sup_{z\in K} \|\hat{U}_{\alpha_0}(z) - z\| \le \varepsilon$, and $\|\hat{T} - 2\hat{S}_{\alpha_0}\hat{T}\| \le \|\hat{T}\| + \varepsilon$. We get that

$$\|i_Z T - 2 i_Z \hat{U}_{\alpha_0} T\| \le \|i_Z T - 2\hat{S}_{\alpha_0} i_Z T\| + 2\|\hat{S}_{\alpha_0} i_Z T - i_Z \hat{U}_{\alpha_0} T\| \le \|\hat{T}\| + 3\varepsilon.$$

Hence $\|T - 2\hat{U}_{\alpha_0} T\| \le \|T\| + 3\varepsilon$, and the result follows from the local characterization of u-ideals Theorem 4.1.1.

□

Remark 4.3.7. If $\mathcal{F}(Y,Z)$ is a u-ideal in $\mathcal{W}(Y,Z)$ for every Banach space Y and subspace Z of X with the AP, then $\mathcal{F}(Y,X)$ is not necessarily a u-ideal in $\mathcal{W}(Y,X)$ for every Banach space Y. Indeed, for $1 < p < \infty$, choose a subspace X of ℓ_p such that X does not have the AP (cf. e.g. [23, p. 91]). X cannot be complemented and hence is not an ideal in ℓ_p. It is probably well known that $\mathcal{F}(Y,\ell_p)$ is a u-ideal in $\mathcal{W}(Y,\ell_p)$ for all Banach spaces Y. (It can be proved by using that the standard basis of ℓ_p is 1-unconditional and then Theorem 4.3.2 (g).) By Proposition 4.3.6 $\mathcal{F}(Y,Z)$ is a u-ideal in $\mathcal{W}(Y,Z)$ for every subspace Z of X with the AP. But X does not have the AP so $\mathcal{F}(Y_0,X)$ is not even an ideal in $\mathcal{W}(Y_0,X)$ for some Banach space Y_0 by [16, Theorem 3.3].

Let X be a Banach space. In the next theorem we want to study when $\mathcal{F}(X,Y)$ is a u-ideal in $\mathcal{W}(X,Y)$ for all Banach spaces Y. In Theorem 6.5 in [11] it was proved that (a) $\mathcal{K}(X,Y)$ is a u-ideal in $\mathcal{L}(X,Y)$ for all Banach spaces Y is equivalent to (c) there is a net $(T_\alpha) \subset \mathcal{K}(X,X)$ with $\limsup_\alpha \|I - 2T_\alpha\| \le 1$ such that $T_\alpha x \to x$ for all $x \in X$ and $T_\alpha^* x^* \to x^*$ for all $x^* \in X^*$ which in turn is equivalent to (e) X has the metric compact approximation property and X has property (wM^*). Note that the equivalence of (c) and (e) follows from the equivalence of (3°) and (2°) in Corollary 4.5 in [29] by taking $a = 1$ and $B = \{-2\}$. In all these statements $\mathcal{K}(X,X)$ (resp. $\mathcal{K}(X,Y)$) may be replaced by $\mathcal{F}(X,X)$ (resp. $\mathcal{F}(X,Y)$) (see the text after Corollary 4.6 in [29]).

Theorem 4.3.8. *Let X be a Banach space. The following statements are equivalent.*

(a) *$\mathcal{F}(X,Y)$ is a u-ideal in $\mathcal{W}(X,Y)$ for every Banach space Y.*

(b) *$\mathcal{F}(X,Y)$ is a u-ideal in $\mathcal{W}(X,Y)$ for every reflexive Banach space Y.*

(c) *$\mathcal{F}(X,Y)$ is a u-ideal in span$(\mathcal{F}(X,Y),\{T\})$ for every $T \in \mathcal{W}(X,Y)$ and for every reflexive Banach space Y.*

(d) *For every reflexive Banach space Y there exists a Hahn-Banach extension operator $\Psi \in I\!B(\mathcal{F}(X,Y), \mathcal{W}(X,Y))$ such that for every $T \in \mathcal{W}(X,Y)$ there is a net $(T_\alpha) \subset \mathcal{F}(X,Y)$ with $\limsup_\alpha \|T - 2T_\alpha\| \le \|T\|$ such that $T_\alpha \to \Psi^*(T) = T$ weak* in $\mathcal{F}(X,Y)^{**}$.*

(e) *For every weakly compact compact set $K \subset X^*$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\lim_\alpha \sup_{x^* \in K} \|x^* - 2S_\alpha^* x^*\| \le \sup_{x^* \in K} \|x^*\|$ such that $S_\alpha^* \to I_{X^*}$ uniformly on compact subsets of K.*

(f) *For every Banach space Y and $T \in \mathcal{W}(X,Y)$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $\limsup_\alpha \|T - 2TS_\alpha\| \le \|T\|$ and $S_\alpha^* \to I_{X^*}$ uniformly on compact sets in X^*.*

(g) *For every Banach space Y and $T \in \mathcal{W}(X,Y)$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $\limsup_\alpha \|T - 2TS_\alpha\| \le \|T\|$ and $S_\alpha^* \to I_{X^*}$ in the strong operator topology.*

(h) *For every reflexive Banach space Y and $T \in \mathcal{W}(X,Y)$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $\limsup_\alpha \|T - 2TS_\alpha\| \le \|T\|$ and $S_\alpha^* T^* \to T^*$ in the strong operator topology.*

Proof. If Y is a reflexive Banach space, we have isometries $\mathcal{F}(X,Y) = \mathcal{F}(Y^*, X^*)$ and $\mathcal{W}(X,Y) = W(Y^*, X^*)$. Using this observation, Theorem 4.3.8, for reflexive spaces Y, follows from Theorem 4.3.2.

It now suffices to show that the statements in (a) and (f) hold whenever they hold for reflexive spaces Y. Indeed, to see that (a) holds we can use the local characterization of u-ideals in Theorem 4.1.1 and an argument similar to (g) $\Rightarrow$ (a) in Theorem 4.3.2 (use (4.1.2) instead of (4.1.1)).

To see that (f) holds we put $[Z, \Phi, J] = \mathrm{DFJP}(\mathrm{span}(\{T\})$ where Y is a Banach space and $T \in \mathcal{W}(X,Y)$. Since Z is reflexive and $J \in \mathcal{W}(X,Z)$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup_\alpha \|J - 2JS_\alpha\| \le \|J\| = 1$ such that $S_\alpha^* \to I_{X^*}$ uniformly on compact sets in X^*. Finally, write $\limsup_\alpha \|T - 2TS_\alpha\| \le \limsup_\alpha \|\Phi(T)\| \|J - 2JS_\alpha\| \le \|T\|$ and we are done. □

Remark 4.3.9. By [16, Theorem 3.4] we get that $\mathcal{F}(\ell_1, Y)$ is an ideal in $\mathcal{W}(\ell_1, Y)$ for every Banach space Y. In Remark 4.3.3 we noticed that $\mathcal{F}(\ell_1, \hat{\ell}_2)$ is not a u-ideal in $\mathcal{W}(\ell_1, \hat{\ell}_2)$ where $\hat{\ell}_2$ is the equivalent renorming of ℓ_2 constructed by Oja in [25]. Thus ℓ_1 does not fulfill statement (a) in Theorem 4.3.8.

Note that Proposition 2.3 in [22] for M-ideals also holds for u-ideals by using the local characterization of u-ideals in Theorem 4.1.1 instead of the 3-ball-property used in [22, Proposition 2.3] (see [13, Theorem 6.17], [7, Theorem I.2.2] or [22, Theorem 2.1]). Thus if a dual space X^* contains a copy of c_0, then $\mathcal{F}(\ell_1, Y)$ is a u-ideal in $\mathcal{W}(\ell_1, Y)$ whenever $\mathcal{F}(X,Y)$ is a u-ideal in $\mathcal{W}(X,Y)$. If $\hat{\ell}_2$ is the equivalently renormed version of ℓ_2 constructed by Oja, it follows from the preceding paragraph that $\mathcal{F}(X, \hat{\ell}_2)$ fails to be a u-ideal in $\mathcal{W}(X, \hat{\ell}_2)$ whenever X^* contains a copy of c_0.

Remark 4.3.10. Recall that a u-ideal Z in X is *strict* if the u-complement of $Z^\perp$ in X^* is a norming subspace for X, i.e. if $\varphi(Z^*)$ is a norming subspace of X^* where $\varphi \in I\!B(Z,X)$ is the unconditional Hahn-Banach extension operator.

If Y is a reflexive Banach space and $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,X)$ then it is in fact a strict u-ideal. This is easily seen from the proof of Proposition

4.2.5. Indeed, in this case there is a unique Hahn-Banach extension operator $\Phi \in \mathrm{HB}(\mathcal{F}(Y,X), \mathcal{W}(Y,X))$ which is of the form $\Phi = I_{X^*} \otimes I_Y$. Since $B_{X^*} \otimes B_Y \subset \mathcal{W}(Y,X)^*$ is norming for $\mathcal{W}(Y,X)$ the claim follows. Similarly by Corollary 4.2.6, if Y is reflexive, then $\mathcal{F}(X,Y)$ is a strict u-ideal in $\mathcal{W}(X,Y)$ whenever it is a u-ideal.

If X is a Banach space it follows from [16, Theorem 3.4] and [12, Proposition 2.5] that $\mathcal{F}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$ for every Banach space Y if and only if $\mathcal{F}(Z,Y)$ is an ideal in $\mathcal{W}(Z,Y)$ for every Banach space Y and for every separable ideal Z in X. For u-ideals we have the following result.

Proposition 4.3.11. *Let X be a Banach space. If $\mathcal{F}(X,Y)$ is a u-ideal in $\mathcal{W}(X,Y)$ for every Banach space Y, then $\mathcal{F}(Z,Y)$ is a u-ideal in $\mathcal{W}(Z,Y)$ for every ideal Z in X and Banach space Y.*

Proof. Let Y be a Banach space and let Z be an ideal in X with corresponding Hahn-Banach extension operator $\varphi \in \mathrm{HB}(Z,X)$. Let G be a finite dimensional subspace of $\mathcal{W}(Z,Y)$ and define the map $L : G \to \mathcal{W}(X,Y)$ by

$$L(T) = T^{**} \circ \varphi^*|_X, \quad T \in G.$$

Let $\varepsilon > 0$. By the local characterization of u-ideals, Theorem 4.1.1, there is an operator $M : L(G) \to \mathcal{F}(X,Y)$ such that $M(S) = S$ for every $S \in \mathcal{F}(X,Y) \cap L(G)$ and $\|S - 2M(S)\| \leq (1+\varepsilon)\|S\|$ for every $S \in L(G)$. Now define an operator $N : G \to \mathcal{F}(Z,Y)$ by

$$N(T) = M(L(T)) \circ i_Z.$$

It is straightforward to verify that the operator N fulfills (d) in Theorem 4.1.1 and the result follows. ☐

4.4 $\mathcal{F}(Y,X)$ as a u-ideal in $\mathcal{K}(Y,X^{**})$ and $\mathcal{W}(Y,X^{**})$

From [17, Theorem 5.1] and [19, Proposition 2.10] we have the following result.

Proposition 4.4.1 (Lima and Oja)**.** *Let X be a closed subspace of a Banach space Y. If $\mathcal{F}(Z,X)$ is a u-ideal in $\mathcal{K}(Z,Y)$ for every reflexive Banach space Z, then X is a u-ideal in Y.*

The next result tells us more.

Proposition 4.4.2. *Let X be a closed subspace of a Banach space Y and let Z be a reflexive Banach space. Assume $\mathcal{F}(Z,X)$ is a u-ideal in $\mathcal{K}(Z,Y)$ with unconditional extension operator Ψ. Then X is a u-ideal in Y with unconditional extension operator ψ satisfying*

$$\Psi(x^* \otimes z) = (\psi x^*) \otimes z$$

for all $z \in Z$ and $x^* \in X^*$.

Moreover, if the above assumption holds for every separable reflexive Banach space Z, then $\psi^*|_Y$ is in the w^*-closure of $\mathcal{F}(Y,X)$ in $\mathcal{L}(Y,X^{**})$.

Proof. We proceed as in the proof of [18, Theorem 2.3]. Let $\Psi \in \mathbb{HB}(\mathcal{F}(Z,X),\mathcal{K}(Z,Y))$ be the unconditional Hahn-Banach extension operator and denote the corresponding ideal projection on $\mathcal{K}(Z,Y)^*$ by P_Ψ. Since Z is reflexive, it follows from [18, Theorem 1.3] that there exist $\{\psi_i : i = 1, ..., n\} \subset \mathbb{HB}(X,Y)$ such that

$$Z = \sum_{i=1}^{n} \oplus_1 Z_{\Psi\psi_i}, \qquad Z_{\Psi\psi_i} \neq \{0\} \text{ for all } 1 \leq i \leq n,$$

where

$$Z_{\Psi\psi_i} = \{z \in Z : \Psi(x^* \otimes z) = (\psi_i x^*) \otimes z, \forall x^* \in X^*\}.$$

Let (P_{ψ_i}) be the corresponding ideal projections on Y^*. It now follows that for $z \in Z_{\Psi\psi_i}$ and $y^* \in Y^*$

$$\begin{aligned}\|z\|\|y^*\| = \|y^* \otimes z\| &\geq \|(I - 2P_\Psi)(y^* \otimes z)\| = \|y^* \otimes z - 2P_\Psi(y^* \otimes z)\| \\ &= \|y^* \otimes z - 2(P_{\psi_i} y^*) \otimes z\| = \|(y^* - 2P_{\psi_i} y^*) \otimes z\| = \|z\|\|y^* - 2P_{\psi_i} y^*\|.\end{aligned}$$

Hence every ψ_i is unconditional and by uniqueness, see Proposition 4.2.2, they are all equal. With $\psi = \psi_i$ we have $Z = Z_{\Psi\psi}$.

Furthermore, if $\mathcal{F}(Z,X)$ is a u-ideal in $\mathcal{K}(Z,X)$ for all separable reflexive Z, then by Lemma 2.1 in [20] there is for every such Z and $T \in \mathcal{K}(Z,Y)$ a net (T_α) in $\mathcal{F}(Z,X)$ with $\sup_\alpha \|T_\alpha\| \leq \|T\|$ such that $T_\alpha^* \to T^*\psi$ in the strong operator topology. By boundedness we may also assume that $\langle u, T_\alpha \rangle \to \langle u, T \rangle$ for all $u \in X^* \hat{\otimes}_\pi Z$.

Choose $u = \sum_n x_n^* \otimes y_n \in X^* \hat{\otimes}_\pi Y$ and assume that $\sum_n \|x_n^*\| = 1$ and $1 \geq \|y_n\| \to 0$ and put $[Z, J] = \text{DFJP}(\overline{\text{conv}}\{\pm y_n : n = 1, ..., \infty\})$. Then Z is a separable reflexive Banach space and $J \in \mathcal{K}(Z,Y)$ with $\|J\| \leq 1$. Pick a net $(J_\alpha) \subset \mathcal{F}(Z,X)$ with $\sup_\alpha \|J_\alpha\| \leq \|J\|$ such that $J_\alpha^* \to J^*\psi$ uniformly on compact sets. As in the proof of (c)$\Rightarrow$ (d) in Theorem 4.3.2 we may assume that each $J_\alpha^* = J^* S_\alpha^*$ for some $S_\alpha \in \mathcal{F}(Y,X)$. Now choose $\varepsilon > 0$ and let $z_n \in B_Z$ such that $y_n = Jz_n$. Since $J_\alpha^* \to J^*\psi$ uniformly on compact sets, it follows that there is an operator $S \in \mathcal{F}(Y,X)$ such that

$$\varepsilon > |\sum_{n=1}^{\infty} \langle J^* S^* x_n^*, z_n \rangle - \sum_{n=1} \langle J^* \psi x_n^*, z_n \rangle | = | \sum_{n=1} \langle x_n^*, S y_n \rangle - \sum_{n=1} \langle x_n^*, \psi^* y_n \rangle |.$$

Hence $\psi^*|_Y$ is in the w^*-closure of $\mathcal{F}(Y,X)$ in $\mathcal{L}(Y, X^{**})$.

□

Remark 4.4.3. If $Y = X^{**}$ in Proposition 4.4.2 we actually have that $\psi^*|_{X^{**}}$ is in the weak*-closure of set $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**}, X^{**})$. In this case $J^*(X^*)$ and not just $J^*(X^{***})$ is norm-dense in Z^* (see the proof of [10, Proposition 2.1]). Thus we can write each $J_\alpha^* = J^* S_\alpha^*$ for some S_α in $\mathcal{F}(X,X)$ (and not only in $\mathcal{F}(X^{**}, X)$).

Let X be a Banach space. From Theorem 4.3.1 we have that $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y, X^{**})$ for every Banach space Y if and only if $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y, X^{**})$ for every reflexive Banach space Y. The next results contain other characterizations of these statements.

Theorem 4.4.4. *Let X be a Banach space. The following statements are equivalent.*

(a) *$\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X^{**})$ for every Banach space Y.*

(b) *X is a u-ideal in its bidual with unconditional Hahn-Banach extension operator $\psi \in H\!B(X, X^{**})$ such that for every Banach space Y and $T \in \mathcal{W}(Y, X^{**})$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leq \|T\|$ such that $S_\alpha^{**}T \to \psi^* T$ weak* in $\mathcal{L}(Y, X^{**})$.*

(c) *There exists a Hahn-Banach extension operator $\psi \in H\!B(X, X^{**})$ such that for every Banach space Y and $T \in \mathcal{W}(Y, X^{**})$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leq \|T\|$ such that $S_\alpha^{**}T \to \psi^* T$ weak* in $\mathcal{L}(Y, X^{**})$.*

(d) *For every weakly compact compact set $K \subset X^{**}$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\lim_\alpha \sup_{x^{**} \in K} \|x^{**} - 2S_\alpha^{**}x^{**}\| \leq \sup_{x^{**} \in K} \|x^{**}\|$ such that $S_\alpha \to I_X$ uniformly on compact subsets of $K \cap X$.*

(e) *For every Banach space Y and $T \in \mathcal{W}(Y, X^{**})$, there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leq \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X.*

(f) *For every reflexive Banach space Y and $T \in \mathcal{W}(Y, X^{**})$, there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leq \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X.*

Proof. (a) $\Rightarrow$ (b). Let Y be a Banach space and let $T \in \mathcal{W}(Y, X^{**})$. Put $G = \operatorname{span}(\{T\})$ and let $[Z, J, \Phi] = \mathrm{DFJP}(G)$. Now Z is reflexive and $J \in \mathcal{W}(Z, X^{**})$ is of norm 1. Let $\Psi : \mathcal{F}(Z, X)^* \to \mathcal{W}(Z, X^{**})^*$ be the unconditional Hahn-Banach extension operator. As in the proof of Proposition 4.4.2 we can show that X is a u-ideal in X^{**} with $\psi \in H\!B(X, X^{**})$ unconditional such that $\Psi(x^* \otimes z) = \psi(x^*) \otimes z$ for every $x^* \in X^*$ and $z \in Z$. By Theorem 4.1.1 there is a net $(J_\alpha) \subset \mathcal{F}(Z, X)$ such that $\limsup_\alpha \|J - 2J_\alpha\| \leq 1$ and $J_\alpha \to \Psi^*(J)$ weak*. Since $J^*(X^*)$ is norm dense in Z^* we can assume that each $J_\alpha = S_\alpha^{**}J$ where $(S_\alpha) \subset \mathcal{F}(X, X)$. Since $\|T - 2S_\alpha^{**}T\| = \|J\Phi(T) - 2S_\alpha^{**}J\Phi(T)\| \leq \|T\|\|J - 2S_\alpha^{**}J\|$ we get $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leq \|T\|$.

Let $u = \sum_n x_n^* \otimes y_n \in X^* \hat{\otimes}_\pi Y$. Then $v = \sum_n x_n^* \otimes (\Phi(T)y_n) \in X^* \hat{\otimes}_\pi Z$. We get that

$$\begin{aligned}\langle u, \psi^* T\rangle &= \sum_n \langle \psi x_n^*, J\Phi(T)y_n\rangle = \langle \Psi(v), J\rangle = \langle v, \Psi^*(J)\rangle \\ &= \lim_\alpha \langle v, S_\alpha^{**}J\rangle = \lim_\alpha \sum_n \langle x_n^*, S_\alpha^{**}Ty_n\rangle = \lim_\alpha \langle u, S_\alpha^{**}T\rangle.\end{aligned}$$

This shows that $S_\alpha^{**}T \to \psi^* T$ weak* in $\mathcal{L}(Y, X^{**})$.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d) is similar to the proof of (c) $\Rightarrow$ (d) in Theorem 4.3.2.

(d) $\Rightarrow$ (e) is similar to the proof of (d) $\Rightarrow$ (e) in Theorem 4.3.2.

(e) $\Rightarrow$ (f) is trivial.

(f) $\Rightarrow$ (a) is similar to the proof of (f) $\Rightarrow$ (a) in Theorem 4.3.2. □

Remark 4.4.5. Note that $X = c_0$ fulfills Theorem 4.4.4 since c_0 an M_∞ space (see [7] p. 306) and [7, Proposition 5.6].

Theorem 4.4.6. *Let X be a Banach space. The following statements are equivalent.*

(a) *$\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{K}(Y,X^{**})$ for every Banach space Y.*

(b) *X is a u-ideal in X^{**} with unconditional Hahn-Banach extension ψ such that $\psi^*|_{X^{**}}$ is in the weak*-closure of the $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$.*

(c) *X is a u-ideal in its bidual with unconditional Hahn-Banach extension operator $\psi \in I\!H\!B(X,X^{**})$ such that for every Banach space Y and $T \in \mathcal{K}(Y,X^{**})$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \le \|T\|$ such that $S_\alpha^{**}T \to \psi^*T$ weak* in $\mathcal{L}(Y,X^{**})$.*

(d) *For every Banach space Y and $T \in \mathcal{K}(Y,X^{**})$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \le \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X.*

(e) *For every separable reflexive Banach space Y and $T \in \mathcal{K}(Y,X^{**})$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \le \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X.*

Proof. (a) $\Rightarrow$ (b) follows from Proposition 4.4.2.

(b) $\Rightarrow$ (c). Let Y be a Banach space and let $T \in \mathcal{K}(Y,X^{**})$. Put $G = \mathrm{span}(\{T\})$ and write $[Z,J,\Phi] = \mathrm{DFJP}(G)$. Now Z is reflexive and $J \in \mathcal{K}(Z,X^{**})$ has norm one. Let $\psi \in I\!H\!B(X,X^{**})$ be the unconditional Hahn-Banach extension operator and choose a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $S_\alpha^{**} \to \psi^*|_{X^{**}}$ weak* in $\mathcal{L}(X^{**},X^{**})$. Since Z is reflexive, $\mathcal{K}(Z,X^{**})^*$ is a quotient of $X^{***}\hat{\otimes}_\pi Z$ by [4, Theorem 1] of Feder and Saphar. Now let $\varepsilon > 0$ and let $u \in X^{***}\hat{\otimes}_\pi Z$. Choose a representation $\sum_{n=1}^\infty x_n^{***} \otimes z_n$ for u such that $\sum_{n=1}^\infty \|x_n^{***}\|\|z_n\| \le \|u\|_\pi + \varepsilon$ and write $x_n^* = x_n^{***}|_X$. We get that

$$|\langle u, J - 2S_\alpha^{**}J\rangle| = |\sum_{n=1}^\infty \langle x_n^{***}, (J - 2S_\alpha^{**}J)z_n\rangle| = |\sum_{n=1}^\infty \langle x_n^{***} - 2S_\alpha^* x_n^*, Jz_n\rangle|$$
$$\to |\sum_{n=1}^\infty \langle x_n^{***} - 2\psi x_n^*, Jz_n\rangle \le \sum_{n=1}^\infty \|x_n^{***}\|\|Jz_n\| \le \|u\|_\pi + \varepsilon.$$

Hence $\mathrm{conv}(J - 2S_\alpha^{**}J)$ can not be strongly separated from $B_{\mathcal{K}(Z,X^{**})}$. By taking successive convex combinations we get a new net, also denoted (S_α), such that $\limsup_\alpha \|J - 2S_\alpha^{**}J\| \le 1$. Thus

$$\limsup_\alpha \|T - 2S_\alpha^{**}T\| \le \limsup_\alpha \|\Phi(T)\|\|J - 2S_\alpha^{**}J\| \le \|T\|.$$

Obviously $S_\alpha^{**}T \to \psi^*T$ weak* in $\mathcal{L}(Y,X^{**})$.

(c) $\Rightarrow$ (d). Argue as in the proof of (d) $\Rightarrow$ (e) in Theorem 4.4.4.

(d) $\Rightarrow$ (e) is trivial.

(e) $\Rightarrow$ (a). Argue as in the proof of (g) $\Rightarrow$ (a) in Theorem 4.3.2. □

Remark 4.4.7. In [10, Proposition 2.1] it is proved that $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X^{**})$ for every Banach space Y if and only if $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{K}(Y,X^{**})$ for every Banach space Y. This fails if we replace "ideal" with "u-ideal". Indeed, if we let $X = \hat{\ell}_2$, the equivalent renorming of ℓ_2 obtained by Oja (see Remark 4.3.3), then we have a counterexample. This proves that the statements in Theorem 4.4.6 are strictly weaker than those in Theorem 4.4.4.

The next result shows that $\mathcal{F}(Y,X)$ being a u-ideal in $\mathcal{W}(Y,X^{**})$ for all Banach spaces Y is inherited by some subspaces of X.

Proposition 4.4.8. *Suppose $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,X^{**})$ for every Banach space Y and let $\varphi \in I\!H\!B(X,X^{**})$ be the unconditional Hahn-Banach extension operator. Then $\mathcal{F}(Y,Z)$ is a u-ideal in $\mathcal{W}(Y,Z^{**})$ for every Banach space Y and ideal Z in X such that $\varphi^*(Z^{\perp\perp}) \subset Z^{\perp\perp}$.*

Proof. Let Y be a reflexive Banach space and let Z be an ideal in X such that $\varphi^*(Z^{\perp\perp}) \subset Z^{\perp\perp}$. Denote by $i_Z : Z \to X$ the natural embedding. Since $\varphi^*(Z^{\perp\perp}) \subset Z^{\perp\perp}$, it follows from Theorem 4.2.4 that Z is a u-ideal in its bidual with an unconditional extension operator $\psi \in I\!H\!B(Z,Z^{**})$ such that $i_Z^{**}\psi^*|_{Z^{**}} = \varphi^* i_Z^{**}$. From Theorem 4.4.6 we have $\varphi^*|_{X^{**}}$ in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$. By the Principle of Local Reflexivity it is routine to check that $\psi^*|_{Z^{**}}$ is in the weak*-closure of $\mathcal{L}(Z^{**},Z^{**})$.

Choose a compact subset K of Z and an operator $T \in \mathcal{W}(Y,Z^{**})$. Put $\hat{T} = i_Z^{**} \circ T \in \mathcal{W}(Y,X^{**})$. By Theorem 4.4.4 there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup_\alpha \|\hat{T} - 2S_\alpha^{**}\hat{T}\| \le \|\hat{T}\| = \|T\|$ such that $S_\alpha^{**}\hat{T} \to \varphi^*|_{X^{**}}\hat{T}$ weak* in $\mathcal{L}(X^{**},X^{**})$. From the first paragraph there is a net $(U_i) \subset \mathcal{F}(Z,Z)$ such that $U_i^{**} \to \psi^*|_{Z^{**}}$ weak* in $\mathcal{L}(Z^{**},Z^{**})$. Assume (S_α) and (U_i) have the same index set. Thus we will write (U_α) for the net in $\mathcal{F}(Z,Z)$. Note that $U_\alpha \to I_Z$ uniformly on compact sets in Z. Now let $u = \sum_n x_n^* \otimes y_n \in \mathcal{F}(Y,X)^*$ and $T_\alpha = S_\alpha^{**} i_Z^{**} T - i_Z^{**} U_\alpha^{**} T$. From this we get that

$$\begin{aligned}
\langle u, T_\alpha \rangle &= \sum_n \langle x_n^*, (S_\alpha^{**} i_Z^{**} - i_Z^{**} U_\alpha^{**})(Ty_n) \rangle \\
&= \sum_n \langle x_n^*, S_\alpha^{**}(i_Z^{**} T y_n) \rangle - \sum_n \langle i_Z^* x_n^*, U_\alpha^{**}(Ty_n) \rangle \\
&\to \sum_n \langle x_n^*, \varphi^*(i_Z^{**} T y_n) \rangle - \sum_n \langle i_Z^* x_n^*, \psi^*(Ty_n) \rangle = 0.
\end{aligned}$$

Hence $T_\alpha \to 0$ weakly in $\mathcal{F}(Y,X)$. Consequently a suitable net of convex combinations of T_α converges in norm to 0. Thus there exist α_0, $\hat{S}_{\alpha_0} \in \operatorname{co}\{S_\alpha^{**} : \alpha > \alpha_0\}$, and $\hat{U}_{\alpha_0} \in \operatorname{co}\{U_\alpha^{**} : \alpha > \alpha_0\}$ such that $\|\hat{T} - 2\hat{S}_{\alpha_0}\hat{T}\| \le \|\hat{T}\| + \varepsilon$, $\sup_{z \in K} \|\hat{U}_{\alpha_0} z - z\| \le \varepsilon$, and $\|\hat{S}_{\alpha_0} i_Z^{**} T - i_Z^{**}\hat{U}_{\alpha_0} T)\| \le \varepsilon$. We get

$$\|i_Z^{**}T - 2i_Z^{**}\hat{U}_{\alpha_0}T\| \le \|i_Z^{**}T - 2\hat{S}_{\alpha_0} i_Z^{**}T\| + 2\|\hat{S}_{\alpha_0} i_Z^{**}T - i_Z^{**}\hat{U}_{\alpha_0}T\| \le \|\hat{T}\| + 3\varepsilon.$$

Hence $\|T - 2\hat{U}_{\alpha_0}T\| \le \|T\| + 3\varepsilon$, and the result follows. □

In [21] Lima and Oja introduced and studied the weak metric approximation property. Following Lima and Oja a Banach space X is said to have the *weak metric approximation property (weak MAP)* if for every Banach space Y and operator $T \in \mathcal{W}(X,Y)$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\sup_\alpha \|TS_\alpha\| \le \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact subsets in X. It is easy to see that the MAP implies the weak MAP. In [31, Corollary 1] it is shown that the weak MAP and the MAP are indeed equivalent for a Banach space for which either its dual or its bidual has the RNP.

Lima proved in [10] that X has the weak MAP if and only if $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{K}(Y,X^{**})$ for every Banach space Y. Based on this, it is natural to guess that an "unconditional version" of the weak MAP could be the property that for every Banach space Y and operator $T \in \mathcal{K}(X,Y)$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup_\alpha \|T - 2TS_\alpha\| \le \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X. As remarked below, this property is strictly weaker than the statements in Theorem 4.4.6.

Proposition 4.4.9. *Let X be a Banach space. The following statements are equivalent.*

(a) *For every Banach space Y and operator $T \in \mathcal{K}(X,Y)$, there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $\limsup_\alpha \|T - 2TS_\alpha\| \le \|T\|$ and $S_\alpha \to I_X$ uniformly on compact sets.*

(b) *For every reflexive Banach space Y and operator $T \in \mathcal{K}(X,Y)$, there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $\limsup_\alpha \|T - 2TS_\alpha\| \le \|T\|$ and $TS_\alpha \to T$ uniformly on compact sets.*

(c) *There is a Hahn-Banach extension operator $\psi \in \mathit{HB}(X, X^{**})$ with $\|I_{X^{**}} - 2\psi^*{}_{|X^{**}}\| = 1$ such that $\psi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**}, X^{**})$.*

Proof. (a) $\Rightarrow$ (b) is trivial.
(b) $\Rightarrow$ (c). The proof is essentially that of [10, Proposition 2.5].
(c) $\Rightarrow$ (a) is similar to Theorem 4.4.6 (c) $\Rightarrow$ (d). □

Remark 4.4.10. If $\psi \in \mathit{HB}(X, X^{**})$ is an unconditional extension operator then $\|I_{X^{**}} - 2\psi^*|_{X^{**}}\| = \|I_{X^{***}} - 2\psi k_X^*\| = 1$. To see this, first note that $1 = \|I_{X^{***}} - 2\psi k_X^*\| = \|I_{X^{****}} - 2k_X^{**}\psi^*\|$. Write the identity operator on the dual X^* as $I_{X^*} = k_X^* k_{X^*}$ and the identity operator on bidual X^{**} as $I_{X^{**}} = k_{X^*}^* k_{X^{**}}$. By taking adjoints we obtain from the first equality that $I_{X^{**}} = (I_{X^*})^* = k_{X^*}^* k_X^{**}$. It follows that

$$\begin{aligned}\|I_{X^{**}} - 2\psi^* k_{X^{**}}\| &= \|I_{X^{**}} - 2I_{X^{**}}\psi^* k_{X^{**}}\| \\ &= \|k_{X^*}^* k_{X^{**}} - 2k_{X^*}^* k_X^{**}\psi^* k_{X^{**}}\| \le 1\end{aligned}$$

Proposition 4.4.11. *Let X be a Banach space. If every equivalent renorming of X is a u-ideal in its bidual, then X is a strict u-ideal in its bidual.*

Proof. Let $x^{***} \in X^{***}$, $x^* = k_X^*(x^{***})$, and let $\varepsilon > 0$. By [11, Lemma 2.4] there is an equivalent renorming X_1 of X which is locally uniformly rotund at x^* such that $B_X \subseteq B_{X_1} \subseteq B_X(0, 1+\varepsilon)$. Let $|\cdot|$ be the norm on X_1 and let $P : X_1^{***} \to X_1^{***}$ be the u-ideal projection. Then $P(x^{***}) = x^*$ and

$$\|x^{***} - 2x^*\| \le |x^{***} - 2x^*| = |x^{***} - 2P(x^{***})| \le |x^{***}| \le (1+\varepsilon)\|x^{***}\|$$

which shows that $\|I - 2\pi\| = 1$ where $\pi = k_{X^*} k_X^*$ so X is a strict u-ideal in its bidual. □

Remark 4.4.12. The statements in Proposition 4.4.9 are strictly weaker than those in Theorem 4.4.6. Indeed, as noted in [5] (see p. 29) ℓ_1 is not a strict u-ideal in its bidual. Thus it follows from Proposition 4.4.11 that there exists an equivalent renorming, $\hat{\ell_1}$, of ℓ_1 for which $\hat{\ell_1}$ is not a u-ideal in its bidual. Since $\hat{\ell_1}$ has the AP, Proposition 4.4.9 (c) is fulfilled with $\psi = k_{\hat{\ell}_1^*}$.

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