Relaxation Schemes for degenerate parabolic convection-diffusion equations with a discontinuous coefficient

Master Thesis in Applied Mathematics

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November 20, 2008

Preface

I want to thank all the people who have been supportive during my master study.

I would like to thank Kenneth W. Karlsen who introduced me to an interesting application of relaxation models.
Especially I thank my supervisor Henrik Kalisch who helped me finish the thesis with his valuable suggestions and comments, and for providing help with the implementations.

David Rabenorolahy.
Bergen, november 2008.

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Introduction

The theory of partial differential equations (PDE) is a subject that has found its way into all branches of science and engineering due to its wide range of applications. Numerical calculation is commonplace today in fields where it was virtually unknown before 1950. Some consider the celebrated 1928 paper of Courant, Friedrichs and Lewy as the birthdate of the modern theory of numerical methods for partial differential equations. The algebraic solution of finite difference approximations is best accomplished by some iteration procedure. Finite difference approximations for derivatives were already in use by Euler in 1768 and various schemes have been proposed to accelerate the convergence of the iteration.

Accurate modelling of the interaction between convective and diffusive processes is a challenging task in the numerical approximation of PDE. This is partly because of the problems themselves, their great variety and widespread occurrence. Mathematical models that involve a combination of convective and diffusive processes are among the area of prime research interest and widespread in all of science, engineering and other fields where mathematical modelling is important. Very often the dimensionless parameter that measures the relative strength of the diffusion is quite small, so one often meets with situations where thin boundary and interior layers are present and singular perturbation problems arise. There are many physical systems in which parabolic equations are coupled to hyperbolic equations so that two (or more) transport phenomena must be calculated simultaneously. Problems which incorporate ideal fluid motion and some other transport process, such as heat transfer, have mathematical models which are coupled equations of mixed parabolic-hyperbolic type. In all such circumstances difficulties will be experienced with standard numerical approximations. Thus a very large literature has built up over the last few decades on a variety of techniques for analysing and overcoming these difficulties.

Discontinuous solutions do not satisfy the partial differential equation in the classical sense at all points, since the derivatives are not defined at discontinuities. We have to define what we mean by a solution to a conservation law in this case.
Since the partial differential equation continues to hold except at discontinuities, we supply the differential equations by additional "jump condition" that must be satisfied across discontinuities. We may derive additional conditions using the integral form of the conservation law since the integral form continues to be valid even for discontinuous solutions. Unfortunately integral forms are more difficult to work with than differential equations, especially in terms of discretizations schemes. Another mathematical difficulty is possible nonuniqueness of solutions . Often there is more than one solution to the conservation law with the same initial data. This is a consequence due to the physical effects ignored because equations are only simplified models of reality. To obtain unique and hopefully physical correct solutions, we have to supply an additional condition. This so-called entropy condition will help us to pick the correct so-called weak solution to the original partial differential equations. In general it is not possible to derive exact solutions to these equations, so we need to devise and study numerical methods for their approximate solu-

tion. The general theory of numerical methods for nonlinear PDEs applies in particular to systems of conservation laws, but there are several reasons for studying this particular class of equations. Many practical problems in science and engineering involve conserved quantities and lead to PDEs of this class. As noted above, there are special difficulties like shock formation associated with these time dependent systems of nonlinear partial differential equations. When we attempt to calculate weak solutions numerically, we face several problems. Most important, the discretization schemes for the PDE must be able to handle discontinuities in the solution. Ideally we would like to have a simple numerical method producing sharp approximations to discontinuous solutions without excessive smearing. However, simple methods like the first order upwind schemes typically produce excessive numerical smearing.

Relaxation approximation to nonlinear partial differential equations have been introduced on the basis of the replacement of the equations with a suitable semilinear hyperbolic system with stiff relaxation terms. Relaxation schemes are a class of nonoscillatory numerical schemes for systems of conservation laws proposed by Jin and Xin [22]. They are motivated by relaxation models for flow which are not in thermodynamic equilibrium, i.e. they constitute more general and more accurate models of certain physical phenomena. The main advantage of numerically solving the relaxation model over the original conservation laws lies in the simple structure of the linear characteristic fields and the localised lower order term. In particular, the semilinear nature of the relaxation system gives a new way to develop numerical schemes that are simple, general and Riemann solver free. The Riemann solver is more accurate, but the price to pay is that the numerical methods become complicated to implement and time-consuming. The approach is inspired by relaxation schemes where the nonlinearity inside the equation is replaced by a semilinearity. This reduction is carried out in order to obtain numerical schemes that are easy to implement, also for more general and complex problems.

In Chapter 1 we start out by looking at the Initial Value Problem for a one-dimensional scalar nonlinear degenerate parabolic convection-diffusion equation. We introduce the mixed parabolic-hyperbolic problem before moving to the numerical methods we will use to approximate the Initial Value problem. The chapter includes also some basic definitions and notations, assumptions on the data of the parabolic convection-diffusion equation. In Chapter 2 we establish the existence of weak solutions. Chapter 3 presents the Relaxation Schemes for both the pure hyperbolic case and the diffusion equation. In Chapter 4 we derive $\mathbf{L}^{\infty}$ and $\mathbf{L}^{\mathbf{1}}$ estimates on the approximate solutions of the relaxation system and we state the Entropy solutions. Then we develop the numerical schemes in Chapter 5, while a convergence result is proved in Chapter 6. In Chapter 7 we present numerical experiments. Finally, we summarise the conclusions and look at possible improvements and further work.

Chapter 1

Initial Value Problem

The aim of this work is to analyse from both a theoretical and computational point of view the relaxation schemes to approximate the Initial Value problem for a one-dimensional scalar nonlinear degenerate parabolic convection-diffusion equations of the type

$$\begin{cases} u_t + f(\gamma(x), u)_x = B(u)_{xx}, & (x,t) \in \mathbf{\Pi_T} = \mathbf{R} \times (\mathbf{0}, \mathbf{T}). \\ u(x,0) = u_0(x) & x \in \mathbf{R}. \end{cases} \tag{1.1}$$

The special aspect for this problem is the combination between the convective part and the diffusion part $B(u)_{xx}$. The nonlinear convective flux function depends explicitly on spatial location through the coefficient $\gamma(x)$, that may be discontinuous. The diffusion function $B(u)_{xx}$ is allowed to be strongly degenerate, in the sense that $B'(\cdot) \geq 0$.

When we list the assumptions for the problem in section (1.3), we will see that the closely related hyperbolic conservation laws with a discontinuous coefficient will also be included. The purely convective version of (1.1) is obtained when $B'(u) \equiv 0$, which means that the diffusion part degenerates, i.e. B'(u) may vanish for some values for u.

$$u_t + f(\gamma(x), u)_x = 0. \tag{1.2}$$

Parabolic Convection-Diffusion equations (1.1) are of great importance since they govern a variety of physical phenomena. To name a few of the interesting problems of the type (1.1), we mention fluid mechanics, flow in porous media, sedimentation-consolidation processes. A physical model corresponding to the convective version of (1.1) is the model of car traffic flow in a highway. The spatially varying coefficient γ corresponds to changing road conditions. We mention also applications modeling the displacement of oil in a reservoir by water and polymer. Multiphase flow problems in porous materials give rise to somewhat difficult systems of conservation laws. One important application area is secondary oil recovery, in which water is pumped down one well in an effort to force more oil out of

other wells, see [25]. The convection-diffusion equations also arise in front propagation and financial modeling.

Computation of certain physical problems, for example a fluid flow, requires numerical resolution of the small scales in order to track the underlying physical properties in great detail. This is rather demanding and often cannot be done satisfactorily. Thus one would be satisfied if critical properties are captured (at the macroscopic level). In such a case, it is desirable to design *underresolved* numerical methods. We aim to develop schemes that allow the use of underresolved discretization, $\Delta t >> \epsilon$. We will develop numerical methods that are suitable for underresolved calculation, meaning that one can still capture the macroscopic physical behaviour without numerically solving the small scale by using mesh size and time step much larger than the small scale parameter. Such a solution is referred to as the underresolved solution.
Using the same notations as Jin and Xin [22], we call the discretization of the Relaxation systems Relaxing Schemes, which depends on ϵ and the artificial variable v. We also derive zero relaxation limit of these Relaxing Schemes and call the limiting schemes the Relaxed Schemes, obtained in the limit $\epsilon \to 0$. By applying the Chapman-Enskog expansion to the relaxing schemes (for fixed grids and $\epsilon \to 0$) we can also formally derive the relaxed schemes that are the leading order approximation of the relaxing schemes in the small ϵ limit. These relaxed schemes are consistent and stable discretizations of the original conservation laws. Here by Relaxation Schemes we indicate both the relaxing Schemes and the relaxed schemes. When ϵ is very small the relaxing schemes and the relaxed schemes produce essentially the same results.

We will propose a diffusive relaxation approximation for the nonlinear parabolic diffusive equation, based on the same idea used on hyperbolic conservation laws. A splitting method approach to the problem will also be considered. Several relaxation approximation to partial differential equations have been recently proposed. We have seen earlier numerical approaches that work for relaxation systems where both the relaxation term and the convection term are stiff, schemes that work independently on ϵ.
The idea in this work is to study how these diffusive relaxation schemes perform when we use them on a mixed convection-diffusion problem where the flux function has a discontinuous coefficient and the diffusion part may degenerate, which then changes the problem to a purely hyperbolic case. We will concentrate the study on the stiff regime for the system, where $\epsilon << 1$, and the flux function is convex. We will also propose the relaxed schemes for these numerical approximations and compare their ability to capture the parabolic behaviour.
In the hyperbolic case, we will study a different approach proving the convergence of the approximate solution $\mathbf{u(x,t)}$. We attempt to establish convergence of the relaxation approximation (3.1) using the Singular Mapping approach. In the literatures analysing numerical approximations and in some of the papers cited in this work, convergence of numerical methods for conservation laws with discontinuous coefficients has been estab-

lished by the singular mapping approach and compensated compactness method. Herein we use instead the singular mapping to study the behaviour of the relaxed scheme since the leading order behaviour of the relaxing schemes is governed by the relaxed schemes as $\epsilon \to 0^+$. We will show that the piecewise constant function constructed by the solution of the relaxed scheme converges to the entropy solution of the problem.

1.1 Basic Definitions and Notation

In this section we define notations that will be used in the text and we introduce some definitions.

supp f: The support of a continuous function $\mathbf{f(x)}$ defined on $\mathbf{R^n}$ is the closure of the set of points where $\mathbf{f(x)}$ is nonzero: $\mathbf{supp\ f} = \mathbf{x} \in \mathbf{R^n} : \mathbf{f(x)} \neq \mathbf{0}$.

A set in $\mathbf{R^n}$ is bounded if it is contained in a ball $\mathbf{B_R(0)}$ with $\mathbf{R}$ sufficiently large. The closed bounded sets in $\mathbf{R^n}$ are the compact sets. If **supp f** is bounded, we say **f** has compact support and denote such functions by $\mathbf{C_0(R^n)}$.
sup: The supremum norm: $\|\mathbf{v}\|_\infty = \mathbf{sup}|\mathbf{v(x)}|$.

We have a domain $\mathbf{\Omega} \subset \mathbf{R^n}$:

$\mathbf{L^1(\Omega)}$: A function **f** defined on a domain $\mathbf{\Omega}$ is integrable if $\int_{\mathbf{\Omega}} |\mathbf{f(x)}|\mathbf{x}$ is defined and finite. We denote all such functions by $\mathbf{L^1(\Omega)}$.
$\mathbf{L^1_{loc}(\Omega)}$: When we consider a larger space. Functions which are "locally" integrable: Integrable on any compact subset of $\mathbf{\Omega}$, but not necessarily integrable at the boundary of $\mathbf{\Omega}$ or at infinity.
$\mathbf{L^\infty(\Omega)}$: measurable real valued functions which are bounded. $\mathbf{C(\Omega)}$ and $\mathbf{C^1(\Omega)}$: We denote the continuous functions on $\mathbf{\Omega}$ by $\mathbf{C(\Omega)}$, and those whose first order derivatives are also continuous by $\mathbf{C^1(\Omega)}$. Similarly $\mathbf{C^k(\Omega)}$ denotes the functions having all derivatives up to the order **k** continuous on $\mathbf{\Omega}$.
Lipschitz continuity : We use this property when we require a certain amount of smoothness in the function. A function $\mathbf{f(u,t)}$ is Lipschitz continuous in **u** over some range of **t** and **u**, if there exists some constant L>0 so that

$$|f(u,t) - f(u^*,t)| \leq L|u - u^*|$$

for all **u** and $\mathbf{u}^*$.
This is slightly stronger than mere continuity, which only requires that
$|f(u) - f(u^*)| \to 0 \quad$ as $u \to u^*$.
If $\mathbf{f(u,t)}$ is differentiable with respect to **u** and the derivative $\mathbf{f_u} = \partial\mathbf{f}/\partial\mathbf{u}$ is bounded then we can take $\mathbf{L} = \mathbf{max}|\mathbf{f_u}(\mathbf{u},\mathbf{t})|$.
Lipschitz constant: The size of the Lipschitz constant is important when we intend to solve the problem numerically since our numerical approximation will almost certainly produce a value $\mathbf{u^n}$ at time $\mathbf{t_n}$ that is not exactly equal to the true value $\mathbf{u(t_n)}$. Hence we are

on a different curve than the true solution.
The Lipschitz constant gives an indication of whether solution curves that start close together can be expected to stay close together or to diverge rapidly.

Nonlinear Stability. When we attempt to solve nonlinear conservation laws numerically we run into additional difficulties not seen in the linear equation. Moreover, the nonlinearity makes everything harder to analyse. For nonlinear problems the method might be "nonlinearly unstable", i.e., unstable on the nonlinear problem even though linearised versions appear to be stable. Often oscillations will trigger nonlinear instabilities. In order to prove a convergence result we must define an appropriate notion of stability. Convergence notions and discussions done by Lax and Wendroff suggest that we can hope to correctly approximate discontinuous weak solutions to the conservation law by using a conservative method. The theorem does not guarantee convergence, nontheless, this is a very powerful and important theorem, for it says that we can have confidence in solutions we compute.

Theorem 1.1.1. *(LAX-WENDROFF) Consider a sequence of grids indexed by l= 1,2,..., with mesh parameters* $k_l, h_l \to 0$ *as* $l \to \infty$. *Let* $U_l(x,t)$ *denote the numerical approximation computed with a consistent and conservative method on the lth grid. Suppose that* $\mathbf{U_l}$ *converges to a function u as* $l \to \infty$. *Then u(x,t) is a weak solution of the conservation law.*

The above theorem suggests that we can hope to correctly approximate discontinuous weak solutions to the conservative law by using a conservative method. Lax and Wendroff proved that this is true, at least in the sense that if we converge to some function $\mathbf{u(x,t)}$ as the grid is refined, through some sequence $\mathbf{k_l}, \mathbf{h_l} \to \mathbf{0}$, then this function will in fact be a weak solution of the conservation law. But the Lax-Wendroff [10] theorem does not say anything about whether the method converges, only that if a sequence of approximations converges then the limit is a weak solution. To guarantee convergence, we need some form of stability.
The Lax Equivalence Theorem can no longer be used to prove convergence since that approach relies heavily on linearity. For nonlinear problems the primary tools used to prove convergence is *compactness*. We will define this concept and indicate its use for our goals of defining stability and proving convergence.
In relation to our goals of defining stability and proving convergence, we will use the important property that compactness guarantees the existence of convergent subsequences and combine it with the Lax-Wendroff Theorem. To get convergence of the whole sequence in question and not just the subsequence, we need to combine the suitable compactness argument with a uniqueness result.

If $\mathbf{K}$ is a compact set in some normed space, then any infinite sequence of elements of $\mathbf{K}$,$\{k_1, k_2, k_3, \ldots\}$ contains a subsequence which converges to an element of $\mathbf{k}$. This means that from the original sequence we can, by selecting certain elements from this sequence, construct a new infinite sequence

$\{k_{i1}, k_{i2}, k_{i3}, \ldots\} \qquad (i_1 < i_2 < i_3 < \ldots)$, which converges to some element $\mathbf{k} \in \mathbf{K}$.

$\| k_{ij} - k \| \to 0 \qquad$ when $\quad j \to \infty$.

Total Variation Stability.
Let $\mathbf{U_k}$ denote the numerical approximation generated by a numerical method in conservative form. We measure the global error in our approximation by the distance from $\mathbf{U_k(x,t)}$ to the set of all weak solutions
$\mathcal{W} = \{\omega{:}\omega(x,t)$ is a weak solution to the conservation law$\}$. To measure this distance we need a norm, for example the 1-norm over the finite time interval [0,T]. The global error is then defined by

$$dist(U_k, \mathcal{W}) = inf_{\omega \in \mathcal{W}} \|U_k - \omega\|_{1,T}.$$

If $\mathbf{U_k}$ is generated by a numerical method in conservation form, consistent with the conservation law, and if the method is, stable in some appropriate sense, then $\mathbf{dist}(\mathbf{U_k}, \mathcal{W}) \to 0$ as $k \to 0$. In situations where there is a unique physically relevant weak solution satisfying some entropy condition, we would show convergence to this particular weak solution.

In order to obtain a compact set in $\mathbf{L_1}$, we put a bound on the total variation of the functions.
The set

$$\{v \in L, TV(v) \leq \mathbf{R}, \text{and} \quad supp(v) \subset [-M, M]\}$$

is a compact set, M,R > 0, and any sequence of functions with uniformly bounded total variation and support must contain convergent subsequences.

TV denotes the Total Variation Function. Per definition,

$$TV(v) = \sup \sum_{j=1}^{N} |v(x_j) - v(x_{j-1})|$$

where the supremum is taken over all subdivisions of the real line $-\infty = x_0 < x_1 < ... < x_N = \infty$. For the total variation to be finite v must approach constant values $v_{\pm\infty}$ as $x \to \pm\infty$.

Since the numerical approximations $\mathbf{U_k}$ are functions of $\mathbf{x}$ and $\mathbf{t}$, we need to bound the Total Variation in both space and time. Defining the total variation over the time interval [0,T] by

$$TV_T(u) = \limsup_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T \int_{-\infty}^{\infty} |u(x+\epsilon, t) - u(x,t)| dx dt + \limsup_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T \int_{-\infty}^{\infty} |u(x, t+\epsilon) - u(x,t)| dx dt.$$

The set $\mathbf{K}$ defined above can be shown to be a compact set in $\mathbf{L_{1,T}}$,

$$\mathbf{K} = u \in L_{1,T} : TV_T(u) \leq R \quad \text{and} \quad (u(\cdot,t)) \subset [-M,M] \quad \forall t \in [0,T]. \tag{1.3}$$

We will say that a numerical method is total variation stable, TV-stable, if all the approximations $\mathbf{U_k}$ for $\mathbf{k < k_o}$ lie in some fixed set of the form (1.3). If $\mathbf{U_k}$ is generated by a numerical method in conservation form with a Lipschitz continuous numerical flux, consistent with a conservation law, then the method is TV-stable if $TV(U^n)$ is uniformly bounded for all n, k with $\mathbf{k < k_o}$, $nk \leq T$.
Total Variation Stability guarantees convergence in the sense that $\mathbf{dist}(\mathbf{U_k}, \mathcal{W}) \rightarrow 0$ as $k \rightarrow 0$. One way to ensure TV-stability is to require that the total variation be nonincreasing as time evolves, so that the total variation at any time is uniformly bounded by the total variation of the initial data.

1.2 Assumptions on the data of the problem (1.1)

Let us detail the assumptions that we need to impose on the data of the the problem (1.1). For the nonlinear degenerate parabolic convection-diffusion initial value problem we keep time $\mathbf{T > 0}$ fixed. $\mathbf{u(x,t)}$ is the scalar unknown function that is sought, and the flux function $\mathbf{f}(\gamma, \mathbf{u})$, the coefficient $\gamma(\mathbf{x})$, the diffusion function $\mathbf{B}$ and the initial function $\mathbf{u_0}$ are given functions to be detailed.
For the coefficient γ, we assume that

$$\gamma(x) \in [\underline{\gamma}, \overline{\gamma}] \quad \forall x \in \mathbf{R}, \text{ for some constants } \quad \underline{\gamma}, \overline{\gamma}, |\gamma(\mathbf{x})| > \mathbf{0} \text{ a.e on } \quad \mathbf{R}.$$

We assume that γ belongs to the Bounded Variation of $\mathbf{R}$, denoted $\mathbf{BV(R)}$, and also allowed to be discontinuous. The convection part of (1.1) depends explicitly on the spatial location through $\gamma(\mathbf{x})$ and this dependency may be discontinuous. The coefficient $\gamma(\mathbf{x})$ varies in space and is assumed to be piecewise $\mathbf{C^1}$ with finitely many jumps in γ and γ', located in $\zeta_\mathbf{1} < \zeta_\mathbf{2} < \ldots < \zeta_\mathbf{M}$.

For the convective flux function $\mathbf{f}$, we assume that

$$\mathbf{f} : \mathbf{R} \rightarrow \mathbf{R}; \qquad f(\gamma,0) = f_0 \in R \quad \forall \gamma, \quad f(\gamma,1) = f_1 \in R \quad \forall \gamma. \tag{1.4}$$

We look in the interval [0,1] and the purpose of this assumption is to guarantee that a solution initially in the interval [0,1] remains in [0,1] for all subsequent times. Assume that $\mathbf{f}$ is Lipschitz continuous in each variable:

$$|f(\gamma,u) - f(\gamma,v)| \leq \|f_u\||u-v| \quad \forall \gamma, \forall u,v \in \mathbf{U}.$$

For a compact set $\mathbf{U}$. The solution $\mathbf{u}$ is essentially bounded, providing us with

$$u(x,t) \in \mathbf{U}, \forall (x,t) \in \mathbf{\Pi}_T, f \in Lip\Big([\underline{\gamma}, \overline{\gamma}] \times [0,1]\Big).$$

With this assumption the partial derivatives $\mathbf{f}_\gamma, \mathbf{f}_\mathbf{u}$, exist almost everywhere.

$$\|f_u(\gamma,u)\| > 0, \quad \|f_\gamma(\gamma,u)\| > 0 \quad \text{for almost all} \quad u \in [0,1],$$

and $\|f_\gamma\|_\infty$ and $\|f_u\|_\infty$ are Lipschitz constants of $\mathbf{f}$ w.r.t γ and $\mathbf{u}$. Let

$$f_u^+(\gamma,u) = max(0, f_u(\gamma,u)), \quad f_u^-(\gamma,u) = min(0, f_u(\gamma,u)).$$

We require the technical assumption that $\mathbf{f_u}$ is Lipschitz continuous as a function of γ, with Lipschitz constant $\mathbf{L_{u\gamma}}$.
It follows then that $\mathbf{f_u^+}$ and $\mathbf{f_u^-}$ are also Lipschitz continuous in γ with the same Lipschitz constant. Lipschitz constants measure how much flux functions change. We adopt the assumptions from [14, 15].
Assume that for each $\gamma \in [\underline{\gamma}, \overline{\gamma}]$, there exist a unique maximum $\mathbf{u}^*(\gamma) \in [0,1]$ such that $f(\gamma,.)$ is strictly increasing for $u < u^*(\gamma)$ and $f(\gamma,.)$ is strictly decreasing for $u > u^*(\gamma)$.

We assume that the diffusion function $\mathbf{B}$ belongs to $Lip([0,1])$ with Lipschitz constant $\|\mathbf{B'}\|$.
The degenerate parabolicity condition holds:

$$B(\cdot) \in C^2[0,1];\ B(\cdot) \text{ is nondecreasing with B}(0) = 0.$$

This condition is why (1.1) is refered to as a mixed hyperbolic-parabolic problem. The condition is general enough to include as a special case of (1.1) the hyperbolic conservation law with discontinuous coefficient.
We make the following simplifying assumption.
Suppose B degenerates, is constant on a finite set of disjoint intervals:

$$B'(r) = 0 \quad \forall r \in \bigcup_{i=1}^{K} [\alpha_i, \beta_i] = \Gamma \qquad where \quad \alpha_i < \beta_i \qquad i = 1:K, K \geq 1.$$

On these intervals, (1.1) acts as a pure hyperbolic equation.
$\mathbf{B}$ is non-degenerate off these intervals, which means that $\mathbf{B}$ is strictly increasing and (1.1) behaves as a parabolic problem on

$$[0,1] \setminus \Gamma. \quad B'(r) > 0 \quad \forall r \notin \bigcup_{i=1}^{M} [\alpha_i, \beta_i].$$

The maximum $\mathbf{u}^*(\gamma)$ is assumed to lie either in $\mathbf{\Gamma}$, or lies in the closure of $\mathbf{[0,1] \setminus \Gamma \forall \gamma}$; $\mathbf{Max\ B'(r) > 0, r \in [0,1]}$.

Assume that the integrable bounded initial function $\mathbf{u_0}$ satisfies

$$\left.\begin{aligned} u_0 \in L^1(R) \cap BV(R), u_0(x) \in [0,1] \forall x \in \mathbf{R}, \\ B(u_0) \quad \text{is absolutely continuous on} \quad \mathbf{R}, \\ B(u_0)_x \in \mathbf{BV}(\mathbf{R}). \end{aligned}\right\} \tag{1.5}$$

$\mathbf{B}(\mathbf{u_0})$ being absolutely continuous demands that any jump in $\mathbf{u_0}$ must be contained within one of the intervals $[\alpha_\mathbf{i}, \beta_\mathbf{i}]$ where $\mathbf{B}$ is constant.
Independently of the smoothness of γ , if (1.1) is allowed to degenerate at certain points, that is, $\mathbf{B}'(\mathbf{r}) = \mathbf{0}$ for some values of $\mathbf{r}$, we cannot expect the solution to be smooth. We must look at weak solutions. On the other hand, if $\mathbf{B}'(\mathbf{r})$ is zero on an interval $[\alpha_\mathbf{i}, \beta_\mathbf{i}]$, weak solutions may be discontinuous and they are not uniquely determined by their initial data. Consequently, an entropy condition must be imposed to single out the physically correct solution. We will here assume that there exist at least one interval $[\alpha_\mathbf{i}, \beta_\mathbf{i}]$ on which $\mathbf{B}'$ is zero. Which means that equation (1.1) may possess discontinuous solutions.

Chapter 2

Existence of weak solutions

In this section we will present the main results that establish the existence of weak solutions to the Cauchy problem for a one-dimensional scalar degenerate parabolic equation with a flux function that depends explicitly on the spatial position trough a coefficient $\gamma(\mathbf{x})$ that may be discontinuous. The proof can be found in [13] .
Independently of the smoothness of $\gamma(\mathbf{x})$, if (1.1) is allowed to degenerate (become zero) at certain points, that is $\mathbf{B}'(\mathbf{r}) = \mathbf{0}$ for some values of $\mathbf{r}$, solutions are not necessarily smooth and we need to find weak solutions.
The basic idea is to multiply (1.1) by a smooth test function, integrate one or more times over the domain, and then use integration by parts to move derivatives off the function $\mathbf{u}$ and on to the smooth test function. The result is an equation involving fewer derivatives on $\mathbf{u}$, and hence requiring less smoothness.
A weak solution is defined as follows

Definition 2.0.1. *A function $u(x,t) \in L^1(\Pi_T) \bigcap L^\infty(\Pi_T)$ is a weak solution of the Initial Value Problem (1.1) if it satisfies the following conditions:*

i) $\mathtt{B}(u)$ *is continuous and* $\mathtt{B}(u)_x \in \mathrm{L}^\infty(\Pi_T)$.
ii) For all test functions $\phi \in \mathcal{D}(\Pi_T)$ such that $\phi|_{t=T} = 0$,

$$\iint_{\Pi_T} \Big(u\phi_t + (f(\gamma(x),u) - B(u)_x)\phi_x \Big) dxdt + \int_{\mathrm{R}} u_0(x)\phi(x,0) = 0. \tag{2.1}$$

On the other hand, if $\mathbf{B}'(\mathbf{r}) = \mathbf{0}$ is zero on an interval $[\alpha, \beta]$, weak solutions may be discontinuous and therefore not uniquely determined by their initial data. As a consequence, an entropy condition must be imposed to single out the physically correct solution.
If $\gamma(\mathbf{x})$ is sufficiently smooth, a weak solution $\mathbf{u}(\mathbf{x},\mathbf{t})$ satisfies the entropy condition if, see [14, 15], all convex C^2 entropy functions $\eta(\mathbf{u})$, $\eta : R \to R$, and corresponding entropy fluxes $\mathbf{q}(\gamma(\mathbf{x}),\mathbf{u})$,

$$\eta(u)_t + (q(\gamma(x),u))_x + r(u)_{xx} + \gamma'(x)\Big[\eta'(u)f_\gamma(\gamma(x),u) - q_\gamma(\gamma(x),u)\Big] \leq 0 \in \mathcal{D}'(\mathbf{\Pi}_T), \tag{2.2}$$

where (η, q) is is the convex $\mathbf{C^2}$ entropy-entropy flux pair and $r : R \to R$ is defined by $r'(u) = \eta'(u)B'(u)$.

For the hyperbolic part, the entropy condition is given by

$$\eta(u)_t + q(\gamma(x), u)_x + \gamma'(x)(\eta'(u)f_\gamma(\gamma(x), u) - q_\gamma(\gamma(x), u)) \leq 0 \tag{2.3}$$

$$\text{where} \quad q_u(\gamma(x), u) = \eta'(u)f_u(\gamma(x), u).$$

We can show this inequality.
In the first part of equation (2.3) we suppose that the entropy function $\eta(u)$ satisfies a conservation law of the form

$$\eta(u)_t + q(\gamma(x), u)_x = 0, \tag{2.4}$$

for some entropy flux q.

Then we obtain from this, for smooth u,

$$\eta'(u)u_t + q_\gamma(\gamma(x), u)\gamma'(x)u_x = 0. \tag{2.5}$$

In the second part, we recall that the hyperbolic equation (1.2) can be written as

$$u_t + f_\gamma(\gamma(x), u)\gamma'(x)u_x = 0. \tag{2.6}$$

We multiply with $\eta'(u)$,

$$\eta'(u)u_t + \eta'(u)f_\gamma(\gamma(x), u)\gamma'(x)u_x = 0, \tag{2.7}$$

and compare with (2.5),

$$q_\gamma(\gamma(x), u)\gamma'(x)u_x = \eta'(u)f_\gamma(\gamma(x), u)\gamma'(x)u_x,$$

$$\Rightarrow \quad \gamma'(x)[\eta'(u)f_\gamma(\gamma(x), u) - q_\gamma(\gamma(x), u)] \leq 0.$$

Here we find again the definition for q,
$q : \mathbf{R} \to \mathbf{R}$,

$$q_u(\gamma(x), u) = \eta'(u)f_u(\gamma(x), u),$$

and we recover the entropy condition for the hyperbolic part. By standard limiting argument, the former entropy condition implies the Kružkov-type entropy condition given in, see e.g. [14].

$$|u - c|_t + \Big[sign(u-c)(f(\gamma(x), u) - f(\gamma(x), c))\Big]_x + |B(u) - B(c)|_{xx} + \gamma'(x)sign(u-c)f_\gamma(\gamma(x), c) \leq 0 \tag{2.8}$$

holds in $\mathcal{D}'(\mathbf{\Pi}_T)$, for all $c \in \mathbf{R}$. The Sign function is defined by

$$\mathbf{Sign(s)} = \begin{cases} -1 & s < 0, \\ 0 & s = 0, \\ 1 & s < 0. \end{cases}$$

The entropy solution described above breaks down when $\gamma(\mathbf{x})$ is discontinuous. We can use a Kružkov-type entropy inequality, see e.g. [15, 8] to find a definition for entropy solution for the situation where $\gamma(\mathbf{x})$ is discontinuous. This condition is often more convenient to work with in the sense that it combines the definition of a weak solution with that of the entropy condition. The spatially varying coefficient $\gamma(\mathbf{x})$ has finitely many jumps in γ and γ', located at $\xi_\mathbf{1} < \xi_\mathbf{2} < ... < \xi_\mathbf{M}$. The following definition is suggested in [15].

Definition 2.0.2 (Entropy solution)**.** *A weak solution u of the IVP (1.1) is called an entropy solution, if the following Kružkov-type entropy inequality holds for all $c \in \mathbf{R}$ and all test functions $0 \leq \phi \in \mathcal{D}(\Pi_T)$:*

$$\begin{aligned} &\iint_{\Pi_T} \Big(|u-c|\phi_t + sign(u-c)(f(\gamma(x),u) - f(\gamma(x),c))\phi_x + |B(u)-B(c)|\phi_{xx} \Big) dtdx \\ &- \iint_{\Pi_T \setminus {\xi_m}_{m=1}^M} sign(u-c) f(\gamma(x),c)_x \phi dtdx \\ &+ \int_0^T \sum_{m=1}^M |f(\gamma(\xi_m+),c) - f(\gamma(\xi_m-),c)|\phi(\xi_m,t)dt \geq 0. \end{aligned} \tag{2.9}$$

As was mentioned in the introduction for this section, we will only present the existence results and the main theorems. The main reference for the existence proof of a weak solution of (1.1) is the recent paper by Karlsen, Risebro and Towers [13]. They prove also the uniqueness of the constructed weak solution.
They aim at proving existence of a weak solution to (1.1) when $\gamma(\mathbf{x})$ may depend discontinuously on $\mathbf{x}$. They derive their results using the assumption that $\mathbf{f}(\gamma(\mathbf{x}), \mathbf{u})$ is of multiplicative form $\gamma(\mathbf{x})\mathbf{f}(\mathbf{u})$. This form will simplify slightly some of the formulas.
Existence of a weak solution is proved by passing to the limit as $\epsilon \downarrow 0$ in a suitable sequence $\{u_\epsilon\}_{\epsilon>0}$ of smooth approximations solving the problem above with the flux $\gamma(\mathbf{x})\mathbf{f}(\cdot)$ replaced by $\gamma_\epsilon(\mathbf{x})\mathbf{f}(\cdot)$ and the diffusion function $\mathbf{B}(\cdot)$ replaced by $\mathbf{B}_\epsilon(\cdot)$, where $\gamma_\epsilon(\cdot)$ is smooth and $\mathbf{B}'_\epsilon(\cdot) > 0$. In their paper the existence of a weak solution is proved by establishing convergence of a suitable sequence of smooth functions solving regularised problems. Let $\omega_\epsilon \in \mathbf{C_0^\infty}(\mathbf{R})$ be a nonnegative function satisfying

$$\omega(x) = \omega(-x), \quad \omega(x) \equiv 0 \quad for \quad |z| \geq 1, \int_\mathcal{R} \omega(z)dz = 1.$$

For $\epsilon > 0$, let $\omega_\epsilon(x) = \frac{1}{\epsilon}\omega(\frac{x}{\epsilon})$ and introduce the "smoothed" coefficient

$$\gamma_\epsilon = \omega_\epsilon \star \gamma.$$

Define the approximate initial function

$$u_{0\epsilon} = \omega_\epsilon \star u_0.$$

Observe that $u_{0\epsilon} \in \mathbf{C}^\infty(\mathbf{R})$ and
$u_{0\epsilon} \to u_0$ a.e in $\mathbf{R}$ and in $\mathbf{L^P}(\mathbf{R})$ for any $p \in [1, \infty)$ as $\epsilon \downarrow 0$.
We then let u_ϵ be the unique classical solution of the uniformly parabolic problem

$$\begin{cases} \partial_t u_\epsilon + \partial_x(\gamma_\epsilon(x) f(u_\epsilon)) = \partial_x^2 B_\epsilon(u_\epsilon), (x,t) \in \Pi_T, \\ u_\epsilon(x,0) = u_{0\epsilon}(x), x \in \mathbf{R}, \end{cases} \tag{2.10}$$

where $\mathbf{B}_\epsilon(\mathbf{u}) = \mathbf{B}(\mathbf{u}) + \epsilon\mathbf{u}$.
Roughly speaking, their main theorem can be stated as follows:
The sequence of $\{u_\epsilon\}_{\epsilon>0}$ converges strongly in $\mathbf{L^1}$ to a weak solution u of (1.1). Furthermore, a subsequence of $\{B_\epsilon(u_\epsilon)\}_{\epsilon>0}$ converges uniformly on compact sets to a *Hölder* continuous function that coincides with B(u) a.e.
Since $\gamma(\cdot)$ may be discontinuous, the total variation $|u_\epsilon|_{BV}$ cannot be bounded uniformly with respect to $\epsilon > 0$. The lack of variation bound prevents an application of the standard Bounded Variation (BV) compactness argument to $\{u_\epsilon\}_{\epsilon>0}$. To circumvent this analytical difficulty, they establish instead strong compactness of the diffusion function $\{B_\epsilon(u_\epsilon)\}_{\epsilon>0}$ as well as the "total flux" $\{\gamma_\epsilon(x)f(u_\epsilon) - \partial_x B_\epsilon(u_\epsilon)\}_{\epsilon>0}$. This strong compactness together with some a priori estimates on the "total flux" will make it possible for them to use the compensated compactness method to obtain the desired strong convergence. The first lemma gives uniform $\mathbf{L^1}$ and $\mathbf{L}^\infty$ estimates on u_ϵ.

Lemma 2.0.1. *There exists a constant $C > 0$, independent of ϵ, such that $\|u_\epsilon(\cdot,t)\|_{\mathbf{L^1(R)}}$, $\|u_\epsilon(\cdot,t)\|_{\mathbf{L}^\infty(\mathbf{R})} \le C$, for all $t \in (0,T)$.*

The next lemma provides us with a uniform $\mathbf{L^2(\Pi_T)}$ space and time translation estimate on $\mathbf{B}(\mathbf{u}_\epsilon)$, and hence strong $\mathbf{L^2_{loc}}$ compactness of $\{B(u_\epsilon)\}_{\epsilon>0}$. This lemma will be used to pass to the limit in the nonlinear diffusion term.

Lemma 2.0.2. *There exists a constant $C > 0$ which depends on T but not ϵ such that*

$$\|B(u_\epsilon(\cdot+y,\cdot+\tau)) - B(u_\epsilon(\cdot,\cdot))\|_{L^2(\Pi_{T-\tau})} \le C(|y| + \sqrt{\tau}), \forall y \in (R) \quad \textit{and} \quad \forall \tau \ge 0. \tag{2.11}$$

In particular, we have that $\{B(u_\epsilon)\}_{\epsilon>0}$ is strongly compact in $\mathbf{L^2_{loc}(\Pi_T)}$.

Before we can state the fundamental theorem in the theory of Compensated Compactness, we recapitulate the results they use from the compensated compactness method used to prove the existence of weak solution. For a nice overview of applications of the compensated compactness method to hyperbolic conservation laws we refer to Chen.
Let $\mathcal{M}(\mathbf{R^n})$ denote the space of bounded Radon measure on $\mathbf{R}^n$ and

$$\mathbf{C}_0(\mathbf{R}^n) = \{\Psi \in \mathbf{C}(\mathbf{R}^n) : \lim_{|x|\to\infty} \Psi(x) = 0$$

}.

If $\mu \in \mathcal{M}(\mathbf{R^n})$,

then

$$\langle \mu, \Psi \rangle = \int_{\mathbf{R}^n} \Psi d\mu, \forall \Psi \in \mathbf{C_0}(\mathbf{R}^n).$$

Recall that $\mu \in \mathcal{M}(\mathbf{R^n})$ if and only if $|\langle \mu, \Psi \rangle| \leq \mathbf{C}\|\mathbf{\Psi}\|_{\mathbf{L}^\infty(\mathbf{R^n})}$ $\forall \Psi \in \mathbf{C_0}(\mathbf{R}^n)$. We define

$$\|\mu\|_{\mathcal{M}(\mathbf{R^n})} = sup\{|\langle \mu, \Psi \rangle| : \Psi \in \mathbf{C_0}(\mathbf{R}^n), \|\Psi\|_{\mathbf{L}^\infty(\mathbf{R^n})} \leq 1\}.$$

The space $(\mathcal{M}(\mathbf{R^n}), \|\cdot\|_{\mathcal{M}(\mathbf{R^n})})$ is a Banach space and it is isometrically isomorphic to the dual space of $(\mathbf{C}_0(\mathbf{R}^n), \|\cdot\|_{\mathbf{L}^\infty(\mathbf{R^n})})$, while we define the space of probability measures $\mathrm{Prob}(\mathbf{R}^n)$ as

$$Prob(\mathbf{R}^n) = \{\mu \in \mathcal{M}(\mathbf{R^n}) : \mu \quad \text{is nonnegative and} \quad \|\mu\|_{\mathcal{M}(\mathbf{R^n})} = 1\}.$$

Then we can state the fundamental theorem in the theory of compensated compactness.

Theorem 2.0.1. *Let $\mathbf{K} \subset \mathbf{R}$ be a bounded open set and $u_\epsilon : \Pi_T \to \mathbf{K}$. Then there exists a family of probability measures $\{\nu_{(x,t)}(\lambda) \in Prob(\mathbf{R}^n)\}_{(x,t)\in\Pi_T}$ (depending weak-$\star$ measurably on (x,t)) such that*

$$\nu_{(x,t)} \subset \overline{\mathbf{K}} \quad \textit{for a.e.} \quad (x,t) \in \Pi_T.$$

Furthermore, for any continuous function $\Phi : \mathbf{K} \to \mathbf{R}$, we have along a subsequence

$$\Phi(u_\epsilon) \overset{\star}{\rightharpoonup} \overline{\mathbf{\Phi}} \quad \textit{in} \quad \mathbf{L}^\infty(\mathbf{\Pi_T}) \quad \textit{as} \quad \epsilon \downarrow 0,$$

where (the exceptional set depends possibly on Φ)

$$\overline{\mathbf{\Phi}}(x,t) := \langle \nu_{(x,t)}, \Phi \rangle = \int_{\mathbf{R}} \Phi(\lambda) \quad d\nu_{(x,t)}(\lambda) \quad \textit{for a.e.} \quad (x,t) \in \Pi_T.$$

In the literature, $\nu_{(\mathbf{x},\mathbf{t})}$ is often referred to as a Young measure. Theorem 2.0.1 provides us with a representation formula for weak limits in terms of nonlinear functions and Young measures. A uniformly bounded sequence $\{u_\epsilon\}_{\epsilon>0}$ converges to $\mathbf{u}$ a.e. on $\mathbf{\Pi_T}$ if and only if the corresponding Young measure $\nu_{(\mathbf{x},\mathbf{t})}$ reduces to a Dirac measure located at $\mathbf{u}(\mathbf{x},\mathbf{t})$, i.e., $\nu_{(x,t)} = \delta_{u(x,t)}$. We have the following "reduction" result:

Lemma 2.0.3. *Let $\mathbf{K} \subset \mathcal{R}$ be a bounded open set and $u_\epsilon : \Pi_T \to \mathbf{K}$. Suppose that $u_\epsilon \overset{\star}{\rightharpoonup} u$ in $\mathbf{L}^\infty(\mathbf{\Pi_T})$. Suppose also that for any pair of (not necessarily convex) $\mathbf{C^2}$ functions $\eta_1, \eta_2 : \mathbf{R} \to \mathbf{R}$, we have for a subsequence*

$$\gamma(x)q_1(u_\epsilon)\eta_2(u_\epsilon) - \eta_1(u_\epsilon)\gamma(x)q_2(u_\epsilon) \overset{\star}{\rightharpoonup} \gamma(x)\overline{q_1\eta_2} - \overline{\eta_1}\gamma(x)\overline{q_2} \quad \textit{in} \quad \mathbf{L}^\infty(\mathbf{\Pi_T}) \quad \textit{as} \quad \epsilon \downarrow 0,$$

(2.13)

where $q_i : \mathbf{R} \to \mathbf{R}$ *is defined by* $q_i'(u) = \eta_i'(u)f'(u)$, $i = 1,2$. *Then for a subsequence*

$$\gamma(x)f(u_\epsilon) \overset{\star}{\rightharpoonup} \gamma(x)f(u) \quad in \quad \mathbf{L}^\infty(\mathbf{\Pi_T}) \quad as \quad \epsilon \downarrow 0.$$

Furthermore, if $\gamma(x) \neq 0$ *for a.e.* $x \in \mathbf{R}$ *and there is no interval on which* $f(\cdot)$ *is linear, then a subsequence of* $\{u_\epsilon\}_{\epsilon>0}$ *converges to* u *a.e. on* Π_T.

Remark. If $\gamma(\cdot) = 0$ on a set of non-zero measure, then it is not possible to conclude that (a subsequence of) u_ϵ converges strongly to u nor that $f(u_\epsilon) \overset{\star}{\rightharpoonup} f(u)$ in $\mathbf{L}^\infty(\mathbf{\Pi_T})$.

Theorem 2.0.2. *Suppose that* $\{u_\epsilon\}_{\epsilon>0} \subset \mathbf{L}^\infty(\mathbf{\Pi_T})$ *uniformly in* ϵ. *Suppose also that for any* $\mathbf{C^2}$ *function* $\eta : \mathbf{R} \to \mathbf{R}$, *the subsequence of distributions* $\{\partial_t\eta(u_\epsilon) + \partial_x(\gamma(x)q(u_\epsilon))\}_{\epsilon>0}$ *lies in a compact subset of* $\mathbf{H^{-1}_{loc}}(\mathbf{\Pi_T})$, *where* $q : \mathbf{R} \to \mathbf{R}$ *is defined by* $q'(u) = \eta'(u)f'(u)$. *Then along a subsequence* $u_\epsilon \overset{\star}{\rightharpoonup} u$ *in* $\mathbf{L}^\infty(\mathbf{\Pi_T})$ *as* $\epsilon \downarrow 0$, $\gamma(x)f(u_\epsilon) \overset{\star}{\rightharpoonup} \gamma(x)f(u)$ *in* $\mathbf{L}^\infty(\mathbf{\Pi_T})$ *as* $\epsilon \downarrow 0$. *Furthermore, if* $\gamma(x) \neq 0$ *for a.e.* $x \in \mathbf{R}$ *and there is no interval on which* $f(\cdot)$ *is linear, then a subsequence of* $\{u_\epsilon\}_{\epsilon>0}$ *converges to* u *a.e. on* Π_T.

From Lemma 2.0.1 we know that $M := \|u_\epsilon\|_{\mathbf{L}^\infty(\mathbf{\Pi_T})} \leq 1$ (uniformly in ϵ). Let

$$K = \max_{\lambda\in[0,1]} |B(\lambda)| = B(1).$$

For any function $\Phi \in \mathbf{C}([0, K])$, we then have

$$\|\Phi(B(u_\epsilon))\|_{\mathbf{L}^\infty(\mathbf{\Pi_T})} \leq \mathbf{C},$$

so that along a subsequence

$$\Phi(B(u_\epsilon)) \overset{\star}{\rightharpoonup} \overline{\Phi} \quad \text{in} \quad \mathbf{L}^\infty(\mathbf{\Pi_T}),$$

and, from Theorem 2.0.1,

$$\overline{\Phi}(x,t) = \int_{\mathbf{R}} \Phi(B(\lambda)) \quad d\nu_{(x,t)}(\lambda), \quad \forall (x,t) \in \Pi_T \setminus N_\Phi, \tag{2.14}$$

For some exceptional set $\mathbf{N_\Phi}$ that depends on Φ and $|\mathbf{N_\Phi}| = 0$. One can choose a sequence $\{\Phi_j\}_{j=1}^\infty \subset \mathbf{C}([\mathbf{0}, \mathbf{K}])$ that is dense in $\mathbf{C}([\mathbf{0}, \mathbf{K}])$ and set

$$N = \bigcup_{j=1}^{\infty} N_{\Phi j}.$$

Then $|N| = 0$ and (2.14) holds at any point $(x,t) \in \Pi_T \setminus N$ for each $\Phi \in \mathbf{C}([\mathbf{0}, \mathbf{K}])$. From Lemmas 2.0.1 and 2.0.2, we have that $B(u_\epsilon)$ converges along a subsequence to some

function $\overline{B}$ a.e. on Π_T. Let u denote the $\mathbf{L^\infty(\Pi_T})$ weak-$\star$ limit of $\{u_\epsilon\}_{\epsilon>0}$. We can assume without loss of generality that

$$u(x,t) = \int_{\mathbf{R}} \lambda \quad d\nu_{(x,t)}(\lambda) \quad \forall (x,t) \in \Pi_T \setminus N. \tag{2.15}$$

For $\xi \in [0,K]$, define the functions

$$l(\xi) = \min\{\lambda \in [0,1] : B(\lambda) = \xi\}, L(\xi) = \max\{\lambda \in [0,1] : B(\lambda) = \xi\}. \tag{2.16}$$

Furthermore,

$$l(B(\lambda)) \leq \lambda \leq L(B(\lambda)) \text{for all} \lambda \in [0,1],$$
$$l(B(\lambda)) = \lambda = L(B(\lambda)) \text{ for a.e.} \lambda \in [0,1].$$

We need also the measurable sets

$$H := \{(x,t) \in \Pi_T \setminus N : l(B(u(x,t))) < L(B(u(x,t)))\},$$
$$P := \{(x,t) \in \Pi_T \setminus N : l(B(u(x,t))) = L(B(u(x,t)))\}.$$

The statement that $\overline{B} = B(u(x,t))$ for all $(x,t) \in \Pi_T \setminus N$ implies that $\{u_\epsilon\}_{\epsilon>0}$ converges to u a.e. on P. The proof of this claim is classical. Let $K := P \bigcap [a,b]$ for any $a,b \in \mathbf{R}$, and note that $u_\epsilon^2 \overset{\star}{\rightharpoonup} u^2$ in $\mathbf{L^\infty(K)}$. Then we have

$$\iint_K (u_\epsilon - u)^2 dtdx = \iint_K (u_\epsilon^2 - 2u_\epsilon u + u^2) dtdx \to 0 \quad \text{as} \quad \epsilon \downarrow 0, \tag{2.17}$$

for which the claim follows.
We sum up the compactness properties of the diffusion part of (2.10). A subsequence of $\{B(u_\epsilon)\}_{\epsilon>0}$ converges strongly to B(u) in $\mathbf{L^2_{loc}(\Pi_T})$, where u is the $\mathbf{L^\infty(\Pi_T})$ weak-$\star$ limit of $\{u_\epsilon\}_{\epsilon>0}$. Furthermore, $B(u) \in \mathbf{L^\infty(\Pi_T)} \cap \mathbf{L^2(0,T;H^1(R))}$.
The next Lemma provides us with a series of priori estimates that imply strong compactness of the "total flux" sequence $\{\gamma_\epsilon(x) f(u_\epsilon) - \partial_x B_\epsilon(u_\epsilon)\}_{\epsilon>0}$. These a priori estimates only hold if the initial function u_0 satisfies, in addition to (1.5), the stronger regularity condition

$$|\gamma(x) f(u_0) - \partial_x B(u_0)|_{BV(\mathbf{R})} < \infty. \tag{2.18}$$

Lemma 2.0.4. *Suppose that (2.18) holds and introduce the function*

$$v_\epsilon(x,t) = \gamma_\epsilon(x) f(u_\epsilon) - \partial_x B_\epsilon(u_\epsilon).$$

There exists a constant C>0, independent of ϵ, such that for all $t \in (0,T)$

$$(i) \quad \|v_\epsilon(\cdot,t)\|_{\mathbf{L^\infty(R)}} \leq C,$$
$$(ii) \quad |v_\epsilon(\cdot,t)|_{BV(\mathbf{R})} \leq C,$$
$$(iii) \quad \|v_\epsilon(\cdot,t+\tau) - v_\epsilon(\cdot,t)\|_{\mathbf{L^1(R)}} \leq C\sqrt{\tau}, \quad \forall \tau \geq 0.$$

In particular, we have that $\{v_\epsilon\}_{\epsilon>0}$ is strongly compact in $\mathbf{L^1_{loc}(\Pi_T})$.

The main result is the following theorem.

Theorem 2.0.3. *Suppose that the conditions imposed on the assumptions hold. Then there exists a weak solution (in the sense of Definition 2.0.1) of the Cauchy problem (1.1). Furthermore, u can be constructed as the strong limit of the sequence* $\{u_\epsilon\}_{\epsilon>0}$*, where* u_ϵ *solves the regularised problem (2.10).*
Let v be another weak solution constructed as the strong limit of the sequence $\{v_\epsilon\}_{\epsilon>0}$*, where* v_ϵ *solves the regularised problem (2.10) corresponding to initial data* v_0*. Then*

$$\int_{\mathbf{R}} |u(x,t) - v(x,t)| \quad dx \leq \int_{\mathbf{R}} |u_0(x) - v_0(x)| \quad dx. \tag{2.19}$$

Consequently, the constructed weak solution u of (1.1) is unique. Suppose that the initial function u_0 *satisfies the additional regularity condition stated in (2.18). Then the constructed weak solution u has the following regularity properties:*

$$(i)|(\gamma(x)f(u) - \partial_x B(u))(\cdot,t)|_{BV(\mathbf{R})} \leq C, \quad \forall t \in (0,T),$$
$$(ii)\|u(\cdot,t+\tau) - u(\cdot,t)\|_{\mathbf{L^1(R)}} \leq C\tau, \quad \forall \tau \geq 0.$$

In the pure hyperbolic case, Theorem 2.0.3 (i) implies that the total variation of f(u) is finite if $u_0 \in \mathbf{BV(R)}$, although the total variation of u need not be finite.
It is worthwhile mentioning that if $B(\cdot)$ is strictly increasing we do not need the compensated compactness method to get strong convergence of $\{u_\epsilon\}_{\epsilon>0}$.

Chapter 3

Relaxation Schemes

3.1 Relaxation approximation for the hyperbolic case

The basic idea is based on replacing the nonlinear convection-diffusion equation with a semilinear system, using a stiff relaxation term containing the discontinuous flux function $\mathbf{f}(\gamma(\mathbf{x}), \mathbf{u})$. The schemes proposed in this work are based on the same idea at the basis of the wellknown relaxation schemes for hyperbolic conservation laws by Jin and Xin [22]. They introduced a prototype model that bears many critical properties of more general nonlinear hyperbolic systems with relaxation. Before introducing the framework of Relaxation Schemes for the parabolic equation, we will start with the hyperbolic case. As mentioned in the assumptions, (1.1) behaves like a hyperbolic problem on intervals

$$B'(r) = 0, \quad \forall r \in \bigcup_{i=1}^{K} [\alpha_i, \beta_i] = \Gamma.$$

In these intervals, (1.1) can be approximated by a 2x2 semilinear hyperbolic system with a stiff relaxation term containing the discontinuous flux function $\mathbf{f}(\gamma(\mathbf{x}), \mathbf{u})$.
An additional variable $\mathbf{v(x,t)}$ and a positive parameter ϵ are introduced and the following relaxation system is obtained.

$$\begin{cases} u_t^\epsilon + v_x^\epsilon = 0, \\ v_t^\epsilon + \lambda^2 u_x^\epsilon = \dfrac{1}{\epsilon}(f(\gamma(x), u^\epsilon) - v^\epsilon). \end{cases} \tag{3.1}$$

- $\epsilon > 0$ is the relaxation time.
- λ, a positiv constant, satisfies the subcharacteristic condition:

  $$0 < max_{\gamma,u} |f_u(\gamma, u)| < \lambda.$$

From the given data, the maximum is taken over the set

$$(\gamma, u) \in [\underline{\gamma}, \overline{\gamma}] \times [\underline{u}, \overline{u}].$$

We choose the initial condition for the system (3.1)

$$u^\epsilon(x,0) = u_0(x), \quad v^\epsilon(x,0) = f(\gamma(x), u_0(x)). \tag{3.2}$$

We can rearrange the second equation in system (3.1).

$$\epsilon[v_t^\epsilon + \lambda^2 u_x^\epsilon] = [f(\gamma(x), u^\epsilon) - v^\epsilon],$$

And as $\epsilon \to 0^+$ we obtain

$$v = f(\gamma(x), u). \tag{3.3}$$

Substituting this expression in the first equation of the relaxation system (3.1), we recover the hyperbolic conservation law.

$$u_t + f(\gamma(x), u)_x = 0.$$

The state satisfying (3.3) is called the local equilibrium. In the limit $\epsilon \to \mathbf{0}$, solving the Relaxation system is equivalent to solving the hyperbolic case of the problem.
The characteristic speeds of the local system must be interlaced with the characteristic speeds of the relaxing system to ensure the stability of the limit. The same condition is true for the 2×2 semilinear case to ensure that the local relaxation approximation is dissipative. This condition is referred to as the subcharacteristic condition.

To understand better this approximation, we can present a Chapman-Enskog type expansion [17, 18] for the relaxation system (3.1).
Roughly, the difference between this expansion and the classical Hilbert expansion (also asymptotic expansion) lies in that the Hilbert expansion expands the solution, while the CE-expansion expands the equation.
Let us do a Chapman-Enskog expansion for the relaxation system. We suppose for the moment that u^ϵ, v^ϵ and $\gamma(x)$ are smooth functions and make the ansatz

$$v^\epsilon = f(\gamma(x), u^\epsilon) + \epsilon \tilde{v}^\epsilon + o(\epsilon^2),$$

for some $\tilde{v}^\epsilon$. We can rewrite the second equation of the relaxation system (3.1) as

$$\begin{aligned} v_t + \lambda^2 u_x &= \frac{1}{\epsilon}[f(\gamma(x), u^\epsilon) - v^\epsilon], \\ \epsilon v_t + \epsilon\lambda^2 u_x &= f(\gamma(x), u^\epsilon) - v^\epsilon, \\ v^\epsilon &= f(\gamma(x), u^\epsilon) - \epsilon[v_t + \lambda^2 u_x] \end{aligned}$$

Which means that

$$v^\epsilon = f(\gamma(x), u^\epsilon) + o(\epsilon). \tag{3.4}$$

From the first equation of (3.1), we have

$$\frac{\partial u}{\partial t} = -\frac{\partial v}{\partial x}. \tag{3.5}$$

Differentiating (3.4) with respect to time, $\frac{\partial}{\partial t}$.

$$\begin{aligned} v_t^\epsilon &= \frac{\partial f}{\partial u}[\gamma(x), u^\epsilon]\frac{\partial u}{\partial t}, \\ v_t^\epsilon &= f_u(\gamma(x), u^\epsilon)u_t^\epsilon + o(\epsilon), \end{aligned}$$

then substituting, using (3.5) we obtain

$$v_t^\epsilon = -f_u(\gamma(x), u^\epsilon)v_x + o(\epsilon).$$

We find v_x in the former expression using (3.4). We differentiate with respect to x.

$$\begin{aligned} v_t^\epsilon &= -f_u(\gamma(x), u^\epsilon) \cdot \frac{\partial}{\partial x}[f(\gamma(x), u^\epsilon) + o(\epsilon)] + o(\epsilon), \\ v_t^\epsilon &= -f_u(\gamma(x), u^\epsilon)\Big[\frac{\partial f}{\partial \gamma}\gamma'(x) + \frac{\partial f}{\partial u}u_x^\epsilon + o(\epsilon)\Big] + o(\epsilon), \\ &= -f_u(\gamma(x), u^\epsilon)\Big[f_\gamma(\gamma(x), u^\epsilon)\gamma'(x) + f_u(\gamma(x), u^\epsilon)u_x^\epsilon + o(\epsilon)\Big] + o(\epsilon), \\ v_t^\epsilon &= -[f_u(\gamma(x), u^\epsilon)]^2 u_x^\epsilon - f_u(\gamma(x), u^\epsilon)f_\gamma(\gamma(x), u^\epsilon)\gamma'(x) + o(\epsilon). \end{aligned}$$

Inserted into the second equation of (3.1)

$$\begin{aligned} v^\epsilon &= f(\gamma(x), u^\epsilon) - \epsilon[v_t^\epsilon + \lambda^2 u_x^\epsilon], \\ v^\epsilon &= f(\gamma(x), u^\epsilon) - \epsilon\Big[-[f_u(\gamma(x), u^\epsilon)^2)u_x^\epsilon - f_u(\gamma(x), u^\epsilon)f_\gamma(\gamma(x), u^\epsilon)\gamma'(x) + o(\epsilon)] + \lambda^2 u_x^\epsilon\Big], \\ v^\epsilon &= f(\gamma(x), u^\epsilon) - \epsilon\Big[\lambda^2 - (f_u(\gamma(x), u^\epsilon))^2 u_x^\epsilon - f_u(\gamma(x), u^\epsilon)f_\gamma(\gamma(x), u^\epsilon)\gamma'(x) + o(\epsilon)\Big]. \end{aligned}$$

Plugging into the first equation of (3.1), we need $\frac{\partial}{\partial x}v^\epsilon$:

$$\begin{aligned} \frac{\partial}{\partial x}v^\epsilon &= f_\gamma(\gamma(x), u^\epsilon)\gamma'(x) + f_u(\gamma(x), u^\epsilon)u_x^\epsilon - \epsilon[\lambda^2 - (f_u(\gamma(x), u^\epsilon))^2 u_{xx}^\epsilon \\ &\quad + \epsilon[-f_u(\gamma(x), u^\epsilon)f_\gamma(\gamma(x), u^\epsilon)\gamma'(x)]_x, \\ \frac{\partial}{\partial x}v^\epsilon &= [f(\gamma(x), u^\epsilon) - \epsilon f_u(\gamma(x), u^\epsilon)f_\gamma(\gamma(x), u^\epsilon)\gamma'(x)]_x - \epsilon[\lambda^2 - (f_u(\gamma(x), u^\epsilon))^2 u_x^\epsilon]_x, \end{aligned}$$

$$\Rightarrow \quad u_t^\epsilon + \Big(f(\gamma(x), u^\epsilon - \epsilon f_u(\gamma(x), u^\epsilon)f_\gamma(\gamma(x), u^\epsilon)\gamma'(x)\Big)_x = \epsilon\Big((\lambda^2 - (f_u(\gamma(x), u^\epsilon))^2 u_x^\epsilon)\Big)_x. \tag{3.6}$$

This is the first order correction to (1.1). The equation (3.6) governs the asymptotic behaviour of the relaxation system (3.1) either as time approaches infinity or as the relaxation rate ϵ tends to zero. The right hand side contains a second derivative of u and hence represents a dissipation (viscous) term. The coefficient represents the coefficient of viscosity, therefore, the relaxation system provides a vanishing viscosity model to the original conservation law. We observe also that (3.6) contains an $\mathbf{O}(\epsilon)$ diffusion correction as well as an $\mathbf{O}(\epsilon)$ convection correction. For the coefficient to be positive and to ensure that this equation is parabolic, the following condition should be satisfied,

$$\lambda^2 \geq (f_u(\gamma(x), u))^2 \Leftrightarrow -\lambda \leq (f_u(\gamma(x), u) \leq \lambda. \tag{3.7}$$

This is referred to as the the **subcharacteristic condition**. The constant λ in the relaxation system (3.1) should be chosen in such a way that the Condition (3.7) is satisfied. Chen, Levermore and Liu [5] show that if the Subcharacteristic Condition is always satisfied, then solutions of the system tend to solutions of the equilibrium equation as the relaxation time tends to zero. The fact that the first order correction to the original system has a dissipative structure implies that the numerical solutions to the relaxation system should also converge to the entropy solution of the original system.

3.2 Relaxation approximation to the nonlinear convection-diffusion equation

We now extend the previous approach to nonlinear parabolic equation. We propose Diffusive Relaxation Schemes for the numerical approximation of nonlinear parabolic equations. The schemes proposed here are based on the same idea at the basis of the relaxation schemes for the hyperbolic conservation laws. The relaxation system read

$$\begin{cases} u_t + v_x = 0, \\ v_t + \dfrac{1}{\epsilon^2} B(u)_x = -\dfrac{1}{\epsilon^2}(v - f(\gamma(x), u)). \end{cases} \tag{3.8}$$

With initial data

$$u^\epsilon(x, 0) = u_0(x), \quad v^\epsilon(x, 0) = f(\gamma(x), u_0(x)). \tag{3.9}$$

The positive parameter ϵ has physical dimensions of time and represents the so-called relaxation time for the system and the limit problem for $\epsilon \to 0$ is called diffusive limit. The relaxation term is stiff, which means that $\epsilon << 1$. That is, the relaxation time is much shorter than the time it takes, for example, for the hyperbolic wave to propagate over a gradient length.The study is concentrated on the stiff regime. It is immediately recognisable that system (3.8) has the form used to construct relaxation schemes for conservation laws by Jin and Xin [22]. Theoretical justification for the passage from (3.1) to (1.2) was made

in [5, 16], while the numerical discretization for such problems was studied in [9]. In such a problem only the source term is stiff, thus a proper splitting of an explicit convection and an implicit source term suffices to give a scheme with a formal uniform accuracy in ϵ. The new formulation (3.8), due to its close relation with (3.1), allows us to use some of the numerical techniques used to solve (3.1). However, we face additional difficulties here because the convection step is also stiff, e.g.see [3, 2].
In the small relaxation limit, $\epsilon \to 0$, the relaxation system (3.8) can be approximated to leading order by

$$\begin{cases} v = f(\gamma(x), u) - B(u)_x, \\ u_t + f(\gamma(x), u)_x = B(u)_{xx}. \end{cases} \tag{3.10}$$

The state satisfying the first equation of (3.10) is called local equilibrium. The relaxation system (3.8) has two characteristic variables

$$v + \frac{\sqrt{B'(u)}}{\epsilon}, \quad v - \frac{\sqrt{B'(u)}}{\epsilon},$$

that travel with characteristic speeds

$$\pm\frac{\sqrt{B'(u)}}{\epsilon}.$$

Since the equilibrium equation is of parabolic type, the main stability condition for the relaxation system is, see [3],

$$|f'(\gamma(x), u)|^2 \leq \frac{B'(u)}{\epsilon^2}, \tag{3.11}$$

which is the **subcharacteristic condition**, a necessary condition for convergence to equilibrium. We expect to find condition (3.11) verified in the relaxation limit. As long as the solutions for the limit equation are smooth, the stability in a suitable norm and the convergence of the problem as the relaxation parameter ϵ tends to zero can be completely justified. Unfortunately, in the general case the solutions of the equilibrium equation (3.10) may become discontinuous in a finite time. For the system $\mathbf{B}'(\mathbf{u}) > \mathbf{0}$, $f'(\gamma(x), u)$ is the characteristic speed, such that (3.11) reeds

$$\frac{-\sqrt{B'(u)}}{\epsilon} < f'(\gamma(x), u) < \frac{\sqrt{B'(u)}}{\epsilon}. \tag{3.12}$$

Note that the local equilibrium approximation (3.10), which has the characteristic speed $\mathbf{f}'(\gamma(\mathbf{x}), \mathbf{u})$, will exceed the characteristic speeds of the original system unless (3.12) is satisfied.
The consistency of the approximation would be satisfied, if only to preserve the proper causality. Hence (3.12) will be referred to as a Stability Criterion. By considering a

Chapman-Enskog expansion for the relaxation system, we get a better understanding of the argument.

Let $\Big(u^\epsilon(x,t), v^\epsilon(x,t)\Big)$ be a family of solutions to

$$\begin{cases} u_t^\epsilon + v_x^\epsilon = 0, \\ v_t^\epsilon + \dfrac{1}{\epsilon^2} B(u^\epsilon)_x = -\dfrac{1}{\epsilon^2}(v^\epsilon - f(\gamma(x), u^\epsilon)). \end{cases} \tag{3.13}$$

From the second equation of (3.13),

$$\begin{aligned} \epsilon^2 v_t^\epsilon + B(u^\epsilon)_x &= -(v^\epsilon - f(\gamma(x), u^\epsilon)), \\ v^\epsilon - f(\gamma(x), u^\epsilon) &= -\epsilon^2 v_t^\epsilon - B(u^\epsilon)_x, \\ v^\epsilon &= f(\gamma(x), u^\epsilon) - \epsilon^2 v_t^\epsilon - B(u^\epsilon)_x, \\ v^\epsilon &= f(\gamma(x), u^\epsilon) - B(u^\epsilon)_x - \epsilon^2 v_t^\epsilon, \end{aligned} \tag{3.14}$$

We start with the usual ansatz:

$$v^\epsilon = f(\gamma(x), u^\epsilon) - B(u^\epsilon)_x + o(\epsilon^2), \tag{3.15}$$

differentiating with respect to time,

$$v_t^\epsilon = f_{u^\epsilon}(\gamma(x), u_t^\epsilon - B_{xt}(u^\epsilon) + o(\epsilon^2).$$

From the first equation of (3.13), we have
$u_t^\epsilon = -v_x^\epsilon, then$

$$v_t^\epsilon = f_{u^\epsilon}(\gamma(x), u^\epsilon)(-v_x^\epsilon) - B_{xt}(u^\epsilon) + o(\epsilon^2). \tag{3.16}$$

We find v_x^ϵ from (3.15).

$$\begin{aligned} \frac{\partial}{\partial x} v^\epsilon &= \frac{\partial}{\partial x}[f(\gamma(x), u^\epsilon) - B(u^\epsilon)_x + o(\epsilon^2)], \\ v_x^\epsilon &= \frac{\partial f}{\partial \gamma}(\gamma(x), u^\epsilon)\gamma'(x) + \frac{\partial f}{\partial u^\epsilon}(\gamma(x), u^\epsilon) u_x^\epsilon - B_{xx}(u^\epsilon) + o(\epsilon^2). \end{aligned}$$

Substituting into (3.16).

$$\begin{aligned} v_t^\epsilon &= f_{u^\epsilon}(\gamma(x), u^\epsilon)[f_\gamma(\gamma(x), u^\epsilon)\gamma'(x) + f_{u^\epsilon}(\gamma(x), u^\epsilon) u_x^\epsilon - B_{xx}(u^\epsilon) + o(\epsilon^2)] - B_{xt}(u^\epsilon) + o(\epsilon^2), \\ v_t^\epsilon &= -f_{u^\epsilon}(\gamma(x), u^\epsilon) f_\gamma(\gamma(x), u^\epsilon)\gamma'(x) - f_{u^\epsilon}(\gamma(x), u^\epsilon)^2 u_x^\epsilon + f_{u^\epsilon}(\gamma(x), u^\epsilon) B_{xx}(u^\epsilon) - B_{xt}(u^\epsilon) + o(\epsilon^2), \end{aligned}$$

Put into (3.14).

$$\begin{aligned} v^\epsilon = f(\gamma(x), u^\epsilon) &- B(u^\epsilon)_x + \epsilon^2 f_{u^\epsilon}(\gamma(x), u^\epsilon) f_\gamma(\gamma(x), u^\epsilon)\gamma'(x) \\ &+ \epsilon^2 f_{u^\epsilon}(\gamma(x), u^\epsilon)^2 u_x^\epsilon - \epsilon^2 f_{u^\epsilon}(\gamma(x), u^\epsilon) B_{xx}(u^\epsilon) + B_{xt}(u^\epsilon) + o(\epsilon^2). \end{aligned}$$

We differentiate now with respect to space.

$$\frac{\partial}{\partial x}v^\epsilon = f_\gamma(\gamma(x),u^\epsilon)\gamma'(x)+f_{u^\epsilon}(\gamma(x),u^\epsilon)u_x^\epsilon-B_{xx}(u^\epsilon)+\epsilon^2\frac{\partial}{\partial x}[f_{u^\epsilon}(\gamma(x),u^\epsilon)\gamma'(x)]+\epsilon^2\frac{\partial}{\partial x}[f_{u^\epsilon}(\gamma(x),u^\epsilon)^2u_x^\epsilon]$$

$$-\epsilon^2\frac{\partial}{\partial x}[f_{u^\epsilon}(\gamma(x),u^\epsilon)B_{xx}(u^\epsilon)]+\epsilon^2\frac{\partial}{\partial x}(B_{xt}(u^\epsilon))+o(\epsilon^2),$$

$$\frac{\partial}{\partial x}v^\epsilon = \Big[f(\gamma(x),u^\epsilon)-\epsilon^2 f_{u^\epsilon}(\gamma(x),u^\epsilon)f_\gamma(\gamma(x),u^\epsilon)\gamma'(x)\Big]_x-B_{xx}(u^\epsilon)+\epsilon^2\frac{\partial}{\partial x}\Big[f_{u^\epsilon}(\gamma(x),u^\epsilon)^2u_x^\epsilon\Big]$$

$$-\epsilon^2\frac{\partial}{\partial x}\Big[f_{u^\epsilon}(\gamma(x),u^\epsilon)B_{xx}(u^\epsilon)\Big]+\epsilon^2\frac{\partial}{\partial x}(B_{xt}(u^\epsilon))+o(\epsilon^2).$$

Using this in the first equation of (3.13), dropping higher order terms leads to the second order correction $\mathbf{O}(\epsilon^2)$ to the local equilibrium approximation in the form

$$u_t^\epsilon+[f(\gamma(x),u^\epsilon)-\epsilon^2 f_{u^\epsilon}(\gamma(x),u^\epsilon)f_\gamma(\gamma(x),u^\epsilon)\gamma'(x)] = \frac{\partial}{\partial x}[B'(u^\epsilon)-\epsilon^2 f'(\gamma(x),u^\epsilon)^2]u_x^\epsilon. \quad (3.17)$$

The right hand side represent a dissipation. For the equation to be dissipative, following condition should be satisfied:

$$|B'(u^\epsilon)-\epsilon^2 f'(\gamma(x),u^\epsilon)^2| \geq 0,$$
$$|f'(\gamma(x),u^\epsilon)|^2 \leq \frac{B'(u^\epsilon)}{\epsilon^2}. \quad (3.18)$$

This inequality is naturally verified in the limit $\epsilon \to 0$. The equation (3.10) will always possess the positive viscosity if the stability condition (3.18) is satisfied. Thus, in the regime when ϵ^2 is small, the behaviour of the solution to (3.8), the diffusive relaxation system, is governed by (3.10). We call this kinds of relaxation limits the diffusive relaxation limits and we emphasise the fact that the equations of (3.10) provide a link between the relaxation parameter ϵ of the system and the physical viscosity of the limiting equilibrium equations.

Chapter 4

L^∞ and L^1 Estimates

4.1 L^∞ Estimates

In this section we shall establish a uniform supremum norm bound for the solution $(\mathbf{u}^\epsilon, \mathbf{v}^\epsilon)$ of the Cauchy problem $(3.8) - (3.9)$, we consider the following assumptions.

(A1) f is locally Lipschitz continuous function with $f(0) = f'(0) = 0$.

(A2) The functions (u^ϵ, v^ϵ) are uniformly bounded in $L^\infty(\mathbb{R}^2)$ by

$$N_0 = max\Big(\sup_{\epsilon>0} \parallel u_0^\epsilon \parallel_\infty, \sup_{\epsilon>0} \parallel v_0^\epsilon \parallel_\infty \Big).$$

Moreover, the sequence $(u_0^\epsilon, v_0^\epsilon)$ converges in $\mathbf{L^1_{loc}}(\mathbb{R})^\mathbf{2}$ to some $(\overline{u_0}, \overline{v_0}) \in L^\infty(\mathbb{R}^2)$ as $\epsilon \to 0^+$.
We will also use the following supplementary assumption on the initial data:

(A3) For any bounded closed interval $K \subseteq \mathbb{R}$, it holds

$$\lim_{\epsilon \to 0^+} \parallel v_0^\epsilon - f(\gamma(x), u_0^\epsilon) \parallel_{L^1(K)} = 0.$$

We define for any $\mathbf{N > 0}$,

$$F(N) := \sup_{|\zeta| N} |f(\zeta)|, \tag{4.1}$$

$$B(N) := 2N + F(2N), \tag{4.2}$$

and

$$M(N) = \sup_{|\zeta| B(N)} |f'(\zeta)|.$$

we state the global existence and boundedness result for problem (3.8), similar to that in [16].

Lemma 4.1.1. *Assume (A1) and (A2). For any $N_0 > 0$, and $\epsilon > 0$, if*

$$\frac{\sqrt{B'(u)}}{\epsilon} > M(N_0), \tag{4.3}$$

then there exist a unique, globally bounded solution (u^ϵ, v^ϵ) to the relaxation system in $\mathbf{C}([\mathbf{0},\infty);\mathbf{L^1_{loc}}(\mathbb{R}^\mathbf{2}))$ such that

$$\|v^\epsilon \pm \frac{\sqrt{B'(u^\epsilon)}}{\epsilon}u^\epsilon\|_{L^\infty(\mathbb{R}\times(0,\infty))} \leq \frac{\sqrt{B'(u^\epsilon)}}{\epsilon}B(N_0). \tag{4.4}$$

The subcharacteristic inequality

$$|f'(\gamma(x), u^\epsilon)| \leq \frac{\sqrt{B'(u^\epsilon)}}{\epsilon} \tag{4.5}$$

holds for all $\epsilon > 0$ and for almost every $(x,t) \in \mathbf{R}(0,\infty)$.

Some remarks based on the $\mathbf{L}^\infty$ a priori estimate. Consider uniformly bounded solutions $\mathbf{u}^\epsilon = (\mathbf{u}^\epsilon, \mathbf{v}^\epsilon) \in \mathbf{L}^\infty$ of the 2×2 system satisfying the entropy inequality. Assume that the strict stability condition holds and the subcharacteristic speed is monotone almost everywhere for the local variable $\mathbf{u} \in \mathbb{R}$. The stability theory ensures the existence of such a strictly convex entropy. Then $\mathbf{u}^\epsilon$ strongly converges to **(u,v)** and the limit functions**(u(x,t),v(x,t))** are on the equilibrium curve for almost all **(x,t)**, t $>$ 0, where **u(x,t)** is the solution of the Cauchy problem. The initial data may be far from equilibrium but the convergence result indicates that the limit functions **(u,v)** come into local equilibrium as soon as t$>$0. When we can show the compactness of the zero relaxation limit, we also then have an indication that the sequence $\mathbf{u}^\epsilon$ is compact no matter how oscillatory the initial data are.

4.2 L^1Estimates

We derive some a priori uniform stability estimates in $\mathbf{L^1_{loc}}$ for the solutions of the relaxation system (3.8). The main goal is to establish compactness properties of the approximating sequences. Under the assumptions of Lemma 4.1.1, let $\frac{\sqrt{B'(u^\epsilon)}}{\epsilon} > M(N_0)$ and $\epsilon > 0$. Let $(\mathbf{u}^\epsilon, \mathbf{v}^\epsilon)$ be the solution of the Cauchy problem (3.8)-(3.9). The statements from [16, 19] conclude that for any interval $(a,b) \subseteq \mathbf{R}$ and for any $T > 0$ there exists a continuous nondecreasing function $w \in \mathbf{C}([\mathbf{0},\mathbf{T}])$, not depending on ϵ and with $w(0) = 0$, such that, for every $t \in (0,T)$

$$\int_a^b |u^\epsilon(x+h,t) - u^\epsilon(x,t)| + |v^\epsilon(x+h,t) - v^\epsilon(x,t)|dx \leq w(|h|), \quad \text{for any} |h| \leq h_0. \quad (h_0 > 0).$$

(4.6)

Under these assumptions, suppose further that the initial data $(\mathbf{u_0^\epsilon}, \mathbf{v_0^\epsilon})$ are locally of bounded variation. Then there exist a constant $\mathbf{c}$, not depending on ϵ, such that for any interval $(\mathbf{a}, \mathbf{b}) \subseteq \mathbf{R}$ and for every $t \geq 0$

$$\|u^\epsilon(\cdot,t), v^\epsilon(\cdot,t)\|_{BV((a,b))} \quad \leq c\|u_0^\epsilon, v_0^\epsilon\|_{BV\left((a-\frac{\sqrt{B'(u)}}{\epsilon}t, b+\frac{\sqrt{B'(u)}}{\epsilon}t)\right)}. \tag{4.7}$$

$\mathbf{c}$ is a generic constant and may change due to calculations. As a consequence, we have that the sequence $(\mathbf{u}^\epsilon, \mathbf{v}^\epsilon)$ lies in a compact set of $\mathbb{L}^1_{loc}$ for all $t \geq 0$. We can now state the following. The proof for similar statement can be found in [16].

Theorem 4.2.1. *Under the assumptions of Lemma 4.1.1, let* $\frac{\sqrt{B'(u^\epsilon)}}{\epsilon} > M(N_0)$ *and* $\epsilon > 0$. *Let* $(\mathbf{u}^\epsilon, \mathbf{v}^\epsilon)$ *be the solution of the Cauchy problem (3.8)-(3.9). Then for any interval* $(a,b) \subseteq \mathbf{R}$ *and for any* $T > 0$ *there exists a continuous nondecreasing function* $\tilde{w} \in \mathbf{C}([\mathbf{0}, \mathbf{T}])$, *not depending on* ϵ *and with* $\tilde{w}(0) = 0$, *such that for every* $0 \leq t \leq t + \tau \leq T$, $(\tau > 0)$ *it holds that*

$$\int_a^b |u^\epsilon(x, t+\tau) - u^\epsilon(x,t)| \quad dx\tilde{w}(\tau). \tag{4.8}$$

It turns out that, as $\epsilon \to \mathbf{0}^+$, the sequence $(\mathbf{u}^\epsilon)$ converges towards the entropy solution $\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathbf{t})$, in the sense of Natalini [19]. More precisely:

Theorem 4.2.2. *Assume A1-A3 and* (4.3). *And let* (u^ϵ, v^ϵ) *be the global solution to problem* (3.8). *Then there exists a weak solution* u *to* (1.1) *and a subsequence, still denoted* $(u^\epsilon(x,t), v^\epsilon(x,t))$ *such that*

$$u^\epsilon \to u \quad in \quad C\Big([0,\infty); L^1_{loc}(\mathbf{R})\Big), \qquad as \quad \epsilon \to 0^+, \tag{4.9}$$

$$v^\epsilon \to f(\gamma(x), u) \quad in \quad C\Big([0,\infty); L^1_{loc}(\mathbf{R})\Big), as \quad \epsilon \to 0^+. \tag{4.10}$$

4.3 Entropy solutions

We know that weak solutions to the Cauchy problem (1.1) are in general not unique and, to select a reasonable solution, we have to restrict the class of admissible solutions giving the so-called entropy condition.
In this section we shall show that the limit function obtained in section (4.2) when $\epsilon \to 0^+$ is actually an entropy solution. Let us state some subtle details involved in the analysis of the behaviour of entropy pairs of system (3.8). Similar but more complex analysis can be found in [16, 19, 5].
We assume **(A1)** and **(A2)**. Then there exists $\mathbf{N_0} > \mathbf{0}$ such that the stability condition (4.3) holds. We also assume one of the following hypotheses

(i) $\sup_{\mathbf{u}\in\mathbf{R}} |f'(u)| < \infty$;

(ii) $\mathbf{f} \in \mathbf{C^2}$ and there is $\mathbf{M} > \mathbf{0}$ such that if $|u| \geq M$ then $|f''(u)| > 0$.

Then for every $\mathbf{N_0} > \mathbf{0}$ the weak solution $\mathbf{u}$ of (3.8)-(3.9) given by Theorem 4.2.2 is an entropy solution for the same problem. We observe that under these assumptions and due to the uniqueness result for the entropy solutions, the whole sequence $(\mathbf{u}^\epsilon, \mathbf{v}^\epsilon)$ in Theorem 4.2.2 is actually converging. We aim to show in this section that limit solutions $(\mathbf{u}^\epsilon, \mathbf{v}^\epsilon)$ are entropy solutions to (3.13). We recall from previous section that $(\eta, \mathbf{q})$ is the convex $\mathbf{C}^2$ entropy/entropy flux pair for the system (1.1).
For any given entropy pair $(\eta, \mathbf{q})$ for (1.1), we construct an $\mathbf{C}^2$ entropy/entropy flux pair $(\chi, \mathbf{\Psi})$ for the system (3.13) on some open convex set Ω, such that for all functions

$$\chi, \Psi : \quad [\underline{\gamma}, \overline{\gamma}] \times \Omega \to \mathbf{R},$$

$$(\chi(\gamma(x), u^\epsilon, v^\epsilon), \Psi(\gamma(x), u^\epsilon, v^\epsilon))$$

satisfy the compatibility conditions

$$\Psi_u(\gamma, u, v) = \lambda^2 \chi_v(\gamma, u, v), \quad \Psi_v(\gamma, u, v) = \chi_u(\gamma, u, v). \tag{4.11}$$

In addition, on the equilibrium curve $\mathbf{v} = \mathbf{f}(\gamma(\mathbf{x}), \mathbf{u})$, we require that the entropy-entropy flux pair $(\chi, \mathbf{\Psi})$ reduces to entropy/entropy flux pair $(\eta, \mathbf{q})$ for (1.1),

$$\chi(\gamma, u, f(\gamma, u)) = \eta(u) \quad \text{and} \quad \Psi(\gamma, u, f(\gamma, u)) = q(\gamma, u), \quad \forall u \in \Omega. \tag{4.12}$$

We can extend an arbitrary entropy/entropy-flux pair $(\eta, \mathbf{q})$ for (1.1) to an entropy/entropy-flux pair for (3.13) by viewing $(\eta, \mathbf{q})$ as an "equilibrium"entropy/entropy-flux pair for (3.13). The idea goes back to [5]. We let $(\eta, \mathbf{q})$ be a strictly convex entropy pair for the local equilibrium equation (3.10). Assume that the stability criterion (3.12)

$$\frac{-\sqrt{B'(u)}}{\epsilon} < f'(\gamma(x), u) < \frac{\sqrt{B'(u)}}{\epsilon}$$

holds on $\mathbf{v} = \mathbf{f}(\gamma(\mathbf{x}), \mathbf{u})$, then there exists a strictly convex entropy pair $(\chi, \mathbf{\Psi})$ for the system (3.13) over an open convex set $\mathbf{\Omega}$ containing the local equilibrium curve $\mathbf{v} = \mathbf{f}(\gamma(\mathbf{x}), \mathbf{u})$, along which it satisfies (4.12). Smooth solutions $(\mathbf{u}^\epsilon, \mathbf{v}^\epsilon)$ of (3.13) satisfy

$$\partial_t \chi(\gamma(x), u^\epsilon, v^\epsilon) + \partial_x \psi(\gamma(x), u^\epsilon, v^\epsilon) = \frac{-1}{\epsilon^2} \partial_v \chi(\gamma(x), u^\epsilon, v^\epsilon)(v^\epsilon - f(\gamma(u^\epsilon, u^\epsilon)), \tag{4.13}$$

by Theorem 3.2 in [5].
If $\gamma(\mathbf{x})$ is smooth, a weak solution $(\mathbf{u}^\epsilon, \mathbf{v}^\epsilon)$ of (3.13) is said to satisfy the entropy condition if

$$\begin{aligned} &\chi(\gamma(x), u^\epsilon, v^\epsilon)_t + \psi(\gamma(x), u^\epsilon, v^\epsilon)_x + r(u)_{xx} - \gamma'(x)(\psi_\gamma(\gamma(x), u^\epsilon, v^\epsilon)) \leq \\ &\frac{1}{\epsilon} \chi_v(\gamma(x), u^\epsilon, v^\epsilon)(f(\gamma(x), u^\epsilon, v^\epsilon) - v^\epsilon) \quad \text{in} \quad \mathcal{D}', \end{aligned} \tag{4.14}$$

where

$$r'(u) = \eta'(u)B'(u). \tag{4.15}$$

As a consequence of these estimates and result, we can state the following result of relaxation, see [5, 16].
Assume $B \in \mathbf{C}^2$, $B'(u) > 0 \quad \forall u \in \mathbf{R}$.
$f \in \mathbf{C}^2, f(u) = 0$.
And the stability condition

$$B'(u) - \epsilon^2 f'(\gamma(x), u)^2 \geq 0. \tag{4.16}$$

Assume the initial data $(\mathbf{u_0^\epsilon}, \mathbf{v_0^\epsilon}) \in \mathbf{L^\infty} \cap \mathbf{L^2}$ and verify (A2). Then there exists a constant $N_0 > 0$, such that if

$$\|(u_0^\epsilon, v_0^\epsilon)\|_{L^\infty} \leq N_0, \tag{4.17}$$

there exists a subsequence, denoted $(\mathbf{u}^\epsilon, \mathbf{v}^\epsilon)$, of global weak solutions to the Cauchy problem (3.8)-(3.9) that converges pointwise almost everywhere,

$$(u^\epsilon(x,t), v^\epsilon(x,t)) \to (u(x,t), v(x,t)), \text{as} \quad \epsilon \to 0^+. \tag{4.18}$$

The limit function $\mathbf{u(x,t)}$ satisfies

i) v(x,t) = f(u(x,t)) for almost all $(x,t) \in \mathbf{R} \times (0,\infty)$ and

ii) u is a weak entropy solution of Cauchy problem (1.1).

To sum up. In the stability theory, the convexity of entropy χ is essential. The existence of strictly convex entropy χ implies the following:
-The local equilibrium system is hyperbolic with a strictly convex entropy pair $(\eta, \mathbf{q}) = (\chi, \mathbf{\Psi})|_{\mathbf{v=f(\gamma,u)}}$.

-The characteristic speeds of the local system are interlaced with the characteristic speeds of the original system (1.1).

-The first order correction is locally dissipative with nonnegative diffusion $\mathbf{B(u)}$.

Chapter 5

Numerical schemes

5.1 Discretization of the relaxation system

We consider in this section the discretization of the proposed relaxation system (3.8). Using the same notations as Jin and Xin [22], we call the discretization the Relaxing Schemes. We will also derive the zero relaxation limit for the scheme. This limit should be a consistent and stable discretization of the equation (1.1), the original conservation laws. Since here, in addition to the stiff source term, the convection term is also stiff, it is necessary to overcome the difficulty with the stiff source term. Special care must be taken to ensure that the schemes possess the correct zero relaxation limit, in the sense that the asymptotic limit that leads from system (3.8) to (3.10) should be preserved (at a discrete level). When we now concentrate on the stiff regime, $\epsilon << 1$, we also face the type of problems for underresolved numerical methods. They are well known to be stable, but may result in spurious numerical solutions totally unphysical. To overcome the difficulty with the diffusive limit we need numerical schemes with the correct diffusive limit. From known numerical methods, when the convective part is kept explicit and the relaxation part implicit, the Courant-Friedrichs-Levy(CFL) Condition is like $\mathbf{\Delta t \approx \epsilon \Delta x}$. Results from [21, 3] state that this is too restrictive and unnecessary near the parabolic(diffusive) regime where $\epsilon < \mathbf{\Delta x}$. A cfl-condition of diffusive type $\mathbf{\Delta t \sim (\Delta x)^2}$ is expected.

5.2 Upwind based discretizations.

The spatial domain R is discretized into cells $I_j = (x_{j-1/2}, x_{j+1/2})$, spatial grid points $x_{j+1/2}$, with mesh width $h_j = x_{j+1/2} - x_{j-1/2}$.
Similarly, the time interval (0,T):
The discrete time level t_n, spaced uniformly with space step $\Delta t = t_{n+1} - t_n$ for n = 0,1,2,...
We will call U_j^n the cell average and $U_{j+1/2}^n$ the nodal (point) value of U at $x = x_{j+1/2}, t = t_n$.

U_j^n is defined by:

$$U_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} U(x,t^n)dx.$$

In order to write a stable discretization to system (3.8) we can use implicit temporal integrators on the stiff terms. The simplest way to do this is to use backward Euler method for both the convection and the source term. In conservative form, the relaxing scheme may take the semi-implicit form, see e.g. [4],

$$\begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\Delta t} + \dfrac{v_{j+1/2}^n - v_{j-1/2}^n}{\Delta x} = 0, \\ \dfrac{v_j^{n+1}-v_j^n}{\Delta t} + \dfrac{1}{\epsilon^2}B'(u)u_x\dfrac{(u_{j+1/2}^{n+1}-u_{j-1/2}^{n+1})}{\Delta x} = -\dfrac{1}{\epsilon^2}(v_j^{n+1} - f(\gamma_j, u_j^{n+1})). \end{cases} \tag{5.1}$$

Finding upwind relaxing fluxes. With the aim to expose the basic ideas more clearly, we need to specify the scheme by relating the nodal flux values (numerical fluxes) $u_{j\pm1/2}^{n+1}, v_{j\pm1/2}^{n+1}$ to the cell averaged values u_j, v_j, also in order to have an economical discretization procedure.
To determine the nodal values for the system (5.1), we apply the first order upwind scheme on the variables **u** and **v** at the generic time $\mathbf{t^n}$, see [21]. We obtain

$$v_{j+1/2}^n = v_{j+1}^n, \quad u_{j+1/2}^n = u_j^n, u_{j-1/2}^n = u_{j-1}^n, v_{j-1/2}^n = v_j^n.$$

The values $u_{j-1/2}^n, v_{j-1/2}^n$ are obtained by translating j to j-1.
The upwind selection technique in [3] gives the nodal values for system (5.1).

$$\begin{aligned} u_{j\pm1/2}^n &= \frac{1}{2}(u_j^n + u_{j\pm1}^n) \pm \frac{\epsilon}{2}(v_j^n - v_{j\pm1}^n), \\ v_{j\pm1/2}^n &= \frac{1}{2}(v_j^n + v_{j\pm1}^n) \pm \frac{1}{2\epsilon}(u_j^n - u_{j\pm1}^n). \end{aligned} \tag{5.2}$$

With these choices of relaxing fluxes, we then propose the following finite difference scheme

$$\begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\Delta t} + \dfrac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} - \dfrac{1}{2\epsilon\Delta x}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) = 0, \\ \dfrac{v_j^{n+1}-v_j^n}{\Delta t} + \dfrac{B'(u)}{2\epsilon^2\Delta x}(u_{j+1}^n - u_{j-1}^n) - \dfrac{B'(u)}{2\epsilon\Delta x}(v_{j+1}^n - 2v_j^n + v_{j-1}^n) = -\dfrac{1}{\epsilon^2}(v_j^{n+1} - f(\gamma_j, u_j^{n+1})). \end{cases} \tag{5.3}$$

As $\epsilon \to 0$, we obtain the equilibrium fluxes

$$v_j^n = \frac{B'(u)(u_{j+1}^n - u_{j-1}^n)}{2\Delta x}.$$

$$\begin{cases} v_{j+1}^n = B'(u_{j+1}^n)(u_{j+2}^n - u_j^n), \\ v_{j-1}^n = B'(u_{j-1}^n)(u_j^n - u_{j-2}^n). \end{cases}$$

We can verify that the use of (5.2) in the discrete equations, for small values of ϵ, leads to the scheme of leading order

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{B'(u_{j+1}^n)(u_{j+2}^n - u_j^n)}{(2\Delta x)^2} - \frac{B'(u_{j-1}^n)(u_j^n - u_{j-2}^n)}{(2\Delta x)^2} - \frac{(u_{j+1}^n - 2u_j^n + u_{j-1}^n)}{2\epsilon\Delta x} = 0. \quad (5.4)$$

5.3 Modified Schemes

In this section we show how it is possible to modify the upwind schemes in order to have the correct asymptotic behaviour and to capture the proper parabolic behaviour. The idea from [3, 4], is to use the upwind selection to the system on the variable $(u \pm j)/2$ instead of the characteristic variables $(u \pm \epsilon j)/2$. This choice leads to the fluxes

$$v_{j\pm1/2}^n = \frac{1}{2}(v_j^n + v_{j\pm1}^n) \pm \frac{1}{2}(u_j^n - u_{j\pm1}^n). \quad (5.5)$$

Applying these fluxes, the equilibrium of the relaxation system (5.1) now reads

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{B'(u_{j+1}^n)(u_{j+2}^n - u_j^n)}{(2\Delta x)^2} - \frac{B'(u_{j-1}^n)(u_j^n - u_{j-2}^n)}{(2\Delta x)^2} - \frac{(u_{j+1}^n - 2u_j^n + u_{j-1}^n)}{2\Delta x} = 0, \quad (5.6)$$

which is a consistent approximation of the equilibrium equation with an accuracy of $\mathcal{O}(\Delta x/2)$. So, the discretization defined by (5.5) applies to the discrete equation (5.1) and has the correct diffusion limit.

5.4 Reformulation of the problem

In the previous sections, we have studied the diffusive behaviour of upwind schemes. In particular we saw how it is possible to construct upwind schemes which are also able to capture the correct asymptotic behaviour. We proposed a possible solution to this problem based on upwind fluxes. However, in practice, the implicit time integrator may present several limitations like the gain of stability is partially offset by the loss of accuracy, typical of implicit schemes in the context of wave propagation phenomena. On the contrary, the use of explicit schemes leads to a Courant-Friedrichs-Levy (CFL) condition of the type $\mathbf{\Delta t \sim \epsilon \Delta x}$ which is to restrictive when the equilibrium equations are of the hyperbolic type and unnecessary near the parabolic regime where we expect a stability condition like $\mathbf{\Delta t \sim (\Delta x)^2}$. Since both the above options have advantages as well as drawbacks, it is natural to look for a scheme with mixed character. We will in this section look for a reformulated problem and compare the results with the our proposed relaxing scheme (5.3).

In particular, the special structure of the reformulated problem enables one to solve it numerically without using either modified upwind schemes spatially or nonlinear systems of algebraic equations solvers at each time step.
We will discuss further the additional problem due to the stiff convection term. The idea is to reformulate the problem for the relaxation schemes to conservation laws by properly combining the stiff component of the convection term into the relaxation term. We need to overcome this difficulty with an adequate and economical discretization procedure. We want the schemes also to have the correct diffusion limit. The strategy is based on splitting the equations into a hyperbolic conservation law for convection and a possibly degenerate parabolic equation for diffusion. We define a hyperbolic solver for the conservation law, while the diffusion equation is solved implicitly. We need a splitting method that is unconditionally stable in the sense that the splitting time step $\mathbf{\Delta t}$ is not limited by the space discretization $\mathbf{\Delta x}$. And finally, we want our approach to handle the convection-diffusion combination, including the purely hyperbolic case.

5.4.1 Operator splitting methods

First it is interesting to point out how earlier numerical approaches that work for relaxation systems with stiff source terms apply to these problems. We demonstrate the popular operator splitting method where we do a proper splitting of an explicit convection step from an implicit source term.
I. The usual splitting applied to system (3.8). We split into convection part

$$\begin{cases} u_t + v_x = 0, \\ v_t + \dfrac{1}{\epsilon^2}\partial_x B(u) = 0, \end{cases}, \tag{5.7}$$

and the relaxation part

$$\begin{cases} u_t = 0, \\ v_t = -\dfrac{1}{\epsilon^2}(v - f(\gamma(x), u)). \end{cases} \tag{5.8}$$

In the zero relaxation (or diffusion) limit, $\epsilon \to 0^+$, the system (3.8) is approximated to leading order by

$$v - f(\gamma(x), u) = 0, \frac{\partial u}{\partial t} = 0, \frac{\partial v}{\partial t} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0. \tag{5.9}$$

We obtain the equilibrium equations

$$\begin{cases} f(\gamma(x), u)_x = 0, \\ \dfrac{\partial u}{\partial x} = 0. \end{cases} \tag{5.10}$$

We can see that this splitting is inconsistent with (3.10). So this splitting will give poor results near the parabolic region. The usual splitting do not suffice to give a scheme with uniform accuracy in ϵ.

II. We try to overcome this problem. As discussed in the introduction of this section, the key idea is to write (3.8) in the form, see [3, 21, 1],

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \nu \partial_x B(u) = -\dfrac{1}{\epsilon^2}(v - f(\gamma(x), u)), \end{cases} \tag{5.11}$$

A suitable parameter ν is introduced, where $\nu \leq \mathbf{1}/\epsilon^{\mathbf{2}}$ is a nonnegative parameter. The parameter ν allows to move the stiff terms without losing the hyperbolicity of the system. It is immediately recognisable that system (5.11) has the form known from usual hyperbolic systems with stiff relaxation term. This new formulation allows us to try the numerical techniques already developed for hyperbolic problems with stiff relaxation [22, 18, 9]. One of the technique in those problems is to split the system into an explicit convection step and an implicit source term. We treat the relaxation step implicitly for better numerical stability.
We split our model (5.11) into two subproblems:

$$\begin{cases} \dfrac{\partial u}{\partial t} = 0, \\ \dfrac{\partial v}{\partial t} = -\dfrac{1}{\epsilon^2}(v - f(\gamma(x), u)). \end{cases} \tag{5.12}$$

And

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial v}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} + \nu \dfrac{\partial}{\partial x} B(u) = 0. \end{cases} \tag{5.13}$$

Now we have a pure nonstiff convection step (5.13) and a stiff source step (5.12), stiff relaxation part.
But at this stage, it is not obvious that this splitting provides any simplification to the challenging numerical solution of the problem (3.8).
The idea now is to solve (5.13) using upwind approximations and (5.12) with a numerical method that possesses the proper diffusive limit.
We will use an explicit scheme for the convection step and solve the source term implicitly. As demonstrated in previous section, when $\epsilon \to \mathbf{0}$, the relaxation step (5.12) always projects the solution to the correct local equilibrium, which is a sufficient condition to

guarantee that the splitting preserves the asymptotic limit from the relaxation system to the convection-diffusion equation.

A first order splitting scheme for the model problem (5.11) is defined by

$$\begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\Delta t}=0, \\ \dfrac{v_j^{n+1}-v_j^n}{\Delta t}=-\dfrac{1}{\epsilon^2}(v_j^{n+1}-f(\gamma(j),u_j^{n+1})). \end{cases} \tag{5.14}$$

And

$$\begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\Delta t}+\dfrac{v_{j+1/2}^n-v_{j-1/2}^n}{\Delta x}=0, \\ \dfrac{v_j^{n+1}-v_j^n}{\Delta t}+\nu\dfrac{B(u_{j+1/2}^n)-B(u_{j-1/2}^n)}{\Delta x}=0. \end{cases} \tag{5.15}$$

The first step here is fully implicit, so in the small relaxation limit, we get the correct local equilibrium $\mathbf{v_j^{n+1}}=\mathbf{f}(\gamma(\mathbf{j}),\mathbf{u_j^{n+1}})$, independent of ν. We apply this equilibrium state into the second step. Then it can be upwinded by using a shock-capturing scheme for the computation of $\mathbf{v^n_{j\pm1/2}}$ and $\mathbf{B(u^n_{j\pm1/2})}$ in (5.15). In order to satisfy the subcharacteristic condition, we have to impose the stability criteria

$$\nu B'(u)\geq f'(\gamma,u)^2. \tag{5.16}$$

To define the scheme we need to relate the equilibrium fluxes $B(u^n_{j\pm1/2})$ to the nodal values for $\mathbf{u}$ and $\mathbf{v}$. First, due to the structure of problem (5.14) and in order to avoid solving systems of algebraic equations in $\mathbf{v}$, we seek for a second order accurate definition of these fluxes independent of the nodal values for $\mathbf{v}$. In fact, this permits one to evaluate the relaxation step explicitly because $\mathbf{u}$ does not change in time in (5.14) .
So we propose to define the equilibrium fluxes $B(u^n_{j\pm1/2})$ of centred form

$$\begin{cases} B(u_{j+1/2}^n)=\dfrac{B'(u_j^n)}{2}(u_{j+1}^n-u_j^n), \\ B(u_{j-1/2}^n)=\dfrac{B'(u_j^n)}{2}(u_j^n-u_{j-1}^n). \end{cases} \tag{5.17}$$

To define the scheme, we can apply the computations

$$\begin{cases} v_{j+1/2}^n=\dfrac{1}{2}(v_j^n+v_{j+1}^n)+\dfrac{1}{2}(u_j^n-u_{j+1}^n), \\ v_{j-1/2}^n=\dfrac{1}{2}(v_j^n+v_{j-1}^n)-\dfrac{1}{2}(u_j^n-u_{j-1}^n). \end{cases} \tag{5.18}$$

We select nodal values by centred schemes to avoid solving nonlinear systems of algebraic equations. In (5.14), we can evaluate the relaxation step explicitly because u does not

change in time, i.e. $\mathbf{u_j^{n+1} = u_j^n}$ during the step. Previous numerical results also show that a robust, economical discretization for the correct numerical behaviour near the parabolic regime should be based on implicit centred schemes. The solution procedure is split into the following two steps.

5.4.2 Convection step

We present the discretization of the convection step (5.15). Since this step is now nonstiff and hyperbolic, a natural choice is to use explicit upwind schemes. We solve (5.15) for $\mathbf{u_j^{n+1}}$ and $\mathbf{v_j^{n+1}}$. We now have

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{\Delta t}{\Delta x}(v_{j+1/2}^n - v_{j-1/2}^n), \\ v_j^{n+1} &= v_j^n - \nu\frac{\Delta t}{\Delta x}(B(u_{j+1/2}^n) - B(u_{j-1/2}^n)). \end{aligned} \tag{5.19}$$

We apply straightforward the upwind schemes (5.18) and the proposed centred fluxes (5.17), the final upwind schemes in the convection step can be written as

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{\Delta t}{2\Delta x}[(v_{j+1}^n - v_{j-1}^n) - (u_{j+1}^n - 2u_j^n + u_{j-1}^n)], \\ v_j^{n+1} &= v_j^n - \nu\frac{B'(u_j^n)}{2}\frac{\Delta t}{\Delta x}[u_{j+1}^n - 2u_j^n + u_{j-1}^n]. \end{aligned} \tag{5.20}$$

The parameter ν can be used as a weight function in the nonstiff regime and can be chosen to depend on the discretization parameters. However, numerical experience, see e.g. [3, 21, 2], shows that in most practical situations the simple choice $\nu = 1$ for $\epsilon \leq 1$ suffices to give accurate and stable discretizations. With the simplest choice of $\nu = 1$, we define $\mu = \frac{\Delta t}{\Delta x}$. The CFL number satisfying $\mu \leq 1$.

5.4.3 Relaxation step

Our goal is to develop a scheme where the source term is treated implicitly for better numerical stability. We take into account that u does not change in time during this step. We use a linear scheme in this step.

$$\begin{aligned} u_j^{n+1} &= u_j^n, \\ v_j^{n+1} &= v_j^n - \frac{\Delta t}{\epsilon^2}(v_j^{n+1} - f(\gamma(j), u_j^{n+1})). \end{aligned} \tag{5.21}$$

Although we have an implicit relaxation term, the new values $\mathbf{u^{n+1}}$ and $\mathbf{v^{n+1}}$ can be updated explicitly since the values of $\mathbf{u_j^{n+1}}$ can be computed from the first equation in

(5.14) and the term in $\mathbf{v_j^{n+1}}$ is linear. In addition, as $\epsilon \to \mathbf{0}$ in (5.21), it projects over the correct local equilibrium

$$v_j^{n+1} = f(\gamma(j), u_j^{n+1}).$$

If we use an implicit diffusion solver, our operator splitting method is unconditionally stable in the sense that the time step Δt is not limited by the space discretization Δx.

5.4.4 Splitting Procedure.

An underlying design principle for many successfull numerical methods for equations such as (1.1), is viscous operator splitting. That is, one splits the time evolution into two partial steps in order to separate the effects of convection and diffusion. The above strategy of splitting the method into a relaxation step and convection step involves a splitting error which makes the method first order accurate in time, irrespective of whatever higher order discretization is used in time and space in both steps.
We shall describe the operator splitting. That is, we will obtain the solution of (1.1) through a composition of solution operator for the convection step and for the degenerate parabolic problem.
Before describing the splitting algorithms in more detail, we define the solution operators for our two different equations. We call $\mathcal{H}(t)$ the approximate solution operator for the convection part, and $\mathcal{S}(t)$ denotes the solution operator for the relaxation step. The viscous splitting method is then based on the following approximation

$$u(x, n\Delta t) \approx [\mathcal{S}(\Delta t)\mathcal{H}(\Delta t)]^n u_0(x).$$

We fix T>0 and $\Delta t > 0$, and let N be such that $N\Delta t = T$. Let $\mathbf{u^n}$ denote the approximate solution to (1.1) at fixed time $t_n = n\Delta t (n = 0, ..., N-1), u^0 = u_0$. We construct approximation u^{n+1} from u_n by the product formula

$$u^{n+1} = [\mathcal{S}_{\Delta t} \circ \mathcal{H}_{\Delta t}]u^n.$$

In applications, the exact solution operators $\mathcal{H}(t)$ and $\mathcal{S}(t)$ are replaced by numerical methods to fully propose a discrete splitting method. Here the solution operator for the convection part is replaced by a solution generated by the upwind schemes (5.20) and the solution operator for the relaxation part is replaced by an implicit centred scheme.
We remark that in applications, Strang splitting technique is often used, e.g. see [22, 12, 9, 18, 21, 20]. In Strang splitting, the relaxation step is solved for a half time step ($\frac{\mathbf{\Delta t}}{\mathbf{2}}$), followed by a convection step for full time step ($\mathbf{\Delta t}$) and then again by a relaxation step for a half time step($\frac{\mathbf{\Delta t}}{\mathbf{2}}$). Resulting in

$$U^{n+1} = \mathcal{S}(\frac{\Delta t}{2})\mathcal{H}(\Delta t)\mathcal{S}(\frac{\Delta t}{2})U^n. \tag{5.22}$$

But Jin [9] demonstrated that even the Strang splitting degenerates to first order accuracy in the limit $\epsilon \to \mathbf{0}$. Even higher order Runge-Kutta methods in the convection step cannot improve the result.
In order to improve the order of accuracy we describe a second order TVD Runge-Kutta splitting scheme introduced by Jin [9] for the Euler scaling. We will apply the second order extension directly for our convection-diffusion splitting approximation. This splitting scheme takes two implicit stiff source steps and two explicit convection steps alternatively. Various applications can be found in [20, 21, 18, 22, 3]. We then apply the second order splitting scheme to the diffusive relaxation system. The second order result is as follows: Given (u_j^n, v_j^n) , (u_j^{n+1}, v_j^{n+1}) are computed by

$$
\begin{aligned}
&u_j^* = u_j^n,\\
&v_j^* = v_j^n + \frac{\Delta t}{\epsilon^2}(v_j^* - f(\gamma(j), u_j^*)),\\
&u_j^1 = u_j^* - \Delta t D_x v_j^*,\\
&v_j^1 = v_j^* - \nu B'(u_j^n)\Delta t D_x u_j^*,\\
&u_j^{**} = u_j^1;\\
&v_j^{**} = v_j^1 - \frac{\Delta t}{\epsilon^2}(v_j^{**} - f(\gamma(j), u_j^{**})) - \frac{2\Delta t}{\epsilon^2}(v_j^* - f(\gamma(j), u_j^*)),\\
&u_j^2 = u_j^{**} - \Delta t D_x v_j^{**},\\
&v_j^2 = v_j^{**} - \nu B'(u_j^n)\Delta t D_x u_j^{**},\\
&u_j^{n+1} = \frac{1}{2}(u_j^n + u_j^2),\\
&v_j^{n+1} = \frac{1}{2}(v_j^n + v_j^2).
\end{aligned}
\tag{5.23}
$$

The convections terms are keeped explicit because, first, one does not need to solve systems of linear algebraic equations that will arise if the convection terms are implicit. Secondly, due to the special structure of the source term, one does not need to solve any systems of nonlinear algebraic equations, in spite of the implicit nonlinear source terms. Since the source terms are treated implicitly, this discretization is stable independent of ϵ, so that the choice of Δt is based only on the usual CFL-condition,
$CFL := \frac{\Delta t}{\Delta x} \leq 1$. We recall that we defined the initial state as the local equilibrium, namely $\mathbf{v}(\mathbf{x}, \mathbf{0}) = \mathbf{f}(\gamma, \mathbf{u}(\mathbf{x}, \mathbf{0}))$, then Jin and Xin [22] show that the variables $\mathbf{v_j^*}$ and $\mathbf{v_j^{**}}$ in (5.23) approximate the local equilibrium $\mathbf{f}(\gamma, \mathbf{u_j^*})$ and $\mathbf{f}(\gamma, \mathbf{u_j^{**}})$ respectively when $\epsilon \to \mathbf{0}$. Then applying $\mathbf{v_j^*} = \mathbf{f}(\gamma, \mathbf{u_j^*})$ and $\mathbf{v_j^{**}} = \mathbf{f}(\gamma, \mathbf{u_j^{**}})$ in u_j^1 and u_j^2 respectively, a second order relaxed scheme is obtained

$$
\begin{aligned}
&u_j^1 = u_j^n - \Delta t D_x v_j^n|_{v_j^n = f(\gamma(j), u_j^n)},\\
&u_j^2 = u_j^1 - \Delta t D_x v_j^1|_{v_j^1 = f(\gamma(j), u_j^1)},\\
&u_j^{n+1} = \frac{1}{2}(u_j^n + u_j^2).
\end{aligned}
\tag{5.24}
$$

To define the schemes explicitly, we use the finite differences

$$\begin{aligned} D_x u_j^* &= \frac{u_{j+1/2}^* - u_{j-1/2}^*}{\Delta x}, \\ D_x v_j^* &= \frac{v_{j+1/2}^* - v_{j-1/2}^*}{\Delta x}. \end{aligned} \tag{5.25}$$

And the numerical fluxes in (5.25) are obtained by using an upwind scheme to the characteristic variables $\mathbf{v} \pm \frac{\sqrt{\mathbf{B'(u)}}}{\epsilon}$,

$$\begin{cases} (v + \dfrac{\sqrt{B'(u)}}{\epsilon})_{j+1/2} = (v + \dfrac{\sqrt{B'(u)}}{\epsilon})_j, \\ (v - \dfrac{\sqrt{B'(u)}}{\epsilon})_{j+1/2} = (v - \dfrac{\sqrt{B'(u)}}{\epsilon})_{j+1}. \end{cases} \tag{5.26}$$

To serve our purpose in the numerical computation, we may use that Jin and Xin [22] and Jin [9] proved that the corresponding second order relaxed scheme to the splitting algorithm is consistent and TVD provided that the subcharacteristic conditions and CFL-condition are satisfied.
Consistency and TVD-property imply convergence, and converges to the limit function **u(x,t)** which is the weak entropy solution of the convection-diffusion problem.

5.5 Pseudocode for the Diffusive Relaxation Scheme (5.3)

In The following pseudocode we propose the procedure to implement the diffusive relaxation scheme for convection-diffusion problems. The method can be solved explicitly since the values of $\mathbf{u_j^n}$ can be updated and computed from the first equation and the source term in $\mathbf{v^{n+1}}$ is linear.

program parabolic
initial $u_0, v_0 = f(u_0)$
integer parameter m, N
real parameter h, eps, k, c1, c2
h $\leftarrow$ (2pi)/N
k $\leftarrow$ (h)^2
$c1 \leftarrow \epsilon/(\epsilon + k)$
c2 $\leftarrow \epsilon k/(\epsilon + k)$
x = 0:h:2pi-h
differentiation matrices M1, M2
M1 $\leftarrow [v_{j+1} - v_{j-1}]; [u_{j+1} - u_{j-1}]$
M2 $\leftarrow [v_{j+1} - 2v_j + v_{j-1}]; [u_{j+1} - 2u_j + u_{j-1}]$

assigne initial condition $U1 = u(x,0); V1 = f(u(x,0))$
collecting values for U $MM = zeros(N, m-1); MM(:,1) = U1$
loop with time step
output U^{n+1}
t=0
for nx= 2 **to** N **do**
t = t+k
U(n+1) $= u^n - kM1(v^n) + (k/\epsilon)M2(u^n)$
V(n+1) $= c1v^n - B'c2M1(u^n) + \epsilon B'c2M2(v^n) + c2f(U(n+1))$
update V and U
end for
collect U-values $MM(:,nx) = U(n+1)$

Chapter 6

Convergence

6.1 The reduced problem

As a motivation for the complex analysis of the parabolic problem, we will look closer to the hyperbolic case incorporated in our convection-diffusion problem, and present some analytical results. We present the underresolved numerical schemes for hyperbolic conservation laws with a discontinuous coefficient and the corresponding relaxed scheme.

In chapter 3, we presented a relaxation system for the hyperbolic case of the IVP, that is without the diffusion term $\mathbf{B(u)}$ and the equation is of the form

$$u_t + f(\gamma(x), u)_x = 0.$$

Now we consider the relaxing scheme for the reduced problem, a first order upwind approximation to the relaxation system (3.1) is given by

$$\begin{cases} \frac{1}{\Delta t}(u_j^{n+1} - u_j^n) + \frac{1}{2\Delta x}(v_{j+1}^n - v_{j-1}^n) - \frac{\lambda}{2\Delta x}(u_{j-1}^n - 2u_j^n + u_{j+1}^n) = 0, \\ \frac{1}{\Delta t}(v_j^{n+1} - v_j^n) + \frac{\lambda^2}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) - \frac{\lambda}{2\Delta x}(v_{j-1}^n - 2v_j^n + v_{j+1}^n) = \frac{1}{\epsilon}(f(\gamma_j, u_j^{n+1}) - v_j^{n+1}). \end{cases} \tag{6.1}$$

We can start the iteration by

$$u_j^0 = \frac{1}{\Delta x}\int_{I_j} u_0(x)dx, \qquad v_j^0 = \frac{1}{\Delta x}\int_{I_j} f(\gamma(x), u_0(x))dx. \tag{6.2}$$

The method is implicit but can be solved explicitly, since the values of $\mathbf{u_j^n}$ can be updated and computed from the first equation and the source term in $\mathbf{v^{n+1}}$ is linear. We do not have to solve a system of equations in order to update $\mathbf{u^n}$ and $\mathbf{v^n}$.

The limiting schemes are called relaxed schemes. The computational results available indicate that the relaxed scheme obtained in the limit $\epsilon \to \mathbf{0}$ provide a quite promising class of new schemes. Since the leading order behaviour of the relaxing schemes is governed by the the relaxed schemes as $\epsilon \to 0^+$, we study the behaviour of the relaxed scheme using **Singular Mapping**.
The relaxed scheme for the reduced problem takes the form

$$\begin{cases} v_j^n = f(\gamma_j, u_j^n), \\ u_j^{n+1} = u_j^n - \dfrac{\mu}{2}[f(\gamma_{j+1}, u_{j+1}^n) - f(\gamma_{j-1}, u_{j-1}^n)] + \dfrac{\mu\lambda}{2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n). \end{cases} \tag{6.3}$$

This is the leading order equation as $\epsilon \to 0^+$. For the relaxed scheme (6.3), we assume the following CFL-condition:

$$\mu\lambda \leq 1, \text{where} \quad \mu = \frac{\Delta t}{\Delta x}. \tag{6.4}$$

And the Subcharacteristic Condition:

$$0 < max_{\gamma,u}|f_u(\gamma, u)| < \lambda. \tag{6.5}$$

Our understanding of the zero relaxation limit of the relaxing scheme is that poor numerical results may be generated if the numerical scheme does not have the correct asymptotic limit.
A scheme for the relaxation system (3.1) is said to have the correct asymptotic limit if, for fixed $\mathbf{\Delta t}$ and $\mathbf{\Delta x}$ as $\epsilon \to \mathbf{0}^+$, the limiting scheme is a good (consistent and stable) discretization of the system (1.2).

6.2 Singular Mapping

In this section, we attempt to establish convergence of the relaxation approximation (3.1) using the **singular mapping approach**.

We present the definition for the approximate solutions.
Let $\Delta x > 0$ and $\Delta t > 0$ be the spatial and temporal discretization parameters. The spatial domain $\mathbf{R}$ is discretized into cells

$$\mathbf{I_j} = [\mathbf{x_{j-1/2}}, \mathbf{x_{j+1/2}}), \text{where} \quad x_k = k\Delta x, \quad \text{for} \quad k = 0, \pm 1/2, \pm 1, \pm 3/2....$$

Similarly, the time interval [0,T] is discretized via $\mathbf{t_n} = \mathbf{n\Delta t}$ for n = 0,...,N. The integer N is chosen such that $N\Delta t = T$, resulting in the strips

$$\mathbf{I^n} = [\mathbf{t_n}, \mathbf{t_{n+1}}).$$

We define $\chi_{\mathbf{j}}(\mathbf{x})$ and $\chi^{\mathbf{n}}(\mathbf{t})$ to be the characteristic function for the intervals $\mathbf{I_j}$ and $\mathbf{I^n}$ respectively.
Let $\chi^{\mathbf{n}}_{\mathbf{j}}(\mathbf{x},\mathbf{t}) = \chi_{\mathbf{j}}(\mathbf{x})\chi^{\mathbf{n}}(\mathbf{t})$ be the characteristic function for the rectangle $\mathbf{R^n_j} = \mathbf{I_j} \times \mathbf{I^n}$. The difference solution $\mathbf{u^n_j}$ generated by the scheme (6.3) is extended to all of $\prod_T$ by defining

$$u^{\Delta}(x,t) = \sum_{n\in Z_N^+}\sum_{j\in Z} \chi_j^n(x,t)u_j^n, \quad (x,t) \in \prod_T .\text{Where} \quad \Delta = (\Delta x, \Delta t). \quad Z_N^+ = 1,..,N. \tag{6.6}$$

For the discontinuous coefficient γ,

$$\gamma^{\Delta}(x) = \sum_{j\in Z} \chi_{j+1/2}(x)\gamma_{j+1/2} \quad x \in \mathbf{R}.$$

$\gamma^{\mathbf{\Delta}}(\mathbf{x})$ is approximated at each cell boundary, resulting in a discretisized version of γ

$$\gamma_{j+1/2} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \gamma(x)dx.$$

This analytical problem is solved (hopefully) by using a transformed variable $\mathbf{\Theta^{\Delta}} = \psi(\gamma^{\mathbf{\Delta}}, \mathbf{u^{\Delta}})$. The idea is to show that the relaxed scheme converges along a subsequence to a weak solution of the IVP by constructing a singular mapping

$$\psi : (\gamma, u) \to (\gamma, \Theta)$$

such that strong compactness for the sequence of transformed functions

$$\Theta^{\Delta}(x,t) = \psi(\gamma^{\Delta}(x), u^{\Delta}(x,t))$$

can be obtained. We are in intervals where $\mathbf{B'(r)} = \mathbf{0}, \forall \mathbf{r} \in \bigcup_{\mathbf{i=1}}^{\mathbf{k}}[\alpha_{\mathbf{i}}, \beta_{\mathbf{i}}] = \mathbf{\Gamma}$. $\mathbf{S}$ is the characteristic function for $\mathbf{u_i} \in [\alpha_{\mathbf{i}}, \beta_{\mathbf{i}}]$.
The singular mapping is defined by

$$\psi(\gamma, u) = \int_0^u S(r)|f_u(\gamma, r)|dr. \tag{6.7}$$

We know from [14] that the singular mapping is designed to be Lipschitz continuous and strictly increasing as a function of $\mathbf{u}$. ψ belongs to $\mathbf{Lip}([\underline{\gamma}, \overline{\gamma}] \times [\mathbf{0}, \mathbf{1}])$. Once the existence of a subsequential limit $\mathbf{\Theta}$ has been established, the invertibility of ψ then allows the corresponding weak solution $\mathbf{u}$ to be recovered from the limit $\mathbf{\Theta}$, with $\mathbf{u^{\Delta}} \to \mathbf{u}$ guaranteed by the continuity of ψ.
Assumptions: Assume the given data are satisfied, and the scheme (6.3) is applied with the parameter μ chosen so that the following CFL-condition is satisfied for each succeeding time step,

$$\mu\lambda \leq 1$$

then the computed solution remain in the interval [0,1] and the scheme (6.3) is monotone.

6.3 Monotonicity

A useful property of the entropy-satisfying weak solution is, if we take two sets of initial data $\mathbf{u_0}$ and $\mathbf{v_0}$, with

$$v_0(x) \geq u_0(x) \quad \forall x,$$

then the respective entropy solutions $\mathbf{u(x,t)}$ and $\mathbf{v(x,t)}$ satisfy

$$v(x,t) \geq u(x,t) \quad \forall x,t.$$

In numerical applications, if the analogous property

$$V_j^n \geq U_j^n \quad \forall j \Rightarrow V_j^{n+1} \geq U_j^{n+1} \quad \forall j$$

holds, then the numerical method $\mathbf{U_j^{n+1} = H(U^n; j)}$ is called a monotone method.
To check that the relaxed scheme is monotone, it suffices to check that

$$\frac{\partial}{\partial u_j^n} H(U^n; j) \geq 0 \quad \text{for all} \quad i, j, U^n.$$

This means that if we increase the value of any $\mathbf{U_j^n}$ then the value of $\mathbf{U_j^{n+1}}$ cannot decrease as a result.

The relaxation scheme defines U_j^{n+1} as a function
$\mathbf{U_j^{n+1} = H(U^n; j)}$,

$$U_j^{n+1} = H_j[u_{j+1}^n, u_j^n, u_{j-1}^n, \gamma_{j+1}, \gamma_{j-1}].$$

- i= j+1.

$$H(U^n; j) = \frac{-\mu}{2}[f(\gamma_{j+1}, u_{j+1}^n)] + \frac{\mu\gamma}{2}(u_{j+1}^n),$$
$$\frac{\partial H}{\partial U_j^n} = \frac{-\mu}{2}[f'(\gamma_{j+1}, u_{j+1}^n)] + \frac{\mu\gamma}{2}.$$

- i = j - 1.

$$H(U^n; j) = \frac{-\mu}{2}[-f(\gamma_{j-1}, u_{j-1}^n)] + \frac{\mu\gamma}{2}(u_{j-1}^n),$$
$$\frac{\partial H}{\partial U_j^n} = \frac{-\mu}{2}[-f'(\gamma_{j-1}, u_{j-1}^n)] + \frac{\mu\gamma}{2}.$$

- i = j.

$$H(U^n; j) = u_j^n - \frac{\mu}{2}[f(\gamma_{j+1}, u_j^n) - f(\gamma_{j-1}, u_j^n)] + \frac{\mu\gamma}{2}(-2u_j^n),$$
$$\frac{\partial H}{\partial U_j^n} = 1 - \frac{\mu}{2}[f'(\gamma_{j+1}, u_j^n) - f'(\gamma_{j-1}, u_j^n)] - \mu\gamma.$$

The CFL-condition (6.4) and the subcharacteristic condition (6.5) guarantee that

$$\frac{\partial}{\partial u_j^n} H(U^n; j) \geq 0 \quad \text{for all} \quad i, j.$$

The relaxed scheme (6.3) is a monotone scheme.
The major drawback with a monotone scheme is that it is at best only first order accurate even in regions where the solution is smooth. In the case where γ is constant, monotonicity implies that the scheme is Total Variation Decreasing(TVD).

6.4 Compactness of approximate solutions $\mathbf{u^{\Delta}}$

We assume that initial data and γ satisfy given assumptions, with the CFL-condition mentioned above.
We will use the $\mathbf{L_1}$-contractive property in the subsequent analysis. Meaning that the relaxed scheme is $\mathbf{L_1}$-contractive if the inequality

$$\sum_j |V_j^{n+1} - U_j^{n+1}|\Delta x \leq \sum_j |V_j^n - U_j^n|\Delta x \tag{6.8}$$

holds for a pair of approximate solutions $\mathbf{u_j^n}$ and v_j^n generated by the scheme. Towers [23] propose and proves the following inequality for the related problem:

$$u_0, v_0 \in L^1 \cap L^\infty,$$
$$\sum_j |u_j^{n+1} - u_j^n|\Delta x \leq \sum_j |u_j^1 - u_j^0|\Delta x. \tag{6.9}$$

Computed solutions $\mathbf{u^{\Delta}}(\cdot, \mathbf{t^n})$ satisfy a uniform $\mathbf{L^1(R)}$ bound for $\mathbf{t^n} \in [\mathbf{0}, \mathbf{T}]$.

$\mathbf{L^1}$-contraction property:
If v^{Δ} is another solution, we have that

$$\|u^{\Delta}(\cdot, t^n) - v^{\Delta}(\cdot, t^n)\|_{L^1(R)} \leq \|u^{\Delta}(\cdot,0) - v^{\Delta}(\cdot,0)\|_{L^1(R)}.$$

Taking into account the jumps in $\mathbf{\Theta}$ at cell-boundaries, due to jumps in $\mathbf{u^{\Delta}}$ and jumps at cell centres, due to jumps in $\gamma^{\mathbf{\Delta}}$.

Total Variation of $\boldsymbol{\Theta}$ is defined by

$$\mathbf{TV}(\boldsymbol{\Theta}) = \underbrace{\sum_j |\Delta_+^u \Theta_j|}_{cell-boundaries} + \underbrace{\sum_j |\Delta_+^\gamma \Theta_{j-1/2}|}_{jumps-at-centres}. \tag{6.10}$$

The second sum , due to jumps in $\gamma^{\boldsymbol{\Delta}}$, is bounded by $\mathbf{TV}(\gamma)$, using

$$|\psi(u,\gamma_1) - \psi(u,\gamma_2)| \leq |\gamma_1 - \gamma_2|,$$

since Lipschitz continuity relationships in $\mathbf{u}$ and γ follow directly from the definition of ψ and conditions imposed on the flux $\mathbf{f}$.
For each value of γ in $[\underline{\gamma}, \overline{\gamma}]$,
$\psi(\cdot,\gamma): \quad [0,1] \to [-\gamma,\gamma]$ is an increasing 1-1 mapping.
We recall that γ is assumed to be bounded and strictly positive, $0 < \underline{\gamma} \leq \gamma(x) \leq \overline{\gamma}$.
Let $\Theta^n(x) = \psi(u^\Delta(x,t^n), \gamma^\Delta(x))$. We then define the following

$$\Delta_+^u \Theta_j = \psi(u_{j+1}, \gamma_{j+1/2}) - \psi(u_j, \gamma_{j+1/2}); \quad \Delta_+^\gamma \Theta_{j-1/2} = \psi(u_j, \gamma_{j+1/2}) - \psi(u_j, \gamma_{j-1/2}). \tag{6.11}$$

Summing over all the jumps:

$$\sum_j \Delta_+^u \Theta_j + \sum_j \Delta_+^\gamma \Theta_{j-1/2} = \varphi, \tag{6.12}$$

$$\sum_j |\Delta_+^u \Theta_j| = \sum_j (\Delta_+^u \Theta_j)_+ - \sum_j (\Delta_+^u \Theta_j)_-, \tag{6.13}$$

$$\sum_j (\Delta_+^u \Theta_j)_+ - \sum_j (\Delta_+^u \Theta_j)_- + \sum_j \Delta_+^\gamma \Theta_{j-1/2} = \varphi, \tag{6.14}$$

$$\sum_j (\Delta_+^u \Theta_j)_+ = \varphi - \sum_j (\Delta_+^u \Theta_j)_- - \sum_j (\Delta_+^\gamma \Theta_{j-1/2}). \tag{6.15}$$

From Lipschitz identity above, jumps in $\gamma^{\boldsymbol{\Delta}}$ is bounded by $TV(\gamma)$, such that

$$\sum_j \Delta_+^\gamma \Theta_{j-1/2} \leq TV(\gamma), \tag{6.16}$$

$$\Rightarrow \sum_j (\Delta_+^u \Theta_j)_+ \leq \varphi - \sum_j (\Delta_+^u \Theta_j)_- + TV(\gamma). \tag{6.17}$$

Now, applying the identities proven in [23], modified for our purpose, we can estimate $-\sum_j (\Delta_+^u \Theta_j)_-$.

From (6.11)

$$\begin{aligned} -\sum(\Delta_+^u\Theta_j) &= \sum_j \psi(u_j,\gamma_{j+1/2}) - \psi(u_{j+1},\gamma_{j+1/2}),\\ -\sum(\Delta_+^u\Theta_j)_- &= \sum_j (\psi(u_j,\gamma_{j+1/2}) - \psi(u_{j+1},\gamma_{j+1/2}))_+. \end{aligned} \tag{6.18}$$

We include $\mathbf{L^1}$-contractiveness and $\mathbf{f^*}$, the single maximum at $\mathbf{u^*} \in (\mathbf{0},\mathbf{1})$, and from the assumptions, we get

$$-\sum(\Delta_+^u\Theta_j)_- \le \frac{1}{\mu f^*}\sum|u_j^1 - u_j^0| + TV(\gamma). \tag{6.19}$$

We need to bound $\sum_j |u_j^1 - u_j^0|$,

$$\sum_j |u_j^1 - u_j^0| \le 2\mu\overline{\gamma}\|f'\|_\infty TV(u_0) + \mu f^* TV(\gamma). \tag{6.20}$$

We substitute into (6.19), to get

$$-\sum(\Delta_+^u\Theta_j)_- \le \frac{2\overline{\gamma}}{f^*}\|f'\|_\infty TV(u_0) + 2TV(\gamma). \tag{6.21}$$

These statements in place we have essentially bounded $\mathbf{TV(\Theta^n)}$ uniformly for all $n \ge 0$, and all $\Delta > 0$. Establishing $\mathbf{L^1}$ compactness of $\mathbf{\Theta^\Delta}$.
Application of propositions

$$R \times [0,\infty) \to [0,1] \quad \text{and} \quad 0 < \underline{\gamma} \le \gamma(x) \le \overline{\gamma}$$

gives uniform bounds on $\|\mathbf{\Theta^\Delta}(\cdot,\mathbf{t})\|_\infty$ and $\|\mathbf{\Theta^\Delta}(\cdot,\mathbf{t})\|_{\mathbf{L^1}}$ for any compact interval, providing a uniform bound on $\mathbf{TV(\Theta^\Delta}(\cdot,\mathbf{t}))$.
By standard compactness arguments applied to the sequence $\mathbf{\Theta^\Delta}$, there is a subsequence, also denoted $\mathbf{\Theta^\Delta}$, which converges in $\mathbf{L^1_{loc}}\prod_{\mathbf{T}}$ to some function
$\mathbf{\Theta} \in \mathbf{L^1_{loc}}(\prod_{\mathbf{T}}) \bigcap \mathbf{L^\infty}(\prod_{\mathbf{T}})$, $\mathbf{\Theta^\Delta} \to \mathbf{\Theta}$.
Let $u(x,t) = \psi^{-1}(\gamma(x),\Theta(x,t))$.
Due to strict monotonicity of $\psi(\cdot,\gamma)$, the function $\mathbf{u}$ is well defined a.e., $\mathbf{u} \in [\mathbf{0},\mathbf{1}]$ a.e., and $\mathbf{u} \in \mathbf{L^1_{loc}}(\prod_{\mathbf{T}}) \bigcap \mathbf{L^\infty}(\prod_{\mathbf{T}})$.
We will now use the fact that $u^\Delta = \psi^{-1}(\gamma^\Delta,\Theta^\Delta)$, to show that $u^\Delta \to u$. An estimate of $|u - u^\Delta|$ is necessary, requiring a bound for $|\psi^{-1}(\gamma,\Theta^\Delta) - \psi^{-1}(\gamma^\Delta,\Theta^\Delta)|$.
Due to the continuity of $\psi^{-\mathbf{1}}$ as a function of its second argument, we can write the result as

$$\begin{aligned} |\psi(\gamma,u^\Delta) - \psi(\gamma,u)| &\le |\psi(\gamma,u^\Delta) - \psi(\gamma^\Delta,u^\Delta)| + |\psi(\gamma^\Delta,u^\Delta) - \psi(\gamma,u)|,\\ &\le \|\psi_\gamma|\gamma - \gamma^\Delta| + |\Theta^\Delta - \Theta|. \end{aligned}$$

Since $\gamma^{\mathbf{\Delta}} \to \gamma$ a.e. and $\mathbf{\Theta^\Delta} \to \mathbf{\Theta}$ a.e. $\psi(\gamma,\mathbf{u^\Delta}) \to \psi(\gamma,\mathbf{u})$ a.e. in $\prod_{\mathbf{T}}$.
And since $\psi(\gamma,\cdot)$ is strictly increasing, it follows that $\mathbf{u^\Delta} \to \mathbf{u}$ boundedly.
The CFL-condition guarantees that the computed solutions $\mathbf{u^\Delta}$ remain within [0,1].
Convergence in $\mathbf{L^1_{loc}}(\prod_{\mathbf{T}})$ follows, and $\mathbf{u}$ is a weak solution of the conservation law.

6.5 Error Estimates

In this section we will present statements and results about convergence rates pointed out in [6, 7, 19, 16] for the relaxation scheme (6.1) by looking at the accuracy of relaxation scheme for solving the conservation law (1.2). This is done by studying the error of approximation $\mathbf{u} - \mathbf{u}^{\epsilon}_{\boldsymbol{\Delta}}$ between the exact solution $\mathbf{u}$ and the numerical solution $\mathbf{u}^{\epsilon}_{\boldsymbol{\Delta}}$ measured in $\mathbf{L^1}$ norm. The parameters ϵ and $\boldsymbol{\Delta x}$ determine the scale of approximation and converge to zero as the scale becomes finer. They call the order of this error in these parameters the convergence rate of the numerical solution generated by relaxation scheme.
To make this point precise, we choose the initial data for (3.1) as

$$(I_1) u_0^{\epsilon} := u_0(x), v_0^{\epsilon} = f(u_0(x)) + K(x)\omega(\epsilon),$$

where $\mathbf{K} \in \mathbf{L}^{\infty} \cap \mathbf{L^1}(\mathbf{R}) \cap \mathbf{BV}(\mathbf{R})$, $\omega : [0, \infty[\rightarrow [0, \infty[$ is continuous, and $\omega(\mathbf{0}) = \mathbf{0}$. Here we allow for an initial error $\mathbf{K}(\mathbf{x})\omega(\epsilon)$ instead of $\mathbf{v_0^{\epsilon}} = \mathbf{f}(\gamma(\mathbf{x}), \mathbf{u_0})$ because we want to see the contribution of this error to the global error. It is possible to consider perturbed data in the $\mathbf{u}$-component, then in the final result an initial error $\|\mathbf{u_0^{\epsilon}} - \mathbf{u_0}\|_{\mathbf{L^1(R)}}$ would persist in time and may prevent the convergence of $\mathbf{u}^{\epsilon}$ to the entropy solution. However, the initial error in the $\mathbf{v}$-component persists only for a short time of order ϵ, thereby it does not prevent the convergence of $\mathbf{u}^{\epsilon}$.
We initialise the relaxation scheme (6.1) by the cell averaging the initial data $(\mathbf{u_0^{\epsilon}}, \mathbf{v_0^{\epsilon}})$ in the usual way

$$(u_j^0, v_j^0) = \frac{1}{\Delta x} \int (u_0^{\epsilon}(x), v_0^{\epsilon}(x)) \chi_j(x) dx. \tag{6.22}$$

$\chi_{\mathbf{j}}(\mathbf{x})$ denotes the indicator function $\chi_{\mathbf{j}}(\mathbf{x}) := \mathbf{1}_{\{|\mathbf{x}-\mathbf{j}\boldsymbol{\Delta}\mathbf{x}| \leq \boldsymbol{\Delta}\mathbf{x}/\mathbf{2}\}}$. We will apply the following notations. The $\mathbf{L^1}$-norm is denoted by $|\cdot\|_{\mathbf{1}}$, and The BV-norm is defined as

$$\|u\|_{BV} = \|u\|_1 + TV(u).$$

For grid functions the total variation is defined by

$$TV(u^n) = \sum_{i \in \mathbb{Z}} |u_i^n - u_{i-1}^n|,$$

and $\|\cdot\|_1$ denotes the discrete $\mathbf{l^1 - norm}$

$$\|u^n\|_1 = \Delta x \sum_{i \in \mathbb{Z}} |u_1^n|.$$

Taking initial data (6.22), we summarise the main convergence rate result by stating the following, see e.g.[6].

Theorem 6.5.1. *Take any* $\mathbf{T > 0}$ *and let the relation* $\mathbf{T = N\Delta t}$ *be satisfied for a suitable* $\mathbf{N} \in \mathbb{N}$ *and time step* Δt*. Further, let* u *be the entropy solution of (1.2) with initial data* $\mathbf{u_0 in L^\infty(R) \cap BV(R)}$*, and let* $(\mathbf{u^N, v^N})$ *be a piecewise constant representation on* $\mathbf{R \times [0, T]}$ *of the approximate solution* $\mathbf{(u_i^n), v_i^n)_{i \in Z, 0 \le n \le N}}$ *generated by the relaxation scheme (6.1) with initial data satisfying* (I_1) *and (6.22). Then for fixed* $\mu = \frac{\Delta t}{\Delta x}$ *satisfying the CFL condition* $\mu\lambda < 1$*, there exists a constant* C_T*, independent of* $\Delta x, \Delta t$*, and* ϵ*, such that*

$$\|u^N - u(\cdot, T)\|_1 \le C_T[\sqrt{\epsilon} + \sqrt{\Delta x}]. \tag{6.23}$$

Theorem (6.5.1) suggests that the accumulation of errors comes from two sources: the relaxation error and the discretization error. To explain the structure of the proof, it may be helpful to consider that the relaxation scheme was designed through two steps, namely the the relaxation step and the discretization step. The basic idea is to investigate the error bound of the two steps separately and then the total convergence rate by combining the relaxation error and the discretization error.
We split the error $e^\epsilon_\Delta = u(\cdot, T) - u^\epsilon_\Delta(\cdot, t_N)$ into a relaxation error e^ϵ with $\|e^\epsilon\|_1 \le C_T\sqrt{\epsilon}$, and a discretization error e_Δ with $\|e_\Delta\|_1 \le C_T\sqrt{\Delta x}$, i.e., we have the decomposition

$$e^\epsilon_\Delta = e^\epsilon + e_\Delta.$$

$$e^\epsilon = u(\cdot, T) - u^\epsilon(\cdot, T),$$
$$e_\Delta = u^\epsilon(\cdot, T) - u_\Delta(\cdot, t_N).$$

We review some assumptions and preliminaries with the specific initial data $(\mathbf{u_0^\epsilon, v_0^\epsilon})$, which will be of use in the error analysis. We make the following assumptions
$(\mathbf{I_2})$ the flux function f is $\mathbf{C^1}$ with f(0) = f'(0) = 0;
$(\mathbf{I_3})$ the initial data satisfy $(u_0^\epsilon, v_0^\epsilon) \in L^1(R) \cap L^\infty \cap BV(R)$ and there exist constants $\rho_0 > 0, M > 0$ not depending on ϵ such that

$$\rho_0 = max\Big(\sup_{\epsilon>0} \|v_0^\epsilon\|_\infty, \sup_{\epsilon>0} \|u_0^\epsilon\|_\infty\Big), \quad \|(u_0^\epsilon, v_0^\epsilon)\|_{BV} := \|u_0^\epsilon\|_{BV} + \|v_0^\epsilon\|_{BV} \le M,$$

and for the flux function $\mathbf{f}$ as well as $\mathbf{K}$ given in $(\mathbf{I_1})$,

$$Lip(f) := \sup_{x \ne y} |\frac{f(x) - f(y)}{- y}| \le M, \|K\|_1 \le M. \quad \square$$

Equipped with assumptions in $(\mathbf{I_1}) - (\mathbf{I_3})$, it has been proved that, as $\epsilon \to 0^+$, our main result on the limit $\epsilon \downarrow 0$ is summarised in the following theorem.

Theorem 6.5.2. *Consider the system (3.1), subject to* $L^\infty(R) \cap BV(R)$*-perturbed initial data satisfying* $(\mathbf{I_1}) - (\mathbf{I_3})$*. Then the global solution* (u^ϵ, v^ϵ) *converges to* $(u, f(\gamma, u))$ *as* $\epsilon \downarrow 0$ *and the following error estimates hold:*

$$\|u^\epsilon(\cdot, t) - u(\cdot, t)\|_1 \le C_T\sqrt{\epsilon}, \tag{6.24}$$

$$\|v^\epsilon(\cdot, t) - f(\gamma, u^\epsilon(\cdot, t))\|_1 \le C_T[e^{\frac{-t}{\epsilon}}\omega(\epsilon) + \epsilon(1 - e^{\frac{-t}{\epsilon}})], \quad 0 \le t \le T. \tag{6.25}$$

Thus, (6.25) reflects two sources of error which are the initial contribution of size $\omega(\epsilon)$ and the relaxation error of order ϵ. However, we mention that the effect of the initial contribution persists only for a short time of order ϵ, and beyond this time the nonequilibrium solution approaches a state close to equilibrium at an exponential rate. The discrete initial data satisfy

$$\begin{aligned} &max(\|u_\Delta^0\|_\infty, v_\Delta^0\|_\infty) \leq \rho_0, \\ &TV(u_\Delta^0) + TV(v_\Delta^0) \leq M, \\ &\|v_\Delta^0 - f(\gamma, u_\Delta^0)\|_1 \leq M\omega(\epsilon). \end{aligned} \tag{6.26}$$

The grid parameters $\mathbf{\Delta x}$ and $\mathbf{\Delta t}$ are assumed to satisfy $\frac{\mathbf{\Delta t}}{\mathbf{\Delta x}} =$ constant. So, since $\mu = \frac{\mathbf{\Delta t}}{\mathbf{\Delta x}}$ is assumed constant, $\mathbf{\Delta x} \to \mathbf{0}$ implies $\mathbf{\Delta t} \to \mathbf{0}$ as well.
We know the usual projection error, of order $\mathbf{\Delta x}$, see e.g. [11].

$$\begin{aligned} &\|u_\Delta^0 - u_0^\epsilon\|_1 \leq \Delta x TV(u_0^\epsilon), \\ &\|v_\Delta^0 - v_0^\epsilon\|_1 \leq \Delta x TV(v_0^\epsilon). \end{aligned} \tag{6.27}$$

As was shown by Natalini and Aregba-Driollet,[19], for a large enough constant λ a uniform bound for the numerical approximations given by scheme (6.1) can be found. More precisely, there exists a positive constant $\overline{M}(\rho_0)$ such that if

$$\lambda > \overline{M}(\rho_0),$$

then the numerical solution satisfies

$$(u_j^n, v_j^n) \in K_{\rho_0} := \{(u, v) \in R^2, |u \pm \frac{v}{\lambda}| \leq \overline{B}(\rho_0)\}, \tag{6.28}$$

where $\overline{B}(\rho_0)$ is a constant depending only on ρ_0. For the proofs of the error properties for our schemes we refer to [7, 6, 16, 19]. We summarise:
Discretization error.
We already have (u^ϵ, v^ϵ) as the weak slution of (3.1), the relaxation system, with initial data $(u_0^\epsilon, v_0^\epsilon)$, and let (u^N, v^N) be a piecewise constant representation of the data (u_j^N, v_j^N) generated by (6.1) starting with (u_Δ^0, v_Δ^0). Then, for any fixed $T = N\Delta T \geq 0$, there is a finite constant C_T independent of $\Delta x, \Delta t$ and ϵ such that

$$\|v^\epsilon(\cdot, T) - v^N\|_1 + \|u^\epsilon(\cdot, T) - u^N\|_1 \leq C_T\sqrt{\Delta x}. \tag{6.29}$$

We remark here that they get an uniform error bound of order $\sqrt{\Delta x}$ in $\mathbf{L^1}$ which is independent of the relaxation parameter ϵ.
Relaxation error. The global solution (u^ϵ, v^ϵ) converges to $(u, f(\gamma, u))$ as $\epsilon \downarrow 0$ and the following error estimate holds:

$$\|u^\epsilon(\cdot, t) - u(\cdot, t)\|_1 \leq C_T\sqrt{\epsilon}. \tag{6.30}$$

This estimate reflects a relaxation error of order ϵ.

In the studies concerning relaxation schemes, some important properties for the numerical scheme were obtained through investigating the reformulated scheme using the Riemann Invariants and Maxwellian functions. We mention properties like the $\mathbf{L}^\infty$ boundedness, the TVD property and the $\mathbf{L^1}$ continuity in time. So, just for curiosity, we present the alternative representation for our equations.

$$R_1^\epsilon = \frac{1}{2}(u^\epsilon - \frac{v^\epsilon}{\lambda}); \quad R_2^\epsilon = \frac{1}{2}(u^\epsilon + \frac{v^\epsilon}{\lambda}).$$

$$M_1(u^\epsilon) = \frac{1}{2}(u^\epsilon - \frac{f(\gamma, u^\epsilon)}{\lambda}); \quad M_2(u^\epsilon) = \frac{1}{2}(u^\epsilon + \frac{f(\gamma, u^\epsilon)}{\lambda}).$$

$M_i(u), i = 1, 2.$

$$\sum_{i=1}^{2} M_i(u) = u; \quad \sum_{i=1}^{2} \lambda_i M_i(u) = f(\gamma, u).$$

We see that

$$u^\epsilon = R_1^\epsilon + R_2^\epsilon, v^\epsilon = \lambda(R_2^\epsilon - R_1^\epsilon).$$

Then we can rewrite the system (3.1) into a kinetic formulation

$$\partial_t R_i^\epsilon + \lambda_i \partial_x R_i^\epsilon = \frac{1}{\epsilon}[M_i(u^\epsilon) - R_i^\epsilon] \quad i = 1, 2.$$

This formulation can be used in the investigation of convergence rates for the relaxation model (3.1) as well as for the corresponding relaxing scheme (6.1).

The Riemann Invariants take the form

$$R_{1,j}^{n,\epsilon} = \frac{1}{2}(u_j^n - \frac{v_j^{n,\epsilon}}{\lambda}); R_{2,j}^{n,\epsilon} = \frac{1}{2}(u_j^n + \frac{v_j^{n,\epsilon}}{\lambda}).$$

And the Maxwellians

$$M_1(u_j^n) = \frac{1}{2}(u_j^n - \frac{f(\gamma(j), u_j^n)}{\lambda}); M_2(u_j^n) = \frac{1}{2}(u_j^n + \frac{f(\gamma(j), u_j^n)}{\lambda}).$$

It follows from the above equations that

$$u_j^{n,\epsilon} = R_{1,j}^{n,\epsilon} + R_{2,j}^{n,\epsilon}; v_j^{n,\epsilon} = \lambda(R_{2,j}^{n,\epsilon} - R_{1,j}^{n,\epsilon}).$$

6.6 Solution procedure for the hyperbolic problem

6.6.1 Pseudocode

A pseudocode to carry out the numerical process is given next. A solution procedure for the hyperbolic problem (6.1) may be accomplished like in the following pseudocode. We need to construct the first order differentiation matrix for the difference schemes. The pseudocode for Newthons method is used to find the exact solution.

program Hyperbolic relaxation
integer parameter m,N
real parameter h, eps, k, c1, c2
h $\leftarrow$ (2*pi)/N
k $\leftarrow$ (h)^2
c1 $\leftarrow$ (eps/(eps+k))
c2 $\leftarrow$ (eps*k)/(eps + k)
x = 0:h:2pi-h)
initial $u(x,0) = u_0, v_0 = f(u_0)$
differentiation matrices M1,M2
M1 $\leftarrow [v_{j+1} - v_{j-1}]; [u_{j+1} - u_{j-1}]$
M2 $\leftarrow [v_{j-i} - 2v_j + v_{j+1}]; [u_{j-1} - 2u_j + u_{j+1}]$
loop with time step
output u(n+1)
for nx = 2 **to** N **do**
t = t+k
u(n+1) = $u^u - kM1v^n + k\lambda M2u^n$
v(n+1) = $c1v^n - epsc2\lambda^2 M1u^n + epsc2\lambda M2v^n + c2f(u(n+1))$
end for
procedure exact solution
function Newton method
$x_0 = 0$
array $(x_i)_{1:N}$
for kk= 1 **to** N **do**
$x_0 = x_0 + h$;
toll = 1e-10
err = toll +1;
while err > toll **do**
fx0=x(kk)-x0-tf(x0)
dfx0 = (fx0)'
xn= x0-fx0/dfx0
err = abs(xn-x0)
x0 = xn
end while

output xi(kk)= xn
end for

6.6.2 Operator splitting for the First order relaxation scheme

We will in this section present another implementation for solving the relaxing scheme (6.1) for the hyperbolic problem. A solution procedure using operator splitting is possible. We split into two steps. For the convection step, we have the expressions for the upwind schemes.

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}[\frac{1}{2}(v_{j+1}^n - v_{j-1}^n) - \frac{\lambda}{2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n)], \tag{6.31}$$

$$v_j^{n+1} = v_j^n - \frac{\Delta t}{\Delta x}[\frac{\lambda^2}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\lambda}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n)]. \tag{6.32}$$

The source step is solved by an implicit method which avoids the time step being dependent on ϵ.

$$v_j^{n+1} = v_j^n - \frac{\Delta t}{\epsilon}(v_j^{n+1} - f(\gamma(j), u_j^{n+1})). \tag{6.33}$$

The implementation of a first order relaxation algorithm to solve (3.1) is carried out following the framework of [20, 22, 9, 18], based on Runge-Kutta type splitting method.

Given (u_j^n, v_j^n),
(u_j^{n+1}, v_j^{n+1}) are computed by

$$u_j^* = u_j^n, \tag{6.34}$$
$$v_j^* = v_j^n - \frac{\Delta t}{\epsilon}(v_j^* - f(\gamma(j), u_j^*)), \tag{6.35}$$
$$u_j^1 = u_j^* - \Delta t D_x v_j^*, \tag{6.36}$$
$$v_j^1 = v_j^* - \Delta t \lambda^2 D_x u_j^*, \tag{6.37}$$
$$u_j^{n+1} = u_j^1, \tag{6.38}$$
$$v_j^{n+1} = v_j^1. \tag{6.39}$$

We define the following finite differences

$$\begin{aligned} D_x u_j^* &= \frac{u_{j+1/2}^* - u_{j-1/2}^*}{\Delta x}, \\ D_x v_j^* &= \frac{v_{j+1/2}^* - v_{j-1/2}^*}{\Delta x}. \end{aligned} \tag{6.40}$$

We use a first order upwind scheme to the characteristic variables $\mathbf{v} \pm \lambda \mathbf{u}$ in order to obtain the numerical fluxes in (6.40), by

$$\begin{cases} (v + \lambda u)_{j+1/2} = (v + \lambda u)_j, \\ (v - \lambda u)_{j+1/2} = (v - \lambda u)_{j+1}. \end{cases} \tag{6.41}$$

To obtain

$$\begin{cases} u_{j+1/2} = \dfrac{1}{2}(u_j + u_{j+1}) - \dfrac{1}{2\lambda}(v_{j+1} - v_j), \\ v_{j+1/2} = \dfrac{1}{2}(v_j + v_{j+1}) - \dfrac{\lambda}{2}(u_{j+1} - u_j). \end{cases} \tag{6.42}$$

Using these schemes, neither algebraic equations nor nonlinear source terms can arise. The first order schemes are stable independent of ϵ, so the choice of $\mathbf{\Delta t}$ is based only on the usual CFL-condition,

$$\lambda^2 \frac{\Delta t}{\Delta x} \leq 1.$$

A splitting method that possesses the discrete analogue of the continuous asymptotic limit is able to capture the correct physical behaviours even if the small relaxation time is not numerically resolved.
We demonstrate that the discretizations above have the correct zero relaxation limit.
The initial data in local equilibrium, $\epsilon << \mathbf{1}$, $\mathbf{v}(\mathbf{x}, \mathbf{0}) = \mathbf{f}(\gamma, \mathbf{u}(\mathbf{x}, \mathbf{0}))$.
This is how we define the initial condition to avoid an initial layer where the solution undergoes sharp change [24].

$$v^n - f(\gamma(j), u^n) = 0 \quad \text{at} \quad t = t^n, \tag{6.43}$$

$$v^* - f(\gamma(j), u^*) = 0 \quad \text{at the intermediate step}, \tag{6.44}$$

$$v^{n+1} - f(\gamma(j), u^{n+1}) = O(\Delta t) \quad \text{at} \quad t = t^{n+1}. \tag{6.45}$$

From (6.31) and (6.43), we have

$$\begin{aligned} v^* - v^n &= -\frac{\Delta t}{\epsilon}(v^* - v^n + v^n - f(\gamma(j), u^*)), \\ &= -\frac{\Delta t}{\epsilon}(v^* - v^n + v^n - f(\gamma(j), u^n)), \\ &= -\frac{\Delta t}{\epsilon}(v^* - v^n). \end{aligned} \tag{6.46}$$

Thus

$$v^* - v^n = 0. \tag{6.47}$$

Applying this result into (6.31) implies

$$v^* - f(\gamma(j), u^*) = 0.$$

We now apply (6.47) in (6.33) and (6.35),

$$v^{n+1} - f(\gamma(j), u^{n+1}) = v^* - f(\gamma(j), u^{n+1}) + O(\Delta t). \tag{6.48}$$

We have that $u^{n+1} - u^* = u^{n+1} - u^n = O(\Delta t)$, so

$$\begin{aligned} v^{n+1} - f(\gamma(j), u^{n+1}) &= v^* - f(\gamma, u^n) + f(\gamma, u^n) - f(\gamma, u^{n+1}) + O(\Delta t), \\ v^{n+1} - f(\gamma(j), u^{n+1}) &= O(\Delta t). \end{aligned} \tag{6.49}$$

This confirms (6.44) and (6.45).

Chapter 7

Numerical Applications

We present numerical examples calculated by the relaxation schemes presented here. We will use MATLAB to implement the schemes, and we consider the relaxing schemes and the corresponding relaxed schemes separately. In the applications, we will apply the multiplicative form of the flux function, namely $\gamma(x)f(u)_x$ instead of $f(\gamma(x), u)_x$. In the hyperbolic problems, we choose $\epsilon = 10^{-08}$ and $\lambda = 1$ in all the computations, and we simplify the discontinuous coefficient to be $\gamma(x) = 1$. A comparison between the direct and explicit solving of relaxing scheme (6.1) and the solution procedure using splitting algorithm is presented. We will also perform some numerical tests with our proposed scheme (5.3) to approximate convection-diffusion problems. We have also calculated the $\mathbf{L^1}$ errors between the exact solutions and the numerical solutions. We consider the error for the relaxing schemes and the relaxed schemes. In our test, we use a very small relaxation parameter ϵ such that the contribution from the relaxation error $\mathbf{e}^\epsilon$ is assumed minimal. Therefore, to calculate the error we apply

$$E = \Delta x \sum_{j=1}^{N} |u(x_j) - u_N(x_j)|.$$

7.1 Linear equation

In this section, we first perform accuracy tests on a linear problem. Numerical examples calculated by the relaxation schemes for the hyperbolic case are presented. The relaxing scheme (6.1) and the relaxed scheme (6.3). The first example is the advection equation. We will compare our methods with the exact solution of the problem.
Example 1. Let us consider the scalar linear hyperbolic equation

$$u_t + au_x = 0, \tag{7.1}$$

with the initial condition u(x,0) = sin(x). The exact solution of the problem is given by

$$u(x,t) = sin(x - at). \tag{7.2}$$

We use periodic boundary condition with varying number of grid points N. We set $\Delta x = (2\pi)/N$, the length of the space step, in the numerical tests. We computed the numerical solutions at time t = 0.3, 0.5 and t = 2.

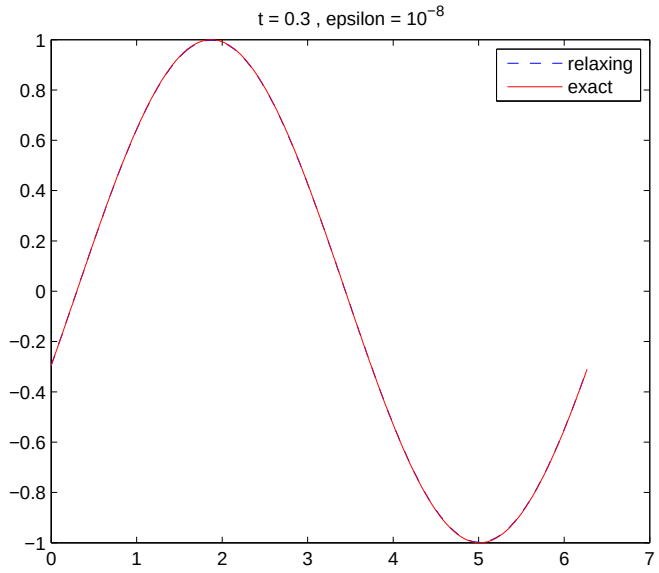

Figure 7.1: Advection equation, $u_t + au_x = 0$. Number of grid points N = 400. m = 200, time steps. $\Delta t = 0.0015, \Delta x = 2 * pi/N$. Comparing the relaxing scheme (6.1) and the exact solution for a = 1. $\epsilon = 10^{-8}$. Plot at t = 0.3.

We can see from the figures (7.1-7.6) that numerical solutions computed by the relaxing scheme (6.1) and the relaxed scheme (6.3) approximate very well the exact solution of the advection equation. These results demonstrate clearly the numerical convergence of the relaxing schemes to the corresponding relaxed scheme as $\epsilon \to 0$, and the fact that the solutions of the relaxed scheme presented here converge to the exact solution of our test problem.
Table 1 and 2 show the computed L^1-error of the difference between the numerical and the exact solution with final time t = 0.2, with varying number of grid points N. The error decreases for both the schemes but we see that the expected ratio (about 1.15) is not reached. We suspect that we do not reach the predicted ratio because of the implementation chosen. Nevertheless, further experiments reveal that the method converges nicely to the correct solution.

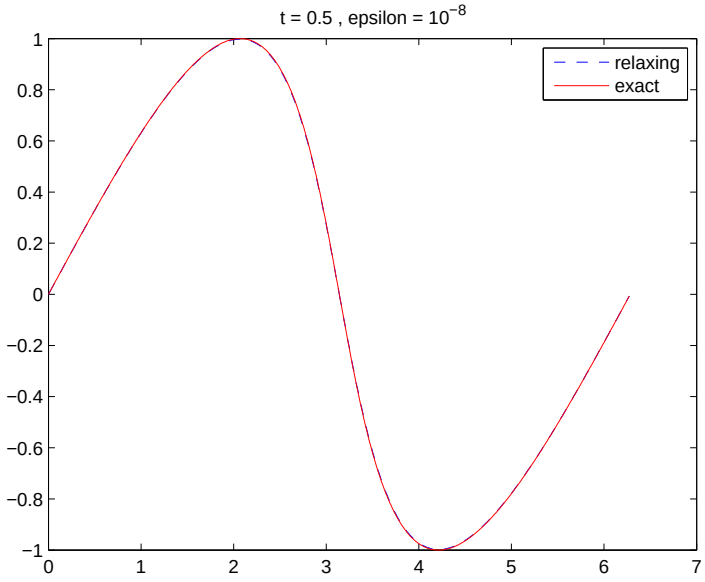

Figure 7.2: Advection equation, $u_t + au_x = 0$. $\Delta t = 0.0017, \Delta x = 2 * pi/N$. N = 600, m = 300. Comparing the relaxing scheme (6.1) and the exact solution. $\epsilon = 10^{-8}$, $\lambda = 1$, a = 1. Plot at t = 0.5.

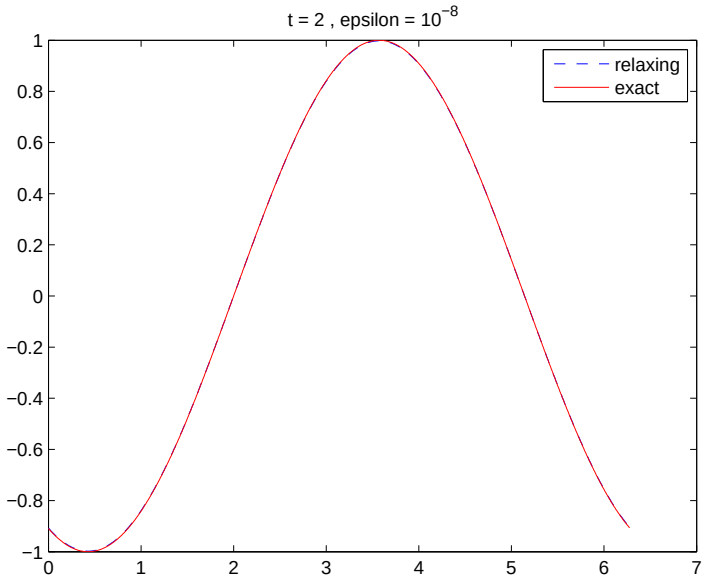

Figure 7.3: Advection equation, $u_t + au_x = 0$. $\Delta t = 0.004, \Delta x = 2 * pi/N$. N = 1000, m = 500. cfl = 0.6366. Comparing the relaxing scheme (6.1) and the exact solution. $\epsilon = 10^{-8}$, $\lambda = 1$, a = 1. Plot at t = 2.

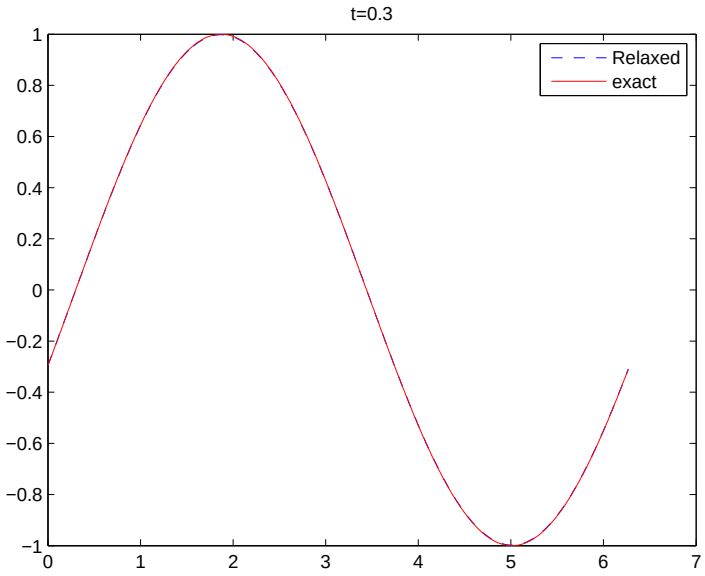

Figure 7.4: Advection equation, $u_t + au_x = 0$. N = 400, m = 200. $\Delta t = 0.0015, \Delta x = 2 * pi/N$. Comparing the relaxed scheme (6.3) with the exact solution. a = 1, plot at t= 0.3.

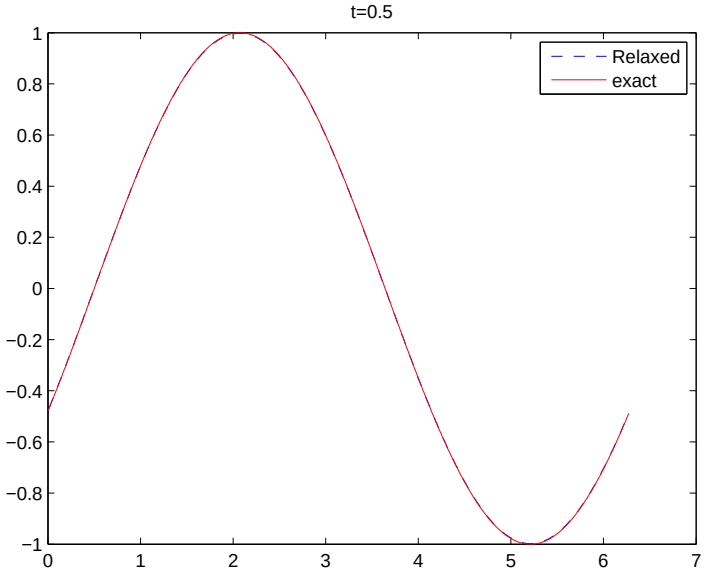

Figure 7.5: Advection equation, $u_t + au_x = 0$. $\Delta t = 0.0017, \Delta x = 2 * pi/N$. N = 600, m = 300. Comparing the relaxed scheme (6.3) and the exact solution. $\lambda = 1$, a = 1 and cfl = 0.6366. Plot at t = 0.5.

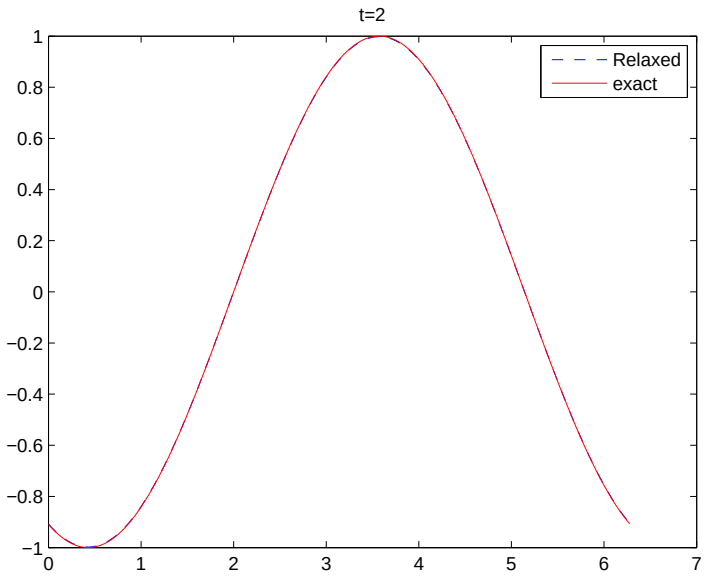

Figure 7.6: Advection equation, $u_t + au_x = 0$. $\Delta t = 0.004, \Delta x = 2 * pi/N$. N = 1000, m = 500. Comparing the relaxed scheme (6.3) and the exact solution. $\lambda = 1$, a = 1. Plot at t = 2.

N	L^1-error	Ratio
16	0.1530	
32	0.0778	1.9681
64	0.0391	1.9909
128	0.0196	1.9962
256	0.0098	1.9996
512	0.0049	2.0029

Table 1. *Discretization error in L^1-norms for the linear advection problem (7.1) at t=0.2 using relaxing schemes (6.1).*

N	L^1-error	Ratio
16	0.1530	
32	0.0778	1.9681
64	0.0391	1.9909
128	0.0196	1.9962
256	0.0098	1.9996
512	0.0049	2.0029

Table 2. *Discretization error in L^1-norms for the linear advection problem (7.1) at t=0.2 using relaxed schemes (6.3).*

7.2 Inviscid Burgers equation

Now we will apply the relaxing scheme (6.1) and the relaxed scheme (6.3) to the inviscid Burgers equation.

Example 2. In this example we approximate solutions to the inviscid Burgers equation,

$$u_t + (\frac{u^2}{2})_x = 0. \tag{7.3}$$

We start with the smooth initial data

$$u(x,0) = 0.5 + sin(x), \quad x \in [0, 2\pi]. \tag{7.4}$$

and we use periodic boundary conditions. We recall that the unique entropy solution of (7.3)-(7.4) is smooth up to the critical time $t_c = 1$. We perform some numerical tests with our relaxing schemes (6.1) and the corresponding relaxed scheme (6.3). We will also present a comparison where we solve the same problem with the first order split method (6.34). We apply Newton's method to find the exact solution for the Burgers equation.

In figures (7.7-7.11), we present the approximate solutions at the pre-shock times when the solution is still smooth. We choose $\epsilon = 10^{-08}$, and plot at various times. As expected all the schemes capture well the correct behaviour given by the inviscid Burgers equation up to the critical time t = 1. To see how fast the numerical solution approximate the exact when we increase the grid points, a small time step is used, $\Delta t = 6 * 10^{-5}$. The L^1-error behaviour is shown in Table 3 for the relaxation scheme (6.1) and in Table 5 for the split relaxation method (6.34)-(6.39). Again we see that the error decreases but the predicted ratio is only partially reached in the table. In any case, we point out that the relaxed scheme proposed is capable of reaching the same accuracy as the relaxing scheme for this problem, as we can see in Table 4. The L^1-error is identical for the methods. We see also that the error is reduced by increasing the number of grid points, thus reducing the time step.
We present here a test with the first order splitting algorithm to see if the splitting indicates a certain advantage compared to the investigated relaxing schemes. We observe

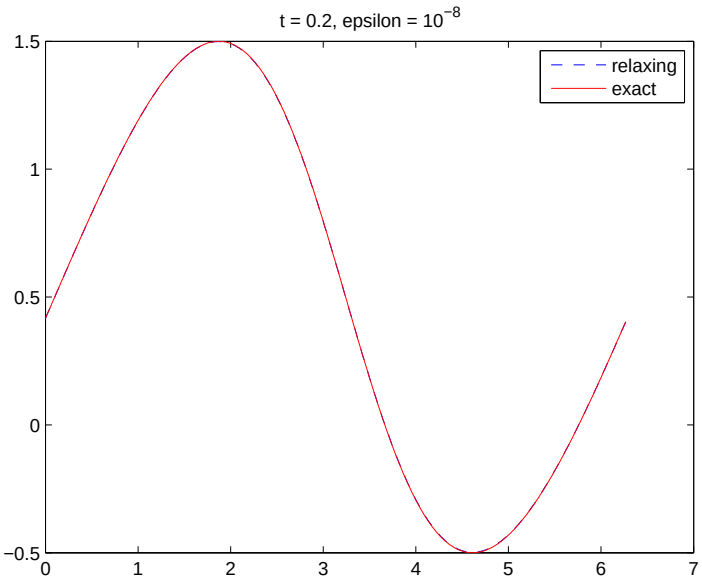

Figure 7.7: Numerical solution of inviscid Burgers equation $u_t + (\frac{u^2}{2})_x = 0$ using the relaxing scheme (6.1), with N=400 space steps and number of timesteps m = 100, $\Delta x = 2 * pi/N$, $\Delta t = 0.002$. Plot at t = 0.2 with $\epsilon = 10^{-8}$.

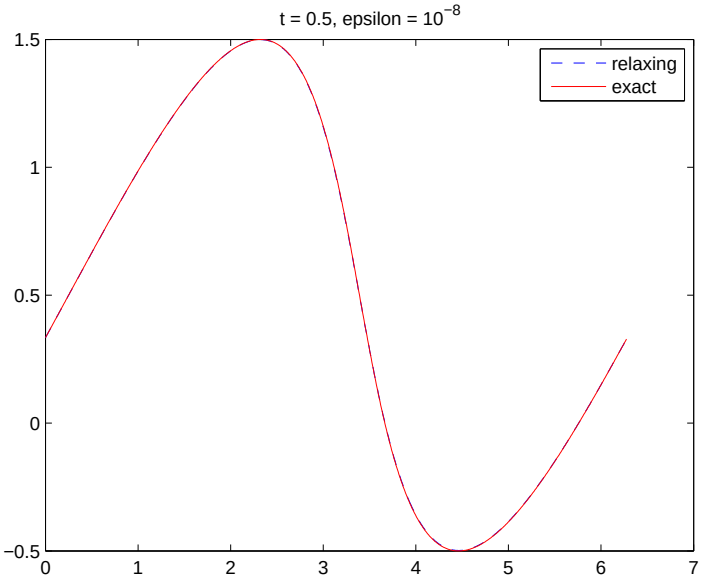

Figure 7.8: Numerical solution of Burgers equation $u_t + (\frac{u^2}{2})_x = 0$ using the relaxing scheme (6.1), with N=800 grid points and m = 200 time steps. $\Delta x = 2 * pi/N$, $\Delta t = 0.0025$. Plot at t = 0.5 with $\epsilon = 10^{-8}$.

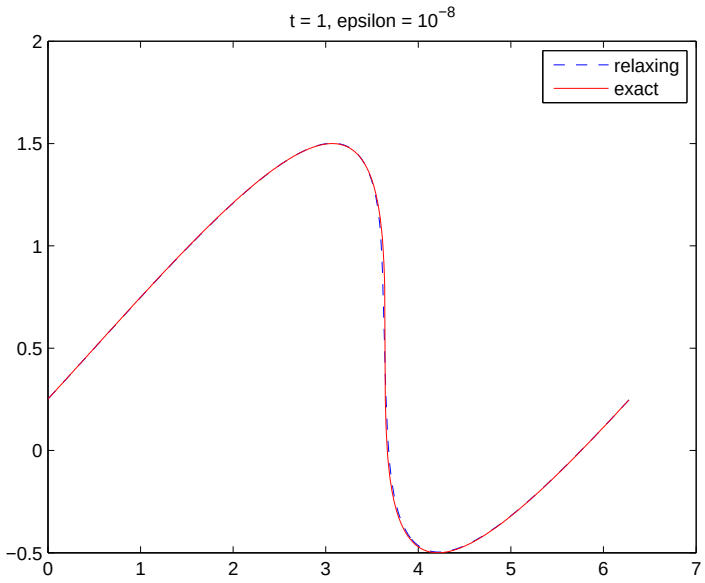

Figure 7.9: Numerical solution of Burgers equation $u_t+(\frac{u^2}{2})_x = 0$ using the relaxing scheme (6.1), with N=800 grid points and m = 200 time steps. $\Delta x = 2 * pi/N$, $\Delta t = 0.005$. Plot at t = 1 with $\epsilon = 10^{-8}$ and cfl= 0.6366.

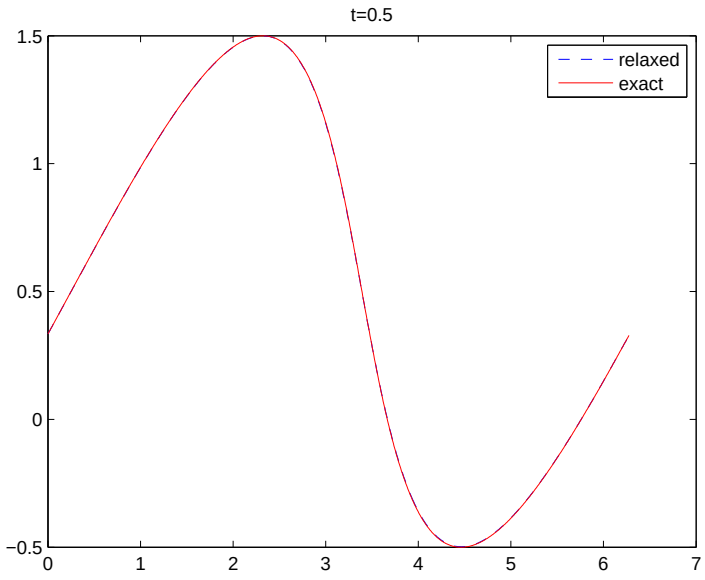

Figure 7.10: Numerical solution of inviscid Burgers equation $u_t + (\frac{u^2}{2})_x = 0$ using the relaxed scheme (6.3), with N=800 space steps and m=200 time steps. $\Delta x = 2 * pi/N$, $\Delta t = 0.0025$, cfl= 0.3183. Plot at t = 0.5.

N	L^1-error	Ratio
4	0.4577	
8	0.2971	1.5407
16	0.1578	1.8828
32	0.0795	1.9843
64	0.0398	1.9984
128	0.0199	1.9993

Table 3. *Discretization error in L^1-norms for the inviscid Burgers equation (7.3) at t=0.2 using relaxing schemes (6.1).*

N	L^1-error	Ratio
4	0.4577	
8	0.2971	1.5407
16	0.1578	1.8828
32	0.0795	1.9843
64	0.0398	1.9984
128	0.0199	1.9993

Table 4. *Discretization error in L^1-norms for the inviscid Burgers equation (7.3) at t=0.2 using relaxed scheme (6.1).*

N	L^1-error	Ratio
4	0.4577	
8	0.2971	1.5407
16	0.1578	1.8825
32	0.0796	1.9838
64	0.0398	1.9973
128	0.0199	1.9972

Table 5. *Discretization error in L^1-norms for the inviscid Burgers equation (7.3) at t=0.2 using split method (6.34).*

that despite the splitting error, the method produces the same results as the relaxing scheme for this problem. Our computational results suggest that the schemes converge with appropriate rate even if some combinations in the implementation do not reach the predicted error reduction in the tables. Other choices can be made in the discretizations discussed to sharpen the results. Our idea is only to carry out the analysis and to illustrate the basic ideas. We can remark that all the results with $\epsilon = 10^{-08}$ can almost be reproduced with about equal quality by using the relaxed schemes. Thus for strictly hyperbolic systems and for the purpose of just solving the conservation laws, one can just use the relaxed schemes, which are easier to implement with more efficiency and much less memory. This concludes also that the relaxing schemes have the correct zero relaxation limit mentioned in the analysis.

7.3 Convection-Diffusion Equation

In this section we consider a numerical example calculated by our proposed Relaxation Schemes (5.3) when the diffusion term B(u) is incorporated, the convection-diffusion problem.

Example 3. Now we test the relaxing schemes (5.3) for the convection-diffusion equation

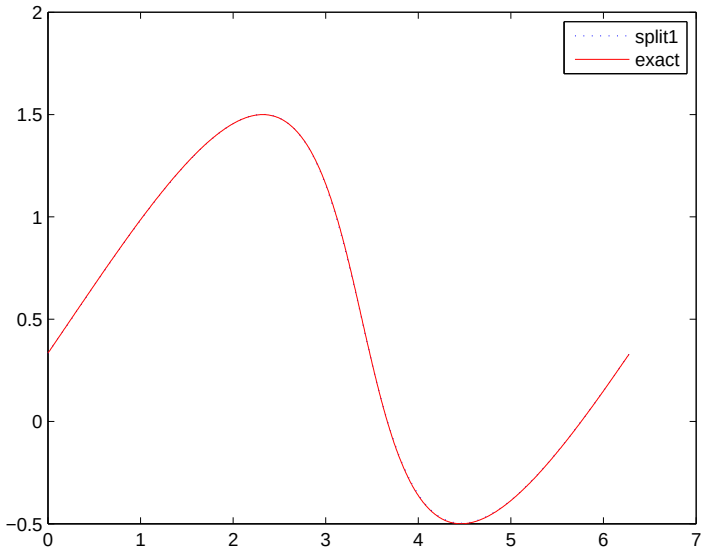

Figure 7.11: Numerical solution of inviscid Burgers equation $u_t + (\frac{u^2}{2})_x = 0$ using the first order split method (6.34), with 200 time steps and N = 800 grid points. $\Delta x = 2 * pi/N$, $\Delta t = 0.0025$, $\epsilon = 10^{-08}$. Plot at t = 0.5 with cfl = 0.3183.

on the viscid Burger equation

$$u_t + uu_x = au_{xx}, a > 0. \tag{7.5}$$

It was first introduced by J.M.Burgers as the simplest model for the differential equations of fluid flow. To find the explicit solution for a > 0, see e.g. the paper by Eberhard Hopf (1950) on pure and applied mathematics, treating the partial differential equation. We introduce a new dependent variable **q(x,t)** into Burgers equation that solves the heat equation $q_t = aq_{xx}$.
For $t > 0$, a is positive and the function

$$u(x,t) = -2a\frac{q_x(x,t)}{q(x,t)}$$

solves the viscid Burger equation. More precisely stated: If u solves (7.5) in an open rectangle **R** of the x,t-plane and if u, u_x, u_{xx} are continuous in **R** then there exists a positive function **q** that solves the heat equation in **R** and for which q, q_x, q_{xx} are continuous in **R**.
In this test we apply the function $q(x,t) = e^{-t}sin(x) + 2$ and a $= 1$, so that $u(x,t) = -2\frac{e^{-t}cos(x)}{e^{-t}sin(x)+2}$ is an exact solution. We choose the initial data to be $u(x,0) = u_0(x) = \frac{cosx}{sinx+2}$. We set $\gamma(x) = 1$ and the flux function is $f(u) = \frac{1}{2}u^2$. We use the definition to initiate v(x,t), v(x,0) $= \gamma(x)f(u_0(x))$. We choose $\epsilon = 0.1$ The boundary conditions are chosen to be periodic.

The numerical solutions computed by our proposed relaxation scheme for convection-diffusion equation for different number of grid points are presented in figures (7.12-7.15). These results demonstrate the performance of our scheme in the rarefied regime where $\epsilon = 0.1, \epsilon > \Delta x$. The numerical solutions match the exact solution very well and capture the parabolic behaviour when we use very fine spatial grids. Visually, there is a good agreement in the figures, but we experienced at t $= 1$ that the numerical solution computed by the scheme is sensitive to the choice of Δt and Δx when we keep $\epsilon = 0.1$. To see if this behaviour persists and to see how the error evaluates, we measure the L^1 error between the exact solution and the approximated one at time t $= 3$ in Table 6.

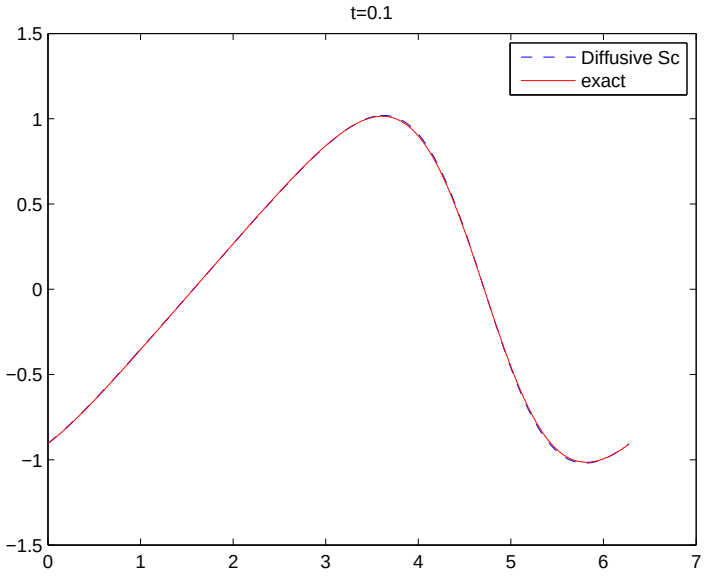

Figure 7.12: Numerical solution of viscid Burgers equation $u_t + (\frac{u^2}{2})_x = u_{xx}$ using the diffusive relaxing scheme (5.3), $\Delta x = 0.009$. $\Delta t = 0.0002$. $\epsilon = 0.1$. Plot at t = 0.1.

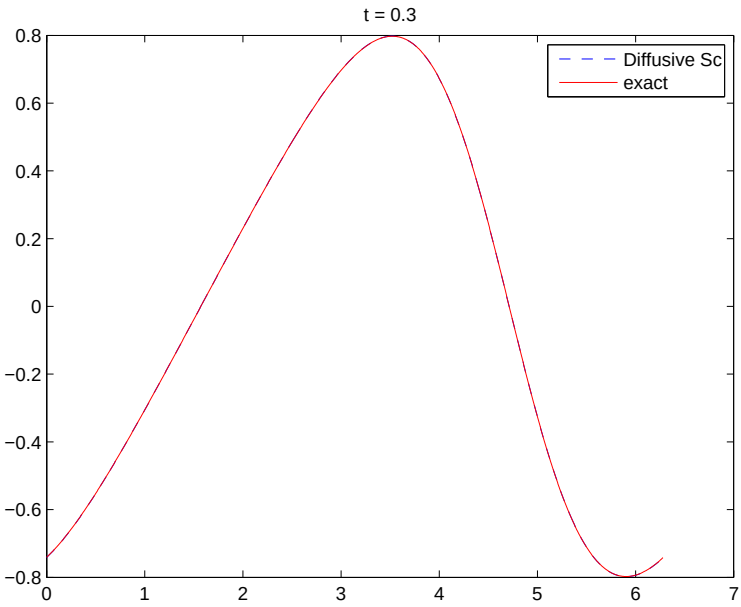

Figure 7.13: Numerical solution of viscid Burgers equation $u_t + (\frac{u^2}{2})_x = u_{xx}$ using the diffusive relaxing scheme (5.3), $\Delta x = 0.003$. $\Delta t = 0.0001$. $\epsilon = 0.1$. Plot at t = 0.3.

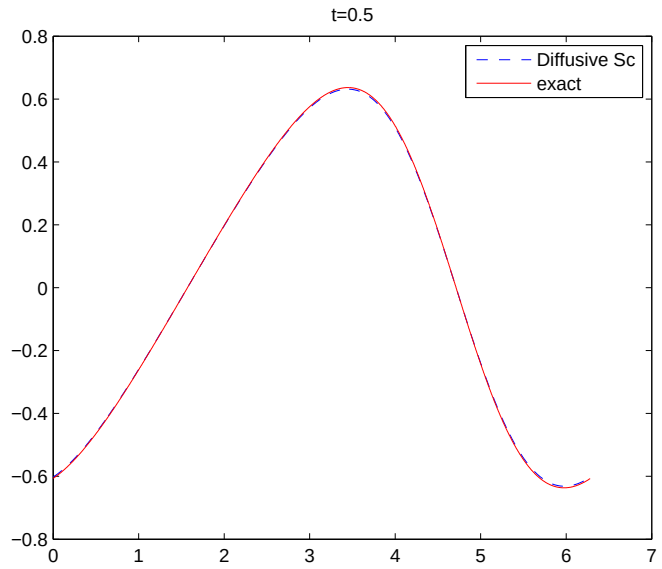

Figure 7.14: Numerical solution of viscid Burgers equation $u_t + (\frac{u^2}{2})_x = u_{xx}$ using the diffusive relaxing scheme (5.3), with $\Delta x = 0.003$. $\Delta t = 0.0001$. $\epsilon = 0.1$. Plot at t = 0.5.

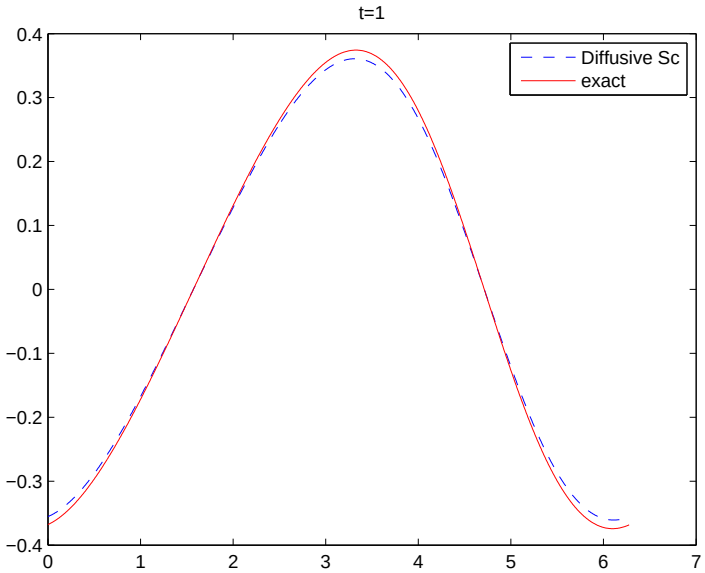

Figure 7.15: Numerical solution of viscid Burgers equation $u_t + (\frac{u^2}{2})_x = u_{xx}$ using the diffusive relaxing scheme (5.3), with $\Delta x = 0.006$. $\Delta t = 0.00025$. $\epsilon = 0.1$. Plot at t = 1.

N	L^1-error	Ratio
40	0.1792	
80	0.1375	1.30
160	0.0897	1.53
320	0.0534	1.68
640	0.0309	1.72
1280	0.0184	1.67

Table 6. *The computed L^1 norms of the difference between the numerical and the exact solution of the viscid Burger equation at final time $t = 3$. We computed the numerical solution with the diffusive relaxation scheme (5.3).*

Chapter 8

Summary and Conclusion

In this thesis a class of numerical schemes based on local relaxation approximation for hyperbolic and parabolic equations have been introduced and analysed. The main feature of this class of schemes is its simplicity and generality. Both numerical experiments and theoretical analysis indicate that the relaxation schemes proposed have great deal of advantages.
For these methods, our aim have just been to concentrate on developing the basic framework. We have analysed both from a theoretical and computational point of view the relaxation schemes to approximate the nonlinear degenerate parabolic equations, and we have incorporated the closely related hyperbolic conservation laws.
In the hyperbolic case, when there is no diffusion, we considered underresolved numerical schemes with a discontinuous coefficient and the corresponding relaxed scheme. The first order upwind approximation to the hyperbolic relaxation system was tested to illustrate the accuracy and the good properties. Our conclusion is that the relaxation schemes seem to give acceptable results for conservation laws even if we simplify the discontinuous coefficient to a constant in the computations. Inspired from recent advances in developing high order relaxed schemes, a theoretical convergence analysis for the relaxed scheme is presented. The results indicate that the relaxed schemes obtained in the limit $\epsilon \longrightarrow 0$ provide a promising class of new schemes. We showed that for hyperbolic systems, one can just use the relaxed schemes instead of the relaxing version.
In the parabolic case, we extended the approach to nonlinear parabolic equations, and we have introduced a way of constructing numerical schemes for equations in the diffusive regime. In our first approach, we concentrate on developing the basic ideas when we propose a new form of the relaxation scheme based on the same idea used on hyperbolic conservation laws since our approach has the same form. Since our goal is just to define the concept, more experiments need to be done, and comparison with other methods have to be made. By using suitable discratization in space and time, we were able to produce some numerical results which indicate the potential of the schemes.
We face additional difficulties here due to the stiff convection part combined with a

discontinuous coefficient in the source term. A reformulation of the problem is tried to see if we can improve the results. The analysis for operator splitting is carried out to see if a higher order of accuracy can be achieved. When applying the splitting procedure, we want to point out a lack of accuracy in the v^{n+1}-component in the convection step. So one need to investigate further the choice of equilibrium fluxes to rectify the problem and rigorous theoretical justification to analyse the behaviour in the diffusive regime. Other ideas could also be incorporated to give more delicate results, such as a different choice of the parameter ν that depend on the discretization parameters, or adaptive mesh refinement and shock tracking techniques.
Our conclusion is that it is not obvious that this splitting provides any simplification or improvement to the numerical solution. However, based on experiments with other splitting methods, we believe this basic framework can be competitive to other methods, easier to implement and no Riemann solvers are necessary.

8.1 Further work

After finishing this thesis, there are still some open questions and room for many further developments.
The prospects in the numerical experiments and theoretical analysis are very encouraging. The relaxation formulation can be used as a platform for developing schemes for hyperbolic conservation laws. They are simpler compared to the existing approaches and are attractive for further research. One possible future research is to design schemes that extends the relaxing schemes with a discontinuous coefficient to higher order schemes and more complicated systems, to demonstrate that the accuracy of relaxations schemes can be increased. An interesting aspect is to construct simple alternative higher order relaxed schemes with applications to more general hyperbolic-parabolic convection-diffusion equations.
When solving the parabolic problem, we focused on the stiff regime, $\epsilon << 1$. It is desirable to develop a class of numerical schemes that can work with a uniform accuracy for all range of ϵ, an accurate and stable discratization for a possibly degenerate convection-diffusion equation with the discontinuous coefficient and the corresponding relaxed schemes.
We would like to investigate further the ability of the proposed splitting to capture shocks with high resolution and avoiding solving nonlinear algebraic systems, but still maintaining all the nice properties of those constructed earlier.

Bibliography

[1] G. Puppo M. Semplice F. Cavalli, G. Naldi. High order relaxation schemes for nonlinear diffusion problems. *arXiv:math.NA/0604572*, v1:19, 2006.

[2] Lorenzo Pareschi Giovanni Naldi and Giuseppe Toscani. Hyperbolic Relaxation Approximation to Nonlinear Parabolic Problems.

[3] L.Pareschi G.Naldi. Numerical Schemes for Hyperbolic Systems of Conservation Laws with stiff diffusive relaxation. *Siam J.Numer.Anal*, 37,No.4:1246–1270, 1997.

[4] L.PARESCHI G.NALDI. Numerical Schemes for Kinetic Equations in Diffusive Regimes. *App.Math.Lett.PERGAMON*, Vol.11,No.2:pp.29–35, 1998.

[5] T.P.Liu G.Q.Chen, C.D.Levermore. Hyperbolic Conservation Laws with stiff ralaxation terms and entropy. *Comm. on Pure and Applied math.*, Vol.XLVII:787–830, 1994.

[6] Gerald Warnecke Hailiang Liu. Convergence Rates for Relaxation Schemes Approximating Conservation Laws .

[7] Jinghua Wang Hailiang Liu and Gerald Warnecke. The Lip+Stabilty and Error Estimates for a Relaxation Scheme. *AMS-MANUSCRIPT VERSION*, 1999.

[8] Nils H. Risebro Helge Holden. Front Tracking for Conservation Laws. 1999.

[9] Shi Jin. Runge Kutta methods for Hyperbolic Conservation Laws with Stiff Relaxation Terms. *J.Comp.PHYS.*, 1995.

[10] Randall J.Leveque. *Numerical Methods for conservation laws*, volume 300. Birkhauser Verlag, 2nd edition, 1992.

[11] Randall J.Leveque. *Finite Difference Methods for Differential Equations*. University of Washington, draft version: course: amath 585-6 edition, 1998.

[12] Steinar Evje. Kenneth Hvistendahl Karlsen. Viscous splitting approximation of mixed hyperbolic-parabolic convection-diffusion equations. *Numer.Math.83*, 1999.

[13] N.H.Risebro K.H.Karlsen and J.D.Towers. On a nonlinear degenerate parabolic transport-diffusion equation with a discontinuous coefficient. *Electronic Journal of Differential Equations*, Vol.2002,No.93,pp.1-23., 2002.

[14] N.H.Risebro K.H.Karlsen and J.D.Towers. Upwind difference approximations for degenerate parabolic convection-diffusion equations with a discontinuous coefficient. *Journal of Numerical Analysis*, 41:623–664, 2002.

[15] N.H.Risebro K.H.Karlsen and J.D.Towers. L^1 stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients. *The Journal of Numerical analysis*, 49, 2003.

[16] Roberto Natalini. Convergence to Equilibrium for the Relaxation Approximation of conservation Laws. *Comm.Pure App.Math*, XLIX:795–823, 1996.

[17] Roberto Natalini. Recent mathematical results on hyperbolic relaxation problems. ., pages 1–66, 1998.

[18] S.V. Raghurama Rao. New Numerical Schemes based on relaxation Systems for Conservation Laws. 2002.

[19] D.E.Driollet Roberto Natalini. Convergence of Relaxation Schemes for Conservation Laws. *Appl.Anal.*, page 28.

[20] Mapundi K.Banda. Mohammad Seaid. Higher-order relaxation schemes for hyperbolic systems of conservation laws. *Fachbereich Mathematik*.

[21] G. Toscani S.Jin, L. Pareschi. Diffusive relaxation schemes for multiscale discrete-velocity kinetic equations. *Siam J. Numer.Anal .*, 35,No.6:2405–2439, 1998.

[22] Z.Xin S.Jin. The Relaxation Schemes for systems of Conservation Laws in Arbitrary Space dimensions. *Comm.Pure App.Math*, 45:235–276, 1995.

[23] John D. Towers. Convergence of a difference scheme for conservation laws with a discontinuous flux. *SIAM J. Numer. Anal.*, 38,No.2:pp.730–742, 2000.

[24] Alan W.BUSH. *Perturbation methods for engineers and scientists*, volume 303. CRC Press,Inc., 1992.

[25] Ãystein Pettersen. Basics of Reservoar Simulation with the Eclipse Reservoar Simulator. 2006.