

Geometry of embedded CR manifolds

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Chapter 1

Introduction

The study of real hypersurfaces in two-dimensional complex space dates to the early 1900s, with publications by Henri Poincaré, Eugenio E. Levi and later Eli Cartan, e.g. [Poi07], [Lev10], [Lev11] and [Car39]. Since then, the mathematics of these hypersurfaces have been generalized in several different ways, leading to what we now call CR geometry. It can be considered a crossroad between three branches of mathematics; the study of partial differential equations, complex analysis in several variables and differential geometry.

The 1950s to present day is considered the modern era of this subject. In 1956, Hans Lewy discovered a deep connection between the theory of several complex variables and partial differential equations ([Lew56]). Interest in the geometric study of CR manifolds is a bit more recent. In the 1970s, Noboru Tanaka and Sidney M. Webster independently discovered a linear connection on strictly pseudoconvex CR manifolds in the publications [Tan75] and [W⁺78]. This connection is a fundamental tool for the geometry of CR manifolds.

Despite there being numerous literature on the topic of CR manifolds, it is difficult to find explicit formulas linking embedded CR manifolds with the corresponding geometry. The goal for this thesis is to provide explicit formulas for some of the important geometric aspects of embedded CR manifolds and highlight both the real and the complex viewpoint.

The structure of the thesis is as follows: Chapter 2 covers some of the important ideas from differential topology and differential geometry that we will use later in this thesis, but the reader is assumed to be familiar with manifolds.

In Chapter 3, linear complex structures are introduced, which gives us a way of defining complex manifolds and holomorphic tangent bundles. CR manifolds are finally introduced in Chapter 4. First, we introduce both abstract and embedded CR manifolds with arbitrary codimension and discuss some of the important features that both abstract and embedded CR manifolds of arbitrary type possess. The later parts of the discussion focus on the case when the manifold in question has codimension one.

In Chapter 5 we put our focus to embedded CR manifolds. We give a few examples of CR manifolds, including the 3-sphere and the Heisenberg group. A lot of focus is put to the case of a general 3-dimensional CR manifold embedded into two-dimensional complex space, where we give explicit formulas for all of the geometric aspects mentioned in Chapter 4.

This thesis contains a handful of original result that are not found in any literature. This includes formulas for functions of the Reeb vector field in Section 5.5, the computation of the Tanaka-Webster connection in Section 5.6 and its associated pseudo-holomorphic curvature for an embedded $\text{CR}(1,1)$ submanifold. Proposition 4.5.4, which unifies two different formulations of the Tanaka-Webster connection is also not found in any literature.

Chapter 2

Differential topology and differential geometry

Manifolds are, loosely speaking, topological spaces that locally look like \mathbb{R}^n ¹. When studying manifolds, one can imagine a hierarchy arranged according to the complexity of the structure. At the ground level, we have *topological manifolds*. From a topological manifold, we can impose a criteria of having a differentiable or smooth atlas, which yields a *smooth manifold*. This allows us to do calculus on our manifolds, such as defining derivatives, tangent vectors, vector fields, and more.

Section 2.1 and Section 2.2 covers some of the important results and terminology from differential topology that we will use in the later chapters. In Section 2.3, we introduce a the Riemannian metric, and then later explore a few of the important aspects of the pair (M, g) , like linear connections and curvature.

For more reading on the subject, one could advice [Lee10], [Lee01] and [Lee18], which covers topological-, smooth- and Riemannian manifolds respectively.

¹Complex manifold, which look locally like \mathbb{C}^n are introduced in Chapter 3

2.1 Sections on vector bundles

2.1.1 Vector fields

In the language of differential geometry, a *vector field* is a section of the tangent bundle. We will write $p \mapsto X(p)$, such that for each $p \in M$, $X(p) \in T_p M$. In local coordinates for some chart, $(x_1, \dots, x_n) \in U \subset M$, we have a basis $\{\partial_{x_1}, \dots, \partial_{x_n}\}$ for the tangent bundle TM . We can write a vector field in the following way:

$$X(p) = \sum_{j=1}^n X_j(p) \partial_{x_j} \Big|_p \quad X_j \in C^\infty(M).$$

Most of the time, we will shorten this expression to $X = \sum_{j=1}^n X_j \partial_{x_j}$. We will primarily be interested in smooth vector fields, which are precisely smooth sections of the tangent bundle. A vector field X is smooth if and only if its component functions X_j are all smooth.

Vector fields acts as derivations on smooth functions, that is $X(f) = df(X)$ and they satisfy the following product rule: $X(fY) = X(f)Y + fXY$. We denote the vector bundle of all smooth sections by $\Gamma(TM)$. In general we have that $XY \neq YX$. The commutator of two vector fields is called the *Lie bracket*.

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad f \in C^\infty(M).$$

The Lie bracket is a map $[\cdot, \cdot] : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$.

Proposition 2.1.1. *The Lie bracket satisfies the following properties for any $X, Y, Z \in \Gamma(TM)$:*

1. $[a_1X + a_2Y, Z] = a_1[X, Z] + a_2[Y, Z]$ and $[X, b_1Y + b_2Z] = b_1[X, Y] + b_2[X, Z]$ for $a_1, a_2, b_1, b_2 \in \mathbb{R}$ (\mathbb{R} -bilinearity)
2. $[X, Y] = -[Y, X]$ (*anti-symmetry*)
3. $[X[Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ (*Jacobi identity*)

The two first properties are direct consequences of the definition and the Jacobi identity is just a matter of writing it out to see that everything will cancel out in pairs.

Proposition 2.1.2 (Product rule for Lie brackets). *We have the following two product rules for the Lie bracket:*

$$[X, fY] = f[X, Y] + X(f)Y \quad [fX, Y] = f[X, Y] - Y(f)X.$$

Proof. Using the product rule for derivations, $X(fY) = X(f)Y + fXY$, we get that

$$[X, fY] := X(fY) - (fY)X = X(f)Y + fXY - fYX = f[X, Y] + X(f)Y.$$

Using the same product rule in the first component, we get

$$[fX, Y] := (fX)Y - Y(fX) = fXY - Y(f)X - fYX = f[X, Y] - Y(f)X.$$

□

2.1.2 Differential forms

Definition 2.1.3. A k -differential form is a smooth section of the k -th exterior power of the cotangent bundle.

$$\alpha : M \rightarrow \bigwedge^k T^*M$$

Locally we have that $\{dx_1, \dots, dx_n\}$ is a basis for T^*M . A k -form will then be of the form

$$\alpha = \sum_I \alpha_I dx_I,$$

where the multi-index set $I = (i_1, \dots, i_k)$ and $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$. We use the terminology *1-form* for a smooth section of the cotangent bundle, and *2-form* for a smooth section of $\bigwedge^2 T^*M$. In general, we may call a k -differential form a k -form. We write $\Omega^k(M) := \Gamma(\bigwedge^k T^*M)$.

There exists a unique \mathbb{R} -linear map

$$d : \Omega^k(M) \rightarrow \Omega^{k+1}(M),$$

such that for a function $f \in C^\infty(M)$,

$$df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j.$$

More generally, for a k -form $\omega = f dx_I$ and for some index set I ,

$$d\omega = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \wedge dx_I.$$

The differential satisfies $d^2 = 0$, inducing a cochain complex called *De Rham complex*. De Rham cohomology connects differential forms to the topology of the manifold. For more information on this topic, see Chapter 11 in [Lee01].

Definition 2.1.4. Let $\alpha \in \Omega^k(M)$. Then the *Lie derivative* is given by

$$(\mathcal{L}_X \alpha)(Y_1, \dots, Y_k) = X\alpha(Y_1, \dots, Y_k) - \alpha([X, Y_1], \dots, Y_k) - \dots - \alpha(Y_1, \dots, [X, Y_k]).$$

Note that for smooth functions f , $\mathcal{L}_X f = Xf$ and for another vector field Y , $\mathcal{L}_X Y = [X, Y]$. In other words, we can think of the Lie derivative as a generalization of both the differential of smooth functions and the Lie bracket.

We define the *interior product* of a differential form by

$$\iota_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M), \quad \iota_X(\omega)(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1}).$$

With the help of the interior product we can write the Lie derivative in a different way, as shown below.

Theorem 2.1.5 (Cartan's magic formula).

$$\mathcal{L}_X \alpha = \iota_X d\alpha + d\iota_X \alpha$$

A proof is given in Chapter 13 of [Lee01]. We will focus on a special application of Cartan's magic formula for a 1-form.

Example 2.1.6. Let α be a 1-form.

$$\begin{aligned} (\mathcal{L}_X \alpha)(Y) &= X\alpha(Y) - \alpha([X, Y]) = d\alpha(X, Y) + d(\alpha(X))(Y) \\ &= d\alpha(X, Y) + Y\alpha(X). \end{aligned}$$

Solving for $d\alpha(X, Y)$, we get that

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]).$$

This expression for $d\alpha$ is very useful, as it allows us express $d\alpha$ using what we know about α . We will apply this formula numerous times to the differential of the pseudo-Hermitian structure, which we will define in Subsection 4.1.3.

2.2 Immersions, submersions and regular values

In this section we introduce the notion of rank of a smooth function between two smooth manifolds. When the rank is maximal, we have either a submersion or an immersion, depending on the dimension of the manifolds. Theorem 2.2.4 is particularly important for us, as it will be used in Chapter 5.

Let $f : M \rightarrow N$ be a smooth function between two smooth manifolds of dimension m and n , respectively. We define the rank of f at p , denoted $\text{rank}_p f$, to be the same as the rank of the linear map df_p .

$$\begin{array}{ccc}
 T_p M & \xrightarrow{df_p} & T_{f(p)} N \\
 \downarrow A_{\varphi_M} & & \downarrow A_{\varphi_N} \\
 \mathbb{R}^m & \xrightarrow{D(\varphi_N \circ f \circ \varphi_M^{-1})(\varphi_M(p))} & \mathbb{R}^n
 \end{array}$$

We say that f is a *submersion* if df_p is surjective for all $p \in M$, or equivalently that $\text{rank} f = n$ for all $p \in M$. We say that f is an *immersion* if df_p is injective for all $p \in M$. An equivalent condition is that $\text{rank} f = m$ for all $p \in M$. If $f : M \rightarrow N$ is a submersion at a point p , we say that $p \in M$ is *regular* and we say that $q \in N$ is a *regular value* if all $p \in f^{-1}(q)$ is regular.

There are many important results related to regular values. We will list some important results related to the rank of a function. See [Dun18] for a proof of these claims.

Proposition 2.2.1. *For $f : M \rightarrow N$, If $\text{rank}_p f = r$, then there exists a neighborhood U_p around p such that $\text{rank}_q f \geq r$ for all $q \in U_p$. Put into words, the rank is locally non-decreasing.*

Theorem 2.2.2 (The inverse function theorem). *A function $f : M \rightarrow N$ is invertible at p if and only if $df_p : T_p M \rightarrow T_{f(p)} N$ is invertible, in which case the inverse of the differential is given by $d(f^{-1})_{f(p)} = (df_p)^{-1}$.*

Corollary 2.2.3. *Let $f : M \rightarrow N$ be a smooth map between two manifolds of dimension n . Then f is a diffeomorphism if and only if f is bijective and df_p is bijective for all $p \in M$. Equivalently $\text{rank}_p f = n$ for all $p \in M$.*

Theorem 2.2.4 (The rank theorem). *Let $[f] : (M, p) \rightarrow (N, q)$ be a germ where $\dim M = m$ and $\dim N = n$, that is, an equivalence class of functions that agree in a neighborhood of p . We let*

$$\text{Pr} : \mathbb{R}^m \rightarrow \mathbb{R}^k, \quad \text{inc} : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$$

denote projection and inclusion, respectively.

1. *If $\text{rank}_p f \geq k$, then for any chart containing $q \in (\varphi_N, U_q)$, there exists a chart (φ_M, U_p) containing p such that*

$$[\text{Pr} \circ \varphi_N \circ f \circ \varphi_M^{-1}] = [\text{Pr}]$$

2. *If $\text{rank}_p f = k$, then there exists charts (φ_M, U_p) and (φ_N, U_q) such that*

$$[\varphi_N \circ f \circ \varphi_M^{-1}] = [\text{inc} \circ \text{Pr}]$$

3. *If $\text{rank}_p f = n = k$, then for any chart (φ_N, U_q) there exists (φ_M, U_p) such that*

$$[\varphi_N \circ f \circ \varphi_M^{-1}] = [\text{Pr}]$$

4. *If $\text{rank}_p f = m = k$, then for any chart (φ_M, U_p) there exists (φ_N, U_q) such that*

$$[\varphi_N \circ f \circ \varphi_M^{-1}] = [\text{inc}]$$

The next result will be very important for us in Chapter 5, as it gives us way of constructing CR submanifolds.

Theorem 2.2.5 (Regular value theorem). *Let $f : M \rightarrow N$ be a smooth function where $\dim M = n + k$ and $\dim N = n$. If $q \in N$ is a regular value and $f^{-1}(q)$ is not empty, then $f^{-1}(q) \subseteq M$ is a k -dimensional smooth submanifold.*

Proof. We let $p \in f^{-1}(q)$. Since q is a regular value by assumption, we know that p is regular. By part 3 of Theorem 2.2.4, for any chart (φ_N, U_q) containing q , there exists a chart (φ_M, U_p) containing p such that $\varphi_N(q) = 0$, $\varphi_M(p) = 0$ and $\varphi_N \circ f \circ \varphi_M^{-1}$ is a local projection $\mathbb{R}^{n+k} \rightarrow \mathbb{R}^n$. If we let $V = U_p \cap f^{-1}(U_q)$, we may write $(f \circ \varphi_N|_V)^{-1}(0) = V \cap f^{-1}(q)$. Applying φ_M , we get

$$\varphi_M(V \cap f^{-1}(q)) = (\varphi_N \circ f \circ \varphi_M^{-1}|_{\varphi(V)})^{-1}(0).$$

By the Rank Theorem, the right-hand side is of the form

$$\{(0, \dots, 0, x_{n+1}, \dots, x_{n+k}) \in \varphi_M(V)\} = (\varphi_M(V) \cap \{0\}^n) \times \mathbb{R}^k.$$

This shows us that $f^{-1}(q)$ is a k -dimensional submanifold. \square

As an immediate consequence, if we consider $N = \mathbb{R}$, $f : M \rightarrow \mathbb{R}$ with a regular value $q \in \mathbb{R}$ gives a submanifold $f^{-1}(q)$ of dimension $m - 1$. A submanifold with dimension $m - 1$ that is embedded into a m -dimensional manifold is called a *hypersurface*. In particular, we will consider manifolds obtained from the preimage of regular values for functions $F : \mathbb{C}^n \rightarrow \mathbb{R}$ in Chapter 5.

2.3 Riemannian manifolds

2.3.1 Riemannian metrics

A priori, the manifolds that we have discussed are topological spaces, and although we have ways of defining differentiability and concepts related to smoothness, we still have no way of describing length, distance, angle, area, volume and curvature. The purpose of Riemannian geometry is to introduce a metric which allows us to define many important geometrical properties of manifolds.

Definition 2.3.1. A *Riemannian metric* is a symmetric 2-tensor

$$g : \Gamma(TM) \times \Gamma(TM) \rightarrow C^\infty(M)$$

which is positive definite, that is $g(X, X) > 0$ for all $X \neq 0$. We may relax the condition of positive definiteness and instead require g to be nondegenerate, meaning that $g(X, Y) = 0$ for all $Y \in \Gamma(TM)$ implies that $X = 0$. In this case we get a *semi-Riemannian metric*².

The pair (M, g) is called a *Riemannian manifold*.

Theorem 2.3.2. A smooth manifold M admits a Riemannian metric.

See [Dun18] for a proof. An important thing to note here is that this theorem guarantees the existence of a metric, but not its uniqueness. The same smooth manifold M can admit several different Riemannian metrics, yielding several different Riemannian manifolds. We will use this result in Section 3.6 to prove that complex manifolds admits a Hermitian metric.

Given a basis for the tangent bundle, $TM = \text{Span}\{\partial_{x_1}, \dots, \partial_{x_n}\}$, we may write a Riemannian metric in the following way:

$$g = \sum_{i,j=1}^n g_{ij} dx_i dx_j, \quad g_{ij} = g(\partial_{x_i}, \partial_{x_j})$$

2.3.2 Connections on Riemannian manifolds

We now introduce the notion of a linear connection on a tangent bundle, which is a crucial tool when studying the geometric properties of a Riemannian manifold. It allows us for example to define curvature, which is introduced in Subsection 2.3.3.

²Semi-Riemannian metrics are also commonly referred to as pseudo-Riemannian metrics. This choice of labeling is a bit unfortunate, since *pseudo* in the context of complex and CR geometry means "on a subbundle", analogously to sub-Riemannian geometry.

Definition 2.3.3 (Linear connection). A *linear connection* is a map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM), \quad (X, Y) \mapsto \nabla_X Y$$

satisfying the following:

1. $\nabla_{fX} Y = f \nabla_X Y$ (C^∞ -linearity in X)
2. $\nabla_X aY = a \nabla_X Y$ (\mathbb{R} -linearity in Y)
3. $\nabla_X fY = (Xf)Y + f \nabla_X Y$ (Product rule in Y)

Intuitively speaking, we can think of the connection as a directional derivatives of vector fields. We say that $\nabla_X Y$ is the covariant derivative of Y in the direction of X . If we have that $\nabla_X Y = 0$ for all $X \in \Gamma(TM)$, we will simply write $\nabla Y = 0$.

Example 2.3.4.

1. If we take the covariant derivative of a function, it coincides with our usual notion of differential. That is, $\nabla_X f = Xf = df(X)$.
2. If we have an endomorphism $J : TM \rightarrow TM$, then we have that $(\nabla_X J)Y = \nabla_X JY - J \nabla_X Y$.
3. For a Riemannian metric g , we have that $\nabla_X g(Y, Z) = Xg(Y, Z) = (\nabla_X g)(Y, Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

Definition 2.3.5. For a Riemannian manifold (M, g) , we say the connection is *compatible with g* if $\nabla g = 0$. Equivalently, we have identity for all $X, Y, Z \in \Gamma(TM)$:

$$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

That the two conditions are equivalent follows immediately from the preceding example. The expression *metric connection* is used for a metric that is compatible with g .

Definition 2.3.6. The torsion tensor is a $\binom{2}{1}$ -tensor $\tau : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$, defined by

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Definition 2.3.7. A connection ∇ is said to be *symmetric* or *torsion-free* if the torsion vanishes, i.e.

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Theorem 2.3.8 (Levi-Civita connection). *There exists a unique connection ∇ on M that is symmetric and compatible with g . This unique connection is called the Levi-Civita connection.*

The Levi-Civita connection can be expressed by the *Koszul formula*:

$$g(\nabla_X Y, Z) = \frac{1}{2} (Xg(Y, Z) + Yg(X, Z) - Zg(X, Y)) \\ + \frac{1}{2} (g([X, Y], Z) - g([Y, Z], X) - g([X, Z], Y)).$$

The first 3 terms of the Koszul formula vanishes in an orthonormal basis, and the 3 last terms vanishes in a coordinate basis. The Levi-Civita connection is a particularly important in the study of Riemannian manifolds. In Section 4.5, we introduce a different connection which is also compatible with the metric, but not torsion-free.

Another useful formula that we will take with us is the Lie derivative of a Riemannian metric.

Example 2.3.9. Let (M, g) be a Riemannian manifold. The Lie derivative of the Riemannian metric is given by

$$\mathcal{L}_X g(Y, Z) = Xg(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]).$$

If we are working with a metric connection, we can use the formula in Definition 2.3.5 to write

$$\mathcal{L}_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g([X, Y], Z) - g(Y, [X, Z]).$$

2.3.3 Curvature in Riemannian geometry

Intuitively speaking, the curvature of a manifold measures how much curves deviate from being straight lines, surfaces deviates from being flat planes, with similar analogues in higher dimensions. There are several different types of curvature in Riemannian geometry, e.g. scalar curvature, Ricci curvature and sectional curvature. We will focus on the sectional curvature and later look at a complex counterpart, called pseudo-holomorphic sectional curvature, in Section 4.5.

What all the aforementioned types of curvature have in common is that they are determined by the Riemann curvature tensor, which we will now introduce.

Definition 2.3.10. The *Riemann curvature tensor* is given by

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM), \quad (X, Y, Z) \mapsto R(X, Y)Z$$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

For two fixed vector fields X, Y , we may also define the *Riemann curvature endomorphism*

$$R(X, Y) : \Gamma(TM) \rightarrow \Gamma(TM), \quad Z \mapsto R(X, Y)Z$$

Proposition 2.3.11. $R(X, Y)Z$ is a $\binom{3}{1}$ -tensor.

Proof. For this proof, we want to prove that $R(X, Y)Z$ is $C^\infty(M)$ -linear in each component.³ We will only consider linearity with respect to smooth functions in the X and Z component, since linearity in Y component follows from that $R(X, Y)Z = -R(Y, X)Z$.

$$R(fX, Y)Z = \nabla_{fX} \nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX, Y]} Z.$$

Let us consider each of the three components, using linearity and the product rule.

$$\nabla_{fX} \nabla_Y Z = f \nabla_X \nabla_Y Z,$$

$$\nabla_Y \nabla_{fX} Z = \nabla_Y f (\nabla_X Z) = (Yf) \nabla_X Z + f \nabla_Y (\nabla_X Z).$$

³That any $C^\infty(M)$ -multilinear map defines a tensor field is known as the *Tensor characterization lemma*. This lemma can be found in [Lee18].

For the last one, we use that since $[X, fY] = (Xf)Y + f[X, Y]$, we have $[fX, Y] = f[X, Y] - (Yf)X$,

$$\nabla_{[X, Y]}Z = \nabla_{f[X, Y] - (Yf)X}Z = \nabla_{f[X, Y]}Z - (Yf)\nabla_X Z.$$

We see that the two terms containing $(Yf)\nabla_X Z$ cancel each other out, and we get

$$R(fX, Y)Z = f\nabla_X \nabla_Y Z - f\nabla_Y (\nabla_X Z) - f\nabla_{[X, Y]}Z = fR(X, Y)Z.$$

We also want to show $C^\infty(M)$ -linearity in the Z component.

$$R(X, Y)fZ = \nabla_X \nabla_Y fZ - \nabla_Y \nabla_X fZ - \nabla_{[X, Y]}fZ.$$

We want to expand each term using the product rule. For the first term, we get

$$\nabla_X (\nabla_Y fZ) = XY(f)Z + Y(f)\nabla_X Z + X(f)\nabla_Y Z + f\nabla_X \nabla_Y Z$$

and

$$\nabla_Y (\nabla_X fZ) = YX(f)Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + f\nabla_Y \nabla_X Z.$$

We see that the terms $X(f)\nabla_Y Z$ and $Y(f)\nabla_X Z$ cancel out. Expanding the third part by using the product rule, we have

$$\nabla_{[X, Y]}fZ = [X, Y](f)Z + f\nabla_{[X, Y]}Z = XY(f)Z - YX(f)Z + f\nabla_{[X, Y]}Z.$$

From this, we see that the terms $XY(f)Z$ and $YX(f)Z$ cancel out, and we get that $R(X, Y)fZ = fR(X, Y)Z$. Since $R(X, Y)Z$ is multilinear in $C^\infty(M)$, we have a tensor field. \square

Proposition 2.3.12 (Symmetries of R).

1. $R(X, Y)Z = -R(Y, X)Z$ for any connection ∇ .
2. $g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$ for any metric connection.
3. $g(R(X, Y)Z, W) + g(R(Y, Z)X, W) + g(R(Z, X)Y, W) = 0$ for any torsion-free connection.
4. $g(R(X, Y)Z, W) = g(R(Z, W)X, Y)$ for the Levi-Civita connection.

The first statement is straight-forward from the definition, but the remaining 3 claims requires a bit of work. In the 2nd and 3rd, one will have to use compatibility with metric and that the torsion vanishes, respectively. The final claim can be proven using the previous 3 properties. For a complete proof, see [Lee18].

Definition 2.3.13 (Sectional curvature). Let ∇ be the Levi-Civita connection. If $\Pi \subseteq T_p M$ is a two-dimensional subspace and u, v is an orthonormal basis for Π . Then the *sectional curvature* is given by

$$K(\Pi) = g(R(u, v)v, u).$$

In general we may have that $\Pi = \text{Span}\{u, v\}$, but u and v are not orthonormal. In that case, the sectional curvature is given by

$$K(\Pi) = \frac{g(R(u, v)v, u)}{|u|^2|v|^2 - g(u, v)^2}.$$

Because of Proposition 2.3.12, the sectional curvature does not depend on the choice of basis. A short discussion on the complex counterpart, pseudo-holomorphic sectional curvature, is given in Section 4.5.

Chapter 3

Complex geometry

Complex geometry is the study of several complex variables, complex manifolds and complex algebraic varieties. The rigid nature of holomorphic functions makes complex geometry more reliant on topological and algebraic methods. It can therefore be considered a crossroad between algebraic geometry and differential geometry. There are many reasons for why one might want to study complex geometry. It has gained some traction due to its applications in physics, particularly conformal field theory and mirror symmetry.

This chapter will mainly focus on the linear and exterior algebra from the viewpoint of complex vector spaces, describe complex manifolds and its tangent bundle, give some insight into similarities and differences between complex manifolds and their real counterparts and briefly discuss Hermitian manifolds.

3.1 Complex linear algebra

This section is dedicated to the linear algebra of real and complex vector spaces, with a focus on linear complex structures and exterior algebra of vector spaces. One may consult Section 1.2 and Section 1.3 in [Huy06] for more details.

3.1.1 Complex structures

Throughout this section, V denotes a finite-dimensional real vector space. An endomorphism $J : V \rightarrow V$ is called a *linear complex structure*¹ if $J^2 = -\text{id}_V$. Note that $J \in \text{GL}(V)$ and that V has to be even-dimensional.

Given a linear complex structure $J : V \rightarrow V$, the vector space V will then admit the structure of a complex vector space. To see this, let $z = x + iy$ be a complex scalar and $v \in V$. Then

$$(x + iy)v = xv + yJ(v).$$

Since J is \mathbb{R} -linear and satisfies $J^2 = -\text{id}_V$, we have

$$((x + iy)(a + ib))v = (x + iy)(av + bJ(v)),$$

and $i(iv) = -v$. This shows that a linear complex structure J induces the structure of a complex vector space on V .

Proposition 3.1.1. *Let V be a finite-dimensional vector space. V admits a linear complex structure if and only if $\dim_{\mathbb{R}} V = n = 2k$. Moreover, the linear complex structure induces a natural orientation.*

Proof. Let J be a linear transformation such that $J^2 = -1$. We get that $\det(J^2) = (\det J)^2 = (-1)^n$. Since V is a real vector space, $\det J$ must be real, but if n is odd, then $(\det J)^2 = -1$, which implies that $\det J = \pm i$. Hence n must be even. Since the linear complex structure induces a complex vector space structure, it is sufficient to note that \mathbb{C}^n has a orientation given by the standard basis of \mathbb{C}^n . \square

For the real vector space V (not necessarily even-dimensional), we define its *complexification* as $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. The subscript tells us that the tensor product is taken over the real numbers. Since V is a vector space over \mathbb{R} , we

¹We will reserve the terms *almost complex structure* and *complex structure* to refer to linear structures on the tangent spaces of a manifold.

may safely omit the subscript when talking about complexifying real vector spaces. Clearly $V \subset V_{\mathbb{C}}$, where V is the part of $V_{\mathbb{C}}$ that is invariant under conjugation: $\overline{(v \otimes \lambda)} = v \otimes \bar{\lambda}$.

Given an almost complex structure $J : V \rightarrow V$, we can extend it $J : V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$. The eigenvalues of J is $\pm i$, and we have the corresponding eigenspaces:

$$V^{1,0} = \{v \in V_{\mathbb{C}} : J(v) = iv\}$$

$$V^{0,1} = \{v \in V_{\mathbb{C}} : J(v) = -iv\}$$

Proposition 3.1.2. *Given (V, J) , the complexified vector space splits:*

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}.$$

Furthermore, we have that $V^{1,0} \cong V^{0,1}$ by conjugation.

Proof. The intersection $V^{1,0} \cap V^{0,1} = \{0\}$, since if $v \in V^{1,0}$, then $J(v) = iv = -iv$ if and only if $v = 0$. The empty intersection implies that $V^{1,0} \oplus V^{0,1} \rightarrow V_{\mathbb{C}}$ is injective. The inverse map is also injective, which is given by

$$v \mapsto \frac{1}{2} \left(v - iJ(v) \oplus v + iJ(v) \right).$$

For the second assertion, let $v \in V_{\mathbb{C}}$ and write $v = x + iy$, $x, y \in V$. Then

$$\overline{(v - iJ(v))} = (x - iy + iJ(x) + J(y)) = (\bar{v} + iJ(v)).$$

We see that conjugation maps elements in $V^{1,0}$ to $V^{0,1}$, and vice versa. \square

Lemma 3.1.3. *If V admits a linear complex structure, then so does its dual V^* . Moreover, the complexification of the dual is equal to the dual of the complexification,*

$$(V^*)_{\mathbb{C}} \cong (V_{\mathbb{C}})^*$$

and the decomposition is given by

$$(V^*)^{1,0} = \{\varphi \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) : \varphi(J(v)) = i\varphi(v)\} = (V^{1,0})^*,$$

$$(V^*)^{0,1} = \{\varphi \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) : \varphi(J(v)) = -i\varphi(v)\} = (V^{0,1})^*.$$

Proof. The complex structure of the dual space V^* is given by

$$J^* : V^* \rightarrow V^* \quad J(\varphi(v)) = \varphi(J(v))$$

The complexification of the dual is simply defined as $(V^*)_{\mathbb{C}} := V^* \otimes \mathbb{C}$. We consider the isomorphism $(V^*)_{\mathbb{C}} \cong \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$ given by $(\varphi_1 \otimes 1 + \varphi_2 \otimes i) \leftrightarrow \varphi_1 + i\varphi_2$. If $\varphi \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$, i.e. $\varphi : V \rightarrow \mathbb{C}$ where φ is \mathbb{R} -linear, we may naturally extend it to a \mathbb{C} -linear map $\varphi : V_{\mathbb{C}} \rightarrow \mathbb{C}$, defined by

$$\varphi(v \otimes z) = z\varphi(v)$$

Hence $\text{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong \text{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) = (V_{\mathbb{C}})^*$. The last assertion follows a similar argument to the decomposition of the complexified vector space $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$.² \square

3.1.2 Exterior algebra on complex vector spaces

Just like in the case of a real vector space, we can define an exterior algebra on a complexified vector space. Since a complexified vector space splits into a direct sum of the two eigenspaces, the resulting exterior algebra is a bit more delicate. Recall that if $\dim_{\mathbb{R}} V = d$, we have the k -th exterior power $\bigwedge^k V$ and the exterior algebra

$$\bigwedge(V) := \bigoplus_{j=0}^d \bigwedge^j V.$$

Given a complexified vector space $V_{\mathbb{C}}$, we can consider the exterior algebra of $V_{\mathbb{C}}$:

$$\bigwedge(V_{\mathbb{C}}) := \bigoplus_{k=0}^d \bigwedge^k(V_{\mathbb{C}})$$

Since $\dim_{\mathbb{R}} V = d = 2n$, we have the decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, with $\dim_{\mathbb{C}} V^{1,0} = \dim_{\mathbb{C}} V^{0,1} = n$. We define

$$\bigwedge^{p,q} V := \bigwedge^p V^{1,0} \otimes_{\mathbb{C}} \bigwedge^q V^{0,1}.$$

Note that we are tensoring over \mathbb{C} ; the exterior products of $V^{1,0}$ and $V^{0,1}$ are taken as exterior products of complex vector spaces, in contrast to the complexification itself. An element $\alpha \in \bigwedge^{p,q} V$ is said to be of bidegree (p, q) .

²A generalized result which follows the same arguments is that $(\text{Hom}_{\mathbb{R}}(V, W))_{\mathbb{C}} \cong \text{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, W_{\mathbb{C}})$.

Proposition 3.1.4.

1. $\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V$
2. $\bigwedge^{p,q} V \subset \bigwedge^{p+q} V_{\mathbb{C}}$
3. $\bigwedge^{p,q} V \cong \bigwedge^{q,p} V$
4. The wedge product maps in the following way $\wedge : \bigwedge^{p,q} V \times \bigwedge^{r,s} V \rightarrow \bigwedge^{p+r,q+s} V$,
 $(\alpha, \beta) \mapsto \alpha \wedge \beta$.

Proof. 1. There is a canonical isomorphism

$$\bigoplus_{p+q=k} \bigwedge^{p,q} V \rightarrow \bigwedge^k V_{\mathbb{C}}.$$

Using the decomposition of $V_{\mathbb{C}}$ and the definition of $\bigwedge^{p,q} V$, we may write this as

$$\bigoplus_{p+q=k} \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1} \rightarrow \bigwedge^k V^{1,0} \oplus \bigwedge^k V^{0,1},$$

which is given by $(v_1 \wedge \dots \wedge v_p) \otimes (w_1 \wedge \dots \wedge w_q) \mapsto (v_1 \wedge \dots \wedge v_p \wedge w_1 \wedge \dots \wedge w_q)$ in each direct summand, thus

$$\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V.$$

2. This part follows from the first assertion. Since $\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V$, it follows that $\bigwedge^{p,q} V \subset \bigwedge^{p+q} V_{\mathbb{C}}$.

3. We have already seen that $V^{1,0} \cong V^{0,1}$, with the isomorphism given by conjugation. Hence,

$$\bigwedge^{p,q} V := \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1} \cong \bigwedge^q V^{0,1} \otimes \bigwedge^p V^{1,0} = \bigwedge^{q,p} V$$

is also given by conjugation.

4. Let $\alpha \in \bigwedge^{p,q} V$ and $\beta \in \bigwedge^{r,s} V$. We write

$$\alpha = (v_1^\alpha \wedge \dots \wedge v_p^\alpha, w_1^\alpha \wedge \dots \wedge w_q^\alpha) \mapsto v_1^\alpha \wedge \dots \wedge v_p^\alpha \wedge w_1^\alpha \wedge \dots \wedge w_q^\alpha,$$

and the same for β ,

$$\beta = (v_1^\beta \wedge \dots \wedge v_p^\beta, w_1^\beta \wedge \dots \wedge w_q^\beta) \mapsto v_1^\beta \wedge \dots \wedge v_p^\beta \wedge w_1^\beta \wedge \dots \wedge w_q^\beta.$$

Taking the wedge product,

$$\alpha \wedge \beta = v_1^\alpha \wedge \dots \wedge v_p^\alpha \wedge w_1^\alpha \wedge \dots \wedge w_q^\alpha \wedge v_1^\beta \wedge \dots \wedge v_p^\beta \wedge w_1^\beta \wedge \dots \wedge w_q^\beta.$$

Which is an element in $\bigwedge^{p+q+r+s} V$. Going the other direction, we have

$$\alpha \wedge \beta = (v_1^\alpha \wedge \dots \wedge v_p^\alpha \wedge v_1^\beta \wedge \dots \wedge v_p^\beta, w_1^\alpha \wedge \dots \wedge w_q^\alpha \wedge w_1^\beta \wedge \dots \wedge w_q^\beta),$$

which is an element in $\bigwedge^{p+r, q+s} V$. □

3.2 Complex manifolds

In this section we give a definition of complex manifolds through holomorphic charts, in a similar manner to how we define smooth manifolds. As a convention used throughout the text, M, N will denote real smooth manifolds, while \mathcal{M}, \mathcal{N} denotes complex manifolds. Hopefully, this will make matters immediately clear when we later talk about real submanifolds of complex space in Chapter 4 and Chapter 5. One may consult [Huy06] for a more details.

We define a *holomorphic atlas* on a manifold M to be a collection $\{(U_j, \varphi_j)\}$ such that $U_j \cong \varphi_j(U_j) \subset \mathbb{C}^n$ is a biholomorphism, i.e. bijective holomorphic with a holomorphic inverse. For it to be biholomorphic we require the transition functions to be a holomorphic map:

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_{ij}) \rightarrow \varphi_j(U_{ij}), \quad (U_{ij} = U_i \cap U_j)$$

There are many equivalent ways of defining holomorphic functions in several variables. For example, we require that a holomorphic function satisfies the Cauchy-Riemann equation in each variable separately. Note that φ_{ij} is just a map between subsets of \mathbb{C}^n . We then define an equivalence relation between two holomorphic atlases; we say that $\mathcal{A}_1 \sim \mathcal{A}_2$ if the transition function between any two charts in \mathcal{A}_1 and \mathcal{A}_2 is holomorphic. Finally, a complex manifold \mathcal{M} is topological manifold equipped with an equivalence class of holomorphic atlases.

Proposition 3.2.1. *A complex manifold \mathcal{M} of dimension n is also a smooth manifold of (real) dimension $2n$.*

Proof. We know from basic properties of holomorphic functions that they are indeed infinitely differentiable, and hence any holomorphic transition function is also smooth. A biholomorphism is therefore necessarily a diffeomorphism, and we can identify \mathbb{C}^n with \mathbb{R}^{2n} . \square

In a similar manner to how we define smooth functions on smooth manifolds, we say a function $f : \mathcal{M} \rightarrow \mathbb{C}$ is holomorphic if $f \circ \varphi_j^{-1}$ is holomorphic, and similarly, a map $f : \mathcal{M} \rightarrow \mathcal{N}$ is holomorphic if $\varphi_{\mathcal{N}} \circ f \circ \varphi_{\mathcal{M}}^{-1}$ is holomorphic.

Due to the rigidity of holomorphic functions, we can not use bump functions to glue together functions and define them globally, since a holomorphic function that vanishes in a neighborhood must vanish identically. It is

therefore not always very useful to consider global holomorphic functions, as seen in Proposition 3.2.4. Therefore we instead consider sheaves of holomorphic functions.

Definition 3.2.2 (Pre-sheaves and sheaves). A *pre-sheaf* \mathcal{F} of vector spaces³ on a topological space X consists of a vector space $\mathcal{F}(U)$ for every open set $U \subset X$ and a linear map $r_{U,V} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ for any $U \subset V \subset X$, that satisfies A1 and A2 below.

$$\text{A1: } r_{U,U} = \text{id}_{\mathcal{F}(U)}$$

$$\text{A2: For any } U \subset V \subset W, \text{ we have } r_{U,V} \circ r_{V,W} = r_{U,W}$$

Furthermore, we let $\bigcup_{i \in I} U_i = U$, where each $U_i \subset X$ is open. A pre-sheaf that also satisfies the following two conditions is a *sheaf*:

$$\text{A3: If } a, b \in \mathcal{F}(U) \text{ with } r_{U_i,U}(a) = r_{U_i,U}(b) \text{ for all } i \in I, \text{ then } a = b.$$

$$\text{A4: If } a_i \in \mathcal{F}(U_i) \text{ for each } i \text{ such that } r_{U_i \cap U_j, U_i}(a_i) = r_{U_i \cap U_j, U_j}(a_j) \text{ for any } j, \text{ then there exists } a \in \mathcal{F}(U) \text{ such that } r_{U_i,U}(a) = a_i \text{ for all } i.$$

Definition 3.2.3 (Stalk). The *stalk* of \mathcal{F} at x is

$$\mathcal{F}_x := \{(U, s) : x \in U \subset X, s \in \mathcal{F}(U)\} / \sim$$

With the equivalence class given by $(U_1, s_1) \sim (U_2, s_2)$ if there exists $U \subset U_1 \cap U_2$ open containing x such that $r_{U,U_1}(s_1) = r_{U,U_2}(s_2)$. This is nothing but the direct limit

$$\mathcal{F}_x = \varinjlim \mathcal{F}(U)$$

We let $\mathcal{O}_{\mathcal{M}}$ denote the *sheaf of holomorphic functions*, with sections $\mathcal{O}_{\mathcal{M}}(U) = \{f : U \rightarrow \mathbb{C} : f \text{ holomorphic}\}$. For each point $p \in \mathcal{M}$, we have an isomorphism of the stalks $\mathcal{O}_{\mathcal{M},p} \cong \mathcal{O}_{\mathbb{C}^n,0}$. In a similar fashion we can define a sheaf of meromorphic functions.

Let us consider the sheaf \mathcal{O} on \mathbb{C} given by

$$\mathcal{O}_{\mathbb{C}}(U) = \{f : U \rightarrow \mathbb{C} : f \text{ is holomorphic}\},$$

with restrictions being the usual restriction of functions. We want to see that $\mathcal{O}_{\mathbb{C}}$ is in fact a sheaf.

³Sheaves and pre-sheaves are much more general. We may replace vector spaces with another category, where each $\mathcal{F}(U)$ is an object in the chosen category and linear map is a morphism in the chosen category.

If $V \subset U$, then $r_{U,V} = \mathcal{O}(U) \rightarrow \mathcal{O}(V)$, $f \mapsto f|_V$. Clearly, $r_{U,U} = \text{id}_{\mathcal{O}_{\mathbb{C}}(U)}$ and $r_{W,V} \circ r_{V,U} = r_{W,U}$ for $W \subset V \subset U$. Hence it is a pre-sheaf. If $U \subseteq \bigcup_{\alpha \in A} U_{\alpha}$ and $f, g \in \mathcal{O}_{\mathbb{C}}(U)$ such that $f|_{U_{\alpha}} = g|_{U_{\alpha}}$ for every $\alpha \in A$, then clearly $f = g$. In fact, we have a much stronger result, namely the identity theorem. f and g only need to coincide on one open neighborhood for them to be identically equal. The last axiom for a sheaf tells us that we can glue together functions $f_{\alpha} \in \mathcal{O}_{\mathbb{C}}(U_{\alpha})$ to a function f such that $f|_{U_{\alpha}} = f_{\alpha}$. This is nothing but analytic continuation of holomorphic functions.

Proposition 3.2.4. *Any global holomorphic function on a compact connected complex manifold \mathcal{M} is constant.*

Proof. Let $f : \mathcal{M} \rightarrow \mathbb{C}$ be a holomorphic function with a maximum attained at a point $z_0 \in \mathcal{M}$. The maximum is attained since \mathcal{M} is compact. We consider a chart (U, φ) containing z_0 . Applying the maximum modulus principle to $f \circ \varphi^{-1}$ on $\varphi(U)$, we get that it must be constant. Since \mathcal{M} is connected, we conclude that f must be constant. \square

An important thing to remember is that for a complex manifold, we are allowed to talk about smooth functions as well as holomorphic functions, since any complex manifold will necessarily be a smooth manifold, as shown in Proposition 3.2.1.

3.3 Vector bundles on complex manifolds

In this section we shall reap the benefits from Section 3.1, where we discussed complex structures and complexification of vector spaces, and apply them to the tangent spaces and tangent bundles of manifolds.

To understand the geometry of a complex manifold \mathcal{M} , we naturally study tangent bundles on \mathcal{M} . A *rank n holomorphic vector bundle* is a triple (\mathcal{M}, E, π) , where E is a complex manifold, $\pi : E \rightarrow \mathcal{M}$ is a holomorphic map, we have local trivializations:

$$\psi_j : \pi^{-1}(U_j) \cong U_j \times \mathbb{C}^n, \quad \mathcal{M} = \bigcup_{j \in \mathbb{N}} U_j,$$

and each fiber $\pi^{-1}(q)$ is a complex vector space. Note that we differentiate between complex vector bundles and holomorphic ones. A complex vector bundle would simply be one where the fibers are complex vector spaces, while a holomorphic vector bundle requires that the projection map is holomorphic.

Since for each holomorphic vector bundle we can identify the fibers with \mathbb{C}^n , we may use any construction on a complex vector space fiberwise on a holomorphic vector bundle. Given a holomorphic vector bundle (\mathcal{M}, E, π) , we can for example Construct the dual bundle $E^* = \prod_{p \in \mathcal{M}} E_p^*$. Complexification, discussed in Section 3.1 is another construction that is very important in the context of complex geometry. The next section explores the complexification of the tangent bundle $T\mathcal{M}$.

We know from Proposition 3.2.1 that an n -dimensional complex manifold can be viewed as a smooth $2n$ -dimensional real manifold. We also know that for any smooth manifold has a tangent bundle. Thus, for any complex manifold \mathcal{M} , it admits a tangent bundle $T\mathcal{M}$. A priori, this tangent bundle is the real tangent bundle of the complex manifold \mathcal{M} , viewed as a smooth manifold.

Since the real dimension of \mathcal{M} is even, the tangent spaces $T_p\mathcal{M}$ admits a linear complex structure at each $p \in \mathcal{M}$,

$$J_p : T_p\mathcal{M} \rightarrow T_p\mathcal{M}, \quad J^2 = -\text{id}_{T_p\mathcal{M}}.$$

Any even-dimensional manifold admits a linear complex structure point-wise. If we have a vector bundle endomorphism $J : T\mathcal{M} \rightarrow T\mathcal{M}$ such that

$J^2 = -1$, then J is called an *almost complex structure* and a pair (\mathcal{M}, J) is called an *almost complex manifold*. Further discussion on almost complex structures and integrable complex structures is postponed until Section 3.5. We will assume that \mathcal{M} is a complex manifold for the rest of this section and the section following this one.

We now take a look at the previously mentioned complexification, where we take each vector space $T_p\mathcal{M}$ and allow for complex coefficients by $(T_p\mathcal{M})_{\mathbb{C}} := T_p\mathcal{M} \otimes_{\mathbb{R}} \mathbb{C}$. Doing this in each fiber gives us the *complexified tangent bundle*,

$$T\mathcal{M}_{\mathbb{C}} = T\mathcal{M} \otimes \mathbb{C}.$$

Furthermore, we may extend the (almost) complex structure J to act on $T\mathcal{M}_{\mathbb{C}}$, defined by

$$J_{\mathbb{C}} : T\mathcal{M}_{\mathbb{C}} \rightarrow T\mathcal{M}_{\mathbb{C}}, \quad J(X + iY) = J(X) + iJ(Y).$$

We have a complex structure on a complexified vector bundle with eigenvalues $\pm i$. By previous assertions, the complexified tangent bundle splits, and we have that

$$T\mathcal{M}_{\mathbb{C}} = T^{1,0}\mathcal{M} \oplus T^{0,1}\mathcal{M}.$$

We call $T^{1,0}\mathcal{M}$ the *holomorphic tangent bundle*, and analogously $T^{0,1}\mathcal{M}$ is the *anti-holomorphic tangent bundle*.

3.4 Local coordinates

In this section we will take a look at how the complex structure works when given local coordinates. We let \mathcal{M} be a complex n -dimensional manifold and (z_1, \dots, z_n) be local coordinates in some chart. Since each $z_j = x_j + iy_j$, in the view of \mathcal{M} as a real $2n$ -dimensional manifold, we have that $(x_1, \dots, x_n, y_1, \dots, y_n)$ are real local coordinates. For any $z \in \mathcal{M}$, a basis for the tangent space $T_z\mathcal{M}$ is given by

$$\{\partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n}\}.$$

and each tangent space admits a natural almost complex structure

$$J : T_z\mathcal{M} \rightarrow T_z\mathcal{M}, \quad \partial_{x_j} \mapsto \partial_{y_j}, \quad \partial_{y_j} \mapsto -\partial_{x_j}.$$

Moreover, we have a dual basis for $(T_z\mathcal{M})^*$, given by

$$\{dx_1, \dots, dx_n, dy_1, \dots, dy_n\},$$

and an almost complex structure on the dual

$$J : (T_z\mathcal{M})^* \rightarrow (T_z\mathcal{M})^*, \quad dx_j \mapsto dy_j, \quad dy_j \mapsto -dx_j.$$

When we consider the complexified tangent bundle $T\mathcal{M}_{\mathbb{C}}$, we have a new basis of sections, called the *Wirtinger derivatives*:

$$T\mathcal{M}_{\mathbb{C}} = \text{Span}\{\partial_{z_1}, \dots, \partial_{z_n}, \partial_{\bar{z}_1}, \dots, \partial_{\bar{z}_n}\},$$

which we may define in terms of the real basis of $T\mathcal{M}$;

$$\partial_{z_j} = \frac{1}{2} (\partial_{x_j} - i\partial_{y_j}), \quad \partial_{\bar{z}_j} = \frac{1}{2} (\partial_{x_j} + i\partial_{y_j}).$$

The factor of $\frac{1}{2}$ comes from the coordinate change $(x, y) \mapsto (z, \bar{z})$, given by

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}.$$

Then using the chain rule,

$$\partial_x = \frac{\partial z}{\partial x} \partial_z + \frac{\partial \bar{z}}{\partial x} \partial_{\bar{z}} = \partial_z + \partial_{\bar{z}},$$

and

$$\partial_y = \frac{\partial z}{\partial y} \partial_z + \frac{\partial \bar{z}}{\partial y} \partial_{\bar{z}} = i(\partial_z - \partial_{\bar{z}}).$$

Solving for ∂_z and $\partial_{\bar{z}}$ yields our desired result. Since we want dz_k such that $d\bar{z}_k(\partial_{\bar{z}_j}) = dz_k(\partial_{z_j}) = \delta_{j,k}$ and $dz_k(\partial_{\bar{z}_j}) = d\bar{z}_k(\partial_{z_j}) = 0$, We get that dz_j and $d\bar{z}_j$ take the following form;

$$dz_j = dx_j + idy_j, \quad d\bar{z}_j = dx_j - idy_j.$$

Writing it all out, we have

$$dz_j(\partial_{z_j}) = dx_j \left(\frac{1}{2} (\partial_{x_j} - i\partial_{y_j}) \right) + idy_j \left(\frac{1}{2} (\partial_{x_j} - i\partial_{y_j}) \right) = \frac{1}{2} + \frac{1}{2} = 1.$$

and

$$d\bar{z}_j(\partial_{\bar{z}_j}) = dx_j \left(\frac{1}{2} (\partial_{x_j} + i\partial_{y_j}) \right) - idy_j \left(\frac{1}{2} (\partial_{x_j} + i\partial_{y_j}) \right) = \frac{1}{2} + \frac{1}{2} = 1.$$

We also get that

$$dz_j(\partial_{\bar{z}_j}) = d\bar{z}_j(\partial_{z_j}) = \frac{1}{2} - \frac{1}{2} = 0,$$

and

$$dz_k(\partial_{z_j}) = d\bar{z}_k(\partial_{\bar{z}_j}) = 0,$$

for $j \neq k$, since $dx_k(\partial_{x_j}) = dy_k(\partial_{y_j}) = 0$. We now want to consider the differential of a smooth function. From Subsection 2.1.2, we have that the real differential is given by

$$df = \sum_{j=1}^n \partial_{x_j} f dx_j + \sum_{j=1}^n \partial_{y_j} f dy_j.$$

Using the expressions for dz_j and $d\bar{z}_j$ and solving for dx_j and dy_j respectively, we get

$$dx_j = \frac{1}{2} (dz_j + d\bar{z}_j), \quad dy_j = \frac{1}{2i} (dz_j - d\bar{z}_j).$$

From this, we can rewrite the differential, df in the following way:

$$\begin{aligned} df &= \sum_{j=1}^n (\partial_{z_j} + \partial_{\bar{z}_j}) f (dz_j + d\bar{z}_j) + i(\partial_{z_j} - \partial_{\bar{z}_j}) f \frac{1}{2i} (dz_j - d\bar{z}_j) \\ &= \sum_{j=1}^n f_{z_j} dz_j + f_{\bar{z}_j} d\bar{z}_j. \end{aligned}$$

Proposition 3.4.1. *An (almost) complex structure J is independent of choice of holomorphic coordinates.*

Proof. Let's suppose (z_1, \dots, z_n) where $z_j = x_j + iy_j$ and (w_1, \dots, w_n) where $w_j = u_j + iv_j$ are both holomorphic coordinates. They must satisfy the Cauchy-Riemann equation

$$\frac{\partial x_j}{\partial u_j} = \frac{\partial y_j}{\partial v_j}, \quad \frac{\partial x_j}{\partial v_j} = -\frac{\partial y_j}{\partial u_j},$$

and therefore

$$J(\partial_{u_k}) = \sum_j \left(\frac{\partial x_j}{\partial u_k} J(\partial_{x_k}) + \frac{\partial y_j}{\partial u_k} J(\partial_{y_k}) \right)$$

Using the way J acts on ∂_{x_k} and ∂_{y_k} , and then using CR equations,

$$J(\partial_{u_k}) = \sum_j \left(\frac{\partial x_j}{\partial u_k} \partial_{y_k} - \frac{\partial y_j}{\partial u_k} \partial_{x_k} \right) = \sum_j \left(\frac{\partial y_j}{\partial v_k} \partial_{y_k} + \frac{\partial x_j}{\partial v_k} \partial_{x_k} \right) = \partial_{v_k}.$$

The same procedure gives us $J(\partial_{v_k}) = -\partial_{u_k}$. □

3.5 Integrability conditions

In Section 3.2 we defined a complex manifold in terms of an equivalence class of holomorphic atlases. Another approach is to define an *integrable complex structure* as any one of the conditions in Proposition 3.5.3 and then define a complex manifold to be a smooth manifold with an integrable complex structure.

Definition 3.5.1. Given an endomorphism $A : TM \rightarrow TM$, the *Nijenhuis tensor* is defined as

$$N_A(X, Y) = -A^2[X, Y] + A([AX, Y] + [X, AY]) - [AX, AY]$$

Proposition 3.5.2. *The Nijenhuis tensor is anti-symmetric $\binom{2}{1}$ -tensor, and given an almost complex structure, it does satisfy*

$$N_J(X, JX) = 0.$$

Proof. Anti-symmetry follows from the anti-symmetry of the Lie brackets. To show C^∞ -linearity, we want to show that $N_A(fX, gY) = fgN_A(X, Y)$. However, due to anti-symmetry it suffices to show that $N_A(fX, Y) = fN_A(X, Y)$. We recall that

$$[fX, Y] = f[X, Y] - Y(f)X.$$

Thus,

$$\begin{aligned} N_A(fX, Y) &= -A^2[fX, Y] + A([AfX, Y] + [fX, AY]) - [AfX, AY] \\ &= fN_A(X, Y) + \left(A^2Y(f)X - AY(Af)X - A^2Y(f)X + AY(Af)X \right) \\ &= fN_A(X, Y). \end{aligned}$$

The last part of the proposition is a straight-forward calculation

$$\begin{aligned} N_J(X, JX) &= -J^2[X, JX] + J([JX, JX] + [X, JJX]) - [JX, JJX] \\ &= [X, JX] + J[X, -X] - [JX, -X] = [X, JX] - J[X, X] + [JX, X] \\ &= [X, JX] + [JX, X] = [X, JX] - [X, JX] = 0. \end{aligned}$$

□

The following proposition gives many different but equivalent criterions for an almost complex structure J to be integrable.

Proposition 3.5.3. *Let (\mathcal{M}, J) be an almost complex manifold. J is integrable (meaning (\mathcal{M}, J) is a complex manifold) if one of the equivalent definitions hold:*

1. $d = \partial + \bar{\partial}$
2. $d\alpha$ has no $(0, 2)$ component for $\alpha \in \Omega^{1,0}(\mathcal{M})$
3. $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$
4. $T^{1,0}\mathcal{M}$ is preserved by Lie brackets (i.e. $[T^{1,0}\mathcal{M}, T^{1,0}\mathcal{M}] \subset T^{1,0}\mathcal{M}$)
5. $T^{0,1}\mathcal{M}$ is preserved by Lie brackets
6. The Niljenhuis tensor vanishes: $N_J(X, Y) \equiv 0$

Proof.

(1) \iff (2): The first direction is straight-forward. Let α be a $(1, 0)$ -form. Then $d\alpha = (\partial + \bar{\partial})\alpha$ has $(2, 0)$ and $(1, 1)$ components. For the converse, we let $\alpha \in \Omega^{p,q}(\mathcal{M})$. Locally, we can write

$$\alpha = fw_{j_1} \wedge \dots \wedge w_{j_p} \wedge w'_{k_1} \wedge \dots \wedge w'_{k_q}.$$

When considering its differential $d\alpha$, it has components df , dw_j and dw'_k . We see that $df \in \Omega^{1,0}(\mathcal{M}) \oplus \Omega^{0,1}(\mathcal{M})$. Using the assumption (1) and using the conjugation on (1); that is the statement $d\alpha$ has no $(2, 0)$ component for $\alpha \in \Omega^{0,1}(\mathcal{M})$. From this we have $dw_j \in \Omega^{2,0}(\mathcal{M}) \oplus \Omega^{1,1}(\mathcal{M})$ and $dw'_k \in \Omega^{1,1}(\mathcal{M}) \oplus \Omega^{0,2}(\mathcal{M})$. Thus,

$$d\alpha \in \Omega^{p+1,q}(\mathcal{M}) \oplus \Omega^{p,q+1}(\mathcal{M}),$$

which implies that $d = \partial + \bar{\partial}$.

(1) \implies (3): First, assuming $d = \partial + \bar{\partial}$, we have

$$d^2 = 0 \implies (\partial + \bar{\partial})^2 = \partial^2\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2 = 0$$

It follows that $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$.

(2) \iff (4) & (5): Let α be a $(0, 1)$ -form and X, Y be holomorphic vector fields, i.e. elements in $T^{1,0}\mathcal{M}$. Using the formula for the differential, we have

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) = -\alpha([X, Y])$$

Since α is a $(0, 1)$ -form, it vanishes on $T^{1,0}\mathcal{M}$. $d\alpha$ has no component of type $(2, 0)$ if and only if $[X, Y]$ is a holomorphic vector field. (2) \iff (5) is obtained by conjugation.

(3) \implies (5): Let α be a $(0, 1)$ -form, hence $d\alpha = \bar{\partial}\alpha$, and we can locally write $\alpha = \bar{\partial}f$. For two anti-holomorphic vector fields $X, Y \in T^{0,1}\mathcal{M}$,

$$\begin{aligned} 0 &= \bar{\partial}^2 f(X, Y) = X\bar{\partial}f(Y) - Y\bar{\partial}f(X) - \bar{\partial}f([X, Y]) \\ &= Xdf(Y) - Ydf(X) - \bar{\partial}f([X, Y]) \\ &= d^2f(X, Y) + df([X, Y]) - \bar{\partial}f([X, Y]) \\ &= \partial f([X, Y]) + \bar{\partial}f([X, Y]) - \bar{\partial}f([X, Y]) \\ &= \partial f([X, Y]). \end{aligned}$$

The $(1, 0)$ -form of the type ∂f generates $\Omega^{1,0}(\mathcal{M})$. Thus, $\partial f([X, Y]) = 0$ implies that $[X, Y]$ is anti-holomorphic, i.e. $[X, Y] \in T^{0,1}\mathcal{M}$.

(6) is the Newlander-Nirenberg theorem. See [Smi19] for details and a proof. \square

Example 3.5.4. On \mathbb{R}^4 with coordinates (x_1, x_2, x_3, x_4) , we define a linear map

$$J : T\mathbb{R}^4 \rightarrow T\mathbb{R}^4$$

such that

$$\begin{aligned} J\partial_{x_1} &= \partial_{x_2}, & J\partial_{x_2} &= -\partial_{x_1}, \\ J\partial_{x_3} &= \partial_{x_4} + f_1\partial_{x_1} + f_2\partial_{x_2}, & J\partial_{x_4} &= -\partial_{x_3} + f_3\partial_{x_1} + f_4\partial_{x_2} \end{aligned}$$

for $f_1, f_2, f_3, f_4 \in C^\infty(\mathbb{R}^4)$. For what functions f_1, f_2, f_3, f_4 is J an almost complex structure, and when is it a complex structure?

If J is an almost complex structure, then

$$-\partial_{x_3} = J^2\partial_{x_3} = J(\partial_{x_4} + f_1\partial_{x_1} + f_2\partial_{x_2}) = -\partial_{x_3} + (f_3 - f_2)\partial_{x_1} + (f_1 + f_4)\partial_{x_2}$$

Thus, $f_2 = f_3$ and $f_1 = -f_4$. The same procedure for $J^2\partial_{x_4} = -\partial_{x_4}$ gives us the same two equations. Another way of seeing this is writing out J as a matrix,

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ f_1 & f_3 & 0 & 1 \\ f_2 & f_4 & -1 & 0 \end{pmatrix}.$$

Squaring J yields

$$J^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ f_3 - f_2 & f_1 + f_4 & -1 & 0 \\ -f_1 - f_4 & f_3 - f_2 & 0 & -1 \end{pmatrix} = -id \iff f_2 = f_3, f_1 = -f_4.$$

For integrability, we want to see that $T^{1,0}\mathbb{R}^4$ is involutive.

$$Z_1 = \partial_{x_1} - iJ\partial_{x_1} = (\partial_{x_1} - i\partial_{x_2})$$

$$Z_2 = \partial_{x_3} - iJ\partial_{x_3} = \partial_{x_3} - i(\partial_{x_4} + f_1\partial_{x_1} + f_2\partial_{x_2})$$

Since Z_1, Z_2 spans the holomorphic bundle $T^{1,0}\mathbb{R}^4$, we only need to consider one bracket;

$$\begin{aligned} [Z_1, Z_2] &= (\partial_{x_1} - i\partial_{x_2})(\partial_{x_3} - i(\partial_{x_4} + f_1\partial_{x_1} + f_2\partial_{x_2})) \\ &\quad - (\partial_{x_3} + i(\partial_{x_4} + f_1\partial_{x_1} + f_2\partial_{x_2}))(\partial_{x_1} - i\partial_{x_2}) \\ &= -iZ_1(f_1\partial_{x_1} + f_2\partial_{x_2}) \\ &= -i(\partial_{x_1}f_1 - i\partial_{x_2}f_1)\partial_{x_1} - i(\partial_{x_1}f_2 - i\partial_{x_2}f_2)\partial_{x_2} = -i\tilde{f}Z_1. \end{aligned}$$

We get

$$\begin{aligned} \tilde{f} &= \partial_{x_1}f_1 - i\partial_{x_2}f_1, \\ \partial_{x_1}f_1 &= \partial_{x_2}f_2, \quad \partial_{x_2}f_1 = -\partial_{x_1}f_2. \end{aligned}$$

We see that $f_1 + if_2$ needs to satisfy the Cauchy-Riemann equations.

It is tempting to think that all even-dimensional and orientable manifolds admit a complex structure or even just an almost complex structure. Unfortunately, that is far away from the truth. Wu Wen-tsün showed in the 50s that a sphere S^{4n} can not admit an almost complex structure. Borrel and Serre proved in [Bor53] that any even-dimensional sphere S^{2n} for $n \geq 4$ does not admit an almost complex structure. The only even-dimensional spheres that we have left are S^2 and S^6 . The complex structure on S^2 is well-studied, and is known as the Riemann sphere. It is still unknown if S^6 admits an integrable complex structure. This is known as the *Hopf problem*.

3.6 Hermitian manifolds

We can think of Hermitian manifolds as the complex analogue to Riemannian manifolds. We give a brief overview of what a Hermitian manifold is and then give an example of a special type of Hermitian manifold called a Kähler manifold.

Definition 3.6.1. Let \mathcal{M} be a complex manifold. A Riemannian metric g on \mathcal{M} is called a *Hermitian metric* if it is compatible with the complex structure, meaning that $g(X, Y) = g(JX, JY)$ for all $X, Y \in T\mathcal{M}$. We call a manifold equipped with a Hermitian metric a *Hermitian manifold*.⁴

In Section 4.3, we will define a Riemannian metric which is compatible with the complex structure on a subbundle, giving us a pseudo-Hermitian metric.

Any complex manifold \mathcal{M} admits a *Hermitian metric*. This is true since \mathcal{M} admits a Riemannian metric \tilde{g} by Theorem 2.3.2. We can define a Hermitian metric

$$g(X, Y) = \frac{1}{2} (\tilde{g}(X, Y) + \tilde{g}(JX, JY))$$

Lemma 3.6.2. *For any bilinear and anti-symmetric form satisfying $\omega(JX, JY) = \omega(X, Y)$, the 2-form $g(X, Y) = \omega(X, JY)$ is symmetric.*

Proof.

$$\begin{aligned} g(X, Y) &= \omega(X, JY) = \omega(JX, J^2Y) = \omega(JX, -Y) \\ &= -\omega(JX, Y) = \omega(Y, JX) = g(Y, X) \end{aligned}$$

□

There is an induced real (1, 1)-form $\omega(X, Y) = g(JX, Y)$, called the *Kähler form*, which locally looks like

$$\omega = \frac{i}{2} \sum_{j,k=1}^n h_{jk} dz_j \wedge d\bar{z}_k$$

where h_{jk} is a Hermitian matrix that is positive definite for all $p \in \mathcal{M}$. A *Kähler structure* is a Hermitian structure g for which the Kähler form ω is closed, i.e. $d\omega = 0$. A *Kähler manifold* is a manifold equipped with a

⁴In the case of compatibility with an almost complex structure, we call the manifold an *almost Hermitian manifold*.

Kähler structure. It is worth mentioning that the form ω is a closed non-degenerate 2-form, which is what we call a *symplectic form*. This means that Kähler manifolds have a compatible triple viewpoint; a complex structure, a Riemannian structure and a symplectic structure. As an example, all Riemann surfaces are necessarily Kähler. Since they have real dimension two and $d\omega$ is a 3-form, it must vanish identically. Another example is \mathbb{C}^n , and more generally, connected subsets $U \subseteq \mathbb{C}^n$. A more exotic and involved example is the complex projective space $\mathbb{C}\mathbb{P}^n$.

Example 3.6.3 ($\mathbb{C}\mathbb{P}^n$). We may describe the projective n -space as the space of complex lines through \mathbb{C}^{n+1} . More precisely,

$$\mathbb{C}\mathbb{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C}^*),$$

where \mathbb{C}^* denotes the punctured plane $\mathbb{C} \setminus \{0\}$. We consider the coordinates

$$[z_0 : \dots : z_n] \sim [\lambda z_0 : \dots : \lambda z_n], \quad \lambda \in \mathbb{C}^*.$$

We have an open cover $U_j = \{[z] \in \mathbb{C}\mathbb{P}^n : z_j \neq 0\}$ for $\mathbb{C}\mathbb{P}^n$, and charts

$$\varphi_j : U_j \rightarrow \mathbb{C}^n, \quad [z_0 : \dots : z_n] \mapsto \frac{1}{z_j} (z_0, \dots, \hat{z}_j, \dots, z_n).$$

The hat means that the coordinate is projected upon and "deleted". If we have a homogeneous polynomial $P : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}$ (meaning all terms have the same degree) and z is a regular value, then $\mathcal{M} := P^{-1}(z)/\mathbb{C}^* \subset \mathbb{C}\mathbb{P}^n$ is a projective hypersurface. We may also define a Kähler metric on $\mathbb{C}\mathbb{P}^n$, making it a Kähler manifold.

$$\omega_j := \frac{i}{2\pi} \partial \bar{\partial} \log \left(\sum_{l=0}^n \left| \frac{z_l}{z_j} \right|^2 \right).$$

When applied to charts, we have

$$\varphi_j(\omega_j) = \frac{i}{2\pi} \partial \bar{\partial} \log \left(1 + \sum_{k=1}^n |w_k|^2 \right).$$

For more details see [DO10], which deals with CR submanifolds of complex projective space.

Chapter 4

CR geometry

The word *CR* in *CR manifold* has a two-fold meaning; the first meaning being Cauchy-Riemann and the second meaning being complex-real. The Cauchy-Riemann equations induces the *tangential Cauchy-Riemann equations* and the *tangential Cauchy-Riemann complex* on a CR manifold. The second meaning is because embedded CR submanifolds are real submanifolds residing in complex space.

CR geometry is a rich mathematical subject which lies in the intersection of several mathematical disciplines: partial differential equations, several complex variables and differential geometry. There are many applications in areas of analysis and geometry, as well as some more surprising ones like number theory, see for example [D04].

The goal for this chapter will be to provide a down-to-earth introduction to CR manifolds and related concepts, with the main focus being on the embedded hypersurface type CR manifolds. Some of the theory will be given in the general case where the CR manifold has arbitrary codimension, and later on we will focus on the case where the codimension $k = 1$, also known as real hypersurfaces. For a comprehensive study, see [DT07], [Bog17] or [BER99].

4.1 CR manifolds

4.1.1 Abstract and embedded CR manifolds

Let M be a smooth manifold with $\dim_{\mathbb{R}} M = m$. In some contexts it is reasonable to ask for less, i.e. that M is a C^k differentiable manifold, but we will always assume M to be smooth. Furthermore, we let $H^{1,0} \subseteq TM_{\mathbb{C}}$ be a subbundle of complex rank n , where $1 \leq n \leq \lfloor \frac{m}{2} \rfloor$. We sometimes write $H^{1,0}M$ to avoid any confusion. We say that $H^{1,0}$ is an *almost CR structure* if it satisfies the following conditions;

$$H^{1,0} \cap H^{0,1} = \{0\}, \quad (H^{0,1} = \overline{H^{1,0}}).$$

If the structure is integrable i.e. $[\Gamma(H^{1,0}), \Gamma(H^{1,0})] \subseteq \Gamma(H^{1,0})$, then we say that it is a *CR structure*. We call n the CR-dimension and $k := m - 2n$ is the CR-codimension, and we say that the CR structure is of type (n, k) . A smooth manifold equipped with a CR structure of type (n, k) is a *CR manifold* of type (n, k) , abbreviated $\text{CR}(n, k)$. When the CR manifold is defined in the way above, where M is not a submanifold of \mathbb{C}^n or another complex manifold, we call it an *abstract CR manifold*. This terminology is used in contrast to *embedded CR manifolds*, where we have M as a submanifold of a complex manifold. The CR manifolds of type $(n - 1, 1)$, called *real hypersurfaces*, are of particular interest to us. In particular, we will take a closer look at embedded $\text{CR}(1, 1)$ manifolds in Chapter 5, that is, 3-dimensional manifolds that are embedded into \mathbb{C}^2 .

Definition 4.1.1. Let $M \subset \mathcal{N}$ be an m -dimensional smooth submanifold of a complex n -dimensional manifold \mathcal{N} . We define the CR structure on M to be

$$H^{1,0} := T^{1,0}\mathcal{N} \cap TM_{\mathbb{C}}, \quad H^{0,1} := T^{0,1}\mathcal{N} \cap TM_{\mathbb{C}},$$

where $T^{1,0}\mathcal{N}$ denotes the holomorphic tangent bundle of \mathcal{N} and $TM_{\mathbb{C}}$ is the complexified tangent bundle of M . We say that M is an *embedded CR manifold*. We will see later that both $H^{1,0}$ and $H^{0,1}$ as defined above are involutive and therefore integrable.

We let $(M, H^{1,0}M)$ and $(N, H^{1,0}N)$ be two CR manifolds of arbitrary type. A smooth map $f : M \rightarrow N$ is called a *CR map* if

$$df_p(H_p^{1,0}M) \subseteq H_{f(p)}^{1,0}N, \quad \forall p \in M,$$

where df_p is the \mathbb{C} -linear extension of the differential of f . We say that $f : M \rightarrow N$ is a *CR isomorphism* if it is a diffeomorphism and a CR map.

A slightly different point of view is presented in for example [LNR19] and [Bog17]. A function $f : M \subset \mathbb{C}^n \times \mathbb{R} \rightarrow \mathbb{C}$ of a real hypersurface M is called a *CR function* if it is holomorphic in the first n variables. We can define a CR map between two embedded CR manifolds $M \subset \mathbb{C}^m \times \mathbb{R}^k$ and $N \subset \mathbb{C}^n \times \mathbb{R}^d$ in a similar way. If each component f_j of the map $f : M \rightarrow N$ is a CR function, then f is a CR map. The study of CR functions is an important part of several complex variables, see for example Bochner's Extension Theorem for CR functions. The theorem can be found in [LT10].

If a CR manifold $(M, H^{1,0})$ is locally CR isomorphic to an embedded CR manifold, we say it is *locally embeddable*. A CR manifold $(M, H^{1,0})$ is *globally embeddable* if there is a CR isomorphism to a submanifold $M' \subset \mathcal{N}$ of a complex manifold \mathcal{N} . For example, any real analytic $\text{CR}(n, k)$ manifold is globally embeddable. This result is due to A. Andreotti and G.A. Fredricks, [AF79].

An important thing to note is that for a CR structure of type $(n, 0)$, we have that $k = m - 2n = 0$. We get that $m = 2n$ and a $\text{CR}(n, 0)$ structure is just a complex structure and $(M, H^{1,0})$ is a complex manifold. The same analogy holds for CR maps between $\text{CR}(n, 0)$ manifold. It turns out that complex manifolds and holomorphic maps (between complex manifolds) are a special case of CR manifolds and CR maps.

4.1.2 The Levi distribution

We turn our attention to a very important distribution for the purposes of the real point of view in CR geometry. The Levi distribution is a real subbundle of TM , which we will see is the maximal J -invariant subbundle of TM .

Definition 4.1.2 (Levi distribution). Let $(M, H^{1,0})$ be a CR manifold of type (n, k) . The *Levi distribution*¹ is a real rank $2n$ subbundle $\mathcal{H} \subseteq TM$ given by

$$\mathcal{H} = \text{Re} (H^{1,0} \oplus H^{0,1}).$$

There is a natural complex structure on \mathcal{H} ,

$$J : \mathcal{H} \rightarrow \mathcal{H}, \quad J(Z + \bar{Z}) = i(Z - \bar{Z})$$

¹It is also known as the *holomorphic tangent bundle* or *complex tangent bundle* in some literature. We will avoid this label, to avoid confusion with the holomorphic tangent bundle of a complex manifold.

for $Z \in H^{1,0}$. An equivalent way of defining the Levi distribution is the following:

$$\mathcal{H} = TM \cap J(TM).$$

We say that a subbundle L is J -invariant² if $X \in L$ implies that $JX \in L$. An important related concept is the part of the tangent bundle where we do not have a complex structure J . We call this bundle the *totally real part*, denoted \mathcal{R} . We have that

$$\mathcal{R} = TM/\mathcal{H}.$$

Lemma 4.1.3. *Let $L \subseteq TM$ be a subbundle. Then $L \cap JL$ is a J -invariant subbundle.*

Proof. Let $X \in L \cap JL$. Then $X \in L$ which implies that $JX \in JL$. On the other hand, since $X \in JL$, it implies that there exists $Y \in L$ such that $JY = X$. We get that $JX = J^2Y = -Y \in L$, and hence $JX \in L \cap JL$. We conclude with that $L \cap JL$ is J -invariant. \square

Proposition 4.1.4. \mathcal{H} is the maximal J -invariant subbundle of TM .

Proof. We use the other definition of the Levi distribution,

$$\mathcal{H} := TM \cap J(TM)$$

By the Lemma 4.1.2, \mathcal{H} is J -invariant. Suppose there exists $\tilde{\mathcal{H}}$ that is J -invariant, i.e. $J\tilde{\mathcal{H}} \subseteq \tilde{\mathcal{H}} \subseteq TM$. Then for $X \in \tilde{\mathcal{H}} \subseteq TM$, we have $Y = JX \in J\tilde{\mathcal{H}}$.

$$-X = J^2X = JY \in J\tilde{\mathcal{H}} \subseteq J(TM)$$

We see that $X \in TM \cap J(TM) = \mathcal{H}$, hence $\tilde{\mathcal{H}} \subseteq \mathcal{H}$. Thus, \mathcal{H} is maximal. \square

In the beginning of this chapter we required a CR structure to be integrable by definition and later defined the embedded CR structure as $H^{1,0} := T^{1,0}\mathcal{N} \cap TM_{\mathbb{C}}$. The next proposition shows us that the embedded CR structure and its conjugate will always be involutive, and hence integrable.

Proposition 4.1.5. *Let $M \subset \mathcal{N}$ be an embedded CR manifold and (\mathcal{N}, J) a complex manifold. Then the CR structure $H^{1,0}$ and its conjugate $H^{0,1}$, as defined in Definition 4.1.1 are integrable.*

²This expression should not be confused with j -invariant in the context of elliptic curves.

Proof. For $X \in \Gamma(\mathcal{H})$, we have that

$$\Gamma(H^{1,0}) = \{X - iJX : X \in \Gamma(\mathcal{H})\}, \quad H^{0,1} = \{X + iJX : X \in \Gamma(\mathcal{H})\}$$

Since M is embedded in \mathcal{N} , J is the complex structure of \mathcal{N} restricted to \mathcal{H} . Let $Z = X - iJX$, $W = Y - iJY \in \Gamma(H^{1,0})$. Then

$$[Z, W] = [X - iJX, Y - iJY] = [X, Y] - [JX, JY] - i([JX, Y] + [X, JY]).$$

Recall that the Nijenhuis tensor for a complex structure vanishes, that is

$$N_J(X, Y) = [X, Y] + J([JX, Y] + [X, JY]) - [JX, JY] = 0.$$

We can use that $J(N_J(X, Y)) = 0$ to obtain

$$J[X, Y] - J[JX, JY] - [JX, Y] - [X, JY] = 0.$$

We can rewrite this as

$$[JX, Y] + [X, JY] = J([X, Y] - [JX, JY]).$$

Using this expression, we write

$$\begin{aligned} [Z, W] &= [X, Y] - [JX, JY] - i([JX, Y] + [X, JY]) \\ &= [X, Y] - [JX, JY] - i(J([X, Y] - [JX, JY])). \end{aligned}$$

Since $X, Y \in \Gamma(\mathcal{H}) \subset \Gamma(TM)$, it follows that $JX, JY \in \Gamma(TM)$ and therefore

$$X' = [X, Y] - [JX, JY] \in \Gamma(TM).$$

On the other hand, we have that $\mathcal{H} \subset J(TM)$ and as a consequence,

$$X' = [X, Y] - [JX, JY] \in \Gamma(J(TM)).$$

We get that $X' \in \Gamma(\mathcal{H})$, and therefore it follows that

$$[Z, W] = [X, Y] - [JX, JY] - i(J([X, Y] - [JX, JY])) = X' - iJX' \in \Gamma(H^{1,0}).$$

This concludes the proof that $H^{1,0}$ is involutive and thus integrable. The proof for $H^{0,1}$ follows the exact same steps. \square

4.1.3 Pseudo-Hermitian structure and the Levi form

This subsection we introduce the notion of pseudo-Hermitian structure, which is closely related to a Hermitian structure, discussed in Section 3.6. We also define a 2-form called the *Levi form*. The degree to which $\mathcal{H}_\mathbb{C}$ fails to be involutive is measured by the Levi form. In codimension $k = 1$, the Levi form will under certain conditions be nondegenerate or positive definite. Since positive definiteness doesn't generalize to higher codimension in an obvious way, we will therefore focus on the $\text{CR}(n - 1, 1)$ manifolds.

Let M be an orientable $\text{CR}(n - 1, 1)$ manifold. We define a subbundle of the cotangent bundle in the following way:

$$E := \{\omega \in T^*M : \mathcal{H} \subseteq \ker(\omega)\}.$$

E is a real line subbundle of the cotangent bundle. Since M is assumed to be orientable and \mathcal{H} is oriented by the complex structure J , it follows that E is orientable. Real line bundle are trivial if and only if they are orientable, hence E is trivial. As a consequence, there exists a non-vanishing globally defined section $\theta : M \rightarrow E$. We call the section θ a *pseudo-Hermitian structure*. An important note is that $\ker(\theta) = \mathcal{H}$.

Definition 4.1.6. Given $(M, H^{1,0}, \theta)$, we define the *Levi form* is as

$$L_\theta(Z, \bar{W}) = -id\theta(Z, \bar{W}), \quad \forall Z, W \in H^{1,0}.$$

We say that $(M, H^{1,0}, \theta)$ is *nondegenerate* if the Levi form L_θ is nondegenerate and if L_θ is positive definite, then $(M, H^{1,0}, \theta)$ is *strictly pseudoconvex*³.

We observe that the pseudo-Hermitian structure isn't unique, meaning that given a $\text{CR}(n - 1, 1)$ manifold, there is a choice to be made in defining θ . Despite there being a choice in defining θ , any two pseudo-Hermitian structures $\theta, \hat{\theta}$ are related by

$$\hat{\theta} = \lambda\theta, \quad \lambda \in C^\infty(M), \lambda(p) \neq 0, \quad \forall p \in M.$$

It follows from the product rule that

$$d\hat{\theta}|_{\wedge^2 \mathcal{H}} = d(\lambda\theta) = d\lambda \wedge \theta + \lambda d\theta|_{\wedge^2 \mathcal{H}}.$$

³Strictly pseudoconvex CR manifolds are closely related to *strictly pseudoconvex domains* in several complex variables. To read more about this, one could consult Chapter 7 in [LT10].

Since $\ker(\theta) = \mathcal{H}$, θ vanishes on $H^{1,0}$ and $H^{0,1}$, thus

$$d\hat{\theta}|_{\wedge^2 \mathcal{H}} = \lambda d\theta|_{\wedge^2 \mathcal{H}}.$$

In the same way, we get that the Levi form of two pseudo-Hermitian structures are related by:

$$L_{\hat{\theta}}|_{\wedge^2 \mathcal{H}} = \lambda L_{\theta}|_{\wedge^2 \mathcal{H}}.$$

The way $d\theta$ and L_{θ} transforms with a change of pseudo-Hermitian structure is closely related to conformal geometry. A *conformal metric* is an equivalence class of metrics $g_1 \sim \lambda^2 g_2$ for $\lambda \in C^\infty(M)$. There is a strong analogy between CR geometry and conformal geometry, which is emphasized in [JL⁺87].

Nondegeneracy is not dependent on choice of θ . We say that nondegeneracy is a *CR invariant property*. Strict pseudoconvexity is not a CR invariant property. Take for example $\lambda = -1$. Then if $\hat{\theta} = \lambda\theta$, then $L_{\hat{\theta}} = -L_{\theta}$, and if L_{θ} is positive definite, then $L_{\hat{\theta}}$ is negative definite.

If M is a nondegenerate CR manifold with a pseudo-Hermitian structure θ , we say that (M, θ) is a *pseudo-Hermitian manifold*. If we have a CR map between two pseudo-Hermitian manifolds $f : (M, \theta_M) \rightarrow (N, \theta_N)$, then

$$f^*\theta_N = \lambda\theta_M,$$

for some $\lambda \in C^\infty(M)$. A CR map $f : (M, \theta_M) \rightarrow (N, \theta_N)$ is called a *pseudo-Hermitian map* if the pullback satisfies

$$f^*\theta_N = c\theta_M, \quad c \in \mathbb{R}.$$

4.2 The Reeb vector field

Recall that for a CR manifold, we have a subbundle $\mathcal{R} \subset TM$ which we call the totally real part of TM . In the special case of a strictly pseudoconvex hypersurface, the totally real part is 1-dimensional and will be spanned by a single vector field.

Theorem 4.2.1. *Let M be a strictly pseudoconvex hypersurface. There exists a unique globally defined non-vanishing vector field $\xi : M \rightarrow TM$ such that*

$$\theta(\xi) = 1, \quad d\theta(\xi, \cdot) = 0.$$

We call ξ the *Reeb vector field*. It is also known in the literature as the *characteristic direction*, [DT07] and sometimes the *bad direction*, [D'A19].

Proof. Existence: Since $\theta \neq 0$, there exists $\tilde{\xi}$ such that $\theta(\tilde{\xi}) = 1$. We know that M admits a Riemannian metric g with the musical isomorphism $\sharp : T^*M \rightarrow TM$. We define

$$\tilde{\xi} = \frac{\theta^\sharp}{|\theta^\sharp|^2} \implies \theta(\tilde{\xi}) = \frac{1}{|\theta^\sharp|^2} \theta(\theta^\sharp) = 1$$

We now have that $\theta(\xi) = \theta(\tilde{\xi}) = 1$. We define $\alpha \in \mathcal{H}^*$ as follows:

$$\alpha = d\theta(\tilde{\xi}, \cdot)|_{\mathcal{H}}.$$

Since $d\theta$ is nondegenerate on \mathcal{H} , the following map is invertible

$$\psi : \mathcal{H} \rightarrow \mathcal{H}^*, \quad v \mapsto d\theta(v, \cdot).$$

If we let $\xi = \tilde{\xi} - \psi^{-1}(\alpha)$, we then have

$$d\theta(\xi, \cdot) = \alpha - \alpha = 0.$$

This proves the existence.

Uniqueness: Suppose ξ_1 and ξ_2 both satisfies the conditions for the Reeb vector field and let $\Xi = \xi_1 - \xi_2$. Then

$$\theta(\Xi) = \theta(\xi_1 - \xi_2) = \theta(\xi_1) - \theta(\xi_2) = 0,$$

which implies that $\Xi \in \mathcal{H}$.

$$\psi(\Xi) = d\theta(\Xi, \cdot) = d\theta(\xi_1, \cdot) - d\theta(\xi_2, \cdot) = 0.$$

Since ψ is invertible, it follows that $\Xi = 0$ and thus $\xi_1 = \xi_2$. This proves the uniqueness. \square

Corollary 4.2.2. *Let $(M, H^{1,0})$ be nondegenerate CR with a pseudo-Hermitian structure θ and the Reeb vector field ξ . We identify \mathcal{R} with $\text{Span}\{\xi\}$. Then*

$$TM = \mathcal{H} \oplus \text{Span}\{\xi\} = \mathcal{H} \oplus \mathcal{R}.$$

Proof. Let $X \in TM$ and define $Y = X - \theta(X)\xi$. Then

$$\theta(Y) = \theta(X - \theta(X)\xi) = \theta(X) - \theta(X)\theta(\xi) = 0$$

This implies that $Y \in \ker \theta = \mathcal{H}$. □

The Reeb vector field is unique to CR geometry. In fact, any contact manifold admits a unique Reeb vector field. This means that any nondegenerate $\text{CR}(n-1, 1)$ manifold is a contact manifold. If the pseudo-Hermitian structure θ is nondegenerate, it defines a contact form. One can read more about this in [DT07].

4.3 The Webster metric

In order for us to talk about the geometry of CR manifolds, we will have to introduce a pseudo-Hermitian metric. In this section we will assume that M is strictly pseudoconvex hypersurface. We define a bilinear form

$$G_\theta(X, Y) = d\theta(X, JY), \quad \forall X, Y \in \mathcal{H}.$$

Recall the Levi form $L_\theta(Z, \bar{W}) = -id\theta(Z, \bar{W})$ is also defined in terms of $d\theta$, and although $L_\theta(Z, \bar{W})$ is defined for complex vector fields Z, W , it turns out that L_θ and G_θ are closely related, as illustrated in the following proposition.

Proposition 4.3.1. *Let $X = \operatorname{Re}(Z)$, $Y = \operatorname{Re}(W)$ for $Z, W \in H^{1,0}$. Then*

$$G_\theta(X, Y) = \frac{1}{2}L_\theta(Z, \bar{W}).$$

Proof.

$$\begin{aligned} G_\theta(X, Y) &= d\theta\left(\frac{1}{2}(Z + \bar{Z}), J\left(\frac{1}{2}(W + \bar{W})\right)\right) = d\theta\left(\frac{1}{2}(Z + \bar{Z}), \frac{i}{2}(W - \bar{W})\right) \\ &= \frac{i}{4}d\theta((Z + \bar{Z}), (W - \bar{W})) = \frac{1}{4}L_\theta(Z, \bar{W}) + \frac{1}{4}L_\theta(W, \bar{Z}) = \frac{1}{2}L_\theta(Z, \bar{W}). \end{aligned}$$

□

In order for us to know that G_θ defines a metric on \mathcal{H} , we need to verify that it is symmetric. Smoothness and bilinearity comes from the definition of $d\theta$ and positive definiteness holds under the assumption that L_θ is positive definite.

Proposition 4.3.2. *$G_\theta(JX, JY) = G_\theta(X, Y)$ for all $X, Y \in \mathcal{H}$ and is therefore symmetric by Lemma 3.6.2.*

Proof. We consider the difference

$$G_\theta(JX, JY) - G_\theta(X, Y) = d\theta(JX, JJY) - d\theta(X, JY) = -d\theta(JX, Y) - d\theta(X, JY)$$

Using Cartan's magic formula, as seen in Example 2.1.6, on $d\theta(X, Y)$ and that $\theta(X) = 0$ for all $X \in \mathcal{H}$, we get that $-d\theta(X, Y) = \theta([X, Y])$, and thus

$$-d\theta(JX, Y) - d\theta(X, JY) = (\theta([JX, Y]) + \theta([X, JY])).$$

Recall that we have the following integrability condition by requiring N_J to vanish:

$$[JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0,$$

Using the equation above, we can write

$$G_\theta(JX, JY) - G_\theta(X, Y) = \theta(J([X, Y] - [JX, JY])) = 0.$$

The last part is obtained using that $\theta \circ J = 0$. This completes the proof that $G_\theta(JX, JY) = G_\theta(X, Y)$. Since the Levi form and the G_θ coincide on \mathcal{H} , we can conclude that L_θ is also symmetric. \square

As it stands right now, G_θ is a symmetric bilinear form, assumed to be positive definite. However, it is only defined on the Levi distribution. By using the direct sum decomposition of the tangent bundle $TM = \mathcal{H} \oplus \mathcal{R}$, we want to extend G_θ to TM .

Definition 4.3.3. Let (M, θ) be a strictly pseudoconvex CR manifold, i.e. with a positive definite Levi form L_θ . We let $\pi_{\mathcal{H}} : TM \rightarrow \mathcal{H}$ be the natural projection to the Levi distribution. We define the *Webster metric* to be

$$g_\theta(X, Y) = G_\theta(\pi_{\mathcal{H}}X, \pi_{\mathcal{H}}Y) + \theta(X)\theta(Y), \quad X, Y \in TM$$

Proposition 4.3.4. *The Webster metric satisfies the following properties:*

$$\begin{aligned} g_\theta(X, Y) &= G_\theta(X, Y), \quad \forall X, Y \in \mathcal{H} \\ g_\theta(X, \xi) &= 0 \\ g_\theta(\xi, \xi) &= 1 \end{aligned}$$

Proof. For $X, Y \in \mathcal{H}$, we have $\theta(X), \theta(Y)$ are both zero and $\pi_{\mathcal{H}}X = X$, $\pi_{\mathcal{H}}Y = Y$, so

$$g_\theta(X, Y) = G_\theta(X, Y).$$

For the second claim, we use that $\pi_{\mathcal{H}}\xi = 0$ and again that $\theta(X) = 0$, and thus

$$g_\theta(X, \xi) = G_\theta(X, 0) + \theta(X)\theta(\xi) = 0.$$

For the last claim, we again use $\pi_{\mathcal{H}}\xi = 0$ and $\theta(\xi) = 1$ to get

$$g_\theta(\xi, \xi) = G_\theta(0, 0) + \theta(\xi)\theta(\xi) = 1.$$

\square

4.4 CR Lie groups and CR Lie algebras

In this section we give a brief overview of CR Lie groups. The motivation behind this is that there are important examples of CR manifold that are CR Lie groups. Particular emphasis is put on quadric manifolds, which are described in Definition 4.4.2.

A Lie group is a smooth manifold with a smooth group operation. They are important examples of manifolds with numerous applications in mathematics and physics. Related to a Lie group is its Lie algebra, which is a vector space \mathfrak{a} equipped with a map $[\cdot, \cdot] : \mathfrak{a} \times \mathfrak{a} \rightarrow \mathfrak{a}$ satisfying the same axioms we have seen for the Lie bracket of vector fields, see Proposition 2.1.1. For more details on Lie groups one may consult [Lee01]. Analogously, CR Lie groups are important examples of CR manifolds.

Let \mathcal{G} be a $(2n + k)$ -dimensional real Lie algebra. An (n, k) -structure on \mathcal{G} is an n -dimensional complex subalgebra $\mathfrak{a} \subset \mathcal{G}_{\mathbb{C}}$ such that $\mathfrak{a} \cap \bar{\mathfrak{a}} = \{0\}$. We say that $(\mathcal{G}, \mathfrak{a})$ is a CR Lie algebra.

We define $\mathfrak{h} = \text{Re}(\mathfrak{a} \oplus \bar{\mathfrak{a}})$, which is an $2n$ -dimensional real subspace (not necessarily a subalgebra). Let $J : \mathfrak{h} \rightarrow \mathfrak{h}$ be defined by $J(Z + \bar{Z}) = i(Z - \bar{Z})$, then \mathfrak{h} is an n -dimensional complex subspace with multiplication defined by $iX = JX$ and $\mathfrak{h} \cong \mathfrak{a}$.

Definition 4.4.1. Let G be a real $(2n + k)$ -dimensional Lie group and $H^{1,0}$ be an $\text{CR}(n, k)$ -structure on G . We say that $(G, H^{1,0})$ is an (n, k) CR Lie group if $H^{1,0}$ is left invariant or equivalently, the group operation is a CR map.

Let $\text{Lie}(G)$ be the Lie algebra of left-invariant vector fields on G . If $k = 1$, $\theta \in \text{Lie}(G)^*$ such that $\ker(\theta) = \mathfrak{h}$. Then θ defines a left-invariant pseudo-Hermitian structure on G .

Definition 4.4.2. The map $q : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^d$ is called a *quadratic form* if it satisfies the following:

1. q is bilinear over \mathbb{C}
2. $q(z, w) = q(w, z)$ for all $z, w \in \mathbb{C}^n$
3. $\overline{q(z, w)} = q(\bar{z}, \bar{w})$ for all $z, w \in \mathbb{C}^n$

When given a quadratic form $q : \mathbb{C}^{n-d} \times \mathbb{C}^{n-d} \rightarrow \mathbb{C}^d$, we can define

$$M = \{(z, w) \in \mathbb{C}^d \times \mathbb{C}^{n-d} : q(w, \bar{w}) = \text{Im}(z)\}.$$

M is called a *quadric manifold*. The requirements (2) and (3) ensures that $q(w, \bar{w}) \in \mathbb{R}^d$, hence M is of real dimension $2n - d$. Furthermore, we define an operation on M ,

$$(z_1, w_1) \star (z_2, w_2) = (z_1 + z_2 + 2iq(w_1, \bar{w}_2), w_1 + w_2)$$

As shown in [Bog17], any generic CR manifold, i.e. any CR manifold with $\dim_{\mathbb{R}} \mathcal{H} = 2n - 2k$, can be approximated at the origin by a quadric manifold. We can think of quadric manifolds as models for a general CR submanifold.

Proposition 4.4.3. *The operation \star defines a group structure on $\mathbb{C}^n \times \mathbb{C}^n$, which restricts to $M \times M$.*

Proof. Associativity follows from addition in \mathbb{C}^n , and the identity is simply just $(0, 0) \in \mathbb{C}^d \times \mathbb{C}^{n-d}$. Given an element (z, w) , its inverse is given as $(z, w)^{-1} = (-z + 2iq(w, \bar{w}), -w)$. What is left is to show that the group restricted to $M \times M \rightarrow M$ is closed and for all $(z, w) \in M$, there is a unique $(z, w)^{-1} \in M$. First let us assume $(z_1, w_1), (z_2, w_2) \in M$. Then

$$(z_1, w_1) \star (z_2, w_2) = (z_1 + z_2 + 2iq(w_1, \bar{w}_2), w_1 + w_2) = (z_3, w_3)$$

Using the bilinearity of q ,

$$q(w_3, \bar{w}_3) = q(w_1, \bar{w}_1) + q(w_1, \bar{w}_2) + q(w_2, \bar{w}_1) + q(w_2, \bar{w}_2)$$

By symmetry, $q(w_2, \bar{w}_1) = q(\bar{w}_1, w_2)$ and by the conjugation of q ,

$$q(\bar{w}_1, w_2) = \overline{q(w_1, \bar{w}_2)}.$$

Thus, we have

$$q(w_3, \bar{w}_3) = \text{Im}(z_1 + z_2) + q(w_1, \bar{w}_2) + \overline{q(w_1, \bar{w}_2)} = \text{Im}(z_1 + z_2) + 2\text{Re}(q(w_1, \bar{w}_2))$$

and since $\text{Im}(2iq(w_1, \bar{w}_2)) = 2\text{Re}(q(w_1, \bar{w}_2))$, we arrive at

$$q(w_3, \bar{w}_3) = \text{Im}(z_3).$$

Now let us assume $(z, w) \in M$. We want to show that $(z, w)^{-1}$ must also be in M , i.e. that $(z, w)^{-1} = (z', w')$ satisfies

$$q(w', \bar{w}') = \text{Im}(z').$$

We use the previously mentioned inverse,

$$(z, w)^{-1} = (-z + 2iq(w, \bar{w}), -w).$$

The left-hand side of the equation, can be written as

$$q(w', \bar{w}') = q(-w, -\bar{w}') = q(w, \bar{w}),$$

by using the bilinearity of q . We write $z = x + iy$. Then the left-hand side of the equation becomes

$$\text{Im}(z') = \text{Im}(-z + 2iq(w, \bar{w})) = \text{Im}(-x - iy + 2iy) = y = \text{Im}(z).$$

Hence, $(z, w)^{-1} \in M$ for all $(z, w) \in M$. This completes the proof. \square

Furthermore, $\star : M \times M \rightarrow M$ is smooth, hence (M, \star) is a Lie group. It is proven in [LMN07] that any compact Lie group K of odd dimension admits a left-invariant $CR(n-1, 1)$ structure, meaning any compact odd-dimensional Lie group admits a CR Lie group structure.

Proposition 4.4.4. *Given a quadric manifold (M, q) , the vector fields*

$$Z_j = \partial_{w_j} + 2i \sum_{l=1}^d \frac{\partial q_l}{\partial w_j} \partial_{z_l}, \quad 1 \leq j \leq n-d,$$

together with \bar{Z}_j ($1 \leq j \leq n-d$) and vector fields spanning the real part of M :

$$\xi_j = \partial_{x_j}, \quad 1 \leq j \leq d.$$

The vector fields Z_j, \bar{Z}_j and ξ_j are all left-invariant vector fields, and $H^{1,0} = \text{Span}_{\mathbb{C}}\{Z_j\}$.

For a proof of this claim, see [Bog17]. In the particular case of a quadric $CR(1, 1)$ -manifold, we have the vector fields Z, \bar{Z} , given by

$$\bar{Z} = \partial_w + 2i \frac{\partial q}{\partial w} \partial_z, \quad Z = \partial_{\bar{w}} - 2i \frac{\partial q}{\partial \bar{w}} \partial_{\bar{z}}$$

and the Reeb vector field given by $\xi = \partial_x$. In the next chapter we will take a closer look at a common example of a quadric CR manifold, the Heisenberg group.

4.5 Connection and curvature on CR manifolds

4.5.1 The Tanaka-Webster connection

The problem we run into with the Levi-Civita connection (Definition 2.3.8) on CR manifolds is that it does not preserve \mathcal{H} or J . That is, $\nabla J \neq 0$ and given $X, Y \in \Gamma(\mathcal{H})$, $\nabla_X Y$ is not necessarily in $\Gamma(\mathcal{H})$. We therefore introduce a unique linear connection for CR manifolds, called the Tanaka-Webster connection, which preserves \mathcal{H} and J . After we have introduced the Tanaka-Webster connection, we will define the associated pseudo-holomorphic sectional curvature.

Recall that given a linear connection ∇ , the torsion tensor field is given by

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

We say that ∇ is *torsion-free* if τ vanishes, and that a connection is *compatible with the metric g* if $\nabla g = 0$. The Tanaka-Webster connection will be compatible with g , but not torsion-free. Instead, we define a new condition for the torsion.

Definition 4.5.1. We say that ∇ has *pure torsion* if

1. $\tau(X, Y) = d\theta(X, Y)\xi$
2. $(\tau(\xi, JX)) + J(\tau(\xi, X)) = 0$

Theorem 4.5.2 (Tanaka-Webster). *Let (M, θ, g_θ) be a nondegenerate CR manifold and let J be the complex structure on \mathcal{H} extended to TM , with $J\xi = 0$. Then there exists a unique linear connection ∇ on M satisfying:*

1. $\nabla_X Y \in \Gamma(\mathcal{H})$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(\mathcal{H})$ (we say that \mathcal{H} is parallel with respect to ∇).
2. $\nabla J = 0, \quad \nabla g_\theta = 0$.
3. The torsion τ of ∇ is pure.

We call this connection ∇ the *Tanaka-Webster connection*. The associated 1-form $\tau(\xi, \cdot)$ is called the *pseudo-Hermitian torsion* of ∇ .

Proof. The proof is long and very technical. A proof based on Tanaka's work can be found in [DT07]. One can also consult the original monograph written by Tanaka, [Tan75] or the paper written by Webster, [W⁺78]. \square

In [Tan89], Tanno generalized the Tanaka-Webster connection to a contact Riemannian manifold.

4.5.2 Pseudo-holomorphic sectional curvature

We now turn our attention back to the topic of curvature on manifolds, this time in the context of complex manifolds and CR manifolds. Just like many other aspects of Riemannian geometry, sectional curvature has a complex counterpart. We refer to the article [Bar07] for an interpretation of the pseudo-holomorphic sectional curvature.

Definition 4.5.3. Let (\mathcal{M}, J, g) be given, $E \subseteq TM$ and $g(v, w) = g(Jv, Jw)$. For any non-zero $v \in E$, we define $S = \text{Span}\{v, Jv\}$. The *pseudo-holomorphic sectional curvature*⁴ is defined by

$$\kappa(S) = \frac{g(R(v, Jv)Jv, v)}{|v|^4}$$

If $E = TM$, we call it *holomorphic sectional curvature*.

Since we are working with CR manifolds, the natural choice for vector bundle $E = \mathcal{H}$. In the next chapter we will give explicit formulas for the Tanaka-Webster connection of a CR(1, 1) manifold and the associated pseudo-holomorphic sectional curvature.

4.5.3 Extending the connection

Whenever the Tanaka-Webster connection is mentioned in literature, a complex formulation is usually given. That is, everything is formulated for vector fields $Z \in H^{1,0}$ and $\bar{W} \in H^{0,1}$. The aim of this subsection is to unify the approach using real vector fields given in Subsection 4.5.1 and the approach using complex vector fields given in Chapter 1.2 in [DT07].

If ∇ is an linear connection on a real vector bundle E then we can extend it to the complexified vector bundle $E_{\mathbb{C}} = E \otimes \mathbb{C}$ and complex vector fields by linearity. We want to do this for ∇ on the complexified tangent bundle

$$TM_{\mathbb{C}} = H^{1,0} \oplus H^{1,0} \oplus \text{Span}\{\xi\} \oplus \text{Span}\{i\xi\}.$$

Using the properties of ∇ , we want to show an equivalent condition for τ to have pure torsion.

⁴Also called the *Pseudo-Hermitian sectional curvature*, see e.g [DT07].

Proposition 4.5.4. *Let $(\tau(\xi, JX)) + J(\tau(\xi, X)) = 0$. The condition $\tau(X, Y) = d\theta(X, Y)\xi$ is equivalent to*

1. $\tau(Z, W) = 0$
2. $\tau(Z, \overline{W}) = 2iL_\theta(Z, \overline{W})\xi$

For $Z, W \in H^{1,0}$.

Proof. We let $Z = X - iJX$ and $W = Y - iJY$.

$$\begin{aligned}\tau(Z, W) &= \tau(X - iJX, Y - iJY) \\ &= \tau(X, Y) - \tau(JX, JY) - i(\tau(X, JY) + \tau(JX, Y))\end{aligned}$$

Using the definition $\tau(X, Y) = d\theta(X, Y)\xi$ and that $d\theta(X, JY) = -d\theta(JX, Y)$, we get

$$\tau(Z, W) = (d\theta(X, Y) - d\theta(X, Y) - id\theta(X, JY) - id\theta(JX, Y))\xi = 0.$$

For the second claim, we write

$$\begin{aligned}\tau(Z, \overline{W}) &= \tau(X - iJX, Y + iJY) \\ &= \tau(X, Y) + \tau(JX, JY) + i(\tau(X, JY) - \tau(JX, Y)) \\ &= 2(d\theta(X, Y) + d\theta(X, iJY))\xi \\ &= 2i(L_\theta(X, Y) - L_\theta(X, iJY)) \\ &= 2iL_\theta(X - iJX, Y + iJY)\xi \\ &= 2iL_\theta(Z, \overline{W})\xi.\end{aligned}$$

For the other direction of both claims, one can do each step in reverse and use that $(\tau(\xi, JX)) + J(\tau(\xi, X)) = 0$. \square

Chapter 5

Embedded CR submanifolds

Now that we have introduced CR manifolds and some of its geometry, we can finally look deeper into the case of embedded CR submanifolds. First, we show various examples: a totally real submanifold, a complex submanifold, a flat hypersurface and an example of nonconstant rank, which is not a CR submanifold.

We then show a more general example of 3-dimensional CR manifold, described by an arbitrary defining function. We get a general formula for the psuedo-Hermitian structure, the Levi form and the Webster metric. We then dedicate a section each for two particularly important examples, the sphere S^3 and the Heisenberg group.

In Section 5.5, we revisit the topic of the Reeb vector field and give a formula for the component functions of ξ . Section 5.6 further explores the Tanaka-Webster connection, proving important properties and giving a formula for the connection in the 3-dimensional case, as well as a formula for the associated pseudo-holomorphic sectional curvature. The last section serves as a brief summary of how to calculate all the aforementioned parts of an embedded $\text{CR}(1, 1)$ manifold.

5.1 Examples of embedded CR manifolds

The first two examples will highlight the two extreme cases for our Levi distribution: when $\mathcal{H} = \{0\}$ and when $\mathcal{H} = TM$.

Example 5.1.1 (Totally real submanifold). A CR manifold is called *totally real* if $\mathcal{H} = \{0\}$. If we take

$$M = \{z \in \mathbb{C}^2 : y_1 = y_2 = 0\}$$

Written in real coordinates, we have

$$M = \{(x_1, 0, x_2, 0) \in \mathbb{R}^4\}$$

The tangent bundle is then spanned by ∂_{x_j} , that is,

$$TM = \text{Span}\{\partial_{x_1}, \partial_{x_2}\}.$$

But we have that $J(\partial_{x_j}) = \partial_{y_j}$, thus we get that $TM \cap JTM = \{0\}$.

Example 5.1.2 (Complex submanifold).

$$M = \{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}$$

It is not difficult to see that our manifold is just the complex plane embedded into \mathbb{C}^2 . Following the arguments from the previous example, we have

$$TM = \text{Span}\{\partial_{x_1}, \partial_{y_1}\}$$

and since $J\partial_{x_1} = \partial_{y_1}$, $J\partial_{y_1} = -\partial_{x_1}$, we get that $JTM = TM$ and hence $\mathcal{H} = TM \cap JTM = TM$. In general, if $\mathcal{H} = TM$, then M is a complex submanifold.

Example 5.1.3 (The flat hyperplane). Consider the function

$$F(z_1, z_2) = \text{Im}(z_2), \quad M = F^{-1}(0).$$

The function F acts as a projection in the second coordinate. The manifold M has dimension 3 and non-vanishing differential, i.e. it is a hypersurface. However, we will see that we are not guaranteed to have a strictly pseudoconvex or even nondegenerate pseudo-Hermitian structure. Using that $\text{Im}(z_2) = y_2$ and $dz_2 = dx_2 + idy_2$, we get that the differential is given by

$$dF = dy_2,$$

and the psuedo-Hermitian structure is given by

$$\theta = -dx_2.$$

We immediately see that $d\theta = 0$, and as a consequence, $L_\theta = 0$. CR submanifolds with $L_\theta = 0$ are called *Levi flat*.

Example 5.1.4 (An example of non-constant rank on \mathcal{H}). We give an example showing that we will not always get a CR-structure. Let us consider a subset of the $2n - 1$ -sphere where the n -th coordinate has no imaginary part. $\|z\|^2$ is short-hand notation for $\sum_{j=1}^n |z_j|^2 = \sum_{j=1}^n z_j \bar{z}_j$. We consider the following:

$$M = \{z \in \mathbb{C}^n : \|z\|^2 = 1, \text{Im}(z_n) = 0\}.$$

Let p_1 be the point given by $z_1 = z_2 = \dots = z_{n-2} = 0$, $z_{n-1} = 1$, $z_n = 0$. Written in real coordinates,

$$x_1 = \dots = x_{n-2} = 0, \quad x_{n-1} = 1, \quad x_n = 0, \quad y_1 = \dots = y_n = 0.$$

We get that $\partial_{x_{n-1}}$ is the normal at p_1 and

$$T_{p_1}M = \text{Span}\{\partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_{n-2}}, \partial_{y_{n-2}}, \partial_{y_{n-1}}, \partial_{x_n}\}.$$

Since $J(\partial_{x_n}) = \partial_{y_n}$ and $J(\partial_{y_{n-1}}) = -\partial_{x_{n-1}}$, we get that

$$\mathcal{H}_{p_1} = \text{Span}\{\partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_{n-2}}, \partial_{y_{n-2}}\}.$$

and we see that $\dim_{\mathbb{R}} \mathcal{H}_{p_1} = 2n - 4$. By choosing $p_2 \in M$ given by $z_1 = z_2 = \dots = z_{n-1} = 0$, $z_n = 1$, we get that

$$T_{p_2}M = \mathcal{H}_{p_2}M = \text{Span}\{\partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_{n-1}}, \partial_{y_{n-1}}\}.$$

In this case we have that $\dim_{\mathbb{R}} \mathcal{H}_{p_2} = 2n - 2$. In conclusion, we see that the dimension of \mathcal{H}_p might vary, but we require the CR structure and the Levi distribution to be of constant rank. This example is not a CR manifold since it fails to have constant rank.

5.2 CR manifold of an arbitrary defining function

We will let $F : \mathbb{C}^2 \rightarrow \mathbb{R}$ be an arbitrary defining function. We let $M = F^{-1}(p)$, and assume p to be a regular value. The differential in complex coordinates is then given by

$$dF = F_{z_1} dz_1 + F_{z_2} dz_2 + F_{\bar{z}_1} d\bar{z}_1 + F_{\bar{z}_2} d\bar{z}_2.$$

Here F_{z_j} is short-hand notation for $\partial_{z_j} F$, and similarly $F_{\bar{z}_1} = \partial_{\bar{z}_1} F$. Under the assumption that p is a regular value, the differential dF is non-vanishing on M , as the Theorem 2.2.5 guarantees that M is in fact a smooth manifold. A quick exercise in complex differentials and Wirtinger derivatives gives us the differential in real coordinates:

$$F_z dz + F_{\bar{z}} d\bar{z} = F_z(dx + idy) + F_{\bar{z}}(dx - idy) = (\partial_z + \partial_{\bar{z}})F dx + i(\partial_z - \partial_{\bar{z}})F dy,$$

and we get that (as expected),

$$dF = F_{x_1} dx_1 + F_{y_1} dy_1 + F_{x_2} dx_2 + F_{y_2} dy_2.$$

We want to describe the real tangent bundle TM and the Levi distribution $\mathcal{H} = \ker \theta$. Since M is described in terms of F , we get that the tangent bundle $TM = \ker dF$. Since we can write $\mathcal{H} = TM \cap JTM$, we have that $\mathcal{H} = \ker \theta$.

In complex coordinates, we write $\theta(\cdot) = dF(J\cdot)$ in the following way:

$$\theta = i(F_{z_1} dz_1 + F_{z_2} dz_2 - F_{\bar{z}_1} d\bar{z}_1 - F_{\bar{z}_2} d\bar{z}_2).$$

The pseudo-Hermitian structure is a real 1-form, despite the imaginary unit i in this expression. By using Wirtinger derivatives, one can show that θ can be written as a real differential,

$$\theta = F_{x_1} dy_1 - F_{y_1} dx_1 + F_{x_2} dy_2 - F_{y_2} dx_2.$$

The CR structure and its conjugate is given by

$$H^{1,0} = TM_{\mathbb{C}} \cap T^{1,0}\mathbb{C}^2 = \text{Span}_{\mathbb{C}}\{Z = F_{z_2} \partial_{z_1} - F_{z_1} \partial_{z_2}\}.$$

$$H^{0,1} = TM_{\mathbb{C}} \cap T^{0,1}\mathbb{C}^2 = \text{Span}_{\mathbb{C}}\{\bar{Z} = F_{\bar{z}_2} \partial_{\bar{z}_1} - F_{\bar{z}_1} \partial_{\bar{z}_2}\}.$$

When we take the differential of the pseudo-Hermitian structure θ , we get the following:

$$d\theta = \frac{1}{2}(\partial + \bar{\partial}) (i(F_{z_1} dz_1 + F_{z_2} dz_2 - F_{\bar{z}_1} d\bar{z}_1 - F_{\bar{z}_2} d\bar{z}_2)).$$

Some of the terms will naturally cancel out, since $dz_j \wedge dz_j = d\bar{z}_j \wedge d\bar{z}_j = 0$. Moreover, the terms $dz_1 \wedge dz_2$ and $dz_2 \wedge dz_1$ cancel out, since $dz_1 \wedge dz_2 = -dz_2 \wedge dz_1$, and the same happens for the anti-holomorphic differentials. We are left with only the mixed wedge products consisting of both holomorphic and anti-holomorphic differentials:

$$d\theta = 2i \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} d\bar{z}_k \wedge dz_j,$$

where $F_{\bar{z}_k, z_j} = \partial_{\bar{z}_k} \partial_{z_j} F$. In the previous chapter we defined the Levi form to be $L_\theta(Z, \bar{W}) = -id\theta(Z, \bar{W})$ for $Z, W \in H^{1,0}$. We use the expression for $d\theta$ to write the Levi form as follows:

$$\begin{aligned} L_\theta(Z, \bar{W}) &= -id\theta(Z, \bar{W}) = 2 \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} d\bar{z}_k \wedge dz_j(Z, \bar{W}) \\ &= 2 \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} (d\bar{z}_k(Z) dz_j(\bar{W}) - d\bar{z}_k(\bar{W}) dz_j(Z)). \end{aligned}$$

Since $d\bar{z}_k(Z) = dz_j(\bar{W}) = 0$, we get the expression

$$L_\theta(Z, \bar{W}) = -2 \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} (dz_j \otimes d\bar{z}_k)(Z \otimes \bar{W}).$$

We want to do the same thing for the symmetric tensor G_θ , which will give us an expression for the Webster metric. Using the fact that $d\bar{z}_k(Y) = d\bar{z}_k(iJY)$, we can write

$$\begin{aligned} G_\theta(X, Y) &= d\theta(X, JY) = - \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} d\bar{z}_k(Y + iJY) dz_j(X - iJX). \\ &= -4 \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} d\bar{z}_k(Y) dz_j(X) = -4 \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} (dz_j \otimes d\bar{z}_k)(X \otimes Y). \end{aligned}$$

Since $G_\theta(X, Y)$ is real, we can do the following

$$\begin{aligned} G_\theta(X, Y) &= -2 \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} (dz_j \otimes d\bar{z}_k)(X \otimes Y) \\ &\quad - 2 \sum_{j,k=1}^2 \overline{F_{\bar{z}_k, z_j} (dz_j \otimes d\bar{z}_k)(X \otimes Y)} \end{aligned}$$

$$\begin{aligned}
&= -2 \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} (dz_j \otimes d\bar{z}_k)(X \otimes Y) - 2 \sum_{j,k=1}^2 F_{\bar{z}_j, z_k} (d\bar{z}_j \otimes dz_k)(X \otimes Y) \\
&= -2 \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} (dz_j \otimes d\bar{z}_k + d\bar{z}_k \otimes dz_j)(X \otimes Y)
\end{aligned}$$

Using that $\alpha\beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$, we can write this as

$$= -4 \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} dz_j d\bar{z}_k (X \otimes Y)$$

Writing $dz_j = dx_j + idy_j$, we can do the following calculation:

$$dz_j \otimes d\bar{z}_k + d\bar{z}_k \otimes dz_j = (dx_j + idy_j) \otimes (dx_k - idy_k) + (dx_k - idy_k) \otimes (dx_j + idy_j)$$

We have

$$dz_j \otimes d\bar{z}_k = dx_j \otimes dx_k + dy_j \otimes dy_k + idy_j \otimes dx_k - idx_j \otimes dy_k,$$

$$d\bar{z}_k \otimes dz_j = dx_k \otimes dx_j + dy_k \otimes dy_j + idx_k \otimes dy_j - idy_k \otimes dx_j.$$

Adding them together and using the symmetric product yields

$$dz_j \otimes d\bar{z}_k + d\bar{z}_k \otimes dz_j = dx_j dx_k + dy_j dy_k + i(dx_k dy_j - dx_j dy_k)$$

We can therefore write G_θ in two ways:

$$\begin{aligned}
G_\theta &= -2 \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} dz_j d\bar{z}_k \\
&= -2 \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} (dx_j dx_k + dy_j dy_k + i(dx_k dy_j - dx_j dy_k)).
\end{aligned}$$

Lastly, we want an orthonormal basis for TM . This will help us when calculating the functions for the Reeb vector field and calculating the Tanaka-Webster connection. We let $\tilde{Y} = \frac{Z + \bar{Z}}{2} \in \Gamma(\mathcal{H})$, where $Z \in H^{1,0}$ is the vector field that spans the CR distribution, as described earlier. Note that \tilde{Y} is a non-vanishing vector field. We define

$$Y = \frac{\tilde{Y}}{\|\tilde{Y}\|} = \frac{\tilde{Y}}{\sqrt{G_\theta(\tilde{Y}, \tilde{Y})}}$$

By the symmetry of g_θ , we have that

$$g_\theta(Y, Y) = g_\theta(JY, JY) = \|Y\|_{g_\theta}^2 = 1.$$

Using the definition of the Webster metric and Cartan magic formula (as seen in Example 2.1.6) on $d\theta$, we see that

$$g_\theta(Y, JY) = -d\theta(Y, Y) = -\theta([Y, Y]) = 0.$$

We also know by Proposition 4.3.4 that

$$g_\theta(\xi, \xi) = 1, \quad g_\theta(\xi, Y) = g_\theta(\xi, JY) = 0.$$

Thus we have that $\{Y, JY, \xi\}$ constitutes an orthonormal basis on TM .

5.3 The 3-sphere

5.3.1 Hopf fibrations

We want to explore S^3 and the Hopf fibration. The Hopf fibration is the following sequence:

$$S^1 \hookrightarrow S^3 \hookrightarrow \mathbb{C}\mathbb{P}^1 \cong S^2.$$

We start of by identifying $S^1 \subset \mathbb{C}$, $S^3 \subset \mathbb{C}^2$ and $S^2 \subset \mathbb{C} \times \mathbb{R}$. The spheres are the points with distance equal to 1 from the origin. The first map $S^1 \hookrightarrow S^3$ is just inclusion. We then define the map

$$p(z_1, z_2) = (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2).$$

We need to check that $p(z_1, z_2) \in S^2$ for any $(z_1, z_2) \in S^3$.

$$\sqrt{p(z_1, z_2)^2} = 4|z_1|^2|\bar{z}_2|^2 + |z_1|^4 - 2|z_1|^2|z_2|^2 + |z_2|^4 = (|z_1|^2 + |z_2|^2)^2 = 1.$$

We have that $p(z_1, z_2) = p(w_1, w_2)$ if and only if $(w_1, w_2) = (\lambda z_1, \lambda z_2)$ for $|\lambda|^2 = 1$, i.e. $\lambda \in S^1$. As a consequence, for all $q \in S^2$, $p^{-1}(q) \cong S^1$.

The Hopf fibration defines a fiber bundle $p : S^3 \rightarrow S^2$ with fibers $p^{-1}(q) \cong S^1$. Moreover, we have local trivializations $p^{-1}(U) \cong U \times S^1$, for some neighborhood $U \subset S^2$.

Another view of S^3 is as a Lie group, shown in [Lee01]. Let $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. The three left-invariant vector fields forms a basis for the tangent bundle of S^3 :

$$X_1 = -x_2\partial_{x_1} + x_1\partial_{x_2} - x_4\partial_{x_3} + x_3\partial_{x_4},$$

$$X_2 = -x_3\partial_{x_1} + x_4\partial_{x_2} + x_1\partial_{x_3} - x_2\partial_{x_4},$$

$$X_3 = -x_4\partial_{x_1} - x_3\partial_{x_2} + x_2\partial_{x_3} + x_1\partial_{x_4}.$$

5.3.2 S^3 as a CR manifold

We begin with the defining function

$$F : \mathbb{C}^2 \rightarrow \mathbb{R}, \quad z = (z_1, z_2) \mapsto \|z\|^2$$

We know that $F^{-1}(1) = S^3$. Moreover, by using the Hermitian inner product on \mathbb{C}^2 , we can write the function in the following way:

$$F(z_1, z_2) = \|z\|^2 = \langle z_1, z_1 \rangle + \langle z_2, z_2 \rangle = z_1\bar{z}_1 + z_2\bar{z}_2$$

Writing it in this way allows us to easily write the differential,

$$dF = \bar{z}_1 dz_1 + z_1 d\bar{z}_1 + \bar{z}_2 dz_2 + z_2 d\bar{z}_2$$

and

$$\theta = i(\bar{z}_1 dz_1 - z_1 d\bar{z}_1 + \bar{z}_2 dz_2 - z_2 d\bar{z}_2).$$

Using the differential again, we get

$$d\theta = i(d\bar{z}_1 \wedge dz_1 - dz_1 \wedge d\bar{z}_1 + d\bar{z}_2 \wedge dz_2 - dz_2 \wedge d\bar{z}_2).$$

Using the anti-symmetry of the wedge product, we get that

$$d\theta = 2i(d\bar{z}_1 \wedge dz_1 + d\bar{z}_2 \wedge dz_2).$$

By using the results from Section 5.2, we can then write

$$L_\theta(Z, \bar{W}) = -2(dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2)(Z, \bar{W})$$

and

$$G_\theta(X, Y) = -2(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2)(X \otimes Y).$$

We have vector fields Z, \bar{Z} given by

$$Z = \bar{z}_2 \partial_{z_1} - \bar{z}_1 \partial_{z_2}, \quad \bar{Z} = z_2 \partial_{\bar{z}_1} - z_1 \partial_{\bar{z}_2},$$

$$Y = \frac{Z + \bar{Z}}{2} = \frac{1}{2}(x_2 \partial_{x_1} - y_2 \partial_{y_1} - x_1 \partial_{x_2} + y_1 \partial_{y_2}).$$

Using that $G_\theta(X, Y) = \frac{1}{2}L_\theta(Z, \bar{W})$, we get that both

$$L_\theta(Z, \bar{Z}) = -2\|z\|^2, \quad G_\theta(Y, Y) = -\|z\|^2$$

By recalibrating the pseudo-Hermitian structure, i.e. letting $\theta_0 = -\theta$, we have a strictly pseudoconvex CR manifold with $\{Y, JY\}$ being an orthonormal basis for \mathcal{H} .

5.4 The Heisenberg group

5.4.1 Classical theory

The Heisenberg group in classical literature is usually said to be the spaces of matrices on the form

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix},$$

with matrix multiplication as the group operation. If the coefficients $a, b, c \in \mathbb{R}$, we refer to it as the *continuous Heisenberg group*, $H_3(\mathbb{R})$. If we have coefficients $a, b, c \in \mathbb{Z}$, it is referred to as the *discrete Heisenberg group*, $H_3(\mathbb{Z})$. Its Lie algebra consists of matrices on the form

$$\begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}$$

which is spanned by the basis

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A quick exercise in linear algebra gives us the relations $[X, Y] = Z$, $[X, Z] = [Y, Z] = 0$. We can generalize the 3-dimensional Heisenberg group to dimension $2n + 1$, by letting a and b be row and column vectors in \mathbb{R}^n .

The Heisenberg group is also a common example of a sub-Riemannian manifold. We have a 1-form

$$\theta = dz - \frac{1}{2}(xdy - ydx).$$

Then $\ker \theta = H$ defines a distribution, and $(H_3(\mathbb{R}), H, g)$ is a sub-Riemannian manifold.

5.4.2 CR geometry of the Heisenberg group

Now we will approach the Heisenberg group from the point of view of CR geometry. The first thing to do is to realize the Heisenberg group as a hypersurface in complex space. We can accomplish this by considering the following function:

$$F : \mathbb{C} \times \mathbb{C}^{n-1} \rightarrow \mathbb{R}, \quad (z, w) \mapsto \operatorname{Im}(z) - \|w\|^2$$

and set

$$\mathbb{H}_{2n+1} = F^{-1}(0).$$

¹ This description of the Heisenberg group is a quadric manifold, and we will see that it is a CR Lie group. Recall that for a quadric map $q : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^d$,

$$M = \{(z, w) \in \mathbb{C}^d \times \mathbb{C}^{n-d} : q(w, \bar{w}) = \text{Im}(z)\}$$

M is a quadric manifold with group operation

$$(z_1, w_1) \star (z_2, w_2) = (z_1 + z_2 + 2iq(w_1, \bar{w}_2), w_1 + w_2).$$

Observe that $q(z, w) = \sum_{j=1}^{n-1} z_j w_j$, and hence $q(w, \bar{w}) = \|w\|^2$, and for $(z, w) \in \mathbb{H}_{2n+1}$, $\|w\|^2 = \text{Im}(z)$, the very definition of a quadric manifold. We will focus on the 3-dimensional case, \mathbb{H}_3 , where we have a function $F(z, w) = w\bar{w} - \text{Im}(z)$. From this we get the group operation

$$(z_1, w_1) \star (z_2, w_2) = (z_1 + z_2 + 2iw_1\bar{w}_2, w_1 + w_2).$$

The differential of the function can be written in the following way:

$$dF = \bar{w}dw - wd\bar{w} - dy.$$

From here, we can find the psuedo-Hermitian structure

$$\theta = dx + i(wd\bar{w} - \bar{w}dw).$$

and using the differential again, we have

$$d\theta = 2i(dw \wedge d\bar{w}) = -2i(d\bar{w} \wedge dw).$$

This expression is similar to the one we had for S^3 , except it differs by a minus sign and we have only one component $d\bar{w} \wedge dw$.

$$L_\theta(Z, \bar{W}) = 2(dw \otimes d\bar{w})(Z, \bar{W}),$$

$$G_\theta(X, Y) = 2dwd\bar{w}(X \otimes Y).$$

As a consequence of Proposition 4.4.4, the left-invariant holomorphic and anti-holomorphic vector fields which spans $H^{1,0}$ and $H^{0,1}$ are given by

$$Z = \partial_w + 2i\bar{w}\partial_x, \quad \bar{Z} = \partial_{\bar{w}} - 2iw\partial_x.$$

¹Some literature, for example [DT07] uses the notation \mathbb{H}_n to denote the Heisenberg group $\mathbb{C}^n \times \mathbb{R} \subset \mathbb{C}^{n+1}$. This can be a bit misleading, since the CR manifold in question has real dimension $2n + 1$.

We see that $L_\theta(Z, \bar{Z}) = 2$, and therefore $G_\theta(Y, Y) = 1$ where $Y = \operatorname{Re}(Z)$. It is straight-forward to see that $\theta(\partial_x) = 1$ and $d\theta(\partial_x, \cdot) = 0$, so we can conclude by the uniqueness of the Reeb vector field that $\xi = \partial_x$.

The left translation of the Heisenberg group $l_{(w,t)} : \mathbb{H}_3 \rightarrow \mathbb{H}_3$ are given by

$$l_{\mathbb{C}}(\eta, s) = w + \eta, \quad l_{\mathbb{R}}(\eta, s) = t + s + 2iw\bar{\eta}$$

The function l is holomorphic in the component $l_{\mathbb{C}}$, and hence the left translations are CR map, which tells us that \mathbb{H}_3 is a CR Lie group. See [DT07] for details regarding the left translations.

The choice of CR manifolds in these last two sections is not arbitrary, but two very important examples of CR manifolds. For example, the *Schoen-Webster theorem* states that if M is a strictly pseudoconvex CR manifold whose automorphism group nonproperly, then M is either the standard sphere or the Heisenberg group, [KM09]. Another reason to look into the Heisenberg group in the context of CR manifolds is the *Lewy operator* \bar{Z} , discovered by Hans Lewy in [Lew56]. It has some interesting nonsolvability properties related to the tangential Cauchy-Riemann complex, discussed in [Bog17].

5.5 Calculating the Reeb vector field

In Section 4.2 we introduced the Reeb vector field and proved its existence and uniqueness. In this chapter we will find an explicit form for the Reeb vector field for an embedded CR(1,1)-manifold. Recall the two conditions it satisfies:

$$\theta(\xi) = 1, \quad d\theta(\xi, \cdot) = 0.$$

The first condition tells us that ξ has a component that is not in $\Gamma(\mathcal{H})$, since $\mathcal{H} = \ker \theta$. In general we also need a component of ξ that is in \mathcal{H} , which ensures the second condition. We will write the Reeb vector field in the following form:

$$\xi = \tilde{\xi} + \xi_\theta = \beta_0 \xi' + \beta_1 Y + \beta_2 JY.$$

The vector fields Y and JY are assumed to be orthonormal, as discussed in the end of Section 5.2, and $\beta_j : M \rightarrow \mathbb{R}$ for $j = 0, 1, 2$ are real-valued smooth functions. In this section we are concerned with finding β_0 , β_1 and β_2 .

5.5.1 Calculating β_0

To avoid confusion between linear connections and gradients, the notation $\text{grad}^{1,0}$ and $\text{grad}^{0,1}$ is used for the holomorphic and anti-holomorphic gradient respectively. That is,

$$\text{grad}^{1,0}F = \sum_{j=1}^n \partial_{z_j} F \partial_{z_j} \quad \text{grad}^{0,1}F = \sum_{j=1}^n \partial_{\bar{z}_j} F \partial_{\bar{z}_j}$$

As we will see in the calculations of Theorem 5.5.1, it turns out that $\tilde{\xi}$ takes the form

$$\tilde{\xi} = \beta_0 i(\text{grad}^{1,0}F - \text{grad}^{0,1}F).$$

Recall that the pseudo-Hermitian structure is then given by the differential dF :

$$\theta = i(F_{z_1} dz_1 + F_{z_2} dz_2 - F_{\bar{z}_1} d\bar{z}_1 - F_{\bar{z}_2} d\bar{z}_2)$$

Now, we are interested in what form the real-valued function $\beta_0 : M \rightarrow \mathbb{R}$ takes.

Theorem 5.5.1. *The normalizing function β_0 takes the form*

$$\beta_0 = -\frac{1}{\left(\sum_{j=1}^2 F_{z_j}^2 + F_{\bar{z}_j}^2\right)}.$$

Equivalently, using the real partial derivatives, we can write

$$\beta_0 = -\frac{2}{\left(\sum_{j=1}^2 (F_{x_j}^2 - F_{y_j}^2)\right)}.$$

Proof. We want to compute $\theta(\tilde{\xi})$ using

$$\begin{aligned} \theta(\tilde{\xi}) &= \theta(i\beta_0(\text{grad}^{1,0}F - \text{grad}^{0,1}F)) \\ &= i(F_{z_1}dz_1 + F_{z_2}dz_2 - F_{\bar{z}_1}d\bar{z}_1 - F_{\bar{z}_2}d\bar{z}_2)(i\beta_0(\text{grad}^{1,0}F - \text{grad}^{0,1}F)) \\ &= -\beta_0 \left(\sum_{j=1}^2 F_{z_j}^2 + F_{\bar{z}_j}^2 \right) = 1 \end{aligned}$$

Solving for β_0 , we get

$$\beta_0 = -\frac{1}{\left(\sum_{j=1}^2 F_{z_j}^2 + F_{\bar{z}_j}^2\right)}.$$

We can choose to write this in terms of real coordinates as well.

$$F_{z_j}^2 = \left(\frac{1}{2}(\partial_{x_j} - i\partial_{y_j})F\right)^2 = \frac{1}{4}(F_{x_j}F_{x_j} - 2iF_{x_j}F_{y_j} - F_{y_j}F_{y_j}),$$

and similarly,

$$F_{\bar{z}_j}^2 = \left(\frac{1}{2}(\partial_{x_j} + i\partial_{y_j})F\right)^2 = \frac{1}{4}(F_{x_j}F_{x_j} + 2iF_{x_j}F_{y_j} - F_{y_j}F_{y_j}).$$

Adding them together, we get

$$F_{z_j}^2 + F_{\bar{z}_j}^2 = \frac{1}{2}(F_{x_j}F_{x_j} - F_{y_j}F_{y_j}).$$

Thus, we can write β_0 in the following way:

$$\beta_0 = -\frac{2}{\left(\sum_{j=1}^2 (F_{x_j}^2 - F_{y_j}^2)\right)}.$$

□

5.5.2 Calculating β_1 and β_2

Now we turn our attention to the two functions β_1 and β_2 that will ensure us that $d\theta(\xi, X) = 0$ for any vector field X . We begin by proving an important property of the pseudo-Hermitian structure.

Lemma 5.5.2. *Despite Lie brackets not being C^∞ -linear, we still have*

$$\theta([\xi, f_1X_1 + f_2X_2]) = f_1\theta([\xi, X_1]) + f_2\theta([\xi, X_2]).$$

Furthermore, we also have that

$$\theta([\beta_1Y + \beta_2JY, X]) = \beta_1\theta([Y, X]) + \beta_2\theta([JY, X]).$$

Proof. Since $[X, Y + Z] = [X, Y] + [X, Z]$, we can write

$$\theta([\xi, f_1X_1 + f_2X_2]) = \theta([\xi, f_1X_1] + [\xi, f_2X_2]) = \theta([\xi, f_1X_1]) + \theta([\xi, f_2X_2]).$$

Now using the product rule for the Lie brackets, we get

$$= \theta(\xi(f_1)X_1 + f_1[\xi, X_1]) + \theta(\xi(f_2)X_2 + f_2[\xi, X_2]).$$

Since $\xi(f_1)X_1$ and $\xi(f_2)X_2$ vanishes on θ , we get

$$\theta([\xi, f_1X_1 + f_2X_2]) = f_1\theta([\xi, X_1]) + f_2\theta([\xi, X_2]).$$

The second assertion is obtained through the same arguments. The product rule for the Lie bracket gives us

$$\begin{aligned} \theta([\beta_1Y, X]) + \theta([\beta_2JY, X]) &= \theta(\beta_1[Y, X] - X(\beta_1)Y) + \theta(\beta_2[JY, X] - X(\beta_2)JY) \\ &= \theta(\beta_1[Y, X]) + \theta(\beta_2[JY, X]) = \beta_1\theta([Y, X]) + \beta_2\theta([JY, X]). \end{aligned}$$

□

Theorem 5.5.3. *The two remaining functions of the Reeb vector field is given by the following formulas:*

$$\beta_1 = \theta([\tilde{\xi}, JY]), \quad \beta_2 = -\theta([\tilde{\xi}, Y]).$$

Proof. Using Cartan's magic formula,

$$d\theta(\xi, X) = \xi\theta(X) - X\theta(\xi) - \theta([\xi, X]) = -\theta([\xi, X]) = 0.$$

Since $d\theta(\xi, \xi) = 0$, we can take X to be a section in \mathcal{H} . We therefore write

$$\xi = \tilde{\xi} + \beta_1 Y + \beta_2 JY, \quad X = f_1 Y + f_2 JY.$$

Proceeding with the calculations and applying Lemma 5.5.2, we can write

$$\begin{aligned} -\theta([\xi, X]) &= -\theta([\tilde{\xi} + \beta_1 Y + \beta_2 JY, X]) \\ &= -\theta([\tilde{\xi}, X]) - \beta_1 \theta([Y, X]) - \beta_2 \theta([JY, X]). \end{aligned}$$

When we now write $X = f_1 Y + f_2 JY$, we see that $\beta_1 \theta([Y, X]) = \beta_1 \theta([Y, f_1 Y + f_2 JY])$, but since $[Y, Y] = 0$, we get

$$\beta_1 \theta([Y, X]) = \beta_1 f_2 \theta([Y, JY]).$$

Similarly, we have

$$\beta_2 \theta([JY, X]) = \beta_2 f_1 \theta([JY, Y]).$$

Using Cartan's magic formula again, we can compute $\theta([Y, JY]) = -\theta([JY, Y])$.

$$\theta([Y, JY]) = -d\theta(Y, JY) = -g_\theta(Y, Y) = -1.$$

Our equation now reduced down to

$$d\theta(\xi, X) = -\theta([\tilde{\xi}, X]) + \beta_1 f_2 - \beta_2 f_1.$$

We expand the first part of the equation and again use the preceding lemma,

$$\theta([\tilde{\xi}, X]) = \theta([\tilde{\xi}, f_1 Y + f_2 JY]) = f_1 \theta([\tilde{\xi}, Y]) + f_2 \theta([\tilde{\xi}, JY]).$$

We then finally have that

$$d\theta(\xi, X) = -f_1 \theta([\tilde{\xi}, Y]) - f_2 \theta([\tilde{\xi}, JY]) + \beta_1 f_2 - \beta_2 f_1 = 0.$$

Solving for β_1 and β_2 , respectively, we have

$$\beta_1 = \theta([\tilde{\xi}, JY]), \quad \beta_2 = -\theta([\tilde{\xi}, Y]).$$

□

5.6 Calculating the Tanaka-Webster connections

We have previously introduced a connection on CR manifolds called the Tanaka-Webster connection. In this part we will first show some important properties of the Tanaka-Webster connection. Then we will write the connection in terms of connection 1-forms and find a formula for the connection forms for a strictly pseudoconvex CR 3-manifold.

5.6.1 Additional properties of ∇

Proposition 5.6.1. *The Tanaka-Webster connection satisfies the following:*

$$\nabla_X(JY) = J\nabla_X Y.$$

Proof. By definition, $(\nabla_X J)Y = \nabla_X(JY) - J\nabla_X Y$. Since we require $\nabla J = 0$, we get $\nabla_X JY - J\nabla_X Y = 0$ and thus $\nabla_X JY = J\nabla_X Y$. \square

Proposition 5.6.2.

1. $\nabla_X Y_1 \in \Gamma(\mathcal{R})$ for any $X \in \Gamma(TM)$ and $Y_1 \in \Gamma(\mathcal{R})$ (i.e. \mathcal{R} is parallel with respect to ∇)
2. $\nabla \xi = 0$
3. $\nabla \theta = 0$
4. $\nabla d\theta = 0$

Proof. (1): Let $Y_1 \in \Gamma(\text{Span}\{\xi\}) = \Gamma(\mathcal{R})$, $X \in \Gamma(TM)$ and $Y_2 \in \Gamma(\mathcal{H})$. We have that

$$g_\theta(\nabla_X Y_1, Y_2) = Xg_\theta(Y_1, Y_2) - g_\theta(Y_1, \nabla_X Y_2) = 0,$$

because $\nabla_X Y \in \Gamma(\mathcal{H})$, $\nabla g_\theta = 0$ and $g_\theta(X, \xi) = 0$. Thus, $\nabla_X Y_1 \in \Gamma(\mathcal{R})$.

(2): We know from Proposition 4.3.4 that $g_\theta(\xi, \xi) = 1$. Therefore we have that

$$g_\theta(\nabla_X \xi, \xi) = \frac{1}{2} Xg_\theta(\xi, \xi) = 0.$$

Since we know that $\nabla_X \xi \in \Gamma(\mathcal{R})$, we get that $\nabla_X \xi = g_\theta(\nabla_X \xi, \xi)\xi = 0$.

(3): Since \mathcal{H} is parallel with respect to ∇ , we get that

$$\nabla_X \theta(Y) = X\theta(Y) - \theta(\nabla_X Y) = 0.$$

For ξ , we get the following:

$$\nabla_X \theta(\xi) = X\theta(\xi) - \theta(\nabla_X \xi) = 0.$$

Using the fact that $\nabla_X \xi = 0$. Hence $\nabla \theta = 0$.

(4): Let $X \in \Gamma(TM)$, $Y_1, Y_2 \in \Gamma(\mathcal{H})$.

$$\nabla_X d\theta(Y_1, Y_2) = X d\theta(Y_1, Y_2) - d\theta(\nabla_X Y_1, Y_2) - d\theta(Y_1, \nabla_X Y_2).$$

By using the definition of the Webster metric $g_\theta(X, Y) = d\theta(X, JY)$, we can write $d\theta(X, Y) = -g_\theta(X, JY)$:

$$= -X g_\theta(Y_1, JY_2) + g_\theta(\nabla_X Y_1, JY_2) + g_\theta(Y_1, J\nabla_X Y_2) = g_\theta(Y_1, -\nabla_X JY_2 + J\nabla_X Y_2).$$

By Proposition 5.6.1, we have that $\nabla_X JY = J\nabla_X Y$, so

$$= g_\theta(Y_1, -\nabla_X JY_2 + \nabla_X JY_2) = 0.$$

Since \mathcal{H} and \mathcal{R} are parallel with respect to ∇ and $d\theta(\xi, Y) = 0$ for $Y \in \Gamma(\mathcal{H})$,

$$\nabla_X d\theta(\xi, Y_1) = X d\theta(\xi, Y_1) - d\theta(\nabla_X \xi, Y_1) - d\theta(\xi, \nabla_X Y_1) = 0.$$

□

To summarize, here are the properties that we know the Tanaka-Webster connection satisfies:

1. \mathcal{H} is parallel with respect to ∇ .
2. $\nabla J = 0$.
3. $\nabla g_\theta = 0$.
4. $\tau(X, Y) = d\theta(X, Y)\xi$ for all $X, Y \in \mathcal{H}$.
5. $\tau_\xi \circ J + J \circ \tau_\xi = 0$.
6. $\nabla_X JY = J\nabla_X Y$.
7. $\text{Span}\{\xi\}$ is parallel with respect to ∇ .
8. $\nabla \xi = 0$.
9. $\nabla \theta = 0$.
10. $\nabla d\theta = 0$.

5.6.2 Connection 1-forms

Throughout this section, $\{Y, JY, \xi\}$ denotes an orthonormal basis for TM . By using linearity and the product rule for the connection, if we know the connection in the basis $\{Y, JY, \xi\}$, we know it for any vector field of TM . We now introduce two 1-forms $\Gamma_1, \Gamma_2 : TM \rightarrow \mathbb{R}$ in the orthonormal basis $\{Y, JY, \xi\}$. We don't need a connection form for ξ , since $\nabla\xi = 0$. These connection forms uniquely determine the Tanaka-Webster connection:

$$\nabla_X Y = \Gamma_1(X)Y + \Gamma_2(X)JY + 0\xi = \Gamma_1(X)Y + \Gamma_2(X)JY.$$

Using the property $\nabla_X JY = J\nabla_X Y$, we get

$$\nabla_X JY = -\Gamma_2(X)Y + \Gamma_1(X)JY.$$

Proposition 5.6.3. *The first connection 1-form vanishes, i.e.*

$$\Gamma_1(X) = 0.$$

Proof. Want to use compatibility with metric, i.e. $\nabla g_\theta = 0$.

$$\nabla_X g_\theta(Y, Y) = g_\theta(\nabla_X Y, Y) + g_\theta(Y, \nabla_X Y) = 2g_\theta(\nabla_X Y, Y).$$

Using the definition of the connection and that $\{Y, JY\}$ is an orthonormal basis, we can write

$$\nabla_X g_\theta(Y, Y) = 2g_\theta(\Gamma_1(X)Y + \Gamma_2(X)JY, Y) = 2\Gamma_1(X).$$

Since $g_\theta(Y, Y) = 1$, we know that $\nabla_X g_\theta(Y, Y) = 0$. Thus, $\Gamma_1(X) = 0$. \square

In order for us to determine ∇ , we now only need to know $\Gamma_2(Y)$, $\Gamma_2(JY)$ and $\Gamma_2(\xi)$.

Theorem 5.6.4. *We have the following formulas for the connection 1-form in the direction of Y and JY :*

$$\Gamma_2(Y) = -g_\theta([Y, JY], Y), \quad \Gamma_2(JY) = -g_\theta([Y, JY], JY).$$

Proof. We will use the torsion $\tau(Y, JY)$ of ∇ to determine $\Gamma_2(Y)$.

$$\tau(Y, JY) := \nabla_Y JY - \nabla_{JY} Y - [Y, JY] = -\Gamma_2(Y)Y - \Gamma_2(JY)JY - [Y, JY].$$

Now using that in an orthonormal basis, $x = \sum_{j=1}^n \langle x, x_j \rangle x_j$, we can write

$$[Y, JY] = g_\theta([Y, JY], Y)Y + g_\theta([Y, JY], JY)JY + g_\theta([Y, JY], \xi)\xi.$$

The last term can be simplified in the following way:

$$\begin{aligned} g_\theta([Y, JY], \xi) &= G_\theta(\pi_{\mathcal{H}}[Y, JY], \pi_{\mathcal{H}}\xi) + \theta([Y, JY])\theta(\xi) \\ &= \theta([Y, JY]) = -d\theta(Y, JY) = -g_\theta(Y, Y) = -1. \end{aligned}$$

Hence $g_\theta([Y, JY], \xi)\xi = -\xi$. On the other hand, we know that τ has pure torsion, so

$$\tau(Y, JY) = d\theta(Y, JY)\xi = g_\theta(Y, Y)\xi = \xi.$$

Using the definition of the Webster metric. Through the definition of torsion and the condition that the torsion is pure, we have

$$\begin{aligned} \xi = \tau(Y, JY) &:= -\Gamma_2(Y)Y - \Gamma_2(JY)JY - [Y, JY] \\ &= -\Gamma_2(Y)Y - \Gamma_2(JY)JY - g_\theta([Y, JY], Y)Y - g_\theta([Y, JY], JY)JY + \xi. \end{aligned}$$

The ξ on each side cancels out as expected, since \mathcal{H} is required to be parallel. Rearranging the equation yields

$$\Gamma_2(Y)Y + \Gamma_2(JY)JY = -g_\theta([Y, JY], Y)Y - g_\theta([Y, JY], JY)JY,$$

and we see that

$$\Gamma_2(Y) = -g_\theta([Y, JY], Y), \quad \Gamma_2(JY) = -g_\theta([Y, JY], JY).$$

□

We still have to find an expression for $\Gamma_2(\xi)$. As a short-hand notation, we write $\tau_\xi = \tau(\xi, \cdot)$, which is the pseudo-Hermitian torsion of the Tanaka-Webster connection.

Lemma 5.6.5. *We have the following formula:*

$$J[\xi, Y] = [\xi, JY].$$

Proof. For Y , we have

$$\tau_\xi(Y) = \tau(\xi, Y) = \nabla_\xi Y - \nabla_Y \xi - [\xi, Y] = \Gamma_2(\xi)JY - [\xi, Y]$$

Similarly for JY ,

$$\tau_\xi(JY) = \nabla_\xi JY - \nabla_{JY} \xi - [\xi, JY] = -\Gamma_2(\xi)Y - [\xi, JY]$$

Since $J\nabla_X Y = \nabla_X JY$, we get that $J[\xi, Y] = [\xi, JY]$.

□

Theorem 5.6.6. *We have the following formula for the connection 1-form in the direction of ξ :*

$$\Gamma_2(\xi) = g_\theta([\xi, Y], JY).$$

Proof. To find $\Gamma_2(\xi)$, we will use the definition of the Lie derivative.

$$\mathcal{L}_\xi g_\theta(Y, JY) = \xi g_\theta(Y, JY) - g_\theta([\xi, Y], JY) - g_\theta(Y, [\xi, JY]).$$

Compatibility with metric gives us that

$$\xi g_\theta(Y, JY) = g_\theta(\nabla_\xi Y, JY) + g_\theta(Y, \nabla_\xi JY) = g_\theta(\Gamma_2(\xi)JY, JY) + g_\theta(-\Gamma_2(\xi)Y, Y).$$

On one hand, we have that $g_\theta(\Gamma_2(\xi)Y, Y) = g_\theta(\Gamma_2(\xi)JY, JY) = \Gamma_2(\xi)$, so we get

$$\begin{aligned} \mathcal{L}_\xi g_\theta(Y, JY) &= g_\theta(\Gamma_2(\xi)JY, JY) + g_\theta(-\Gamma_2(\xi)Y, Y) - g_\theta([\xi, Y], JY) - g_\theta(Y, [\xi, JY]) \\ &= \Gamma_2(\xi) - \Gamma_2(\xi) - g_\theta([\xi, Y], JY) - g_\theta(Y, [\xi, JY]) = -g_\theta([\xi, Y], JY) - g_\theta(Y, [\xi, JY]). \end{aligned}$$

Since $[\xi, JY] = J[\xi, Y]$, we have that

$$\begin{aligned} -g_\theta([\xi, Y], JY) - g_\theta(Y, [\xi, JY]) &= -g_\theta([\xi, Y], JY) - g_\theta(J[\xi, Y], Y) \\ &= -g_\theta([\xi, Y], JY) - g_\theta(-[\xi, Y], JY) = 0. \end{aligned}$$

On the other hand, we the condition that τ_ξ is pure tells us that $\tau_\xi \circ J + J \circ \tau_\xi = 0$, which is equivalent to $J \circ \tau_\xi \circ J = \tau_\xi$;

$$J\tau_\xi(JY) = J(-\Gamma_2(\xi)Y - [\xi, JY]) = -\Gamma_2(\xi)JY + [\xi, Y]$$

We can write the same Lie derivative in the following way:

$$\begin{aligned} \mathcal{L}_\xi g_\theta(Y, JY) &= g_\theta(\Gamma_2(\xi)JY, JY) + g_\theta(-\Gamma_2(\xi)Y, Y) - g_\theta([\xi, Y], JY) - g_\theta(Y, [\xi, JY]) \\ &= g_\theta(\Gamma_2(\xi)JY - [\xi, Y], JY) + g_\theta(-\Gamma_2(\xi)JY + [\xi, Y], JY) \\ &= g_\theta(\tau_\xi(Y), JY) + g_\theta(J\tau_\xi(JY), JY) = 2g_\theta(\tau_\xi(Y), JY). \end{aligned}$$

Putting these two equations together, we have that

$$\begin{aligned} 0 &= 2g_\theta(\tau_\xi(Y), JY) = 2g_\theta(\Gamma_2(\xi)JY - [\xi, Y], JY) \\ &= 2g_\theta(\Gamma_2(\xi)JY, JY) - 2g_\theta([\xi, Y], JY). \end{aligned}$$

Thus, we finally come to the conclusion that

$$\Gamma_2(\xi) = g_\theta([\xi, Y], JY).$$

□

5.6.3 Pseudo-holomorphic curvature of \mathcal{H}

We will be using the same basis as in the previous section; $\{Y, JY, \xi\}$. The goal is to determine an expression of the pseudo-holomorphic sectional curvature of the Levi distribution with respect to the Tanaka-Webster connection, in the case of CR(1, 1) manifold.

Lemma 5.6.7. *The Riemann curvature tensor $R(Y, JY)JY$ of the Tanaka-Webster connection is given by*

$$R(Y, JY)JY = JY(\Gamma_2(Y))Y - Y(\Gamma_2(JY))Y + \Gamma_2([Y, JY])Y.$$

Proof. By definition, the Riemann curvature tensor is given by

$$R(Y, JY)JY = \nabla_Y \nabla_{JY} JY - \nabla_{JY} \nabla_Y JY - \nabla_{[Y, JY]} JY.$$

Recall that $\nabla_X Y = \Gamma_2(X)Y$. We have that the first part of the curvature tensor is given by

$$\begin{aligned} \nabla_Y \nabla_{JY} JY &= -\nabla_Y \Gamma_2(JY)Y \\ &= -(Y(\Gamma_2(JY))Y + \Gamma_2(JY)\nabla_Y Y) \\ &= -(Y(\Gamma_2(JY))Y + \Gamma_2(JY)\Gamma_2(Y)JY) \end{aligned}$$

and

$$\begin{aligned} \nabla_{JY} \nabla_Y JY &= -\nabla_{JY} \Gamma_2(Y)Y \\ &= -(JY(\Gamma_2(Y))Y + \Gamma_2(Y)\nabla_{JY} JY) \\ &= -(JY(\Gamma_2(Y))Y + \Gamma_2(Y)\Gamma_2(JY)JY) \end{aligned}$$

For the third part of the curvature tensor, we get

$$\nabla_{[Y, JY]} JY = -\Gamma_2([Y, JY])Y.$$

Adding them all together, we have

$$R(Y, JY)JY = JY(\Gamma_2(Y))Y - Y(\Gamma_2(JY))Y + \Gamma_2([Y, JY])Y.$$

□

Theorem 5.6.8. *The pseudo-holomorphic sectional curvature of \mathcal{H} with respect to the Tanaka-Webster connection is given by*

$$\begin{aligned} \kappa(\mathcal{H}) &= -JY g_\theta([Y, JY], Y) + Y g_\theta([Y, JY], JY) \\ &\quad - g_\theta([Y, JY], Y)^2 - g_\theta([Y, JY], JY)^2 - g_\theta([\xi, Y], JY). \end{aligned}$$

Proof. We want to prove this by using the formula for pseudo-holomorphic sectional curvature from Section 4.5. Since Y and JY are unit vector fields, the formula simplifies to

$$\kappa(\mathcal{H}) = g_\theta(R(Y, JY)JY, Y)$$

Recall that

$$\begin{aligned}\Gamma_2(Y) &= -g_\theta([Y, JY], Y) \\ \Gamma_2(JY) &= -g_\theta([Y, JY], JY) \\ \Gamma_2(\xi) &= g_\theta([\xi, Y], JY)\end{aligned}$$

Using these formulas and Lemma 5.6.7, we get that

$$\begin{aligned}\kappa(\mathcal{H}) &= g_\theta(JY(\Gamma_2(Y))Y, Y) - g_\theta(Y(\Gamma_2(JY))Y, Y) + g_\theta(\Gamma_2([Y, JY])Y, Y) \\ &= JY(\Gamma_2(Y)) - Y(\Gamma_2(JY)) + \Gamma_2([Y, JY]) \\ &= -JYg_\theta([Y, JY], Y) + Yg_\theta([Y, JY], JY) + \Gamma_2([Y, JY])\end{aligned}$$

We again use the fact that we can decompose the bracket $[Y, JY]$ with respect to the orthonormal basis, giving us

$$[Y, JY] = g_\theta([Y, JY], Y)Y + g_\theta([Y, JY], JY)JY - \xi.$$

When substituting this expression and using linearity of Γ_2 , we have get

$$\Gamma_2([Y, JY]) = -g_\theta([Y, JY], Y)^2 - g_\theta([Y, JY], JY)^2 - g_\theta([\xi, Y], JY).$$

When we add the parts together we get

$$\begin{aligned}\kappa(\mathcal{H}) &= -JYg_\theta([Y, JY], Y) + Yg_\theta([Y, JY], JY) \\ &\quad - g_\theta([Y, JY], Y)^2 - g_\theta([Y, JY], JY)^2 - g_\theta([\xi, Y], JY).\end{aligned}$$

□

5.7 Summary

We end the chapter with a short summary of how to calculate the important properties of a CR(1, 1) manifold. We start by letting

$$F : \mathbb{C}^2 \longrightarrow \mathbb{R}.$$

We assume that F is smooth and $dF \neq 0$. The differential can be written as

$$dF = \sum_{j=1}^2 F_{z_j} dz_j + F_{\bar{z}_j} d\bar{z}_j$$

1. The pseudo-Hermitian structure is given as

$$\theta = i \left(\sum_j^2 F_{z_j} dz_j - F_{\bar{z}_j} d\bar{z}_j \right).$$

(One may multiply θ by a smooth non-vanishing function to ensure the Levi form and Webster metric to be positive definite.)

2. The differential of the psuedo-Hermitian structure:

$$d\theta = 2i \left(\sum_{j,k=1}^2 F_{z_j, \bar{z}_k} d\bar{z}_k \wedge dz_j \right).$$

3. The CR structure is spanned by a single holomorphic vector field Z :

$$H^{1,0} = \text{Span}_{\mathbb{C}}\{Z = F_{z_2} \partial_{z_1} - F_{z_1} \partial_{z_2}\}, \quad H^{0,1} = \overline{H^{1,0}}.$$

4. Both L_θ and G_θ are defined by using $d\theta$.

$$L_\theta(Z, \bar{W}) = -2 \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} dz_j \otimes d\bar{z}_k (Z \otimes \bar{W}), \quad Z, W \in H^{1,0}.$$

We can write G_θ using the symmetric product, either in terms of dz_j and $d\bar{z}_j$ or in terms of dx_j and dy_j :

$$\begin{aligned} G_\theta &= -4 \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} dz_j d\bar{z}_k \\ &= -4 \sum_{j,k=1}^2 F_{\bar{z}_k, z_j} (dx_j dx_k + dy_j dy_k) + i(dx_k dy_j - dx_j dy_k) \end{aligned}$$

5. We let $\tilde{Y} = \frac{Z+\bar{Z}}{2} \in \mathcal{H}$. We can define a unit vector field by letting $Y = \frac{\tilde{Y}}{\|\tilde{Y}\|} = \frac{\tilde{Y}}{\sqrt{G_\theta(\tilde{Y}, \tilde{Y})}}$. Then $\{Y, JY\}$ defines an orthonormal basis for \mathcal{H} .
6. Given an orthonormal basis $\{Y, JY\}$ for \mathcal{H} , The Reeb vector field takes the form

$$\xi = \tilde{\xi} + \beta_1 Y + \beta_2 JY,$$

where

$$\tilde{\xi} = \beta_0 i(\text{grad}^{1,0} F - \text{grad}^{0,1} F).$$

We have that the β_0 is given by the following formula:

$$\beta_0 = -\frac{1}{\left(\sum_{j=1}^2 F_{z_j}^2 + F_{\bar{z}_j}^2\right)}.$$

We also have that β_1 and β_2 are given by these formulas:

$$\beta_1 = \theta([\tilde{\xi}, JY]), \quad \beta_2 = -\theta([\tilde{\xi}, Y]).$$

7. With the Reeb vector field we can define a Riemannian metric on TM :

$$g_\theta(X, Y) = G_\theta(\pi_{\mathcal{H}}X, \pi_{\mathcal{H}}Y) + \theta(X)\theta(Y), \quad X, Y \in TM.$$

8. The Tanaka-Webster connection can be expressed with a single connection 1-form:

$$\nabla_X Y = \Gamma_2(X)JY.$$

In the direction of $Y, JY \in \mathcal{H}$, we have

$$\Gamma_2(Y) = -g_\theta([Y, JY], Y), \quad \Gamma_2(JY) = -g_\theta([Y, JY], JY).$$

In the direction of the Reeb vector field we have

$$\Gamma_2(\xi) = g_\theta([\xi, Y], JY).$$

9. The pseudo-holomorphic sectional curvature is of the Levi distribution \mathcal{H} is given

$$\begin{aligned} \kappa(\mathcal{H}) = & -JY g_\theta([Y, JY], Y) + Y g_\theta([Y, JY], JY) \\ & - g_\theta([Y, JY], Y)^2 - g_\theta([Y, JY], JY)^2 - g_\theta([\xi, Y], JY). \end{aligned}$$

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