Sub-Riemannian geometry of spheres and rolling of manifolds

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Preface

The structure of the thesis is as follows:

Chapter 1. We give an abbreviated review covering the history of sub-Riemannian geometry and some other related geometric and algebraic structures that play an important role in this work. The mathematical prerequisites that are treated include contact, CR and qCR manifolds, as well as principal bundles and the real division algebras of quaternions and octonions.

Chapter 2. We discuss the main results of the present thesis, based on the tools described in the previous chapter. The results presented in this thesis belong to two topics in sub-Riemannian geometry namely, the sub-Riemannian structures on odd dimensional spheres arising from their structure as principal S^1 or S^3-bundles, and the kinematic system of a manifold rolling on another manifold without twisting or slipping.

In the former, we compare the horizontal distributions on odd dimensional spheres arising from different points of view and we prove that they coincide. This allows us to explicitly determine the sub-Riemannian geodesics in each case and obtain several geometric corollaries. In addition we find a geodesic equation in the first quaternionic $H-$type group $\mathbf{H}^1$ by using variational arguments, and we describe the intrinsic sub-Laplacian and heat kernel of the sphere S^7 with respect to the contact distribution.

In the latter, we introduce the notion of intrinsic rolling and we show that all the relevant information of the dynamics is contained in this coordinate-free definition. We study the controllability problem in some examples and afterwards we study the existence of intrinsic rollings under various hypotheses. We finish this chapter with a summary and a list of open questions related to the results obtained here that will be dealt with in future research.

Chapter 3. We include four papers, two of which are accepted for publication, one is submitted and one is in preparation.

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Contents

Chapter 1

Introduction

This chapter contains some preliminaries in sub-Riemannian geometry as well as presenting the main results of this thesis. In Section 1.1 we give a brief historical account of the most relevant developments in sub-Riemannian geometry that are of importance for us. Section 1.2 consists of an overview of the thesis, contextualizing the relevance of the results here presented. Finally, Section 1.3 presents some mathematical prerequisites and notations that are necessary for understanding the main ideas of this thesis.

1.1 Historical background

It is fair to say that sub-Riemannian geometry, as an area of differential geometry and global analysis of importance on its own, was born with the paper [37] and its addendum [38]. In this reference it is possible to find the introduction of some important concepts for the theory, such as the cometric or the sub-Riemannian Christoffel symbols, as well as the solution to early problems of the theory. The articles [7, 15, 27] were also fundamental for the growth of the theory into an independent field of research. Certainly some of the techniques, the main examples of the theory and some applications can be traced back to very early stages of differential geometry; for example, the problem of the sphere rolling on the plane as a quintessential example of a non-holonomic system can be found in the scientific literature as far back as the late 19th and early 20th century [11, 12]. In any case, sub-Riemannian geometry owes its existence to the –nowadays elementary– idea of non-integrable constraints, and thus, it has received increasing interest in recent years in applied disciplines such as robotics, control theory, financial mathematics and diverse physical theories.

Sub-Riemannian geometry can be thought of as a *generalized Riemannian geometry* in the sense that we admit some of the eigenvalues of the metric to be infinite: some directions in the tangent spaces are forbidden as velocity vectors of curves. The curves whose velocity vectors almost everywhere satisfying the constraints are usually referred to as *horizontal* or *admissible*. With this point of view, sub-Riemannian geometry can be considered as an application of the well-known *penalization* methods to differential geometry [28]. In this new framework, problems of existence of admissible trajectories and search for optimal ones become much harder than in classical Riemannian geometry, but nevertheless they offer new techniques and sometimes even challenges to try to extend some of the old results to more general situations.

The existence of admissible trajectories, also known as the *accessibility problem*, has been known to differential geometers and control theorists for a long time. An early particular solution –but nevertheless widely employed nowadays– is the Chow-Rashevskiĭ theorem [13, 30], by which any two points can be joined by an admissible curve, as long as the space of admissible directions forms a completely non-integrable distribution. It is interesting to note that this result contains as a particular case the celebrated Kalman rank condition for controllability of linear systems, see for instance [35]. The complete solution of the accessibility problem was found in [39] via a control theoretic approach. The search for optimal solutions requires more subtle distinctions and it is currently a subject of active research, see for example [8, 9, 10, 24]. In fact, the fundamental question of the smoothness of minimizers, remains open and its solution is known only in a few particular cases.

Finally, as a way of stressing the benefits of the symbiosis between sub-Riemannian geometry and geometric control theory, let us briefly discuss the particular example of rolling manifolds without slipping or twisting. The well-known two dimensional version of this mechanical system has been important in robotics, see [26], while its higher dimensional formulation has shown to be very convenient when dealing with interpolation problems, see [19]. The first time this dynamics was presented in the higher dimensional context was in [33], where the definition is given for submanifolds of $\mathbb{R}^n$. The questions of existence of rollings and of controllability of the system in dimensions bigger or equal than three have been usually treated in a case-by-case approach, see [6, 18, 23, 40], mostly employing the geometry and mechanics of Euclidean space and techniques of control theory on Lie groups. An interesting fact to remark is that the particular case of rolling a manifold over Euclidean space, known as *development* has been previously addressed by geometers and probability theorists. In geometry it is used to obtain the tangent space in any point, once the tangent space in one point is known [33], and in probability it is used in order to define Brownian motion on a manifold [17, 21].

1.2 Overview of the thesis

The aim of this thesis is to study two topics in sub-Riemannian geometry: the sub-Riemannian structures on odd dimensional spheres arising from their structure as principal S^1 or S^3-bundles and the kinematic system of a manifold rolling on another manifold without twisting or slipping.

A first important result concerning the first of the problems treated in this thesis is the equivalence between sub-Riemannian structures on S^3 arising from several geometrical contexts. More precisely, the CR geometry, the contact geometry, the principal S^1-bundle structure and the Lie group structure of S^3 can be given a common sub-Riemannian framework by understanding the holomorphic tangent space as the contact distribution, and this distribution as an Ehresmann connection for the Hopf fibration.

This vector bundle is trivializable via the Lie group action of S^3 to itself. Interestingly, the first three constructions coincide for all odd dimensional spheres in the classical settings. Some of the previous results were extended to the case of the action of S^3 on spheres S^{4n+3} as subsets of $(n+1)-$dimensional quaternionic space. In all of these cases, the sub-Riemannian metric is the one induced by the standard Riemannian metric on the corresponding sphere. As a consequence of this equivalences, formulas for normal sub-Riemannian geodesics as "twisted" Riemannian geodesics are presented and, moreover, it is possible to detect when these geodesics are closed curves by determining whether certain parameters are rational or not, extending a similar result for S^3.

In addition, using variational arguments, it is possible to emulate a sub-Riemannian geodesic equation for the quaternionic $H-$type group $\mathbf{H}^1$ which resembles an analogous one for the case of three dimensional Sasakian pseudo-Hermitian manifolds.

The problem of constructing an intrinsic sub-Laplacian for the contact S^7 is also addressed. We show that the sub-Laplacian of S^7, considered as an S^1-bundle, corresponds to a sum of squares of vector fields. Such a result seems to be false for S^7 with the structure of an S^3-bundle. The above characterization allows us to find a convenient realization of the sub-Riemannian heat kernel of S^7, considered as an S^1-bundle, as a composition of the Riemannian heat kernel of S^7 and the unbounded operator of heat flow along the Reeb vector field for negative times.

Concerning the problem of rolling manifolds, we construct the configuration space of the mechanical system as an $\mathrm{SO}(n)-$bundle, which captures the information determined by the geometry of the manifolds. Additionally we construct an $\mathrm{SO}(N-n)-$bundle, which captures the information determined by the imbedding. With this definitions we are able to formulate the no-twisting and no-slipping conditions in terms of a distribution on the imbedding independent $\mathrm{SO}(n)-$bundle, giving rise to the notion of an intrinsic rolling. These conditions can be seen as direct generalizations of an analogous construction for surfaces.

An important result in this direction is that given an intrinsic rolling of two manifolds of dimension n and concrete isometric imbedding of these into $\mathbb{R}^N$, then there exists a unique extrinsic rolling corresponding to the given intrinsic rolling, up to the initial configuration of the rolling.

Having this appropriate coordinate-free setting, it is possible to address questions related to controllability and geometric behavior of rollings in a better way. For example it is possible to show that the sphere S^n rolling on $\mathbb{R}^n$ is a controllable system in contrast to the case of the group of Euclidean rigid motions $\mathrm{SE}(3)$ rolling on its Lie algebra $\mathfrak{se}(3)$, which in fact induces a foliation of its 27-dimensional configuration space, where the leaves have dimension 12. Finally, we present conditions for the existence of intrinsic rollings under different assumptions, in terms of the generalized geodesic curvatures of the rolling curves on each of the manifolds.

1.3 Mathematical prerequisites

The aim of this section is to present briefly some general ideas, which will satisfy a twofold purpose: to set the context for some of the results in this thesis, as well as to fix notational conventions.

1.3.1 Sub-Riemannian manifolds

Let M be an $n-$ dimensional connected smooth manifold. A smooth subbundle of the tangent bundle $TM \to M$ is called a *horizontal distribution* or simply a *distribution*. If $\mathcal{H}$ is a distribution then, for each $p \in M$, $\mathcal{H}_p$ denotes the *fiber* of $\mathcal{H}$ at p. The dimension of $\mathcal{H}_p$ is the *rank* of the distribution at p.

It is important to stress that the definition for distribution employed in this thesis corresponds to the one used in the context of differential geometry. In problems motivated by control theory or analysis, for example [25, 34], it is often convenient to define a distribution D as a map $D : M \to \mathrm{Grass}(TM)$, where $\mathrm{Grass}(TM)$ denotes the Grassmanian bundle of TM, such that $D(p)$ is a vector subspace of T_pM for all p and then require extra conditions. Though this setting contains the aforementioned definition as a particular case, its generality will be unnecessary for the purposes of the present work.

One of the main goals of sub-Riemannian geometry is to study curves that are admissible in a certain sense. To be precise, an absolutely continuous curve $\gamma : [0, 1] \to M$ is called *admissible* or *horizontal* if $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$ almost everywhere.

A distribution $\mathcal{H}$ is said to be *bracket generating* if the Lie-hull of its sections $\mathrm{Lie}(\mathcal{H})$ equals T_pM at each $p \in M$. To be more precise, define the following vector bundles

$$\mathcal{H}^1 = \mathcal{H}, \qquad \mathcal{H}^{r+1} = [\mathcal{H}^r, \mathcal{H}] + \mathcal{H}^r \quad \text{for } r \geq 1,$$

where $[\mathcal{H}, \mathcal{H}^k] = \mathrm{span}\{[X, Y] : X \text{ is a section of } \mathcal{H}, Y \text{ is a section of } \mathcal{H}^k\}$. This vector bundles naturally give rise to the flag

$$\mathcal{H} = \mathcal{H}^1 \subseteq \mathcal{H}^2 \subseteq \mathcal{H}^3 \subseteq \ldots.$$

A distribution is bracket generating if for all $p \in M$ there is an $r(p) \in \mathbb{Z}^+$ such that

$$\mathcal{H}_p^{r(p)} = T_pM. \tag{1.3.1}$$

If the dimensions $\dim \mathcal{H}_p^r$ do not depend on p for any $r \geq 1$, we say that $\mathcal{H}$ is a regular distribution. The least r such that (1.3.1) is satisfied is called the *step* of $\mathcal{H}$.

One of the core results used in order to relate the notion of path-connectedness by means of horizontal curves and the assumption that $\mathcal{H}$ is a bracket generating distribution is the following theorem, usually referred to as Chow-Rashevskiĭ theorem.

Theorem 1 ([13, 30]) *Let M be a connected manifold. If a distribution $\mathcal{H} \subset TM$ is bracket generating, then any two points in M can be joined by a horizontal path.*

With all of this at hand, we can define what a sub-Riemannian manifold is. A *sub-Riemannian structure* on a manifold M is a pair $(\mathcal{H}, \langle\cdot,\cdot\rangle_{sR})$, where $\mathcal{H}$ is a bracket generating distribution and $\langle\cdot,\cdot\rangle_{sR}$ a fiber inner product defined on $\mathcal{H}$, which varies smoothly from point to point. A *sub-Riemannian manifold* is a triple $(M, \mathcal{H}, \langle\cdot,\cdot\rangle_{sR})$, where $(\mathcal{H}, \langle\cdot,\cdot\rangle_{sR})$ is a sub-Riemannian structure on M. If the sub-Riemannian structure is clear from the context, we say that M is a sub-Riemannian manifold. In this setting, the *length* of a horizontal curve $\gamma : [0,1] \to M$ is

$$\ell(\gamma) := \int_0^1 \|\dot{\gamma}(t)\|_{sR}\, dt,$$

where $\|\dot{\gamma}(t)\|_{sR} = \langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle_{sR}^{1/2}$ whenever $\dot{\gamma}(t)$ exists.

Thus, it is possible to define the *sub-Riemannian distance* $d(p,q) \in [0,+\infty)$ between two points $p, q \in M$ by $d(p,q) := \inf \ell(\gamma)$, where the infimum is taken over all absolutely continuous horizontal curves joining p to q. An absolutely continuous horizontal curve that realizes the distance between two points is called a *horizontal length minimizer*.

In order to define what a normal geodesic is in the sub-Riemannian context, let us digress briefly about the underlying Hamiltonian formalism. The sub-Riemannian metric $\langle\cdot,\cdot\rangle_{sR}$ defines a linear mapping $\beta_p : T_p^*M \to T_pM$, referred to as the cometric, by requiring that:

- The image of T_p^*M under β_p is $\mathcal{H}_p$.
- The equality $\langle X, \beta_p\lambda\rangle_{sR} = \lambda(X)$ holds for all $X \in \mathcal{H}_p$, $\lambda \in T_p^*M$.

Observe that β_p induces a bilinear form $\tilde{\beta}_p : T_p^*M \times T_p^*M \to \mathbb{R}$. An important feature of the cometric β_p is that its dual map $\beta_p^* : T_p^*M \to T_p^{**}M \cong T_pM$ coincides with β_p, where $\cong$ denotes the the inverse of the canonical isomorphism of evaluation. On the other hand, two important features of the induced map $\tilde{\beta}_p$ are that it is symmetric and nonnegative definite, meaning that $\tilde{\beta}_p(\lambda,\mu) = \tilde{\beta}_p(\mu,\lambda)$ and that $\tilde{\beta}_p(\lambda,\mu) \geq 0$ for all $\lambda, \mu \in T_p^*M$.

Given the cometric $\beta_p : T_p^*M \to \mathcal{H}_p$ we have the Hamiltonian function

$$H(p,\lambda) = \frac{1}{2}\lambda(\beta_p(\lambda))$$

on T^*M. Considering a trivializing neighborhood U_p around $p \in M$ for the subbundle $\mathcal{H}$, one can find a smooth local orthonormal basis $X_1, \ldots, X_k$ with respect to $\langle\cdot,\cdot\rangle_{sR}$. The associated sub-Riemannian Hamiltonian is given by

$$H(q,\lambda) = \frac{1}{2}\sum_{m=1}^{k} \lambda(X_m(q))^2,$$

where $(q,\lambda) \in T^*U_p$ and $\lambda \notin \ker \beta_p$. A *normal geodesic* corresponds to the projection to

$U_p \subset M$ of the solution of the Hamiltonian system

$$\dot{q}_i = \frac{\partial H}{\partial \lambda_i} \tag{1.3.2}$$

$$\dot{\lambda}_i = -\frac{\partial H}{\partial q_i}, \tag{1.3.3}$$

where (q_i, λ_i) are the coordinates in the cotangent bundle of M.

On the contrary from the Riemannian case, for sub-Riemannian manifolds there can be found examples of curves minimizing the length functional but not being the solution of the Hamiltonian system (**??**). These curves are known as *abnormal geodesics* and to this day it is still unknown whether all of them are smooth or not.

1.3.2 Contact manifolds

A manifold M of dimension $2n + 1$ is said to be a *contact manifold* if there is a one form ω such that $\omega \wedge (d\omega)^n$ never vanishes. The subbundle $\xi = \ker \omega$ is usually called *contact distribution*. We can define ω equivalently by requiring that $d\omega$ defines a symplectic form on ξ, see [14]. This explains why contact manifolds must have odd dimension, since it is a well known fact from linear algebra that symplectic forms can only exist in even dimensional vector spaces.

An important construction in contact geometry is the *Reeb vector field*, which is a nowhere vanishing vector field R uniquely determined by the condition $d\omega(R, \cdot) = 0$ and $\omega(R) = 1$.

As a consequence of Cartan's formula for one forms, the contact distribution ξ is always a bracket generating distribution of step two.

1.3.3 Principal bundles

Let Q and M be two manifolds. For a submersion $\pi : Q \to M$ with fiber $Q_p = \pi^{-1}(p)$ through $p \in M$, the *vertical space* at $q \in Q$ is given by $T_q Q_{\pi(q)}$ and is denoted by V_q. Note that $V_q = \ker d_q\pi$. In this context, an *Ehresmann connection* for $\pi : Q \to M$ is a distribution $\mathcal{H} \subset TQ$ which is everywhere transverse to the vertical space, that is:

$$V_q \oplus \mathcal{H}_q = T_q Q \quad \text{for every } q \in Q.$$

Let us assume that the submersion $\pi : Q \to M$ is a fiber bundle with fiber G, where G is a Lie group acting on Q on the right. We say that π is a *principal $G-$bundle* with connection $\mathcal{H}$ if the following conditions hold: G acts freely and transitively on each fiber, the group orbits are the fibers of $\pi : Q \to M$, and the action of G on Q preserves the connection $\mathcal{H}$. Observe that the second condition implies that M is isomorphic to Q/G and that π is the canonical projection.

Let us denote the Lie algebra of G by $\mathfrak{g}$, and the corresponding exponential map by $\exp_G : \mathfrak{g} \to G$. For the principal $G-$bundle $\pi : Q \to M$, the *infinitesimal generator* for the group action is the map $\sigma_q : \mathfrak{g} \to T_qQ$ defined by

$$\sigma_q(\xi) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} q \exp_G(\epsilon\xi)$$

for $q \in Q$ and $\xi \in \mathfrak{g}$. In the case of a principal $G-$bundle, for each $q \in Q$ the infinitesimal generator σ_q is an isomorphism between the vertical space V_q and $\mathfrak{g}$. We refer to its inverse as the $\mathfrak{g}-$*valued connection one form.*

1.3.4 CR and qCR manifolds

Let W be a real vector space. A linear map $J : W \to W$ is called an *almost complex structure map* if $J \circ J = -I$, where $I : W \to W$ is the identity map. In the case $W = T_p\mathbb{R}^{2n}$, $p = (x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$, we say that the standard almost complex structure for W is defined by setting

$$J_n(\partial_{x_j}) = \partial_{y_j}, \qquad J_n(\partial_{y_j}) = -\partial_{x_j}, \qquad 1 \leq j \leq n.$$

For a smooth real submanifold M of $\mathbb{C}^n$ and a point $p \in M$, in general the tangent space T_pM is not invariant under the almost complex structure map J_n for $T_p\mathbb{C}^n \cong T_p\mathbb{R}^{2n}$. For a point $p \in M$, the *complex* or *holomorphic tangent space* of M at p is the vector space

$$H_pM = T_pM \cap J_n(T_pM).$$

Note that H_pM is the largest subspace of T_pM which is invariant under the action of J_n.

It can be shown, see [5], that if M is a real submanifold of $\mathbb{C}^n$ of real dimension $2n - d$, then

$$2n - 2d \leq \dim_{\mathbb{R}} H_pM \leq 2n - d,$$

and $\dim_{\mathbb{R}} H_pM$ is an even number.

A real submanifold M of $\mathbb{C}^n$ is said to have a *CR structure* if $\dim_{\mathbb{R}} H_pM$ does not depend on $p \in M$. In particular, every smooth real hypersurface S embedded in $\mathbb{C}^n$ satisfies $\dim_{\mathbb{R}} H_pS = 2n - 2$, therefore S is a CR manifold.

The definition of qCR manifolds, as quaternionic analogues of CR manifolds, is more involved and requires some extra care in the hypotheses. For the moment, let us assume that M is a manifold of dimension $4n + 3$. For a triple of linearly independent 1-forms $\omega = (\omega_1, \omega_2, \omega_3)$ we define a triple of 2-forms $\rho = (\rho_1, \rho_2, \rho_3)$ as follows:

$$\rho_1 = d\omega_1 - 2\varepsilon\, \omega_2 \wedge \omega_3, \quad \rho_2 = d\omega_2 + 2\, \omega_3 \wedge \omega_1 \quad \text{and} \quad \rho_3 = d\omega_3 + 2\, \omega_1 \wedge \omega_2,$$

where $\varepsilon = \pm 1$.

The triple ω is called an $\varepsilon-$quaternionic CR structure ($\varepsilon = \pm 1$), if the associated 2-forms ρ_α, $\alpha = 1, 2, 3$ satisfy the following conditions :

1. They are non degenerate on the codimension three distribution $\mathcal{H} = \ker \omega_1 \cap \ker \omega_2 \cap \ker \omega_3$ and have the same 3-dimensional kernel V ,

2. The three fields of endomorphisms J_α of the distribution $\mathcal{H}$, defined by

$$J_1 = -\varepsilon(\rho_3|_{\mathcal{H}})^{-1} \circ \rho_2|_{\mathcal{H}}, \quad J_2 = (\rho_1|_{\mathcal{H}})^{-1} \circ \rho_3|_{\mathcal{H}} \quad \text{and} \quad J_3 = (\rho_2|_{\mathcal{H}})^{-1} \circ \rho_1|_{\mathcal{H}},$$

anti-commute and satisfy the $\varepsilon-$quaternionic relations

$$J_2^2 = -\varepsilon J_2^2 = -\varepsilon J_3^2 = -1 \quad \text{and} \quad J_2 J_3 = -\varepsilon J_1.$$

For $\varepsilon = -1$, the $\varepsilon-$quaternionic CR structure is called a qCR structure and for $\varepsilon = +1$ the $\varepsilon-$quaternionic CR structure is called para-qCR structure. The manifold M with an $\varepsilon-$quaternionic CR structure is called $\varepsilon-$quaternionic CR manifold.

For more details on the geometry and examples of qCR manifolds, see [3].

1.3.5 Quaternions and octonions

The set of quaternions $\mathbb{H}$ is an associative real division algebra of dimension four, generated by the so-called quaternion units i, j, k and 1, that is

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\},$$

where the quaternion units satisfy the Hamilton relations

$$i^2 = j^2 = k^2 = ijk = -1.$$

The *conjugate* of a quaternion $q = a+bi+cj+dk \in \mathbb{H}$ is the quaternion $\bar{q} = a-bi-cj-dk$. The *norm* of q is the real number

$$|q| = (a^2 + b^2 + c^2 + d^2)^{1/2} = q\bar{q}^{1/2}.$$

The real part of q is a and the imaginary part of q is $(b, c, d) \in \mathbb{R}^3$. Observe that the quaternions of norm one form a Lie group which is diffeomorphic to the three dimensional sphere S^3.

The *quaternionic exponential* is defined by

$$e^{ai+bj+ck} = \cos\sqrt{a^2 + b^2 + c^2} + \sin\sqrt{a^2 + b^2 + c^2} \cdot \frac{ai + bj + ck}{\sqrt{a^2 + b^2 + c^2}},$$

for $a, b, c \in \mathbb{R}$.

The projective quaternionic space is the $4n-$dimensional manifold

$$\mathbb{H}P^n = \frac{\mathbb{H}^{n+1} \setminus \{(0, \dots, 0)\}}{\sim},$$

where $\sim$ is the equivalence relation

$$v \sim w \quad \text{if and only if} \quad v = w \cdot \lambda,$$

for some $\lambda \in \mathbb{H} \setminus \{0\}$. Note that $\mathbb{H}P^1$ is diffeomorphic to the four dimensional sphere S^4. This can be shown by constructing a stereographic projection completely analogous to the complex one.

The set of octonions $\mathbb{O}$ is a non-associative real division algebra of dimension eight, generated by the so-called octonion units $e_1, \ldots, e_7$ and 1, that is

$$\mathbb{O} = \{a_0 + a_1 e_1 + \ldots + a_7 e_7 : a_0, \ldots, a_7 \in \mathbb{R}\},$$

where the octonion units satisfy the relations in Table 1.1.

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Table 1.1: Multiplication table for the octonion units.

The *conjugate* of an octonion $o = a_0 + a_1 e_1 + \ldots + a_7 e_7 \in \mathbb{O}$ is the octonion $\bar{o} = a_0 - a_1 e_1 - \ldots - a_7 e_7$. The *norm* of o is the real number

$$|o| = (a_0^2 + \ldots + a_7^2)^{1/2} = o\bar{o}^{1/2}.$$

The real part of o is a_0 and the imaginary part of o is $(a_1, \ldots, a_7) \in \mathbb{R}^7$. The octonions of norm one **do not** form a Lie group, but as a manifold they are diffeomorphic to the seven dimensional sphere S^7.

Analogously to the construction for quaternions, it is possible to define the *octonionic projective* line $\mathbb{O}P^1$ and the *Cayley plane* $\mathbb{O}P^2$. As commented in [4], the definition of $\mathbb{O}P^n$ makes sense only when $n \leq 2$. As in the case of quaternions, there is a stereographic projection that corresponds to a diffeomorphism between $\mathbb{O}P^1$ and the sphere S^8.

Chapter 2

Main Results

2.1 Sub-Riemannian geometry of odd dimensional spheres

This is the main interest of Papers A and B. We study the distributions of codimension one on the spheres S^{2n+1} arising from its CR, contact and principal S^1-bundle structures. We show that these distributions coincide. Moreover in the case of S^3 as a Lie group, the same distribution is shown to be generated by left invariant vector fields. Similar considerations follow for the spheres S^{4n+3}, where the distributions arising from the qCR structure and the principal S^3-action on S^{4n+3} coincide. As a consequence of these equivalences we find explicit formulas for the sub-Riemannian geodesics in all of the above mentioned cases.

2.1.1 Codimension one case

Let us consider the complex hypersurface

$$S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + \ldots + |z_n|^2 = 1\}, \tag{2.1.1}$$

which corresponds to an odd dimensional unit sphere. For the rest of this subsection, we will denote $z_j = x_j + iy_j$, where $x_j, y_j \in \mathbb{R}$ for all $j = 0, \ldots, n$.

An important fact that holds immediately from (2.1.1) is that S^{2n+1} is naturally endowed with the structure of a CR manifold. Additionally, S^{2n+1} is also a contact manifold since

$$\omega_{n+1} = -y_0 dx_0 + x_0 dy_0 - \ldots - y_n dx_n + x_n dy_n$$

is a contact form. In fact one has the following Theorem.

Theorem 2 (Paper A) *The one-form ω_{n+1} satisfies*

$$(d\omega_{n+1})^n \wedge \omega_{n+1} = n! \cdot 2^n \mathrm{dvol}_{S^{2n+1}},$$

where $\mathrm{dvol}_{S^{2n+1}}$ is the Riemannian volume form for S^{2n+1} with respect to the usual Riemannian metric $\langle \cdot, \cdot \rangle$.

Note that if we give S^{2n+1} the usual Riemannian metric $\langle \cdot, \cdot \rangle$, then its holomorphic tangent space and the contact distribution coincide, since both of them correspond to the orthogonal complement of the vector field

$$V_{n+1} = -y_0 \partial_{x_0} + x_0 \partial_{y_0} - \ldots - y_n \partial_{x_n} + x_n \partial_{y_n}.$$

In fact, V_{n+1} is the Reeb vector field for the chosen contact structure of S^{2n+1}.

Additionally, we see from (2.1.1) that S^{2n+1} has a natural $U(1)$ action given by

$$(z_0, \ldots, z_n) \mapsto (\lambda z_0, \ldots, \lambda z_n)$$

where $\lambda \in U(1)$ denotes a complex number of norm one. This action induces the projection

$$\begin{array}{rccc} H_{\mathbb{C}} & : & S^{2n+1} & \to & \mathbb{C}P^n \\ & & (z_0, \ldots, z_n) & \mapsto & [z_0 : \ldots : z_n] \end{array}$$

which is a principal $U(1)-$bundle. This projection is sometimes called generalized Hopf fibration.

The Ehresmann connection corresponding to the orthogonal complement of the vertical space for the submersion $H_{\mathbb{C}}$ coincides with the contact distribution and complex tangent space of S^{2n+1}, since the vector field V_{n+1} is tangent to the fibers of the projection $H_{\mathbb{C}}$. The respective $\mathfrak{su}(1)-$valued connection form is given by $i\omega_{n+1}$.

Now let us consider horizontal curves with respect to the distribution $\mathcal{H} = \ker \omega_{n+1}$. Since it is a contact distribution, it is bracket generating of step two and thus, by restricting the metric $\langle \cdot, \cdot \rangle$ of TS^{2n+1} to $\mathcal{H}$, we obtain a sub-Riemannian manifold $(S^{2n+1}, \mathcal{H}, \langle \cdot, \cdot \rangle_{sR})$, where $\langle \cdot, \cdot \rangle_{sR}$ denotes the restricted metric.

In this context we have the following characterization of normal sub-Riemannian geodesics.

Theorem 3 (Paper B) *Let $p \in S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + \ldots + |z_n|^2 = 1\}$ and $v \in T_pS^{2n+1}$. If $\gamma_R(t) = (z_0(t), \ldots, z_n(t))$ is the great circle satisfying $\gamma_R(0) = p$ and $\dot{\gamma}_R(0) = v$, then the corresponding sub-Riemannian geodesic is given by*

$$\gamma(t) = \left(z_0(t) e^{-it\langle v, V_{n+1} \rangle}, \ldots, z_n(t) e^{-it\langle v, V_{n+1} \rangle} \right). \tag{2.1.2}$$

A remarkable fact is that in the case of dimension three all the above mentioned structures coincide with the left invariant sub-Riemannian structure induced by the Lie group multiplication of S^3. More precisely, S^3 is a Lie group isomorphic to the symplectic group $Sp(1)$ consisting of quaternions of norm one. By right translating the canonical basis at the identity of the group $(1, 0) \in S^3$, we obtain the vector fields

$$\begin{aligned} V(y) &= -y_0 \partial_{x_0} + x_0 \partial_{y_0} - y_1 \partial_{x_1} + x_1 \partial_{y_1}, \\ X(y) &= -x_1 \partial_{x_0} + y_1 \partial_{y_0} + x_0 \partial_{x_1} - y_0 \partial_{y_1}, \\ Y(y) &= -y_1 \partial_{x_0} - x_1 \partial_{y_0} + y_0 \partial_{x_1} + x_0 \partial_{y_1}, \end{aligned}$$

which are orthonormal with respect to $\langle \cdot, \cdot \rangle$. It can be easily seen that $V = V_2$ as defined before, and thus the distribution $\text{span}\{X, Y\}$ coincides with the corresponding distribution $\mathcal{H}$ constructed above. Though not in the form of a theorem, this equivalence can be viewed as one of the main accomplishments of Paper A.

In the case of dimension seven, S^7 can be considered as the set of octonions of norm one. In a similar way as in the case of S^3, the sub-Riemannian geometry of S^7 can be described by means of octonion multiplication, even though this structure does not endow S^7 with a Lie group structure. Straightforward calculations give the orthonormal vector fields

$$\begin{aligned}
Y_1(z) &= -y_0\partial_{x_0} + x_0\partial_{y_0} - y_1\partial_{x_1} + x_1\partial_{y_1} - y_2\partial_{x_2} + x_2\partial_{y_2} - y_3\partial_{x_3} + x_3\partial_{y_3},\\
Y_2(z) &= -x_1\partial_{x_0} + y_1\partial_{y_0} + x_0\partial_{x_1} - y_0\partial_{y_1} - x_3\partial_{x_2} + y_3\partial_{y_2} + x_2\partial_{x_3} - y_2\partial_{y_3},\\
Y_3(z) &= -y_1\partial_{x_0} - x_1\partial_{y_0} + y_0\partial_{x_1} + x_0\partial_{y_1} + y_3\partial_{x_2} + x_3\partial_{y_2} - y_2\partial_{x_3} - x_2\partial_{y_3},\\
Y_4(z) &= -x_2\partial_{x_0} + y_2\partial_{y_0} + x_3\partial_{x_1} - y_3\partial_{y_1} + x_0\partial_{x_2} - y_0\partial_{y_2} - x_1\partial_{x_3} + y_1\partial_{y_3},\\
Y_5(z) &= -y_2\partial_{x_0} - x_2\partial_{y_0} - y_3\partial_{x_1} - x_3\partial_{y_1} + y_0\partial_{x_2} + x_0\partial_{y_2} + y_1\partial_{x_3} + x_1\partial_{y_3},\\
Y_6(z) &= -x_3\partial_{x_0} + y_3\partial_{y_0} - x_2\partial_{x_1} + y_2\partial_{y_1} + x_1\partial_{x_2} - y_1\partial_{y_2} + x_0\partial_{x_3} - y_0\partial_{y_3},\\
Y_7(z) &= -y_3\partial_{x_0} - x_3\partial_{y_0} + y_2\partial_{x_1} + x_2\partial_{y_1} - y_1\partial_{x_2} - x_1\partial_{y_2} + y_0\partial_{x_3} + x_0\partial_{y_3},
\end{aligned}$$

by right translating the canonical basis at $(1, 0, 0, 0) \in S^7$ using octonion multiplication. Here the well-known fact that S^7 is parallelizable can be seen explicitly. It can be easily seen that $Y_1 = V_4$ as defined before, and thus the distribution $\text{span}\{Y_2, \ldots, Y_7\}$ coincides with the corresponding distribution $\mathcal{H}$ as constructed above.

2.1.2 Codimension three case

Let us consider the quaternionic analogues of Subsection 2.1.1

$$S^{4n+3} = \{(q_0, \ldots, q_n) \in \mathbb{H}^{n+1} : |q_0|^2 + \ldots + |q_n|^2 = 1\}. \tag{2.1.3}$$

For the rest of this subsection, we will denote $q_s = x_s + iy_s + jz_s + kw_s$, where $x_s, y_s, z_s, w_s \in \mathbb{R}$ for all $s = 0, \ldots, n$.

As before, the spheres S^{4n+3} can be naturally endowed with the structure of a qCR manifold, see for example [3]. An important difference is that in this situation, S^{4n+3} possesses three independent contact forms, namely

$$\omega^1_{n+1} = -y_0dx_0 + x_0dy_0 + w_0dz_0 - z_0dw_0 - \ldots - y_ndx_n + x_ndy_n + w_ndz_n - z_ndw_n,$$

$$\omega^2_{n+1} = -z_0dx_0 - w_0dy_0 + x_0dz_0 + y_0dw_0 - \ldots - z_ndx_n - w_ndy_n + x_ndz_n + y_ndw_n,$$

$$\omega^3_{n+1} = -w_0dx_0 + z_0dy_0 - y_0dz_0 + x_0dw_0 - \ldots - w_ndx_n - z_ndy_n + y_ndz_n + x_ndw_n.$$

In a similar way, the spheres S^{4n+3} have a natural right $Sp(1)$ action given by

$$(q_0, \ldots, q_n) \mapsto (q_0 \cdot \lambda, \ldots, q_n \cdot \lambda)$$

where $\lambda \in Sp(1)$ denotes a quaternion of norm one. This action induces the projection

$$\begin{array}{cccc} H_{\mathbb{H}} & : & S^{4n+3} & \to & \mathbb{H}P^n \\ & & (q_0, \ldots, q_n) & \mapsto & [q_0 : \ldots : q_n] \end{array},$$

which is a principal $Sp(1)-$bundle. This projection is sometimes called *quaternionic Hopf fibration.*

Observe that if we give S^{4n+3} the usual Riemannian metric $\langle \cdot, \cdot \rangle$, then the Ehresmann connection corresponding to the orthogonal complement of the vertical space for the submersion $H_{\mathbb{H}}$ coincides with the distribution $\mathcal{H} = \ker \omega^1_{n+1} \cap \ker \omega^2_{n+1} \cap \ker \omega^3_{n+1}$, since the vector fields

$$V^1_{n+1}(p) = -y_0\partial_{x_0} + x_0\partial_{y_0} + w_0\partial_{z_0} - z_0\partial_{w_0} - \ldots - y_n\partial_{x_n} + x_n\partial_{y_n} + w_n\partial_{z_n} - z_n\partial_{w_n},$$

$$V^2_{n+1}(p) = -z_0\partial_{x_0} - w_0\partial_{y_0} + x_0\partial_{z_0} + y_0\partial_{w_0} - \ldots - z_n\partial_{x_n} - w_n\partial_{y_n} + x_n\partial_{z_n} + y_n\partial_{w_n},$$

$$V^3_{n+1}(p) = -w_0\partial_{x_0} + z_0\partial_{y_0} - y_0\partial_{z_0} + x_0\partial_{w_0} - \ldots - w_n\partial_{x_n} - z_n\partial_{y_n} + y_n\partial_{z_n} + x_n\partial_{w_n},$$

are tangent to the fibers of the projection $H_{\mathbb{H}}$. The respective $\mathfrak{sp}(1)-$valued connection form is given by $A = i\omega^1_{n+1} + j\omega^2_{n+1} + k\omega^3_{n+1}$. Observe that V^1_{n+1}, V^2_{n+1} and V^3_{n+1} are the corresponding Reeb vector fields determined by the contact forms $\omega^1_{n+1}, \omega^2_{n+1}$ and ω^3_{n+1}.

Now let us consider horizontal curves with respect to the distribution $\mathcal{H}$. Since S^{4n+3} is a qCR manifold, the distribution $\mathcal{H}$ is bracket generating of step two and thus, by restricting the metric $\langle \cdot, \cdot \rangle$ of TS^{4n+3} to $\mathcal{H}$, we obtain a sub-Riemannian manifold $(S^{4n+3}, \mathcal{H}, \langle \cdot, \cdot \rangle_{sR})$, where $\langle \cdot, \cdot \rangle_{sR}$ denotes the restricted metric.

In this context we have the following characterization of normal sub-Riemannian geodesics.

Theorem 4 (Paper B) *Let $p \in S^{4n+3} = \{(q_0, \ldots, q_n) \in \mathbb{H}^{n+1} : |q_0|^2 + \ldots + |q_n|^2 = 1\}$ and $v \in T_pS^{4n+3}$. If $\gamma_R(t) = (q_0(t), \ldots, q_n(t))$ is the great circle satisfying $\gamma_R(0) = p$ and $\dot{\gamma}_R(0) = v$, then the corresponding sub-Riemannian geodesic is given by*

$$\gamma(t) = \left(q_0(t) \cdot e^{-tA(v)}, \ldots, q_n(t) \cdot e^{-tA(v)}\right). \tag{2.1.4}$$

Note that the curve $e^{-tA(v)}$ corresponds to the Riemannian geodesic in S^3 starting at the identity of the group $e = (1, 0, 0, 0)$, with initial velocity vector

$$(0, -\omega^1_{n+1}(v), -\omega^2_{n+1}(v), -\omega^3_{n+1}(v)).$$

In this case an analogy to the remark at the end of Subsection 2.1.1 is not straightforward. It turns out that in the case of dimension seven, the vector fields V^1_2, V^2_2 and V^3_2 coincide with the vector fields $\frac{1}{2}[Y_5, Y_4]$, $\frac{1}{2}[Y_6, Y_4]$ and $\frac{1}{2}[Y_5, Y_6]$ respectively, thus the distribution in this case can be seen as the orthogonal complement to the Lie brackets between Y_4, Y_5 and Y_6. A major inconvenience is that a global basis of such a distribution is unknown, however, it can be completely described by appropriately constructed charts. This corresponds to Theorem 5 in Paper A.

2.2 Curvature of sub-Riemannian geodesics in $\mathbf{H}^1$

In recent years, mathematicians have seen the need for trying to obtain geodesic differential equations for different model sub-Riemannian manifolds. The case of the three dimensional Heisenberg group with its usual sub-Riemannian structure was studied in [31] and as was noted in [20], the same proof holds for all three dimensional Sasakian pseudo-Hermitian sub-Riemannian manifolds, up to obvious modifications. It is of special interest to observe that these differential equations are naturally connected with the Riemannian concept of curvature.

Note that similar geodesic equations were found in [32], which hold for all pseudo-Hermitian manifolds. An important difference is that the affine connection considered , since the affine connection considered there is the Webster connection and, in general, its torsion does not vanish.

2.2.1 Case of S^3

By studying the first variation of the sub-Riemannian length functional it is possible to obtain a geodesic equation in the three dimensional case, assuming that the sub-Riemannian manifold satisfies additional conditions on its geometry.

In this case it is fundamental to consider only variations that are admissible. More precisely, a *variation* of a curve $\gamma : [a, b] \to M$ is a C^2-map $\tilde{\gamma} : I_1 \times I_2 \to M$, where I_1, I_2 are open intervals, $0 \in I_2$ and $\tilde{\gamma}(s, 0) = \gamma(s)$. The variation curves $\tilde{\gamma}(\cdot, \varepsilon)$ for fixed ε are often denoted by $\gamma_\varepsilon(\cdot)$. A variation γ_ε of a horizontal curve γ is called *admissible* if all curves $\gamma_\varepsilon : I_1 \to M$ are horizontal, $\gamma_\varepsilon(a) = \gamma(a)$ and $\gamma_\varepsilon(b) = \gamma(b)$ for all $\varepsilon \in I_2$.

In this context the following result has essentially been proved in [20, 31].

Proposition 1 (Hurtado, Ritoré, Rosales) *Let $(M, H, \langle\cdot,\cdot\rangle_{sR})$ be a Sasakian pseudo-Hermitian sub-Riemannian manifold of dimension 3. Let $\gamma : I \to M$ be a C^2 horizontal curve parameterized by arc-length. Then γ is a critical point of length functional for any admissible variation if and only if there is $\lambda \in \mathbb{R}$ such that γ satisfies the second order ordinary differential equation*

$$\nabla_{\dot{\gamma}}\dot{\gamma} + 2\lambda J(\dot{\gamma}) = 0, \tag{2.2.1}$$

where ∇ is the Levi-Civita connection associated to the Sasakian metric, and $J : H \to H$ is the almost complex structure.

The parameter λ is called the curvature of γ since in the case of the three dimensional Heisenberg group, after projecting it to $\mathbb{R}^2$ via the orthogonal projection

$$\begin{array}{ccc} \mathbb{R}^2 \times \mathbb{R} & \to & \mathbb{R}^2 \\ (z, t) & \mapsto & z \end{array},$$

then λ becomes precisely the curvature of the projected curve in $\mathbb{R}^2$. In the case of S^3, the parameter λ is the curvature of the projection of γ to S^2 via the Hopf fibration

$$\begin{array}{ccc} S^3 & \to & S^2 \\ (z, w) & \mapsto & (2z\bar{w}, |z|^2 - |w|^2) \end{array},$$

where $S^2 \subset \mathbb{C} \times \mathbb{R}$.

An interesting result in this direction is a geometric realization of λ in terms of the initial velocity of the great circle which determines it.

Proposition 2 (Paper B) *The curvature of the sub-Riemannian geodesic*

$$\gamma(t) = e^{-i\langle v, V_2\rangle t}\gamma_R(t)$$

in S^3, parameterized by arc-length, equals $\langle v, V_2\rangle$.

In [20] it is given a criterion saying that a sub-Riemannian geodesic in S^3 is a closed curve if and only if $\lambda/\sqrt{1+\lambda^2}$ is a rational number. By Proposition 2, this criterion can be written in terms of the initial velocity of the great circle giving rise to the corresponding sub-Riemannian geodesic. This gives us the feeling that a more general statement should hold. In fact, we have the following result.

Proposition 3 (Paper B) *Let $\gamma : \mathbb{R} \to S^{2n+1}$ be a complete sub-Riemannian geodesic parameterized by arc-length, with initial velocity $v \in T_pS^{2n+1}$, where S^{2n+1} is considered as a principal S^1-bundle. Then γ is closed if and only if*

$$\frac{\langle v, V_{n+1}\rangle}{\sqrt{1+\langle v, V_{n+1}\rangle^2}} \in \mathbb{Q}.$$

Similarly, if $\gamma : \mathbb{R} \to S^{4n+3}$ be a complete sub-Riemannian geodesic parameterized by arc-length, with initial velocity $v \in T_pS^{4n+3}$, where S^{4n+3} is considered as a principal S^3-bundle. Then γ is closed if and only if

$$\frac{\langle v, V^1_{n+1}\rangle}{\|v\|^2}, \frac{\langle v, V^2_{n+1}\rangle}{\|v\|^2}, \frac{\langle v, V^3_{n+1}\rangle}{\|v\|^2} \in \mathbb{Q}.$$

2.2.2 Case of $\mathbf{H}^1$

When trying to extend Proposition 1 to the case of manifolds of dimension 7 with a sub-Riemannian structure of corank three, one finds some technical problems not present in the case of dimension 3. The first step in this direction is studying the first quaternionic $H-$type group $\mathbf{H}^1$, see [9].

Let us consider the 4×4 matrices $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$, given by

$$\mathcal{I} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{K} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Note that $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$ are a fixed representation of the quaternion units, that is if $\mathcal{U}$ denotes the identity matrix of size 4×4, then $\operatorname{span}\{\mathcal{U}, \mathcal{I}, \mathcal{J}, \mathcal{K}\}$ is isomorphic to $\mathbb{H}$ as algebras via the isomorphism

$$\varphi : \operatorname{span}\{\mathcal{U}, \mathcal{I}, \mathcal{J}, \mathcal{K}\} \to \mathbb{H}$$

given by $\varphi(\mathcal{U}) = 1, \varphi(\mathcal{I}) = i, \varphi(\mathcal{J}) = j, \varphi(\mathcal{K}) = k$ and extended by linearity.

The seven dimensional quaternionic $H-$type group $\mathbf{H}^1$ corresponds to the manifold $\mathbb{R}^4 \oplus \mathbb{R}^3$ with the group operation $\circ$ defined by

$$(x, z) \circ (x', z') = \left(x + x', z_{\mathcal{I}} + z'_{\mathcal{I}} + \frac{1}{2} x'^T \mathcal{I} x, z_{\mathcal{J}} + z'_{\mathcal{J}} + \frac{1}{2} x'^T \mathcal{J} x, z_{\mathcal{K}} + z'_{\mathcal{K}} + \frac{1}{2} x'^T \mathcal{K} x \right)$$

where $z = (z_{\mathcal{I}}, z_{\mathcal{J}}, z_{\mathcal{K}}) \in \mathbb{R}^3$, x, x' are column vectors in $\mathbb{R}^4$ and x^T, x'^T are the corresponding row vectors obtained by transposition.

The Lie algebra $\mathfrak{h}^1$ corresponding to $\mathbf{H}^1$ is spanned by the left invariant vector fields

$$\begin{aligned}
X_1(x,z) &= \frac{\partial}{\partial x_1} + \frac{1}{2}\left(+x_2 \frac{\partial}{\partial z_{\mathcal{I}}} - x_4 \frac{\partial}{\partial z_{\mathcal{J}}} - x_3 \frac{\partial}{\partial z_{\mathcal{K}}} \right), \\
X_2(x,z) &= \frac{\partial}{\partial x_2} + \frac{1}{2}\left(-x_1 \frac{\partial}{\partial z_{\mathcal{I}}} - x_3 \frac{\partial}{\partial z_{\mathcal{J}}} + x_4 \frac{\partial}{\partial z_{\mathcal{K}}} \right), \\
X_3(x,z) &= \frac{\partial}{\partial x_3} + \frac{1}{2}\left(+x_4 \frac{\partial}{\partial z_{\mathcal{I}}} + x_2 \frac{\partial}{\partial z_{\mathcal{J}}} + x_1 \frac{\partial}{\partial z_{\mathcal{K}}} \right), \\
X_4(x,z) &= \frac{\partial}{\partial x_4} + \frac{1}{2}\left(-x_3 \frac{\partial}{\partial z_{\mathcal{I}}} + x_1 \frac{\partial}{\partial z_{\mathcal{J}}} - x_2 \frac{\partial}{\partial z_{\mathcal{K}}} \right), \\
Z_{\mathcal{I}}(x,z) &= \frac{\partial}{\partial z_{\mathcal{I}}}, \quad Z_{\mathcal{J}}(x,z) = \frac{\partial}{\partial z_{\mathcal{J}}}, \quad Z_{\mathcal{K}}(x,z) = \frac{\partial}{\partial z_{\mathcal{K}}}.
\end{aligned}$$

at a point $(x, z) = (x_1, x_2, x_3, x_4, z_{\mathcal{I}}, z_{\mathcal{J}}, z_{\mathcal{K}}) \in \mathbf{H}^1$. A Riemannian metric $\langle \cdot, \cdot \rangle$ in $\mathbf{H}^1$ is declared so that $X_1, \dots, X_4, Z_{\mathcal{I}}, \dots, Z_{\mathcal{K}}$ is an orthonormal frame at each $(x, z) \in \mathbf{H}^1$. The sub-Riemannian structure on $\mathbf{H}^1$ we are interested in is defined by the left invariant distribution $\mathcal{D} = \operatorname{span}\{X_1, X_2, X_3, X_4\}$ and the restriction of the metric previously defined.

In this context, the following theorem holds.

Theorem 5 (Paper B) *Let $\gamma : [a, b] \to \mathbf{H}^1$ be a horizontal curve, parameterized by arc length. Then γ is a critical point of the sub-Riemannian length functional if and only if there exist $\lambda_{\mathcal{I}}, \lambda_{\mathcal{J}}, \lambda_{\mathcal{K}} \in \mathbb{R}$ satisfying the second order differential equation*

$$\nabla_{\dot\gamma}\dot\gamma - 2 \sum_{r=\mathcal{I},\mathcal{J},\mathcal{K}} \lambda_r J_r(\dot\gamma) = 0. \tag{2.2.2}$$

where ∇ is the Levi-Civita connection associated to the Riemannian metric previously defined and $J_{\mathcal{I}}, J_{\mathcal{J}}, J_{\mathcal{K}} : \mathcal{D} \to \mathcal{D}$ are the almost complex structures

$$J_r(X) = 2\nabla_X Z_r, \quad r = \mathcal{I}, \mathcal{J}, \mathcal{K},$$

2.3 Intrinsic sub-Laplacian of S^7

The notion of hypoelliptic operators in connection with sub-Riemannian geometry first came up in the seminal paper [16], where the celebrated Hörmander condition is introduced. Interestingly enough, the condition for hypoellipticity of differential operators in the form of a sum of squares is equivalent to the condition of the Chow-Rashevskiĭ theorem for the corresponding vector fields. During several decades, a mayor interest in CR geometry and in the study of Carnot groups was the analytic properties of sub-Laplacians satisfying Hörmander condition, regardless of eventual dependence on the choice of coordinates. This technical inconvenience was solved in [1] by using the notion of Popp's measure to define a sub-Riemannian divergence, which naturally defines an intrinsic sub-Laplacian maintaining its hypoelliptic character.

Employing the aforementioned construction, we show that the intrinsic sub-Laplacian of S^7, considered as a principal S^1-bundle, is a sum of squares and as a corollary we see that the heat kernel of the sub-Laplacian commutes with the heat flow in the direction of the Reeb vector field.

2.3.1 Popp's measure for contact manifolds

In this subsection, we briefly come back to the general situation in which $(M, \mathcal{H}, \langle \cdot, \cdot \rangle_{sR})$ is a sub-Riemannian manifold, where $\mathcal{H}$ is a bracket generating distribution of rank k, and M has dimension n. As observed in [1], for analytical reasons it is convenient to assume that $\mathcal{H}$ is a regular distribution. Recall that $\mathcal{H}$ is a regular distribution if the so-called growth vector

$$(\dim \mathcal{H}_q, \dim \mathcal{H}_q^2, \dots, \dim \mathcal{H}_q^k)$$

does not depend on the point $q \in M$. The reason for this assumption will be made clear in Subsection 2.3.2.

The construction of a sub-Laplacian defined intrinsically required a correct definition of divergence in the sub-Riemannian setting. On the other hand, an appropriate definition of sub-Riemannian divergence required the knowledge of a volume form capturing the geometric information of the bracket generating distribution at the level of vector fields.

The main idea in this context is to note that a flag

$$F : F_1 \subset F_2 \subset \cdots \subset F_k = E$$

of vector subspaces of a vector space E induces a canonical isomorphism

$$\bigwedge^n E^* \xrightarrow{\cong} \bigwedge^n \mathrm{Gr}\,(F)^*\,,$$

where $\mathrm{Gr}\,(F)$ is the graded vector space associated to the flag F defined as

$$\mathrm{Gr}\,(F) = F_1 \oplus F_2/F_1 \oplus \cdots \oplus F_k/F_{k-1}.$$

For the construction of the isomorphism, see [28, Chapter 10].

Given the fact that a surjection from an inner product space to a vector space induces an inner product on the target, we see that if $\mathcal{H}$ is a bracket generating distribution, then the map

$$\begin{array}{ccc} \bigotimes^j \mathcal{H}_q & \to & (\mathcal{H}^j/\mathcal{H}^{j-1})_q \\ v_1 \otimes v_2 \otimes \cdots \otimes v_j & \mapsto & [v_1, [v_2, [\cdots, v_j] \cdots]] \end{array}$$

endows the space $\mathrm{Gr}\,(\mathcal{H})_q$ with an inner product, arising from the inner product on $\bigoplus_{j=1}^k \bigotimes^j \mathcal{H}_q$ inherited from $\langle \cdot, \cdot \rangle_{sR}$. Since any finite dimensional inner product space possesses a natural volume form, there is a canonical isomorphism

$$\bigwedge^n \mathrm{Gr}\,(\mathcal{H})_q^* \xrightarrow{\cong} \mathbb{R}\,,$$

which, up to sign, induces a well-defined element in $\bigwedge^n T_q^* M$. The corresponding $n-$form, denoted by μ_{sR} is called the Popp measure of the sub-Riemannian manifold $(M, \mathcal{H}, \langle \cdot, \cdot \rangle_{sR})$.

An important example is provided by contact manifolds. As it is observed in Paper B, in the case of Riemannian contact manifolds of dimension $2n+1$, the $n-$form μ_{sR} locally takes the form

$$\mu_{sR} = \pi_1 \wedge \ldots \wedge \pi_{2n} \wedge \pi_{2n+1},$$

where $\pi_1, \ldots, \pi_{2n}$ is a dual basis for a local orthonormal frame of the contact distribution and π_{2n+1} is dual to the Reeb vector field. In particular, given the trivialization of TS^7 presented in Subsection 2.1.1, we have a global description of Popp's measure for S^7 with distribution of corank 1, given by

$$dY_1 \wedge \ldots \wedge dY_7.$$

2.3.2 Sub-Laplacian and heat kernel for S^7

In [1] the definition of an intrinsic sub-Laplacian is introduced by generalizing the Riemannian concepts of gradient and divergence to the sub-Riemannian context. The horizontal gradient $\nabla_{sR} f$ is defined by the equation

$$\langle \nabla_{sR} f(p), v \rangle_{sR} = d_p f(v),$$

and the sub-Riemannian divergence $\mathrm{div}_{sR} X$ of a horizontal vector field X by

$$\mathrm{div}_{sR} X \mu_{sR} = L_X \mu_{sR},$$

where μ_{sR} is Popp's volume form, and L_X denotes the Lie derivative in the direction of X. Note that the bracket generating condition is essential for the choice of Popp's measure, and thus, essential for the definition of the sub-Riemannian divergence.The intrinsic sub-Laplacian $\Delta_{sR} f$ is given by

$$\Delta_{sR} f = \mathrm{div}_{sR}(\nabla_{sR} f). \tag{2.3.1}$$

If the distribution under consideration is regular, this operator is hypoelliptic. As it is noted in [1], the Grushin plane gives an example of a bracket generating distribution which is not regular and its intrinsic sub-Laplacian is not hypoelliptic.

Let $\{X_1, \ldots, X_k\}$ be a local orthonormal basis of $\mathcal{H}$ and consider the corresponding dual basis $\{dX_1, \ldots, dX_k\}$. It is possible to find vector fields $\{X_{k+1}, \ldots, X_n\}$ such that the vector fields $\{X_1, \ldots, X_n\}$ span TM and such that Popp's volume form is locally given by

$$\mu_{sR} = dX_1 \wedge \ldots \wedge dX_k \wedge dX_{k+1} \wedge \ldots \wedge dX_n. \tag{2.3.2}$$

In this setting, the sub-Laplacian $\Delta_{sR} f$ can be written explicitly as

$$\Delta_{sR} f = \sum_{r=1}^{k} \left(L_{X_r}^2 f + L_{X_r} f \sum_{s=1}^{n} dX_s([X_r, X_s]) \right). \tag{2.3.3}$$

Thus we have the following result.

Theorem 6 (Paper B) *Let $\mathcal{H}$ be the contact distribution for S^7 and $\langle \cdot, \cdot \rangle_{sR}$ the restriction of the usual Riemannian metric in $\mathbb{R}^8$ to $\mathcal{H}$. Then the intrinsic sub-Laplacian of $(S^7, \mathcal{H}, \langle \cdot, \cdot \rangle_{sR})$ is given by the sum of squares*

$$\Delta_{sR} = \sum_{a=2}^{7} Y_a^2.$$

An important fact that the operator Δ_{sR} on S^7 satisfies is that it commutes with the operator defined by the Reeb vector field Y_1. In other words, the following result holds.

Theorem 7 (Paper B) *The operators Δ_{sR} and Y_1^2 commute.*

Theorem 7 has as a consequence a characterization of the sub-Riemannian heat kernel in S^7 in terms of the Riemannian heat kernel and the unbounded operator of heat flow along the Reeb vector field for negative times.

Corollary 1 *Denoting by Δ_{S^7} the Riemannian Laplacian in S^7 with respect to the usual Riemannian structure, we have that*

$$e^{-t\Delta_{sR}} = e^{-t\Delta_{S^7}} e^{tY_1^2}.$$

2.4 Intrinsic rolling of manifolds

The aim of this section is to provide a coordinate-free version of the definition given in [33] for the system of two connected Riemannian manifolds of dimension $n \geq 2$, one of which is rolling on the other one without slipping or twisting. The so-called controllability question asks if any given configuration of the manifolds can be obtained from any other one by an appropriate rolling. The idea is to find the answer to the controllability question from geometric information of the manifolds.

The fact that the kinematic constraints of no-slipping and no-twisting can be understood as a distribution of rank n over the manifold corresponding to the configuration space of the system, which is a manifold of dimension $\frac{n(n+3)}{2}$, transforms the controllability question into a sub-Riemannian problem: if we know that the rolling distribution is bracket generating, then the system is controllable. We can give an answer to the converse statement in the analytic category. If the manifolds and the distribution are analytic, the converse of the Chow-Rashevskiĭ theorem is true, thus controllability holds if and only if the distribution is bracket generating. This section treats the results obtained in Paper C.

2.4.1 Extrinsic definition of rolling

Let M and $\widehat{M}$ be connected oriented Riemannian manifolds of dimension n. Let ι and $\widehat{\iota}$ be orientation preserving isometric imbeddings of M and $\widehat{M}$ into $\mathbb{R}^{n+\nu}$ for an appropriate choice of ν, which exist due to [29]. Here $\mathbb{R}^{n+\nu}$ is equipped with the standard Euclidean metric and standard orientation. Let us identify the abstract manifolds M and $\widehat{M}$ with their image under the corresponding imbeddings. These imbeddings of M and $\widehat{M}$ into $\mathbb{R}^{n+\nu}$ split the tangent space of $\mathbb{R}^{n+\nu}$ into direct sums:

$$\begin{aligned} T_x\mathbb{R}^{n+\nu} &= T_xM \oplus T_xM^{\perp}, \quad x \in M, \\ T_{\widehat{x}}\mathbb{R}^{n+\nu} &= T_{\widehat{x}}\widehat{M} \oplus T_{\widehat{x}}\widehat{M}^{\perp}, \quad \widehat{x} \in \widehat{M}. \end{aligned} \tag{2.4.1}$$

A notational convention that will be used throughout this section is that a vector $v \in T_x\mathbb{R}^{n+\nu}$, $x \in M$, will be written as $v = v^{\top} + v^{\perp}$, where $v^{\top} \in T_xM$ and $v^{\perp} \in T_xM^{\perp}$.

Let ∇ denote the Levi-Civita connection on M or on $\widehat{M}$. The context will indicate on which manifold the connection is defined. The Levi-Civita connection on $\mathbb{R}^{n+\nu}$ is denoted by $\overline{\nabla}$. If X and Y are tangent vector fields on M, then

$$\nabla_X Y(x) = \left(\overline{\nabla}_{\bar{X}}\bar{Y}(x)\right)^{\top}, \qquad x \in M,$$

where $\bar{X}$ and $\bar{Y}$ are any local extensions to $\mathbb{R}^{n+\nu}$ of X and Y, respectively. Similarly, if Υ is a normal vector field on M and X is a tangent vector field on M, then the normal connection is defined by

$$\nabla^{\perp}_X \Upsilon(x) = \left(\overline{\nabla}_{\bar{X}}\bar{\Upsilon}(x)\right)^{\perp}, \qquad x \in M,$$

where $\bar{\Upsilon}$ is any local extension to $\mathbb{R}^{n+\nu}$ of Υ. Equivalent statements hold for $\widehat{M}$. Capital Latin letters X, Y, Z denote tangent vector fields and capital Greek letters Υ, Ψ denote normal vector fields. We denote by $\frac{D}{dt}$ the covariant derivative associated to the Levi-Civita connection on M or on $\widehat{M}$, and similarly $\frac{D^{\perp}}{dt}$ denotes the covariant derivative corresponding to the normal connection.

The following definition is a reformulation of rolling as presented in [33, Appendix B] with the additional condition of preserving orientation.

Definition 1 *A rolling of M on $\widehat{M}$ without slipping or twisting is an absolutely continuous curve $(x, g) : [0, \tau] \to M \times \mathrm{Isom}^+(\mathbb{R}^{n+\nu})$ satisfying the following conditions:*

(i) $\widehat{x}(t) := g(t)x(t) \in \widehat{M}$,

(ii) $dg(t)T_{x(t)}M = T_{\widehat{x}(t)}\widehat{M}$,

(iii) No slip condition: $\dot{\widehat{x}}(t) = dg(t)\dot{x}(t)$, *for almost every t.*

(iv) No twist condition (tangential part):

$$dg(t)\frac{D}{dt}Z(t) = \frac{D}{dt}dg(t)Z,$$

for any tangent vector field $Z(t)$ along $x(t)$ and almost every t.

(v) No twist condition (normal part):

$$dg(t)\frac{D^{\perp}}{dt}\Psi(t) = \frac{D^{\perp}}{dt}dg(t)\Psi(t),$$

for any normal vector field $\Psi(t)$ along $x(t)$ and almost every t.

(vi) $d_{x(t)}g(t)|_{T_{x(t)}M} : T_{x(t)}M \to T_{\widehat{x}(t)}\widehat{M}$ *is orientation preserving.*

Here $\mathrm{Isom}^+(\mathbb{R}^{n+\nu})$ denotes the space of affine oriented isometries of $\mathbb{R}^{n+\nu}$, that is, the Lie group $\mathrm{SE}(n+\nu)$.

2.4.2 Intrinsic definition of rolling manifolds

Let us introduce the configuration space Q, which can be thought of as all the relative positions in which M can be tangent to $\widehat{M}$. Define the $\mathrm{SO}(n)-$bundle over $M \times \widehat{M}$ by

$$Q = \left\{ q \in \mathrm{Isom}_0^+(T_xM, T_{\widehat{x}}\widehat{M}) \,\middle|\, x \in M,\, \widehat{x} \in \widehat{M} \right\}. \tag{2.4.2}$$

Here $\mathrm{Isom}_0^+(V, \widehat{V})$ denotes the group of linear orientation preserving isometries between the oriented inner product spaces V and $\widehat{V}$. We denote by $\pi : Q \to M \times \widehat{M}$ the canonical projection. In what follows, Q will denote indistinctly both the configuration space of the kinematic problem and the $\mathrm{SO}(n)-$bundle (2.4.2). It is important to stress that the $\mathrm{SO}(n)-$bundle Q is, in general, not principal for $n \geq 3$.

Fixing a pair of imbeddings ι and $\widehat{\iota}$ of M and $\widehat{M}$ into $\mathbb{R}^{n+\nu}$, respectively, we can construct an $\mathrm{SO}(\nu)-$bundle $P_{\iota,\widehat{\iota}}$ of isometries of the normal tangent space over $M \times \widehat{M}$. This construction is analogous to Q, but restricted to the normal directions of the imbeddings. The aforementioned bundle is given by

$$P_{\iota,\widehat{\iota}} := \left\{ p \in \mathrm{Isom}_0^+(T_xM^{\perp}, T_{\widehat{x}}\widehat{M}^{\perp}) \,\middle|\, x \in M, \widehat{x} \in \widehat{M} \right\}. \tag{2.4.3}$$

The dimension of $P_{\iota,\widehat{\iota}}$ is $2n + \frac{\nu(\nu-1)}{2}$. Observe that Q is invariant of the choice of imbeddings, while $P_{\iota,\widehat{\iota}}$ is not. As in the case of Q, the $\mathrm{SO}(\nu)-$bundle $P_{\iota,\widehat{\iota}}$ is, in general, not principal for $\nu \geq 3$.

Proposition 4 (Paper C) *If a curve $(x, g) : [0, \tau] \to M \times \mathrm{Isom}^+(\mathbb{R}^{n+\nu})$ satisfies (i)-(vi) of Definition 1, then the mapping*

$$t \mapsto (dg(t)|_{T_{x(t)}M}, dg(t)|_{T_{x(t)}M^\perp}) =: (q(t), p(t)),$$

defines a curve in $Q \oplus P_{\iota,\widehat{\iota}}$ with the following properties:

(I) no slip condition: $\dot{\widehat{x}}(t) = q(t)\dot{x}(t)$ for almost every t.

(II) no twist condition (tangential part): $q(t)\frac{D}{dt}Z(t) = \frac{D}{dt}q(t)Z(t)$ for any tangent vector field $Z(t)$ along $x(t)$ and almost every t.

(III) no twist condition (normal part): $p(t)\frac{D^\perp}{dt}\Psi(t) = \frac{D^\perp}{dt}p(t)\Psi(t)$ for any normal vector field $\Psi(t)$ along $x(t)$ and almost every t.

Conversely, if $(q, p) : [0, \tau] \to Q \oplus P_{\iota,\widehat{\iota}}$ is an absolutely continuous curve satisfying (I)–(III), then there exists a unique rolling $(x, g) : [0, \tau] \to M \times \mathrm{Isom}^+(\mathbb{R}^{n+\nu})$, such that $dg(t)|_{T_{x(t)}M} = q(t)$ and $dg(t)|_{T_{x(t)}M^\perp} = p(t)$.

Proposition 4 is the first step in a coordinate free version of the rolling manifolds problem, since it allows to split the information of the rolling in the tangent directions and the normal directions with respect to the imbedding. This motivates the following definition.

Definition 2 *An extrinsic rolling of M on $\widehat{M}$ is an absolutely continuous curve*

$$(q, p) : [0, \tau] \to Q \oplus P_{\iota,\widehat{\iota}}$$

satisfying (I)-(III) in Proposition 4.

Note that Proposition 4 means that a rolling in the sense of Definition 1 uniquely defines an extrinsic rolling and vice versa. Restricting conditions *(I)–(III)* to the bundle Q gives a purely intrinsic definition of rolling.

Definition 3 *An intrinsic rolling of M on $\widehat{M}$ is an absolutely continuous curve*

$$q : [0, \tau] \to Q,$$

satisfying the following conditions: if $x(t) = \mathrm{pr}_M q(t)$ and $\widehat{x}(t) = \mathrm{pr}_{\widehat{M}} q(t)$, then

(I') no slip condition: $\dot{\widehat{x}}(t) = q(t)\dot{x}(t)$ for almost all t;

(II') no twist condition: $q(t)\frac{D}{dt}Z(t) = \frac{D}{dt}q(t)Z(t)$ for any tangent vector field $Z(t)$ along $x(t)$ and almost every t.

Observe that condition *(II')* implies that $Z(t)$ is a parallel tangent vector field along $x(t)$, if and only if $q(t)Z(t)$ is parallel along $\widehat{x}(t)$ for almost all t. In this sense, Definition 3 naturally generalizes the definition given in [2, Chapter 24] for 2-dimensional Riemannian manifolds imbedded in $\mathbb{R}^3$.

A natural question is whether an intrinsic rolling extends uniquely to an extrinsic one dependent on given imbeddings. This is answered by the following theorem.

Theorem 8 (Paper C) *Let $q : [0,\tau] \to Q$ be an intrinsic rolling and let $(x_0, \widehat{x}_0) \in M \times \widehat{M}$ be the point $\mathrm{pr}_{M\times\widehat{M}}\, q(0)$. Assume $\iota : M \to \mathbb{R}^{n+\nu}$ and $\widehat{\iota} : \widehat{M} \to \mathbb{R}^{n+\nu}$ are given imbeddings.*

Then, given an initial configuration p_0 in the fiber of $P_{\iota,\widehat{\iota}}$ over $(x_0, \widehat{x}_0)$, there exists a unique rolling $(q,p) : [0,\tau] \to Q \oplus P_{\iota,\widehat{\iota}}$ satisfying $p(0) = p_0$.

2.4.3 Distribution

The goal of this subsection is to find a distribution corresponding to the restrictions of no-twisting and no-slipping. This distribution is called *rolling distribution.* Let $U \subset M$ denote a neighborhood around x, such that both bundles $TM \to M$ and $TM^\perp \to M$ become trivial when restricted to U. Define $\widehat{U} \subset \widehat{M}$ around $\widehat{x}$ similarly. Let $\{e_j\}_{j=1}^n$, $\{\epsilon_\lambda\}_{\lambda=1}^\nu$, $\{\hat{e}_i\}_{i=1}^n$ and $\{\hat{\epsilon}_\kappa\}_{\kappa=1}^\nu$ denote positively oriented orthonormal frames of $TM|_U$, $TM^\perp|_U$, $T\widehat{M}|_{\widehat{U}}$ and $T\widehat{M}^\perp|_{\widehat{U}}$, respectively. There is a trivialization

$$\begin{array}{rcl} (Q \oplus P_{\iota,\widehat{\iota}})|_{U\times\widehat{U}} & \xrightarrow{h} & U \times \widehat{U} \times \mathrm{SO}(n) \times \mathrm{SO}(\nu), \\ (q,p) & \mapsto & (x, \widehat{x}, A, B), \end{array} \tag{2.4.4}$$

given by

$$\begin{aligned} x = \mathrm{pr}_U(q,p), \qquad & \widehat{x} = \mathrm{pr}_{\widehat{U}}(q,p), \\ A = (a_{ij})_{i,j=1}^n &= (\langle qe_j, \hat{e}_i\rangle)_{i,j=1}^n\,, \\ B = (b_{\kappa\lambda})_{\kappa,\lambda=1}^\nu &= (\langle p\epsilon_\lambda, \hat{\epsilon}_\kappa\rangle)_{\kappa,\lambda=1}^\nu\,. \end{aligned}$$

Let us consider the following global left invariant basis of $T\mathrm{SO}(n)$ defined by

$$W_{ij}(A) = \sum_{r=1}^n \left(a_{ri}\frac{\partial}{\partial a_{rj}} - a_{rj}\frac{\partial}{\partial a_{ri}} \right),$$

where $A = (a_{ij}) \in \mathrm{SO}(n)$.

Definition 4 *For a vector field X on M, we define the vector fields $\mathcal{V}(X)$ and $\mathcal{V}^\perp(X)$ on $Q \oplus P_{\iota,\widehat{\iota}}$, such that under any local trivialization h as in (2.4.4) and for any pair of isometries $(q,p) \in (Q \oplus P_{\iota,\widehat{\iota}})_{(x,\widehat{x})}$, the following hold*

$$dh\left(\mathcal{V}(X)(q,p)\right) = \sum_{i<j} \Big(\big\langle \nabla_{X(x)} e_j(x), e_i(x)\big\rangle - \sum_{s=1}^n a_{sj} \left\langle \nabla_{qX(x)}\hat{e}_s(\widehat{x}), qe_i(x)\right\rangle \Big) W_{ij}(A),$$

$$dh\left(\mathcal{V}^{\perp}(X)(q,p)\right) = \sum_{\kappa<\lambda}\Big(\left\langle \nabla^{\perp}_{X(x)}\epsilon_\lambda(x), \epsilon_\kappa(x)\right\rangle - \sum_{\alpha=1}^{\nu} b_{\alpha\lambda}\left\langle \nabla^{\perp}_{qX(x)}\hat{\epsilon}_\alpha(\widehat{x}), p\epsilon_\kappa(x)\right\rangle\Big)W_{\kappa\lambda}(B),$$

for $A=(a_{ij})\in \mathrm{SO}(n)$, $B=(b_{\kappa\lambda})\in\mathrm{SO}(\nu)$ *and* $(x,\widehat{x})\in M\times\widehat{M}$.

The vector fields $\mathcal{V}(X)$ and $\mathcal{V}^{\perp}(X)$ allow us to define the rolling distributions for the case of intrinsic and extrinsic rolling. This definition is contained in the following proposition.

Proposition 5 (Paper C) *A curve* $(q(t),p(t))$ *in* $Q\oplus P_{\iota,\widehat{\iota}}$ *is a rolling if and only if it is almost everywhere tangent to the distribution* E*, defined by*

$$E_{(q,p)} = \{X_0 + qX_0 + \mathcal{V}(X_0)(q,p) + \mathcal{V}^{\perp}(X_0)(q,p) | X_0\in T_xM\},$$

where $(q,p)\in(Q\oplus P_{\iota,\widehat{\iota}})_{(x,\widehat{x})}$*, the fiber of* $Q\oplus P_{\iota,\widehat{\iota}}\to M\times\widehat{M}$ *over* $(x,\widehat{x})\in M\times\widehat{M}$.

Moreover, a curve $q(t)$ *in* Q *is an intrinsic rolling if and only if it is almost everywhere tangent to the distribution* D

$$D_q = \{X_0 + qX_0 + \mathcal{V}(X_0)(q) |\, X_0\in T_xM\},$$

for each $q\in Q_{(x,\widehat{x})}$.

2.4.4 Examples of controllability

In this subsection we briefly discuss two examples of manifolds rolling: The group SE(3) rolling on its Lie algebra $\mathfrak{se}(3)$ and the $n-$dimensional sphere S^n rolling on $\mathbb{R}^n$. The first of these examples is non-controllable while the second one is completely controllable. More details concerning both examples can be found in Paper C.

SE(3) **rolling on** $\mathfrak{se}(3)$

Let SE(3) be the group of orientation preserving isometries of $\mathbb{R}^3$. We consider the case of SE(3), endowed with a left invariant metric that will be defined later, rolling over its tangent space $T_1\mathrm{SE}(3)=\mathfrak{se}(3)$ at the identity, with the restricted metric.

Give SE(3) coordinates as follows. For any $x\in\mathrm{SE}(3)$ there exist $C=(c_{ij})\in\mathrm{SO}(3)$ and $r=(r_1,r_2,r_3)\in\mathbb{R}^3$, such that $x=(C,r)$ acts via

$$x(y) = Cy + r,\qquad \text{for all } y\in\mathbb{R}^3.$$

The tangent space of SE(3) at $x=(C,r)$ is spanned by the left invariant vector fields

$$e_1 = Y_1 = \frac{1}{\sqrt{2}}\left(C\cdot\frac{\partial}{\partial c_{12}} - C\cdot\frac{\partial}{\partial c_{21}}\right) = \frac{1}{\sqrt{2}}\sum_{j=1}^{3}\left(c_{j1}\frac{\partial}{\partial c_{j2}} - c_{j2}\frac{\partial}{\partial c_{j1}}\right) \tag{2.4.5}$$

$$e_2 = Y_2 = \frac{1}{\sqrt{2}}\left(C\cdot\frac{\partial}{\partial c_{13}} - C\cdot\frac{\partial}{\partial c_{31}}\right) = \frac{1}{\sqrt{2}}\sum_{j=1}^{3}\left(c_{j1}\frac{\partial}{\partial c_{j3}} - c_{j3}\frac{\partial}{\partial c_{j1}}\right) \tag{2.4.6}$$

$$e_3 = Y_3 = \frac{1}{\sqrt{2}}\left(C \cdot \frac{\partial}{\partial c_{23}} - C \cdot \frac{\partial}{\partial c_{32}}\right) = \frac{1}{\sqrt{2}} \sum_{j=1}^{3} \left(c_{j2} \frac{\partial}{\partial c_{j3}} - c_{j3} \frac{\partial}{\partial c_{j2}}\right) \tag{2.4.7}$$

$$e_{k+3} = X_k = C \cdot \frac{\partial}{\partial r_k} = \sum_{j=1}^{3} c_{jk} \frac{\partial}{\partial r_j}, \qquad k = 1, 2, 3. \tag{2.4.8}$$

Define a left invariant metric on SE(3) by declaring the vectors $e_1, \dots, e_6$ to form an orthonormal basis. We will determine whether this system is controllable or not. Note that $Q = \mathrm{SE}(3) \times \mathbb{R}^6 \times \mathrm{SO}(6)$, because both manifolds SE(3) and $\mathbb{R}^6$ are Lie groups, so their tangent bundles are trivial.

By Proposition 5 and Definition 4, the distribution D over Q is spanned by

$$\begin{aligned} Z_1 &= Y_1 + qY_1 + \tfrac{1}{2\sqrt{2}} W_{23} + \tfrac{1}{\sqrt{2}} W_{45}, \\ Z_2 &= Y_2 + qY_2 - \tfrac{1}{2\sqrt{2}} W_{13} + \tfrac{1}{\sqrt{2}} W_{46}, \\ Z_3 &= Y_3 + qY_3 + \tfrac{1}{2\sqrt{2}} W_{12} + \tfrac{1}{\sqrt{2}} W_{56}, \end{aligned} \tag{2.4.9}$$

$$K_1 = X_1 + qX_1, \qquad K_2 = X_2 + qX_2, \qquad K_3 = X_3 + qX_3. \tag{2.4.10}$$

The distribution D defined by the vector fields (2.4.9) and (2.4.10) is not bracket generating, indeed

$$\begin{aligned} D^2 &= D \oplus \mathrm{span}\,\{W_{12}, W_{13}, W_{23}\}, \\ D^3 &= D^2 \oplus \mathrm{span}\{qY_1, qY_2, qY_3\}, \\ D^4 &= D^3, \end{aligned} \tag{2.4.11}$$

and so $\dim D^2 = 9$, $\dim D^k = 12$ for all $k \geq 3$ and the step of D is 3.

The fact that the system is not controllable follows from a version of the Orbit theorem in the analytic category, see [2, Chapter 5]. In this case, the tangent space of the orbits and the Lie hull of D coincide. Since the manifold Q and the distribution D are analytic, the non-controllability of the system is equivalent to the fact that D is not bracket generating.

S^n rolling on $\mathbb{R}^n$

It is proved in [40] that rolling S^n on $\mathbb{R}^n$, $n \geq 2$, as submanifolds imbedded in $\mathbb{R}^{n+1}$ is a controllable system, i.e. any configuration can be obtained from an initial one by rolling without twisting or slipping. To prove this result the author first rewrites the kinematic equations as a right-invariant control system without drift, evolving on a connected Lie group G, in order to apply a theorem for controllability on Lie groups proved by Jurdjevic and Sussmannn [22] which reduces the controllability issue to proving that the rolling distribution generates the Lie algebra of G. This theorem results from an adaptation to Lie groups of the Chow-Rashevskiĭ theorem. Here we briefly discuss the fact that the distribution D from Definition 4 is bracket generating in this case, thus obtaining the controllability of this system directly as consequence of the Chow-Rashevskiĭ theorem.

Consider the unit sphere S^n as the submanifold of the Euclidean space $\mathbb{R}^{n+1}$,

$$S^n = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} |\, x_0^2 + \cdots + x_n^2 = 1\right\},$$

with the induced metric.

For an arbitrary point $\tilde{x} = (\tilde{x}_0, \ldots, \tilde{x}_n) \in S^n$, at least one of the coordinates $\tilde{x}_0, \ldots, \tilde{x}_n$ does not vanish. Assume that $\tilde{x}_n \neq 0$, and consider the neighborhood $U = \{(x_0, \ldots, x_n) \in S^n |\, \pm x_n > 0\}$, where the choice of the $\pm$ sign corresponds to the sign of $\tilde{x}_n$. To simplify the notation, we define the following functions on U

$$s_j(x) = \sum_{r=j}^{n} x_r^2 .$$

These functions are always strictly positive on U, and we use them to define an orthonormal basis of TU. We will write simply s_j instead of $s_j(x)$, since dependence of x is clear from the context. Define the following vector fields on U

$$e_j = \sqrt{\frac{s_j}{s_{j-1}}} \left(-\frac{\partial}{\partial x_{j-1}} + \frac{x_{j-1}}{s_j} \sum_{r=j}^{n} x_r \frac{\partial}{\partial x_r} \right), \qquad j = 1, \ldots, n. \tag{2.4.12}$$

These vector fields form an orthonormal basis of the tangent space over U.

Consider the vector fields $X_k = e_k + qe_k + \mathcal{V}(e_k)$ which generate the distribution D, introduced in Proposition 5, restricted to U. We have the explicit form

$$X_k(x, \hat{x}, A) = e_k(x) + \sum_{i=1}^{n} a_{ik} \hat{e}_i(\widehat{x}) - \sum_{i=1}^{k-1} \frac{x_{i-1}}{\sqrt{s_{i-1} s_i}} W_{ik}(A).$$

In order to determine the commutators $[X_k, X_l]$, let us assume that $k > l$. Lengthy calculations show that

$$[X_k, X_l] = \frac{x_{l-1}}{\sqrt{s_{l-1} s_l}} X_k - W_{lk}.$$

Define the vector fields Y_{lk}, for $l < k$, by

$$Y_{lk} := [X_l, X_k] + \frac{x_{l-1}}{\sqrt{s_{l-1} s_l}} X_k = W_{lk}.$$

Finally, let $Z_1 = [Y_{12}, X_2] = \sum_{i=1}^{n} a_{i1} \hat{e}_i$, and

$$Z_k = [X_1, Y_{1k}] = \sum_{i=1}^{n} a_{ik} \hat{e}_i \,, \qquad k = 2, \ldots, n.$$

We conclude that the entire tangent space TQ is spanned by $\{X_k\}_{k=1}^n, \{Y_{lk}\}_{1 \leq l < k \leq n}$ and $\{Z_k\}_{k=1}^n$. Hence, D is a regular bracket generating distribution of step 3, which implies that the system of rolling S^n over $\mathbb{R}^n$ without slipping or twisting is completely controllable.

2.5 Existence of rollings

Paper D of the present thesis deals with problems of existence of intrinsic rollings as defined in the previous section. This is a work in progress, here we show some of the results obtained so far.

In the literature of probability theory, see for instance [17, 21], the notion of development is the main tool to construct Brownian motions on Riemannian manifolds. This procedure starts with a curve $x : [0, \tau] \to M$ and an initial rolling configuration $q_0 = (x_0, \widehat{x}_0, A_0) \in Q$ of M rolling on $\mathbb{R}^n$. The development along x is equivalent to constructing a curve $\widehat{x} : [0, \tau] \to \mathbb{R}^n$ as follows:

$$\widehat{x}(t) = q_0 \circ \exp_{x_0}^{-1} \circ x(t), \tag{2.5.1}$$

where $\exp_{x_0}$ denotes the Riemannian exponential mapping of M at $x_0 = x(0)$. Note that $\widehat{x}(0) = \widehat{x}_0$. Using this curve, we can define $A : [0, \tau] \to \mathrm{SO}(n)$ as follows. Let $X_1, \ldots, X_n$ be an orthonormal basis of $T_{x_0}M$ and $\widehat{X}_i = q_0 X_i$ be the corresponding orthonormal basis of $T_{\widehat{x}_0}\mathbb{R}^n$. By parallel translating both bases along x and $\widehat{x}$, we define the vector fields $X_i(t)$ and $\widehat{X}_i(t)$ along x and $\widehat{x}$ respectively. The map $A(t)$ is defined as the isometry mapping $X_i(t)$ to $\widehat{X}_i(t)$. Note that by construction $A(0) = A_0$.

With these notations, we have the following result.

Proposition 6 (Paper D) *Let $x : [0, \tau] \to M$ be a geodesic in M and let $\widehat{x} : [0, \tau] \to \mathbb{R}^n$ be defined by equation* (2.5.1)*. The curve*

$$\begin{array}{cccc} q & : & [0,\tau] & \to & Q \cong U \times \widehat{U} \times \mathrm{SO}(n) \\ & & t & \mapsto & (x(t), \widehat{x}(t), A(t)) \end{array} \tag{2.5.2}$$

defined for a sufficiently small time τ and a sufficiently small neighborhood $U \times \widehat{U}$ of $(x_0, \widehat{x}_0) \in M \times \mathbb{R}^n$ is an intrinsic rolling.

This shows that given a geodesic in M, it is possible to construct an intrinsic rolling over $\mathbb{R}^n$. A natural step is to discuss the existence of an intrinsic rolling of two manifolds, M and $\widehat{M}$, following given trajectories $x : [0, \tau] \to M$ and $\widehat{x} : [0, \tau] \to \widehat{M}$. More precisely, the problem asks whether a rolling of the form

$$\begin{array}{cccc} q & : & [0,\tau] & \to & Q \\ & & t & \mapsto & (x(t), \widehat{x}(t), A(t)) \end{array} \tag{2.5.3}$$

exists, whenever the curves $x : [0, \tau] \to M$ and $\widehat{x} : [0, \tau] \to \widehat{M}$ are given.

The following theorem gives necessary and sufficient conditions for the existence of the map (2.5.3) in the case of Riemannian surfaces.

Theorem 9 (Paper D) *Let M and $\widehat{M}$ be two Riemannian connected surfaces. Let $x : [0, \tau] \to M$ and $\widehat{x} : [0, \tau] \to \widehat{M}$ be two curves of class C^1 parameterized by arc-length and geodesic curvatures $k_g(t)$ and $\widehat{k}_g(t)$ respectively. Then, there is a rolling along x and $\widehat{x}$ if and only if $k_g(t) = \widehat{k}_g(t)$.*

An interesting consequence of Theorem 9 is the following corollary.

Corollary 2 (Paper D) *With the notation and hypotheses of Theorem 9, let us assume that* $\widehat{M} = \mathbb{R}^2$ *with the usual Riemannian structure and the curves* $x : [0, \tau] \to M$ *and* $\widehat{x} : [0, \tau] \to \mathbb{R}^2$ *are simple loops, where* $x(0) = x(\tau)$ *and* $\widehat{x}(0) = \widehat{x}(\tau)$*. Let* α *be the angle between* $\dot{x}(0)$ *and* $\dot{x}(\tau)$*, then*

$$\int_0^\tau k_g(t)dt = \alpha.$$

In order to find a condition similar to the one in Theorem 9 in the case of rolling manifolds of higher dimension, it is necessary to find the correct analog to the geodesic curvature. The definition that is suitable in this context can be found in [36, pp. 21–32].

Then the analog of Theorem 9 in higher dimension is the following.

Theorem 10 (Paper D) *Let* M *and* $\widehat{M}$ *be two Riemannian manifolds of dimension* n*, and let* $x : [0, \tau] \to M$ *and* $\widehat{x} : [0, \tau] \to \widehat{M}$ *be two curves of class* C^n *parametrized by arc-length. Suppose that both* x *and* $\widehat{x}$ *have* n *well defined Frenet vector fields and* $n-1$ *geodesic curvatures* $\{\kappa_j\}_{j=1}^{n-1}$ *and* $\{\widehat{\kappa}_j\}_{j=1}^{n-1}$ *respectively. Then there exists a rolling along* $x(t)$ *and* $\widehat{x}(t)$ *if and only if*

$$\kappa_j = \pm\widehat{\kappa}_j, \qquad j = 1, \ldots, n. \tag{2.5.4}$$

As a consequence of Theorem 10 for sufficiently small τ, Proposition 6 and the uniqueness of rollings, see [33, p. 381], we have the following corollary.

Corollary 3 (Paper D) *With the notation and hypotheses of Theorem 10, consider a given initial configuration for a rolling* $(x_0, \widehat{x}_0, q_0) \in Q$*, where* $x_0 = x(0)$ *and* $\widehat{x}_0 = \widehat{x}(0)$*, and assume* x *is a geodesic in* M*. Then, for sufficiently small values of* τ*, the equality*

$$\widehat{x}(t) = q_0 \circ \exp_{x_0}^{-1} \circ\, x(t)$$

holds if and only if $\kappa_j = \pm\widehat{\kappa}_j$*.*

2.6 Future research

Based on the results described in the previous sections, some related questions arise and are posed here as open problems to be treated in the future.

- Detailed study of the intrinsic sub-Laplacian and heat kernel of general sub-Riemannian principal bundles, in terms of the geometry of the fibers and base spaces.
- Formulation of a geodesic equation for S^7 endowed with its qCR structure.
- Connectivity by geodesics in general sub-Riemannian principal bundles.
- Applications of intrinsic rollings to interpolation theory and controllability.
- Geometric characterization of abnormal extremals in rolling problems.
- Study of the motion planning problem for rollings of higher dimensional manifolds.
- Extend the results of existence of rollings to more general cases.

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Chapter 3

Papers A-D

- **Paper A**: M. Godoy Molina, I. Markina. *Sub-Riemannian geometry of parallelizable spheres.* To appear in Revista Matemática Iberoamericana, Issue 3, 2011.

- **Paper B**: M. Godoy Molina, I. Markina. *Sub-Riemannian geodesics and heat operator on odd dimensional spheres.* Submitted.

- **Paper C**: M. Godoy Molina, E. Grong, F. Silva Leite, I. Markina. *An intrinsic formulation of the rolling manifolds problem.* Accepted in a shorter version in Journal of Control and Dynamical Systems.

- **Paper D**: M. Godoy Molina, E. Grong. *Geometric conditions for existence of intrinsic rollings.* In preparation.

3.1 Paper A

SUB-RIEMANNIAN GEOMETRY OF PARALLELIZABLE SPHERES

MAURICIO GODOY MOLINA
IRINA MARKINA

Abstract. The first aim of the present paper is to compare various sub-Riemannian structures over the three dimensional sphere S^3 originating from different constructions. Namely, we describe the sub-Riemannian geometry of S^3 arising through its right action as a Lie group over itself, the one inherited from the natural complex structure of the open unit ball in $\mathbb{C}^2$ and the geometry that appears when it is considered as a principal S^1-bundle via the Hopf map. The main result of this comparison is that in fact those three structures coincide.

We present two bracket generating distributions for the seven dimensional sphere S^7 of step 2 with ranks 6 and 4. The second one yields to a sub-Riemannian structure for S^7 that is not widely present in the literature until now. One of the distributions can be obtained by considering the CR geometry of S^7 inherited from the natural complex structure of the open unit ball in $\mathbb{C}^4$. The other one originates from the quaternionic analogous of the Hopf map.

1. Introduction

One of the main objectives of classical sub-Riemannian geometry is to study manifolds which are path-connected by curves admissible in a certain sense. In order to define what does admissibility mean in this context, we begin by setting notations that will be used throughout this paper. Let M be a smooth connected manifold of dimension n, together with a smooth distribution $\mathcal{H} \subset TM$ of rank $2 \leq k < n$. Such vector bundles are often called *horizontal* in the literature. An absolutely continuous curve $\gamma : [0, 1] \to M$ is called *admissible* or *horizontal* if $\dot{\gamma}(t) \in \mathcal{H}$ a.e.

Distributions satisfying the condition that their Lie-hull equals the whole tangent space of the manifold at each point play a central role in the search for horizontally path-connected manifolds. Such vector bundles are said

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to satisfy the *bracket generating condition*. To be more precise, define the following real vector bundles

$$\mathcal{H}^1 = \mathcal{H}, \qquad \mathcal{H}^{r+1} = [\mathcal{H}^r, \mathcal{H}] + \mathcal{H}^r \quad \text{for } r \geq 1,$$

which naturally give rise to the flag

$$\mathcal{H} = \mathcal{H}^1 \subseteq \mathcal{H}^2 \subseteq \mathcal{H}^3 \subseteq \dots .$$

Then we say that a distribution is bracket generating if for all $x \in M$ there is an $r(x) \in \mathbb{Z}^+$ such that

$$\mathcal{H}_x^{r(x)} = T_x M. \tag{1}$$

If the dimensions $\dim \mathcal{H}_x^r$ do not depend on x for any $r \geq 1$, we say that $\mathcal{H}$ is a regular distribution. The least r such that (1) is satisfied is called the step of $\mathcal{H}$. We will focus on regular distributions of step 2. In [18] the reader can find detailed definitions and broad discussion about terminology.

The following classical result shows the precise relation between the notion of path-connectedness by means of horizontal curves and the assumption that $\mathcal{H}$ is a bracket generating distribution.

Theorem 1 ([11, 23]). *Let M be a connected manifold. If a distribution $\mathcal{H} \subset TM$ is bracket generating, then any two points in M can be joined by a horizontal path.*

We recall the definition of sub-Riemannian manifold.

Definition 1. *A sub-Riemannian structure over a manifold M is a pair $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, where $\mathcal{H}$ is a bracket generating distribution and $\langle \cdot, \cdot \rangle$ a fiber inner product defined on $\mathcal{H}$. In this setting, the length of an absolutely continuous horizontal curve $\gamma : [0, 1] \to M$ is*

$$\ell(\gamma) := \int_0^1 \|\dot{\gamma}(t)\| dt,$$

where $\|\dot{\gamma}(t)\|^2 = \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle$ whenever $\dot{\gamma}(t)$ exists. The triple $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ is called sub-Riemannian manifold.

Thereby, restricting our considerations to connected sub-Riemannian manifolds endowed with bracket generating distributions, it is possible to define the notion of sub-Riemannian distance between two points.

Definition 2. *The sub-Riemannian distance $d(p, q) \in [0, +\infty)$ between two points $p, q \in M$ is given by $d(p, q) := \inf \ell(\gamma)$, where the infimum is taken over all absolutely continuous horizontal curves joining p to q.*

An absolutely continuous horizontal curve that realizes the distance between two points is called a horizontal length minimizer.

Remark: The connectedness of M and the fact that $\mathcal{H}$ is bracket generating, assure that $d(p, q)$ is a finite nonnegative number. Nevertheless the bracket generating hypothesis, required for the previous definition, is possible to be weakened. In fact, in [25] the author finds a necessary and sufficient requirement to horizontal path-connectedness for a manifold in terms of the corresponding distribution. Clearly, this theorem contains, as a particular case, the bracket generating condition.

Historically, the first examples of sub-Riemannian manifolds that have been considered were Lie groups, see e.g. [2, 6, 9, 14, 17]. Due to its algebraic structure, it is sufficient to define appropriate distributions at the identity of the group. Right (or left) translations allow to find globally defined bracket generating distributions of right (or left) invariant vector fields. An important role has been played by considering domains in Euclidean spaces with special algebraic structures (such as the Heisenberg groups, $\mathbb{H}-$type groups as their natural generalizations to Clifford algebras, Engel groups, Carnot groups, etc.). Particular attention have had the three dimensional unimodular Lie groups which were studied, for example, in [2, 6, 14] and the Heisenberg group, see [13]. The main purpose of this communication is to present recent results concerning different sub-Riemannian structures of the second simplest family of examples of manifolds, namely, spheres. The main tool for the study of sub-Riemannian structures on spheres arise from the $G-$principal bundle structure given by the Hopf fibrations. We are also inspired by the article [26], where the close relation between the Hopf map and physical applications is presented.

The following celebrated theorem in topology, see [1], gives a very strong restriction on the problem of finding globally defined sub-Riemannian structures over spheres.

Theorem 2 (Adams)**.** *Let $S^{n-1} = \{x \in \mathbb{R}^n : \|x\|^2 = 1\}$ be the unit sphere in $\mathbb{R}^n$, with respect to the usual Euclidean norm $\|\cdot\|$. Then S^{n-1} has precisely $\varrho(n)-1$ linearly independent, globally defined and non vanishing vector fields, where $\varrho(n)$ is defined in the following way: if $n = (2a+1)2^b$ and $b = c + 4d$ where $0 \leq c \leq 3$, then $\varrho(n) = 2^c + 8d$.*

In particular, two classical consequences follow: S^1, S^3 and S^7 are the only spheres with maximal number of linearly independent globally defined non vanishing vector fields, and the even dimensional spheres have no globally defined and non vanishing vector fields.

The condition that a manifold M has maximal number of linearly independent globally defined non vanishing vector fields is usually rephrased as saying that M is *parallelizable*. The fact that S^1, S^3 and S^7 are the only parallelizable spheres was proved in [7] and that the even dimensional spheres have no globally defined and non vanishing vector fields is a consequence of the Hopf index theorem, see [27].

This theorem permits to conclude at least two major points of discussion: there is no possible global basis of a sub-Riemannian structure for spheres with even dimension and it is impossible to find a globally defined basis for bracket generating distributions, except for S^3 and S^7. The fact that S^3 and S^7 can be seen as the set of quaternions and octonions of unit length will play a core role in many arguments throughout this paper.

The main results that we present here are: a comparison between three sub-Riemannian structures of S^3 and the constructions for two different sub-Riemannian structures for S^7. More specifically, the first result can be summarized as an equivalence between the sub-Riemannian geometry of S^3 arising through its right Lie group action over itself as the set of unit quaternions, the one inherited from the natural complex structure of the open unit ball in $\mathbb{C}^2$ and the geometry that appears when considering the Hopf map as a principal S^1-bundle. Notice that this structure admits a tangent cone isomorphic to the one dimensional Heisenberg group in the sense of Gromov-Margulis-Mitchell-Mostow construction of the tangent cone [15, 20, 21, 22]. With respect to the second result, by considering the CR structure of S^7 inherited from the natural complex structure of the open unit ball in $\mathbb{C}^4$, we obtain a 2-step bracket generating distribution of rank 6. This construction is intimately related to the Hopf fibration $S^1 \to S^7 \to \mathbb{C}P^3$, in the sense that the holomorphic tangent space defining the CR structure is an Ehresmann connection, that is, the orthogonal complement to the vertical space defined by the Hopf fibration as a principal S^1-bundle. This fact is generalized to all odd-dimensional spheres and, moreover, it implies that the tangent cone for $(2n+1)-$dimensional spheres is isomorphic to the $n-$dimensional Heisenberg group. Making use of the quaternionic analogue of the Hopf map $S^3 \to S^7 \to S^4$, we present another 2-step bracket generating distribution that has rank 4. We conclude that the sphere S^7 admits at least two different sub-Riemannian structures. The tangent cone, in the first case, is isomorphic to the 3-dimensional Heisenberg group, and in the second case it has the structure of the quaternionic Heisenberg-type group with 3-dimensional center [9]. In both cases we present the basis of the horizontal distribution that is very useful in future studies of geodesics and hypoelliptic operators related to the sub-Riemannian geometry of spheres. We would like to note that, in the case of the rank 6 distribution the given basis is globally defined, while in the case of the rank 4 distribution the search for a globally defined basis will be analyzed in a forthcoming paper. It is also expected that S^7 with the structure induced by the quaternionic Hopf fibration satisfies the conditions of a qCR$-$manifold in the sense of [3].

2. S^3 as a sub-Riemannian manifold

Throughout this paper $\mathbb{H}$ will denote the quaternions, that is, $\mathbb{H} = (\mathbb{R}^4, +, \circ)$ where $+$ stands for the usual coordinate-wise addition in $\mathbb{R}^4$ and $\circ$ is a non-commutative product given by the formula

$$\begin{aligned} &(x_0 + x_1 i + x_2 j + x_3 k) \circ (y_0 + y_1 i + y_2 j + y_3 k) = \\ &= (x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3) + (x_1y_0 + x_0y_1 - x_3y_2 + x_2y_3)i + \\ &+ (x_2y_0 + x_3y_1 + x_0y_2 - x_1y_3)j + (x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3)k. \end{aligned}$$

It is important to recall that $\mathbb{H}$ is a non-commutative, associative and normed real division algebra. Let $q = t + ai + bj + ck \in \mathbb{H}$, then the conjugate of q, is given by

$$\bar{q} = t - ai - bj - ck.$$

We define the norm $|q|$ of $q \in \mathbb{H}$ by $|q|^2 = q\bar{q}$.

The realization of the sphere S^3 as the set of unit quaternions, produces a Lie group structure induced by quaternion multiplication.

The multiplication rule in $\mathbb{H}$ induces a right translation $R_y(x)$ of an element $x = x_0 + x_1 i + x_2 j + x_3 k$ by the element $y = y_0 + y_1 i + y_2 j + y_3 k$. The right invariant basis vector fields are defined as $Y(y) = (R_y(x))_* Y(0)$, where $Y(0)$ are the basis vectors at the unity of the group. The matrix corresponding to the tangent map $(R_y(x))_*$, obtained by the multiplication rule, becomes

$$(R_y(x))_* = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ -y_1 & y_0 & -y_3 & y_2 \\ -y_2 & y_3 & y_0 & -y_1 \\ -y_3 & -y_2 & y_1 & y_0 \end{pmatrix}.$$

Calculating the action of $(R_y(x))_*$ in the basis of unit vectors of $\mathbb{R}^4$ we get the four vector fields

$$\begin{aligned} N(y) &= y_0\partial_{y_0} + y_1\partial_{y_1} + y_2\partial_{y_2} + y_3\partial_{y_3}, \\ V(y) &= -y_1\partial_{y_0} + y_0\partial_{y_1} - y_3\partial_{y_2} + y_2\partial_{y_3}, \\ X(y) &= -y_2\partial_{y_0} + y_3\partial_{y_1} + y_0\partial_{y_2} - y_1\partial_{y_3}, \\ Y(y) &= -y_3\partial_{y_0} - y_2\partial_{y_1} + y_1\partial_{y_2} + y_0\partial_{y_3}. \end{aligned}$$

It is easy to see that $N(y)$ is the unit normal to S^3 at $y \in S^3$ with respect to the usual Riemannian structure $\langle \cdot, \cdot \rangle$ in $T\mathbb{R}^4$. Moreover, for any $y \in S^3$

$$\langle N(y), V(y) \rangle_y = \langle N(y), X(y) \rangle_y = \langle N(y), Y(y) \rangle_y = 0$$

and

$$\langle N(y), N(y) \rangle_y = \langle V(y), V(y) \rangle_y = \langle X(y), X(y) \rangle_y = \langle Y(y), Y(y) \rangle_y = 1.$$

Since the matrix

$$\begin{pmatrix} -y_1 & y_0 & -y_3 & y_2 \\ -y_2 & y_3 & y_0 & -y_1 \\ -y_3 & -y_2 & y_1 & y_0 \end{pmatrix}$$

has rank three, we conclude that the vector fields $\{V(y), X(y), Y(y)\}$ form an orthonormal basis of T_yS^3 with respect to $\langle\cdot,\cdot\rangle_y$, for any $y \in S^3$.

Observing that $[X, Y] = 2V$, we see that the distribution span$\{X, Y\}$ is bracket generating, therefore it satisfies the hypothesis of Theorem 1. The geodesics of the left invariant sub-Riemannian structure of S^3 are determined in [10], while in [17] the same results are achieved by considering the right invariant structure of S^3.

Notice that the distribution span$\{X, Y\}$ can also be defined as the kernel of the contact one form

$$\omega = -y_1\,dy_0 + y_0\,dy_1 - y_3\,dy_2 + y_2\,dy_3.$$

Remark: It is easy to see that $[V, Y] = 2X$ and $[X, V] = 2Y$, therefore the distributions span$\{Y, V\}$ and span$\{X, V\}$ are also bracket generating. The corresponding contact forms are

$$\theta = -y_2\,dy_0 + y_3\,dy_1 + y_0\,dy_2 - y_1\,dy_3$$

and

$$\eta = -y_3\,dy_0 - y_2\,dy_1 + y_1\,dy_2 + y_0\,dy_3$$

respectively. This means that there is a priori no natural choice of a sub-Riemannian structure on S^3 generated by the Lie group action of multiplication of quaternions. Any choice that can be made, will produce essentially the same geometry.

3. S^3 as a CR manifold

Consider S^3 as the boundary of the unit ball B^4 on $\mathbb{C}^2$, that is, as the hypersurface

$$S^3 := \{(z, w) \in \mathbb{C}^2 : z\bar{z} + w\bar{w} = 1\}.$$

The sphere S^3 cannot be endowed with a complex structure, but nevertheless it possess a differentiable structure compatible with the natural complex structure of the ball $B^4 = \{(z, w) \in \mathbb{C}^2 : z\bar{z} + w\bar{w} < 1\}$ as an open set in $\mathbb{C}^2$. We will show that this differentiable structure over the sphere S^3 (CR structure) is equivalent to the sub-Riemannian one considered in the previous section. We begin by recalling the definition of a CR structure, according to [5].

Definition 3. *Let W be a real vector space. A linear map $J : W \to W$ is called an almost complex structure map if $J \circ J = -I$, where $I : W \to W$ is the identity map.*

In the case $W = T_p\mathbb{R}^{2n}$, $p = (x_1, y_1, \dots, x_n, y_n \in \mathbb{R}^{2n}$, we say that the standard almost complex structure for W is defined by setting

$$J_n(\partial_{x_j}) = \partial_{y_j}, \qquad J_n(\partial_{y_j}) = -\partial_{x_j}, \qquad 1 \leq j \leq n.$$

For a smooth real submanifold M of $\mathbb{C}^n$ and a point $p \in M$, in general the tangent space T_pM is not invariant under the almost complex structure map J_n for $T_p\mathbb{C}^n \cong T_p\mathbb{R}^{2n}$. We are interested in the largest subspace invariant under the action of J_n.

Definition 4. *For a point $p \in M$, the complex or holomorphic tangent space of M at p is the vector space*

$$H_pM = T_pM \cap J_n(T_pM).$$

In this setting, the following result takes place, see [5].

Lemma 1. *Let M be a real submanifold of $\mathbb{C}^n$ of real dimension $2n - d$. Then*

$$2n - 2d \leq \dim_{\mathbb{R}} H_pM \leq 2n - d,$$

and $\dim_{\mathbb{R}} H_pM$ is an even number.

A real submanifold M of $\mathbb{C}^n$ is said to have a CR structure if $\dim_{\mathbb{R}} H_pM$ does not depend on $p \in M$. In particular, by Lemma 1, every smooth real hypersurface S embedded in $\mathbb{C}^n$ satisfies $\dim_{\mathbb{R}} H_pS = 2n - 2$, therefore S is a CR manifold. This fact applies to every odd dimensional sphere.

The question addressed now is to describe H_pS^3. By the discussion in the previous paragraph, H_pS^3 can be seen as a complex vector space of complex dimension one. This description is achieved by considering the differential form

$$\omega = \bar{z}dz + \bar{w}dw$$

and observing that $\ker \omega$ is precisely the set we are looking for. Straightforward calculations show that

$$\ker \omega = \operatorname{span}\{\bar{w}\partial_z - \bar{z}\partial_w\}.$$

In real coordinates this corresponds to

$$\bar{w}\partial_z - \bar{z}\partial_w = \frac{1}{2}(-X + iY),$$

where X and Y were defined in Section 2. It is important to remark that this is precisely the maximal invariant J_2-subspace of T_pS^3, namely

$$J_2(X) = Y, \qquad J_2(Y) = -X,$$

then $J_2(\operatorname{span}\{X, Y\}) = \operatorname{span}\{X, Y\}$, but $J_2(V) = -N \notin T_pS^3$ for any point $p \in S^3$. Therefore, the distribution corresponding to the right invariant action of S^3 over itself is the same to its one dimensional (complex) CR structure.

Remark: The distribution associated to the anti-holomorphic form

$$\bar{\omega} = zd\bar{z} + wd\bar{w}$$

is the conjugate to the previous one and isomorphic and isomorphic to the 2-dimensional real distribution $\mathcal{H}$. More explicitly:

$$\ker \omega = \operatorname{span}\{-w\partial_{\bar{z}} + z\partial_{\bar{w}}\}$$

and in real coordinates this corresponds to

$$-w\partial_{\bar{z}} + z\partial_{\bar{w}} = \frac{1}{2}(X + iY).$$

The same almost complex structure as the previously described can be obtained by means of the covariant derivative of S^3 considered as a smooth Riemannian manifold embedded in $\mathbb{R}^4$. Namely, in [17] it is introduced the mapping $J(Z) = \nabla_Z V$, for $Z \in TS^3$, were ∇ denotes the Levi-Civita connection on the tangent bundle to S^3 and V is the vector field defined in Section 2.

4. S^3 as principal bundle

In this section we describe how the structure of a principal S^1-bundle over S^3 induces a bracket generating distribution on S^3. Namely, it is possible to consider S^3 as a S^1-space, according to the action

$$\lambda \cdot (z, w) = (\lambda z, \lambda w),$$

where $\lambda \in S^1 = \{\lambda \in \mathbb{C} : |\lambda|^2 = 1\}$ and $(z, w) \in S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$.

Consider the Hopf map $h : S^3 \to S^2$ as a principal S^1-bundle, see [16, 19], given explicitly by

$$h(z, w) = (|z|^2 - |w|^2, 2z\bar{w}),$$

where $S^2 = \{(x, \zeta) \in \mathbb{R} \times \mathbb{C} : x^2 + |\zeta|^2 = 1\}$. Clearly, h is a submersion of S^3 onto S^2, and it is a bijection between S^3/S^1 and S^2, where S^3/S^1 is understood as the orbit space of the S^1-action over S^3, previously defined.

Let $p = (x_0, \zeta_0) \in S^2$. It is easy to verify that $h^{-1}(p) = (z_0, w_0) \bmod S^1$, where (z_0, w_0) is one preimage of p under h. Consider the great circle

$$\gamma_p(t) = e^{2\pi i t}(z_0, w_0), \qquad t \in [0, 1],$$

in S^3, that projects to p under the Hopf map. Consider the tangent vector field, defined by

$$\dot{\gamma}_p(t) = 2\pi i e^{2\pi i t}(z_0, w_0) \in T_{\gamma_p(t)} S^3.$$

We write the curve γ_p and the map $d_{\gamma_p(t)}h$ in real coordinates, then

$$\begin{aligned} \gamma_p(t) = (z(t), w(t)) &= (x_0(t) + ix_1(t), x_2(t) + ix_3(t)) \\ &= (x_0(t), x_1(t), x_2(t), x_3(t)) \end{aligned}$$

and

$$[d_{\gamma_p(t)}h] = 2\begin{pmatrix} x_0(t) & x_1(t) & -x_2(t) & -x_3(t) \\ x_2(t) & x_3(t) & x_0(t) & x_1(t) \\ -x_3(t) & x_2(t) & x_1(t) & -x_0(t) \end{pmatrix}.$$

Thus, the Hopf map induces the following action over the vector field $\dot{\gamma}_p(t)$:

$$[d_{\gamma_p(t)}h]\dot{\gamma}_p(t) = 4\pi\begin{pmatrix} x_0(t) & x_1(t) & -x_2(t) & -x_3(t) \\ x_2(t) & x_3(t) & x_0(t) & x_1(t) \\ -x_3(t) & x_2(t) & x_1(t) & -x_0(t) \end{pmatrix}\begin{pmatrix} \dot{x}_0(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, if $[d_{\gamma_p(t)}h]$ is a full rank matrix, we would have characterized the kernel of it, by

$$\ker d_{\gamma_p(t)}h = \operatorname{span}\{\dot{\gamma}_p(t)\}. \tag{2}$$

Notice that, using the notation of Section 2, the following identity holds

$$\dot{\gamma}_p(t) = 2\pi V(\gamma_p(t)). \tag{3}$$

To see that the matrix $[d_{\gamma_p(t)}h]$ is full rank, observe that

$$[d_{\gamma_p(t)}h][d_{\gamma_p(t)}h]^t = 4I_3,$$

where I_3 denotes the identity matrix of size 3×3. This implies that $[d_{\gamma_p(t)}h]$ is full rank.

Before describing how the Hopf map induces a horizontal distribution, it is necessary to present some definitions found for example in [22, Chapter 11].

Definition 5 (Ehresmann Connection)**.** *Let M and Q be two differentiable manifolds, and let $\pi : Q \to M$ be a submersion. Denoting by $Q_m = \pi^{-1}(m)$ the fiber through $m \in M$, the* vertical space *at q is the tangent space at the fiber $Q_{\pi(q)}$ and it is denoted by V_q.*

An Ehresmann connection *for the submersion $\pi : Q \to M$ is a distribution $\mathcal{H} \subset TQ$ which is everywhere transversal to the vertical, that is:*

$$V_q \oplus \mathcal{H}_q = T_qQ.$$

We apply Definition 5 to the map h in order to define the Ehresmann connection. Since we know that $\ker d_ph = \operatorname{span}\{V(p)\}$, for every $p \in S^3$ by (2) and (3), and moreover,

$$\langle X(p), V(p)\rangle_p = \langle Y(p), V(p)\rangle_p = \langle X(p), Y(p)\rangle_p = 0,$$

where $\langle\cdot,\cdot\rangle_p$ stands for the usual Riemannian structure defined at $p \in S^3$, we see that

$$\mathcal{H}_p = \operatorname{span}\{X(p), Y(p)\} \tag{4}$$

is an Ehresmann connection for the submersion $h : S^3 \to S^2$ with $V(p)$ as a vertical space.

Definition 6. *Let G be a Lie group acting on Q and $\pi : Q \to M$ a submersion, with Ehresmann connection $\mathcal{H}$, which is a fiber bundle with fiber G. The submersion π is called a principal $G-$bundle with connection, if the following conditions hold:*

- *G acts freely and transitively on fibers,*
- *the group orbits are the fibers of $\pi : Q \to M$ (thus M is isomorphic to Q/G and π is the canonical projection) and*
- *the $G-$action on Q preserves the horizontal distribution $\mathcal{H}$;*

We conclude that the Hopf fibration is a principal S^1-bundle with connection $\mathcal{H}$, defined by (4).

Definition 7. *A sub-Riemannian metric $\langle\cdot,\cdot\rangle$ on the principal G-bundle $\pi : G \to M$ is called a metric of bundle type if the inner product $\langle\cdot,\cdot\rangle$ on the horizontal distribution $\mathcal{H}$ is induced from a Riemannian metric on M.*

The sub-Riemannian metric $\langle\cdot,\cdot\rangle|_{\mathcal{H}}$, obtained by restricting the usual Riemannian metric of S^3 to the distribution $\mathcal{H}$ is, by construction, a metric of bundle type.

Thus the Hopf map indicates, in a topological way, how to make a natural choice of the horizontal distribution $\mathcal{H}$ that was not obvious when we considered the right action of S^3 over itself.

Remark: Observe that the considered vector fields coincide with the right invariant vector fields. This phenomenon does not appear when we change the right action to the left action of S^3 over itself.

5. Tangent vector fields for S^7

In Sections 5 to 7 we will study different sub-Riemannian structures over the sphere S^7, using the ideas of Sections 2 to 4. As a result, we obtain two structurally different types of horizontal distributions. One of them of rank 6 and other of the rank 4. Moreover, as we shall see, the sub-Riemannian structure induced by the CR structure and quaternionic analogue of the Hopf map are essentially different. We start from the construction of a convenient basis of tangent vector fields to S^7.

The multiplication of unit octonions is not associative, therefore S^7 is not a group in a contrast with S^3. Nevertheless, we still able to use the multiplication law in order to find global tangent vector fields. To do this, we present a multiplication table for the basis vectors of $\mathbb{R}^8$. This non-associative multiplication gives rise to the division algebra of octonions

$$\mathbb{O} = \operatorname{span}\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}.$$

According to Table 1, the formula for the product of two octonions is presented in Subsection 8.1 of the Appendix. This multiplication rule induces

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

TABLE 1. Multiplication table for the basis of $\mathbb{O}$.

a matrix representation of the right octonion multiplication, given explicitly by:

$$(R_y(x))_* = \begin{pmatrix} y_0 & -y_1 & -y_2 & -y_3 & -y_4 & -y_5 & -y_6 & -y_7 \\ y_1 & y_0 & y_3 & -y_2 & y_5 & -y_4 & -y_7 & y_6 \\ y_2 & -y_3 & y_0 & y_1 & y_6 & y_7 & -y_4 & -y_5 \\ y_3 & y_2 & -y_1 & y_0 & y_7 & -y_6 & y_5 & -y_4 \\ y_4 & -y_5 & -y_6 & -y_7 & y_0 & y_1 & y_2 & y_3 \\ y_5 & y_4 & -y_7 & y_6 & -y_1 & y_0 & -y_3 & y_2 \\ y_6 & y_7 & y_4 & -y_5 & -y_2 & y_3 & y_0 & -y_1 \\ y_7 & -y_6 & y_5 & y_4 & -y_3 & -y_2 & y_1 & y_0 \end{pmatrix}.$$

We are able to find globally defined tangent vector fields which are invariant under the right product rule. We proceed by analogy with Section 2. The explicit formulae are given in Subsection 8.2 of the Appendix. The vector fields $\{Y_0, \ldots, Y_7\}$ form a frame for $T\mathbb{R}^8$ and, as in Subsection 8.2, the vector fields $\{Y_1, \ldots, Y_7\}$ form a frame for TS^7. More explicitly, we have that the following identities hold

$$\langle Y_i(y), Y_j(y)\rangle_y = \delta_{ij}, \qquad y \in S^7, \qquad i, j \in \{0, 1, \ldots, 7\},$$

where $\langle \cdot, \cdot \rangle$ is the standard Riemannian structure over $\mathbb{R}^8$ and δ_{ij} stands for Kronecker's delta.

Remark: Recall that in contrast with quaternions, the matrix representation $(R_y(x))_*$ of right octonion multiplication is only a convenient way of writing the formula presented in Subsection 8.1 of the Appendix. For quaternions this is actually a representation of quaternion product, but it cannot be such for octonions since they are non-associative and matrix multiplication is associative.

6. CR structure and the Hopf map on S^7

In [22, Chapter 11] it is briefly discussed the general idea of studying a sub-Riemannian geometry for odd dimensional spheres via the higher Hopf fibrations. Namely, consider $S^{2n+1} = \{z \in \mathbb{C}^{n+1} : \|z\|^2 = 1\}$, then the S^1-action on S^{2n+1} given by

$$\lambda \cdot (z_0, \dots, z_n) = (\lambda z_0, \dots, \lambda z_n),$$

for $\lambda \in S^1$ and $(z_0, \dots, z_n) \in S^{2n+1}$, induces the well-known principal S^1-bundle $S^1 \to S^{2n+1} \xrightarrow{H} \mathbb{C}P^n$ given explicitly by

$$S^{2n+1} \ni (z_0, \dots, z_n) \mapsto H(z_0, \dots, z_n) = [z_0 : \dots : z_n] \in \mathbb{C}P^n,$$

where $[z_0 : \dots : z_n]$ denotes homogeneous coordinates. This map is called higher Hopf fibration. The kernel of the map $h\colon S^{2n+1} \to \mathbb{C}P^n$ produces the vertical space and a transversal to the vertical space distribution gives the Ehresmann connection. We show that the vertical space is always given by an action of standard almost complex structure on the normal vector field to S^{2n+1}, and the Ehresmann connection coincides with the holomorphic tangent space at each point of S^{2n+1}.

Theorem 2 asserts that any odd dimensional sphere has at least one globally defined non vanishing tangent vector field. If the dimension of the sphere is of the form $4n+1$, then it has only one globally defined non vanishing tangent vector field. In the case that the dimension of the sphere is of the form $4n+3$, then the sphere admits at least three globally defined non vanishing vector fields. Any sphere S^{2n+1} possesses the vector field

$$V_{n+1}(y) = -y_1 \partial_{y_0} + y_0 \partial_{y_1} - y_3 \partial_{y_2} + \ldots - y_{2n+2} \partial_{y_{2n+1}} + y_{2n+1} \partial_{y_{2n+2}}.$$

Observe that this vector field has appeared already in two opportunities: the vector field V in Sections 2, 3 and 4 corresponds to V_2; and the vector field Y_1 in Subsection 8.2 of the Appendix corresponds to V_4.

The vector field V_{n+1} encloses valuable information concerning the CR structure of S^{2n+1}. We know by Lemma 1 that, as a smooth hypersurface in $\mathbb{C}^{n+1}$ the sphere S^{2n+1} admits a holomorphic tangent space of dimension

$$\dim_{\mathbb{R}} H_p S^{2n+1} = 2n$$

for any point $p \in S^{2n+1}$. The following lemma implies the description of $H_p S^{2n+1}$ as the orthogonal complement to V_{n+1}.

Lemma 2. *Let W be an Euclidean space of dimension $n+2$, $n \geq 1$, and inner product $\langle \cdot, \cdot \rangle_W$. Consider an orthogonal decomposition $W = \operatorname{span}\{X, Y\} \oplus_\perp \widetilde{W}$ with respect to $\langle \cdot, \cdot \rangle_W$ and an orthogonal endomorphism $A : W \to W$ such that*

$$A(\operatorname{span}\{X, Y\}) = \operatorname{span}\{X, Y\},$$

then $\widetilde{W}$ is an invariant space under the action of A, i.e.

$$A(\widetilde{W}) = \widetilde{W}.$$

Proof. Let $v \in \widetilde{W}$, then for any $\alpha, \beta \in \mathbb{R}$ it is clear that

$$\langle Av, \alpha X + \beta Y\rangle_W = \langle v, A^t(\alpha X + \beta Y)\rangle_W = \langle v, A^{-1}(\alpha X + \beta Y)\rangle_W.$$

Since $A(\operatorname{span}\{X, Y\}) = \operatorname{span}\{X, Y\}$, there exist $a, b \in \mathbb{R}$ such that

$$A^{-1}(\alpha X + \beta Y) = aX + bY,$$

and therefore

$$\langle Av, \alpha X + \beta Y\rangle_W = \langle v, aX + bY\rangle_W = 0,$$

which implies that $Av \in \widetilde{W}$. □

As an application of Lemma 2, it is possible to obtain the explicit characterization of the previously mentioned space H_pS^{2n+1}.

Lemma 3. *The vector space H_pS^{2n+1} is the orthogonal complement to the vector $V_{n+1}(p)$ in T_pS^{2n+1}, for any $p \in S^{2n+1}$.*

Proof. Consider the vector space

$$W_p = \operatorname{span}\{N_{n+1}(p)\} \oplus_\perp T_pS^{2n+1} \cong T_p\mathbb{R}^{2n+2},$$

where $N_{n+1}(p)$ is the normal vector to S^{2n+1} at p. The standard almost complex structure map $J_{n+1} : W_p \to W_p$ is orthogonal. Moreover

$$J_{n+1}(V_{n+1}(p)) = -N_{n+1}(p), \quad J_{n+1}(N_{n+1}(p)) = V_{n+1}(p).$$

Using the decomposition $W_p = \widetilde{W}_p \oplus_\perp \operatorname{span}\{V_{n+1}(p), N_{n+1}(p)\}$, it is possible to apply Lemma 2 in order to conclude that $\widetilde{W}_p$, which is the orthogonal complement to $V_{n+1}(p)$ in T_pS^{2n+1}, is invariant under J_{n+1}. Since $\dim_{\mathbb{R}} \widetilde{W}_p = 2n$, we conclude that $\widetilde{W}_p = H_pS^{2n+1}$. □

Remark: The space HS^{2n+1} can also be described as the kernel of the one-form

$$\theta_{n+1} = \bar{z}_0 dz_0 + \ldots + \bar{z}_n dz_n.$$

Indeed, consider $X \in HS^{2n+1}$, then by straightforward calculations we have

$$\theta_{n+1}(X) = \langle X, N_{n+1}\rangle + i\langle X, V_{n+1}\rangle = 0. \tag{5}$$

Lemma 3 provides a horizontal distribution of rank $2n$ for the spheres S^{2n+1}, by considering the holomorphic tangent space. The goal now is to prove that this distribution is bracket generating. In order to do this, let us state a simple result establishing the bracket generating property for an arbitrary contact manifold.

Lemma 4. *Let M be a $(2n+1)-$dimensional contact manifold with contact form ω, then $\xi = \ker \omega$ is a bracket generating distribution of rank $2n$ and step 2.*

Proof. Recall Cartan's formula for a differential one-form ω, namely

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]), \tag{6}$$

for all $X, Y \in TM$. See [8] for the general formulation. It follows from (6) that ξ is Frobenius integrable if and only if $d\omega(X,Y) = 0$ for all $X, Y \in \xi$. Thus, if ω is a contact form, then $d\omega(X,Y) \neq 0$ for all $X, Y \in TM$ and, therefore ξ is not Frobenius integrable. This implies the bracket generating property for ξ, since if $[X,Y](p) \notin \xi_p$ at any point $p \in M$ for some $X(p), Y(p) \in \xi_p$ then $\operatorname{span}\{[X,Y](p)\} \oplus \xi_p = T_pM$. $\square$

By Lemma 4, to prove that HS^{2n+1} is bracket generating, it is sufficient to find a contact one-form ω_{n+1} such that $HS^{2n+1} = \ker \omega_{n+1}$. In order to achieve this, consider

$$\omega_{n+1} = \operatorname{Im} \theta_{n+1} = -y_1 dy_0 + y_0 dy_1 - \ldots - y_{2n+1} dy_{2n} + y_{2n} dy_{2n+1} \tag{7}$$

defined on S^{2n+1}. By (5), the relation $HS^{2n+1} = \ker \omega_{n+1}$ holds immediately.

Theorem 3. *The one-form ω_{n+1} defined in (7) is a contact form. More specifically, ω_{n+1} satisfies*

$$(d\omega_{n+1})^n \wedge \omega_{n+1} = n! \cdot 2^n \mathrm{dvol}_{S^{2n+1}},$$

where $\mathrm{dvol}_{S^{2n+1}}$ is the volume form for S^{2n+1}.

Proof. We observe that

$$d\omega_{n+1} = 2(dy_0 \wedge dy_1 + \ldots + dy_{2n} \wedge dy_{2n+1}).$$

Now, recalling the multinomial formula

$$(x_1 + \ldots + x_m)^p = \sum_{i_1+\ldots+i_m=p} \binom{p}{i_1 \cdots i_m} x_1^{i_1} \cdot \ldots \cdot x_m^{i_m},$$

where $\binom{p}{i_1 \cdots i_m}$ denotes the multinomial coefficient $\dfrac{p!}{i_1! \cdot \ldots i_m!}$. Then

$$(d\omega_{n+1})^n = 2^n \sum_{i_0+\ldots+i_n=n} \binom{n}{i_0 \cdots i_n} (dy_0 \wedge dy_1)^{i_0} \wedge \ldots \wedge (dy_{2n} \wedge dy_{2n+1})^{i_n} = \tag{8}$$

$$= n! \cdot 2^n \sum_{j=0}^{n} (dy_0 \wedge dy_1) \wedge \ldots \wedge (\widehat{dy_{2j}} \wedge \widehat{dy_{2j+1}}) \wedge \ldots \wedge (dy_{2n} \wedge dy_{2n+1}), \tag{9}$$

where $\widehat{dy_k}$ means that this term is ommited. The fact that the differential one-forms are grouped in pairs in (8), permits us to use the multinomial formula. Equality (9) holds since in the summation the only non-zero terms are those when $i_0, \ldots, i_n \in \{0,1\}$ and $i_0 + \ldots + i_n = n$. In this case

$$\binom{n}{i_0 \cdots i_n} = \frac{n!}{0! \cdot 1! \cdot \ldots \cdot 1!} = n!.$$

Taking the exterior power of ω_{n+1} and expression (9) we see that

$$\begin{aligned}(d\omega_{n+1})^n \wedge \omega_{n+1} &= n! \cdot 2^n \sum_{j=0}^{2n+1} (-1)^j y_j dy_0 \wedge \ldots \wedge \widehat{dy_j} \wedge \ldots \wedge dy_{2n+1} \\ &= n! \cdot 2^n \mathrm{dvol}_{S^{2n+1}}.\end{aligned}$$

□

The following corollary holds, by Lemma 4 and Theorem 3.

Corollary 1. *The holomorphic tangent bundle HS^{2n+1} is a bracket generating distribution of step 2 and rank $2n$.*

An important consequence of Theorem 3 follows by considering a classical result by G. Darboux, see [12]. In modern terms, this theorem asserts that every $(2n+1)-$dimensional contact manifold is locally the $n-$dimensional Heisenberg group. This means precisely that the tangent cone of S^{2n+1} as a sub-Riemannian manifold with distribution HS^{2n+1} and metric induced by the usual Euclidean metric in $\mathbb{R}^{2n+2}$ is isomorphic to the $n-$dimensional Heisenberg group.

It is necessary to remark that in general there is no globally defined basis for HS^{2n+1}. By Theorem 2, this is only possible for S^3 and S^7. A basis for this distribution in the case of S^3 was already discussed in Section 2. Here we present an explicit proof that shows the bracket generating property of the basis of HS^7 invariant under right octonion multiplication. A similar proof and other considerations concerning the hypoelliptic nature of the sub-Laplacian associated with the distribution HS^7 can be found in [4].

Theorem 4. *The subbundle $\mathcal{H} = \mathrm{span}\{Y_2, \ldots, Y_7\} = HS^7$ of TS^7 is a bracket generating distribution of rank 6 and step 2.*

Proof. Define the following vector fields

$$\begin{aligned} v_{41}(y) &= -y_4\partial_{y_0} + y_5\partial_{y_1} + y_0\partial_{y_4} - y_1\partial_{y_5}, \\ v_{42}(y) &= y_6\partial_{y_2} - y_7\partial_{y_3} - y_2\partial_{y_6} + y_3\partial_{y_7}, \\ v_{51}(y) &= -y_5\partial_{y_0} - y_4\partial_{y_1} + y_1\partial_{y_4} + y_0\partial_{y_5}, \\ v_{52}(y) &= -y_7\partial_{y_2} - y_6\partial_{y_3} + y_3\partial_{y_6} + y_0\partial_{y_7}, \end{aligned}$$

and observe that $v_{41} + v_{42} = Y_4$ and $v_{51} + v_{52} = Y_5$. By straightforward calculations we see that

$$\langle v_{41}(y), Y_0(y)\rangle_y = \langle v_{42}(y), Y_0(y)\rangle_y = \langle v_{51}(y), Y_0(y)\rangle_y = \langle v_{52}(y), Y_0(y)\rangle_y = 0,$$

$$\langle v_{41}(y), Y_1(y)\rangle_y = \langle v_{42}(y), Y_1(y)\rangle_y = \langle v_{51}(y), Y_1(y)\rangle_y = \langle v_{52}(y), Y_1(y)\rangle_y = 0,$$

which implies that $v_{41}, v_{42}, v_{51}, v_{52} \in \operatorname{span}\{Y_2, \dots, Y_7\}$. The following commutation relation

$$[v_{41}, v_{51}] + [v_{42}, v_{52}] = -2Y_1$$

implies that the distribution $\mathcal{H}$ is bracket generating of step 2. $\square$

Remark: It is possible to repeat the previous argument with other pairs of vector fields. For example, if instead of Y_4 and Y_5 we employ Y_2 and Y_3, we can consider the vector fields

$$\begin{aligned} v_{21}(y) &= -y_2\partial_{y_0} + y_3\partial_{y_1} + y_0\partial_{y_2} - y_1\partial_{y_3}, \\ v_{22}(y) &= -y_6\partial_{y_4} + y_7\partial_{y_5} + y_4\partial_{y_6} - y_5\partial_{y_7}, \\ v_{31}(y) &= -y_3\partial_{y_0} - y_2\partial_{y_1} + y_1\partial_{y_2} + y_0\partial_{y_3}, \\ v_{32}(y) &= y_7\partial_{y_4} + y_6\partial_{y_5} - y_5\partial_{y_6} - y_4\partial_{y_7}, \end{aligned}$$

satisfy $v_{21} + v_{22} = Y_2$, $v_{31} + v_{32} = Y_3$ and

$$[v_{21}, v_{31}] - [v_{21}, v_{31}] = -2Y_1.$$

We can proceed in a similar way if we use Y_6 and Y_7.

We conclude this section by proving that the line bundle $\operatorname{span}\{V_{n+1}\}$ is the vertical space for the submersion given by the Hopf fibration $S^1 \to S^{2n+1} \xrightarrow{H} \mathbb{C}P^n$. This implies that the distribution $\mathcal{H}$ defined in Theorem 4 is an Ehresmann connection for H. To achieve this, we recall that the charts defining the holomorphic structure of $\mathbb{C}P^n$ are given by the open sets

$$U_k = \{[z_0 : \cdots : z_n] : z_k \neq 0\},$$

together with the homeomorphisms

$$\begin{array}{rccc} \varphi_k & : & U_k & \to & \mathbb{C}^n \\ & & [z_0 : \ldots : z_n] & \mapsto & (\frac{z_0}{z_k}, \ldots, \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \cdots, \frac{z_n}{z_k}). \end{array}$$

Then, without loss of generality we will assume that $n = 3$ and we will develop the explicit calculations for $k = 0$. The other cases can be treated similarly.

Using the chart (U_0, φ_0) defined above, we have the map

$$\begin{array}{rccc} \varphi_0 \circ H & : & S^7 & \to & \mathbb{C}^3 \\ & & (z_0, z_1, z_2, z_3) & \mapsto & (\frac{z_1}{z_0}, \frac{z_2}{z_0}, \frac{z_3}{z_0}), \end{array}$$

which in real coordinates can be written as

$$\begin{aligned} \varphi_0 \circ H(x_0, \dots, x_7) &= \left(\frac{x_0x_2 + x_1x_3}{x_0^2 + x_1^2}, \frac{x_0x_3 - x_1x_2}{x_0^2 + x_1^2}, \frac{x_0x_4 + x_1x_5}{x_0^2 + x_1^2}, \right. \\ & \left. \frac{x_0x_5 - x_1x_4}{x_0^2 + x_1^2}, \frac{x_0x_6 + x_1x_7}{x_0^2 + x_1^2}, \frac{x_0x_7 - x_1x_6}{x_0^2 + x_1^2} \right). \end{aligned}$$

The differential of this mapping is given by the matrix

$$d(\varphi_0 \circ H) =$$

$$\begin{pmatrix}
\frac{(x_1^2-x_0^2)x_2-2x_0x_1x_3}{(x_0^2+x_1^2)^2} & \frac{(x_0^2-x_1^2)x_3-2x_0x_1x_2}{(x_0^2+x_1^2)^2} & \frac{x_0}{x_0^2+x_1^2} & \frac{x_1}{x_0^2+x_1^2} & 0 & 0 & 0 & 0 \\
\frac{(x_1^2-x_0^2)x_3+2x_0x_1x_2}{(x_0^2+x_1^2)^2} & \frac{(x_1^2-x_0^2)x_2-2x_0x_1x_3}{(x_0^2+x_1^2)^2} & -\frac{x_1}{x_0^2+x_1^2} & \frac{x_0}{x_0^2+x_1^2} & 0 & 0 & 0 & 0 \\
\frac{(x_1^2-x_0^2)x_4-2x_0x_1x_5}{(x_0^2+x_1^2)^2} & \frac{(x_0^2-x_1^2)x_5-2x_0x_1x_4}{(x_0^2+x_1^2)^2} & 0 & 0 & \frac{x_0}{x_0^2+x_1^2} & \frac{x_1}{x_0^2+x_1^2} & 0 & 0 \\
\frac{(x_1^2-x_0^2)x_5+2x_0x_1x_4}{(x_0^2+x_1^2)^2} & \frac{(x_1^2-x_0^2)x_4-2x_0x_1x_5}{(x_0^2+x_1^2)^2} & 0 & 0 & -\frac{x_1}{x_0^2+x_1^2} & \frac{x_0}{x_0^2+x_1^2} & 0 & 0 \\
\frac{(x_1^2-x_0^2)x_6-2x_0x_1x_7}{(x_0^2+x_1^2)^2} & \frac{(x_0^2-x_1^2)x_7-2x_0x_1x_6}{(x_0^2+x_1^2)^2} & 0 & 0 & 0 & 0 & \frac{x_0}{x_0^2+x_1^2} & \frac{x_1}{x_0^2+x_1^2} \\
\frac{(x_1^2-x_0^2)x_7+2x_0x_1x_6}{(x_0^2+x_1^2)^2} & \frac{(x_1^2-x_0^2)x_6-2x_0x_1x_7}{(x_0^2+x_1^2)^2} & 0 & 0 & 0 & 0 & -\frac{x_1}{x_0^2+x_1^2} & \frac{x_0}{x_0^2+x_1^2}
\end{pmatrix}.$$

By straightforward calculations, we know that

$$\det([d(\varphi_0 \circ H)][d(\varphi_0 \circ H)]^t) = (x_0^2 + x_1^2)^{-8} = |z_0|^{-16} \neq 0,$$

therefore, the matrix $d(\varphi_0 \circ H)$ has rank 6 or equivalently:

$$\dim_{\mathbb{R}} \ker d(\varphi_0 \circ H) = 2.$$

Moreover, since

$$d(\varphi_0 \circ H)(N_{n+1}) = d(\varphi_0 \circ H)(V_{n+1}) = 0,$$

by direct calculations, we conclude

$$\ker d(\varphi_0 \circ H) = \operatorname{span}\{N_{n+1}, V_{n+1}\}.$$

This implies that

$$\ker dH = \operatorname{span}\{V_{n+1}\}.$$

7. Application of the first quaternionic Hopf map

Trying to imitate the work already done for S^3, we find through the quaternionic Hopf bundle $S^3 \to S^7 \to S^4$ a natural choice of horizontal distributions. We consider the quaternionic Hopf map given by

$$\begin{array}{cccccc} h & : & S^7 & \to & S^4 \\ & & (z,w) & \mapsto & (|z|^2 - |w|^2, 2z\bar{w}) \end{array}, \tag{10}$$

which can be written in real coordinates as:

$$\begin{aligned} h(x_0, \ldots, x_7) = (x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2, \\ 2(x_0x_4 + x_1x_5 + x_2x_6 + x_3x_7), 2(-x_0x_5 + x_1x_4 - x_2x_7 + x_3x_6), \\ 2(-x_0x_6 + x_1x_7 + x_2x_4 - x_3x_5), 2(-x_0x_7 - x_1x_6 + x_2x_5 + x_3x_4)). \end{aligned} \tag{11}$$

The differential map dh is the following:

$$dh = 2 \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & -x_4 & -x_5 & -x_6 & -x_7 \\ x_4 & x_5 & x_6 & x_7 & x_0 & x_1 & x_2 & x_3 \\ -x_5 & x_4 & -x_7 & x_6 & x_1 & -x_0 & x_3 & -x_2 \\ -x_6 & x_7 & x_4 & -x_5 & x_2 & -x_3 & -x_0 & x_1 \\ -x_7 & -x_6 & x_5 & x_4 & x_3 & x_2 & -x_1 & -x_0 \end{pmatrix}.$$

Since none of the commutators $[Y_i, Y_j]$, $i, j = 1, \dots, 7$ coincides with the Y_k, $k = 1, \dots, 7$, we look for the kernel of dh among the commutators Y_{ij}, $i, j = 1, \dots, 7$. We found that $[dh]Y_{45} = [dh]Y_{46} = [dh]Y_{56} = 0$. Define $V = \{Y_{45}, Y_{46}, Y_{56}\}$.

Our next step is to find the horizontal distribution $\operatorname{span}\{\mathcal{H}\}$ that is transversal to $\operatorname{span}\{V\}$ and bracket generating: $\operatorname{span}\{\mathcal{H}\}_p \oplus \operatorname{span}\{V\}_p = T_pS^7$ for all $p \in S^7$. To begin with we define five basis for horizontal distributions, that we will work with

$$\mathcal{H}_0 = \{Y_{47}, Y_{57}, Y_{67}, W\},$$

$$\mathcal{H}_1 = \{Y_{34}, Y_{35}, Y_{36}, Y_{37}\}, \quad \mathcal{H}_2 = \{Y_{24}, Y_{25}, Y_{26}, Y_{27}\},$$

$$\mathcal{H}_3 = \{Y_{14}, Y_{15}, Y_{16}, Y_{17}\}, \quad \mathcal{H}_4 = \{Y_{04}, Y_{05}, Y_{06}, Y_{07}\},$$

where the vector field W will be defined later and the notation $Y_{0k} = Y_k$ is chosen for convenience. The numeration is valid only for this section.

We collect some useful information about sets $\mathcal{H}_m$, $m = 0, \dots, 4$, that we will exploit later.

1. All vector fields inside $\mathcal{H}_m$, $m = 0, 1, 2, 3, 4$ are orthonormal (we do not count W before we precise it).
2. All of collections $\mathcal{H}_m$, $m = 0, 1, 2, 3, 4$ are bracket generating with the following commutator relations:

$$\frac{1}{2}[Y_{j4}, Y_{j5}] = Y_{45}, \quad \frac{1}{2}[Y_{j4}, Y_{j6}] = Y_{46}, \quad \frac{1}{2}[Y_{j5}, Y_{j6}] = Y_{56}, \quad j = 0, 1, 2, 3,$$

$$\frac{1}{2}[Y_{47}, Y_{57}] = Y_{45}, \quad \frac{1}{2}[Y_{47}, Y_{67}] = Y_{46}, \quad \frac{1}{2}[Y_{57}, Y_{67}] = Y_{56}.$$

3. We aim to calculate the angles between the vector fields from $\mathcal{H}_m$, $m = 0, 1, 2, 3, 4$ and between vector fields from $\mathcal{H}_m$ and V. Beforehand, we introduce the following notations for the coordinates on the sphere S^4 given by the Hopf map $S^3 \to S^7 \to S^4$.

$$\begin{aligned} a_{00} &= y_0^2 + y_1^2 + y_2^2 + y_3^2 - y_4^2 - y_5^2 - y_6^2 - y_7^2, \\ a_{11} &= 2(y_0y_4 + y_1y_5 + y_2y_6 + y_3y_7), \\ a_{22} &= 2(-y_0y_5 + y_1y_4 - y_2y_7 + y_3y_6), \\ a_{33} &= 2(-y_0y_6 + y_1y_7 + y_2y_4 - y_3y_5), \\ a_{44} &= 2(-y_0y_7 - y_1y_6 + y_2y_5 + y_3y_4). \end{aligned} \tag{12}$$

The first index of a_{mk} reflects the number of the collection $\mathcal{H}_m$, where they will appear and the second one is related to the number of the coordinate on S^4.

We start from $\mathcal{H}_0$ and calculate the inner products:

$$\langle Y_{45}, Y_{67}\rangle = -\langle Y_{46}, Y_{57}\rangle = \langle Y_{56}, Y_{47}\rangle = a_{00}. \tag{13}$$

All other vector fields are orthogonal. We continue for $\mathcal{H}_1$.

$$\begin{array}{lllll} \langle Y_{45}, Y_{36}\rangle & = & -\langle Y_{46}, Y_{35}\rangle \;=\; \langle Y_{56}, Y_{34}\rangle & = & a_{11} \\ \langle Y_{45}, Y_{37}\rangle & = & 2(-y_0y_5 + y_1y_4 + y_2y_7 - y_3y_6) & = & a_{12} \\ \langle Y_{46}, Y_{37}\rangle & = & 2(-y_0y_6 - y_1y_7 + y_2y_4 + y_3y_5) & = & a_{13} \\ \langle Y_{56}, Y_{37}\rangle & = & 2(y_0y_7 - y_1y_6 + y_2y_5 - y_3y_4) & = & a_{14}. \end{array} \tag{14}$$

All other vector fields in $\mathcal{H}_1 \cup V$ are orthogonal. For the set $\mathcal{H}_2$ we see the following:

$$\begin{array}{lllll} -\langle Y_{45}, Y_{26}\rangle & = & \langle Y_{46}, Y_{25}\rangle \;=\; -\langle Y_{56}, Y_{24}\rangle & = & a_{22} \\ \langle Y_{45}, Y_{27}\rangle & = & 2(y_0y_4 + y_1y_5 - y_2y_6 - y_3y_7) & = & a_{21} \\ \langle Y_{46}, Y_{27}\rangle & = & 2(-y_0y_7 + y_1y_6 + y_2y_5 - y_3y_4) & = & a_{24} \\ \langle Y_{56}, Y_{27}\rangle & = & 2(-y_0y_6 - y_1y_7 - y_2y_4 - y_3y_5) & = & a_{23} \end{array} \tag{15}$$

The other products between vector fields from $\mathcal{H}_2 \cup V$ vanish. For $\mathcal{H}_3$ the situation is similar.

$$\begin{array}{lllll} \langle Y_{45}, Y_{16}\rangle & = & -\langle Y_{46}, Y_{15}\rangle \;=\; \langle Y_{56}, Y_{14}\rangle & = & a_{33} \\ \langle Y_{45}, Y_{17}\rangle & = & 2(-y_0y_7 - y_1y_6 - y_2y_5 - y_3y_4) & = & a_{34} \\ \langle Y_{46}, Y_{17}\rangle & = & 2(-y_0y_4 + y_1y_5 - y_2y_6 + y_3y_7) & = & a_{31} \\ \langle Y_{56}, Y_{17}\rangle & = & 2(-y_0y_5 - y_1y_4 + y_2y_7 + y_3y_6) & = & a_{32} \end{array} \tag{16}$$

All other vector fields from $\mathcal{H}_3 \cup V$ are orthogonal. For the last collection $\mathcal{H}_4$ we obtain.

$$\begin{array}{lllll} \langle Y_{45}, Y_{06}\rangle & = & -\langle Y_{46}, Y_{05}\rangle \;=\; \langle Y_{56}, Y_{04}\rangle & = & a_{44} \\ \langle Y_{45}, Y_{07}\rangle & = & 2(y_0y_6 - y_1y_7 + y_2y_4 - y_3y_5) & = & a_{43} \\ \langle Y_{46}, Y_{07}\rangle & = & 2(-y_0y_5 - y_1y_4 - y_2y_7 - y_3y_6) & = & a_{42} \\ \langle Y_{56}, Y_{07}\rangle & = & 2(y_0y_4 - y_1y_5 - y_2y_6 + y_3y_7) & = & a_{41} \end{array} \tag{17}$$

with the rest of the product vanishing.

We notice some relations between the coefficients a_{mk}. The coordinates on S^4 possesses the equality

$$a_{00}^2 + a_{11}^2 + a_{22}^2 + a_{33}^2 + a_{44}^2 = 1. \tag{18}$$

The direct calculations also show

$$\begin{array}{lll} a_{00}^2 + a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{14}^2 & = & 1 \\ a_{00}^2 + a_{21}^2 + a_{22}^2 + a_{23}^2 + a_{24}^2 & = & 1 \\ a_{00}^2 + a_{31}^2 + a_{32}^2 + a_{33}^2 + a_{34}^2 & = & 1 \\ a_{00}^2 + a_{41}^2 + a_{42}^2 + a_{43}^2 + a_{44}^2 & = & 1. \end{array} \tag{19}$$

In other words the sum of the squares of the cosines between vector fields from $\mathcal{H}_m \cup V$, $m = 1,2,3,4$ is equal to $1 - a_{00}^2$. Let us consider 2 cases: $0 < a_{00}^2 \leq 1$ and $a_{00}^2 = 0$.

CASE $0 < a_{00}^2 \leq 1$.

This case corresponds to any point on S^4 except of the set

$$S_1 = \{y_0^2 + y_1^2 + y_2^2 + y_3^2 = y_4^2 + y_5^2 + y_6^2 + y_7^2 = 1/2\}. \tag{20}$$

We observe that the sum of the square of the cosines from (19):

$$\sum_{k=1}^{4} a_{mk}^2 = 1 - a_{00}^2, \quad m = 1,2,3,4$$

belongs to the interval $(0,1)$ and no one of the cosines can be equal to 1. We conclude that each of $\mathcal{H}_m$, $m = 1,2,3,4$, is transverse to V. Particularly, if $a_{00}^2 = 1$ then $\sum_{k=1}^4 a_{mk}^2 = 0$ and $\mathcal{H}_m \perp V$. The latter situation occurs in the antipodal points $(\pm 1,0,0,0,0) \in S^4$ or is to say on the set

$$S_2 = \{y_0^2 + y_1^2 + y_2^2 + y_3^2 = 0,\ y_4^2 + y_5^2 + y_6^2 + y_7^2 = 1\} \cup$$

$$\{y_0^2 + y_1^2 + y_2^2 + y_3^2 = 1,\ y_4^2 + y_5^2 + y_6^2 + y_7^2 = 0\} \in S^7. \tag{21}$$

We also can consider a collection $\mathcal{H}_0$, as a possible horizontal bracket generating distribution, if we choose an adequate vector field W. If $a_{00} \in (0,1)$ we have

$$0 < a_{41}^2 + a_{42}^2 + a_{43}^2 + a_{44}^2 = 1 - a_{00}^2 < 1$$

and none of the products in (17) can give 1. We conclude that Y_{07} can not be collinear to $V = \{Y_{45}, Y_{46}, Y_{56}\}$. Therefore, we choose $W = Y_{07}$. By the same reason we could take Y_{j7}, $j = 1,2,3$. In the case when $a_{00}^2 = 1$ the vector fields Y_{j7}, $j = 0,1,2,3$ are orthogonal to V from (14)- (17) but $\mathcal{H}_0$ is collinear to V from (13) and the collection $\mathcal{H}_0$ with $W = Y_{j7}$ is not transverse to V.

CASE $a_{00}^2 = 0$.

In this case the distribution $\mathcal{H}_0$ nicely serves as a bracket generating if we find a suitable vector field W. Notice that (18) becomes

$$a_{11}^2 + a_{22}^2 + a_{33}^2 + a_{44}^2 = 1. \tag{22}$$

The a_{mm} can not vanish simultaneously. Without lost of generality, we can assume that $a_{44} \neq 0$. Then $a_{41}^2 + a_{42}^2 + a_{43}^2 = 1 - a_{44}^2 < 1$ from (19) and the products (17) imply that Y_{07} is transverse to V and can be used as a vector field W. In the case $a_{44}^2 = 1$ we get that Y_{07} is orthogonal to V. Since $W \perp Y_{j7}$, $j = 4,5,6$ the collection $\mathcal{H}_0$ with any choice of Y_{j7}, $j = 0,\dots,3$ will be orthonormal.

We formulate the latter result in the following theorem

Theorem 5. *Let* (10) *be the quaternionic Hopf map with the vertical space*

$$V = \{Y_{45}, Y_{46}, Y_{56}\},$$

S_1 *and* S_2 *are given by* (20) *and* (21). *Then the Hopf map produces the following Ehresmann connection* $\mathcal{H}_p$, $p \in S^7$*:*

(i) *if* $p \notin S_1$ *then* $\mathcal{H}_p = (\mathcal{H}_m)_p$, *for any choice of* $m = 1, 2, 3, 4$*;*
(ii) *if* $p \notin S_2$ *then* $\mathcal{H}_p = (Y_{47}, Y_{57}, Y_{67}, Y_{j7})_p$, *for any choice of* $j = 0, 1, 2, 3$*;*

and we have respectively

(i) $\operatorname{span}\{(\mathcal{H}_m)\}_p \oplus \operatorname{span}\{V\}_p = T_pS^7,\ m = 1, 2, 3, 4$ *for* $p \in S^7 \setminus S_1$.
(ii) $\operatorname{span}\{Y_{47}, Y_{57}, Y_{67}, Y_{j7}\}_p \oplus \operatorname{span}\{V\}_p = T_pS^7,\ j = 0, 1, 2, 3$ *for* $p \in S^7 \setminus S_2$.

Remark: During the referee process, we were pointed out of the paper [4] where a globally defined basis of the horizontal distribution of rank 4 was constructed considering the Clifford algebra structure of S^7. However, in this case a globally defined basis of the vertical space was not found. In our case, we present a globally defined basis of right invariant vector fields of the vertical space that correspond to the Lie algebra $\mathfrak{su}(2)$ of the S^3-bundle. Nevertheless we did not succeeded in constructing a globally defined basis for the horizontal distribution. The question if both the horizontal distribution and the S^3-fiber are trivializable, remains open.

8. Appendix

8.1. Multiplication of octonions.

Let

$$o_1 = (x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7)$$

and

$$o_2 = (y_0e_0 + y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4 + y_5e_5 + y_6e_6 + y_7e_7)$$

be two octonions. Then we have according to Table 1

$$\begin{aligned}
o_1 \cdot o_2 = {} & (x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7)\circ \\
& (y_0e_0 + y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4 + y_5e_5 + y_6e_6 + y_7e_7) = \\
= {} & (x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7)e_0 + \\
& +(x_1y_0 + x_0y_1 - x_3y_2 + x_2y_3 - x_5y_4 + x_4y_5 + x_7y_6 - x_6y_7)e_1 + \\
& +(x_2y_0 + x_3y_1 + x_0y_2 - x_1y_3 - x_6y_4 - x_7y_5 + x_4y_6 + x_5y_7)e_2 + \\
& +(x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3 - x_7y_4 + x_6y_5 - x_5y_6 + x_4y_7)e_3 + \\
& +(x_4y_0 + x_5y_1 + x_6y_2 + x_7y_3 + x_0y_4 - x_1y_5 - x_2y_6 - x_3y_7)e_4 + \\
& +(x_5y_0 - x_4y_1 + x_7y_2 - x_6y_3 + x_1y_4 + x_0y_5 + x_3y_6 - x_2y_7)e_5 + \\
& +(x_6y_0 - x_7y_1 - x_4y_2 + x_5y_3 + x_2y_4 - x_3y_5 + x_0y_6 + x_1y_7)e_6 +
\end{aligned}$$

$$+(x_7y_0 + x_6y_1 - x_5y_2 - x_4y_3 + x_3y_4 + x_2y_5 - x_1y_6 + x_0y_7)e_7.$$

8.2. **Vector fields.** According to the previous multiplication rule, we have the following unit vector fields of $\mathbb{R}^8$ arising as right invariant vector fields under the octonion product.

$$\begin{aligned}
Y_0(y) &= y_0\partial_{y_0} + y_1\partial_{y_1} + y_2\partial_{y_2} + y_3\partial_{y_3} + y_4\partial_{y_4} + y_5\partial_{y_5} + y_6\partial_{y_6} + y_7\partial_{y_7}\\
Y_1(y) &= -y_1\partial_{y_0} + y_0\partial_{y_1} - y_3\partial_{y_2} + y_2\partial_{y_3} - y_5\partial_{y_4} + y_4\partial_{y_5} - y_7\partial_{y_6} + y_6\partial_{y_7}\\
Y_2(y) &= -y_2\partial_{y_0} + y_3\partial_{y_1} + y_0\partial_{y_2} - y_1\partial_{y_3} - y_6\partial_{y_4} + y_7\partial_{y_5} + y_4\partial_{y_6} - y_5\partial_{y_7}\\
Y_3(y) &= -y_3\partial_{y_0} - y_2\partial_{y_1} + y_1\partial_{y_2} + y_0\partial_{y_3} + y_7\partial_{y_4} + y_6\partial_{y_5} - y_5\partial_{y_6} - y_4\partial_{y_7}\\
Y_4(y) &= -y_4\partial_{y_0} + y_5\partial_{y_1} + y_6\partial_{y_2} - y_7\partial_{y_3} + y_0\partial_{y_4} - y_1\partial_{y_5} - y_2\partial_{y_6} + y_3\partial_{y_7}\\
Y_5(y) &= -y_5\partial_{y_0} - y_4\partial_{y_1} - y_7\partial_{y_2} - y_6\partial_{y_3} + y_1\partial_{y_4} + y_0\partial_{y_5} + y_3\partial_{y_6} + y_2\partial_{y_7}\\
Y_6(y) &= -y_6\partial_{y_0} + y_7\partial_{y_1} - y_4\partial_{y_2} + y_5\partial_{y_3} + y_2\partial_{y_4} - y_3\partial_{y_5} + y_0\partial_{y_6} - y_1\partial_{y_7}\\
Y_7(y) &= -y_7\partial_{y_0} - y_6\partial_{y_1} + y_5\partial_{y_2} + y_4\partial_{y_3} - y_3\partial_{y_4} - y_2\partial_{y_5} + y_1\partial_{y_6} + y_0\partial_{y_7}.
\end{aligned}$$

The vector fields Y_i, $i = 1, \dots, 7$ form an orthonormal frame of T_pS^7, $p \in S^7$, with respect to restriction of the inner product $\langle \cdot, \cdot \rangle$ from $\mathbb{R}^8$ to the tangent space T_pS^7 at each $p \in S^7$.

8.3. **Commutators between vector fields.** Let us denote by $Y_{ij}(y) = \frac{1}{2}[Y_i(y), Y_j(y)]$ the commutator between the right invariant vector fields under the octonion product, described in the previous Subsection, we have the following list:

$$\begin{aligned}
Y_{12}(y) &= y_3\partial_{y_0} + y_2\partial_{y_1} - y_1\partial_{y_2} - y_0\partial_{y_3} + y_7\partial_{y_4} + y_6\partial_{y_5} - y_5\partial_{y_6} - y_4\partial_{y_7}\\
Y_{13}(y) &= -y_2\partial_{y_0} + y_3\partial_{y_1} + y_0\partial_{y_2} - y_1\partial_{y_3} + y_6\partial_{y_4} - y_7\partial_{y_5} - y_4\partial_{y_6} + y_5\partial_{y_7}\\
Y_{14}(y) &= y_5\partial_{y_0} + y_4\partial_{y_1} - y_7\partial_{y_2} - y_6\partial_{y_3} - y_1\partial_{y_4} - y_0\partial_{y_5} + y_3\partial_{y_6} + y_2\partial_{y_7}\\
Y_{15}(y) &= -y_4\partial_{y_0} + y_5\partial_{y_1} - y_6\partial_{y_2} + y_7\partial_{y_3} + y_0\partial_{y_4} - y_1\partial_{y_5} + y_2\partial_{y_6} - y_3\partial_{y_7}\\
Y_{16}(y) &= y_7\partial_{y_0} + y_6\partial_{y_1} + y_5\partial_{y_2} + y_4\partial_{y_3} - y_3\partial_{y_4} - y_2\partial_{y_5} - y_1\partial_{y_6} - y_0\partial_{y_7}\\
Y_{17}(y) &= -y_6\partial_{y_0} + y_7\partial_{y_1} + y_4\partial_{y_2} - y_5\partial_{y_3} - y_2\partial_{y_4} + y_3\partial_{y_5} + y_0\partial_{y_6} - y_1\partial_{y_7}\\
Y_{23}(y) &= y_1\partial_{y_0} - y_0\partial_{y_1} + y_3\partial_{y_2} - y_2\partial_{y_3} - y_5\partial_{y_4} + y_4\partial_{y_5} - y_7\partial_{y_6} + y_6\partial_{y_7}\\
Y_{24}(y) &= y_6\partial_{y_0} + y_7\partial_{y_1} + y_4\partial_{y_2} + y_5\partial_{y_3} - y_2\partial_{y_4} - y_3\partial_{y_5} - y_0\partial_{y_6} - y_1\partial_{y_7}\\
Y_{25}(y) &= -y_7\partial_{y_0} + y_6\partial_{y_1} + y_5\partial_{y_2} - y_4\partial_{y_3} + y_3\partial_{y_4} - y_2\partial_{y_5} - y_1\partial_{y_6} + y_0\partial_{y_7}\\
Y_{26}(y) &= -y_4\partial_{y_0} - y_5\partial_{y_1} + y_6\partial_{y_2} + y_7\partial_{y_3} + y_0\partial_{y_4} + y_1\partial_{y_5} - y_2\partial_{y_6} - y_3\partial_{y_7}\\
Y_{27}(y) &= y_5\partial_{y_0} - y_4\partial_{y_1} + y_7\partial_{y_2} - y_6\partial_{y_3} + y_1\partial_{y_4} - y_0\partial_{y_5} + y_3\partial_{y_6} - y_2\partial_{y_7}\\
Y_{34}(y) &= -y_7\partial_{y_0} + y_6\partial_{y_1} - y_5\partial_{y_2} + y_4\partial_{y_3} - y_3\partial_{y_4} + y_2\partial_{y_5} - y_1\partial_{y_6} + y_0\partial_{y_7}\\
Y_{35}(y) &= -y_6\partial_{y_0} - y_7\partial_{y_1} + y_4\partial_{y_2} + y_5\partial_{y_3} - y_2\partial_{y_4} - y_3\partial_{y_5} + y_0\partial_{y_6} + y_1\partial_{y_7}\\
Y_{36}(y) &= y_5\partial_{y_0} - y_4\partial_{y_1} - y_7\partial_{y_2} + y_6\partial_{y_3} + y_1\partial_{y_4} - y_0\partial_{y_5} - y_3\partial_{y_6} + y_2\partial_{y_7}\\
Y_{37}(y) &= y_4\partial_{y_0} + y_5\partial_{y_1} + y_6\partial_{y_2} + y_7\partial_{y_3} - y_0\partial_{y_4} - y_1\partial_{y_5} - y_2\partial_{y_6} - y_3\partial_{y_7}\\
Y_{45}(y) &= y_1\partial_{y_0} - y_0\partial_{y_1} - y_3\partial_{y_2} + y_2\partial_{y_3} + y_5\partial_{y_4} - y_4\partial_{y_5} - y_7\partial_{y_6} + y_6\partial_{y_7}
\end{aligned}$$

$$
\begin{aligned}
Y_{46}(y) &= y_2\partial_{y_0} + y_3\partial_{y_1} - y_0\partial_{y_2} - y_1\partial_{y_3} + y_6\partial_{y_4} + y_7\partial_{y_5} - y_4\partial_{y_6} - y_5\partial_{y_7} \\
Y_{47}(y) &= -y_3\partial_{y_0} + y_2\partial_{y_1} - y_1\partial_{y_2} + y_0\partial_{y_3} + y_7\partial_{y_4} - y_6\partial_{y_5} + y_5\partial_{y_6} - y_4\partial_{y_7} \\
Y_{56}(y) &= -y_3\partial_{y_0} + y_2\partial_{y_1} - y_1\partial_{y_2} + y_0\partial_{y_3} - y_7\partial_{y_4} + y_6\partial_{y_5} - y_5\partial_{y_6} + y_4\partial_{y_7} \\
Y_{57}(y) &= -y_2\partial_{y_0} - y_3\partial_{y_1} + y_0\partial_{y_2} + y_1\partial_{y_3} + y_6\partial_{y_4} + y_7\partial_{y_5} - y_4\partial_{y_6} - y_5\partial_{y_7} \\
Y_{67}(y) &= y_1\partial_{y_0} - y_0\partial_{y_1} - y_3\partial_{y_2} + y_2\partial_{y_3} - y_5\partial_{y_4} + y_4\partial_{y_5} + y_7\partial_{y_6} - y_6\partial_{y_7}.
\end{aligned}
$$

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3.2 Paper B

SUB-RIEMANNIAN GEODESICS AND HEAT OPERATOR ON ODD DIMENSIONAL SPHERES

MAURICIO GODOY MOLINA
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Abstract. In this article we study the sub-Riemannian geometry of the spheres S^{2n+1} and S^{4n+3}, arising from the principal S^1-bundle structure defined by the Hopf map and the principal S^3-bundle structure given by the quaternionic Hopf map respectively. The S^1 action leads to the classical contact geometry of S^{2n+1}, while the S^3 action gives another type of sub-Riemannian structure, with a distribution of corank 3. In both cases the metric is given as the restriction of the usual Riemannian metric on the respective horizontal distributions. For the contact S^7 case, we give an explicit form of the intrinsic sub-Laplacian and obtain a commutation relation between the sub-Riemannian heat operator and the heat operator in the vertical direction.

1. Introduction

One of the main objectives of classical sub-Riemannian geometry is to study manifolds which are path-connected by curves admissible in a certain sense. Admissibility refers to a constraint on the velocity vector of an absolutely continuous curve $\gamma : [0, 1] \to M$, where M is a smooth connected manifold. More precisely, if $\mathcal{H} \subset TM$ is a smooth distribution, then γ is admissible or horizontal if $\dot{\gamma}(t) \in \mathcal{H}$ a.e. The distribution $\mathcal{H}$ is often called horizontal distribution in the literature.

The idea of studying sub-Riemannian geometry arising from well-behaved fiber bundles was introduced by R. Montgomery in [15], although the Riemannian analogue had been studied many decades before. The idea is the following: given a submersion $\pi : Q \to M$ between two Riemannian manifolds Q and M, where $\dim M < \dim Q$, define a "horizontal" distribution over Q by the pull-back bundle $\pi^*(TM)$ of the tangent bundle of M via π. In the case when we have a principal $G-$action over Q preserving the fibers of the submersion, the manifold M can be identified with the orbits of the

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action and, after some technical assumptions, it is possible to obtain an explicit characterization of sub-Riemannian geodesics.

The aim of the present article is to describe the sub-Riemannian geometry of two sub-Riemannian structures for odd-dimensional spheres. More specifically, we study the sub-Riemannian geometry arising from the contact distribution for the spheres S^{2n+1} with metric given as a restriction of the usual Riemannian metric, and the one arising from the quaternionic Hopf fibration for the spheres S^{4n+3}.

This article is organized as follows. In Section 2, we give some standard definitions of sub-Riemannian geometry which will be needed in the rest of the paper. In Section 3 we give an explicit description of sub-Riemannian geodesics in spheres S^{2n+1} endowed with the standard contact distribution and we study some of their geometric properties. In Section 4 we use the obtained form of geodesics in the case of S^3 to give another interpretation to a result by Hurtado and Rosales in [11]. With this new point of view, we are able to extend their result to contact spheres of an arbitrary odd dimension. Section 5 is the analogue to Sections 3 and 4 for the case of spheres of the form S^{4n+3} endowed with a distribution of corank 3. Section 6 is somewhat different technically, but it is in spirit related to the core of this article. It deals with a geodesic differential equation for the quaternionic $\mathbb{H}-$type group studied in [4], obtained generalizing the techniques in [17]. The reason for studying this equation here is to pose the question of a similar equation for the case of S^7 and a distribution of rank 4. Section 7 consists of the construction of the intrinsic sub-Laplacian for S^7. The main result states that it is the sum of the squares of an orthonormal basis of the horizontal distribution. Finally, Section 8 employs the previous construction to obtain a simple form of the heat operator for S^7 in a similar way as obtained in [3].

2. Preliminaries and notations

2.1. **Sub-Riemannian geometry.** Let us first give some general definitions, which will be adapted to our purposes when it will be necessary. Let M be a smooth connected manifold of dimension n, together with a smooth distribution $\mathcal{H} \subset TM$ of rank k, $2 \leq k < n$. The manifolds of our interest are endowed with distributions satisfying the bracket generating condition, *i.e.* distributions whose Lie hull equals the full tangent bundle of M. To be more precise, define inductively the vector bundles

$$\mathcal{H}^1 = \mathcal{H}, \qquad \mathcal{H}^{r+1} = [\mathcal{H}^r, \mathcal{H}] + \mathcal{H}^r \quad \text{for } r \geq 1,$$

which naturally induce the flag

$$\mathcal{H} = \mathcal{H}^1 \subseteq \mathcal{H}^2 \subseteq \mathcal{H}^3 \subseteq \dots .$$

We say that $\mathcal{H}$ is bracket generating if for all $x \in M$ there is an $r(x) \in \mathbb{Z}^+$ such that

$$\mathcal{H}_x^{r(x)} = T_x M. \tag{1}$$

If the dimensions $\dim \mathcal{H}_x^r$ do not depend on x for any $r \geq 1$, we say that $\mathcal{H}$ is a regular distribution. The least r such that (1) is satisfied is called the step of $\mathcal{H}$. In this paper we will focus on regular distributions of step 2.

A natural question to pose is, given M and $\mathcal{H}$, whether one can join any two points of M via a horizontal curve, *i.e.* an absolutely continuous curve $\gamma : [0,1] \to M$ which satisfies $\dot{\gamma}(t) \in \mathcal{H}$ almost everywhere. A complete answer to this question was given in [19], which shows a deep generalization the celebrated Chow-Rashevskiĭ theorem, see [7, 16], that gives a sufficient condition and can be stated as follows:

Theorem 1. *Let M be a connected manifold and $\mathcal{H} \subset TM$ be a bracket generating distribution, then the set of points that can be connected to $p \in M$ by a horizontal path coincides with M.*

Remark: A slightly more general version of Theorem 1 states that, if M is not connected, then the set of points that can be connected to $p \in M$ by a horizontal path is the connected component containing p. Since we assumed the manifold to be connected, the general formulation is unnecessary.

After these preliminaries, we are ready to specify the class of manifolds of our interest.

Definition 1. *A sub-Riemannian structure over a manifold M is a pair $(\mathcal{H}, \langle\cdot,\cdot\rangle_{sR})$, where $\mathcal{H}$ is a bracket generating distribution and $\langle\cdot,\cdot\rangle_{sR}$ is a fiber inner product defined on $\mathcal{H}$. The triple $(M, \mathcal{H}, \langle\cdot,\cdot\rangle_{sR})$ is called sub-Riemannian manifold.*

In this context, the length of a horizontal curve $\gamma : [0,1] \to M$ is defined to be

$$\ell(\gamma) := \int_0^1 \|\dot{\gamma}(t)\| dt,$$

where $\|\dot{\gamma}(t)\|^2 = \langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle_{sR}$ whenever $\dot{\gamma}(t)$ exists.

This notion of length gives rise to the Carnot-Carathéodory distance $d(p,q)$ between two points $p, q \in M$, given by $d(p,q) := \inf \ell(\gamma)$, where the infimum is taken over all absolutely continuous horizontal curves joining p to q. An absolutely continuous horizontal curve that realizes the distance between two points is called a horizontal length minimizer. It is clear that if $\mathcal{H}$ is bracket generating then $d(p,q)$ is a finite nonnegative number.

Considering a trivializing neighborhood U_p around $p \in M$ for the sub-bundle $\mathcal{H}$, one can find a local orthonormal basis $X_1, \ldots, X_k$ with respect to $\langle\cdot,\cdot\rangle_{sR}$. The associated sub-Riemannian Hamiltonian is given by

$$H(q,\lambda)=\frac{1}{2}\sum_{m=1}^{k}\lambda(X_m(q))^2,$$

where $(q,\lambda)\in T^*U_p$. A normal geodesic corresponds to the projection to $U_p\subset M$ of the solution of the Hamiltonian system

$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial \lambda_i}\\ \dot{\lambda}_i &= -\frac{\partial H}{\partial q_i},\end{aligned}$$

where (q_i,λ_i) are the coordinates in the cotangent bundle of M.
Remark: It is possible to define sub-Riemannian geodesics in a more general context. There are many interesting problems related to the classification of such curves, their analytic and geometric properties. In [12] the problem for the case of rank two distributions is studied and essentially solved. Nevertheless, in the case of step two distributions, the general notion of geodesic gives rise to two cases: curves consisting of one point and normal geodesics. Thus, normal geodesics are the only interesting case for our purposes. Note that in this case normal geodesics are local length minimizers, in the sense that any sufficiently small arc of a normal geodesic minimizes the length functional. On the other hand one of the particular features of sub-Riemannian geometry, as the sub-Riemannian Heisenberg group exemplifies, is that it is possible to find arbitrarily close points that can be joined by normal geodesics with different lengths.

2.2. **Sub-Riemannian principal bundles.** Our first goal is to recall a full characterization of normal geodesics in the case of sub-Riemannian principal bundles. As a direct application we obtain an explicit formula for the sub-Riemannian geodesics on odd-dimensional spheres, with respect to distributions of corank 1 and 3 in Sections 3 and 5 respectively. For the sake of completeness we recall some definitions and notations given in [15].

For a submersion $\pi:Q\to M$ with fiber $Q_m=\pi^{-1}(m)$ through $m\in M$, the vertical space at $q\in Q$ is given by $T_qQ_{\pi(q)}$ and it is denoted by V_q. In this context, an Ehresmann connection for $\pi:Q\to M$ is a distribution $\mathcal{H}\subset TQ$ which is everywhere transversal to the vertical space, that is:

$$V_q\oplus\mathcal{H}_q=T_qQ \quad \text{for every } q\in Q.$$

Let us assume that a Lie group G acts on Q in such a way that $\pi:Q\to M$ becomes a fiber bundle with fiber G. We say that the submersion π is a principal $G-$bundle with connection $\mathcal{H}$ if the following conditions hold: G acts freely and transitively on each fiber, the group orbits are the fibers of $\pi:Q\to M$, and the $G-$action on Q preserves the connection $\mathcal{H}$. Observe that the second condition implies that M is isomorphic to Q/G and π is the canonical projection. We will refer to the connection $\mathcal{H}$ as the horizontal distribution.

For the rest of this section, let us denote the Lie algebra of G by $\mathfrak{g}$, and the corresponding exponential map by $\exp_G : \mathfrak{g} \to G$.

Definition 2. *For the principal $G-$bundle $\pi : Q \to M$, the infinitesimal generator for the group action is the map $\sigma_q : \mathfrak{g} \to T_qQ$ defined by*

$$\sigma_q(\xi) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} q \exp_G(\epsilon\xi)$$

for $q \in Q$ and $\xi \in \mathfrak{g}$. If the metric $\langle\cdot,\cdot\rangle$ in Q is $G-$invariant, we have a well-defined bilinear form

$$\mathbb{I}_q(\xi,\eta) = \langle\sigma_q\xi, \sigma_q\eta\rangle \quad , \quad \xi,\eta \in \mathfrak{g},$$

which is called the moment of inertia tensor at q.

The $G-$invariant Riemannian metric on Q is said to be of constant bi-invariant type if its moment of inertia tensor $\mathbb{I}_q$ is independent of $q \in Q$. Recall also that, in the case of a principal $G-$bundle, for each $q \in Q$ the infinitesimal generator σ_q is an isomorphism between the vertical space V_q and $\mathfrak{g}$. We refer to its inverse as the $\mathfrak{g}$ valued connection one form.

With all of these at hand, we can state the main tool required in this section. This will imply almost immediately Corollaries 1 and 2 which are of core importance in the present paper. The proof of the following theorem can be found in [15].

Theorem 2 (Horizontal Geodesics for Principal Bundles)**.** *Let $\pi : Q \to M$ be a principal $G-$bundle with a Riemannian metric of constant bi-invariant type. Let $\mathcal{H}$ be the induced connection, with $\mathfrak{g}$ valued connection one form A. Let $\exp_R$ be the Riemannian exponential map, so that $\gamma_R(t) = \exp_R(tv)$ is the Riemannian geodesic through q with velocity vector $v \in T_qQ$. Then any horizontal lift γ of the projection $\pi\circ\gamma_R$ is a normal sub-Riemannian geodesic and is given by*

$$\gamma(t) = \exp_R(tv)\exp_G(-tA(v))$$

where $\exp_G : \mathfrak{g} \to G$ is the exponential map of G. Moreover, all normal sub-Riemannian geodesics can be obtained in this way.

Remark: In Theorem 2, the sub-Riemannian geodesics are considered with respect to the metric induced by restricting $\langle\cdot,\cdot\rangle$ to $\mathcal{H}$. Recall that constant bi-invariant metrics must be $G-$invariant.

3. Sub-Riemannian Geodesics on S^{2n+1}

In the case of odd dimensional spheres S^{2n+1}, embedded as the boundary of the unit ball in $\mathbb{C}^{n+1}$, there is a natural action of $S^1 \cong U(1)$ on it, via componentwise multiplication by a complex number of norm 1. This action induces the well known Hopf fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$, which forms a principal S^1-bundle with connection $\mathcal{H}$ given by the orthogonal complement to the vector field

$$V_{n+1}(p) = -y_0\partial_{x_0} + x_0\partial_{y_0} - \ldots - y_n\partial_{x_n} + x_n\partial_{y_n} \tag{2}$$

at each $p = (x_0, y_0, \ldots, x_n, y_n) \in S^{2n+1}$, with respect to the usual Riemannian metric of S^{2n+1} as embedded in $\mathbb{R}^{2(n+1)} \cong \mathbb{C}^{n+1}$. In [9] it is shown that this distribution coincides with the holomorphic tangent space HS^{2n+1} of S^{2n+1} thought as an embedded CR manifold and that it also coincides with the contact distribution given by $\ker\omega$ with respect to the contact form

$$\omega = -y_0dx_0 + x_0dy_0 - \ldots - y_ndx_n + x_ndy_n.$$

Note that the components of the vector $V_{n+1}(p)$ are the same as in the $\mathfrak{u}(1)$ action $i \cdot p$.

As a direct application of Theorem 2, it is possible to describe all sub-Riemannian geodesics for the sphere S^{2n+1} as a sub-Riemannian manifold equipped with connection $\mathcal{H}$ and with metric restricted from $\mathbb{R}^{2(n+1)}$. By the results discussed in [9], the holomorphic tangent space for S^{2n+1} is the distribution induced by the principal S^1-bundle given by the Hopf fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ with $\mathfrak{u}(1)-$valued connection form $A(v) = i\langle v, V_{n+1}\rangle$, $v \in T_pS^{2n+1}$, V_{n+1} denotes $V_{n+1}(p)$ and $\langle\cdot,\cdot\rangle$ stands for the standard inner product in $\mathbb{R}^{2(n+1)}$. Moreover, the usual Riemannian structure on S^{2n+1} is of constant bi-invariant type, since we have

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} q\exp_{\mathfrak{u}(1)}(\epsilon\xi) = \alpha i \cdot q = \alpha V_{n+1}(q),$$

for any $q \in S^{2n+1}$ and $\xi = i\alpha \in \mathfrak{u}(1)$. Therefore, the inertia tensor is given by

$$\mathbb{I}_q(i\alpha, i\tilde{\alpha}) = \langle\alpha V_{n+1}(q), \tilde{\alpha}V_{n+1}(q)\rangle = \alpha\tilde{\alpha},$$

which does not depend of the point.

By Theorem 2, we have the following result.

Corollary 1. *Let $p \in S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + \ldots + |z_n|^2 = 1\}$ and $v \in T_pS^{2n+1}$. If $\gamma_R(t) = (z_0(t), \ldots, z_n(t))$ is the great circle satisfying $\gamma_R(0) = p$ and $\dot{\gamma}_R(0) = v$, then the corresponding sub-Riemannian geodesic is given by*

$$\gamma(t) = \left(z_0(t)e^{-it\langle v,V_{n+1}\rangle}, \ldots, z_n(t)e^{-it\langle v,V_{n+1}\rangle}\right). \tag{3}$$

In order to analyze in more details formula (3), let us introduce some notations and the necessary setup. Recall that the Riemannian geodesic starting at $p \in S^n$ with velocity $v \in T_pS^n$ of any sphere S^n as a submanifold of $\mathbb{R}^{n+1}$, with the standard Riemannian structure, is given by:

$$\gamma_R(t) = p\cos(\|v\|t) + \frac{v}{\|v\|}\sin(\|v\|t), \tag{4}$$

where $\|v\|^2 = \langle v, v\rangle$. In the case of our interest, a great circle $\gamma_R(t)$ in S^{2n+1} as a submanifold of $\mathbb{R}^{2(n+1)} \cong \mathbb{C}^{n+1}$ will be written in complex notation as $\gamma_R(t) = (z_0(t), \ldots, z_n(t))$. For notational simplicity, the action of $\lambda \in S^1$ over $(p_0, \ldots, p_n) \in S^{2n+1}$ is denoted by $\lambda \cdot p = (\lambda p_0, \ldots, \lambda p_n)$. Let us write $\gamma(0) = \gamma_R(0) = p = (a_0 + ib_0, \ldots, a_n + ib_n) \in S^{2n+1}$ and $\dot\gamma_R(0) = v = (\alpha_0 + i\beta_0, \ldots, \alpha_n + i\beta_n) \in T_pS^{2n+1}$. Observe that $V_{n+1}(\gamma(t)) = i \cdot \gamma(t)$. As above, $V_{n+1} = V_{n+1}(\gamma(0))$.

Remark: In the subsequent calculations, the notation $\langle\cdot,\cdot\rangle_H$ will denote the standard Hermitian product in $\mathbb{C}^{n+1}$. We recall that the standard inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^{2(n+1)}$ satisfies

$$\operatorname{Re}\langle\cdot,\cdot\rangle_H = \langle\cdot,\cdot\rangle.$$

Theorem 2 assures that γ is a horizontal curve, *i.e.* $\langle\dot\gamma(t), V_{n+1}(\gamma(t))\rangle = 0$, nevertheless it is possible to check directly this by straightforward calculations. Since some of the computations will appear later, it is convenient to write them down. First notice that

$$\begin{aligned}
\langle\dot\gamma(t), V_{n+1}(\gamma(t))\rangle_H &= \langle(-i\langle v, V_{n+1}\rangle\gamma_R(t) + \dot\gamma_R(t))e^{-i\langle v, V_{n+1}\rangle t},\\
&\quad\; ie^{-i\langle v, V_{n+1}\rangle t}\gamma_R(t)\rangle_H\\
&= -\langle v, V_{n+1}\rangle\langle\gamma_R(t), \gamma_R(t)\rangle_H - i\langle\dot\gamma_R(t), \gamma_R(t)\rangle_H\\
&= -\langle v, V_{n+1}\rangle - i\langle\dot\gamma_R(t), \gamma_R(t)\rangle_H.
\end{aligned}$$

Thus the problem is now to determine the value of

$$\langle\dot\gamma_R(t), \gamma_R(t)\rangle_H = \sum_{k=0}^{n} \dot z_k(t)\overline{z_k(t)}.$$

By straightforward calculations, it is easy to see that

$$\begin{aligned}
\sum_{k=0}^{n} \dot z_k(t)\overline{z_k(t)} &= (\cos^2(\|v\|t) - \sin^2(\|v\|t))\sum_{k=0}^{n}(a_k\alpha_k + b_k\beta_k) +\\
&\quad +i\sum_{k=0}^{n}(a_k\beta_k - b_k\alpha_k)\\
&= \langle p, v\rangle\cos(2\|v\|t) + i\langle v, V_{n+1}\rangle\\
&= i\langle v, V_{n+1}\rangle, \qquad (5)
\end{aligned}$$

yielding to $\langle\dot\gamma(t), V_{n+1}(\gamma(t))\rangle_H = 0$, which implies the horizontality of the curve $\gamma(t)$.

Let us now address the problem of connecting two points in S^{2n+1} by sub-Riemannian geodesics. We know by Theorem 1 that it is possible to find a horizontal curve $\Gamma : [0, T] \to S^{2n+1}$ such that

$$\Gamma(0) = p \quad \text{and} \quad \Gamma(T) = q, \tag{6}$$

for any pair $p, q \in S^{2n+1}$ and all fixed time parameter $T > 0$. A natural question to ask is whether Γ can be taken as a geodesic in (6). Due to the complexity of the problem, we will give a partial answer to it. It is important to remark that Proposition 1 is a direct analogue of the result obtained in [5, Theorem 1] in the particular case of $n = 1$, *i.e.* for the three dimensional sphere.

Proposition 1. *The set of sub-Riemannian geodesics arising from great circles $\gamma_R(t)$ such that $\dot\gamma_R(0) \in \mathcal{H} = \ker\omega$ is diffeomorphic to $\mathbb{C}P^n$.*

Proof. In this case any sub-Riemannian geodesic starting at $p \in S^{2n+1}$ with initial velocity $v \in \mathcal{H} \subset T_pS^{2n+1}$ coincides with the corresponding great circle, since the condition $\dot\gamma_R(0) \in \mathcal{H} = \ker\omega$ is equivalent to $\langle v, V_{n+1}\rangle = 0$, thus

$$\gamma(t) = p\cos(\|v\|t) + \frac{v}{\|v\|}\sin(\|v\|t)$$

whose loci is uniquely determined by the point $[v] \in \mathbb{C}P^n$. □

Observe that this $\mathbb{C}P^n$ can be seen as a submanifold of S^{2n+1} which is transversal to V_{n+1} along the fiber containing p. As remarked in [5] for S^3, this can be seen as a sophisticated analogue of the horizontal space at the identity in the $(2n+1)-$dimensional Heisenberg group.

Let us conclude this discussion with an interesting result which will be of importance in the following Section. This can be thought of as a sort of Pythagoras theorem for contact spheres.

Proposition 2. *For a horizontal sub-Riemannian geodesic of the form*

$$\gamma(t) = \left(z_0(t)e^{-it\langle v,V_{n+1}\rangle}, \dots, z_n(t)e^{-it\langle v,V_{n+1}\rangle}\right)$$

the following equation holds

$$\|\dot\gamma(t)\|^2 + \langle v, V_{n+1}\rangle^2 = \|v\|^2.$$

Thus, its velocity is constant and its sub-Riemannian length for $t \in [a, b]$ is $\ell(\gamma) = (b - a)\sqrt{\|v\|^2 - \langle v, V_{n+1}\rangle^2}$.

Proof. By straightforward calculations, we have

$$\begin{aligned}
\langle\dot\gamma(t), \dot\gamma(t)\rangle_H &= \langle(-i\langle v, V_{n+1}\rangle\gamma_R(t) + \dot\gamma_R(t))e^{-i\langle v,V_{n+1}\rangle t}, \\
&\qquad (-i\langle v, V_{n+1}\rangle\gamma_R(t) + \dot\gamma_R(t))e^{-i\langle v,V_{n+1}\rangle t}\rangle_H \\
&= \langle v, V_{n+1}\rangle^2\langle\gamma_R(t), \gamma_R(t)\rangle_H + \langle\dot\gamma_R(t), \dot\gamma_R(t)\rangle_H \\
&\qquad +\langle v, V_{n+1}\rangle(i\langle\dot\gamma_R, \gamma_R\rangle_H - i\langle\gamma_R, \dot\gamma_R\rangle_H) \\
&= \langle v, V_{n+1}\rangle^2 + \|v\|^2 - 2\langle v, V_{n+1}\rangle^2.
\end{aligned}$$

Here we have used equation (5). The proposition follows. □

Remark: According to Proposition 2, the condition that a curve $\gamma(t) = e^{-it\langle v, V_{n+1}\rangle}\gamma_R(t)$ is parameterized by arclength is equivalent to require that $\|v\|^2 = 1 + \langle v, V_{n+1}\rangle^2$.

4. Curvature of sub-Riemannian geodesics on S^3

In [11], the authors describe the horizontal geodesics of the three dimensional sphere with respect to its contact distribution, obtaining an explicit expression for these curves. The key tool to achieve this is the following proposition.

Proposition 3. *Let $\gamma : I \to S^3$ be a C^2 horizontal curve parameterized by arc-length. Then γ is a critical point of length for any admissible variation if and only if there is $\lambda \in \mathbb{R}$ such that γ satisfies the second order ordinary differential equation*

$$\nabla_{\dot\gamma}\dot\gamma + 2\lambda J(\dot\gamma) = 0, \tag{7}$$

where ∇ is the Levi-Civita connection and J is the standard almost complex structure on S^3.

The authors call the parameter λ above the *curvature* of γ, since after projecting it via the Hopf fibration, λ becomes precisely the curvature of the projected curve in S^2. Note that the curves with zero curvature are precisely the horizontal great circles. It is our purpose to find an explicit expression for λ in terms of known parameters of the sub-Riemannian geodesics of S^3, as presented in Corollary 1.

Proposition 4. *The curvature of the sub-Riemannian geodesic*

$$\gamma(t) = e^{-i\langle v, V_2\rangle t}\gamma_R(t)$$

in S^3, parameterized by arc-length, equals $\langle v, V_2\rangle$.

Proof. The Lie group structure of S^3 as the set of unit quaternions, induces the globally defined vector fields

$$\begin{array}{rcl} V(p) & = & -y_1\partial_{x_1} + x_1\partial_{y_1} - y_2\partial_{x_2} + x_2\partial_{y_2}, \\ X(p) & = & -x_2\partial_{x_1} + y_2\partial_{y_1} + x_1\partial_{x_2} - y_1\partial_{y_2}, \\ Y(p) & = & -y_2\partial_{x_1} - x_2\partial_{y_1} + y_1\partial_{x_2} + x_1\partial_{y_2}, \end{array} \tag{8}$$

at $p = (x_1, y_1, x_2, y_2) \in S^3$, which are orthonormal with respect to the usual Riemannian structure of $\mathbb{R}^3$. Observe that $V(p) = V_2(p)$ as defined in (2).

Let $p = (x_1, y_1, x_2, y_2) = \gamma(0) \in S^3$ be the initial point of γ and let $v = (v_{x_1}, v_{y_1}, v_{x_2}, v_{y_2}) = \dot\gamma_R(0) \in T_pS^3$ be the initial velocity of the corresponding great circle. By direct calculation, we have

$$\dot\gamma(t) = f_X(t)X(\gamma(t)) + f_Y(t)Y(\gamma(t)), \tag{9}$$

where, denoting $\alpha = \langle v, X\rangle, \beta = \langle v, Y\rangle$, we have

$$f_X(t) = \alpha\cos(2t\langle v, V\rangle) + \beta\sin(2t\langle v, V\rangle),$$

$$f_Y(t) = \beta\cos(2t\langle v, V\rangle) - \alpha\sin(2t\langle v, V\rangle).$$

It follows from this decomposition that

$$J(\dot\gamma(t)) = -f_Y(t)X(\gamma(t)) + f_X(t)Y(\gamma(t)). \tag{10}$$

It remains to determine the term $\nabla_{\dot\gamma}\dot\gamma$. It is well-known that for submanifolds of $\mathbb{R}^n$, the vector field $\nabla_{\dot\gamma}\dot\gamma$ corresponds to the projection of the second derivative $\ddot\gamma$ to the tangent space of the submanifold. In this case, differentiating (9) we obtain

$$\begin{aligned} \nabla_{\dot\gamma}\dot\gamma &= 2\langle v, V\rangle(f_Y(t)X(\gamma(t)) - f_X(t)Y(\gamma(t))) \\ &= -2\langle v, V\rangle\, J(\dot\gamma(t)). \end{aligned}$$

The proposition follows. □

Remark: Note that in case $p = (1, 0, 0, 0) \in S^3$, a great circle starting at p with velocity vector $v = (0, v_{y_1}, v_{x_2}, v_{y_2}) \in T_pS^3$ is given by

$$\gamma_R(t) = \left(\cos(\|v\|t), \frac{v_{y_1}}{\|v\|}\sin(\|v\|t), \frac{v_{x_2}}{\|v\|}\sin(\|v\|t), \frac{v_{y_2}}{\|v\|}\sin(\|v\|t)\right).$$

Then, the corresponding sub-Riemannian geodesic is

$$\gamma(t) = e^{-iv_{y_1}t}\gamma_R(t), \tag{11}$$

where $v_{x_2}^2 + v_{y_2}^2 = 1$, since the curve is parameterized by arc-length. It follows that the curvature is given by $\langle v, V_2\rangle = v_{y_1}$.

In [11] the problem of existence of closed sub-Riemannian geodesics is also discussed. Their result is that a complete geodesic γ in S^3 parameterized by arc-length, with curvature λ is closed if and only if $\lambda/\sqrt{1+\lambda^2} \in \mathbb{Q}$. This result can be generalized to any odd dimensional sphere.

Proposition 5. *Let $\gamma : \mathbb{R} \to S^{2n+1}$ be a complete sub-Riemannian geodesic parameterized by arc-length, with initial velocity $v \in T_pS^{2n+1}$. Then γ is closed if and only if*

$$\frac{\langle v, V_{n+1}\rangle}{\sqrt{1 + \langle v, V_{n+1}\rangle^2}} \in \mathbb{Q}.$$

Proof. The curve $\gamma : \mathbb{R} \to S^{2n+1}$ is closed if and only if for some $T > 0$

$$p = e^{-i\langle v, V_{n+1}\rangle T}\left(p\cos(\|v\|T) + \frac{v}{\|v\|}\sin(\|v\|T)\right).$$

Since $v \in T_pS^{2n+1}$, we know that v is orthogonal to the vector joining $0 \in \mathbb{R}^{2n+2}$ to p, with respect to the usual Riemannian structure of $\mathbb{R}^{2n+2}$. This means that $\sin(\|v\|T) = 0$, which forces $T = k\pi/\|v\|$, $k \in \mathbb{Z}$.

To complete the argument, we only need to see that

$$\pm e^{-ik(\langle v, V_{n+1}\rangle/\|v\|)\pi}p = p$$

if and only if

$$\frac{\langle v, V_{n+1}\rangle}{\|v\|} = \frac{\langle v, V_{n+1}\rangle}{\sqrt{1+\langle v, V_{n+1}\rangle^2}} \in \mathbb{Q},$$

where we have used the remark after Proposition 2. $\square$

5. Sub-Riemannian Geodesics on S^{4n+3}

Let us consider the sphere S^{4n+3} embedded as the boundary of the unit ball in $(n+1)-$dimensional quaternionic space $\mathbb{H}^{n+1}$. As usual, let us denote the quaternionic units as i, j, and k. There is a natural right action of $Sp(1) \cong S^3$ on $\mathbb{H}^{n+1}$, via componentwise multiplication by a quaternion of norm one. This action induces a quaternionic Hopf fibrations $S^3 \to S^{4n+3} \to \mathbb{H}P^n$, given by

$$\begin{array}{cccc} H & : & S^{4n+3} & \to & \mathbb{H}P^n \\ & & (q_0,\ldots,q_n) & \mapsto & [q_0:\ldots:q_n]. \end{array}$$

This submersion forms a principal S^3-bundle with connection given by the orthogonal complement to the vector fields

$$V^1_{n+1}(p) = -y_0\partial_{x_0}+x_0\partial_{y_0}+w_0\partial_{z_0}-z_0\partial_{w_0}-\ldots-y_n\partial_{x_n}+x_n\partial_{y_n}+w_n\partial_{z_n}-z_n\partial_{w_n},$$

$$V^2_{n+1}(p) = -z_0\partial_{x_0}-w_0\partial_{y_0}+x_0\partial_{z_0}+y_0\partial_{w_0}-\ldots-z_n\partial_{x_n}-w_n\partial_{y_n}+x_n\partial_{z_n}+y_n\partial_{w_n},$$

$$V^3_{n+1}(p) = -w_0\partial_{x_0}+z_0\partial_{y_0}-y_0\partial_{z_0}+x_0\partial_{w_0}-\ldots-w_n\partial_{x_n}-z_n\partial_{y_n}+y_n\partial_{z_n}+x_n\partial_{w_n},$$

at each $p = (x_0, y_0, z_0, w_0\ldots, x_n, y_n, z_n, w_n) \in S^{4n+3}$, with respect to the usual Riemannian metric of S^{4n+3} as embedded in $\mathbb{R}^{4(n+1)} \cong \mathbb{H}^{n+1}$. It is easy to see that the following commutation relations hold for $V^1_{n+1}, V^2_{n+1}, V^3_{n+1}$

$$[V^1_{n+1}, V^2_{n+1}] = 2V^3_{n+1}, \quad [V^2_{n+1}, V^3_{n+1}] = 2V^1_{n+1}, \quad [V^1_{n+1}, V^3_{n+1}] = -2V^2_{n+1}.$$

Thus one recovers the fact that $\text{span}\{V^1_{n+1}(p), V^2_{n+1}(p), V^3_{n+1}(p)\}$ is isomorphic as Lie algebra to $\mathfrak{sp}(1)$, the Lie algebra associated to S^3.

It is a well established fact that this distribution is bracket generating. In fact, the geometry of this spheres S^{4n+3} is known to be a quaternionic analogue of CR-geometry, see [2]. Note that the components of the vector $V^1_{n+1}(p)$ are the same as in the $\mathfrak{sp}(1)$ action $p\cdot i$. Similar statements hold for $V^2_{n+1}(p)$, $V^3_{n+1}(p)$ and $p\cdot j$, $p\cdot k$ respectively.

In order to apply Theorem 2 in this situation, it is necessary to specify the $\mathfrak{sp}(1)-$valued connection form associated to the submersion H. In this case, the connection form is given by

$$A(v) = i\langle v, V^1_{n+1}\rangle + j\langle v, V^2_{n+1}\rangle + k\langle v, V^3_{n+1}\rangle$$

where $v \in T_pS^{2n+1}$, V^α_{n+1} denotes $V^\alpha_{n+1}(p)$ $(\alpha = 1, 2, 3)$ and $\langle\cdot,\cdot\rangle$ stands for the standard inner product in $\mathbb{R}^{4(n+1)}$. Moreover, the usual Riemannian structure on S^{4n+3} is of constant bi-invariant type, since for any $q \in S^{4n+3}$ and $\xi = i\alpha + j\beta + k\gamma \in \mathfrak{sp}(1)$, $\alpha, \beta, \gamma \in \mathbb{R}$ we have

$$\begin{aligned} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} q \exp_{\mathfrak{sp}(1)}(\epsilon\xi) &= \alpha q \cdot i + \beta q \cdot j + \gamma q \cdot k \\ &= \alpha V^1_{n+1}(q) + \beta V^2_{n+1}(q) + \gamma V^3_{n+1}(q). \end{aligned}$$

Therefore, the inertia tensor is given by

$$\mathbb{I}_q(i\alpha + j\beta + k\gamma, i\tilde{\alpha} + j\tilde{\beta} + k\tilde{\gamma}) =$$
$$= \langle \alpha V_{n+1}(q)\beta V^2_{n+1}(q) + \gamma V^3_{n+1}(q), \tilde{\alpha} V_{n+1}(q)\tilde{\beta} V^2_{n+1}(q) + \tilde{\gamma} V^3_{n+1}(q)\rangle =$$
$$= \alpha\tilde{\alpha} + \beta\tilde{\beta} + \gamma\tilde{\gamma},$$

which does not depend of the point.

As for Corollary 1, we have the following result.

Corollary 2. *Let $p \in S^{4n+3} = \{(u_0, \ldots, u_n) \in \mathbb{H}^{n+1} : |u_0|^2 + \ldots + |u_n|^2 = 1\}$ and $v \in T_pS^{4n+3}$. If $\gamma_R(t) = (u_0(t), \ldots, u_n(t))$ is the great circle satisfying $\gamma_R(0) = p$ and $\dot{\gamma}_R(0) = v$, then the corresponding sub-Riemannian geodesic is given by*

$$\gamma(t) = \left(u_0(t) \cdot e^{-tA(v)}, \ldots, u_n(t) \cdot e^{-tA(v)}\right). \tag{12}$$

In Corollary 2, the quaternionic exponential is defined by

$$e^{ai+bj+ck} = \cos\sqrt{a^2+b^2+c^2} + \sin\sqrt{a^2+b^2+c^2} \cdot \frac{ai+bj+ck}{\sqrt{a^2+b^2+c^2}},$$

for $a, b, c \in \mathbb{R}$. Note that the curve $e^{-tA(v)}$ is simply the Riemannian geodesic in S^3 starting at the identity of the group $e = (1, 0, 0, 0)$, with initial velocity vector $(0, -\langle v, V^1_{n+1}\rangle, -\langle v, V^2_{n+1}\rangle, -\langle v, V^3_{n+1}\rangle)$.

Corollary 2 implies immediate analogues to Proposition 1 and to Proposition 5, which we state for the sake of completeness. Proofs are adaptations of the aforementioned Propositions.

Proposition 6. *The set of sub-Riemannian geodesics in S^{4n+3} arising from great circles $\gamma_R(t)$ such that $\dot{\gamma}_R(0)$ is orthogonal to V^1_{n+1}, V^2_{n+1} and V^3_{n+1} is diffeomorphic to $\mathbb{H}P^n$.*

Proposition 7. *Let $\gamma : \mathbb{R} \to S^{4n+3}$ be a complete sub-Riemannian geodesic parameterized by arc-length, with initial velocity $v \in T_pS^{2n+1}$. Then γ is closed if and only if*

$$\frac{\langle v, V^1_{n+1}\rangle}{\|v\|^2}, \frac{\langle v, V^2_{n+1}\rangle}{\|v\|^2}, \frac{\langle v, V^3_{n+1}\rangle}{\|v\|^2} \in \mathbb{Q}.$$

In analogy with Proposition 2, let us consider a similar statement in the case of the spheres S^{4n+3}.

Proposition 8. *For a horizontal sub-Riemannian geodesic of the form*

$$\gamma(t) = \big(w_0(t) \cdot e^{-tA(v)}, \dots, w_n(t) \cdot e^{-tA(v)}\big)$$

the following equation holds

$$\|\dot\gamma(t)\|^2 + \|A(v)\|^2 = \|v\|^2,$$

where $\|A(v)\|^2 = \langle v, V^1_{n+1}\rangle^2 + \langle v, V^2_{n+1}\rangle^2 + \langle v, V^3_{n+1}\rangle^2$.

Proof. Recall that if γ is a sub-Riemannian geodesic, then the length of the velocity vector $\|\dot\gamma(t)\|$ does not depend on t. Thus without loss of generality we can assume $t = 0$. Let us introduce the following notation

$$\begin{aligned} p &= \gamma(0) = (x_0, y_0, z_0, w_0, \dots, x_n, y_n, z_n, w_n) \in S^{4n+3}, \\ v &= \dot\gamma_R(0) = (v_{x_0}, v_{y_0}, v_{z_0}, v_{w_0}, \dots, v_{x_n}, v_{y_n}, v_{z_n}, v_{w_n}) \in T_pS^{4n+3}. \end{aligned}$$

Differentiating equation (12) and evaluating at $t = 0$, we have

$$\dot\gamma(0) = v - \langle v, V^1_{n+1}\rangle V^1_{n+1} - \langle v, V^2_{n+1}\rangle V^2_{n+1} - \langle v, V^3_{n+1}\rangle V^3_{n+1}.$$

The orthogonality of the vector fields $V^1_{n+1}, V^2_{n+1}, V^3_{n+1}$ implies the desired relation.

□

6. Curvature of sub-Riemannian geodesics on $\mathbf{H}^1$

The proof of Proposition 3 is given in [17] for the case of the three dimensional Heisenberg group. As mentioned in [11], the proof for the case of the sub-Riemannian three dimensional sphere is basically the same. The authors have pointed out, in private communication, that the same result holds for all three Sasakian pseudo-Hermitian manifolds.

Note that that if M is either the Heisenberg group of topological dimension 3 or the sphere S^3, with Reeb vector field R, then the quotient vector bundle

$$TM/\mathrm{span}\{R\} \to M$$

is trivial. We have not been able to show that the corresponding vector bundle

$$TS^7/\mathrm{span}\{V^1_2, V^2_2, V^3_3\} \to S^7$$

is trivial, which makes difficult to find an analogous argument to the one employed in [11].

The main goal of this section is to find an analogue to Proposition 3 for the Gromov-Margulis-Mitchell-Mostow tangent cone of S^7, see [10, 13, 14, 15], which corresponds to the seven dimensional quaternionic $H-$type group $\mathbf{H}^1$, as presented in [4]. Observe that the idea of studying the tangent cone before the sub-Riemannian manifold of interest corresponds to the case in [17], since the three dimensional Heisenberg group is the tangent cone to the sub-Riemannian S^3. We will study whether this method extends to S^7 in a forthcoming paper.

6.1. **The quaternionic $H-$type group $\mathbf{H}^1$.** Let us consider the 4×4 matrices $\mathcal{I},\mathcal{J}$ and $\mathcal{K}$, given by

$$\mathcal{I}=\begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix},\quad \mathcal{J}=\begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix},$$

$$\mathcal{K}=\begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}.$$

Note that $\mathcal{I},\mathcal{J}$ and $\mathcal{K}$ are a fixed representation of the quaternion units, *i.e.* if $\mathcal{U}$ denotes the identity matrix of size 4×4, then $\operatorname{span}\{\mathcal{U},\mathcal{I},\mathcal{J},\mathcal{K}\}\cong\mathbb{H}$ as algebras via the isomorphism

$$\varphi:\operatorname{span}\{\mathcal{U},\mathcal{I},\mathcal{J},\mathcal{K}\}\to\mathbb{H}$$

given by $\varphi(\mathcal{U})=1,\varphi(\mathcal{I})=i,\varphi(\mathcal{J})=j,\varphi(\mathcal{K})=k$ and extended by linearity.

The seven dimensional quaternionic $H-$type group $\mathbf{H}^1$ corresponds to the manifold $\mathbb{R}^4\oplus\mathbb{R}^3$ with the group operation $\circ$ defined by

$$(x,z)\circ(x',z')=\Big(x+x',z_{\mathcal{I}}+z'_{\mathcal{I}}+\frac{1}{2}x'^T\mathcal{I}x,$$
$$z_{\mathcal{J}}+z'_{\mathcal{J}}+\frac{1}{2}x'^T\mathcal{J}x,z_{\mathcal{K}}+z'_{\mathcal{K}}+\frac{1}{2}x'^T\mathcal{K}x\Big)$$

where x,y,z are column vectors and x'^T,y'^T,z'^T are row vectors in $\mathbb{R}^4$.

The Lie algebra $\mathfrak{h}^1$ corresponding to $\mathbf{H}^1$ is spanned by the left invariant vector fields

$$X_1(x,z)=\frac{\partial}{\partial x_1}+\frac{1}{2}\left(+x_2\frac{\partial}{\partial z_{\mathcal{I}}}-x_4\frac{\partial}{\partial z_{\mathcal{J}}}-x_3\frac{\partial}{\partial z_{\mathcal{K}}}\right),$$
$$X_2(x,z)=\frac{\partial}{\partial x_2}+\frac{1}{2}\left(-x_1\frac{\partial}{\partial z_{\mathcal{I}}}-x_3\frac{\partial}{\partial z_{\mathcal{J}}}+x_4\frac{\partial}{\partial z_{\mathcal{K}}}\right),$$
$$X_3(x,z)=\frac{\partial}{\partial x_3}+\frac{1}{2}\left(+x_4\frac{\partial}{\partial z_{\mathcal{I}}}+x_2\frac{\partial}{\partial z_{\mathcal{J}}}+x_1\frac{\partial}{\partial z_{\mathcal{K}}}\right),$$
$$X_4(x,z)=\frac{\partial}{\partial x_4}+\frac{1}{2}\left(-x_3\frac{\partial}{\partial z_{\mathcal{I}}}+x_1\frac{\partial}{\partial z_{\mathcal{J}}}-x_2\frac{\partial}{\partial z_{\mathcal{K}}}\right),$$
$$Z_{\mathcal{I}}(x,z)=\frac{\partial}{\partial z_{\mathcal{I}}},\quad Z_{\mathcal{J}}(x,z)=\frac{\partial}{\partial z_{\mathcal{J}}},\quad Z_{\mathcal{K}}(x,z)=\frac{\partial}{\partial z_{\mathcal{K}}}.$$

at a point $(x,z)=(x_1,x_2,x_3,x_4,z_{\mathcal{I}},z_{\mathcal{J}},z_{\mathcal{K}})\in\mathbf{H}^1$. A Riemannian metric $\langle\cdot,\cdot\rangle$ in $\mathbf{H}^1$ is declared so that $X_1,\dots,X_4,Z_{\mathcal{I}},\dots,Z_{\mathcal{K}}$ is an orthonormal frame at each $(x,z)\in\mathbf{H}^1$. The sub-Riemannian structure on $\mathbf{H}^1$ we are interested in is defined by the left invariant distribution $\mathcal{D} = \operatorname{span}\{X_1, X_2, X_3, X_4\}$ and the restriction of the metric previously defined.

Observe that $\mathcal{D}$ is bracket generating of step two. In fact, we have the commutator relations

$$\begin{array}{rcccl} [X_1, X_2] & = & [X_3, X_4] & = & -Z_{\mathcal{I}}, \\ [X_2, X_3] & = & [X_1, X_4] & = & Z_{\mathcal{J}}, \\ [X_1, X_3] & = & [X_4, X_2] & = & Z_{\mathcal{K}}. \end{array} \tag{13}$$

All the remaining commutators between the chosen basis of $\mathfrak{h}^1$ vanish.

From the well-known Koszul formula for the Levi-Civita connection associated to the metric $\langle\cdot,\cdot\rangle$

$$\langle Z, \nabla_Y X\rangle = \frac{1}{2}(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle -$$

$$-\langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle - \langle [X, Y], Z\rangle),$$

see for example [8], the orthonormality of the basis $\{X_1, \ldots, X_4, Z_{\mathcal{I}}, \ldots, Z_{\mathcal{K}}\}$, and equations (13) we get that

$$\langle X_b, \nabla_{X_a} Z_r\rangle = -\frac{1}{2}\langle [X_a, X_b], Z_r\rangle, \quad \langle Z_s, \nabla_{X_a} Z_r\rangle = 0,$$

for any $a, b = 1, \ldots, 4$, $r, s = \mathcal{I}, \mathcal{J}, \mathcal{K}$. This translates to the equation

$$\nabla_{X_a} Z_r = -\frac{1}{2}\sum_{b=1}^{4}\langle [X_a, X_b], Z_r\rangle X_b, \tag{14}$$

which reduces to the following identities

$$\nabla_{X_1} Z_{\mathcal{I}} = \frac{1}{2}X_2, \quad \nabla_{X_2} Z_{\mathcal{I}} = -\frac{1}{2}X_1, \quad \nabla_{X_3} Z_{\mathcal{I}} = \frac{1}{2}X_4, \quad \nabla_{X_4} Z_{\mathcal{I}} = -\frac{1}{2}X_3,$$

$$\nabla_{X_1} Z_{\mathcal{J}} = -\frac{1}{2}X_4, \quad \nabla_{X_2} Z_{\mathcal{J}} = -\frac{1}{2}X_3, \quad \nabla_{X_3} Z_{\mathcal{J}} = \frac{1}{2}X_2, \quad \nabla_{X_4} Z_{\mathcal{J}} = \frac{1}{2}X_1,$$

$$\nabla_{X_1} Z_{\mathcal{K}} = -\frac{1}{2}X_3, \quad \nabla_{X_2} Z_{\mathcal{K}} = \frac{1}{2}X_4, \quad \nabla_{X_3} Z_{\mathcal{K}} = \frac{1}{2}X_1, \quad \nabla_{X_4} Z_{\mathcal{K}} = -\frac{1}{2}X_2.$$

Therefore, it follows that the maps $J_r : \mathcal{D} \to \mathcal{D}$ defined by

$$J_r(X) = 2\nabla_X Z_r, \quad r = \mathcal{I}, \mathcal{J}, \mathcal{K},$$

are almost complex structures. Note that the equation

$$\langle J_r(U_1), U_2\rangle + \langle U_1, J_r(U_2)\rangle = 0 \tag{15}$$

holds for every $r = \mathcal{I}, \mathcal{J}, \mathcal{K}$ and every $U_1, U_2 \in \mathcal{D}$. Note in particular that equation (15) implies that $\langle U, J_r(U)\rangle = 0$ for all $U \in \mathcal{D}$.

6.2. **A variational argument.** Consider a manifold M and let $\mathcal{H} \subset TM$ be a distribution. A variation of a curve $\gamma : [a, b] \to M$ is a C^2-map $\tilde{\gamma} : I_1 \times I_2 \to M$, where I_1, I_2 are open intervals, $0 \in I_2$ and $\tilde{\gamma}(s, 0) = \gamma(s)$. In what follows, we will denote $\tilde{\gamma}(s, \varepsilon) = \gamma_\varepsilon(s)$.

Let W_ε be the vector field along γ_ε given by

$$W_\varepsilon(s) = \frac{\partial \gamma_\tau(s)}{\partial \tau}\bigg|_{\tau=\varepsilon} = \frac{\partial \gamma}{\partial \tau}(s, \varepsilon).$$

Note that the vector fields W_ε and $\dot{\gamma}_\varepsilon$ commute

$$[W_\varepsilon, \dot{\gamma}_\varepsilon] = \left[\frac{\partial \gamma}{\partial \varepsilon}(s, \varepsilon), \frac{\partial \gamma}{\partial s}(s, \varepsilon)\right] = \left[\frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial s}\right]\gamma(s, \varepsilon) = 0.$$

A variation γ_ε of a horizontal curve γ is called admissible if all curves $\gamma_\varepsilon : I_1 \to M$ are horizontal, $\gamma_\varepsilon(a) = \gamma(a)$ and $\gamma_\varepsilon(b) = \gamma(b)$ for all $\varepsilon \in I_2$. Observe that for an admissible variation of γ, the vector field W_0 vanishes at the endpoints of γ: $W_0(\gamma(a)) = W_0(\gamma(b)) = 0$.

Let us study an admissible variation γ_ε of a horizontal curve γ in the case of $\mathbf{H}^1$, with the Riemannian metric defined in the previous Subsection. Since the variation is admissible, we have

$$\langle \dot{\gamma}_\varepsilon, Z_{\mathcal{I}} \rangle = \langle \dot{\gamma}_\varepsilon, Z_{\mathcal{J}} \rangle = \langle \dot{\gamma}_\varepsilon, Z_{\mathcal{K}} \rangle = 0.$$

In what follows, for an arbitrary vector field X on $\mathbf{H}^1$, we will denote by X_H and X_V the orthogonal projections of X to the horizontal distribution $\mathcal{D} \subset T\mathbf{H}^1$ and the vertical bundle $\mathrm{span}\{Z_{\mathcal{I}}, Z_{\mathcal{J}}, Z_{\mathcal{K}}\}$ respectively.

The horizontality conditions $\langle \dot{\gamma}_\varepsilon, Z_r \rangle = 0$, for $r = \mathcal{I}, \mathcal{J}, \mathcal{K}$, yield

$$\begin{aligned}
0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \langle \dot{\gamma}_\varepsilon, Z_r \rangle &= \langle \nabla_{W_0} \dot{\gamma}, Z_r \rangle + \langle \dot{\gamma}, \nabla_{W_0} Z_r \rangle \\
&= \langle \nabla_{\dot{\gamma}} W_0, Z_r \rangle + \langle \dot{\gamma}, \nabla_{W_{0_H}} Z_r \rangle \\
&= \dot{\gamma}\langle W_0, Z_r \rangle - \langle W_0, \nabla_{\dot{\gamma}} Z_r \rangle + \langle \dot{\gamma}, J_r(W_{0_H}) \rangle \\
&= \dot{\gamma}\langle W_0, Z_r \rangle - \langle W_{0_H}, J_r(\dot{\gamma}) \rangle - \langle J_r(\dot{\gamma}), W_{0_H} \rangle \\
&= \dot{\gamma}\langle W_0, Z_r \rangle - 2\langle W_{0_H}, J_r(\dot{\gamma}) \rangle,
\end{aligned}$$

where we have used equation (15) and $\nabla_{Z_s} Z_r = 0$.

In fact the converse statement also holds.

Lemma 1. *Let W be any C^1 vector field along γ such that $W(\gamma(a)) = W(\gamma(b)) = 0$ and that satisfies*

$$0 = \dot{\gamma}\langle W, Z_r \rangle - 2\langle W_H, J_r(\dot{\gamma}) \rangle.$$

Then there exists an admissible variation γ_ε of γ such that

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \gamma(s, \varepsilon) = W.$$

Proof. Let us decompose $W = f\dot\gamma + \widetilde{W}$, with $\widetilde{W} \perp \dot\gamma$ and $f(\gamma(a)) = f(\gamma(b)) = 0$. With this definition, we have

$$\langle W, \dot\gamma\rangle = f, \quad \langle W, J_r(\dot\gamma)\rangle = \langle \widetilde{W}, J_r(\dot\gamma)\rangle, \quad \langle W, Z_r\rangle = \langle \widetilde{W}, Z_r\rangle.$$

Observe that the term $f\dot\gamma$ will not contribute to any admissible variation, therefore we can assume that $W \perp \dot\gamma$. Let $s \in I_1$ and $\varepsilon > 0$ sufficiently small. Define the mapping

$$F(s,\varepsilon) = \exp_{\gamma(s)}(\varepsilon W(s)),$$

where exp is the exponential map associated to the metric $\langle\cdot,\cdot\rangle$ of $\mathbf{H}^1$.

If W is horizontal in some nonempty interval $I \subset I_1$, then $W = W_H$ and also $\langle W_H, J_r(\dot\gamma)\rangle = \frac{1}{2}\dot\gamma\langle W_H, Z_r\rangle = 0$. This implies $W_H = \lambda(p)\dot\gamma$, but since $W_H \perp \dot\gamma$, then $W_H = 0$.

If $W(s_0)$ is not horizontal, then $F(s,\varepsilon)$ defines locally a surface which is foliated by horizontal curves and it is transversal to the horizontal distribution, since it contains curves in nonhorizontal directions. This implies there exists a C^2 function $g(s,\varepsilon)$ such that

$$\gamma_\varepsilon(s) = \exp_{\gamma(s)}(g(s,\varepsilon)W(s))$$

is a horizontal curve. Choosing g such that $\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} f(s_0,\varepsilon) = 1$, we obtain an admissible variation γ_ε of γ with associated vector field W. $\square$

With this result at hand, we can formulate the main theorem of this section.

Theorem 3. *Let $\gamma : [a,b] \to \mathbf{H}^1$ be a horizontal curve, parameterized by arc length. Then γ is a critical point of the length functional if and only if there exist $\lambda_{\mathcal{I}}, \lambda_{\mathcal{J}}, \lambda_{\mathcal{K}} \in \mathbb{R}$ satisfying the second order differential equation*

$$\nabla_{\dot\gamma}\dot\gamma - 2\sum_{r=\mathcal{I},\mathcal{J},\mathcal{K}} \lambda_r J_r(\dot\gamma) = 0. \tag{16}$$

Proof. Let $\gamma : I = [a,b] \to \mathbf{H}^1$ be a horizontal curve, parameterized by arc length, and let γ_ε be an admissible variation of γ, with vector field U. The first variation of the length functional, see [6], is given by

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L(\gamma_\varepsilon) = -\int_I \langle \nabla_{\dot\gamma}\dot\gamma, U\rangle. \tag{17}$$

Suppose γ is a critical point of the first variation, that is

$$\int_I \langle \nabla_{\dot\gamma}\dot\gamma, U\rangle = 0.$$

The condition $\|\dot{\gamma}\| = 1$ implies $\langle \nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}\rangle = 0$. Since $\langle \dot{\gamma}, Z_r\rangle = 0$ for $r = \mathcal{I}, \mathcal{J}, \mathcal{K}$, we have

$$\begin{aligned} 0 = \dot{\gamma}\langle \dot{\gamma}, Z_r\rangle &= \langle \nabla_{\dot{\gamma}}\dot{\gamma}, Z_r\rangle + \langle \dot{\gamma}, \nabla_{\dot{\gamma}} Z_r\rangle \\ &= \langle \nabla_{\dot{\gamma}}\dot{\gamma}, Z_r\rangle + \langle \dot{\gamma}, J_r(\dot{\gamma})\rangle \\ &= \langle \nabla_{\dot{\gamma}}\dot{\gamma}, Z_r\rangle. \end{aligned}$$

Therefore, counting dimensions

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \sum_{r=\mathcal{I},\mathcal{J},\mathcal{K}} g_r(\gamma) J_r(\dot{\gamma}). \tag{18}$$

In order to prove that the functions g_r are constant, fix three C^1 functions $f_{\mathcal{I}}, f_{\mathcal{J}}, f_{\mathcal{K}} : I \to \mathbb{R}$ such that $f_r(a) = f_r(b) = 0$ and $\displaystyle\int_I f_r = 0$ for $r = \mathcal{I}, \mathcal{J}, \mathcal{K}$. Consider a vector field $\tilde{U}$ such that $\tilde{U}_H = \sum_{r=\mathcal{I},\mathcal{J},\mathcal{K}} f_r J_r(\dot{\gamma})$ and $\langle \tilde{U}, Z_r\rangle(s) = 2\int_a^s f_r(t)dt$. We claim that $\tilde{U}$ satisfies

$$\dot{\gamma}\langle \tilde{U}, Z_r\rangle = 2\langle \tilde{U}_H, J_r(\dot{\gamma})\rangle,$$

for $r = \mathcal{I}, \mathcal{J}, \mathcal{K}$. To see this, observe that

$$\dot{\gamma}\langle \tilde{U}, Z_r\rangle = \frac{d}{ds}\left(2\int_a^s f_r(t)dt\right) = 2f_r(s)$$

and also

$$2\langle \tilde{U}_H, J_r(\dot{\gamma})\rangle = 2\left\langle \sum_{s=\mathcal{I},\mathcal{J},\mathcal{K}} f_s J_s(\dot{\gamma}), J_r(\dot{\gamma})\right\rangle = 2f_r(s).$$

Thus, by Lemma 1, we can conclude that $\tilde{U}$ is a vector field for an admissible variation of γ. By the variational identity (17), we obtain the equality

$$0 = \int_I \langle \nabla_{\dot{\gamma}}\dot{\gamma}, \tilde{U}\rangle = \sum_{r=\mathcal{I},\mathcal{J},\mathcal{K}} \int_I f_r \langle \nabla_{\dot{\gamma}}\dot{\gamma}, J_r(\dot{\gamma})\rangle,$$

which is valid for any three functions with mean zero. This implies that the functions $\langle \nabla_{\dot{\gamma}}\dot{\gamma}, J_r(\dot{\gamma})\rangle$ are constant, and thus we obtain equation (16), for suitable constants $\lambda_{\mathcal{I}}, \lambda_{\mathcal{J}}, \lambda_{\mathcal{K}} \in \mathbb{R}$.

Conversely, let us assume that γ is a horizontal curve, such that $\|\dot{\gamma}\| = 1$ and it satisfies the differential equation (16), for some $\lambda_{\mathcal{I}}, \lambda_{\mathcal{J}}, \lambda_{\mathcal{K}} \in \mathbb{R}$. We need to show that

$$\int_I \langle \nabla_{\dot{\gamma}}\dot{\gamma}, U\rangle = 0$$

for any C^1-smooth vector field U, vanishing at the endpoints of γ and satisfying

$$\dot{\gamma}\langle U, Z_r\rangle = 2\langle U_H, J_r(\dot{\gamma})\rangle,$$

where $r = \mathcal{I}, \mathcal{J}, \mathcal{K}$.

Let us write $U = U_H + U_V = U_H + \sum_{r=\mathcal{I},\mathcal{J},\mathcal{K}} g_r Z_r$, where $g_r(\gamma(a)) = g_r(\gamma(b)) = 0$, then

$$\begin{aligned}\int_I \langle \nabla_{\dot\gamma}\dot\gamma, U\rangle &= -2\sum_{r=\mathcal{I},\mathcal{J},\mathcal{K}} \lambda_r \int_I \langle J_r(\dot\gamma), U\rangle = -2\sum_{r=\mathcal{I},\mathcal{J},\mathcal{K}} \lambda_r \int_I \langle J_r(\dot\gamma), U_H\rangle \\ &= -\sum_{r=\mathcal{I},\mathcal{J},\mathcal{K}} \lambda_r \int_I \dot\gamma\langle U, Z_r\rangle = -\sum_{r=\mathcal{I},\mathcal{J},\mathcal{K}} \lambda_r \int_I \dot\gamma\langle U_V, Z_r\rangle \\ &= -\sum_{r=\mathcal{I},\mathcal{J},\mathcal{K}} \lambda_r \int_I \dot\gamma(g_r) = -\sum_{r=\mathcal{I},\mathcal{J},\mathcal{K}} \lambda_r \int_a^b \frac{d}{dt}(g_r(\gamma(t))) = 0.\end{aligned}$$

□

7. The intrinsic sub-Laplacian for S^7 with growth vector $(6,1)$

In [1] the authors presented an intrinsic form of the sub-Laplacian, by means of Popp's measure μ_{sR}, introduced in [15]. The aim of this section is to construct this differential operator for the case of S^7 endowed with the contact distribution, introduced in Section 3.

7.1. **Construction of the intrinsic sub-Laplacian.** Let $(M, \mathcal{H}, \langle\cdot,\cdot\rangle_{sR})$ be a sub-Riemannian manifold, where $\mathcal{H}$ is a regular distribution. The basic idea is to define the intrinsic sub-Laplacian $\Delta_{sR} f$ of a function $f: M \to \mathbb{R}$ of class C^2, in analogy to the Riemannian case. To do this, let us define the horizontal gradient $\nabla_{sR} f$ by the equation

$$\langle \nabla_{sR} f(p), v\rangle_{sR} = d_p f(v), \tag{19}$$

and the sub-Riemannian divergence $\mathrm{div}_{sR} X$ of a horizontal vector field X by

$$\mathrm{div}_{sR} X \mu_{sR} = L_X \mu_{sR}, \tag{20}$$

where $\mu_{sR} \in \bigwedge^n(T^*M)$ is a fixed non-vanishing $n-$form, known as Popp's volume form, and L_X denotes the Lie derivative in the direction of X. The intrinsic sub-Laplacian is given by

$$\Delta_{sR} f = \mathrm{div}_{sR}(\nabla_{sR} f). \tag{21}$$

For full details about its construction, see [1, 15].

Remark: In the Riemannian case this definition coincides with the classical definition of the Laplacian, see for example [18]. As pointed out in [1], the regularity hypothesis over the distribution cannot be avoided since for example, in the case of the Grushin plane, the operator (21) is not hypoelliptic.

Let $\{X_1, \ldots, X_k\}$ be a local orthonormal basis of $\mathcal{H} \subset TM$ and consider the corresponding dual basis $\{dX_1, \ldots, dX_k\}$. It is possible to find vector fields $\{X_{k+1}, \ldots, X_n\}$ such that $\operatorname{span}\{X_1, \ldots, X_n\} = TM$ and such that Popp's volume form is locally given by

$$\mu_{sR} = dX_1 \wedge \ldots \wedge dX_k \wedge dX_{k+1} \wedge \ldots \wedge dX_n. \tag{22}$$

In this setting, the sub-Laplacian $\Delta_{sR} f$ can be written explicitly as

$$\Delta_{sR} f = \sum_{r=1}^{k} \left(L^2_{X_r} f + L_{X_r} f \sum_{s=1}^{n} dX_s([X_r, X_s]) \right). \tag{23}$$

7.2. **Examples.** The case of the intrinsic sub-Laplacian for S^3 is implied by the following result, characterizing Popp's volume form for contact manifolds of dimension 3.

Proposition 9 ([1, 15])**.** *Let M be a three dimensional orientable contact manifold with a sub-Riemannian metric defined on its contact distribution. Let $\{X_1, X_2\}$ a local orthonormal frame for its contact distribution. Let $X_3 = [X_1, X_2]$ and $\{dX_1, dX_2, dX_3\}$ be the dual basis to $\{X_1, X_2, X_3\}$. Then the form $dX_1 \wedge dX_2 \wedge dX_3$ is an intrinsic volume form.*

In particular, for the sphere S^3 endowed with the contact distribution generated by the globally defined vector fields (8), with commutator

$$[X, Y](x) = 2V(x) = 2(-x_1 \partial_{x_0} + x_0 \partial_{x_1} - x_3 \partial_{x_2} + x_2 \partial_{x_3}),$$

Popp's volume form, as constructed above, is $2dX \wedge dY \wedge dV$, and the intrinsic sub-Laplacian is given by

$$\Delta_{sR} f = (X^2 + Y^2) f.$$

In general, we can extend the previous result to construct locally Popp's volume form over contact manifolds of arbitrary dimension. Let M be a contact manifold of dimension $2n+1$, with contact form ω and contact distribution $\xi = \ker \omega$. The distribution ξ is bracket generating of step two, see [9]. Assume that M has a Riemannian metric g such that, in a neighborhood of each $p \in M$, there is an orthonormal basis $B = \{v_1, \ldots, v_{2n}, v_{2n+1}\}$ for T_pM satisfying $\xi_p = \operatorname{span}\{v_1, \ldots, v_{2n}\}$. Following the construction in [15] we have that Popp's volume form in this case is given locally by

$$\mu_{sR} = \pi_1 \wedge \ldots \wedge \pi_{2n+1}, \tag{24}$$

where $B^* = \{\pi_1, \ldots, \pi_{2n+1}\}$ is the dual basis for B.

In the case of the contact structure of S^7, let us consider the vector fields $X_1, \ldots, X_7$ presented in the Appendix. Since the vector fields X_1 and V_4 from equation (2) coincide, the contact distribution on S^7 introduced in Section 3 corresponds to

$$\mathcal{H} = \ker \omega = \operatorname{span}\{X_2, \ldots, X_7\}.$$

In this context we have the following

Theorem 4. *Let $\mathcal{H}$ be the contact distribution for S^7 and $\langle\cdot,\cdot\rangle_{sR}$ the restriction of the usual Riemannian metric in $\mathbb{R}^8$ to $\mathcal{H}$. Then the intrinsic sub-Laplacian of $(S^7,\mathcal{H},\langle\cdot,\cdot\rangle_{sR})$ is given by the sum of squares*

$$\Delta_{sR}=\sum_{a=2}^{7}X_a^2.$$

Proof. The construction of Popp's measure leads to the globally defined n form

$$\mu_{sR}=dX_1\wedge\ldots\wedge dX_7,$$

which is precisely the Riemannian volume form of S^7. Simple calculations show that

$$dX_b([X_a,X_b])=\langle X_b,[X_a,X_b]\rangle_{sR}=0,\quad a=2,\ldots,7\quad b=1,\ldots 7. \tag{25}$$

The theorem follows from formula (23). □

Remark: A complete list of the commutators $[X_a,X_b]$, for $a<b$, can be found in [9, Section 8]. This list can be used to check equation (25) directly.

8. Heat operator for S^7 with growth vector $(6,1)$

The aim of this section is to show that the above constructed operator Δ_{sR} commutes with the operator X_1^2. A similar observation was exploited to study the heat operator for the sub-Riemannian structure of $SU(2)\cong S^3$ in [3].

The main result of this Section is formulated as follows.

Theorem 5. *The operators Δ_{sR} and X_1^2 commute.*

Proof. Let us introduce the following change of coordinates for S^7:

$$\begin{array}{rcl} x_0+ix_1 &=& e^{i\xi_1}\cos\eta_1\cos\psi\\ x_2+ix_3 &=& e^{i\xi_2}\sin\eta_1\cos\psi\\ x_4+ix_5 &=& e^{i\xi_3}\cos\eta_2\sin\psi\\ x_6+ix_7 &=& e^{i\xi_4}\sin\eta_2\sin\psi \end{array} \tag{26}$$

By the chain rule, the symbol of the sub-Laplacian $\Delta_{sR}=X_2^2+\ldots+X_7^2$ is a quadratic form with matrix

$$\begin{pmatrix} h_1(\eta_1,\psi) & -1 & -1 & -1 & 0 & 0 & 0\\ -1 & h_2(\eta_1,\psi) & -1 & -1 & 0 & 0 & 0\\ -1 & -1 & h_3(\eta_2,\psi) & -1 & 0 & 0 & 0\\ -1 & -1 & -1 & h_4(\eta_2,\psi) & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \sec^2\psi & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \csc^2\psi & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the coefficient functions h_1,h_2,h_3 and h_4 are given by

$$
\begin{aligned}
h_1(\eta_1,\psi) &= -\frac{\sec^2(\eta_1)\sec^2(\psi)}{8}\Big(-6+2\cos(2\eta_1)+\cos(2(\eta_1-\psi))+\\
&\qquad +2\cos(2\psi)+\cos(2(\eta_1+\psi))\Big),\\
h_2(\eta_1,\psi) &= \frac{\csc^2(\eta_1)\sec^2(\psi)}{8}\Big(6+2\cos(2\eta_1)+\cos(2(\eta_1-\psi))-\\
&\qquad -2\cos(2\psi)+\cos(2(\eta_1+\psi))\Big),\\
h_3(\eta_2,\psi) &= \frac{\sec^2(\eta_2)\csc^2(\psi)}{8}\Big(6-2\cos(2\eta_2)+\cos(2(\eta_2-\psi))+\\
&\qquad +2\cos(2\psi)+\cos(2(\eta_2+\psi))\Big),\\
h_4(\eta_2,\psi) &= -\frac{\csc^2(\eta_2)\csc^2(\psi)}{8}\Big(-6-2\cos(2\eta_2)+\cos(2(\eta_2-\psi))-\\
&\qquad -2\cos(2\psi)+\cos(2(\eta_2+\psi))\Big).
\end{aligned}
$$

Observe that $h_1,\dots,h_4$ are independent of $\xi_1,\dots,\xi_4$. On the other hand, the vector field X_1, written in the new coordinates, becomes

$$X_1 = \partial_{\xi_1} + \partial_{\xi_2} + \partial_{\xi_3} + \partial_{\xi_4}.$$

Since the coefficients of Δ_{sR} are independent of the variables ξ_1,ξ_2,ξ_3 and ξ_4, it is clear that the operators Δ_{sR} and X_1 commute. The Theorem follows. □

Let us denote by $e^{-t\Delta_{sR}}$ the semigroup of operators acting on $L^2_{\mu_{sR}}$, with infinitesimal generator Δ_{sR}. The operator $e^{-t\Delta_{sR}}$ is known as the sub-Riemannian heat operator. As a consequence of Theorem 5, we get the announced result.

Corollary 3. *Denoting by Δ_{S^7} the Laplace-Beltrami operator in S^7 with respect to the usual Riemannian structure, we have that*

$$e^{-t\Delta_{S^7}} = e^{-t(\Delta_{sR}+X_1^2)} = e^{-t\Delta_{sR}}e^{-tX_1^2}.$$

Proof. Since $\Delta_{S^7} = \Delta_{sR}+X_1^2$, we have by the commutativity of the operators

$$e^{-t\Delta_{S^7}} = e^{-t(\Delta_{sR}+X_1^2)} = e^{-t\Delta_{sR}}e^{-tX_1^2}, \tag{27}$$

yielding to the stated result. □

The theory of unbounded operators allows us to rephrase the result in Corollary 3 as:

Corollary 4. *The sub-Riemannian heat operator $e^{-t\Delta_{sR}}$ is given by*

$$e^{-t\Delta_{sR}} = e^{-t\Delta_{S^7}}e^{tX_1^2}.$$

9. Appendix: Tangent vector fields to S^7

Octonion multiplication induces the following orthonormal basis of TS^7 with respect to the restriction of the inner product $\langle \cdot, \cdot \rangle$ from $\mathbb{R}^8$ to the tangent space T_pS^7 at each $p \in S^7$.

$$
\begin{aligned}
X_1(x) &= -x_1\partial_{x_0} + x_0\partial_{x_1} - x_3\partial_{x_2} + x_2\partial_{x_3} - x_5\partial_{x_4} + x_4\partial_{x_5} - x_7\partial_{x_6} + x_6\partial_{x_7} \\
X_2(x) &= -x_2\partial_{x_0} + x_3\partial_{x_1} + x_0\partial_{x_2} - x_1\partial_{x_3} - x_6\partial_{x_4} + x_7\partial_{x_5} + x_4\partial_{x_6} - x_5\partial_{x_7} \\
X_3(x) &= -x_3\partial_{x_0} - x_2\partial_{x_1} + x_1\partial_{x_2} + x_0\partial_{x_3} + x_7\partial_{x_4} + x_6\partial_{x_5} - x_5\partial_{x_6} - x_4\partial_{x_7} \\
X_4(x) &= -x_4\partial_{x_0} + x_5\partial_{x_1} + x_6\partial_{x_2} - x_7\partial_{x_3} + x_0\partial_{x_4} - x_1\partial_{x_5} - x_2\partial_{x_6} + x_3\partial_{x_7} \\
X_5(x) &= -x_5\partial_{x_0} - x_4\partial_{x_1} - x_7\partial_{x_2} - x_6\partial_{x_3} + x_1\partial_{x_4} + x_0\partial_{x_5} + x_3\partial_{x_6} + x_2\partial_{x_7} \\
X_6(x) &= -x_6\partial_{x_0} + x_7\partial_{x_1} - x_4\partial_{x_2} + x_5\partial_{x_3} + x_2\partial_{x_4} - x_3\partial_{x_5} + x_0\partial_{x_6} - x_1\partial_{x_7} \\
X_7(x) &= -x_7\partial_{x_0} - x_6\partial_{x_1} + x_5\partial_{x_2} + x_4\partial_{x_3} - x_3\partial_{x_4} - x_2\partial_{x_5} + x_1\partial_{x_6} + x_0\partial_{x_7}.
\end{aligned}
$$

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3.3 Paper C

AN INTRINSIC FORMULATION OF THE ROLLING MANIFOLDS PROBLEM

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ABSTRACT. We present an intrinsic formulation of the kinematic problem of two $n-$dimensional manifolds rolling one on another without twisting or slipping. We determine the configuration space of the system, which is an $\frac{n(n+3)}{2}-$dimensional manifold. The conditions of no-twisting and no-slipping are encoded by means of a distribution of rank n. We compare the intrinsic point of view versus the extrinsic one. We also show that the kinematic system of rolling the n-dimensional sphere over $\mathbb{R}^n$ is controllable. In contrast with this, we show that in the case of $SE(3)$ rolling over $\mathfrak{se}(3)$ the system is not controllable, since the configuration space of dimension 27 is foliated by submanifolds of dimension 12.

1. INTRODUCTION

Rolling surfaces without slipping or twisting is one of the classical kinematic problems that in recent years has again attracted the attention of mathematicians due to its geometric and analytic richness. The kinematic conditions of rolling without slipping or twisting are described by means of motion on a configuration space being tangential to a smooth sub-bundle that we call a distribution. The precise definition of the mentioned motion in the case of two n-dimensional manifolds imbedded in $\mathbb{R}^m$, given for example in [11], involves studying the behavior of the tangent bundles of the manifolds and the normal bundles induced by the imbeddings. This approach leads to significant simplifications, for instance, it suffices to study the case in which the still manifold is the $n-$dimensional Euclidean space. The drawback is that the geometric descriptions depend strongly on the imbedding under consideration.

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So far, however, little attempts have been made to formulate this problem intrinsically. An early enlightening formulation is given in [2], in which the authors study the case of two abstract surfaces rolling in the above described manner. This is achieved by means of an intrinsic version of the moving frame method of Élie Cartan which, for this case, coincides with the classical intrinsic study of surfaces, see [12]. One of the important results established in [2] is the non-integrability property of the rank two distribution corresponding to no-twisting and no-slipping restrictions, namely, if the two surfaces have different Gaussian curvature, then the distribution is of Cartan-type, see [4]. A control theoretic approach to the same problem, studied in [1], has the advantage that the kinematic restrictions are written explicitly as vector fields on appropriate bundles.

We present a generalization of the kinematic problem for two $n-$dimensional abstract manifolds rolling without twisting or slipping via an intrinsic formulation. We define the configuration space of the system, which is an $\frac{n(n+3)}{2}-$dimensional manifold and which is a direct analogue to the one found in the references [2] and [1]. We give several equivalent definitions of rolling motion involving intrinsic characteristics and those that depend only on imbedding and discuss their relations. This new definitions permit to determine the imbedding-independent information contained in the extrinsic definition of the rolling bodies problem presented in [11]. Moreover, we relax the smoothness condition of the rolling map up to absolutely continuity. This allows to enlarge the class of mappings under consideration, still giving the possibility to apply the fundamental theorems of differential geometry and control theory without changing drastically the main classical ideas of rolling maps. The conditions of no-twisting and no-slipping define a distribution of rank n in the tangent bundle of the configuration space. We write explicitly the distribution as a local span of vector fields defined on the configuration space. We test the bracket generating condition of the above mentioned distribution on the known example [14] of rolling the n-dimensional sphere over the n-dimensional Euclidean space and the special group of Euclidean rigid motions SE(3) rolling over $\mathfrak{se}(3)$. As a result we obtain the controllability of the first system and the non controllability of the latter.

The structure of the present paper is the following. Section 2 is an introductory section where we collect necessary definitions and discuss the motivation for the reformulations of kinematic conditions of no-twisting and no-slipping for the rolling problem. We present two formulations and show their equivalence. Section 3 gives a good starting point for comparing different approaches, known in the literature for 2-dimensional rolling manifolds. In Section 4 we give the main formulation of extrinsic rolling as a curve on a configuration space defined as a direct sum of fiber bundles over the Cartesian product of the two rolling manifolds and we prove the equivalence of the new extrinsic definition of rolling with the previous ones and deduce the *intrinsic* definition of a rolling map. We also prove a theorem distinguishing the imbedding independent information contained in the definition of extrinsic rolling. Section 5 is dedicated to the construction of two distributions in the tangent bundle of the configuration space. These distributions encode the no-twisting and no-slipping kinematic conditions of the extrinsic and intrinsic rollings. These rollings can be written as curves in the configuration spaces tangent to the corresponding distributions. In Sections 6 and 7 we present detailed calculations for the two aforementioned examples: rolling the n-dimensional sphere over the n-dimensional Euclidean space and rolling SE(3) over $\mathfrak{se}(3)$. In the first case the distribution is bracket generating, coinciding with the result obtained in [14]. In the second case we obtain that the configuration space, of dimension 27, is foliated by 12 dimensional submanifolds.

2. Definition of rolling map for manifolds imbedded in Euclidean space

2.1. Rolling without twisting or slipping for imbedded manifolds. We start from the classical definition of rolling without slipping or twisting of one manifold over another manifold inside the Euclidean space.

Let us start with some notations. Throughout this paper, M and $\widehat{M}$ will always be oriented connected Riemannian manifolds of dimension n. By the well known result of Nash, see [8], there are isometric imbeddings of M and $\widehat{M}$, denoted by ι and $\widehat{\iota}$ respectively, into $\mathbb{R}^{n+\nu}$ for an appropriate choice of ν. Here and in what follows $\mathbb{R}^{n+\nu}$ will always be equipped with the standard Euclidean metric and standard orientation. As long as there is no possibility for confusion, we will identify the abstract manifolds M and $\widehat{M}$ with their images under the corresponding imbeddings. The imbedding of M into $\mathbb{R}^{n+\nu}$ splits the tangent space of $\mathbb{R}^{n+\nu}$ into a direct sum:

$$T_x\mathbb{R}^{n+\nu} = T_xM \oplus T_xM^{\perp}, \quad x \in M. \tag{1}$$

In general, any objects (points, curves, ...) related to the manifold $\widehat{M}$ will be marked by a hat ($\widehat{\ }$) on top, objects related to M will be free of it, while terms related to the ambient $\mathbb{R}^{n+\nu}$ space carry a bar ($\bar{}$). We use $\mathrm{Isom}(M)$ for the group of isometries of M, and $\mathrm{Isom}^+(M)$ for the group of sense preserving isometries.

We start by given the definition of rolling without twisting and slipping as found in [11].

Definition 0. *Let $M, \widehat{M}$ be submanifolds of $\mathbb{R}^{n+\nu}$. Then, a differentiable map $g : [0, \tau] \to \mathrm{Isom}(\mathbb{R}^{n+\nu})$ satisfying the following conditions for any*

$t \in [0,\tau]$ is called a rolling M on $\widehat{M}$ without slipping or twisting. The rolling conditions:

- *there is a piecewise smooth curve $x : [0,\tau] \to M$, such that*
  - *$g(t)x(t) \in \widehat{M}$,*
  - *$T_{g(t)x(t)}\left(g(t)M\right) = T_{g(t)x(t)}\widehat{M}$.*
- *Furthermore, the curve $\widehat{x}(t) := g(t)x(t)$ satisfies the following*
  - *no-slip condition:*
$$\dot{g}(t)g(t)^{-1}\widehat{x}(t) = 0,$$
  - *no-twist condition, tangential part:*
$$d(\dot{g}(t)g(t)^{-1})T_{\widehat{x}(t)}\widehat{M} \subseteq T_0(\dot{g}(t)g(t)^{-1}\widehat{M})^{\perp},$$
  - *no-twist condition, normal part:*
$$d(\dot{g}(t)g(t)^{-1})T_{\widehat{x}(t)}\widehat{M}^{\perp} \subseteq T_0(\dot{g}(t)g(t)^{-1}\widehat{M}).$$

Remark 1. In the previous definition, we explicitly state that $g : [0,\tau] \to \mathrm{Isom}(\mathbb{R}^{n+\nu})$ is differentiable. This is not stated in [11], but conditions containing $\dot{g}$ are required to hold for all t. Also, a minor inaccuracy in the no-twisting conditions is corrected.

It is clear that Definition 0 is of extrinsic nature. Thus, in order to obtain an intrinsic formulation of the rolling problem, we want to change the original definition as follows:

(1) *Making $x(t)$ part of the data of the rolling:* The reason is to give a local character to conditions of rolling without twisting or slipping. This will emphasize the dependence of the rolling not just on the isometry g but also on a curve x along which the rolling of M on $\widehat{M}$ can be realized. In some particular cases, this may lead to small changes in terminology. The following example illustrates these ideas.

Example 1. Consider the submanifolds of $\mathbb{R}^3$, defined by
$$M = \left\{(\bar{x}_1, \sin\theta, 1-\cos\theta) \in \mathbb{R}^3 |\ \bar{x}_1 \in \mathbb{R},\ \theta \in [0,2\pi)\right\},$$
$$\widehat{M} = \left\{(\bar{x}_1, \bar{x}_2, 0) \in \mathbb{R}^3 |\ \bar{x}_1, \bar{x}_2 \in \mathbb{R}, \right\}.$$
The rolling map
$$g(t) : \bar{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} \mapsto \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2\cos t + (\bar{x}_3 - 1)\sin t + t \\ -\bar{x}_2 \sin t + (\bar{x}_3 - 1)\cos t + 1 \end{pmatrix},$$
describes the rolling of the infinite cylinder M on $\widehat{M}$ along the $\bar{x}_2$-axis with constant speed 1. Then there is an infinite choice of curves $x(t) \in M$, given by
$$x(t) = (\bar{x}_1, \sin t, 1-\cos t), \quad \bar{x}_1 \in \mathbb{R}$$

along which the rolling g can be realized. However, if we make $x(t)$ as part of the data, then each choice of the curve $x(t)$ will correspond to different rollings $\big(x(t), g(t)\big)$.

(2) *Relaxing the differentiability conditions for $g(t)$:* We think that the conditions of differentiability of $g(t)$ for all $t \in [0, \tau]$ and piecewise smoothness of $x(t)$ are too restrictive. The requirement that $(x, g) : [0, \tau] \to M \times \mathrm{Isom}(\mathbb{R}^{n+\nu})$ is absolutely continuous or Lipschitz seems more natural, since this allows us to implement results from control theory, see Subsection 3.2. In this context, absolute continuity of a curve $\big(x(t), g(t)\big)$ on $M \times \mathrm{Isom}(\mathbb{R}^{n+\nu})$ is considered with respect to the parameter t, as in [1, Chapter 2].

(3) *Introducing orientability assumptions:* In order to have a connected configuration space, we exploit the orientability assumption of M and $\widehat{M}$. Since, as mentioned before, the rolling conditions will be local, we may choose an orientable neighborhood of the starting point even on any non-orientable manifold. We will use this to impose some practical restrictions to the definition of a rolling.

- Since $g(t)$ is continuous, it is either always orientation preserving or orientation reversing isometry of $\mathbb{R}^{n+\nu}$ for all t. Given a rolling $g(t)$ of M on $\widehat{M}$, we may assume that $g(t)$ is always orientation preserving by changing the orientation of $\mathbb{R}^{n+\nu}$. To obtain an orientation preserving rolling from an orientation reversing rolling $g(t)$ of M on $\widehat{M}$, pick any constant orientation reversing isometry g_0 of $\mathbb{R}^{n+\nu}$. Then $g_0 g(t)$ is an orientation preserving rolling of M on $g_0(\widehat{M})$.
- It is intuitively clear that for a fixed t, $d_{x(t)}g(t)$ maps elements from $T_{x(t)}M$ to $T_{\widehat{x}(t)}\widehat{M}$ and elements from $T_{x(t)}M^\perp$ to $T_{\widehat{x}(t)}\widehat{M}^\perp$ (for more details see Subsection 2.2). Hence, the matrix form of $d_{x(t)}g(t)$ splits in the following way:

$$
\begin{array}{ccc}
 & \begin{array}{cc} T_{x(t)}M & T_{x(t)}M^\perp \end{array} & \\
d_{x(t)}g(t) \;=\; & \begin{pmatrix} A(t) & 0 \\ 0 & B(t) \end{pmatrix} & \begin{array}{l} T_{\widehat{x}(t)}\widehat{M} \\ T_{\widehat{x}(t)}\widehat{M}^\perp. \end{array}
\end{array}
$$

  Since $g(t)$ is orientation preserving, both linear maps $d_{x(t)}g(t)|_{T_{x(t)}M}$ and $d_{x(t)}g(t)|_{T_{x(t)}M^\perp}$ are either orientation preserving or orientation reversing. By continuity, $d_{x(t)}g(t)|_{T_{x(t)}M}$ is either orientation preserving or orientation reversing for all t. We will require that $d_{x(t)}g(t)|_{T_{x(t)}M}$ is always orientation preserving. If $d_{x(t)}g(t)|_{T_{x(t)}M}$ is orientation reversing, pick any constant orientation preserving isometry $g_0 : \mathbb{R}^{n+\nu} \to \mathbb{R}^{n+\nu}$ so that

$$dg_0|_{T\widehat{M}} : T\widehat{M} \to T(g_0\widehat{M})$$

is orientation reversing. It is sufficient to show that it reverses the orientation at one point in order to show that it reverses orientation at all points due to the fact that M is oriented. Then $g_0 g(t)$ will be a rolling of M on $g_0\widehat{M}$ which is orientation preserving on the tangent space at $x(t)$.

Implementing the above changes to Definition 0, we obtain the following, from which several equivalent reformulations will be presented later.

Definition 1. *A rolling of M on $\widehat{M}$ without twisting or slipping is an absolutely continuous curve $(x, g) : [0, \tau] \to M \times \mathrm{Isom}^+(\mathbb{R}^{n+\nu})$, satisfying the following conditions:*

(i) $\widehat{x}(t) := g(t)x(t) \in \widehat{M}$, *for all* $t \in [0, \tau]$.
(ii) $T_{\widehat{x}(t)}(g(t)M) = T_{\widehat{x}(t)}\widehat{M}$, *for all* $t \in [0, \tau]$.
(iii) *No slip condition:* $\dot{g}(t) \circ g^{-1}(t)\widehat{x}(t) = 0$, *for almost every* t.
(iv) *No twist condition (tangential part):*
$$d(\dot{g}(t) \circ g^{-1}(t))(T_{\widehat{x}(t)}\widehat{M}) \subseteq T_0(\dot{g}(t) \circ g^{-1}(t)\widehat{M})^{\perp},$$
for almost every t.
(v) *No twist condition (normal part):*
$$d(\dot{g}(t) \circ g^{-1}(t))(T_{\widehat{x}(t)}\widehat{M}^{\perp}) \subseteq T_0(\dot{g}(t) \circ g^{-1}(t)\widehat{M}),$$
for almost every t.
(vi) $d_{x(t)}g(t)|_{T_{x(t)}M} : T_{x(t)}M \to T_{\widehat{x}(t)}\widehat{M}$ *is orientation preserving, for all* $t \in [0, \tau]$.

We omit, from now on, the words “without twisting or slipping”, just writing “a rolling of M on $\widehat{M}$”. Furthermore, for given curves $x(t)$ and $\widehat{x}(t)$ in M and $\widehat{M}$, respectively, the expression ”a rolling of M on $\widehat{M}$ along $x(t)$ and $\widehat{x}(t)$” will mean a rolling $(x, g) : [0, \tau] \to M \times \mathrm{Isom}^+(\mathbb{R}^{n+\nu})$ so that $g(t)x(t) = \widehat{x}(t)$.

Remark 2. The definitions we will be working with ignore physical restrictions given by the actual shapes of the manifolds. Intuitively, if we think of the manifolds in Definition 1 as physically touching along the curves $x(t)$ and $\hat{x}(t)$ and rolling according to the isometry $g(t)$, then we cannot rule out the possibility that there might be non-tangential intersections between the manifolds other than the contact points.

Example 2. Consider the imbedded surface
$$M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1^2 - x_2^2 + x_3 = 0,\ x_1^2 + x_2^2 < 1\},$$
and $\widehat{M} = \mathbb{R}^2$, imbedded as an affine plane. Assume that both manifolds M and $\widehat{M}$ carried the induced metric. We can clearly define a rolling of M on $\widehat{M}$ in terms of Definition 1, but there is no way to connect the saddle point

in M with any point in $\widehat{M}$ without there being intersections between the surfaces.

2.2. **First reformulation.** We aim to give a definition of the rolling of M on $\widehat{M}$ in a way that is more fruitful for future considerations. We fix some notations first. According to the splitting (1), any vector $v \in T_x\mathbb{R}^{n+\nu}$, $x \in M$, can be written uniquely as the sum $v = v^\top + v^\perp$, where $v^\top \in T_xM$ is tangent to M at x, while $v^\perp \in T_xM^\perp$ is normal. Analogous projections can be defined for $\widehat{M}$.

Let ∇ denote the Levi-Civita connection on M or on $\widehat{M}$. The context will indicate on which manifold the connection is defined. The "ambient" Levi-Civita connection on $\mathbb{R}^{n+\nu}$ is denoted by $\overline{\nabla}$. Note that if X and Y are tangent vector fields on M, then

$$\nabla_X Y(x) = \left(\overline{\nabla}_{\bar{X}}\bar{Y}(x)\right)^\top, \qquad x \in M,$$

where $\bar{X}$ and $\bar{Y}$ are any local extensions to $\mathbb{R}^{n+\nu}$ of the vector fields X and Y, respectively. Similarly, if Υ is a normal vector field on M and X is a tangent vector field on M, then the normal connection is defined by

$$\nabla^\perp_X \Upsilon(x) = \left(\overline{\nabla}_{\bar{X}}\bar{\Upsilon}(x)\right)^\perp, \qquad x \in M,$$

where $\bar{\Upsilon}$ is any local extension to $\mathbb{R}^{n+\nu}$ of the vector field Υ. Equivalent statements hold for $\widehat{M}$. If no confusions arise, we will use capital Latin letters X, Y, Z to denote tangent vector fields and capital Greek letters Υ, Ψ for notation of normal vector fields.

For a fixed value of $x \in M$ and a fixed vector field Y, the vector $\nabla_X Y(x)$ only depends on the value of $X(x)$. Therefore, for $v \in T_xM$, we will use $\nabla_v Y$ or $\nabla_v Y(x)$ to mean $\nabla_X Y(x)$, where X is an arbitrary vector field satisfying $X(x) = v$. We will use the same convention when ∇ is interchanged with $\nabla^\perp$.

If $Z(t)$ is a vector field along $x(t)$, we will use $\frac{D}{dt}Z(t)$ to denote the covariant derivative (corresponding to ∇) of $Z(t)$ along $x(t)$, and for any normal vector field $\Psi(t)$ along $x(t)$, $\frac{D^\perp}{dt}\Psi(t)$ denotes the normal covariant derivative (see [7, p. 119]). Recall that if M is imbedded isometrically into $\mathbb{R}^{n+\nu}$, then

$$\frac{D}{dt}Z(t) = \left(\frac{d}{dt}Z(t)\right)^\top, \qquad \frac{D^\perp}{dt}\Psi(t) = \left(\frac{d}{dt}\Psi(t)\right)^\perp,$$

where $Z(t)$ and $\Psi(t)$ are tangential and normal vector fields, respectively, along a curve in M.

We say that a tangent vector $Y(t)$ along an absolutely continuous curve $x(t)$ is parallel if $\frac{D}{dt}Z(t) = 0$ for almost every t. Notice that it is possible to define the notion of parallel transport even though the derivative $\dot{x}(t)$ exists only almost everywhere, see, e. g., Existence and Uniqueness Theorem in [10, Appendix C]. Namely, for any absolutely continuous curve $x : [0, \tau] \to M$

and for any $v \in T_{x(t_0)}M$, $0 \leq t_0 \leq \tau$, there exists a unique absolutely continuous tangent vector field $Z(t)$ along $x(t)$, such that $Z(t)$ is parallel and satisfies $Z(t_0) = v$.

We say that a normal vector field $\Psi(t)$ along $x(t)$ is *normal parallel* if $\frac{D^\perp}{dt}(t)\Psi = 0$ for almost every t. A normal analogue of parallel transport is defined likewise.

We are now ready to give a new formulation of the rolling map.

Definition 2. *A rolling of M on $\widehat{M}$ without slipping or twisting is an absolutely continuous curve $(x, g) : [0, \tau] \to M \times \mathrm{Isom}^+(\mathbb{R}^{n+\nu})$ satisfying the following conditions:*

(i') $\widehat{x}(t) := g(t)x(t) \in \widehat{M}$,

(ii') $dg(t)T_{x(t)}M = T_{\widehat{x}(t)}\widehat{M}$,

(iii') *No slip condition:* $\dot{\widehat{x}}(t) = dg(t)\dot{x}(t)$, *for almost every t.*

(iv') *No twist condition (tangential part):*

$$dg(t)\frac{D}{dt}Z(t) = \frac{D}{dt}dg(t)Z,$$

for any tangent vector field $Z(t)$ along $x(t)$ and almost every t.

(v') *No twist condition (normal part):*

$$dg(t)\frac{D^\perp}{dt}\Psi(t) = \frac{D^\perp}{dt}dg(t)\Psi(t),$$

for any normal vector field $\Psi(t)$ along $x(t)$ and almost every t.

(vi') $d_{x(t)}g(t)|_{T_{x(t)}M} : T_{x(t)}M \to T_{\widehat{x}(t)}\widehat{M}$ *is orientation preserving.*

Lemma 1. *Definitions 1 and 2 are equivalent.*

Proof. Since (i) and (i') coincide, we begin by proving the equivalence of (ii) and (ii'). Restricting the action of $g(t)$ to M, we observe that the differential $d_{x(t)}g(t)$ maps $T_{x(t)}M$ into $T_{g(t)x(t)}\left(g(t)M\right)$ by definition, and hence (ii) holds if and only if (ii') holds.

In order to prove the equivalence between (iii) and (iii') we write a curve $g(t)$ in $\mathrm{Isom}^+(\mathbb{R}^{n+\nu})$ as

$$g(t) : \bar{x} \mapsto \bar{A}(t)\bar{x} + \bar{r}(t), \qquad \bar{x} \in \mathbb{R}^{n+\nu},$$

where $\bar{A} : [0, \tau] \to \mathrm{SO}(n + \nu)$ and $\bar{r} : [0, \tau] \to \mathbb{R}^{n+\nu}$. Thus $d_{\bar{x}}g(t)v = \bar{A}(t)v$, $v \in T_{\bar{x}}\mathbb{R}^{n+\nu}$, and we get

$$\begin{aligned}\dot{g}(t) \circ g^{-1}(t)\,\widehat{x}(t) &= \dot{g}(t)x(t) = \dot{\bar{A}}(t)x(t) + \dot{\bar{r}}(t)\\ &= \frac{d}{dt}\left(\bar{A}(t)x(t) + \bar{r}(t)\right) - \bar{A}(t)\dot{x}(t) = \dot{\widehat{x}}(t) - dg(t)\dot{x}(t).\end{aligned}$$

whenever $\dot{x}(t)$ is defined. Hence $\dot{g}(t) \circ g^{-1}(t)\widehat{x}(t) = 0$ almost everywhere if and only if $\dot{\widehat{x}}(t) = dg(t)\dot{x}(t)$ almost everywhere.

Before we continue with the final two conditions, notice that (ii') actually states that both $dg(t)(T_{x(t)}M) = T_{\widehat{x}(t)}\widehat{M}$ and $dg(t)(T_{x(t)}M^{\perp}) = T_{\widehat{x}(t)}\widehat{M}^{\perp}$ hold due to the splitting (1). Hence, the inverse differential $dg^{-1}(t) = (dg(t))^{-1}$ also maps tangent vectors to tangent vectors and normal vectors to normal vectors. This allows us to restate (iv) and (v) as the conditions

$$\big(d\dot{g}(t)v^{\top}\big)^{\top} = 0, \quad \text{and} \quad \big(d\dot{g}(t)v^{\perp}\big)^{\perp} = 0,$$

holding for almost every t and for any $v \in T_{x(t)}\mathbb{R}^{n+\nu}$, decomposed as the sum of $v^{\top} \in T_{x(t)}M$ and $v^{\perp} \in T_{x(t)}M^{\perp}$ via the splitting (1). We calculate

$$\begin{aligned} 0 &= (d\dot{g}(t)\,Z(t))^{\top} = \left(\frac{d}{dt}\big(dg(t)\,Z(t)\big) - dg(t)\left(\frac{d}{dt}Z(t)\right)\right)^{\top} \\ &= \frac{D}{dt}dg(t)\,Z(t) - dg\frac{D}{dt}Z(t) \end{aligned}$$

for any tangent vector field $Z(t)$ along $x(t)$, for any value of t where $\dot{x}(t)$ is defined. By similar calculations, using a normal vector field $\Psi(t)$ along $x(t)$, we obtain

$$dg(t)\frac{D^{\perp}}{dt}\Psi(t) = \frac{D^{\perp}}{dt}dg(t)\,\Psi(t).$$

□

Remark 3. The following observations are useful for the understanding of the nature of a rolling map.

- The proof of Lemma 1 shows that indeed condition (ii') is equivalent to the statement
$$dg(t)T_{x(t)}M^{\perp} = T_{\widehat{x}(t)}\widehat{M}^{\perp}.$$
- Condition (iv') is equivalent to the requirement that any tangent vector field $Z(t)$ is parallel along $x(t)$ if and only if $dg(t)Z(t)$ is parallel along $\widehat{x}(t)$. As a consequence, this condition is automatically satisfied in the case of one dimensional manifolds.
- We can reformulate (v') in terms of normal parallel vector fields. Namely, condition (v') is equivalent to the statement that any normal vector field $\Psi(t)$ is normal parallel along $x(t)$ if and only if $dg(t)\Psi(t)$ is normal parallel vector field along $\widehat{x}(t)$. Thus, if the manifolds are imbedded into Euclidean space and the codimension is one (i.e. $\nu = 1$), condition (v') always holds.

3. Previous intrinsic descriptions of rolling maps dimension 2

The aim of this Section is to present the different intrinsic formulations of a rolling map appearing in literature for two dimensional manifolds. The two best known formulations are given in [1, 2]. We start by introducing the configuration space of the rolling for the general case of n dimensional manifolds and then proceed to describe the previously mentioned two dimensional situation.

3.1. **Frame bundles and bundles of isometries.** Let $\mathrm{SO}(V, \widehat{V})$ denote the collection all linear orientation preserving isometries between the oriented inner product spaces V and $\widehat{V}$. We write simply $\mathrm{SO}(V)$ for the group $\mathrm{SO}(V, V)$.

For any pair M and $\widehat{M}$ of oriented connected Riemannian n-dimensional manifolds, we introduce a manifold Q of all the relative positions in which M can be tangent to $\widehat{M}$. This $\mathrm{SO}(n)-$fiber bundle over $M \times \widehat{M}$ is defined by

$$Q = \left\{ q \in \mathrm{SO}(T_x M, T_{\widehat{x}} \widehat{M}) \,\middle|\, x \in M,\, \widehat{x} \in \widehat{M} \right\}$$

and can be considered as the configuration space of the rolling.

The configuration space Q can be also described in the following way. Let F and $\widehat{F}$ be the oriented orthonormal frame bundle frame bundles of M and $\widehat{M}$, respectively. If $f_1, \ldots, f_n$ is an oriented orthonormal frame at x, then each such frame may be considered as a mapping $f \in \mathrm{SO}(\mathbb{R}^n, T_x M)$, where $\mathbb{R}^n$ has the Euclidean structure and standard orientation, and

$$f\big(\underbrace{0, \ldots, 0, 1, 0, \ldots, 0}_{1 \text{ in the } j-\text{th place}}\big) = f_j. \tag{2}$$

This gives us an obvious action of $\mathrm{SO}(\mathbb{R}^n) = \mathrm{SO}(n)$ on the right, inducing a principal $\mathrm{SO}(n)$-bundle structure on F. On the fiber over each point x, we also have a left action by $\mathrm{SO}(T_x M)$. This group is isomorphic to $\mathrm{SO}(n)$, although not canonically when $n \geq 3$. Therefore we have no natural left action of $\mathrm{SO}(n)$ on Q. Similar considerations holds for $\widehat{F}$.

Consider $F \times \widehat{F}$ as a bundle over $M \times \widehat{M}$ with $\mathrm{SO}(n)$ acting diagonally on the fibers. Then, we can identify Q with $(F \times \widehat{F})/\,\mathrm{SO}(n)$ by the following map. Let f be a frame in F at $x \in M$ and similarly let $\hat{f}$ be a frame in $\widehat{F}$ at $\widehat{x} \in \widehat{M}$. Then to each equivalence class $(f, \hat{f}) \cdot \mathrm{SO}(n)$ we associate the mapping $q \in \mathrm{SO}(T_x M, T_{\widehat{x}} \widehat{M})$, so that

$$\hat{f} = q \circ f, \tag{3}$$

that is, the mapping satisfying $\hat{f}_j = q f_j$ for $j = 1, \ldots, n$. Clearly, this construction does not depend on the choice of a representative of an equivalence class of $(F \times \widehat{F})/\,\mathrm{SO}(n)$. Conversely, given an isometry, there exists a unique equivalence class of frames satisfying (3).

Except for the case when $n = 2$, Q does not possess the structure of a principal $\mathrm{SO}(n)-$bundle in a natural way. However, Q does look locally like a trivial product of $M \times M$ and $\mathrm{SO}(n)$. Let U be a neighborhood in M, such that F trivializes over U. Then there is a section e of $F|_U$, that is, a smooth function on U so that for any $x \in U$, $e(x) \in \mathrm{SO}(\mathbb{R}^n, T_x M)$. Another way to see this is that, if we define vectors $e_j(x)$ in a similar way as in (2) for $e(x)$, then each mapping $e_j : x \mapsto e_j(x)$ is a vector field on U. These vector fields have the property that for any $x \in U$,

$$e_1(x), \ldots, e_n(x),$$

is a positively oriented orthonormal basis of $T_x M$. On this section of $F|_U$, we can define a left action of $\mathrm{SO}(n)$ on $F|_U$ relative to e. This can be done by using $e(x) : \mathbb{R}^n \to T_x M$ at any point $x \in U$, to give an isomorphism of $\mathrm{SO}(n)$ and $\mathrm{SO}(T_x M)$. The corresponding action takes the following form. If $f \in F_x, x \in U$ is any other frame, and

$$f_j = \sum_{i=1}^{n} f_{ij} e_i(x),$$

then the left action of $A = (a_{ij})_{i,j=1}^n \in \mathrm{SO}(n)$ relative to e is defined by

$$A \cdot f_j = \sum_{i,k=1}^{n} f_{ij} a_{ki} e_k, \qquad j = 1, \ldots, n.$$

From this we can locally define a left and right action on Q. Let U and $\widehat{U}$ be neighborhoods in M and $\widehat{M}$ respectively, so that both frame bundles trivialize over these neighborhoods. Let $e : U \to F|_U$ and $\hat{e} : \widehat{U} \to \widehat{F}|_{\widehat{U}}$ be sections. Then the left action of $A \in \mathrm{SO}(n)$ with respect to $\hat{e}$, is defined so that if $\hat{f}_j = q f_j$ for $j = 1, \ldots, n$, then

$$A \cdot \hat{f}_j = (A \cdot q) f_j,$$

where the left action of A on $\hat{f}_j$ is defined with respect to $\hat{e}$. Similarly, the right action with respect to e is defined by

$$\hat{f}_j = (q \cdot A)\big(A^{-1} \cdot f_j\big).$$

Remark that if $A_0 = (\langle \hat{e}_i, q e_j \rangle)_{i,j=1}^n$, then

$$(\langle \hat{e}_i, (A \cdot q) e_j \rangle)_{i,j=1}^n = A A_0, \qquad \text{and} \qquad (\langle \hat{e}_i, (q \cdot A) e_j \rangle)_{i,j=1}^n = A_0 A.$$

Since Q is an $\mathrm{SO}(n)-$fiber bundle over $M \times \widehat{M}$, it has dimension $\frac{n(n+3)}{2}$ as a manifold.

3.2. **Agrachev-Sachkov formulation of rolling surfaces.** A previous definition of a rolling map can be found in [1], where only 2-dimensional manifolds imbedded into $\mathbb{R}^3$ are considered. Although it only deals with the imbedded case, the definition of the rolling is intrinsic in the sense that it does not depend on the imbedding.

The configuration space for rolling one surface on another is Q, which is now 5-dimensional, since M and $\widehat{M}$ are 2-dimensional. A rolling is then an absolutely continuous curve $q : [0, \tau] \to Q$ satisfying the following: if $x(t)$

and $\widehat{x}(t)$ are the projections of $q(t)$ into M and $\widehat{M}$ then the following two conditions are satisfied:

- no slip condition: $\dot{\widehat{x}} = q(t)\,\dot{x}(t)$ for almost every $t \in [0, \tau]$;
- no twist condition: $Z(t)$ is a parallel tangent vector field along $x(t)$ if and only if $q(t)Z(t)$ is a parallel tangent vector field along $\widehat{x}(t)$.

Notice that there is no condition corresponding to the normal no-twist, since the manifolds here have codimension 1. In Section 4 we will show how this definition fits into our Definition 2.

The no-slip and no-twist conditions can be described by means of a distribution D in the tangent bundle of Q. By distribution, we mean a smooth subbundle of the tangent bundle. Then the "no slip − no twist" condition will correspond to the requirement $\dot{q}(t) \in D_{q(t)}$ for almost every t. The distribution D has the following local description. In any sufficiently small neighborhood $U \subset M$ of $y \in M$ we pick a pair of tangent vector field e_1, e_2, such that $\{e_1(x), e_2(x)\}$ is a positively oriented orthonormal basis for every $x \in U$. Define $\hat{e}_1, \hat{e}_2$ in a similar way in a sufficiently small neighborhood $\widehat{U}$. Since the rotation group $SO(2)$ has dimension 1, we simply need to know the relative angle θ to describe q with respect to the frames given by $\{e_1, e_2\}$ and $\{\hat{e}_1, \hat{e}_2\}$. More precisely, θ is defined by

$$\begin{aligned} q\, e_1 &= \cos\theta \hat{e}_1 + \sin\theta \hat{e}_2, \\ q\, e_2 &= -\sin\theta \hat{e}_1 + \cos\theta \hat{e}_2. \end{aligned}$$

Thus, if $\pi : Q \to M \times \widehat{M}$ is the natural projection, then any $q \in \pi^{-1}(U \times \widehat{U})$ is uniquely determined by the coordinates $(x, \widehat{x}, \theta), (x, \widehat{x}) \in U \times \widehat{U}$.

Let $c_1, c_2, \widehat{c}_1$ and $\widehat{c}_2$ be the so-called "structural constants", defined by the commutation relations

$$[e_1, e_2] = c_1 e_1 + c_2 e_2, \qquad [\hat{e}_1, \hat{e}_2] = \widehat{c}_1 \hat{e}_1 + \widehat{c}_2 \hat{e}_2.$$

Define the vector fields X_1 and X_2 on $\pi^{-1}(U \times \widehat{U})$ by

$$\begin{aligned} X_1 &= e_1 + \cos\theta \hat{e}_1 + \sin\theta \hat{e}_2 + (-c_1 + \widehat{c}_1 \cos\theta + \widehat{c}_2 \sin\theta) \frac{\partial}{\partial\theta}, \\ X_2 &= e_2 - \sin\theta \hat{e}_1 + \cos\theta \hat{e}_2 + (-c_2 - \widehat{c}_1 \sin\theta + \widehat{c}_2 \cos\theta) \frac{\partial}{\partial\theta}. \end{aligned} \tag{4}$$

Then $D|_{\pi^{-1}(U \times \widehat{U})}$ is spanned by X_1, X_2.

The connectivity by a curve tangent to the distribution D is the principal problem. More precisely, given two different states $q_0, q_1 \in Q$, we ask whether there exists a rolling motion $q : [0, \tau] \to Q$, such that $q(0) = q_0$ and $q(\tau) = q_1$? The advantage of the formulation of no slipping and no twisting conditions in terms of a distribution, is that the question of connectivity may be reformulated through admissible sets or orbits in control theory.

Given a distribution D on an arbitrary manifold Q, a curve $q : [0, \tau] \to Q$ is said to be *horizontal* (or *admissible*) with respect to D if q is an absolutely continuous curve satisfying $\dot{q}(t) \in D$ for almost every t. The *orbit* of D at a point q_0 is the set of all points $q_1 \in Q$ so that there exists a curve $q : [0, \tau] \to Q$, with $q(0) = q_0$ and $q(\tau) = q_1$, which is horizontal with respect to D. We denote this set by $\mathcal{O}_{q_0}(D)$. It is clear that if $q_1 \in \mathcal{O}_{q_0}(D)$, then $\mathcal{O}_{q_0}(D) = \mathcal{O}_{q_1}(D)$. The Orbit Theorem [6, 13] asserts that $\mathcal{O}_{q_0}$ is an immersed submanifold of Q and describes the tangent space of the orbit in terms of the diffeomorphisms of Q. A precise statement using the chronological exponential and a broad discussion about the Orbit Theorem can be found in Chapter 5 of [1].

Also, define the flag associated to the distribution D inductively by

$$D^1 = D \text{ and } D^{i+1} = D + [D, D^i].$$

We say that D has step $k \geq 2$ at q if k is the maximal integer, so that

$$D_q^{k-1} \subsetneq D_q^k = D_q^{k+1}.$$

If $D_q^k = D_q$ for any integer k, we say that D has step 1 at q. The Orbit Theorem then tells us that $D_q^k \subseteq T_q\mathcal{O}_{q_0}(D)$, where k is the step at $q \in \mathcal{O}_{q_o}(D)$. In particular, if Q is connected and there is an integer k such that $D^k = TQ$, then $\mathcal{O}_{q_0}(D) = Q$. The previous result is known as the Chow-Rashevskiĭ theorem [5, 9] and the distribution D is called bracket generating.

We will use the expression that D has step k if D has step k for any $q \in Q$. Remark that if D is of step k, and there is a local basis of vector fields of D^k in a neighborhood around any point in Q, then

$$D_q^k = T_q\mathcal{O}_{q_0}(D).$$

We now go back to the intrinsic definition presented in [1], where Q is the described 5-dimensional configuration space and D is spanned locally by (4). This definition can be restated as following: a curve $q(t) \colon [0, \tau] \to Q$ is the rolling map of M on $\widehat{M}$ if it is tangent to D. The main result of [1], is the following description of orbits of D. Let $\varkappa(x)$ and $\widehat{\varkappa}(\widehat{x})$ denote the Gaussian curvature of M at x and of $\widehat{M}$ at $\widehat{x}$, respectively.

Theorem 1. *For any $q_0 \in Q$, the orbit at q_0 satisfies* $\dim \mathcal{O}_{q_0}(D) = 2$ *if and only if* $\varkappa(\mathrm{pr}_M q) - \widehat{\varkappa}(\mathrm{pr}_{\widehat{M}} q) = 0$ *for every* $q \in \mathcal{O}_{q_0}(D)$. *Otherwise,* $\dim \mathcal{O}_{q_0}(D) = 5$.

Remark 4. In contrast to the definition in [11], the definition in [1] deals with absolutely continuous curves. The advantage of this, is the ability to apply the Orbit theorem and the Chow-Rashevskiĭ theorem. This was one of the reasons for us to define a rolling map in terms of absolutely continuous curves. Remark that all these theorems also hold if we consider Lipschitz curves instead of absolutely continuous. Hence, we always may interchange "absolutely continuous" with "Lipschitz" for all considerations in the present paper.

3.3. **Bryant-Hsu formulation of rolling surfaces.** In [2] the authors give an intrinsic formulation to the problem of rolling two abstract surfaces M and $\widehat{M}$ with respect to each other. The main tool in this formulation is Cartan's general method of moving frames, that is, determining canonical forms on an appropriate $\mathrm{SO}(2)-$bundle.

Let M and $\widehat{M}$ be two connected oriented Riemannian manifolds of dimension 2. Consider the respective frame bundles F, $\widehat{F}$. Then, as discussed in Subsection 3.1, the configuration space Q for this kinematic system can be identified with $(F \times \widehat{F})/\,\mathrm{SO}(2)$. The conditions of no twisting and no slipping can be understood by means of the canonical one-forms $\alpha_1, \alpha_2, \alpha_{21}$ on F and $\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\alpha}_{21}$ on $\widehat{F}$. Recall, that these forms satisfy the structure equations

$$\begin{aligned} d\alpha_1 &= \alpha_{21} \wedge \alpha_2, & d\widehat{\alpha}_1 &= \widehat{\alpha}_{21} \wedge \widehat{\alpha}_2, \\ d\alpha_2 &= -\alpha_{21} \wedge \alpha_1, & d\widehat{\alpha}_2 &= -\widehat{\alpha}_{21} \wedge \widehat{\alpha}_1, \\ d\alpha_{21} &= \varkappa\, \alpha_1 \wedge \alpha_2, & d\widehat{\alpha}_{21} &= \widehat{\varkappa}\, \widehat{\alpha}_1 \wedge \widehat{\alpha}_2, \end{aligned}$$

where $\varkappa$ and $\widehat{\varkappa}$ are the Gauss curvatures of M and $\widehat{M}$ respectively, see [12, Chapter 7].

The rank two distribution D over Q corresponding to the "no slip – no twist" conditions is the push-forward of the vector fields, solving the Pfaffian equations

$$\alpha_1 - \widehat{\alpha}_1 = \alpha_2 - \widehat{\alpha}_2 = \alpha_{21} - \widehat{\alpha}_{21} = 0, \tag{5}$$

under the natural projection $\pi : F \times \widehat{F} \to Q$. At the points where $\varkappa - \widehat{\varkappa} \neq 0$ the distribution D is of Cartan type, that is, the distributions

$$D^2 = D + [D, D] \quad \text{and} \quad D^3 = D^2 + [D, D^2]$$

have rank 3 and 5 respectively, see [2]. This implies that, under the condition $\varkappa - \widehat{\varkappa} \neq 0$, the distribution D is bracket generating of step 3. To see under which conditions D is of Cartan type, define the following one-forms over the product $F \times \widehat{F}$

$$\theta_1 = \frac{1}{2}(\alpha_1 - \widehat{\alpha}_1), \quad \theta_2 = \frac{1}{2}(\alpha_2 - \widehat{\alpha}_2), \quad \theta_3 = \frac{1}{2}(\alpha_{21} - \widehat{\alpha}_{21}),$$

$$\omega_1 = \frac{1}{2}(\alpha_1 + \widehat{\alpha}_1), \quad \omega_2 = \frac{1}{2}(\alpha_2 + \widehat{\alpha}_2),$$

and observe that the following identities hold:

$$\begin{aligned}
d\theta_1 &= \theta_3 \wedge \omega_2 + \frac{1}{2}(\alpha_{21} + \widehat{\alpha}_{21}) \wedge \theta_2, \\
d\theta_2 &= -\theta_3 \wedge \omega_1 - \frac{1}{2}(\alpha_{21} + \widehat{\alpha}_{21}) \wedge \theta_1, \\
d\theta_3 &= \frac{1}{2}(\varkappa - \widehat{\varkappa})\omega_1 \wedge \omega_2 + \frac{1}{2}\Big((\varkappa + \widehat{\varkappa})(\omega_1 \wedge \theta_2 - \omega_2 \wedge \theta_1) + (\varkappa - \widehat{\varkappa})\theta_1 \wedge \theta_2\Big).
\end{aligned}$$

Denote by $\mathcal{D} = \ker\theta_1 \cap \ker\theta_2 \cap \ker\theta_3$ the space of solutions of the system (5) and let $X = (X_1, X_2), Y = (Y_1, Y_2), Z = (Z_1, Z_2)$ be a local basis of $\mathcal{D}$ chosen such that

$$\begin{array}{cccc}
\alpha_1(X_1) = 1, & \alpha_2(X_1) = 0, & \widehat{\alpha}_1(X_2) = 1, & \widehat{\alpha}_2(X_2) = 0, \\
\alpha_1(Y_1) = 0, & \alpha_2(Y_1) = 1, & \widehat{\alpha}_1(Y_2) = 0, & \widehat{\alpha}_2(Y_2) = 1, \\
\alpha_1(Z_1) = 0, & \alpha_2(Z_1) = 0, & \widehat{\alpha}_1(Z_2) = 0, & \widehat{\alpha}_2(Z_2) = 0.
\end{array}$$

Observe that for a sufficiently small open neighborhood $U \times \widehat{U} \subset M \times \widehat{M}$ of $(p, \widehat{p})$, the differential of the projection π is

$$\begin{array}{rcccc}
d_{((p,C),(\widehat{p},\widehat{C}))}\pi & : & T_pU \times \mathfrak{so}(2) \times T_{\widehat{p}}\widehat{U} \times \mathfrak{so}(2) & \to & T_pU \times T_{\widehat{p}}\widehat{U} \times \mathfrak{so}(2) \\
 & & (x, A, y, B) & \mapsto & (x, y, A - B)
\end{array}$$

for any $C, \widehat{C} \in \mathrm{SO}(2)$ and where $T_C\,\mathrm{SO}(2)$, $T_{\widehat{C}}\,\mathrm{SO}(2)$ and $T_{C\widehat{C}^{-1}}\,\mathrm{SO}(2)$ are identified with $\mathfrak{so}(2)$ in the usual manner. By the construction of the canonical forms on the frame bundles, it is clear that $X, Y \notin \ker d\pi$, whereas it is possible to choose locally Z such that $Z \in \ker d\pi$. Thus since $\ker d\pi$ has dimension one, we have locally

$$\ker d\pi = \mathrm{span}\{Z\}.$$

This implies that a local description of D is given by

$$D = \mathrm{span}\{d\pi(X), d\pi(Y)\}.$$

Recall Cartan's formula for a differential one form η and any two local vector fields v, w, given by

$$d\eta(v, w) = v(\eta(w)) - w(\eta(v)) - \eta([v, w]).$$

In our case, the previous equation implies the following equalities

$$\begin{aligned}
d\theta_1(X,Y) &= -\theta_1([X,Y]) = 0,\\
d\theta_1(X,Z) &= -\theta_1([X,Z]) = 0,\\
d\theta_1(Y,Z) &= -\theta_1([Y,Z]) = 0,\\
d\theta_2(X,Y) &= -\theta_2([X,Y]) = 0,\\
d\theta_2(X,Z) &= -\theta_2([X,Z]) = 0,\\
d\theta_2(Y,Z) &= -\theta_2([Y,Z]) = 0,\\
d\theta_3(X,Y) &= -\theta_3([X,Y]) = \frac{1}{2}(\varkappa - \widehat{\varkappa}),\\
d\theta_3(X,Z) &= -\theta_3([X,Z]) = 0,\\
d\theta_3(Y,Z) &= -\theta_3([Y,Z]) = 0,
\end{aligned}$$

It follows from these equations, that $[X,Z]$, $[Y,Z]$ belong to $\mathcal{D}$ and $[X,Y] \notin \mathcal{D}$ if and only if the difference of curvatures $\varkappa - \widehat{\varkappa}$ does not vanish identically. In fact, counting dimensions, we see that $\operatorname{span}\{X,Y,Z,[X,Y]\} = \ker\theta_1 \cap \ker\theta_2$. It is clear from the choice of Z that $[X,Y] \notin \ker d\pi$ since if $[X,Y] = kZ$ for some $k \in \mathbb{R}$, then $d\theta_3(X,Y) = -k\theta_3(Z) = 0$ which contradicts our assumption. This implies that $\operatorname{span}\{d\pi(X), d\pi(Y), d\pi([X,Y])\} = D_1$ is a distribution of rank 3. Analogously we obtain

$$\begin{aligned}
d\theta_1([X,Y],X) &= -\theta_1([[X,Y],X]) = 0,\\
d\theta_1([X,Y],Y) &= -\theta_1([[X,Y],Y]) = \theta_3([X,Y]),\\
d\theta_2([X,Y],X) &= -\theta_2([[X,Y],X]) = -\theta_3([X,Y]),\\
d\theta_2([X,Y],Y) &= 0.
\end{aligned}$$

By similar considerations, we can see that

$$\operatorname{span}\{X,Y,Z,[X,Y],[[X,Y],X],[[X,Y],Y]\} = T(F \times \widehat{F}),$$

which implies that

$$\operatorname{span}\{d\pi(X), d\pi(Y), d\pi([X,Y]), d\pi([[X,Y],X]), d\pi([[X,Y],Y])\} = D_2,$$

is a distribution of rank 5.

These calculations imply that D is of Cartan type whenever $\varkappa - \widehat{\varkappa}$ does not vanish identically. Since the configuration space Q is 5-dimensional, the distribution D is bracket generating and thus, by the Chow-Rashevskiĭ theorem we can completely solve the connectivity problem. In the case when $\varkappa = \widehat{\varkappa}$, the distribution D is integrable and therefore Q is foliated by submanifolds of dimension 2.

It is mentioned in [2], that their construction does not depend on imbedding into Euclidean space, however no attempts are made to compare this definition to the one for imbedded manifolds.

We present a simple example, illustrating the above mentioned approach.

Example 3. Let us consider the problem of the two dimensional sphere S^2 rolling over the Euclidean plane $\mathbb{R}^2$. We can embed these surfaces in the three dimensional Euclidean space $\mathbb{R}^3$ via the parameterizations

$$S^2 = \{(\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi) : -\pi < \theta \leq \pi, -\frac{\pi}{2} < \varphi \leq \frac{\pi}{2}\},$$

$$\mathbb{R}^2 = \{(x, y, 0) : x, y \in \mathbb{R}\}.$$

It follows from straightforward computations that, in this case, we have

$$\alpha_1 = \cos\varphi d\theta, \quad \alpha_2 = d\varphi, \quad \alpha_{21} = \sin\varphi d\theta;$$
$$\widehat{\alpha}_1 = dx, \quad \widehat{\alpha}_2 = dy, \quad \widehat{\alpha}_{21} = 0.$$

Thus, equations (5) take the form

$$\cos\varphi d\theta - dx = d\varphi - dy = \sin\varphi d\theta = 0.$$

It is easy to see that

$$d\alpha_{21} = \cos\varphi d\theta \wedge d\varphi = \alpha_1 \wedge \alpha_2, \qquad d\widehat{\alpha}_{21} = 0,$$

from which it follows that $\varkappa = 1$ and $\widehat{\varkappa} = 0$. Since the difference of the Gaussian curvatures does not vanish identically, we obtain the well-known result that it is always possible to achieve any configuration from a given one by rolling the sphere over the plane without slipping or twisting.

4. Intrinsic rolling

4.1. **Reformulation of the rolling motion in terms of bundles.** Both formulations of rolling maps given in [1] and [2] only use the configuration space as a manifold of isometries of tangent spaces of M and $\widehat{M}$, without taking into account the imbedding into an ambient space. However, neither of these descriptions attempts to give any justifications for why the ambient space may be ignored, nor do they attempt to compare the intrinsic definition and the extrinsic definition given for imbedded manifolds in [11]. We would like to find a reformulation of Definition 2 in such a way that the conditions (i')-(vi') are stated both in terms of intrinsic conditions given on Q and some additional conditions given on another bundle, that carries the information on imbedding.

The conditions imposed over a rolling (x, g) by Definitions 1 and 2 are nontrivial in normal directions for the imbedding of the manifolds with codimension ν greater than 1. So, it is natural to suppose that the total configuration space of the rolling dynamics will have a normal component which will takes care of the action of g on the normal bundle. Therefore, we make the following analogue construction, as we did for Q, in order to construct a fiber bundle over $M \times \widehat{M}$ of isometries of the normal tangent space. We start from a pair of imbeddings $\iota : M \to \mathbb{R}^{n+\nu}$ and $\widehat{\iota} : \widehat{M} \to \mathbb{R}^{n+\nu}$, given as initial data. Let Φ be the principal $\mathrm{SO}(\nu)-$bundle over M, such that the fiber over a point $x \in M$ consists of all positively oriented orthonormal frames $\{\epsilon_\lambda(x)\}_{\lambda=1}^{\nu}$ spanning $T_xM^\perp$. Let $\widehat{\Phi}$ be the principal $\mathrm{SO}(\nu)-$bundle similarly defined on $\widehat{M}$. Likewise we did in Section 3.1, identifying $(F \times \widehat{F})/\,\mathrm{SO}(n)$ with

$$Q = \left\{ q \in \mathrm{SO}(T_xM, T_{\widehat{x}}\widehat{M}) \,\middle|\, x \in M, \widehat{x} \in \widehat{M} \right\}, \tag{6}$$

we identify $(\Phi \times \widehat{\Phi})/\,\mathrm{SO}(\nu)$ with

$$P_{\iota,\widehat{\iota}} := \left\{ p \in \mathrm{SO}(T_xM^\perp, T_{\widehat{x}}\widehat{M}^\perp) \,\middle|\, x \in M, \widehat{x} \in \widehat{M} \right\}. \tag{7}$$

As for Q, the space $P_{\iota,\widehat{\iota}}$ is not in general a principal $\mathrm{SO}(\nu)-$bundle, but there are "local" left and right actions defined similarly as on Q in Section 3.1. We notice and reflect it in notations that Q is invariant of imbeddings, while $P_{\iota,\widehat{\iota}}$ is not.

By abuse of notation, we will use $Q \oplus P_{\iota,\widehat{\iota}}$ for the fiber bundle over $M \times \widehat{M}$, so that the fiber over $(x, \widehat{x}) \in M \times \widehat{M}$, is $Q_{(x,\widehat{x})} \times P_{\iota,\widehat{\iota}(x,\widehat{x})}$.

Proposition 1. *If a curve $(x, g) : [0, \tau] \to M \times \mathrm{Isom}^+(\mathbb{R}^{n+\nu})$ satisfies (i')-(vi') of Definition 2, then the mapping*

$$t \mapsto (dg(t)|_{T_{x(t)}M}, dg(t)|_{T_{x(t)}M^\perp}) =: (q(t), p(t)),$$

defines a curve in $Q \oplus P_{\iota,\widehat{\iota}}$ with the following properties:

(I) *no slip condition: $\dot{\widehat{x}}(t) = q(t)\dot{x}(t)$ for almost every t.*
(II) *no twist condition (tangential part): $q(t)\frac{D}{dt}Z(t) = \frac{D}{dt}q(t)Z(t)$ for any tangent vector field $Z(t)$ along $x(t)$ and almost every t.*
(III) *no twist condition (normal part): $p(t)\frac{D^\perp}{dt}\Psi(t) = \frac{D^\perp}{dt}p(t)\Psi(t)$ for any normal vector field $\Psi(t)$ along $x(t)$ and almost every t.*

Conversely, if $(q, p) : [0, \tau] \to Q \oplus P_{\iota,\widehat{\iota}}$ is an absolutely continuous curve satisfying (I)-(III), then there exists a unique rolling $(x, g) : [0, \tau] \to M \times \mathrm{Isom}^+(\mathbb{R}^{n+\nu})$, such that $dg(t)|_{T_{x(t)}M} = q(t)$ and $dg(t)|_{T_{x(t)}M^\perp} = p(t)$.

Proof. Assume that $(x, g) : [0, \tau] \to M \times \mathrm{Isom}^+(\mathbb{R}^{n+\nu})$ is a rolling map satisfying (i')-(vi'). The statements (i') and (ii') assure that

$$\begin{aligned} &dg(t)|_{T_{x(t)}M} \in \mathrm{SO}(T_{x(t)}M, T_{\widehat{x}(t)}\widehat{M}) \quad \text{and} \\ &dg(t)|_{T_{x(t)}M^\perp} \in \mathrm{SO}(T_{x(t)}M^\perp, T_{\widehat{x}(t)}\widehat{M}^\perp). \end{aligned} \tag{8}$$

Since $dg(t)$ must be orientation preserving in $\mathbb{R}^{n+\nu}$ we conclude that both of the mappings (8) are either orientation reversing or orientation preserving. The additional requirement (vi') implies that (q, p) is orientation preserving. The conditions (I)-(III) correspond to the conditions (iii')-(v').

Conversely, if we have a curve $(q(t), p(t))$ in $Q \oplus P_{\iota,\widehat{\iota}}$ with projection $(x(t), \widehat{x}(t))$ into the product manifold $M \times \widehat{M}$, then we may construct the isomorphism $g \in \mathrm{Isom}^+(\mathbb{R}^{n+\nu})$ in the following way. We write $g(t) : \bar{x} \mapsto \bar{A}(t)\bar{x} + \bar{r}(t), \bar{A}(t) \in \mathrm{SO}(n+\nu)$, where $\bar{A}(t) = dg(t)$ is determined by the conditions

$$dg(t)|_{T_{x(t)}M} = q(t)|_{T_{x(t)}M}, \qquad dg(t)|_{T_{x(t)}M^\perp} = p(t)|_{T_{x(t)}M^\perp}.$$

Then

$$\mathrm{Image}\, dg(t)|_{T_{x(t)}M} = T_{\widehat{x}(t)}\widehat{M}, \qquad \mathrm{Image}\, dg(t)|_{T_{x(t)}M^\perp} = T_{\widehat{x}(t)}\widehat{M}^\perp.$$

The vector $\bar{r}(t)$ is determined by $\bar{r}(t) = \widehat{x}(t) - \bar{A}(t)x(t)$. □

The one-to-one correspondence between rolling maps and absolutely continuous curves in $Q \oplus P_{\iota,\widehat{\iota}}$, satisfying (I)-(III), naturally leads to a definition of a rolling map in terms of these bundles.

Definition 3. *A rolling of M on $\widehat{M}$ without slipping or twisting is an absolutely continuous curve $(q,p) : [0,\tau] \to Q \oplus P_{\iota,\widehat{\iota}}$ such that $(q(t), p(t))$ satisfies*

(I) *no slip condition: $\dot{\widehat{x}}(t) = q(t)\dot{x}(t)$ for almost every t,*
(II) *no twist condition (tangential part): $q(t)\frac{D}{dt}Z(t) = \frac{D}{dt}q(t)Z(t)$ for any tangent vector field $Z(t)$ along $x(t)$ and almost every t,*
(III) *no twist condition (normal part): $p(t)\frac{D^\perp}{dt}\Psi(t) = \frac{D^\perp}{dt}p(t)\Psi(t)$ for any normal vector field $\Psi(t)$ along $x(t)$ and almost every t.*

A purely intrinsic definition of a rolling is deduced from Definition 3, by restricting it to the bundle Q. This concept naturally generalizes the definition given in [1] for 2-dimensional Riemannian manifolds imbedded into $\mathbb{R}^3$ and we use the term *intrinsic rolling* for this object.

Definition 4. *An intrinsic rolling of two n-dimensional oriented Riemannian manifolds M on $\widehat{M}$ without slipping or twisting is an absolutely continuous curve $q : [0,\tau] \to Q$, satisfying the following conditions: if $x(t) = \mathrm{pr}_M\, q(t)$ and $\widehat{x}(t) = \mathrm{pr}_{\widehat{M}}\, q(t)$, then*

(I') *no slip condition: $\dot{\widehat{x}}(t) = q(t)\dot{x}(t)$ for almost all t,*
(II') *no twist condition: $Z(t)$ is a parallel tangent vector field along $x(t)$, if and only if $q(t)Z(t)$ is parallel along $\widehat{x}(t)$ for almost all t.*

4.2. **Comparing rolling and intrinsic rolling along the same curves.** Suppose that the projection of a rolling map into $M \times \widehat{M}$ is a fixed pair of curves. Questions that naturally arise are:

- If $(q_1(t), p_1(t))$ and $(q_2(t), p_2(t))$ are two rollings of M on $\widehat{M}$, along $x(t)$ and $\widehat{x}(t)$, how do they relate to one another? How many of the properties of the rolling are fixed by choosing paths?
- Suppose that an intrinsic rolling $q(t)$ and two imbeddings, $\iota : M \to \mathbb{R}^{n+\nu}$ and $\widehat{\iota} : \widehat{M} \to \mathbb{R}^{n+\nu}$, are given. When can the intrinsic rolling $q(t)$ be extended to a rolling $(q(t), p(t))$? Is this extension unique?

Before we start working with this, let us consider the following simple example, where the different imbeddings are easy to picture.

Example 4. Let us consider $\widehat{M} = \mathbb{R}$, with the usual Euclidean structure, and $M = S^1$, with the subspace metric, when considered as the unit circle in $\mathbb{R}^2$, with positive orientation counterclockwise. Let $x : [0, \tau] \to S^1$ be written as $x(t) = e^{i\varphi(t)}$, $\varphi : [0, \tau] \to \mathbb{R}$ being an absolutely continuous function. Since $\mathrm{SO}(1)$ is just the trivial group, $Q \cong M \times \widehat{M}$. It is clear from the no-slipping condition that

$$\widehat{x}(t) = \widehat{x}(0) + \varphi(t) - \varphi(0).$$

Without loss of of generality, we may assume $\widehat{x}(0) = \varphi(0) = 0$. We consider the possible rollings under different imbeddings. In the following cases, e_1 and $\hat{e}_1$ will always be positively oriented unit basis vectors for TM and $T\widehat{M}$ respectively (when they are seen as sub-bundles of $T\mathbb{R}^{1+\nu}$ restricted to either M or $\widehat{M}$), while $\{\epsilon_\lambda\}_{\lambda=1}^{\nu}$ and $\{\hat{\epsilon}_\kappa\}_{\kappa=1}^{\nu}$ are positively oriented bases of $TM^\perp$ and $T\widehat{M}^\perp$. The coordinates of $\mathbb{R}^{1+\nu}$ will be denoted by $(\bar{x}_1, \dots, \bar{x}_n)$.

Case 1: Let us consider the simplest example, with

$$\iota_1 : M \to \mathbb{R}^2, \qquad \iota_1 : e^{i\varphi} \mapsto (\sin\varphi, 1 - \cos\varphi),$$
$$\widehat{\iota}_1 : \widehat{M} \to \mathbb{R}^2, \qquad \widehat{\iota}_1 : \widehat{x} \mapsto (\widehat{x}, 0).$$

Then

$$e_1(e^{i\varphi}) = \cos\varphi \frac{\partial}{\partial \bar{x}_1}(\iota_1(e^{i\varphi})) + \sin\varphi \frac{\partial}{\partial \bar{x}_2}(\iota_1(e^{i\varphi})),$$
$$\epsilon_1(e^{i\varphi}) = -\sin\varphi \frac{\partial}{\partial \bar{x}_1}(\iota_1(e^{i\varphi})) + \cos\varphi \frac{\partial}{\partial \bar{x}_2}(\iota_1(e^{i\varphi})),$$
$$\hat{e}_1(\widehat{x}) = \frac{\partial}{\partial \bar{x}_1}(\widehat{\iota}_1(\widehat{x})), \qquad \hat{\epsilon}_1(\widehat{x}) = \frac{\partial}{\partial \bar{x}_2}(\widehat{\iota}_1(\widehat{x})).$$

Here, also $\mathrm{SO}(\nu)$ is trivial, so there is so there is only one way to roll.

Case 2: We do the same imbeddings as above, only increasing the codimension by one.

$$\iota_2 : M \to \mathbb{R}^3, \qquad \iota_2 : e^{i\varphi} \mapsto (\sin\varphi, 1 - \cos\varphi, 0),$$
$$\widehat{\iota}_2 : \widehat{M} \to \mathbb{R}^3, \qquad \widehat{\iota}_2 : \widehat{x} \mapsto (\widehat{x}, 0, 0).$$

Then

$$e_1(e^{i\varphi}) = \cos\varphi \frac{\partial}{\partial \bar{x}_1}(\iota_2(e^{i\varphi})) + \sin\varphi \frac{\partial}{\partial \bar{x}_2}(\iota_2(e^{i\varphi}),$$
$$\epsilon_1(e^{i\varphi}) = -\sin\varphi \frac{\partial}{\partial \bar{x}_1}(\iota_2(e^{i\varphi})) + \cos\varphi \frac{\partial}{\partial \bar{x}_2}(\iota_2(e^{i\varphi})),$$
$$\epsilon_2(e^{i\varphi}) = \frac{\partial}{\partial \bar{x}_3}(\iota_2(e^{i\varphi})),$$

$$\hat{e}_1(\widehat{x}) = \frac{\partial}{\partial \bar{x}_1}(\widehat{\iota_2}(\widehat{x})), \qquad \hat{\epsilon}_1(\widehat{x}) = \frac{\partial}{\partial \bar{x}_2}(\widehat{\iota_2}(\widehat{x})), \qquad \hat{\epsilon}_2(\widehat{x}) = \frac{\partial}{\partial \bar{x}_3}(\widehat{\iota_2}(\widehat{x})).$$

Now we know that the matrix representation B of $p(t)$ with respect to the bases $\{e_\lambda\}_{\lambda=1}^{\nu}$ and $\{\hat{e}_\kappa\}_{\kappa=1}^{\nu}$, can be represented as

$$B = \begin{pmatrix} \langle \hat{e}_1, p(t)e_1 \rangle & \langle \hat{e}_1, p(t)e_2 \rangle \\ \langle \hat{e}_2, p(t)e_1 \rangle & \langle \hat{e}_2, p(t)e_2 \rangle \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{pmatrix} \in \mathrm{SO}(2).$$

We calculate the restrictions of $\theta(t)$ given by (III).

$$p(t)\frac{D}{dt}\epsilon_1(x(t)) = p(t)\nabla^{\perp}_{\dot{x}(t)}\epsilon_1 = 0 = \frac{D^{\perp}}{dt}p(t)\epsilon_1(x(t))$$
$$= -\dot{\theta}(t)(\sin\theta(t)\hat{\epsilon}_1 + \cos\theta(t)\hat{\epsilon}_1) + \cos\theta(t)\nabla^{\perp}_{\dot{\hat{x}}(t)}\hat{\epsilon}_1 - \sin\theta(t)\nabla^{\perp}_{\dot{\hat{x}}(t)}\hat{\epsilon}_2$$
$$= -\dot{\theta}(t)(\sin\theta(t)\hat{\epsilon}_1 + \cos\theta(t)\hat{\epsilon}_2),$$

for almost every t, so $\theta(t)$ is a constant.

Case 3: We continue with $\nu = 2$, but change the imbedding of $\widehat{M}$ to a spiral.

$$\iota_2 : M \to \mathbb{R}^3, \qquad \iota_2 : e^{i\varphi} \mapsto (\sin\varphi, 1-\cos\varphi, 0),$$
$$\widehat{\iota_3} : \widehat{M} \to \mathbb{R}^3, \qquad \iota_3 : \widehat{x} \mapsto \frac{1}{\sqrt{2}}(\cos\widehat{x}, \sin\widehat{x}, \widehat{x}).$$

Then

$$e_1(e^{i\varphi}) = \cos\varphi\frac{\partial}{\partial \bar{x}_1}(\iota_2(e^{i\varphi})) + \sin\varphi\frac{\partial}{\partial \bar{x}_2}(\iota_2(e^{i\varphi}),$$
$$\epsilon_1(e^{i\varphi}) = -\sin\varphi\frac{\partial}{\partial \bar{x}_1}(\iota_2(e^{i\varphi})) + \cos\varphi\frac{\partial}{\partial \bar{x}_2}(\iota_2(e^{i\varphi})),$$
$$\epsilon_2(e^{i\varphi}) = \frac{\partial}{\partial \bar{x}_3}(\iota_2(e^{i\varphi})),$$
$$\hat{e}_1(\widehat{x}) = \frac{1}{\sqrt{2}}\left(-\sin\widehat{x}\frac{\partial}{\partial \bar{x}_1}(\widehat{\iota_2}(\widehat{x})) + \cos\widehat{x}\frac{\partial}{\partial \bar{x}_2}(\widehat{\iota_3}(\widehat{x})) + \frac{\partial}{\partial \bar{x}_3}(\widehat{\iota_3}(\widehat{x}))\right),$$
$$\hat{\epsilon}_1(\widehat{x}) = \frac{1}{\sqrt{2}}\left(-\sin\widehat{x}\frac{\partial}{\partial \bar{x}_1}(\widehat{\iota_2}(\widehat{x})) + \cos\widehat{x}\frac{\partial}{\partial \bar{x}_2}(\widehat{\iota_3}(\widehat{x})) - \frac{\partial}{\partial \bar{x}_3}(\widehat{\iota_3}(\widehat{x}))\right),$$
$$\hat{\epsilon}_2(\widehat{x}) = -\cos\widehat{x}\frac{\partial}{\partial \bar{x}_1}(\widehat{\iota_2}(\widehat{x})) - \sin\widehat{x}\frac{\partial}{\partial \bar{x}_2}(\widehat{\iota_3}(\widehat{x})).$$

We have the same matrix representation of $p(t)$,

$$B = \begin{pmatrix} \langle \hat{e}_1, p(t)e_1 \rangle & \langle \hat{e}_1, p(t)e_2 \rangle \\ \langle \hat{e}_2, p(t)e_1 \rangle & \langle \hat{e}_2, p(t)e_2 \rangle \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{pmatrix} \in \mathrm{SO}(2).$$

We calculate the restrictions of $\theta(t)$ given by (III).

$$p(t)\nabla^{\perp}_{\dot{x}(t)}\epsilon_1 = 0 = \frac{D^{\perp}}{dt}p(t)\epsilon_1$$
$$= -\dot{\theta}(t)(\sin\theta(t)\hat{\epsilon}_1 + \cos\theta(t)\hat{\epsilon}_1) + \cos\theta(t)\nabla^{\perp}_{\dot{\hat{x}}(t)}\hat{\epsilon}_1 - \sin\theta(t)\nabla^{\perp}_{\dot{\hat{x}}(t)}\hat{\epsilon}_2$$

$$= \left(\frac{\dot{\widehat{x}}(t)}{\sqrt{2}} - \dot{\theta}(t) \right) (\sin \theta(t)\hat{\epsilon}_1 + \cos \theta(t)\hat{\epsilon}_2),$$

so $\theta(t) = \theta_0 + \frac{1}{\sqrt{2}}\widehat{x}(t)$. So now, the circle M will rotate along the spiral $\widehat{M}$, but its path is determined by the initial angle. Notice also that if we define a new orthonormal frame of $T\widehat{M}^\perp$ by

$$\widehat{\Upsilon}_1 = \cos\left(\frac{\widehat{x}}{\sqrt{2}}\right)\hat{\epsilon}_1 - \sin\left(\frac{\widehat{x}}{\sqrt{2}}\right)\hat{\epsilon}_2,$$

$$\widehat{\Upsilon}_2 = \sin\left(\frac{\widehat{x}}{\sqrt{2}}\right)\hat{\epsilon}_1 + \cos\left(\frac{\widehat{x}}{\sqrt{2}}\right)\hat{\epsilon}_2,$$

then $p(t)$ becomes a constant matrix with respect to the bases ϵ_1, ϵ_2 and $\widehat{\Upsilon}_1, \widehat{\Upsilon}_2$.

We see that for cases above, the intrinsic rolling $t \mapsto (e^{i\varphi(t)}, \varphi(t))$ either uniquely induces a rolling, or the rolling is determined by an initial configuration of the normal tangent spaces given by $\theta(0) = \theta_0$. Note also that we are able to find a choice of bases so that $p(t)$ is constant with respect to this basis. Notice that these bases consist of normal parallel vector fields.

We continue to work with oriented manifolds M and $\widehat{M}$ imbedded in $\mathbb{R}^{n+\nu}$ and containing curves $x(t)$ and $\widehat{x}(t)$, respectively. In the remaining of this section we will use the following notations: $\{e_j(t)\}_{j=1}^n$ will be a collection of parallel tangent vector fields along $x(t)$ that forms an orthonormal basis for $T_{x(t)}M$ at each point of M, $\{\epsilon_\lambda(t)\}_{\lambda=1}^\nu$ will be a collection of normal parallel vector fields along $x(t)$ forming an orthonormal basis for $T_{x(t)}M^\perp$. We know that we can construct such vector fields by parallel transport and normal parallel transport along $x(t)$. Parallel frames $\{\hat{e}_i\}_{i=1}^n$ and $\{\hat{\epsilon}_\kappa\}_{\kappa=1}^\nu$ will be defined similarly along $\widehat{x}(t)$. Recall that Latin indices $i, j, \dots$ always go from 1 to n, while Greek ones $\kappa, \lambda, \dots$ vary from 1 to ν.

The following lemma reflects that a rolling map preserves parallel vector fields. Namely, the image of a parallel frame over M has constant coordinates in a parallel frame over $\widehat{M}$.

Lemma 2. *A curve $(q(t), p(t))$ in $Q \oplus P_{\iota,\hat{\iota}}$ in the fibers over $(x(t), \widehat{x}(t))$, satisfies (II) and (III) if and only if the matrices $A(t) = (a_{ij}(t))$ and $B(t) = (b_{\kappa\lambda}(t))$, defined by*

$$a_{ij}(t) = \hat{e}_i^*(t)q(t)e_j(t), \qquad b_{\kappa\lambda}(t) = \hat{\epsilon}_\kappa^*(t)p(t)\epsilon_\lambda(t),$$

are constant.

Proof. Let $(q(t), p(t))$ be an absolutely continuous curve. Then we have $\langle \hat{e}_i, \dot{\hat{e}}_j \rangle = \langle e_i, \dot{e}_j \rangle = 0$ and

$$\dot{a}_{ij}(t) = \langle \dot{\hat{e}}_i, q(t)e_j \rangle + \left\langle \hat{e}_i, \frac{d}{dt}(q(t)e_j) \right\rangle$$

by the product rule. The vectors $q(t)^{-1}\hat{e}_i$, $q(t)e_j$ are tangent, so $\langle q(t)^{-1}\hat{e}_i, \dot{e}_j\rangle = \langle \dot{\hat{e}}_i, q(t)e_j\rangle = 0$ and

$$\begin{aligned}
\dot{a}_{ij}(t) &= \langle \hat{e}_i, \dot{q}(t)e_j\rangle + \langle \hat{e}_i, q(t)\dot{e}_j\rangle + \langle \dot{\hat{e}}_i, q(t)e_j\rangle \\
&= \langle \hat{e}_i, \dot{q}(t)e_j\rangle + \langle q(t)^{-1}\hat{e}_i, \dot{e}_j\rangle = \left\langle \hat{e}_i, \frac{d}{dt}\big(q(t)e_j\big) - q(t)\dot{e}_j \right\rangle \\
&= \left\langle \hat{e}_i, \frac{D}{dt}q(t)e_j - q(t)\frac{D}{dt}e_j \right\rangle = 0.
\end{aligned}$$

So (II) holds if and only if $\dot{a}_{ij}(t) = 0$. Similar result holds for the basis of the normal tangent bundle. □

The following two theorems give the answer to the questions raised at the beginning of this section.

Theorem 2. *Let $q : [0,\tau] \to Q$ be a given intrinsic rolling map without slipping or twisting with the projection $\mathrm{pr}_{M\times\widehat{M}}\, q_0(t) = (x(t), \widehat{x}(t))$. Define the vector spaces*

$$V = \{v(t) \textit{ is a parallel vector field along } x(t), \textit{ and } \langle v(t), \dot{x}(t)\rangle = 0 \textit{ for a.e. } t\},$$

$$\widehat{V} = \left\{\widehat{v}(t) \textit{ is a parallel vector field along } \widehat{x}(t), \textit{ and } \langle \widehat{v}(t), \dot{\widehat{x}}(t)\rangle = 0 \textit{ for a.e. } t\right\},$$

with the inner product and orientation induced by the metric and orientation on M and $\widehat{M}$, respectively. Then the two vector spaces have the same dimension, and if we denote this dimension by k, the following holds.

(a) *The map q is the unique intrinsic rolling of M on $\widehat{M}$ along $x(t)$ and $\widehat{x}(t)$ if and only if $k \leq 1$.*
(b) *If $k \geq 2$, all the rollings along $x(t)$ and $\widehat{x}(t)$ differ from q by an element in $\mathrm{SO}(\widehat{V})$.*

Remark 5. Both the inner product and orientation are preserved under parallel transport. Hence, for any pair $v, w \in V$, the value of $\langle v(t), w(t)\rangle$ remains constant for any t. The metric on M therefore induces a well defined inner product on V. Similarly, we can say that a collection of vector fields is positively oriented if it has this property for one value of t (and consequently for all values of t). Similar considerations hold for $\widehat{V}$.

Proof. Pick frames of parallel vector fields $\{e_i\}_{i=1}^n$ and $\{\hat{e}_i\}_{i=1}^n$ along $x(t)$ and $\widehat{x}(t)$, respectively, such that $q(t)e_i = \hat{e}_i$. This is possible due to Lemma 2. We also choose the frames in a way that the k first vector fields are orthogonal to $\dot{\widehat{x}}$. Notice that $e_1, \dots, e_k$ then forms a basis for V, and $\hat{e}_1, \dots, \hat{e}_n$ for $\widehat{V}$.

Writing $\dot{\widehat{x}} = \sum_{i=1}^n \dot{\widehat{x}}_i(t)\hat{e}_i(t)$ and $\dot{x} = \sum_{i=1}^n \dot{x}_i(t)e_i(t)$, we get $\dot{\widehat{x}}_i(t) = \dot{x}_i(t)$ and $\dot{\widehat{x}}_1(t) = \cdots = \dot{\widehat{x}}_k(t) = 0$. So, if $\widetilde{q}$ is any other rolling, then $A = (a_{ij}) =$

$(\langle \hat{e}_i(t), \widetilde{q}(t)\, e_j(t)\rangle)$ is clearly of the form

$$A = \begin{pmatrix} A' & 0 \\ 0 & \mathbf{1}_{n-k} \end{pmatrix}, \qquad A' \in \mathrm{SO}(k), \tag{9}$$

where $\mathbf{1}_{n-k}$ is the $\big((n-k)\times(n-k)\big)$-unit matrix. This will be unique if k is 0 or 1.If $k \geq 2$, there is more freedom, since it is not determined how an arbitrarily rolling $\widetilde{q}$ should map V into $\widehat{V}$.

The converse also holds, that is, for any matrix A on the form (9), there is a rolling corresponding to it. □

Theorem 3. *Let $q : [0,\tau] \to Q$ be an intrinsic rolling and let $\iota : M \to \mathbb{R}^{n+\nu}$ and $\widehat{\iota} : \widehat{M} \to \mathbb{R}^{n+\nu}$ be given imbeddings. Then, given an initial configuration $p_0 \in (P_{\iota,\widehat{\iota}})_{(x_0,\widehat{x}_0)}$, $(x_0, \widehat{x}_0) = \mathrm{pr}_{M\times\widehat{M}}\, q(0)$, there exists a unique rolling $(q,p) : [0,\tau] \to Q \oplus P_{\iota,\widehat{\iota}}$ satisfying $p(0) = p_0$.*

Proof. We pick normal parallel frames $\{\epsilon_\lambda(t)\}_{\lambda=1}^{\nu}$ and $\{\hat{\epsilon}_\kappa(t)\}_{\kappa=1}^{\nu}$ along $x(t)$ and $\widehat{x}(t)$, respectively. Let $B_0 \in \mathrm{SO}(\nu)$ be defined by

$$B = (b_{\kappa\lambda}) = \left(\langle \widehat{\epsilon}_\kappa(0), p_0\, \epsilon_\lambda(0)\rangle\right).$$

Then $p(t)$ must satisfy

$$b_{\kappa\lambda} = \langle \widehat{\epsilon}_\kappa(t), p(t)\, \epsilon_\lambda(t)\rangle,$$

by Lemma 2, and it is uniquely determined by this. □

Remark 6. Define the vector spaces

$$E = \{\epsilon(t) \text{ is a normal parallel vector field along } x(t)\}.$$

$$\widehat{E} = \{\widehat{\epsilon}(t) \text{ is a normal parallel vector field along } \widehat{x}(t)\},$$

with inner product and orientation induced by $TM^\perp$ and $T\widehat{M}^\perp$ respectively, in a similar way to what we described in remark 5. For a fixed intrinsic rolling q, a way of describing any extrinsic rolling (q,p) extending q is to say that it is determined up to a left action of $\mathrm{SO}(\widehat{E})$ or, equivalently, it is determined up to a right action of $\mathrm{SO}(E)$. Both $\mathrm{SO}(E)$ and $\mathrm{SO}(\widehat{E})$ are isomorphic to $\mathrm{SO}(\nu)$, but not canonically.

Corollary 1. *Assume that $x(t)$ is a geodesic in M. Then there exists an intrinsic rolling of M on $\widehat{M}$ along $(x(t), \widehat{x}(t))$ if and only if $\widehat{x}(t)$ is a geodesic with the same speed as $x(t)$. Moreover, if $n \geq 2$, and if $\widehat{V}$ is defined as in Theorem 2, then*

$$\dim \widehat{V} = n-1,$$

and all the rollings along $x(t)$ and $\widehat{x}(t)$ differ by an element in $\mathrm{SO}(\widehat{V})$.

Proof. Taking into account the equality $\frac{D}{dt}\dot{\widehat{x}}(t) = \frac{D}{dt}q(t)\dot{x}(t) = q(t)\frac{D}{dt}\dot{x}(t)$, we conclude that if $x(t)$ is a geodesic then $\widehat{x}(t)$ is also geodesic. In order to satisfy (I) we need to require that the speed of $\dot{\widehat{x}}(t)$ is the same as the speed of $\dot{x}(t)$. Conversely, the equality of speeds implies condition (I).

We start the construction of rolling map by choosing $e_1(t) = \frac{\dot{x}(t)}{\langle \dot{x}(t), \dot{x}(t)\rangle}$ that is parallel along $x(t)$. The remaining $n-1$ parallel vector fields we pick up in a way that they form an orthonormal basis together with $e_1(t)$ along the curve $x(t)$. We repeat the same construction for a parallel frame $\{\hat{e}_i(t)\}_{i=1}^n$ along $\widehat{x}(t)$. Define the intrinsic rolling $q(t)$ by

$$\begin{aligned} \hat{e}_1^*(t)\, q(t)\, e_j(t) = \hat{e}_j^*(t)\, q(t)\, e_1(t) = \delta_{1,j}, \\ A' = \left(\hat{e}_{i+1}^*(t)\, q(t)\, e_{j+1}(t)\right)_{i,j=1}^{n-1}, \end{aligned} \tag{10}$$

where $A' \in \mathrm{SO}(n-1)$ will be a constant matrix. Conversely, we can construct a rolling by formulas (10) starting from $A' \in \mathrm{SO}(n-1)$. ☐

5. Distributions for rolling and intrinsic rolling maps

The aim of this Section is to formulate the kinematic conditions of no-slipping and no-twisting in terms of a distribution. In this setting, a rolling will be an absolutely continuous curve almost everywhere tangent to this distribution.

5.1. **Local trivializations of Q.** Let $\pi : Q \oplus P_{\iota,\widehat{\iota}} \to M \times \widehat{M}$ denote the canonical projection. Consider a rolling $\gamma(t) = (q(t), p(t))$, then $\pi \circ \gamma(t) = (x(t), \widehat{x}(t))$. Given an arbitrary t_0 in the domain of $\gamma(t)$, let U and $\widehat{U}$ denote neighborhoods of $x(t_0)$ and $\widehat{x}(t_0)$ in M and $\widehat{M}$, respectively, such that the both bundles $TM \to M$ and $TM^\perp \to M$ trivialize being restricted to U. In the same way we chose $\widehat{U}$, such that both $T\widehat{M} \to \widehat{M}$ and $T\widehat{M}^\perp \to \widehat{M}$ trivialize when they are restricted to $\widehat{U}$. This implies that the bundle $Q \oplus P_{\iota,\widehat{\iota}} \to M \times \widehat{M}$, trivializes when it is restricted to $U \times \widehat{U}$. To see this, let $\{e_j\}_{j=1}^n$, $\{\epsilon_\lambda\}_{\lambda=1}^\nu$, $\{\hat{e}_i\}_{i=1}^n$ and $\{\hat{\epsilon}_\kappa\}_{\kappa=1}^\nu$ denote positively oriented orthonormal bases of vector fields of $TM|_U$, $TM^\perp|_U$, $T\widehat{M}|_{\widehat{U}}$ and $T\widehat{M}^\perp|_{\widehat{U}}$, respectively. Then there is a trivialization

$$\begin{array}{rcl} Q \oplus P_{\iota,\widehat{\iota}}|_{U\times\widehat{U}} & \overset{h}{\to} & U \times \widehat{U} \times \mathrm{SO}(n) \times \mathrm{SO}(\nu) \\ (q,p) & \mapsto & (x, \widehat{x}, A, B), \end{array} \tag{11}$$

given by projections

$$x = \mathrm{pr}_U(q,p), \qquad \widehat{x} = \mathrm{pr}_{\widehat{U}}(q,p),$$

$$A = (a_{ij})_{i,j=1}^n = \left(\langle q e_j, \hat{e}_i\rangle\right)_{i,j=1}^n, \qquad B = (b_{\kappa\lambda})_{\kappa,\lambda=1}^\nu = \left(\langle p\epsilon_\lambda, \hat{\epsilon}_\kappa\rangle\right)_{\kappa,\lambda=1}^\nu.$$

The domain of γ can be chosen connected, containing t_0, and such that its image lies in $\pi^{-1}(U \times \widehat{U})$. Let us identify $\gamma(t)$ with its image under the trivialization given by $(x(t), \widehat{x}(t), A(t), B(t))$.

Each of the requirements (I)-(III) can be written as restrictions to $\dot{\gamma}(t)$. We will show, that all admissible values of $\dot{\gamma}(t)$ form a distribution; that is a smooth sub-bundle, of $T(Q \oplus P_{\iota,\widehat{\iota}})$. We will use the local trivializations to describe this distribution.

5.2. **The tangent space of** $\mathrm{SO}(n)$**.** Let U and $\widehat{U}$ be as in Section 5.1. Then we get in trivialization

$$T\pi^{-1}(U \times \widehat{U}) = TU \times T\widehat{U} \times T\,\mathrm{SO}(n) \times T\,\mathrm{SO}(\nu).$$

The decomposition requires that we present a detailed description of the tangent space of $\mathrm{SO}(n)$ in terms of left and right invariant vector fields.

We start by considering the imbedding of $\mathrm{SO}(n)$ in $\mathrm{GL}(n)$, the group of invertible real $n \times n$ matrices. Denote the matrix entries of a matrix A by (a_{ij}) and the transpose matrix by A^t. Then, differentiating the condition $A^t A = \mathbf{1}$, we obtain

$$T\,\mathrm{SO}(n) = \bigcap_{i \leq j} \ker \omega_{ij}, \qquad \omega_{ij} = \sum_{r=1}^{n} \left(a_{rj}\, da_{ri} + a_{ri}\, da_{rj}\right).$$

It is clear that the tangent space at the identity 1 of $\mathrm{SO}(n)$ is spanned by

$$W_{ij}(1) := \frac{\partial}{\partial a_{ij}} - \frac{\partial}{\partial a_{ji}}, \qquad 1 \leq i < j \leq n.$$

We denote $\mathfrak{so}(n) = \mathrm{span}\{W_{ij}(1)\}$ following the classical notation. We use the left translation of these vector to define

$$W_{ij}(A) := A \cdot W_{ij}(1) = \sum_{r=1}^{n} \left(a_{ri} \frac{\partial}{\partial a_{rj}} - a_{rj} \frac{\partial}{\partial a_{ri}} \right) \tag{12}$$

as global left invariant basis of $T\,\mathrm{SO}(n)$. Note that the left and right action in $T\,\mathrm{SO}(n)$ is described by

$$A \cdot \frac{\partial}{\partial a_{ij}} = \sum_{r=1}^{n} a_{ri} \frac{\partial}{\partial a_{rj}} \qquad\qquad \frac{\partial}{\partial a_{ij}} \cdot A = \sum_{s=1}^{n} a_{js} \frac{\partial}{\partial a_{is}}.$$

We have the following formula to switch from left to right translation

$$\begin{aligned} A \cdot \frac{\partial}{\partial a_{ij}} &= \sum_{r=1}^{n} a_{ri} \frac{\partial}{\partial a_{rj}} = \sum_{l,r=1}^{n} a_{ri} \delta j, l \frac{\partial}{\partial a_{rl}} = \sum_{l,r,s=1}^{n} a_{ri} a_{si} a_{sl} \frac{\partial}{\partial a_{rl}} \\ &= \sum_{r,s=1}^{n} a_{ri} a_{si} \left(\frac{\partial}{\partial a_{rs}} \cdot A \right), \end{aligned}$$

and the other way around,

$$\frac{\partial}{\partial a_{ij}} \cdot A = \sum_{s=1}^{n} a_{js} \frac{\partial}{\partial a_{is}} = \sum_{l,s=1}^{n} a_{js} \delta_{i,l} \frac{\partial}{\partial a_{ls}} = \sum_{l,r,s=1}^{n} a_{js} a_{ir} a_{lr} \frac{\partial}{\partial a_{ls}}$$

$$= \sum_{r,s=1}^{n} a_{js} a_{ir} \left(A \cdot \frac{\partial}{\partial a_{rs}} \right).$$

Therefore, the right invariant basis of $T\,\mathrm{SO}(n)$ can be written as

$$W_{ij}(1) \cdot A = \mathrm{Ad}(A^{-1}) W_{ij}(A) = \sum_{r<s} (a_{ir} a_{js} - a_{jr} a_{is}) W_{rs}(A).$$

If we let W_{ij} be defined (12) also when i is not less then j, (so $W_{ij} = -W_{ji}$) then the bracket relations are given by

$$[W_{ij}, W_{kl}] = \delta_{j,k} W_{il} + \delta_{i,l} W_{jk} - \delta_{i,k} W_{jl} - \delta_{j,l} W_{ik}.$$

5.3. **Distributions.** Now we are ready to rewrite the kinematic conditions (I)-(III) as a distribution. Let $\gamma(t)$ be a rolling satisfying the conditions (I)-(III). Consider it image under the trivializations. Then

$$\dot{\gamma}(t) = \dot{x}(t) + \dot{\widehat{x}}(t) + \sum_{i,j=1}^{n} \dot{a}_{ij} \frac{\partial}{\partial a_{ij}} + \sum_{\kappa,\lambda=1}^{\nu} \dot{b}_{\kappa\lambda} \frac{\partial}{\partial b_{\kappa\lambda}}. \tag{13}$$

If we denote $\dot{x}(t)$ by $Z(t)$, then (I) holds if and only if $\dot{\widehat{x}}(t) = q(t)Z(t)$.

We want, basing on conditions (II) and (III), write the last two terms in (13) in right invariant basis of corresponding tangent spaces of $\mathrm{SO}(n)$ and $\mathrm{SO}(\nu)$. We start from (II) and remark that

$$q(t) e_j = \sum_{i=1}^{n} a_{ij}(t) \hat{e}_i, \quad \text{and} \quad q^{-1}(t) \hat{e}_i = \sum_{j=1}^{n} a_{ij}(t) e_j$$

for orthonormal bases $\{e_j\}_{j=1}^{n}$ and $\{\widehat{e}_j\}_{j=1}^{n}$. Condition (II) holds if and only if $q \frac{D}{dt} e_j(x(t)) = \frac{D}{dt} q e_j(x(t))$ for $j = 1, \ldots, n$, that yields

$$0 = \left\langle q \frac{D}{dt} e_j(x(t)) - \frac{D}{dt} q e_j(x(t)), \hat{e}_i \right\rangle$$

$$= \left\langle \nabla_{Z(t)} e_j, q^{-1} \hat{e}_i \right\rangle - \left\langle \sum_{l=1}^{n} \dot{a}_{lj} \hat{e}_l, \hat{e}_i \right\rangle - \left\langle \sum_{l=1}^{n} a_{lj} \nabla_{qZ(t)} \hat{e}_l, \hat{e}_i \right\rangle.$$

$$= \sum_{l=1}^{n} a_{il} \left\langle \nabla_{Z(t)} e_j, e_l \right\rangle - \dot{a}_{ij} - \sum_{l=1}^{n} a_{lj} \left\langle \nabla_{qZ(t)} \hat{e}_l, \hat{e}_i \right\rangle$$

for every $i, j = 1, \dots, n$. Hence, the third term in (13) can be written as follows

$$\begin{aligned}
\sum_{i,j=1}^{n} \dot{a}_{ij} \frac{\partial}{\partial a_{ij}} &= \sum_{i,j=1}^{n} \left(\sum_{l=1}^{n} a_{il} \left\langle \nabla_{Z(t)} e_j, e_l \right\rangle - \sum_{l=1}^{n} a_{lj} \left\langle \nabla_{qZ(t)} \hat{e}_l, \hat{e}_i \right\rangle \right) \frac{\partial}{\partial a_{ij}} \\
&= \sum_{j,l=1}^{n} \left\langle \nabla_{Z(t)} e_j, e_l \right\rangle A \cdot \frac{\partial}{\partial a_{lj}} - \sum_{i,l=1}^{n} \left\langle \nabla_{qZ(t)} \hat{e}_l, \hat{e}_i \right\rangle \frac{\partial}{\partial a_{il}} \cdot A \\
&= \sum_{i,j=1}^{n} \left\langle \nabla_{Z(t)} e_j, e_i \right\rangle A \cdot \frac{\partial}{\partial a_{ij}} - \sum_{i,j,r,s=1}^{n} a_{ir} a_{js} \left\langle \nabla_{qZ(t)} \hat{e}_j, \hat{e}_i \right\rangle A \cdot \frac{\partial}{\partial a_{rs}} \\
&= \sum_{i,j=1}^{n} \left(\left\langle \nabla_{Z(t)} e_j, e_i \right\rangle - \sum_{s=1}^{n} a_{sj} \left\langle \nabla_{qZ(t)} \hat{e}_s, \sum_{r=1}^{n} a_{ri} \hat{e}_r \right\rangle \right) A \cdot \frac{\partial}{\partial a_{ij}} \\
&= \sum_{i,j=1}^{n} \left(\left\langle \nabla_{Z(t)} e_j, e_i \right\rangle - \left\langle \nabla_{qZ(t)} q e_j, q e_i \right\rangle \right) A \cdot \frac{\partial}{\partial a_{ij}} \qquad (14)
\end{aligned}$$

The coefficients in the basis $A \cdot \frac{\partial}{\partial_{ij}}$ in the sum (14) are skew symmetric, from the property of the Levi-Civita connection. Now we can write

$$\sum_{i,j=1}^{n} \dot{a}_{ij} \frac{\partial}{\partial a_{ij}} = \sum_{i<j} \left(\left\langle \nabla_{Z(t)} e_j, e_i \right\rangle - \left\langle \nabla_{qZ(t)} q e_j, q e_i \right\rangle \right) W_{ij}(A). \tag{15}$$

Written in a right invariant basis, we obtain

$$\sum_{i,j=1}^{n} \dot{a}_{ij} \frac{\partial}{\partial a_{ij}} = \sum_{i<j} \left(\left\langle \nabla_{Z(t)} q^{-1} \hat{e}_j, q^{-1} \hat{e}_i \right\rangle - \left\langle \nabla_{qZ(t)} \hat{e}_j, \hat{e}_i \right\rangle \right) \mathrm{Ad}(A^{-1}) W_{ij}(A). \tag{16}$$

Similarly, (III) holds if and only if

$$\begin{aligned}
\sum_{\kappa,\lambda=1}^{\nu} \dot{b}_{\kappa\lambda} \tfrac{\partial}{\partial b_{\kappa\lambda}} &= \sum_{\kappa<\lambda} \left(\left\langle \nabla^{\perp}_{Z(t)} \epsilon_\lambda, \epsilon_\kappa \right\rangle - \left\langle \nabla^{\perp}_{qZ(t)} p\epsilon_\lambda, p\epsilon_\kappa \right\rangle \right) W_{\kappa\lambda}(B). \\
&= \sum_{\kappa<\lambda} \left(\left\langle \nabla^{\perp}_{Z(t)} p^{-1} \hat{\epsilon}_\lambda, p^{-1} \hat{\epsilon}_\kappa \right\rangle - \left\langle \nabla^{\perp}_{qZ(t)} \hat{\epsilon}_\lambda, \hat{\epsilon}_\kappa \right\rangle \right) \mathrm{Ad}(B^{-1}) W_{\kappa\lambda}(B).
\end{aligned} \tag{17}$$

Definition 5. *If X is a vector field on M, then let us define $\mathcal{V}(X)$ and $\mathcal{V}^{\perp}(X)$ the vector fields on $Q \oplus P_{\iota,\hat{\iota}}$, such that under any local trivialization h as in* (11) *and any $(q,p) \in \pi^{-1}(x)$ they satisfy*

$$dh\left(\mathcal{V}(X)(q,p)\right) = \sum_{i<j} \left(\left\langle \nabla_{X(x)} e_j, e_i \right\rangle - \left\langle \nabla_{qX(x)} q e_j, q e_i \right\rangle \right) W_{ij}(A). \tag{18}$$

$$dh\left(\mathcal{V}^{\perp}(X)(q,p)\right) = \sum_{\kappa<\lambda} \left(\left\langle \nabla^{\perp}_{X(x)} \epsilon_\lambda, \epsilon_\kappa \right\rangle - \left\langle \nabla^{\perp}_{qX(x)} p\epsilon_\lambda, p\epsilon_\kappa \right\rangle \right) W_{\kappa\lambda}(B). \tag{19}$$

Notice that if $Y(x) = X(x) = X_0 \in T_xM$, then $\mathcal{V}(Y)(q,p) = \mathcal{V}(X)(q,p)$ for every $(q,p) \in (Q \oplus P_{\iota,\widehat{\iota}})_x$. Hence, we may define $\mathcal{V}(X_0)(q,p)$ whenever $X_0 \in T_xM$ and $(q,p) \in (Q \oplus P_{\iota,\widehat{\iota}})_x$. Also notice that the map $X \mapsto \mathcal{V}(X)$ is linear. The same holds for $\mathcal{V}^\perp$.

Remark 7. Notice that, at first glace, it may seem that all of the coefficients of $W_{ij}(A)$ and $W_{\kappa\lambda}(A)$ in (18) and (19) vanish from conditions (II) and (III). This is not true, however. Even though, for any tangential vector field X

$$\frac{D}{dt}X(x(t)) = \nabla_{\dot{x}(t)}X(x(t)),$$

in general, $\nabla_{q\dot{x}(t)}q(t)e_j$ does not coincide with $\frac{D}{dt}q(t)e_j(x(t))$. To see this, notice that

$$\frac{D}{dt}a_{sj}\hat{e}_s(\widehat{x}(t)) = \dot{a}_{sj}\hat{e}_s(\widehat{x}(t)) + a_{sj}\nabla_{\dot{\widehat{x}}(t)}\hat{e}_s(\widehat{x}(t)) = \dot{a}_{sj}\hat{e}_s(\widehat{x}(t)) + a_{sj}\nabla_{q\dot{x}(t)}\hat{e}_s(\widehat{x}(t)),$$

while

$$\nabla_{q\dot{x}(t)}a_{sj}\hat{e}_s(x(t)) = a_{sj}\nabla_{q\dot{x}(t)}\hat{e}_s(x(t)).$$

Similar relations hold for $\frac{D^\perp}{dt}$.

We may now sum up our considerations that have been made in this Section in the following result.

Proposition 2. *A curve $(q(t), p(t))$ in $Q \oplus P_{\iota,\widehat{\iota}}$ is a rolling if and only if it is a horizontal curve with respect to the distribution E, defined by*

$$E_{(q,p)} = \left\{X_0 + qX_0 + \mathcal{V}(X_0)(q,p) + \mathcal{V}^\perp(X_0)(q,p) \,|\, X_0 \in T_xM\right\}, \quad (q,p) \in (Q \oplus P_{\iota,\widehat{\iota}})_x.$$

If we use the same symbol to denote the restriction of $\mathcal{V}(X)$ to Q, we also have

Proposition 3. *A curve $q(t)$ in Q is an intrinsic rolling if and only if it is a horizontal curve with respect to the distribution D, defined by*

$$D_q = \{X_0 + qX_0 + \mathcal{V}(X_0)(q) \,|\, X_0 \in T_xM\}, \quad q \in Q_x.$$

6. A controllable example: S^n rolling over $\mathbb{R}^n$

6.1. **Formulation of the rolling.** We want to illustrate the properties of the distributions, by proving that the unit sphere S^n in $\mathbb{R}^{n+1}$ rolling over $\mathbb{R}^n$ is a completely controllable system, by showing that the distribution D is bracket generating. This result was obtained in [14], but we want to present this example here in order to illustrate the advantages of our theory.

Consider the unit sphere S^n as the submanifold of the Euclidean space $\mathbb{R}^{n+1}$,

$$S^n = \left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \,|\, x_0^2 + \dots + x_n^2 = 1\right\},$$

with the induced metric.

For an arbitrary point $\tilde{x} = (\tilde{x}_0, \dots, \tilde{x}_n) \in S^n$, at least one of the coordinates $\tilde{x}_0, \dots, \tilde{x}_n$ does not vanish. Without lost of generality, we may assume that $\tilde{x}_n \neq 0$, and consider the neighborhood

$$U = \{(x_0, \dots, x_n) \in S^n | \pm x_n > 0\},$$

where the choice of the $\pm$ sign depends on the sign of $\tilde{x}_n$. To simplify the notation, we define the following functions on U

$$s_j(x) = \sum_{r=j}^{n} x_r^2 .$$

These functions are always strictly positive on U, and we use them to define an orthonormal basis of TU. We will write simply s_j instead of $s_j(x)$, since dependence of x is clear from the context. Define the following vector fields on U

$$e_j = \sqrt{\frac{s_j}{s_{j-1}}} \left(-\frac{\partial}{\partial x_{j-1}} + \frac{x_{j-1}}{s_j} \sum_{r=j}^{n} x_r \frac{\partial}{\partial x_r} \right), \qquad j = 1, \dots, n. \tag{20}$$

These vector fields form an orthonormal basis of the tangent space over U. We set $\hat{e}_i = \dfrac{\partial}{\partial \widehat{x}_i}$ to be the standard basis of $\mathbb{R}^n$.

Before proceeding with the necessary calculations, let us state two technical Lemmas whose proofs can be found in section 6.2 and 6.3.

Lemma 3. *Let* $1 \leq i < j \leq n$. *Then*

$$\langle \nabla_{e_k} e_j, e_i \rangle = -\frac{x_{i-1}\delta_{k,j}}{\sqrt{s_{i-1}s_i}} = -\langle \nabla_{e_k} e_i, e_j \rangle ,$$

for any $k = 1, \dots, n$.

Remark 8. The properties of the connection ∇ have the following consequences:

- The compatibility of ∇ with the metric and $\langle e_i, e_j \rangle = \delta_{i,j}$, imply that
$$\langle \nabla_{e_k} e_j, e_i \rangle = -\langle \nabla_{e_k} e_i, e_j \rangle .$$
In particular, $\langle \nabla_{e_k} e_i, e_i \rangle = 0$.
- The symmetry of ∇, imply that if $l < k$, then
$$[e_k, e_l] = \nabla_{e_k} e_l - \nabla_{e_l} e_k = \sum_{i=1}^{n} \langle \nabla_{e_k} e_l - \nabla_{e_l} e_k, e_i \rangle e_i = \frac{x_{l-1}}{\sqrt{s_{l-1}s_l}} e_k .$$

Lemma 4. *For* $k, l = 1, 2, \dots, n$

$$e_k \left(\frac{x_{l-1}}{\sqrt{s_{l-1}s_j}} \right) = \begin{cases} 0 & k > l \\ -\frac{1}{s_k} & k = l \\ & \\ -\frac{x_{k-1}x_{l-1}}{\sqrt{s_{k-1}s_k s_k s_{j-1}}} & k < l. \end{cases}$$

It is a direct consequence of the choice of the vector fields $\hat{e}_k$ that $\nabla_{\hat{e}_k}\hat{e}_l = 0$, and $[\hat{e}_k, \hat{e}_l] = 0$ for all $k, l = 1, \dots, n$.

Consider the vector fields $X_k = e_k + qe_k + \mathcal{V}(e_k)$ which generate the distribution $\mathcal{D}$, introduced in Proposition 3, restricted to U. In this case, we have the explicit form

$$X_k(x, \hat{x}, A) = e_k(x) + \sum_{i=1}^{n} a_{ik}\hat{e}_i(\widehat{x}) - \sum_{i=1}^{k-1} \frac{x_{i-1}}{\sqrt{s_{i-1}s_i}} W_{ik}(A).$$

In order to determine the commutators $[X_k, X_l]$, let us assume that $k > l$. Then

$$[X_k, X_l] = [e_k, e_l] - \sum_{i=1}^{k-1}\sum_{j=1}^{n} \frac{x_{i-1}}{\sqrt{s_{i-1}s_i}} W_{ik} a_{jl}\hat{e}_j + \sum_{j-1}^{l-1}\sum_{i=1}^{n} \frac{x_{j-1}}{\sqrt{s_{j-1}s_j}} W_{jl} a_{ik}\hat{e}_i$$

$$-\sum_{j=1}^{l-1} e_k\left(\frac{x_{j-1}}{\sqrt{s_{j-1}s_j}}\right) W_{jl} + \sum_{i=1}^{k-1} e_l\left(\frac{x_{i-1}}{\sqrt{s_{i-1}s_i}}\right) W_{ik} + \sum_{i=1}^{k-1}\sum_{j=1}^{l-1} \frac{x_{i-1}x_{j-1}}{\sqrt{s_{i-1}s_i s_{j-1}s_{j-1}}} [W_{ik}, W_{jl}]$$

$$= \frac{x_{l-1}}{\sqrt{s_{l-1}s_l}} e_k - \sum_{i=1}^{k-1}\sum_{j=1}^{n} \frac{x_{i-1}}{\sqrt{s_{i-1}s_i}} \left(a_{ji}\delta_{k,l} - a_{jk}\delta_{i,l}\right)\hat{e}_j + \sum_{j-1}^{l-1}\sum_{i=1}^{n} \frac{x_{j-1}}{\sqrt{s_{j-1}s_j}} \left(a_{ij}\delta_{l,k} - a_{il}\delta_{j,k}\right)\hat{e}_i$$

$$-\frac{1}{s_l} W_{lk} - \sum_{i=l+1}^{k-1} \frac{x_{i-1}x_{l-1}}{\sqrt{s_{l-1}s_l s_{i-1}s_i}} W_{ik} + \sum_{i=1}^{k-1}\sum_{j=1}^{l-1} \frac{x_{i-1}x_{j-1}}{\sqrt{s_{i-1}s_i s_{j-1}s_j}} (-\delta_{i,l}W_{jk} + \delta_{i,j}W_{lk})$$

$$= \frac{x_{l-1}}{\sqrt{s_{l-1}s_l}} \left(e_k + \sum_{j=1}^{n} a_{jk}\hat{e}_j - \sum_{i=l+1}^{k-1} \frac{x_{i-1}}{\sqrt{s_{i-1}s_i}} W_{ik} - \sum_{j=1}^{l-1} \frac{x_{j-1}}{\sqrt{s_{j-1}s_j}} W_{jk} \right)$$

$$-\frac{1}{s_l} W_{lk} + \sum_{j=1}^{l-1} \frac{x_{j-1}^2}{s_{j-1}s_j} W_{lk}$$

$$= \frac{x_{l-1}}{\sqrt{s_{l-1}s_l}} \left(e_k + \sum_{j=1}^{n} a_{jk}\hat{e}_j - \sum_{i=1}^{k-1} \frac{x_{i-1}}{\sqrt{s_{i-1}s_i}} W_{ik} \right) + \left(-\frac{1}{s_l} + \frac{x_{l-1}^2}{s_{l-1}s_l} + \sum_{j=1}^{l-1} \left(\frac{1}{s_j} - \frac{1}{s_j} \right) \right) W_{lk}$$

$$= \frac{x_{l-1}}{\sqrt{s_{l-1}s_l}} X_k - W_{lk}.$$

Define the vector fields Y_{lk}, for $l < k$, by

$$Y_{lk} := [X_l, X_k] + \frac{x_{l-1}}{\sqrt{s_{l-1}s_l}} X_k = W_{lk}.$$

Finally, let

$$Z_1 = [Y_{12}, X_2] = \sum_{i=1}^{n} a_{i1}\hat{e}_i,$$

$$Z_k = [X_1, Y_{1k}] = \sum_{i=1}^{n} a_{ik}\hat{e}_i\,, \qquad k = 2, \dots, n.$$

We conclude that the entire tangent space is spanned by $\{X_k\}_{k=1}^n$, $\{Y_{lk}\}_{1\le l<k\le n}$ and $\{Z_k\}_{k=1}^n$. Hence, D is a regular bracket generating distribution of step 3, which implies that the system of rolling S^n over $\mathbb{R}^n$ is completely controllable.

6.2. **Proof of Lemma 3.** The proof of this Lemma is rather technical and it consists mostly of rewriting formulas in an appropriate way. We begin with some observations.

- $s_{j-1} = x_{j-1}^2 + s_j$.
- If H is the Heaviside function

  $$H(x) = \begin{cases} 1 & \text{when } x \ge 0 \\ 0 & \text{when } x < 0 \end{cases},$$

  then

  $$\frac{\partial}{\partial x_k} s_j = 2x_k H(k-j).$$

- For any integer j,

  $$H(j) = \delta_{0,j} + H(j-1).$$

- Due to the identity

  $$\left\langle \frac{\partial}{\partial x_k}, e_i \right\rangle = \sqrt{\frac{s_i}{s_{i-1}}} \left(-\delta_{k,i-1} + \frac{x_k x_{i-1} H(k-i)}{s_i} \right),$$

  we obtain

  $$\begin{aligned} \left\langle \sum_{r=k}^{n} x_r \frac{\partial}{\partial x_r}, e_i \right\rangle &= \sqrt{\frac{s_i}{s_{i-1}}} \sum_{r=k}^{n} \left(-x_r \delta_{r,i-1} + \frac{x_{i-1} x_r^2 H(r-i)}{s_i} \right) \\ &= x_{i-1} \sqrt{\frac{s_i}{s_{i-1}}} \left(\frac{s_{\max\{k,i\}}}{s_i} - H(i-k-1) \right) \\ &= x_{i-1} \sqrt{\frac{s_i}{s_{i-1}}} \left(\delta_{i,k} + \frac{s_k H(k-i-1)}{s_i} \right). \end{aligned}$$

Step 1: Finding $\overline{\nabla}_{\frac{\partial}{\partial x_k}} e_j$. We calculate

$$\frac{\partial}{\partial x_k}\sqrt{\frac{s_j}{s_{j-1}}} = \frac{x_k H(k-j)}{\sqrt{s_{j-1}s_j}} - x_k H(k-j+1)\sqrt{\frac{s_j}{s_{j-1}^3}}$$

$$= x_k H(k-j)\sqrt{\frac{s_j}{s_{j-1}}}\left(\frac{1}{s_j} - \frac{1}{s_{j-1}}\right) - x_k\delta_{k,j-1}\sqrt{\frac{s_j}{s_{j-1}^3}}$$

$$= \sqrt{\frac{s_j}{s_{j-1}}}\left(\frac{x_k x_{j-1}^2 H(k-j)}{s_{j-1}s_j} - \frac{x_k\delta_{k,j-1}}{s_{j-1}}\right)$$

and get

$$\overline{\nabla}_{\frac{\partial}{\partial x_k}} e_j = \left(\frac{x_k x_{j-1}^2 H(k-j)}{s_{j-1}s_j} - \frac{x_k\delta_{k,j-1}}{s_{j-1}}\right) e_j$$

$$+\sqrt{\frac{s_j}{s_{j-1}}}\left(\frac{\delta_{k,j-1}}{s_j}\sum_{r=j}^n x_r\frac{\partial}{\partial x_r} - \frac{2x_{j-1}x_k H(k-j)}{s_j^2}\sum_{r=j}^n x_r\frac{\partial}{\partial x_r} + \frac{x_{j-1}H(k-j)}{s_j}\frac{\partial}{\partial x_k}\right)$$

$$= \left(\frac{x_k x_{j-1}^2 H(k-j)}{s_{j-1}s_j} - \frac{2x_k H(k-j)}{s_j} - \frac{x_k\delta_{k,j-1}}{s_{j-1}}\right) e_j$$

$$+\sqrt{\frac{s_j}{s_{j-1}}}\left(\frac{\delta_{k,j-1}}{s_j}\sum_{r=j}^n x_r\frac{\partial}{\partial x_r} - \frac{2x_k H(k-j)}{s_j}\frac{\partial}{\partial x_{j-1}} + \frac{x_{j-1}H(k-j)}{s_j}\frac{\partial}{\partial x_k}\right)$$

$$= -\left(x_k H(k-j)\frac{s_{j-1}+s_j}{s_{j-1}s_j} + \frac{x_k\delta_{k,j-1}}{s_{j-1}}\right) e_j$$

$$+\frac{1}{s_j}\sqrt{\frac{s_j}{s_{j-1}}}\left(\delta_{k,j-1}\sum_{r=j}^n x_r\frac{\partial}{\partial x_r} + H(k-j)\left(x_{j-1}\frac{\partial}{\partial x_k} - 2x_k\frac{\partial}{\partial x_{j-1}}\right)\right)$$

Step 2: Calculating $\overline{\nabla}_{e_k} e_j$. Using Step 1 and formula (20), we are able to compute

$$\overline{\nabla}_{e_k} e_j = \sqrt{\frac{s_k}{s_{k-1}}}\left(-\overline{\nabla}_{\frac{\partial}{\partial x_{k-1}}} e_j + \frac{x_{k-1}}{s_k}\sum_{l=k}^n x_l\overline{\nabla}_{\frac{\partial}{\partial x_l}} e_j\right)$$

$$= \sqrt{\frac{s_k}{s_{k-1}}}\left(\left(x_{k-1}H(k-j-1)\frac{s_{j-1}+s_j}{s_{j-1}s_j} + \frac{x_{k-1}\delta_{k,j}}{s_{j-1}}\right) e_j\right.$$

$$-\frac{1}{s_j}\sqrt{\frac{s_j}{s_{j-1}}}\left(\delta_{k,j}\sum_{r=j}^n x_r\frac{\partial}{\partial x_r} + H(k-j-1)\left(x_{j-1}\frac{\partial}{\partial x_{k-1}} - 2x_{k-1}\frac{\partial}{\partial x_{j-1}}\right)\right)$$

$$+\frac{x_{k-1}}{s_k}\left(-\left(s_{\max\{j,k\}}\frac{s_{j-1}+s_j}{s_{j-1}s_j} + \frac{x_{j-1}^2 H(j-k-1)}{s_{j-1}}\right) e_j\right.$$

$$+\frac{1}{s_j}\sqrt{\frac{s_j}{s_{j-1}}}\left(x_{j-1}H(j-k-1)\sum_{r=j}^{n}x_r\frac{\partial}{\partial x_r}+x_{j-1}\sum_{r=\max\{j,k\}}^{n}x_r\frac{\partial}{\partial x_r}-2s_{\max\{j,k\}}\frac{\partial}{\partial x_{j-1}}\right)\Bigg)\Bigg)\Bigg)$$

Step 3: Obtaining $\langle\nabla_{e_k}e_j,e_i\rangle$. We calculate it case by case,

- if $k=j$, then

$$\overline{\nabla}_{e_k}e_k=\sqrt{\frac{s_k}{s_{k-1}}}\Bigg(\frac{x_{k-1}}{s_{k-1}}e_k-\frac{1}{s_k}\sqrt{\frac{s_k}{s_{k-1}}}\sum_{r=k}^{n}x_r\frac{\partial}{\partial x_r}$$
$$+\frac{x_{k-1}}{s_k}\left(-\frac{s_{k-1}+s_k}{s_{k-1}}e_k+\frac{1}{s_k}\sqrt{\frac{s_k}{s_{k-1}}}\left(x_{k-1}\sum_{r=k}^{n}x_r\frac{\partial}{\partial x_r}-2s_k\frac{\partial}{\partial x_{k-1}}\right)\right)\Bigg)$$

$$=\sqrt{\frac{s_k}{s_{k-1}}}\Bigg(\frac{x_{k-1}s_k-x_{k-1}s_k-x_{k-1}s_{k-1}}{s_{k-1}s_k}e_k-\frac{1}{s_k}\sqrt{\frac{s_k}{s_{k-1}}}\sum_{r=k}^{n}x_r\frac{\partial}{\partial x_r}$$
$$+\frac{x_{k-1}}{s_k}\left(e_k-\sqrt{\frac{s_k}{s_{k-1}}}\frac{\partial}{\partial x_{k-1}}\right)\Bigg)$$

$$=-\frac{1}{s_{k-1}}\sum_{r=k-1}^{n}x_r\frac{\partial}{\partial x_r}$$

and so

$$\langle\nabla_{e_k}e_k,e_i\rangle=-\frac{x_{i-1}}{s_{k-1}}\sqrt{\frac{s_i}{s_{i-1}}}\left(\delta_{i,k-1}+\frac{s_{k-1}H(k-i-2)}{s_i}\right)=-\frac{x_{i-1}H(k-i-1)}{\sqrt{s_{i-1}s_i}}$$

- if $k<j$, then

$$\overline{\nabla}_{e_k}e_j=\sqrt{\frac{s_k}{s_{k-1}}}\Bigg(\frac{x_{k-1}}{s_k}\Bigg(-\left(\frac{s_{j-1}+s_j}{s_{j-1}}+\frac{x_{j-1}^2}{s_{j-1}}\right)e_j$$
$$+\frac{1}{s_j}\sqrt{\frac{s_j}{s_{j-1}}}\left(x_{j-1}\sum_{r=j}^{n}x_r\frac{\partial}{\partial x_r}+x_{j-1}\sum_{r=j}^{n}x_r\frac{\partial}{\partial x_r}-2s_j\frac{\partial}{\partial x_{j-1}}\right)\Bigg)\Bigg)$$
$$=\sqrt{\frac{s_k}{s_{k-1}}}\left(\frac{x_{k-1}}{s_k}\left(-2e_j+2e_j\right)\right)=0$$

- if $j<k$, then

$$\overline{\nabla}_{e_k}e_j=\sqrt{\frac{s_k}{s_{k-1}}}\Bigg(\left(x_{k-1}\frac{s_{j-1}+s_j}{s_{j-1}s_j}\right)e_j-\frac{1}{s_j}\sqrt{\frac{s_j}{s_{j-1}}}\left(x_{j-1}\frac{\partial}{\partial x_{k-1}}-2x_{k-1}\frac{\partial}{\partial x_{j-1}}\right)$$
$$+\frac{x_{k-1}}{s_k}\Bigg(-\left(s_k\frac{s_{j-1}+s_j}{s_{j-1}s_j}\right)e_j+\frac{1}{s_j}\sqrt{\frac{s_j}{s_{j-1}}}\left(x_{j-1}\sum_{r=k}^{n}x_r\frac{\partial}{\partial x_r}-2s_k\frac{\partial}{\partial x_{j-1}}\right)\Bigg)\Bigg)$$

$$= \sqrt{\frac{s_k s_j}{s_{k-1}s_{j-1}}}\left(-\frac{x_{j-1}}{s_j}\frac{\partial}{\partial x_{k-1}} + \frac{2x_{k-1}}{s_j}\frac{\partial}{\partial x_{j-1}} + \frac{x_{k-1}x_{j-1}}{s_k s_j}\sum_{r=k}^{n} x_r \frac{\partial}{\partial x_r} - \frac{2x_{k-1}}{s_j}\frac{\partial}{\partial x_{j-1}}\right)$$

$$= \frac{x_{j-1}}{\sqrt{s_{j-1}s_j}} e_k$$

The conclusion is that all the Christoffel symbols $\Gamma^i_{kj} = \langle \nabla_{e_k} e_j, e_i\rangle$ vanish, except for

$$\Gamma^k_{kj} = -\Gamma^j_{kk} = \frac{x_{j-1}}{\sqrt{s_{j-1}s_j}}, \qquad j < k.$$

6.3. **Proof of Lemma 4.** We need to consider three cases. Observe that

$$\frac{\partial}{\partial x_k}\left(\frac{x_{l-1}}{\sqrt{s_{l-1}s_l}}\right) = \frac{\delta_{l-1,k}}{\sqrt{s_{l-1}s_l}} - \frac{1}{s_{l-1}}\frac{x^2_{l-1}\delta_{l-1,k}}{\sqrt{s_{l1}s_l}}$$
$$- \frac{1}{s_{l-1}}\frac{x_{l-1}x_k H(k-l)}{\sqrt{s_{l-1}s_l}} - \frac{1}{s_l}\frac{x_{l-1}x_k H(k-l)}{\sqrt{s_{l-1}s_l}}$$

$$= \frac{1}{\sqrt{s^3_{l-1}s_l}}\left(\delta_{l-1,k}s_l - \frac{(s_l+s_{l-1})x_{l-1}x_k H(k-l)}{s_l}\right),$$

so

$$e_k\left(\frac{x_{l-1}}{\sqrt{s_{l-1}s_l}}\right) = \sqrt{\frac{s_k}{s^3_{l-1}s_l s_{k-1}}}\left(-\left(\delta_{l,k}s_l - \frac{(s_l+s_{l-1})x_{l-1}x_{k-1}H(k-l-1)}{s_l}\right)\right.$$
$$\left. + \frac{x_{k-1}}{s_k}\sum_{r=k}^{n} x_r\left(\delta_{l-1,r}s_l - \frac{(s_l+s_{l-1})x_{l-1}x_r H(r-l)}{s_l}\right)\right).$$

$$= \sqrt{\frac{s_k}{s^3_{l-1}s_l s_{k-1}}}\left(-\delta_{l,k}s_l + \frac{(s_l+s_{l-1})x_{l-1}x_{k-1}H(k-l-1)}{s_l}\right.$$
$$\left. + \frac{x_{k-1}}{s_k}\left(x_{l-1}s_l H(l-1-k) - \frac{(s_l+s_{l-1})x_{l-1}s_{\max\{k,l\}}}{s_l}\right)\right).$$

- If $k = l$, then

$$e_k\left(\frac{x_{k-1}}{\sqrt{s_{k-1}s_k}}\right) = \frac{1}{s^2_{k-1}}\left(-s_k - \frac{x_{k-1}}{s_k}\frac{(s_k+s_{k-1})x_{k-1}s_k}{s_k}\right)$$
$$= -\frac{1}{s^2_{k-1}}\frac{s^2_k + (s_k+s_{k-1})(s_{k-1}-s_k)}{s_k} = -\frac{1}{s_k}.$$

- If $k > l$, then

$$e_k\left(\frac{x_{l-1}}{\sqrt{s_{l-1}s_l}}\right) = \sqrt{\frac{s_k}{s^3_{l-1}s_l s_{k-1}}}\left(\frac{(s_l+s_{l-1})x_{l-1}x_{k-1}}{s_l} - \frac{x_{k-1}}{s_k}\frac{(s_l+s_{l-1})x_{l-1}s_k}{s_l}\right) = 0.$$

- If $k < l$, then

$$e_k\left(\frac{x_{l-1}}{\sqrt{s_{l-1}s_l}}\right) = \sqrt{\frac{s_k}{s_{l-1}^3 s_l s_{k-1}}}\left(\frac{x_{k-1}}{s_k}\left(x_{l-1}s_l - \frac{(s_l+s_{l-1})x_{l-1}s_l}{s_l}\right)\right)$$
$$= -\frac{x_{k-1}x_{l-1}}{\sqrt{s_{k-1}s_k s_{l-1}s_l}}.$$

7. A non-controllable example: SE(3) rolling over $\mathbb{R}^6$

7.1. **Calculation of the dimension of the orbits.** Let SE(3) be the group of orientation preserving isometries of $\mathbb{R}^3$. We consider the case of SE(3), endowed with a left invariant metric defined later, rolling over its tangent space at the identity $T_1\,\mathrm{SE}(3) = \mathfrak{se}(3)$, with metric obtained by restricting the left invariant metric on SE(3) to the identity. Our goal is to determine whether any two points in the configuration space can be joined by a curve tangent to the distribution presented in Definition 5. This problem is equivalent to the controllability of the system, that is, we want to obtain any configuration by rolling without twisting or slipping, from a given an initial configuration.

We give SE(3) coordinates as follows. For any $x \in \mathrm{SE}(3)$ there exist $C = (c_{ij}) \in \mathrm{SO}(3)$ and $r = (r_1, r_2, r_3) \in \mathbb{R}^3$, such that $x = (C, r)$ acts via

$$x(y) = Cy + r, \qquad \text{for all } y \in \mathbb{R}^3.$$

The tangent space of SE(3) at $x = (C, r)$ is spanned by the left invariant vector fields

$$\begin{aligned} e_1 = Y_1 &= \tfrac{1}{\sqrt{2}}\left(C\cdot\tfrac{\partial}{\partial c_{12}} - C\cdot\tfrac{\partial}{\partial c_{21}}\right) \\ &= \tfrac{1}{\sqrt{2}}\textstyle\sum_{j=1}^3\left(c_{j1}\tfrac{\partial}{\partial c_{j2}} - c_{j2}\tfrac{\partial}{\partial c_{j1}}\right) \end{aligned} \tag{21}$$

$$\begin{aligned} e_2 = Y_2 &= \tfrac{1}{\sqrt{2}}\left(C\cdot\tfrac{\partial}{\partial c_{13}} - C\cdot\tfrac{\partial}{\partial c_{31}}\right) \\ &= \tfrac{1}{\sqrt{2}}\textstyle\sum_{j=1}^3\left(c_{j1}\tfrac{\partial}{\partial c_{j3}} - c_{j3}\tfrac{\partial}{\partial c_{j1}}\right) \end{aligned} \tag{22}$$

$$\begin{aligned} e_3 = Y_3 &= \tfrac{1}{\sqrt{2}}\left(C\cdot\tfrac{\partial}{\partial c_{23}} - C\cdot\tfrac{\partial}{\partial c_{32}}\right) \\ &= \tfrac{1}{\sqrt{2}}\textstyle\sum_{j=1}^3\left(c_{j2}\tfrac{\partial}{\partial c_{j3}} - c_{j3}\tfrac{\partial}{\partial c_{j2}}\right) \end{aligned} \tag{23}$$

$$e_{k+3} = X_k = C \cdot \frac{\partial}{\partial r_k} = \sum_{j=1}^{3} c_{jk} \frac{\partial}{\partial r_j} \qquad k = 1, 2, 3. \tag{24}$$

Define a left invariant metric on SE(3) by declaring the vectors $e_1, \ldots, e_6$ to form an orthonormal basis. The mapping

$$\sum_{j=1}^{6} \widehat{x}_j e_j(1) \mapsto (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3, \widehat{x}_4, \widehat{x}_5, \widehat{x}_6) \in \mathbb{R}^6,$$

permits to identify $\mathfrak{se}(3)$ endowed with the induced metric, with $\mathbb{R}^6$ with the Euclidean metric. We write $\hat{e}_k = \frac{\partial}{\partial \widehat{x}_k}$ on $\mathbb{R}^6$ and try to see how the intrinsic rollings of SE(3) on $\mathbb{R}^6$ behave. Note that $Q = \mathrm{SE}(3) \times \mathbb{R}^6 \times \mathrm{SO}(6)$, because both manifolds SE(3) and $\mathbb{R}^6$ are Lie groups, so their tangent bundles are trivial, and $\dim Q = 27$.

Let us denote by ∇ the Levi-Civita connection on SE(3) or $\mathbb{R}^6$ with respect to the corresponding Riemannian metrics defined above. The covariant derivatives $\nabla_{e_i} e_j$ are nonzero only in the following cases

$$\begin{aligned}
\nabla_{Y_1} Y_2 &= -\nabla_{Y_2} Y_1 = -\frac{1}{2\sqrt{2}} Y_3 \\
\nabla_{Y_1} Y_3 &= -\nabla_{Y_3} Y_1 = \frac{1}{2\sqrt{2}} Y_2 \\
\nabla_{Y_2} Y_3 &= -\nabla_{Y_3} Y_2 = -\frac{1}{2\sqrt{2}} Y_1 \\
\nabla_{Y_1} X_k &= \frac{1}{\sqrt{2}} \left(\delta_{2,k} X_1 - \delta_{1,k} X_2 \right) \\
\nabla_{Y_2} X_k &= \frac{1}{\sqrt{2}} \left(\delta_{3,k} X_1 - \delta_{1,k} X_3 \right) \\
\nabla_{Y_3} X_k &= \frac{1}{\sqrt{2}} \left(\delta_{3,k} X_2 - \delta_{2,k} X_3 \right),
\end{aligned}$$

where $\delta_{i,j}$ denotes the Kronecker symbol. On the other hand, it is well-known that $\nabla_{\hat{e}_i} \hat{e}_j = 0$ for any i, j. Proposition 3 and Definition 5 show that the distribution D over Q is spanned by

$$\begin{aligned}
Z_1 &= Y_1 + qY_1 + \tfrac{1}{2\sqrt{2}} W_{23} + \tfrac{1}{\sqrt{2}} W_{45}, \\
Z_2 &= Y_2 + qY_2 - \tfrac{1}{2\sqrt{2}} W_{13} + \tfrac{1}{\sqrt{2}} W_{46}, \\
Z_3 &= Y_3 + qY_3 + \tfrac{1}{2\sqrt{2}} W_{12} + \tfrac{1}{\sqrt{2}} W_{56}, \\
K_1 &= X_1 + qX_1, \\
K_2 &= X_2 + qX_2, \\
K_3 &= X_3 + qX_3.
\end{aligned} \tag{25}$$

In order to determine the controllability of rolling SE(3) over $\mathbb{R}^6$, we employ the Orbit Theorem [6, 13]. In the case of D, defined by the vector fields (25) straightforward calculations yield that the flag associated to D is on the form

$$\begin{array}{rcl} D^2 & = & D \oplus \operatorname{span}\{W_{12}, W_{13}, W_{23}\}, \\ D^3 & = & D^2 \oplus \operatorname{span}\{qY_1, qY_2, qY_3\}, \\ D^4 & = & D^3, \end{array} \tag{26}$$

and so $\dim D^2 = 9$, $\dim D^k = 12$ for all $k \geq 3$ and the step of D is 3.

Let $(x_0, \widehat{x}_0, A_0)$ be an arbitrary point in Q, and let $\mathcal{O}_{(x_0,\widehat{x}_0,A_0)}$ denote the subset of all points in Q which are connected to $(x_0, \widehat{x}_0, A_0)$ by an intrinsic rolling. The Orbit Theorem asserts that, at each point, D^3 is contained in the tangent space of the orbits. However, since we know that D^3 has a local basis, we have the stronger result of

$$T_{(x,\widehat{x},A)}\mathcal{O}_{(x_0,\widehat{x}_0,A_0)} = D^3_{(x,\widehat{x},A)},$$

holding for all $(x, \widehat{x}, A) \in \mathcal{O}_{(x_0,\widehat{x}_0,A_0)}$.

It follows from (26) that $\mathcal{O}_{(x_0,\widehat{x}_0,A_0)}$ has dimension 12. Since $\mathcal{O}_{(x_0,\widehat{x}_0,A_0)}$ is not the entire Q, we conclude that the system is not controllable.

We end this Section with a concrete example of an intrinsic rolling $q(t) = (x(t), \widehat{x}(t), A(t))$, where

$$x(0) = \mathrm{id}_{\mathbb{R}^3}, \quad \widehat{x}(0) = 0, \quad A(0) = \mathbf{1}.$$

Define the curve $x : [0, \tau] \to \mathrm{SE}(3)$ by

$$x(t)y = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) & 0 \\ -\sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \psi(t) \end{pmatrix}, \tag{27}$$

where $\theta(t)$ and $\psi(t)$ are absolutely continuous functions with $\theta(0) = \psi(0) = 0$. Then $\dot{x} = \sqrt{2}\,\dot{\theta}(t)Y_1 + \dot{\psi}(t)X_3$ for almost every t, and the rolling has the form $\dot{q} = \sqrt{2}\,\dot{\theta}(t)Z_1 + \dot{\psi}(t)K_3$, or equivalently

$$\dot{x}(t) = \sqrt{2}\,\dot{\theta}(t)Y_1 + \dot{\psi}(t)X_3 \tag{28}$$

$$\dot{\widehat{x}}(t) = \sqrt{2}\,\dot{\theta}(t)\,qY_1 + \dot{\psi}(t)\,qX_3 \tag{29}$$

$$\dot{A}(t) = \dot{\theta}(t)\left(\frac{1}{2}W_{23}(A) + W_{45}(A)\right) \tag{30}$$

for almost every t. It follows from equation (29) that

$$\widehat{x}(t) = \begin{pmatrix} \sqrt{2}\,\theta(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ \psi(t) \end{pmatrix}.$$

Equation (30) can be written as

$$\dot{A}(t) = A \cdot \left(\frac{\dot{\theta}(t)}{2} W_{23}(1) + \dot{\theta}(t)\, W_{45}(1) \right),$$

which can be solved by exponentiating to obtain

$$A(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\left(\frac{\theta(t)}{2}\right) & \sin\left(\frac{\theta(t)}{2}\right) & 0 & 0 & 0 \\ 0 & -\sin\left(\frac{\theta(t)}{2}\right) & \cos\left(\frac{\theta(t)}{2}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta(t) & \sin\theta(t) & 0 \\ 0 & 0 & 0 & -\sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

7.2. **Imbedding of** $\mathrm{SE}(n)$ **into Euclidean space.** Since it is less obvious how to extend an intrinsic rolling of $\mathrm{SE}(3)$ on $\mathfrak{se}(3)$ to an extrinsic rolling in ambient space, we describe an isometric imbedding of $\mathrm{SE}(n)$ into the Euclidean space $\mathbb{R}^{(n+1)^2}$. Identify an element $\overline{C} \in \mathbb{R}^{(n+1)^2}$ with the matrix

$$\overline{C} = \begin{pmatrix} \overline{c}_{11} & \cdots & \overline{c}_{1,n+1} \\ \vdots & \ddots & \vdots \\ \overline{c}_{n+1,1} & \cdots & \overline{c}_{n+1,n+1} \end{pmatrix}.$$

Define the inner product on $\mathbb{R}^{(n+1)^2}$ by

$$\langle \overline{C}_1, \overline{C}_2 \rangle = \operatorname{trace}\left(\left(\overline{C}_1\right)^t \overline{C}_2 \right).$$

Note that since

$$\langle \overline{C}, \overline{C} \rangle = \sum_{i,j=1}^{n+1} |\overline{c}_{ij}|^2,$$

the metric $\langle \cdot, \cdot \rangle$ coincides with the Euclidean metric. From this we get that $\left\{ \frac{\partial}{\partial \overline{c}_{ij}} \right\}_{i,j=1}^{n+1}$ is an orthonormal basis for the tangent bundle $T\mathbb{R}^{(n+1)^2}$ with respect to $\langle \cdot, \cdot \rangle$.

We define the imbedding of $\mathrm{SE}(3)$ into $\mathbb{R}^{(n+1)^2}$ by

$$\begin{aligned} \iota : \qquad \mathrm{SE}(n) &\to \mathbb{R}^{(n+1)^2} \\ x = (C, r) &\mapsto \overline{C} = \begin{pmatrix} C & r \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This mapping is in fact an isometry of $\mathrm{SE}(n)$ onto its image. To see this, notice that the metrics coincide at the identity, and that the metric of $\mathbb{R}^{(n+1)^2}$, restricted to $\mathrm{Image}\,\iota$, is left invariant under the action of $\mathrm{SE}(n)$. Hence, the metrics on $\mathrm{SE}(n)$ and $\mathrm{Image}\,\iota$ coincide, and ι defines an isometric imbedding.

7.3. **Extrinsic rolling.** We will use the imbedding from Subsection 7.2 to construct an extrinsic rolling of $SE(3)$ over $\mathfrak{se}(3)$ in $\mathbb{R}^{16}$. We use ∂_{ij} to denote $\frac{\partial}{\partial \bar{c}_{ij}}$. For the sake of clarity, we denote by M the image of $SE(3)$ by ι. Then the vector fields spanning TM are

$$e_1 = Y_1 = \frac{1}{\sqrt{2}} \sum_{i=1}^{3} \left(\bar{c}_{i1} \partial_{i2} - \bar{c}_{i2} \partial_{i1} \right),$$

$$e_2 = Y_2 = \frac{1}{\sqrt{2}} \sum_{i=1}^{3} \left(\bar{c}_{i1} \partial_{i3} - \bar{c}_{i3} \partial_{i1} \right),$$

$$e_3 = Y_3 = \frac{1}{\sqrt{2}} \sum_{i=1}^{3} \left(\bar{c}_{i2} \partial_{i3} - \bar{c}_{i3} \partial_{i2} \right),$$

$$e_{3+k} = X_k = \sum_{j=1}^{3} \bar{c}_{ik} \partial_{i4}, \qquad k = 1, 2, 3,$$

where we suppressed $d\iota$ in the notation. We introduce an othonormal basis of $TM^{\perp}$

$$\Upsilon_1 = \frac{1}{\sqrt{2}} \sum_{j=1}^{3} \left(\bar{c}_{j1} \partial_{j2} + \bar{c}_{j2} \partial_{j1} \right),$$

$$\Upsilon_2 = \frac{1}{\sqrt{2}} \sum_{j=1}^{3} \left(\bar{c}_{j1} \partial_{j3} + \bar{c}_{j3} \partial_{j1} \right),$$

$$\Upsilon_3 = \frac{1}{\sqrt{2}} \sum_{j=1}^{3} \left(\bar{c}_{j2} \partial_{j3} + \bar{c}_{j3} \partial_{j2} \right),$$

$$\Psi_\lambda = \sum_{j=1}^{3} \bar{c}_{j\lambda} \partial_{j\lambda}, \qquad \lambda = 1, 2, 3$$

$$\Xi_\lambda = \partial_{4\mu}, \qquad \mu = 1, 2, 3, 4$$

We denote by $\widehat{M}$ the image of $\mathbb{R}^6$ into $\mathbb{R}^{16}$ by the imbedding

$$(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3, \widehat{x}_4, \widehat{x}_5, \widehat{x}_6) \overset{\widehat{\iota}}{\mapsto} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\widehat{x}_1 & \frac{1}{\sqrt{2}}\widehat{x}_2 & \widehat{x}_4 \\ -\frac{1}{\sqrt{2}}\widehat{x}_1 & 0 & \frac{1}{\sqrt{2}}\widehat{x}_3 & \widehat{x}_5 \\ -\frac{1}{\sqrt{2}}\widehat{x}_2 & -\frac{1}{\sqrt{2}}\widehat{x}_3 & 0 & \widehat{x}_6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have the following orthonormal basis of $T\widehat{M}$,

$$\hat{e}_1 = \frac{1}{\sqrt{2}} (\partial_{12} - \partial_{21}),$$

$$\hat{e}_2 = \frac{1}{\sqrt{2}} (\partial_{13} - \partial_{31}),$$

$$\hat{e}_3 = \frac{1}{\sqrt{2}}(\partial_{23} - \partial_{32}),$$

$$\hat{e}_{3+k} = \partial_{k4} \qquad k = 1, 2, 3$$

while the vector fields spanning $T\widehat{M}^{\perp}$,

$$\hat{\epsilon}_1 = \frac{1}{\sqrt{2}}(\partial_{12} + \partial_{21}),$$

$$\hat{\epsilon}_2 = \frac{1}{\sqrt{2}}(\partial_{13} + \partial_{31}),$$

$$\hat{\epsilon}_3 = \frac{1}{\sqrt{2}}(\partial_{23} + \partial_{32}),$$

$$\hat{\epsilon}_{3+\kappa} = \partial_{\kappa\kappa}, \qquad \kappa = 1, 2, 3,$$

$$\hat{\epsilon}_{6+\kappa} = \partial_{4\kappa} \qquad \kappa = 1, 2, 3, 4.$$

In order to extend an intrinsic rolling $q(t)$ with $\pi(q(t)) = (x(t), \widehat{x}(t))$, we will find an orthonormal frame of normal parallel vector fields along $x(t)$ and $\widehat{x}(t)$. Along $\widehat{x}(t)$, we may use the restriction of $\{\hat{\epsilon}_\kappa\}_{\kappa=1}^{10}$. For the curve $x(t)$ the answer is more complicated.

We first study the value of $\nabla^\perp$ for different choices of vector fields.

(1) $\nabla^\perp_X \Xi_\lambda = 0$, for any tangential vector field X.
(2) $\nabla^\perp_{X_k} \Upsilon = 0$, for any normal vector field Υ.
(3) Otherwise

	Υ_1	Υ_2	Υ_3	Ψ_1	Ψ_2	Ψ_3
$\nabla^\perp_{Y_1}$	$\frac{1}{2}(\Psi_1 - \Psi_2)$	$-\frac{1}{2\sqrt{2}}\Upsilon_3$	$\frac{1}{2\sqrt{2}}\Upsilon_2$	$-\frac{1}{2}\Upsilon_1$	$\frac{1}{2}\Upsilon_1$	0
$\nabla^\perp_{Y_2}$	$-\frac{1}{2\sqrt{2}}\Upsilon_3$	$\frac{1}{2}(\Psi_1 - \Psi_3)$	$\frac{1}{2\sqrt{2}}\Upsilon_1$	$-\frac{1}{2}\Upsilon_2$	0	$\frac{1}{2}\Upsilon_2$
$\nabla^\perp_{Y_3}$	$-\frac{1}{2\sqrt{2}}\Upsilon_2$	$\frac{1}{2\sqrt{2}}\Upsilon_1$	$\frac{1}{2}(\Psi_2 - \Psi_3)$	0	$-\frac{1}{2}\Upsilon_3$	$\frac{1}{2}\Upsilon_3$

We use the above relation to construct an extrinsic rolling. We will illustrate this by considering the curve (27).

Since $\dot{x}(t) = \sqrt{2}\,\dot{\theta}(t)Y_1(x(t)) + \dot{\psi}(t)X_3(x(t))$, the vector field

$$\Psi(t) = \sum_{\lambda=1}^{3} \left(\upsilon_\lambda(t)\Upsilon_\lambda(x(t)) + \upsilon_{3+\lambda}(t)\Psi_\lambda(x(t))\right),$$

is normal parallel along $x(t)$ if

$$\left(\dot{\upsilon}_1 - \frac{\dot{\theta}}{\sqrt{2}}(\upsilon_4 - \upsilon_5)\right)\Upsilon_1 + \left(\dot{\upsilon}_2 + \frac{\dot{\theta}}{2}\upsilon_3\right)\Upsilon_2 + \left(\dot{\upsilon}_3 - \frac{\dot{\theta}}{2}\upsilon_2\right)\Upsilon_3$$
$$+ \left(\dot{\upsilon}_4 + \frac{\dot{\theta}}{\sqrt{2}}\upsilon_1\right)\Psi_1 + \left(\dot{\upsilon}_5 - \frac{\dot{\theta}}{\sqrt{2}}\upsilon_1\right)\Psi_2 + \upsilon_6\Psi_3 = 0.$$

Hence we define a parallel orthonormal frame along $x(t)$ by

$$\epsilon_1(t) = \cos\theta \Upsilon_1(x(t)) - \frac{1}{\sqrt{2}}\sin\theta \Psi_1(x(t)) + \frac{1}{\sqrt{2}}\sin\theta\Psi_2(x(t))\,,$$

$$\epsilon_2(t) = \cos\left(\frac{\theta}{2}\right)\Upsilon_2(x(t)) + \sin\left(\frac{\theta}{2}\right)\Upsilon_3(x(t))\,,$$

$$\epsilon_3(t) = -\sin\left(\frac{\theta}{2}\right)\Upsilon_2(x(t)) + \cos\left(\frac{\theta}{2}\right)\Upsilon_3(x(t))\,,$$

$$\epsilon_4(t) = \frac{1}{\sqrt{2}}\sin\theta\Upsilon_1(x(t)) + \frac{\cos\theta+1}{2}\Psi_1(x(t)) + \frac{1-\cos\theta}{2}\Psi_2(x(t))\,,$$

$$\epsilon_5(t) = -\frac{1}{\sqrt{2}}\sin\theta\Upsilon_1(x(t)) + \frac{1-\cos\theta}{2}\Psi_1(x(t)) + \frac{1+\cos\theta}{2}\Psi_2(x(t))\,,$$

$$\epsilon_6(t) = \Psi_3(x(t))\,,$$

$$\epsilon_{6+\lambda}(t) = \Xi_\lambda(x(t))\,, \qquad \lambda = 1,2,3,4.$$

Thus $p(t)$ is represented by a constant matrix in the bases $\{\epsilon_\lambda(t)\}_{\lambda=1}^{10}$ and $\{\hat{\epsilon}_\kappa(t)\}_{\kappa=1}^{10}$. Let us choose $p(t)$ to be the identity in these bases, since this is the configuration given by the imbedding.

The curve $g(t) = (q(t), p(t))$ in $\mathrm{Isom}^+(\mathbb{R}^{16})$ is given by

$$g(t)\overline{x} = \overline{A}\overline{x} + \overline{r}(t),$$

where

$$\overline{A}(t) = \left(\begin{array}{cccccccccc|c}
\cos^2\frac{\theta}{2} & -\frac{\sin\theta}{2} & 0 & 0 & \frac{\sin\theta}{2} & \frac{\cos\theta-1}{2} & 0 & 0 & 0 & 0 & \\
\frac{\sin\theta}{2} & \cos^2\frac{\theta}{2} & 0 & 0 & \sin^2\frac{\theta}{2} & \frac{\sin\theta}{2} & 0 & 0 & 0 & 0 & \\
0 & 0 & \cos\frac{\theta}{2} & 0 & 0 & 0 & \sin\frac{\theta}{2} & 0 & 0 & 0 & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
-\frac{\sin\theta}{2} & \sin^2\frac{\theta}{2} & 0 & 0 & \cos^2\frac{\theta}{2} & -\frac{\sin\theta}{2} & 0 & 0 & 0 & 0 & \mathbf{0}_{10\times 6} \\
\frac{\cos\theta-1}{2} & -\frac{\sin\theta}{2} & 0 & 0 & \frac{\sin\theta}{2} & \cos^2\frac{\theta}{2} & 0 & 0 & 0 & 0 & \\
0 & 0 & -\sin\frac{\theta}{2} & 0 & 0 & 0 & \cos\frac{\theta}{2} & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin\frac{\theta}{2} & \cos\frac{\theta}{2} & \\
\hline
\multicolumn{10}{c|}{\mathbf{0}_{6\times 10}} & \mathbf{1}_6
\end{array}\right),$$

and $\overline{r}(t) = \left(-1, \frac{\theta}{\sqrt{2}}, 0, 0, \frac{\theta}{\sqrt{2}}, -1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0\right)^t$. Here, $\mathbf{0}_{m\times n}$ denotes the zero matrix of size $m\times n$ and $\mathbf{1}_6$ is the identity matrix of size 6×6.

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3.4 Paper D

GEOMETRIC CONDITIONS FOR THE EXISTENCE OF INTRINSIC ROLLINGS

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Abstract. We present necessary and sufficient conditions for the existence of intrinsic rollings of manifolds. Given a curve in one manifold and an initial configuration, the existence of a rolling follows from the construction of the development of a curve. We show that it corresponds to a rolling without slipping or twisting. Given two curves one in each of the rolling manifolds we find conditions under which an intrinsic rolling exists following the curves in terms of generalized geodesic curvatures.

1. Introduction

Rolling surfaces without slipping or twisting is one of the classical kinematic problems that in recent years has again attracted the attention of mathematicians due to its geometric and analytic richness. A very interesting historical account of problems in non-holonomic dynamics can be found in [3] in which the problem of the rolling sphere is presented as one of the first examples of a non-holonomic mechanical system. The interest in this particular case can be traced as far as the late 19th century and early 20th century, for instance, see [5, 6]. Recent developments searching to explain the symmetries of the system can be found in [1, 4] and a detailed exposition of the non-holonomy of the rolling sphere is presented in [11].

The definition of the so-called rolling map, which corresponds to rolling manifolds of dimension higher than two imbedded in $\mathbb{R}^m$ without slipping or twisting, was given for the first time in [14]. This was the starting point of [7] where this extrinsic point of view was shown to be equivalent to a purely intrinsic condition and a condition depending solely on the imbeddings of the manifolds. The extrinsic point of view, which depends on the imbeddings, has been successfully applied in some particular cases, obtaining interpolation results [9] and controllability [12, 16].

In the present article we address the problem of existence of rollings for two abstract manifolds of dimension n. We employ the coordinate-free approach introduced in [7] which allows us to consider the problem with purely intrinsic methods. The existence questions treated in this paper are two:

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finding a rolling along a given curve in one of the manifolds and an initial configuration, and determining conditions for a rolling to exist whenever the projection curves of the rolling are given in both of the manifolds.

This paper is organized as follows. In Section 2 we briefly recall the definition of an intrinsic rolling. In Section 3 we show that the well-known construction of development corresponds to a rolling without slipping or twisting. In Section 4 we find a complete characterization of curves along which surfaces can roll on each other. As a corollary, we obtain a geometric consequence of rolling a surface on $\mathbb{R}^2$ along a loop. Finally in Section 5 we deal with the higher dimensional situation, which requires the introduction of Frenet vector fields in order to generalize the notion of geodesic curvature.

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2. Intrinsic rolling

The aim of this Section is to provide the necessary background and notations of the coordinate-free approach of rolling manifolds without slipping or twisting as presented in [7]. As customary, in the rest of the article we use simply *rolling* to refer to rolling without slipping or twisting.

Let M and $\widehat{M}$ be two connected Riemannian manifolds of dimension n. The configuration space Q for the intrinsic rolling is the $\mathrm{SO}(n)-$bundle

$$Q = \left\{ q \in \mathrm{Isom}^+(T_xM, T_{\widehat{x}}\widehat{M}) \;\middle|\; x \in M, \, \widehat{x} \in \widehat{M} \right\}, \tag{1}$$

where $\mathrm{Isom}^+(V, W)$ stands for the space of linear isometries of the inner-product spaces V and W. As noted in [7], the bundle Q can also be represented as

$$Q = (FM \times F\widehat{M})/\mathrm{SO}(n),$$

where FM denotes the oriented unit frame bundle of M, i.e. the principal $\mathrm{SO}(n)-$bundle where the fiber over a point $x \in M$ is the collection of all oriented orthonormal frames in T_xM, and the quotient is with respect to the diagonal $\mathrm{SO}(n)-$action on the cartesian product of FM and $F\widehat{M}$.

Remark 1. Unless $n = 2$, the $\mathrm{SO}(n)-$bundle Q is **not** principal in the general case.

Denoting by $\mathrm{pr}_M : Q \to M$ the projection onto M and similarly the projection $\mathrm{pr}_{\widehat{M}}$, we have the following definition.

Definition 1. *An intrinsic rolling of M on $\widehat{M}$ is an absolutely continuous curve $q : [0, \tau] \to Q$, satisfying the following conditions: if $x(t) = \mathrm{pr}_M q(t)$ and $\widehat{x}(t) = \mathrm{pr}_{\widehat{M}} q(t)$, then*

(I) *no slip condition:* $\dot{\hat{x}}(t) = q(t)\dot{x}(t)$ *for almost all* t*;*
(II) *no twist condition:*

$$q(t)\frac{D}{dt}Z(t) = \frac{D}{dt}q(t)Z(t)$$

for any vector field $Z(t)$ *along* $x(t)$ *and almost every* t*.*

In the previous definition, the symbol $\frac{D}{dt}$ stands for the covariant derivative associated to the Levi-Civita connection on M or $\widehat{M}$.

The main result in [7] states that given an intrinsic rolling q, isometric imbeddings of M and $\widehat{M}$ into $\mathbb{R}^N$, for a sufficiently big N, and an initial configuration of the imbedded manifolds, there is a unique rolling in the sense of Sharpe [14, Appendix B] yielding to the same dynamics as the original rolling q.

In the following Sections, the letter Q is employed uniquely as the configuration space of the intrinsic rolling for the manifolds under consideration and it will always be considered as the bundle of isometries (1). Similarly, all the manifolds are connected and Riemannian.

3. Construction of a rolling: Development

3.1. **Development.** In the construction of stochastic trajectories on manifolds, the idea of a development plays a central role. Let us shortly recall the definition and construction of a development.

A general frame at $x \in M$ is an isomorphism $f : \mathbb{R}^n \to T_xM$, and denote the set of all general frames at x by $\mathcal{F}_x(M)$. Any general frame at x induces a choice of a basis of T_xM given by

$$f_j := f(\underbrace{0, \ldots, 1, \ldots, 0}_{1 \text{ in the } j-\text{th place}}), \quad j = 1, \ldots, n.$$

The general frame bundle $\mathcal{F}(M) = \coprod_x \mathcal{F}_x(M)$ can be naturally be given the structure of a manifold of dimension $n(n+1)$ with a principal $\mathrm{GL}_n(\mathbb{R})-$structure. The manifold structure of $\mathcal{F}(M)$ is such that the natural projection $\pi : \mathcal{F}(M) \to M$ is a smooth map.

Let M be equipped with an affine connection ∇. A curve into the general frame bundle $f : [0, \tau] \to \mathcal{F}(M)$ is called horizontal if the vector fields $f_j(t)$ are parallel along the curve $\pi \circ f : [0, \tau] \to M$. The tangent vectors of all horizontal curves form a distribution E called the Ehresmann connection associated to ∇. For any point $f \in F(M)$, a horizontal vector $v \in E_f$ is called the horizontal lift of $X \in T_{\pi(f)}M$ at f, if $\pi_* v = X$. Since $\pi_*|_{E_f}$ is an isomorphism of vector spaces, the horizontal lift is well defined. Write $H_X(f)$ to denote the horizontal lift of X at f. Note that, given a differentiable curve $x : [0, \tau] \to M$, the horizontal lift $f(t) = (f_1(t), \ldots, f_n(t))$, where each $f_j(t)$

is parallel along $x(t)$, satisfies the differential equation

$$H_{\dot{x}(t)}(f(t)) = \dot{f}(t). \tag{2}$$

The horizontal curve $f(t)$ solving (2) is only determined up to an initial condition $f_1(0), \dots, f_n(0)$.

Definition 2. *A curve $\widehat{x} : [0,\tau] \to \mathbb{R}^n$, where $\widehat{x}(0) = 0$, is called the anti-development of $x : [0,\tau] \to M$, if there is a horizontal curve $f(t)$, so that $(\pi \circ f)(t) = x(t)$ and*

$$f(t)(\dot{\widehat{x}}(t)) = \dot{x}(t). \tag{3}$$

It is convenient to remark that, using (2), equation (3) is often equivalently written as

$$\dot{f}(t) = H_{f(t)(\dot{\widehat{x}}(t))}(f(t)). \tag{4}$$

For the applications of developments to Brownian motion on manifolds and related topics, the interested reader can consult [8, Chapter 2] and [10]. In the firt reference it is also possible to find the comment that the development corresponds to a rolling with no slipping of M on $\mathbb{R}^n$, but no further interpretation is given.

3.2. **Development as an intrinsic rolling.** The aim of this subsection is to reinterpret the definition of development presented in Subsection 3.1 as an intrinsic rolling, where one of the manifolds is $\mathbb{R}^n$. The main idea is to show that equation (3) is equivalent to the no-slip condition, while the requirement of $f(t)$ being horizontal with respect to E is equivalent to the no-twist condition.

Let $x(t)$ be a differentiable curve in a connected oriented Riemannian manifold M. Let ∇ be the Levi-Civita connection on M and let FM be the oriented unit frame bundle. Note that we can consider the Ehresmann connection E as a subbundle of TFM, since parallel transport preserves the orientation and orthonormality of a frame in FM.

Without loss of generality, we can assume that the tangent bundle of M is trivial. This is possible since our considerations are of local nature. Let $e_1, \dots, e_n$ be a global oriented basis of orthonormal vector fields. This choice induces coordinates in FM given by

$$\begin{array}{ccc} F(M) & \xrightarrow{\cong} & M \times \mathrm{SO}(n) \\ f & \longmapsto & (x\,,\,(f_{ij})) \end{array} \quad \text{whenever} \quad f_j = \sum_{i=1}^n f_{ij} e_i(x). \tag{5}$$

Note that the choice of the basis $\{e_i\}_{i=1}^n$ implies that

$$H_k(f) := H_{e_k}(f) = e_k - \sum_{i,j,r=1}^n f_{rj} \Gamma_{kr}^i \frac{\partial}{\partial f_{ij}},$$

where $\Gamma_{is}^r := \langle e_r, \nabla_{e_i} e_s \rangle$ are the Christoffel symbols of M.

Let $\widehat{M} = \mathbb{R}^n$ with the Euclidean metric and the standard orientation and let $\widehat{x}(t) = (\widehat{x}_1(t), \ldots, \widehat{x}_n(t))$ be the anti-development of $x(t)$. Let

$$\widehat{e}_j = \frac{\partial}{\partial \widehat{x}_j},$$

be the standard basis for $T\widehat{M}$. As in (5), this choice of basis defines a trivialization of $F\widehat{M}$. Let $\widehat{f}(t)$ be a horizontal lift of $\widehat{x}(t)$ to $F(\widehat{M})$. By straightforward calculations, we have that

$$\dot{\widehat{f}}_{ij}(t) = 0, \qquad \widehat{f}_j(t) = \sum_{i=1}^{n} \widehat{f}_{ij}(t)\widehat{e}_i(\widehat{x}(t)), \tag{6}$$

For practical purposes, we will pick the horizontal lift $\widehat{f}(t)$ satisfying the relations $\widehat{f}_j(t) = \widehat{e}_j(\widehat{x}(t))$.

Let us consider the coordinates of the velocity vectors $\dot{x}(t)$ and $\dot{\widehat{x}}(t)$ given by

$$\dot{x}_j(t) = \langle \dot{x}(t), e_j(x(t))\rangle \quad \text{and} \quad \dot{\widehat{x}}_j(t) = \langle \dot{\widehat{x}}, \widehat{e}_j(\widehat{x}(t))\rangle.$$

Note that the relations (3) and (4) imply that

$$\dot{\widehat{x}}_j = \sum_{i=1}^{n} f_{ji}\dot{x}_i, \tag{7}$$

$$f_{ij} = -\sum_{k,r=1}^{n} \dot{x}_k f_{rj}\Gamma^i_{kr}. \tag{8}$$

In this terms, the connection between developments and rollings is rather straightforward. For each $t \in [0, \tau]$, define an isometry $q(t)$ by

$$\widehat{f}(t) = q(t) f(t).$$

If we write $q_{ij} = \langle \widehat{e}_i, q e_j\rangle$, then $q_{ij} = \sum_{r=1}^{n} \widehat{f}_{ir} f_{jr}$. Simple computations allow us to reformulate equation (7) as the no-slip condition

$$\dot{\widehat{x}} = q(t)\,\dot{x}(t),$$

and similarly, equations (6) and (8) imply the relation

$$\dot{q}_{ij} = \sum_{k,r=1}^{n} \dot{x}_k q_{ir}\Gamma^r_{kj},$$

which is equivalent to the no-twist condition, see [7].

3.3. **Construction of a rolling.** The aim of this subsection is to show an explicit construction of the development along a curve $x : [0, \tau] \to M$ for a sufficiently small $\tau > 0$ in terms of the Riemannian exponential of M. As seen in Subsection 3.2, this is essentially the same as constructing an intrinsic rolling of M on $\mathbb{R}^n$ along the curve x starting at a given $q_0 = (x_0, \widehat{x}_0, A_0) \in Q$, where $x(0) = x_0$ and A_0 is an isometry between $T_{x_0}M$ and $T_{\widehat{x}_0}\mathbb{R}^n$.

Let us assume that the curve x is a geodesic in M. This condition is necessary for the proof of Proposition 1 to be valid. The case for general curves will be addressed in future research.

Let $U \times \widehat{U} \subset M \times \mathbb{R}^n$ be a neighborhood of $(x_0, \widehat{x}_0) \in M \times \mathbb{R}^n$ such that the bundle $Q|_{U\times\widehat{U}}$ is trivial and the inverse of Riemannian exponential map at x_0 restricted to U is an isometry. Assume τ is sufficiently small so that $x([0, \tau]) \subset U$.

We construct a curve $\widehat{x} : [0, \tau] \to \mathbb{R}^n$ as follows:

$$\widehat{x}(t) = A_0 \circ \exp_{x_0}^{-1} \circ\, x(t), \tag{9}$$

where $\exp_{x_0}$ denote the Riemannian exponential mapping of M at x_0. Using this curve, we can define a map $A : [0, \tau] \to \mathrm{SO}(n)$ as follows. Let $X_1, \dots, X_n$ be an orthonormal basis of $T_{x_0}M$ and $\widehat{X}_i = q_0 X_i$ be the corresponding orthonormal basis of $T_{\widehat{x}_0}\mathbb{R}^n$. By parallel translating both bases along x and $\widehat{x}$, we define the vector fields $X_i(t)$ and $\widehat{X}_i(t)$ along x and $\widehat{x}$ respectively. The map $A(t)$ is defined as the isometry mapping $X_i(t)$ to $\widehat{X}_i(t)$. Note that by construction $A(0) = A_0$.

With these notations, we have the following result.

Proposition 1. *Let $x : [0, \tau] \to M$ be a geodesic in M and let $\widehat{x} : [0, \tau] \to \mathbb{R}^n$ be defined by equation* (9)*. The curve*

$$\begin{array}{cccc} q & : & [0,\tau] & \to & Q \cong U \times \widehat{U} \times \mathrm{SO}(n) \\ & & t & \mapsto & (x(t), \widehat{x}(t), A(t)) \end{array} \tag{10}$$

defined for a sufficiently small τ, is an intrinsic rolling.

Proof. The no-twist condition is satisfied by construction, thus we only need to check that the no-slip condition

$$\dot{\widehat{x}}(t) = A(t)\,\dot{x}(t)$$

holds. Since x is a geodesic, it can be written locally as $\exp_{x_0}(tv)$, where $v = \dot{x}(0)$. This implies that locally

$$\widehat{x}(t) = A_0(v)\,t,$$

and thus $\dot{\widehat{x}}(t) = A(t)\,\dot{x}(t) = v$. □

4. Existence of intrinsic rollings in dimension 2

The aim of the present section and Section 5 is to discuss the existence of an intrinsic rolling of two manifolds, M and $\widehat{M}$, following given trajectories $x : [0,\tau] \to M$ and $\widehat{x} : [0,\tau] \to \widehat{M}$. More precisely, the problem asks whether a rolling of the form

$$\begin{array}{rcccc} q & : & [0,\tau] & \to & Q \\ & & t & \mapsto & (x(t), \widehat{x}(t), A(t)) \end{array} . \tag{11}$$

exists. Before trying to give sufficient conditions for the general situation, let us see the concrete case of surfaces. We follow the notation in [2].

Let us assume that the curves x and $\widehat{x}$ are parametrized by arc-length. It is clear that requiring x and $\widehat{x}$ to have the same length is a necessary condition for the existence of q as in (11). It is easy to construct examples to see that this is not sufficient. For the case of surfaces this problem has a complete solution, assuming the curves are sufficiently regular, as seen in the following Theorem.

Theorem 1. *Let M and $\widehat{M}$ be two Riemannian connected surfaces. Let $x : [0,\tau] \to M$ and $\widehat{x} : [0,\tau] \to \widehat{M}$ be two curves of class C^2, parameterized by arc-length and geodesic curvatures $k_g(t)$ and $\widehat{k}_g(t)$ respectively. Then, there is a rolling*

$$\begin{array}{rcccc} q & : & [0,\tau] & \to & Q \\ & & t & \mapsto & (x(t), \widehat{x}(t), \theta(t)) \end{array}$$

along x and $\widehat{x}$ if and only if $k_g(t) = \widehat{k}_g(t)$.

Proof. Note that the condition that x and $\widehat{x}$ have the same length assures that, if there is a rolling, the no-slip condition is already satisfied. This means we should prove the no-twist condition only.

Let $v : [0,\tau] \to [0, 2\pi)$ be a curve of class C^1 such that

$$\dot{x}(t) = \cos(v(t))e_1 + \sin(v(t))e_2,$$

where $\{e_1, e_2\}$ is an oriented local orthonormal frame in M, and define the curve $\theta : [0,\tau] \to [0, 2\pi)$ of class C^1 such that

$$\dot{\widehat{x}}(t) = \cos(v(t) - \theta(t))\widehat{e}_1 + \sin(v(t) - \theta(t))\widehat{e}_2,$$

where $\{\widehat{e}_1, \widehat{e}_2\}$ is an oriented local orthonormal frame in $\widehat{M}$.

Define the local normal vector fields

$$N(t) = -\sin(v(t))e_1 + \cos(v(t))e_2,$$

$$\widehat{N}(t) = -\sin(v(t) - \theta(t))\widehat{e}_1 + \cos(v(t) - \theta(t))\widehat{e}_2,$$

and then the geodesic curvatures of x and $\widehat{x}$ have the form

$$k_g(t) = \left\langle \frac{D}{dt}\dot{x}(t), N(t) \right\rangle = \dot{v}(t) - \cos(v(t))\Gamma^1_{12} - \sin(v(t))\Gamma^1_{22}, \tag{12}$$

$$\widehat{k}_g(t) = \left\langle \frac{D}{dt}\dot{\widehat{x}}(t), \widehat{N}(t) \right\rangle = \\ = \dot{v}(t) - \dot{\theta}(t) - \cos(v(t) - \theta(t))\widehat{\Gamma}^1_{12} - \sin(v(t) - \theta(t))\widehat{\Gamma}^1_{22}, \tag{13}$$

where $\Gamma^1_{12}, \Gamma^1_{22}$ are the Christoffel symbols of the basis $\{e_1, e_2\}$, and similarly for $\widehat{\Gamma}^1_{12}, \widehat{\Gamma}^1_{22}$.

From the equations (12) and (13), we see that the equality $k_g(t) = \widehat{k}_g(t)$ is equivalent to the condition

$$\dot{\theta}(t) = \cos(v(t))\Gamma^1_{12} + \sin(v(t))\Gamma^1_{22} - \cos(v(t) - \theta(t))\widehat{\Gamma}^1_{12} - \sin(v(t) - \theta(t))\widehat{\Gamma}^1_{22}$$

which coincides with the no-twist condition in dimension 2 found in [2, Chapter 24]. □

This result has the following Corollary, which is a very interesting geometric consequence and answers a question posed by R. Montgomery [13] in the case that one of the manifolds is $\mathbb{R}^2$ and the trajectories are loops, as seen in Figure 1.

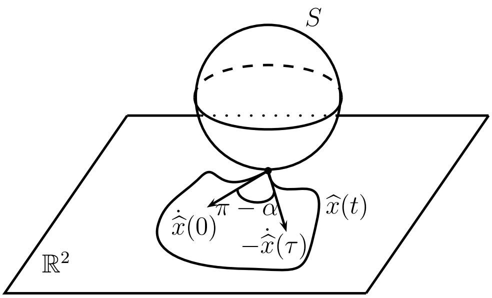

FIGURE 1. A sphere S rolling following a loop $\widehat{x}(t)$ in $\mathbb{R}^2$.

Corollary 1. *With the notation and hypotheses of Theorem 1, assume that $\widehat{M} = \mathbb{R}^2$ with the usual Riemannian structure and the curves $x : [0, \tau] \to M$ and $\widehat{x} : [0, \tau] \to \mathbb{R}^2$ are simple loops, where $x(0) = x(\tau)$ and $\widehat{x}(0) = \widehat{x}(\tau)$. Let α be the angle between $\dot{x}(0)$ and $\dot{x}(\tau)$, then*

$$\int_0^\tau k_g(t)dt = \alpha.$$

Proof. Since $\widehat{M} = \mathbb{R}^2$, we have $\widehat{\Gamma}^1_{12} = \widehat{\Gamma}^1_{22} = 0$, thus

$$k_g(t) = \widehat{k}_g(t) = \dot{v}(t) - \dot{\theta}(t).$$

Since the curve $\theta : [0,\tau] \to [0, 2\pi)$ must be a loop, we have that

$$\int_0^\tau \dot{\theta}(t)dt = 0,$$

which implies that

$$\int_0^\tau k_g(t)dt = \int_0^\tau (\dot{v}(t) - \dot{\theta}(t))dt = v(\tau) - v(0) = \alpha.$$

The Corollary follows. □

5. Existence of intrinsic rollings in dimension n

In order to find a condition similar to the one in Theorem 1 in the case of rolling manifolds of higher dimension, it is necessary to find the correct analog to the geodesic curvature. The definition that is suitable in this context can be found in [15, pp. 21–32].

Let $x(t)$ be a curve of class C^n parametrized by arc length. Consider the following process

- Define $v_1(t) = \dot{x}(t)$.
- If $\frac{D}{dt} v_1 \neq 0$, a.e., define v_2 to be a unit vector field satisfying
$$\frac{D}{dt} v_1(t) = \kappa_1(t) v_2(t),$$
for some function $\kappa_1(t)$ of class C^{n-2}.
- If $\frac{D}{dt} v_{j-1} + \kappa_{j-2} v_{j-2} \neq 0$, a.e., define v_j to be a unit vector field satisfying
$$\frac{D}{dt} v_{j-1}(t) + \kappa_{j-2} v_{j-2}(t) = \kappa_{j-1}(t) v_j(t), \tag{14}$$
for some function $\kappa_{j-1}(t)$ of class C^{n-j}.

Note that $k_1(t)$ is a direct analog to the geodesic curvature. Whenever it exists, the vector field $v_2(t)$ is orthogonal to v_1 as can be seen from the definition of covariant derivative

$$\left\langle v_1(t), \frac{D}{dt} v_1(t) \right\rangle = \frac{1}{2} \frac{d}{dt} \langle v_1(t), v_1(t) \rangle = 0,$$

where the last equality hold since $v_1(t)$ has norm one. Similarly, since $v_2(t)$ has norm one, it follows that

$$\left\langle v_2(t), \frac{D}{dt} v_2(t) \right\rangle = 0. \tag{15}$$

The fact that $\langle v_1(t), v_2(t)\rangle = 0$ implies

$$
(16) \quad 0 = \frac{d}{dt}\langle v_1(t), v_2(t)\rangle = \left\langle v_1(t), \frac{D}{dt} v_2(t)\right\rangle + \left\langle v_2(t), \frac{D}{dt} v_1(t)\right\rangle = \\
= \left\langle v_1(t), \frac{D}{dt} v_2(t)\right\rangle + \kappa_1(t) = \left\langle v_1(t), \frac{D}{dt} v_2(t) + \kappa_1(t) v_1(t)\right\rangle,
$$

Equations (15) and (16) imply that the vector field

$$
\frac{D}{dt} v_2(t) + \kappa_1(t) v_1(t)
$$

is orthogonal to both $v_1(t)$ and $v_2(t)$. By defining the vector fields $v_j(t)$ inductively by (14), we see that $\langle v_i(t), v_j(t)\rangle = \delta_{i,j}$ for all i, j as long as the vector fields exist.

Definition 3. *The vector field $v_j(t)$ is called the the $j-$th Frenet vector field of the curve x. The function $\kappa_j(t)$ is called the $j-$th geodesic curvature of the curve x.*

Remark 2. In the literature it is common to require that κ_{j-1} is positive, see for example [15, Chapter 7B]. We do not adopt this convention since, in that case, it is usual to also require that $\frac{D}{dt} v_{j-1} + \kappa_{j-2} v_{j-2} \neq 0$ for all t. For the purpose of the following results both v_j and κ_{j-1} are only defined up to signs.

Theorem 2. *Let M and $\widehat{M}$ be two Riemannian manifolds of dimension n, and let $x : [0,\tau] \to M$ and $\widehat{x} : [0,\tau] \to \widehat{M}$ be two curves of class C^n, parametrized by arc-length. Suppose that both x and $\widehat{x}$ have n well defined Frenet vector fields and $n-1$ geodesic curvatures $\{\kappa_j\}_{j=1}^{n-1}$ and $\{\widehat{\kappa}_j\}_{j=1}^{n-1}$ respectively. Then there exists a rolling along $x(t)$ and $\widehat{x}(t)$ if and only if*

$$
(17) \qquad \kappa_j = \pm\widehat{\kappa}_j, \qquad j = 1, \ldots, n.
$$

Proof. Write $\{v_j\}_{j=1}^n$ and $\{\widehat{v}_j\}_{j=1}^n$ for the Frenet vector fields along x and $\widehat{x}$.

Assume that there is a rolling $q(t)$ along $x(t)$ and $\widehat{x}(t)$. From the no-slip condition, we know that $q(t)v_1(t) = \widehat{v}_1(t)$. From the no-twist condition and induction, it follows that $q(t)\kappa_{j-1}(t)v_j = \widehat{\kappa}_{j-1}(t)\widehat{v}_j(t)$.

Conversely, assume that (17) holds. By changing the sign of $\widehat{v}_j$, we may assume that $\kappa_j = \widehat{\kappa}_j$ for $j = 1, \ldots, n$. Define

$$
\widehat{v}(t) = q(t)v_j(t).
$$

In order to see that $q(t)$ is a rolling, we need to show that if w is any vector field along $x(t)$, we have

$$
\frac{D}{Dt} q(t)w(t) = q(t)\frac{D}{dt} w(t).
$$

This equality holds since

$$q(t)\frac{D}{dt}w(t) = q(t)\sum_{j=1}^{k}\frac{D}{dt}w_j v_j =$$
$$= \sum_{j=1}^{k}(\dot{w}\widehat{v}_j + w_j(-\kappa_{j-1}\widehat{v}_{j-1} + \kappa_{j+1}\widehat{v}_{j+1})) = \frac{D}{dt}q(t)w(t).$$

This concludes the proof. □

Proposition 2.4 in [14, p. 381] establishes the existence and uniqueness of the rolling map, whenever a curve in one of the manifolds and an initial configuration are given. This proposition, as written in the source, has an innocent error in the formulation, since the result holds only for sufficiently small time intervals. To see this, simply pick

$$M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z - R)^2 = R^2\} \quad \text{and}$$
$$\widehat{M} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y + 2 < 1, z = 0\},$$

where $R > 1/\pi$, the curve in M to be geodesic arc

$$x(t) = (R\sin t, 0, R(1 - \cos t)), \quad t \in [0, \pi],$$

and the initial contact point in $(0, 0, 0)$. As can be easily seen, when rolling along x, we "run out of space", see Figure 2.

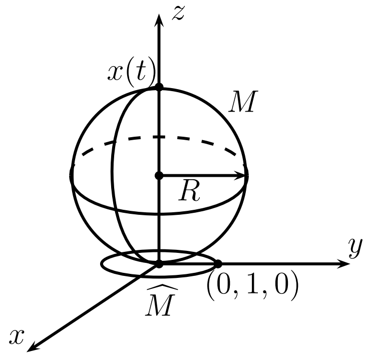

FIGURE 2. An impossible rolling: the manifold $\widehat{M}$ is "too small".

Corollary 2. *With the notation and hypotheses of Theorem 2, consider a given initial configuration for a rolling* $(x_0, \widehat{x}_0, q_0) \in Q$*, where* $x_0 = x(0)$ *and*

$\widehat{x}_0 = \widehat{x}(0)$, and assume x is a geodesic in M. Then, for sufficiently small values of τ, the equality

$$\widehat{x}(t) = q_0 \circ \exp_{x_0}^{-1} \circ x(t)$$

holds if and only if $\kappa_j = \pm\widehat{\kappa}_j$.

Proof. By Proposition 1, for sufficiently small t, the curve $q_0 \circ \exp_{x_0}^{-1} \circ x(t)$ is the projection onto $\mathbb{R}^n$ of a rolling. By uniqueness, it must satisfy $\kappa_j = \pm\widehat{\kappa}_j$ by Theorem 2. The converse holds by similar arguments. □

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