On flow properties of surface waves described by model equations of Boussinesq type

Philosophiae Doctor Thesis

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Preface and Acknowledgements

Preface

This dissertation is submitted as a partial fulfilment of the requirements for the degree Doctor of Philosophy (PhD) at the University of Bergen. This work started in August 2007, and the funding has been provided by the Research Council of Norway and partially by Lånekassen.

During the course of this study, I have been located at the Department of Mathematics, University of Bergen, and at the Environmental Flow Group, at Uni computing, Uni Research As. My main supervisor has been Professor Henrik Kalisch from the Department of Mathematics, while the Department head, Professor Jarle Berntsen has been my co-supervisor.

The overall aim of this thesis is to predict the most important flow properties of free-surface waves using dispersive long-wave models. The focus is on flow properties such as the densities and fluxes of mass, momentum and energy, where the fluid is assumed to be running in a narrow channel. The water pressure is also found as it is vital for expressing the momentum and energy fluxes. Special regard is given to some exact solutions such as solitary and cnoidal waves, and applications such as undular bores.

Outline

This thesis is divided into two parts: Part I gives a general background on the topic plus a summary of the results, while Part II contains the included papers in which detailed descriptions of the theory and results are presented.

Papers included in this thesis

The following papers are included in the thesis:

Paper A: Mechanical Balance Laws for Boussinesq Models of Surface Water Waves, A. Ali and H. Kalisch, accepted for publication in Journal of Nonlinear Science.

Paper B: On the Mass, Momentum and Energy Conservation in the KdV Equation, A. Ali and H. Kalisch.

Paper C: A Dispersive Model for Undular Bores, A. Ali & H. Kalisch, submitted to Analysis and Mathematical Physics.

Paper D: Reconstruction of the Pressure in Long-Wave Models with Constant Vorticity, A. Ali and H. Kalisch.

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I would like to thank my family for their continuous support and encouragement over the years of my studies, and for always believing in me.

Last but not least, I would like to thank my wife Wigdan for always being there for me, without your love, support and understanding I would not have been able to produce this work, special thanks to our wonderful newborn daughter Arwa, you are my inspiration.

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Part I

Background and summary of results

Chapter 1

Introduction

In modeling surface waves, while the model solutions provide fair descriptions of the waves, it is still important to be able to use these descriptions for prediction of the fluid flow properties beneath the surface.

This work attempts to provide some information about the details of fluid flow associated with a given surface wave. In the weakly nonlinear regime, and in particular in the Boussinesq scaling, the derivation of the model equations such as the Boussinesq system and the KdV equation contains clues about the fluid flow below the surface. Once the equations are derived, this information is often discarded. This work explores some implications of this by-product of the derivation.

A parallel development concerns similar work on understanding fluid flow below the surface in the context of the full Euler equations. Even though this appears to be very difficult, recent times have seen much progress in this direction [24, 25, 26, 46], even in the context of rotational flow. These works are based on rigorous mathematical proofs. Numerical work for the full Euler equations has been provided for instance in [38, 50, 75]. More examples of works investigating vorticity effects are [53, 54, 64, 74] and in [35] for linear waves.

In this chapter, we look at the motivation behind this study and give an overview of the problems that have been considered. A general background of the water-wave problem is presented in chapter 2. In chapter 3, the Boussinesq theory including the derivations of the Boussinesq system and KdV equation is reviewed. The last chapter gives a summary of the main results and future work.

1.1 Motivation

A bore is a well known phenomenon that occurs for example in rivers and open-channel flow. River bores are generally due to tides, where a long tidal-wave can steepen and develop a bore if it propagates upstream into a river of funnel-like

shape, and then they are called tidal bores.

Tidal bores have been widely observed in numerous rivers around the world. The largest is the Qiantang river in China where the wave height at the bore front is reported to reach up to 9 m and traveling speed up 11 m/s, and the Severn river in England where there are more than 250 bores a year which can transfer energy for up to 10 Km upstream (National Geography News, Oct. 28, 2010). The phenomenon is also found in the Dordogne river in France and the Lupar Benak river in Malaysia.

Generally, a bore can be defined as a transition between two uniform flows in which an increase in water depth exists. According to Favre's experiments [39] (1935), bores are classified into three different states depending on the ratio of the difference between upstream and downstream flow-depths against the undisturbed depth, denoted as α, these states are:

1. *Undular bores* display smooth oscillations (undulations) behind the bore front with no waves breaking, and occur if α is less than about 0.28.

2. If the ratio α increases beyond 0.28 but not more than 0.75, there will be undulations but with few waves breaking.

3. *Turbulent bores* feature totally turbulence region at the bore front as the value α is beyond 0.75.

An example of the development of undular bores is shown in figure 1.1.

While it is widely believed that the energy loss in turbulent and strong bores is attributed to the observed oscillations and turbulent motion following the bore front, it seems that the precise explanation of the energy loss in weakly undular bores has not been well provided. The classical theory of bores proposed by Lord Rayleigh [69] employs conservation of mass and momentum in the shallow-water model to show that energy cannot be conserved across the bore.

This study is motivated by our desire to understand the precise nature of the energy loss in the undular bores through quantitative evidence. By deriving a formula for the total mechanical energy associated to a dispersive model for the undular bore, we have been able to quantitatively show that the energy loss in the shallow-water theory for undular bores is actually absorbed by the oscillations behind the bore front. These findings led us to a further interesting development, that is, investigating more generally the principles of mass, momentum and energy conservation laws in the dispersive theory.
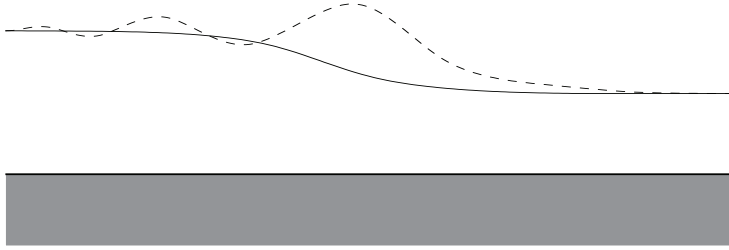
Figure 1.1: The schematic shows the development of an undular bore, where the wave-train is following the bore front, and the wave at the front attains the highest amplitude.

1.2   Energy loss in undular bores

The bore and other long-wave phenomena have been an area of interesting research for many decades. The first theories that have been used for modeling these phenomena are G.B. Airy 1842 [3], G. Stokes 1847 [72], J. Boussinesq 1871 [16], L. Rayleigh 1876 [70], and G. Korteweg& de Vries 1895 [56]. Beside the early experiments by Favre [39] on bores, other laboratory experiments have been done for instance by Binnie and Orkney [8], and recently by Chanson [18] and Koch and Chanson [58]. A development of an undular bore has been investigated in [68] by using dispersive systems, but no energy study was provided.

The first work investigating the energy loss is by Lemoine [62], where it is assumed that the wave-train following the bore front is of sinusoidal shape. Therefore a linear approach is used to calculate the rate of energy radiation and compared with experiments, however only moderate agreement was found. Benjamin and Lighthill [6] argued that the wave-train following the bore front is of cnoidal character. Their results showed that, in order for a wave-train of cnoidal waves to match a uniform stream, there must be a change in the volume flux or momentum flux or energy per unit mass. Later, using a cnoidal wave approximation and Favre's experiments, Sturtevant [73] argued that due to existence of a bottom boundary layer below the bore there must be change in both momentum and energy.

Indeed, dissipation plays a role that may be as important as dispersion in an experimental setting. However, due to the hyperbolic nature of the shallow-water model (non-dispersive), we argue that the energy loss predicted by the shallow-water equations is not due to dissipation mechanism that has not been accounted for, but rather could be looked at as a failure of the shallow-water model to capture the precise transition at the bore front. This argument is quantitatively confirmed

by incorporating dispersion corrections into the shallow-water model.

Let us briefly describe the dispersive system to be used for modeling undular bores. Suppose that a is a typical wave amplitude and ℓ represents a dominant wave-length, and define non-dimensional parameters $\alpha = a/h_0$ and $\beta = h_0^2/\ell^2$, where h_0 is the undisturbed water depth. The Boussinesq theory makes the assumptions that the waves are both of small amplitude and long wave-length, and that there is an approximate balance between non-linearity and dispersion. These assumptions are formulated by requiring that $\alpha << 1$, $\beta << 1$ and $\alpha \approx \beta$. Denoting the non-dimensional surface elevation as $\tilde{\eta}$, the dispersive model is given in the non-dimensional form as

$$\tilde{\eta}_{\tilde{t}} + \tilde{w}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{w})_{\tilde{x}} + \frac{\beta}{6}\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}} = 0,$$
$$\tilde{w}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \alpha\tilde{w}\tilde{w}_{\tilde{x}} + \frac{\beta}{6}\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}} = 0,$$

(1.1)

where $\tilde{w}$ represents the flow horizontal velocity at a non-dimensional height $\sqrt{2/3}$, $\tilde{t}$ and $\tilde{x}$ are the non-dimensional time and horizontal coordinate, respectively. Actually, the shallow-water model can be obtained by taking the limit $\beta \to 0$ in (1.1). The dispersive model (1.1), which is a special case of a family of Boussinesq models, shows that the energy loss is due to the increasing number of oscillating behind the bore front.

In order to use Boussinesq models to study the energy loss, and the mass and momentum conservation principles, the mass, momentum and energy densities, denoted as M, I and E, respectively, and the corresponding fluxes q_M, q_I and q_E have to be developed. This will be done in the general context of the more general Boussinesq systems derived by Bona et al. in [11].

1.3 Mechanical balance laws in Boussinesq theory

We develop the theory by giving a systematic derivation of the mass, momentum and energy densities, and the corresponding fluxes associated with the abcd system introduced in [11]. This system describes the evolution of two-way wave motion at the surface of a fluid body, which in its first order is given by the non-dimensional form

$$\tilde{\eta}_{\tilde{t}} + \tilde{w}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{w})_{\tilde{x}} + a\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}} - b\beta\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{t}} = 0,$$
$$\tilde{w}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \alpha\tilde{w}\tilde{w}_{\tilde{x}} + c\beta\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}} - d\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{t}} = 0\,.$$

(1.2)

The parameters a, b, c and d are given by

$$a = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\lambda, \qquad b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)(1-\lambda),$$
$$c = \frac{1}{2}\left(1 - \theta^2\right)\mu, \qquad d = \frac{1}{2}\left(1 - \theta^2\right)(1-\mu),$$

where λ and μ are modeling parameters which can be chosen freely, and $\tilde{w}$ denotes the non-dimensional horizontal velocity at a non-dimensional depth $0 \leq \theta \leq 1$. The mechanical balance law associated to the system (1.2) for any physically property, such as mass, momentum or energy, can be written in the form

$$\frac{\partial}{\partial \tilde{t}}\tilde{\mathcal{X}} + \frac{\partial}{\partial \tilde{x}}\tilde{q}_\mathcal{X} = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2), \tag{1.3}$$

where $\tilde{\mathcal{X}}$ and $\tilde{q}_\mathcal{X}$ are the non-dimensional density and flux per unit width of the property, respectively. As it will turn out, the derived mechanical balance laws for mass, momentum and energy are correct to the same order of accuracy as the system (1.2), and they have similar forms as (1.3). The derived expressions for the densities and fluxes of the mass, momentum and energy are given in terms of dependent variables η describing the free surface elevation and w representing the horizontal velocity at depth θh_0.

Similar developments exist for the linear and the shallow-water approximations [71], but have not been done for the Boussinesq scaling.

One-directional propagation

In similar developments, we derive the same principles in the case of unidirectional wave propagation, such waves can be modelled by the Korteweg-de Vries (KdV) equation. The KdV equation is popular because it features exact traveling-wave solutions. As this equation is derived under the assumption that there exists a balance between the nonlinear steepening and dispersion effects, waves propagate without changing their shapes. In the non-dimensional variable, the KdV equation has the form

$$\tilde{\eta}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \frac{3}{2}\alpha\tilde{\eta}\tilde{\eta}_{\tilde{x}} + \frac{1}{6}\beta\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}} = 0. \tag{1.4}$$

We examine the case where waves are stationary with respect to a moving frame in addition to both right-moving and left-moving waves scenarios. As will be shown, the derived balance equations are correct to same order as the KdV equation (1.4). In addition to the expressions of the densities and fluxes of mass, momentum and

energy, a formula for the total head is found. All the derived formulas for these physical quantities are given in terms of the principle unknown variable η.

Validation of the results is carried out through comparison with previous works. For example, in the context of stationary cnoidal waves, the mass flux q_M, total head H and momentum flux q_I are compared with the quantities Q, R and S presented in the work by Benjamin-Lighthill [6] on cnoidal waves and bores. As a result, an excellent agreement is found.

In the general Boussinesq theory the momentum and energy fluxes depend on the pressure forces, therefore a formula for water pressure is also derived for both the KdV equation (1.4) and Boussinesq model (1.2).

Pressure approximation in rotational waves

In further investigations, using an asymptotic series for the stream-function, we derive an approximation for the water pressure and the model equation for steady long-waves with constant vorticity. In addition to the vorticity, the solutions of the model equation depend on three constants representing the mass flux Q, the energy per unit mass R and the momentum flux S, which are defined in terms of the rotational flow. Similar equation has been derived in [5]. If the vorticity effects are disregarded, then the model equation becomes equivalent to an equation appeared in [6] used to model stationary cnoidal wave.

We gradually increase the vorticity to study the departure from cnoidal wave and to investigate its affects on the pressure profile. As it will be shown, increasing the vorticity effects will introduce a dip in the pressure and the maximum pressure is no longer found under the wave crest. Moreover, the wave-length may increase or decrease depending on the sign of the vorticity.

Chapter 2

Background

The water-wave problem for surface waves on an inviscid fluid is rich with fascinating mathematics and theories, and has been an active field of research for more than 200 years [29]. In this chapter, we will review some theories that have been used to simulate the wave motion based on different assumptions. In addition we study the background of the problem and present the flow governing equations along with the boundary conditions. We will assume throughout that the fluid is homogeneous (has constant density).

2.1 Euler equations

We start by exploring the governing equations of the fluid motion: the Euler and Navier-Stokes equations. As Euler equations are a special case of the Navier-Stokes equations, we start with the latter.

Navier-Stokes equations

The assumption that the fluid density is constant implies that the fluid is incompressible, and then the three-dimensional motion of the fluid may be described by Navier-Stokes equations. These equations are derived through utilization of Newton's second laws and constitutive relations between the fluid velocity and the stress tensor. Denoting $\vec{U}\,[\mathrm{ms}^{-1}]$ as the velocity field, the compact form of Navier-Stokes equations is given by

$$\frac{D\vec{U}}{Dt} = \frac{\partial \vec{U}}{\partial t} + (\vec{U} \cdot \nabla)\vec{U} = -\frac{1}{\rho}\nabla P + \vec{g} + \nu\nabla^2\vec{U}. \tag{2.1}$$

Here $\vec{g} = (0, 0, -g)\,[\mathrm{m\,s}^{-2}]$ represents the gravitational acceleration, $\rho\,[\mathrm{kg\,m}^{-3}]$ is the density, the water pressure is denoted as $P\,[\mathrm{N\,m}^{-2}]$, $\nu\,[\mathrm{m^2\,s}^{-1}]$ denotes the

kinematic viscosity. The conservation law of mass in a control volume means that the rate of change in mass must be equivalent to the mass net flux, which is expressed in terms the continuity equation as

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{U} = 0, \tag{2.2}$$

however, since ρ is constant as the fluid is assumed incompressible, the continuity equation reduces to

$$\nabla \cdot \vec{U} = 0. \tag{2.3}$$

An important factor that measures the rotation of the fluid flow is called *vorticity*, denoted by $\vec{\Omega}$ can be defined by the curl of the velocity field as

$$\nabla \times \vec{U} = \vec{\Omega}, \tag{2.4}$$

if $\vec{\Omega} = 0$, then the fluid flow is called *irrotational*, otherwise it is called *rotational*. For an incompressible and inviscid fluid, the vorticity remains constant with respect to time and therefore no change in the initial fluid rotation takes place at later time points. This can be shown by taking the curl of equation (2.1) with the substitutions $\nu = 0$, $\nabla \times \vec{U} = \vec{\Omega}$ and $\nabla \times \nabla P = 0$, as a result $\frac{D\vec{\Omega}}{Dt} = 0$, see [30].

Euler equations

When the viscosity effects are neglected, then the Navier-Stokes equations reduce to the Euler equations.

Consider the flow of water is in an open narrow channel with flat bottom placed along the x-direction, where it is assumed that no variation in y-direction as if the wave is long-crested and that the channel extends to infinity in both directions. Therefore the flow may be studied in (x, z) plane. Figure 2.2 describes the problem domain in which x is the direction of wave propagation and z is the vertical coordinate measured from $z = -h_0$ at the flat bottom to $z = \eta(x, t)$ at the free surface. The domain can then be express as $\Lambda(t) = \{(x, z) \in \mathbb{R}^2 | -h_0 \leq z \leq \eta(x, t), 0 < y < 1\}$. Assume that the fluid is inviscid, therefore the flow is governed by the Euler equations given in the form

$$\begin{aligned} U_t + UU_x + WU_z &= -\frac{P_x}{\rho}, \\ W_t + UW_x + WW_z &= -\frac{P_z}{\rho} - g, \end{aligned} \tag{2.5}$$

where U and W are the horizontal and vertical velocities, respectively, while the continuity equation is

$$U_x + W_z = 0. \tag{2.6}$$

We assume that the fluid vorticity is constant denoted by $-\Omega_0$, which is in the (x, z)-plane is defined as

$$(W_x - U_z) = -\Omega_0. \tag{2.7}$$

Now, we have addressed the Euler equations that governed the problem, we proceed further and present the problem along with the boundary conditions for both the rotational and irrotational fluid flows.

Irrotational flow

If we assume irrotational fluid flow, then the identity: $\nabla \times (\nabla \mathcal{K}) = 0$, for any scalar field $\mathcal{K}$, implies that there exists a velocity potential ϕ which satisfies

$$\begin{pmatrix} U \\ W \end{pmatrix} = \nabla \phi.$$

Thus the incompressibility condition (2.6) leads to the Laplace equation

$$\phi_{xx} + \phi_{zz} = 0, \quad \text{in} \ -h_0 < z < \eta(x, t). \tag{2.8}$$

In this context, it will be useful to mention an important equation that relates the pressure, velocity potential and the level in which the pressure can be measured, that is the Bernoulli equation, which is given as

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + \frac{P}{\rho} + gz = C(t) + \mathcal{C}. \tag{2.9}$$

The time-dependent constant $C(t)$ can be eliminated by absorbing it in ϕ, for example by redefining the velocity potential ϕ as $\Phi = \phi - \int_0^t C(s)ds$. Actually Bernoulli equation (2.9) represents a simple integration of the Euler equations (2.5) with the irrotational condition $\Omega_0 = 0$ in (2.7).

Boundary conditions
The bottom boundary condition is given by the requirement that no fluid can penetrate the bottom, which is formulated as

$$\phi_z = 0, \quad \text{on } z = -h_0. \tag{2.10}$$

At the free surface, kinematic and dynamic boundary conditions are imposed. The kinematic condition requires that the fluid particles can never leave the free surface, which is expressed by

$$\eta_t + \phi_x \eta_x - \phi_z = 0, \quad \text{on } z = \eta(x, t). \tag{2.11}$$

And the dynamic condition is given by the requirement that the pressure at the free surface is equal to the atmospheric pressure when neglecting the surface tension, using Bernoulli equation (2.9), this condition is formulated as

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + gz = 0, \quad \text{on } z = \eta(x, t), \tag{2.12}$$

where the atmospheric pressure is small compared to the total pressure P, and therefore assumed to be zero throughout this study.

Rotational flow

If the effects of the vorticity are considered ($\Omega_0 \neq 0$), then the flow becomes rotational, and the problem can be expressed in terms of the stream-function ψ where $(U, W) = (\psi_z, -\psi_x)$. Thus equation (2.7) implies that

$$\Delta \psi = \Omega_0. \tag{2.13}$$

Note that when using the velocity potential ϕ, the velocity in a specific direction is given by taking the derivative of ϕ in the same direction, while the derivative of ψ in a specific direction gives the velocity in a direction $90°$ clockwise from the differential direction.

We assume that the flow in the steady state, therefore all the time derivatives in Euler equations (2.5) are removed. The corresponding form of Bernoulli equation in this case is given [30] by

$$\frac{1}{2}\left(\psi_x^2 + \psi_z^2\right) + \frac{P}{\rho} + gz = \Upsilon(\psi). \tag{2.14}$$

where $\Upsilon(\psi)$ is constant along the streamlines. The problem is formed in terms of the stream-function as

$$\psi_{xx} + \psi_{zz} = \Omega_0, \quad \text{in } -h_0 < z < \eta(x) \tag{2.15}$$

The kinematic condition at the bottom is

$$\psi_x = 0, \quad \text{on } z = -h_0. \tag{2.16}$$

At the free surface, the kinematic condition is given by

$$\psi_z \eta_x + \psi_x = 0, \quad \text{on } z = \eta(x), \tag{2.17}$$

while using (2.14), the dynamic condition is obtained as

$$\frac{1}{2}\left(\psi_x^2 + \psi_z^2\right) + gz = \Gamma, \quad \text{on } z = \eta(x), \tag{2.18}$$

where $\Gamma = \Upsilon(\psi)|_{z=\eta}$ is constant, as (2.14) is evaluated at the free surface streamline.

In the context of modeling the water wave problem using Euler equations, several features of the water flow such as pressure changes in the water column, and mass, momentum and energy conservation properties may be examined. This will be discussed in more details in the next section.

2.2   Conservation integrals

In this section, we define the mass, momentum and energy conservation integrals for the Euler equation in terms of the the velocity potential ϕ. Assume that the fluid is running in a narrow channel of total depth $h(x,t) = h_0 + \eta(x,t)$, therefore, we consider a control volume of unit width delimited by the interval $[x_1, x_2]$ as the one shown in figure (2.1). First we reconstruct an expression for the pressure
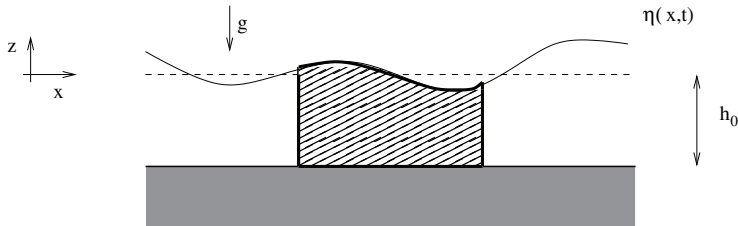
Figure 2.1: The schematic describes a typical control volume used for conservation proprieties of the shallow-water and Boussinesq theories. The bottom and lateral boundary are held fixed, while the upper boundary moves with the fluid free surface.

force, since it is essential for computation of the momentum and energy fluxes. For disturbances near the observer, the pressure can be found from the Bernoulli equation given in the form

$$\phi_t + \frac{1}{2}\left|\nabla\phi\right|^2 = -\frac{P}{\rho} - gz + \mathcal{C}. \tag{2.19}$$

In order to find the constant $\mathcal{C}$, we evaluate this equation at the free surface. Assuming that the surface disturbance is sufficiently localized so that $\eta \to 0$ and $\phi \to const.$ as $x \to \infty$, the constant $\mathcal{C}$ is given by

$$\mathcal{C} = \frac{P_{atm}}{\rho},$$

where P_{atm} denotes the atmospheric pressure which will always be assumed zero in this study. Therefore the pressure P in equation (2.19) can be obtained from

$$P = P_{atm} - \rho g z - \rho \phi_t - \frac{\rho}{2}|\nabla \phi|^2.$$

Mass integral

The total mass in a control volume of unit width defined as in figure 2.1 and delimited by $[x_1, x_2]$ is given by

$$\mathcal{M} = \int_{x_1}^{x_2} \int_{-h_0}^{\eta} \rho \, dz dx.$$

The mass will be conserved if the rate of change in the total mass is equal to the mass net flux, which is formulated as

$$\frac{d}{dt} \int_{x_1}^{x_2} \int_{-h_0}^{\eta} \rho \, dz dx = \left[\int_{-h_0}^{\eta} \rho \, \phi_x(x, z) \, dz \right]_{x_2}^{x_1}.$$

Momentum integral

The total momentum of a fluid of constant density ρ contained in the control volume is given by

$$\mathcal{I} = \int_{x_1}^{x_2} \int_{-h_0}^{\eta} \rho \phi_x \, dz \, dx.$$

Conservation of momentum is given by the requirement that the rate of change of $\mathcal{I}$ is equal to the net influx of momentum through the boundaries corrected to the pressure force. This can be expressed as

$$\frac{d}{dt} \int_{x_1}^{x_2} \int_{-h_0}^{\eta} \rho \phi_x \, dz \, dx = \left[\int_{-h_0}^{\eta} \rho \phi_x^2 \, dz + \int_{-h_0}^{\eta} P \, dz \right]_{x_2}^{x_1}. \qquad (2.20)$$

Energy integral

The total energy mechanical energy in the control volume considered above is defined by

$$\mathcal{E} = \int_{x_1}^{x_2} \int_{-h_0}^{\eta} \left\{ \frac{\rho}{2} |\nabla \phi|^2 + \rho g(z + h_0) \right\} \, dz \, dx, \tag{2.21}$$

which represents the sum of kinetic due to wave motion and potential energy due to surface deformation. The conservation of the total mechanical energy requires that the rate of energy change is equivalent to net energy-flux plus the work done by pressure forces acting on the boundaries. This can be formulated as

$$\frac{d}{dt} \int_{x_1}^{x_2} \int_{-h_0}^{\eta} \left\{ \frac{\rho}{2} |\nabla \phi|^2 + \rho g(z + h_0) \right\} dz dx = $$
$$\left[\int_{-h_0}^{\eta} \left\{ \frac{\rho}{2} |\nabla \phi|^2 + \rho g(z + h_0) \right\} \phi_x dz + \int_{-h_0}^{\eta} \phi_x P dz \right]_{x_2}^{x_1}. \tag{2.22}$$

2.3   Linear theory

Waves are the disturbances that propagate in space and time without displacing the medium, however the energy may be transferred across the medium. The force which is responsible for generating waves is called restoring force, a.e. force tends to bring a deformed system to its equilibrium state. *The surface gravity waves* are waves generated at the free surface of a liquid due to the restoring force of gravity, if such waves exist at the interface between two density-layers of the fluid, they are called *Internal waves* which are common in stratified fluids. However, in this study, our attention is only directed to surface gravity waves in a homogeneous fluid.

Wave parameters
In studies of ocean and river waves, there are important wave parameters that describe the wave such as the wave height, the water depth, the length between two successive crests or troughs and the wave period (time need for the wave to complete full cycle).

In figure 2.2, the free surface elevation is represented by η, the wave-length denoted by L which is defined as the distance between two successive crests, and here a represents the wave amplitude which may be defined as the distance from the crest to the undisturbed depth. The wave period is denoted by T. Therefore,
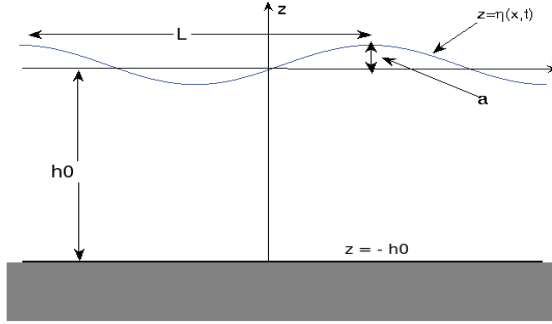
Figure 2.2: The schematic elucidates the geometric setup of the problem domain, where η represents the free surface, L is the wave length, here a denotes the amplitude and h_0 is the the undisturbed depth.

given T and L, the wave speed c is obtained as

$$c = \frac{L}{T},\tag{2.23}$$

which is called the phase speed. If we assume sinusoidal representation, then the free surface η shown in figure 2.2 can have the form

$$\eta(x) = a\cos(kx - \omega t),\tag{2.24}$$

where $k = \frac{2\pi}{L}$ and $\omega = \frac{2\pi}{T}$ are the wave number and the radian frequency, respectively. Here the quantity $(kx - \omega t)$ represents the wave phase, which describes the locations on the fraction of the wave cycle that lies between a trough and a neighbouring crest. The phase speed c can also be given in terms of the wave number and frequency as

$$c = \frac{\omega}{k}.$$

We may also introduce the concept of the *group velocity*, which describes the speed of an envelope of a group of waves propagating as one body. The group velocity, denoted as c_g, is defined as

$$c_g = \frac{d\omega}{dk}.\tag{2.25}$$

Solution of the linear problem

Here, we study the linear water waves theory in the context of the irrotational flow, in particular we consider the problem expressed in section 2.1 and present its solution. The linear wave theory makes the assumptions that waves are of small-amplitude so that the change in the surface elevation with respect to the horizontal dimension, and the horizontal and vertical velocities become small. As a result, all the nonlinear terms are one order smaller than the linear ones, and can then be neglected [57]. Moreover, the terms ϕ_z and ϕ_t can be evaluated at the undisturbed depth $z = 0$ rather than at $z = \eta$. Therefore, the linearized problem is given by Laplace equation

$$\phi_{xx} + \phi_{zz} = 0, \quad \text{in} -h_0 < z < 0, \tag{2.26}$$

with the kinematic boundary conditions

$$\begin{aligned} \phi_z &= 0, \quad \text{on } z = -h_0, \\ \phi_z &= \eta_t, \quad \text{on } z = 0, \end{aligned} \tag{2.27}$$

and the dynamic boundary condition

$$\phi_t = -g\eta, \quad \text{on } z = 0. \tag{2.28}$$

As in [57], if we assume a sinusoidal wave profile in the form

$$\eta(x) = a\cos(kx - \omega t), \tag{2.29}$$

and use only the kinematic boundary conditions (2.27), then the velocity potential solution for Laplace equation (2.26) is obtained as

$$\phi = \frac{a\omega}{k} \frac{\cosh(k(z + h_0))}{\sinh(kh_0)} \sin(kx - \omega t). \tag{2.30}$$

The horizontal and vertical velocities can now be found as

$$U = \phi_x = a\omega \frac{\cosh(k(z + h_0))}{\sinh(kh_0)} \cos(kx - \omega t),$$

$$W = \phi_z = a\omega \frac{\sinh(k(z + h_0))}{\sinh(kh_0)} \sin(kx - \omega t).$$

The total pressure P can be found using the Bernoulli equation and the dispersion relation (2.31) in the form

$$P = -\rho g z + \rho g a \frac{\cosh(k(z + h_0))}{\cosh(kh_0)} \cos(kx - \omega t),$$

where the surface tension and atmospheric pressure are disregarded. In the context of the linear theory, beside the velocity field and pressure distribution, other features such streamlines and particles paths may be found.

Dispersion relation

For linear waves, the *dispersion relation* is the relation $\omega = \omega(k)$, which is important for characterizing waves in the sense that it describes the correlation between the wave-length, frequency and wave speed. Waves are called *dispersive* if waves of different wave-lengths propagate with different speeds. The dispersion relation for the linear surface gravity waves, is given by substituting the solution (2.29)&(2.30) into the dynamic condition (2.28), hence we obtain

$$\omega = \sqrt{gk \tanh kh_0}. \tag{2.31}$$

The wave speed is therefore given by

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{2\pi} L \tanh kh_0}, \tag{2.32}$$

which shows that the waves in this theory are dispersive since the wave speed depends on the wave-length.

As it is mentioned, the above analysis investigates the irrotational characteristics of the linear water waves, however, the rotational features of these waves have been studied for instance in [35].

2.4   Shallow-water theory

The main assumptions in this theory are: the flow is uniform across the depth, the vertical velocity is zero and the wave length is large compared to the water depth (Long-wave approximation). Here, we derive the shallow-water model and the associated flow pressure directly by using the Euler and continuity equations shown in (2.5)&(2.6).

In shallow water, the depth is much smaller than the wave-length, that is $\frac{h_0}{\ell} << 1$, where ℓ as before is a typical wave-length, this assumption yields $\tanh kh_0 \to kh_0$. Therefore, using (2.31), the dispersion relation for linear wave in the shallow-water approximation is found as

$$\omega = \sqrt{gh_0 k^2}, \tag{2.33}$$

and the wave speed for long surface gravity waves c_0 is given by

$$c_0 = \sqrt{gh_0}. \tag{2.34}$$

The uniform horizontal velocity assumption means that the fluid vertical layers are flowing in parallel with the same velocity. Consider the problem domain as illustrated in figure 2.2, and $u(x,t)$ represents the uniform horizontal velocity, then Euler equations (2.5) become

$$u_t + uu_x = \frac{P_x}{\rho},$$
$$0 = -\frac{P_z}{\rho} - g,$$

(2.35)

where all terms consisting of the vertical velocity W and its derivatives are dismissed.

Hydrostatic pressure
In this long-wave theory, as the vertical acceleration is neglected, the water pressure is considered *hydrostatic*, which means that the water pressure at any point inside the fluid is equivalent to the weight of the fluid above that point. The hydrostatic pressure can be obtained by integrating the second equation in (2.35), which gives the hydrostatics pressure in the form

$$P = P_{atm} + \rho g(\eta - z),$$

(2.36)

where P_{atm} is the atmospheric pressure, which is assumed to be zero. Using this expression of the pressure as a substitution in the first equation in (2.35) yields

$$u_t + uu_x + g\eta_x = 0,$$

(2.37)

this equation is one of the two equations representing the shallow-water system. Now, integrating the continuity equation (2.6), we obtain

$$\int_{-h_0}^{\eta} (u_x + 0)\, dz = \frac{\partial}{\partial x}\int_{-h_0}^{\eta} u dz - u\eta_x|_{z=\eta} = 0,$$

where the vertical velocity is considered zero. Therefore, by using the kinematic boundary condition at the free surface

$$\eta_t = -u\eta_x \text{ on } z = \eta,$$

the shallow-water equations emerge in following form

$$\eta_t + h_0 u_x + (u\eta)_x = 0,$$
$$u_t + g\eta_x + uu_x = 0.$$

(2.38)

As can be seen, these equations are obtained using the continuity and Euler equations. Therefore, the shallow-water equations (2.38) arise from the physical principles of mass and momentum conservation, however they do not represents the exact conservation forms for these physical quantities.

Froude number

One of the important criteria in studying water waves is the *Froude Number*, which represents the ratio of the flow horizontal velocity u against the gravity wave speed c_0, given as

$$F_r = \frac{u}{c_0}.$$

If Froude number is greater than unity, then the flow is called *super-critical* which can create a hydraulic jump or bore, however if it less than unity then the flow is called *sub-critical*.

2.5 Balance equations for the shallow-water model

In this section, using the conservation integrals presented in section 2.2 we investigate the mass, momentum and energy conservation laws in the shallow-water system. Denote the total depth as $h(x,t) = h_0 + \eta(x,t)$ in a control volume of unit width delimited by the interval $[x_1, x_2]$ as the one shown in figure 2.1.

The conservation of mass is given by the requirement that the change in the total mass inside the control volume is equal to the net mass-flux, which is formulated as

$$\frac{d}{dt} \int_{x_1}^{x_2} M^0 \, dx = q_M^0 \big|_{x_2}^{x_1}. \tag{2.39}$$

where $M^0 = \rho h$ is the mass density, and $q_M^0 = \rho u h$ is the corresponding flux per unit width. In the same manner, the conservation integral for momentum is given by

$$\frac{d}{dt} \int_{x_1}^{x_2} I^0 \, dx = q_I^0 \big|_{x_2}^{x_1}, \tag{2.40}$$

where $I^0 = \rho u h$ is the momentum density, and $q_I^0 = (\rho u^2 h + \frac{\rho}{2} g h^2)$ is the corresponding flux per unit width. Note that the momentum flux is corrected to pressure forces acting on the boundary. Dividing by $(x_2 - x_1)$ and taking the limit $x_1 \to x_2$

in (2.39) and in (2.40), the conservation laws for mass and momentum are given in the form of the balance equation

$$\frac{\partial}{\partial t}M^0 + \frac{\partial}{\partial x}q_M^0 = 0, \tag{2.41}$$

$$\frac{\partial}{\partial t}I^0 + \frac{\partial}{\partial x}q_I^0 = 0. \tag{2.42}$$

This yields that the conservation laws of mass and momentum are represented by the differential equations:

$$\eta_t + h_0 u_x + (u\eta) = 0\,,$$
$$[u(h_0 + \eta)]_t + [u^2(h_0 + \eta)]_x + g(h_0 + \eta)\eta_x = 0\,.$$

For smooth solutions, the above equations are equivalent to the shallow-water equations (2.38). The conservation of energy is not a separate principle in homogeneous fluids, but follows from the equations of motion [57]. The total energy in the control volume is given by the integral

$$\mathcal{E}(\eta, u) = \frac{\rho}{2}\int_{x_1}^{x_2}\left\{u^2h + gh^2\right\}\,dx, \tag{2.43}$$

where the energy density is $E^0 = u^2h + gh^2$. The energy influx at the boundary corrected to pressure forces is given by

$$q_E^0 = \frac{\rho}{2}u^3h + \rho g u h^2, \tag{2.44}$$

Note that the second term in q_E^0 comprises both the energy flow rate and the work done by the pressure force, and therefore the quantity $q_E^0\big|_{x_2}^{x_1}$ represents the net influx of energy into the control volume. Using similar steps as for the mass and momentum integrals, the energy conservation law is expressed in the form

$$\frac{\partial}{\partial t}E^0 + \frac{\partial}{\partial x}q_E^0 = 0. \tag{2.45}$$

From the above discussion it is clear that the shallow-water model features conservation of mass, momentum and energy for smooth solution and that these principles are written in terms of balance equations, however, to our knowledge, such investigations has not been done before for the Boussinesq theory.

Chapter 3

Boussinesq theory

In this chapter, we present the historical background of Boussinesq theory and recall the derivation of the general Boussinesq system (1.2) and the Korteweg-de Vries equation using non-dimensional asymptotic expansion of the velocity potential. Some traveling-wave solutions for the KdV equation such solitary and cnoidal waves are also presented. In addition, an overview on the work by Benjamin-Lighthill [6] is provided.

3.1 Scott Russell's discovery of the solitary wave

A solitary wave is a wave of a single hump that propagates at a constant speed without changing its shape . Solitary waves were first discovered by Scott Russell in 1834 when he was observing the waves created by a boat moving in the Edinburgh-Glasgow canal. After that he went on and proved the existence of the solitary waves through laboratory experiments. Here, it will be beneficial to mention the much-quoted Russell's description that he included in his 1844 report *'Report on waves'*. He explains:

" I believe I shall best introduce the phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour [14 km/h], preserving its

original figure some thirty feet [9 m] long and a foot to a foot and a half [300 - 450 mm] in height. Its height gradually diminished, and after a chase of one or two miles [23 km] I lost it in the windings of the channel."

He also showed experimentally that the solitary wave speed denoted as C_R can be obtained from the formula

$$C_R^2 = g(h_0 + \eta_0).$$ (3.1)

where η_0 is the solitary wave amplitude. This formula shows that wave with higher amplitude travels faster. For more on this and Russell's experiments, see for example [45]. The early investigations on the solitary waves were associated with some controversy on their existence. However, using the long wave theory ($\beta \to 0$), Boussinesq (1871) and Rayleigh (1876) were the first who were able to mathematically describe Russell's observations. They showed that solitary waves are accurately modelled by the square of the hyperbolic secant function sech^2 and derived his formula for the solitary wave speed, however no equation was provided [33]. The solitary wave represent special solutions of the KdV equation (1.4), and therefore the assumption of small amplitude is required [77].

3.2 Boussinesq systems

Boussinesq models, first developed by Boussinesq in 1872 [15], are derived to model two-way propagation of surface water waves of small amplitude and long wave-length in narrow open channels. These models are derived under the assumptions that there is an approximate balance between nonlinear steepening effects represented by α and dispersive spreading measured by the value of β.

Non-dimensionalization
In order to derive the Boussinesq model we express the irrotational flow problem presented in section 2.1 in the non-physical form, this will introduce the two non-dimensional parameters α and β, measuring the wave amplitude and the wave-length, respectively. The new non-physical parameters are

$$\tilde{x} = \frac{x}{\ell}, \quad \tilde{z} = \frac{z + h_0}{h_0}, \quad \tilde{\eta} = \frac{\eta}{a}, \quad \tilde{t} = \frac{c_0 t}{\ell}, \quad \tilde{\phi} = \frac{c_0}{ga\ell}\phi.$$ (3.2)

Thus, the water-wave problem for irrotational waves in terms of the non-

dimensional variables is given by

$$\beta\tilde{\phi}_{\tilde{x}\tilde{x}} + \tilde{\phi}_{\tilde{z}\tilde{z}} = 0, \text{ in } 0 < \tilde{z} < 1 + \alpha\tilde{\eta}, \tag{3.3}$$

$$\tilde{\phi}_{\tilde{z}} = 0, \text{ on } \tilde{z} = 0, \tag{3.4}$$

$$\tilde{\eta}_{\tilde{t}} + \alpha\tilde{\phi}_{\tilde{x}}\tilde{\eta}_{\tilde{x}} - \frac{1}{\beta}\tilde{\phi}_{\tilde{z}} = 0, \text{ on } \tilde{z} = 1 + \alpha\tilde{\eta}, \tag{3.5}$$

$$\tilde{\eta} + \tilde{\phi}_{\tilde{t}} + \frac{1}{2}\left(\alpha\tilde{\phi}_{\tilde{x}}^2 + \frac{\alpha}{\beta}\tilde{\phi}_{\tilde{z}}^2\right) = 0, \text{ on } \tilde{z} = 1 + \alpha\tilde{\eta}. \tag{3.6}$$

By introducing some restriction on the flow and the parameters α and β, several water wave models have been derived as approximations for this problem, such as the shallow-water model (in the limit $\beta \to 0$) presented in the previous chapter, Boussinesq system and Korteweg-de Vries equation.

In terms of the scaling parameters α and β, the Boussinesq assumptions are given by: $\alpha << 1$, $\beta << 1$ and $\alpha \approx \beta$. Using the scaled Laplace equation (3.3) and the bottom kinematic condition (3.4), the velocity potential may be constructed in the standard form

$$\tilde{\phi} = \sum_{m=0}^{\infty}(-1)^m\frac{\tilde{z}^{2m}}{(2m)!}\frac{\partial^{2m}\tilde{f}}{\partial\tilde{x}^{2m}}\beta^m = \tilde{f} - \frac{\tilde{z}^2}{2}\tilde{f}_{\tilde{x}\tilde{x}}\beta + \frac{\tilde{z}^4}{24}\tilde{f}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}\beta^2 + \mathcal{O}(\beta^3), \tag{3.7}$$

where $\tilde{f}(\tilde{x}, \tilde{t})$ is arbitrary function for which $\tilde{f}_{\tilde{x}} = \tilde{v}$ represents the non dimensional horizontal velocity at the bottom of the channel.

Therefore, the asymptotic expansion (3.7) is used as a substitution in the surface boundary conditions (3.5) and (3.6) so that an approximation system in terms of α and β appears in the form

$$\tilde{\eta}_{\tilde{t}} + \tilde{v}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{v})_{\tilde{x}} - \frac{1}{6}\beta\tilde{v}_{\tilde{x}\tilde{x}\tilde{x}} = \mathcal{O}(\alpha\beta, \beta^2),$$
$$\tilde{\eta}_{\tilde{x}} + \tilde{v}_{\tilde{t}} - \frac{1}{2}\beta\tilde{v}_{\tilde{x}\tilde{x}\tilde{t}} + \alpha\tilde{v}\tilde{v}_{\tilde{x}} = \mathcal{O}(\alpha\beta, \beta^2). \tag{3.8}$$

Now, for expressing this system in terms of $\tilde{w}$, which represents the non-dimensional horizontal velocity at a non-dimensional depth $0 \leq \theta \leq 1$, first we use (3.7) to define $\tilde{w}$ as

$$\tilde{w} = \tilde{\phi}_{\tilde{x}}\big|_{\tilde{z}=\theta} = \tilde{v} - \frac{\theta^2}{2}\tilde{v}_{\tilde{x}\tilde{x}}\beta + \frac{\theta^4}{24}\tilde{v}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}\beta^2 + \mathcal{O}(\beta^3).$$

Then, use the method in [11] to express $\tilde{w}$ in terms of the non-dimensional velocity at the bottom $\tilde{v}$ in the form

$$\tilde{v} = \tilde{w} + \frac{1}{2}\beta\theta^2\tilde{w}_{\tilde{x}\tilde{x}} + \theta^4\frac{5}{24}\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}\beta^2 + \mathcal{O}(\beta^2), \tag{3.9}$$

which can now be used as a substitution in (3.8). Consequently, the new system is obtained as

$$\tilde{\eta}_{\tilde{t}} + \tilde{w}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{w})_{\tilde{x}} + \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}} = \mathcal{O}(\alpha\beta, \beta^2),$$

$$\tilde{w}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \alpha\tilde{w}\tilde{w}_{\tilde{x}} + \frac{1}{2}\left(\theta^2 - 1\right)\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{t}} = \mathcal{O}(\alpha\beta, \beta^2). \tag{3.10}$$

The *Classical Boussinesq system* emerges in the non-dimensional variables when substituting $\theta^2 = \frac{1}{3}$ in (3.10) as

$$\tilde{\eta}_{\tilde{t}} + \tilde{w}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{w})_{\tilde{x}} = 0,$$

$$\tilde{w}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \alpha\tilde{w}\tilde{w}_{\tilde{x}} = \frac{\beta}{3}\tilde{w}_{\tilde{x}\tilde{x}\tilde{t}}. \tag{3.11}$$

The above system is not the one originally developed by Boussinesq [15], but is still commonly known as the classical Boussinesq system. The original Boussinesq system featured the term of the form $\tilde{\eta}_{\tilde{x}\tilde{t}\tilde{t}}$ in the second equation [15, 77]. The general case of the Boussinesq system (3.10) is given in the form

$$\tilde{\eta}_{\tilde{t}} + \tilde{w}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{w})_{\tilde{x}} + \frac{\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\lambda\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}} - \frac{\beta}{2}\left(\theta^2 - \frac{1}{3}\right)(1-\lambda)\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{t}} = \mathcal{O}(\alpha\beta, \beta^2),$$

$$\tilde{w}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \alpha\tilde{w}\tilde{w}_{\tilde{x}} + \frac{\beta}{2}\left(1 - \theta^2\right)\mu\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}} - \frac{\beta}{2}\left(1 - \theta^2\right)(1-\mu)\tilde{w}_{\tilde{x}\tilde{x}\tilde{t}} = \mathcal{O}(\alpha\beta, \beta^2). \tag{3.12}$$

where λ and μ are model parameters which can be chosen freely. The abcd system (1.2) is obtained by neglecting terms of second order in α and β, for which the dimensional form is given as

$$\eta_t + w_x + (\eta w)_x + ah_0^3 w_{xxx} - bh_0^2 \eta_{xxt} + = 0,$$

$$w_t + \eta_x + ww_x + ch_0^2 \eta_{xxx} - dh_0^2 w_{xxt} = 0. \tag{3.13}$$

The general system (3.13) and its higher-order version were reviewed in [11]. The work presented in [11, 12] was important in the sense that it gave a complete classification of systems of Boussinesq type. Special Boussinesq systems such as the coupled BBM system [10] and the Kaup system [51] can be deduced from (3.13) by choosing suitable values for θ, μ and λ.

3.3 The Korteweg-de Vries equation

If the flow is restricted into one direction, then the Boussinesq system reduces to a single equation with only one dependent variable representing the free surface elevation, in the same time, it still incorporates both the non-linearity and dispersions.

This equation was originally derived by the Korteweg-de and Vries in 1895 [56], which is why it is named as the Korteweg-de Vries (KdV) equation. For example, if the flow is considered to the right, one can then look for a solution that approximates the normalized horizontal velocity $\tilde{v}$ in terms of $\tilde{\eta}$ as

$$\tilde{v} = \tilde{\eta} + \alpha \mathcal{A} + \beta \mathcal{B} + \mathcal{O}(\alpha\beta, \beta^2), \tag{3.14}$$

where $\mathcal{A}$ and $\mathcal{B}$ can be found by substituting this estimation in system (3.8). The lowest order approximation $\tilde{\eta}_{\tilde{t}} = -\tilde{\eta}_{\tilde{x}} + \mathcal{O}(\alpha, \beta)$ implies that the time derivative can be replaced by the negative spatial derivative or vice versa. Therefore the system (3.8) will be consistent if

$$\begin{aligned} \mathcal{A} &= -\frac{1}{4}\tilde{\eta}^2, \\ \mathcal{B} &= \frac{1}{3}\eta_{\tilde{x}\tilde{x}}. \end{aligned} \tag{3.15}$$

This consistency leads to the relation

$$\tilde{\eta}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \frac{3}{2}\alpha\tilde{\eta}\tilde{\eta}_{\tilde{x}} + \frac{1}{6}\beta\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}} = \mathcal{O}(\alpha\beta, \beta^2). \tag{3.16}$$

The dimensionless KdV equation presented in (1.4) emerges by neglecting terms of second order in α and β in (3.16). The KdV equation in the physical form can then be obtained as

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + \frac{c_0 h_0^2}{6}\eta_{xxx} = 0, \tag{3.17}$$

while the horizontal velocity at the bottom v has the form

$$v = \frac{c_0}{h_0}\eta - \frac{c_0}{4h_0^2}\eta^2 + \frac{c_0 h_0}{3}\eta_{xx}. \tag{3.18}$$

Similar equations can be derived for left-moving and stationary waves. The BBM equation [7, 13] can be obtained form (3.16) by replacing the third spatial derivative by time derivative using the first order approximation as $\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}} = -\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{t}} + \mathcal{O}(\alpha, \beta)$, and then neglecting terms of second order in α and β. Thus the BBM in the dimensional from is given as

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x - \frac{h_0^2}{6}\eta_{xxt} = 0. \tag{3.19}$$

3.4 Traveling-wave solutions for the KdV equation

Here, we would like to present some traveling-wave solutions for the KdV equation, these waves may be localized (solitary waves) or periodic (cnoidal waves). Assume that the KdV equation (1.4) has traveling-wave solutions in the form

$$\eta(x,t) = \mathcal{F}(\vartheta),$$

where $\vartheta = (x - ct)$ with c representing the constant wave-speed. When substituting this solution in the KdV equation (3.17), and then integrating the new equation, we get

$$(c_0 - c)\mathcal{F} + \frac{3}{4}\frac{c_0}{h_0}\mathcal{F}^2 + \frac{c_0 h_0^2}{6}\mathcal{F}'' = A. \tag{3.20}$$

After multiplying by $\mathcal{F}'$ and integrating again, this equation becomes

$$\frac{c_0 h_0^2}{6}\mathcal{F}'^2 = -\frac{c_0}{2h_0}\mathcal{F}^3 + (c - c_0)\mathcal{F}^2 + A\mathcal{F} + B, \tag{3.21}$$

where A and B are arbitrary constants which arose from the integration. Now, we will examine this equation for the solitary wave case where $A = B = 0$ and for the case of periodic wave of unchanging form.

Solitary wave

The solitary wave solution can be found from equation (3.21) by imposing the boundary conditions that all $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$ approach zero as $\vartheta \to \pm\infty$, hence $A = B = 0$, then we obtain

$$\frac{h_0^3}{3}\mathcal{F}'^2 = \mathcal{F}^2(\eta_0 - \mathcal{F}), \tag{3.22}$$

where $\eta_0 = 2h_0(\frac{c}{c_0} - 1)$. In order to have a real solution, $\mathcal{F}'^2$ has to be positive, which implies that $0 < \mathcal{F} < \eta_0$ and therefore η_0 represents the maximum height of $\mathcal{F}$ equivalent to the solitary wave amplitude. Equation (3.22) can be integrated with utilizing the transformation $\mathcal{F} = \eta_0 \operatorname{sech}^2\theta$, and then the final solitary wave solution will emerge in the form

$$\eta(x,t) = \eta_0 \operatorname{sech}^2\left(\sqrt{\frac{3\eta_0}{4h_0^3}}(x - ct)\right). \tag{3.23}$$

The solitary wave speed C_s in terms of the amplitude is expressed as

$$C_s = c = c_0 \left(1 + \frac{\eta_0}{2h_0}\right).$$

Using mathematical theory and experiments on the solitary waves, the ratio $\frac{\eta_0}{h_0}$ has been shown not to exceed values in the range 0.7-0.78, see [77]. For example, using a Boussinesq model in [9], it was shown that the ratio $\frac{\eta_0}{h_0}$ is limited by a wave-breaking criterion.

Cnoidal wave

Cnoidal waves are periodic solutions for the KdV equation (1.4) which are given by the Jacobian elliptic function cn(x) with modulus r: $0 < r < 1$. These solutions can be obtained by assuming that the cubic in (3.21) has three roots η_1, η_2 and η_3 in descending order, where η_1 and η_2 are the maximum and minimum wave heights, respectively. Then by integrating and using the substitution

$$\mathcal{F} = \eta_1 - (\eta_2 - \eta_1)\sin^2\theta,$$

we finally obtain the cnoidal wave solutions in the form

$$\eta(x,t) = \eta_2 + (\eta_1 - \eta_2)\mathrm{cn}^2\left(\Delta(x - ct); r\right), \tag{3.24}$$

where $\Delta = \sqrt{\frac{3(\eta_1 - \eta_3)}{4h_0^3}}$, and the modulus $r = \frac{\eta_1 - \eta_2}{\eta_1 - \eta_3}$. The wave-length is equal to $\frac{2K(r)}{\Delta}$, where $K(r)$ is the complete elliptic integral of the first kind.

3.5   Benjamin and Lighthill

In [6], Benjamin and Lighthill studied stationary cnoidal waves, such as if the space coordinates are moving with the same wave-speed. It was concluded that a cnoidal-wave solution is uniquely determined by only three constants Q, R and S representing the volume flux per unit width, the total head for which the height is measured from the bottom, and the momentum flux per unit width, respectively. Starting by assuming an asymptotic expansion for the stream-function ψ, they derived the following equation:

$$\frac{Q^2}{3}\left[\frac{d\zeta}{dx}\right]^2 + g\zeta^3 - 2R\zeta^2 + 2S\zeta - Q^2 = 0. \tag{3.25}$$

where $\zeta = h_0 + \eta$ is the total depth. As the flow is assumed irrotational, ψ is harmonic, which is equal to zero at the bottom and holds the value Q at the

free surface streamline. The most significant thing to notice when comparing this equation with equation (3.21) is that, equation (3.25) is derived in such a way that clearly describes the physical meaning of the constants Q, R and S, while in (3.21), such constants are only referred to as constants of integration. The total volume flux per unit width Q is defined by

$$Q = \int_0^\zeta \psi_z dz = \psi|_{z=\zeta}. \tag{3.26}$$

Using the bottom and surface boundary conditions (2.16) and (2.17) we see that

$$\frac{dQ}{dx} = \int_0^\zeta \psi_{xz} dz + \psi_z \zeta_x|_{z=\zeta} = 0, \tag{3.27}$$

which shows that Q is constant with respect to x. The quantity R represents the energy per unit mass (i.e. the total head multiply by g where the height is measured from the bottom to the free surface). Thus R is given by evaluating Bernoulli equation (2.14) at the free surface stream as

$$R = \Upsilon(\psi)|_{z=\zeta} = g\zeta + \frac{1}{2}(\psi_x^2 + \psi_z^2)|_{z=\zeta}. \tag{3.28}$$

Therefore equation (2.14) can be written as

$$\frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_z^2 + gz + \frac{P}{\rho} = R. \tag{3.29}$$

Now, the momentum flux per unit width S (divided by the density and corrected for the pressure force) may be defined as

$$S = \int_0^\zeta (\frac{P}{\rho} + \psi_z^2)dz = \int_0^\zeta \left(R - gz - \frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_z^2 \right) dz. \tag{3.30}$$

In order to show that the momentum flux S is also constant, we perform the following differentiation:

$$\frac{dS}{dx} = \int_0^\zeta (-\psi_x \psi_{xx} + \psi_z \psi_{xz})\, dz + \zeta'[(R - gz - \frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_z^2)]_{z=\zeta}. \tag{3.31}$$

Using the boundary conditions (2.17) and (2.16), and the assumption that ψ is a harmonic function, we find that

$$\begin{aligned}
\frac{dS}{dx} &= -\psi_z^2 \zeta'|_{z=\zeta} + \zeta'[R - gz - \frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_z^2]_{z=\zeta} \\
&= R - [gz + \frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_z^2]_{z=\zeta} = 0.
\end{aligned} \tag{3.32}$$

Therefore, the quantities Q, R and S are all showed to be constants, where no frictional effects are accounted for. Benjamin and Lighthill [6] studied the problem of the energy loss in steady undular bore and argued that the wave-train following the bore front is of cnoidal character. Their results showed that, in order for a wave-train of cnoidal waves to match a uniform stream, there must be a change in the volume flux Q or momentum flux S or the total head R. To illustrate this, consider a uniform stream of depth h_1 and velocity u_1. Then Q, R, and S takes the values

$$Q_1 = u_1 h_1, \quad R_1 = gh_1 + \frac{1}{2}u_1^2, \quad S_1 = u_1^2 h_1 + \frac{1}{2}gh_1^2. \tag{3.33}$$

These values can be used as substitutions in the cubic in (3.25) which can then be factorized as

$$(\zeta - h_1)^2(g\zeta - u_1^2), \tag{3.34}$$

which represents the case of coincident roots. If $gh_1 > u_1^2$ then the Froude number becomes less than unity, and hence the solution is only the uniform sub-critical flow. When $gh_1 < u_1^2$ the flow is super-critical and the only possible solution is the solitary wave as explained in the previous section. If however some change occurs in Q or R or S, then the cubic can have three descending roots, and therefore cnoidal-wave solutions exist.

Chapter 4

Summary of results

In this chapter, we present a summary of the main results of the four papers. In addition, some suggestions for further investigations regarding this study are also given.

4.1 Derivation of balance equations for the Boussinesq system

In paper A, using the general Boussinesq equation (1.2), we derive the mass, momentum and energy densities, and the corresponding fluxes per unit width. Moreover the water pressure is also derived as it is essential in finding the momentum and the energy fluxes. The conservation laws of these physical quantities are correct to the same order as the evolution equations.

In the Boussinesq scaling, as mentioned in the introduction we denote the mass, momentum, energy densities as M, I and E, respectively. The corresponding fluxes per unit width denoted as q_M, q_I and q_E. By converting to the non-physical variables, the corresponding non-dimensional densities are given as

$$\tilde{M} = \frac{M}{\rho h_0}, \quad \tilde{I} = \frac{I}{\rho c_0 h_0}, \quad \tilde{E} = \frac{E}{\rho c_0^2 h_0},$$

while the non-dimensional fluxes are defined as

$$\tilde{q}_M = \frac{q_M}{\rho c_0 h_0}, \quad \tilde{q}_I = \frac{q_I}{\rho c_0^2 h_0}, \quad \tilde{q}_E = \frac{q_E}{\rho c_0^3 h_0}.$$

By using the substitution of the non-dimensional velocity potential $\tilde{\phi}$ in terms of the non-dimensional velocity $\tilde{w}$ given in the derivation of the Boussinesq system (1.2), we derived expressions for $\tilde{M}$, $\tilde{q}_M$, $\tilde{I}$, $\tilde{q}_I$, $\tilde{E}$ and $\tilde{q}_E$ defined above. These

expressions together with the mechanical balance laws are tabulated below.

Conservation of mass
The non-dimensional mass density and flux are derived as

$$\tilde{M} = 1 + \alpha\tilde{\eta},$$

and

$$\tilde{q}_M = \alpha\tilde{w} + \alpha^2\tilde{\eta}\tilde{w} + \frac{1}{2}\alpha\beta\left(\theta^2 - \frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}},$$

respectively. Therefore the mass balance equation is obtained by the relation

$$\frac{\partial}{\partial\tilde{t}}\tilde{M} + \frac{\partial}{\partial\tilde{x}}\tilde{q}_M = \mathcal{O}(\alpha\beta, \beta^2)$$

with disregarding terms of $\mathcal{O}(\alpha\beta, \beta^2)$. It turns out that this balance equation is exactly the same as the first equation in the Boussinesq system (1.2) for $\lambda = 0$. Which means that this system exactly conserves the mass given that $\lambda = 0$.

Conservation of momentum
We found that the non-dimensional momentum density and flux to be

$$\tilde{I} = \alpha\tilde{w} + \alpha^2\tilde{w}\tilde{\eta} + \frac{1}{2}\alpha\beta\left(\theta^2 - \frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}},$$

and

$$\tilde{q}_I = \alpha\tilde{\eta} + \alpha^2\tilde{w}^2 + \frac{\alpha^2}{2}\tilde{\eta}^2 - \frac{\alpha\beta}{3}\tilde{w}_{\tilde{x}\tilde{t}} + \frac{1}{2},$$

respectively, note that the momentum flux $\tilde{q}_I$ is corrected for the pressure force. Then the momentum balance equation is given by neglecting terms of second order in α and β in the relation

$$\frac{\partial}{\partial\tilde{t}}\tilde{I} + \frac{\partial}{\partial\tilde{x}}\tilde{q}_I = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2).$$

Conservation of energy
The non-dimensional energy density and flux are respectively derived in the form

$$\tilde{E} = \alpha\tilde{\eta} + \frac{\alpha^2}{2}\tilde{\eta}^2 + \frac{\alpha^2}{2}\tilde{w}^2,$$

and

$$\tilde{q}_E = \alpha\tilde{w} + 2\alpha^2\tilde{w}\tilde{\eta} + \frac{\alpha\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}}.$$

where the energy flux is corrected for the work done by pressure force. Note that the energy density is defined such that it is equal to zero when there is no wave motion. With these definitions, the energy balance equation is found by neglecting the right hand side in

$$\frac{\partial}{\partial t}\tilde{E} + \frac{\partial}{\partial x}\tilde{q}_E = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2).$$

Using the same method, the mass, momentum, and energy densities and fluxes are also derived for the higher-order Boussinesq system, where similar balance equations are also found, see paper A for more details. Note that a similar study was conducted for the single Boussinesq equation [52], and that [34] provides an alternative method for obtaining an energy equation for a Boussinesq system.

There are other theories which provide different forms of energy formulas and Hamiltonian functions for the KdV and other equations, see [17, 27]. However, here the focus is on mechanical quantities which are valid to the same order of approximation as the evolution equation and not on mathematical conservation proprieties.

4.2   Conservation laws for the KdV equation

In paper B, in similar developments as in paper A, we present the derivation of the mass, momentum and energy densities and fluxes in the context of the KdV theory for unidirectional wave motion. However, we consider three cases: left-moving waves, right-moving waves and the case where the motion is in a moving reference frame in which waves are stationary. The water density is assumed to be unity.

In order to validate the results by comparing them with the physical quantities Q, S, R defined in section 3.5 and investigated by [6], we also derive the total head H. The derived mechanical balance equations for mass, momentum and energy are correct to the same order as the KdV equation.

In this summary, we will only present the expressions in the case where the waves are predominantly moving to right. The non-dimensional total head is defined as $\tilde{H} = H/(gh_0)$. Thus, the expressions for $\tilde{M}$, $\tilde{q}_M$, $\tilde{I}$, $\tilde{q}_I$, $\tilde{E}$ $\tilde{q}_E$ and $\tilde{H}$ along with the mass balance equation are tabulated below.

Mass balance
The expressions for mass density and flux in the the non-dimensional form are

found as

$$\tilde{M} = 1 + \alpha\tilde{\eta},$$

$$\tilde{q}_M = \alpha\tilde{\eta} + \frac{3}{4}\alpha^2\tilde{\eta}^2 + \frac{1}{6}\alpha\beta\tilde{\eta}_{\tilde{x}\tilde{x}}.$$

Therefore, the mass balance law for the KdV equation is given by the relation

$$\frac{\partial}{\partial\tilde{t}}\tilde{M} + \frac{\partial}{\partial\tilde{x}}\tilde{q}_M = \mathcal{O}(\alpha\beta, \beta^2). \tag{4.1}$$

When neglecting terms of $\mathcal{O}(\alpha\beta, \beta^2)$, we find that the mass balance equation (4.1) is exactly the same as the KdV equation (1.4), which means that the KdV equation exactly conserves the fluid mass, see [42].

Momentum balance
For the momentum, the density and flux are derived in the forms

$$\tilde{I} = \alpha\tilde{\eta} + \frac{3}{4}\alpha^2\tilde{\eta}^2 + \frac{\alpha\beta}{6}\tilde{\eta}_{\tilde{x}\tilde{x}},$$

$$\tilde{q}_I = \frac{1}{2} + \alpha\tilde{\eta}^2 + \frac{3}{2}\alpha^2\tilde{\eta}^2 + \frac{\alpha\beta}{3}\tilde{\eta}_{\tilde{x}\tilde{x}}.$$

Energy balance
The non-dimensional energy density and flux are derived as

$$\tilde{E} = \alpha\tilde{\eta} + \alpha^2\tilde{\eta}^2 + \frac{1}{2},$$

$$\tilde{q}_E = \alpha\tilde{\eta} + \frac{7}{4}\alpha^2\tilde{\eta}^2 + \frac{\alpha\beta}{6}\tilde{\eta}_{\tilde{x}\tilde{x}}.$$

The non-dimensional total head with depth measured from the bottom to the free surface is derived in the from

$$\tilde{H} = \frac{\alpha^2}{2}\tilde{\eta}^2 + (1 + \alpha\tilde{\eta}).$$

In order to facilitate the comparison with the work of Benjamin-Lighthill presented in [6], we have to consider stationary cnoidal waves with respect to a moving reference frame in which the mass flux Q is positive. Therefore, we derive the above expressions in the case of waves moving to the left in a moving frame of reference which also travels to the left with the same wave speed. Then the comparison can be achieved.

As an example, we use the cnoidal wave solution shown in figure 4.1(a) to evaluate the dimensional quantities q_M, H, and the average of q_I and compare them with the corresponding quantities Q, R, and S. These comparisons are shown in figure 4.1. As can be seen, this figure shows an excellent agreement with the expressions predicted in [6]. Moreover, the mass flux q_M is always found to be constant.

4.3   Energy balance for undular bores

In paper C, we study the problem of the energy loss in weak undular bores. These bores are commonly modelled by using the shallow-water equation (2.38), where it is assumed that the transition between the upstream and downstream flows is gradual and the water depth is small. The shallow-water system has a well known weak solution with a discontinuity represents an abrupt transition between two uniform flows.

Figure 4.2 depicts the geometric setup of the shock solution, it is considered fluid running in a horizontal channel of unit width and undisturbed depth h_0. The difference between the two uniform depths is a_0 and therefore the ratio $\alpha = \frac{a_0}{h_0}$. The arrows point in the upstream direction as this is the flow direction for river bores.

Consider a control volume of unit width delimited by the interval $[x_1, x_2]$ in x-direction. Assuming conservation of mass and momentum and using the shock solution for the shallow water-model, the loss of energy across the bore may be
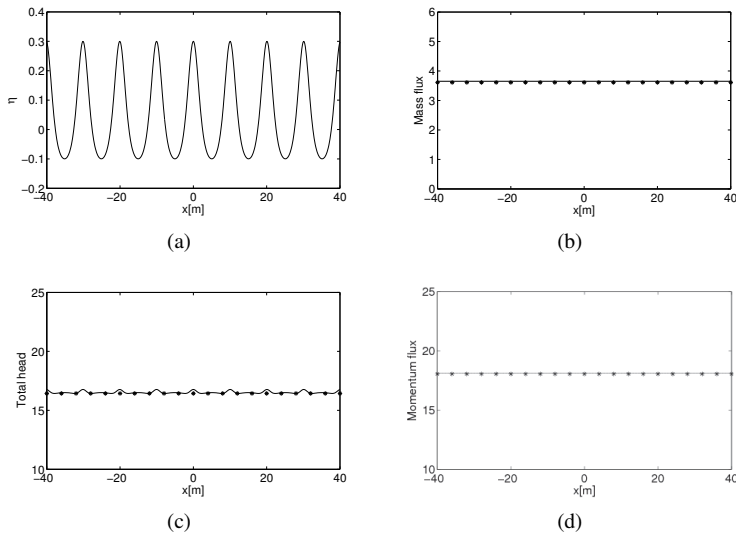

Figure 4.1:   A cnoidal wave solution η is shown in panel (a) where $h_0 = 1.1\,\text{m}$. Panel (b) shows the mass flux q_M vs. Q, panel (c) shows the total head H vs. R, and panel (d) shows $\bar{q}_I$ vs. S, where the solid-lines refer to the plots of q_M, H and $\bar{q}_I$, while the stars refer to the plots of Q, R and S.
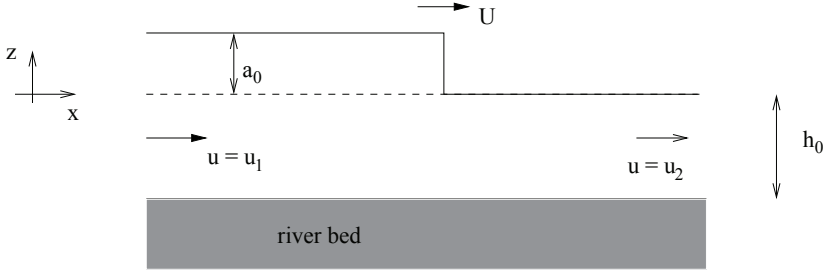
Figure 4.2: Schematic of a discontinuous solution of (2.38). The initial bore height is a_0 and the still water depth is h_0. The velocity jumps from u_1 to u_2, and the front of the bore moves upstream at a velocity U.

computed exactly by the equation

$$\left(q_E^0|_{x=x_1} - q_E^0|_{x=x_2}\right) - \frac{d\mathcal{E}(\eta, u)}{dt} = \frac{a_0^3}{4}\,\rho\sqrt{\tfrac{1}{2}\,g^3\left(\tfrac{1}{h_0} + \tfrac{1}{a_0+h_0}\right)}, \tag{4.2}$$

where q_E^0 is the energy flux and $\mathcal{E}(\eta, u)$ is the total mechanical energy contained in the fluid region between x_1 and x_2, these quantities are defined in (2.43) and (2.44). The right-hand side of equation (4.2) represents the energy lost due the approximate nature of the governing equations (2.38) and the discontinuous solution shown in figure 4.2 . The velocity U of the bore front can be deduced from the shock solutions as

$$U = u_2 + \sqrt{\frac{g}{2h_0}\left(2h_0^2 + 3a_0h_0 + a_0^2\right)}, \tag{4.3}$$

and the initial flow velocity at $x = x_1$ as

$$u_1 = u_2 + \frac{a_0}{a_0 + h_0}\sqrt{\frac{g}{2h_0}\left(2h_0^2 + 3a_0h_0 + a_0^2\right)}. \tag{4.4}$$

In order to capture the energy loss in equation (4.2) we use the dispersive model (1.1) and derive the associated total energy $\tilde{\mathcal{E}}(\tilde{\eta}, \tilde{w})$, which is in the non-dimensional variables found as

$$\tilde{\mathcal{E}}(\tilde{\eta}, \tilde{w}) = \frac{\alpha^2}{2}\int_{x_1/\ell}^{x_2/\ell}\left\{\tilde{w}^2 + \alpha\tilde{\eta}\tilde{w}^2 + \tfrac{\beta}{3}\tilde{w}\tilde{w}_{xx} + \tfrac{\beta}{3}\tilde{w}_x^2\right\}d\tilde{x}$$

$$+ \frac{1}{2}\int_{x_1/\ell}^{x_2/\ell}\left\{1 + 2\alpha\tilde{\eta} + \alpha^2\tilde{\eta}^2\right\}d\tilde{x}.$$

As our desire is to compare the results with the conserved quantities associated to the shallow-water system, higher-order terms are included so that in the limit of small β, the dimensional form of $\tilde{\mathcal{E}}(\tilde{\eta}, \tilde{w})$ reduces correctly to the corresponding expression $\mathcal{E}(\eta, u)$ in the shallow-water theory given in (2.43). These higher-order corrections do not change the order of accuracy of the energy integral, which is correct to second-order in α and β, but they facilitate the comparison with the shallow-water theory. In the dimensional variables, the total energy is given by

$$\mathcal{E}(\eta, w) = \frac{\rho}{2} \int_{x_1}^{x_2} \left((h_0 + \eta)w^2 + g(h_0^2 + 2h_0\eta + \eta^2) + \frac{h_0^3}{3} ww_{xx} \right.$$
$$\left. + \frac{h_0^3}{3} w_x^2 \right) dx.$$

The dispersive system (1.1) in the physical variable is given as

$$\eta_t + h_0 w_x + (w\eta)_x + \frac{h_0^3}{6} w_{xxx} = 0,$$
$$w_t + g\,\eta_x + ww_x + \frac{gh_0^2}{6}\eta_{xxx} = 0. \tag{4.5}$$

Numerical approximation of (4.5) for the undular bore are found using a finite difference method. The well-posedness of the linear version of this system is

α	$(q_E^0\vert_{x=x_1} - q_E^0\vert_{x=x_2})$	$\frac{d}{dt}\mathcal{E}(\eta, w)$	$\frac{d}{dt}\mathcal{E}(\eta, u)$	Difference
	$kg\,m^2/s^3$	$kg\,m^2/s^3$	$kg\,m^2/s^3$	%
0.1	0.836	0.836	0.829	0.84
0.2	2.170	2.187	2.128	1.93
0.3	4.200	4.201	4.005	4.87
0.4	7.036	7.034	6.582	6.48
0.5	10.86	10.87	9.992	7.92
0.6	15.87	15.88	14.38	9.38
0.7	22.26	22.27	19.92	10.51

Table 4.1:   Comparison of the change in energy in the shallow-water and dispersive systems when a strong backflow $u_2 = -2m/s$ is present. Column 1 shows the non-dimensional bore amplitude $\alpha = a_0/h_0$. The net energy flux is shown in Column 2. Columns 3 and 4 display the rate of change in the energy of the dispersive theory and the shallow-water theory, respectively. Column 5 shows the percentage difference between the net energy flux and the rate of change of the shallow-water energy. The dispersive theory gives the correct result to within less than 0.3% error.

found in [40].

Table 4.1 shows that the dispersive model (4.5) successfully captures the energy loss due to the shallow-water approximation within less than 0.3% error. This can be explained by the fact that the dispersive model captures the correct

transition at the bore front and that the energy loss is absorbed by the increasing number of oscillations following the bore front. The results of this paper have been announced in [1].

4.4 Pressure estimation in a flow with constant vorticity

In paper D, we consider the time independent problem for the rotational flow presented in section 2.1 with fluid density equals to unity, and where the vorticity is constant denoted as $-\Omega_0$. We derive the model equation along with an approximation for the water pressure by reconstructing an asymptotic series for the stream-function taking into account the affects of the vorticity. Using the following scaling

$$\tilde{x} = \frac{x}{\ell}, \quad \tilde{z} = \frac{z}{h_0}, \quad \tilde{\zeta} = \frac{\zeta}{h_0}, \quad \tilde{\psi} = \frac{1}{c_0 h_0}\psi, \quad \tilde{\Omega}_0 = \frac{h_0}{c_0}\Omega_0,$$

where ζ is the total depth, we find the asymptotic expansion of ψ in the non-dimensional form

$$\tilde{\psi} = \frac{1}{2}\tilde{z}^2\tilde{\Omega}_0 + \tilde{z}\tilde{f} - \frac{\beta}{3!}\tilde{z}^3\tilde{f}'' + \mathcal{O}(\beta^2).$$

Here, the function $\tilde{f}(\tilde{x})$ represents the non-dimensional velocity at the bottom. The boundary condition that $\psi = Q$ at the free surface streamline $z = \zeta$ implies that $\tilde{f}$ can be written in the form

$$\tilde{f} = \frac{\tilde{Q}}{\tilde{\zeta}} - \frac{1}{2}\tilde{\zeta}\tilde{\Omega}_0 + \frac{\beta}{6}\tilde{\zeta}^2\tilde{f}'' + \mathcal{O}(\beta^2),$$

where $\tilde{Q}$ is the non-dimensional mass flux. Disregarding terms of $\mathcal{O}(\beta^2)$, this equation can be solved for $\tilde{f}$ by inverting the differential operator as the following

$$\tilde{f} = \left(1 - \frac{\beta}{6}\tilde{\zeta}^2\frac{d^2}{dx^2}\right)^{-1}\left(\frac{\tilde{Q}}{\tilde{\zeta}} - \frac{1}{2}\tilde{\zeta}\tilde{\Omega}_0\right)$$
$$= \left(1 + \frac{\beta}{6}\tilde{\zeta}^2\frac{d^2}{dx^2} + \mathcal{O}(\beta^2)\right)\left(\frac{\tilde{Q}}{\tilde{\zeta}} - \frac{1}{2}\tilde{\zeta}\tilde{\Omega}_0\right).$$

The derived model equation, which describes the departure from stationary cnoidal waves, appears in the form

$$\left(Q + \frac{\Omega_0}{2}\zeta^2\right)^2\zeta'^2 = -3\left(\frac{\Omega_0^2}{12}\zeta^4 + g\zeta^3 - (2R - \Omega_0 Q)\zeta^2 + 2S\zeta - Q^2\right), \quad (4.6)$$

where the constants Q, R and S are the same as before but are defined in terms of the rotational flow. A similar equation as (4.6) has been derived in [5]. As in [6], the undisturbed depth h_0 can be defined as $h_0 = g^{-\frac{1}{3}} Q^{\frac{2}{3}}$.

If $\Omega_0 = 0$, this equation reduces to equation (3.25), and will therefore have exact cnoidal wave solutions, otherwise, smooth travelling-wave solutions can be found numerically. The pressure formula is found as

$$P = R - gy - \frac{1}{2}\left(\frac{Q}{\zeta^2} + \frac{\Omega_0}{2}\right)^2 \left(z^2 \zeta'^2 + \zeta^2\right)$$
$$+ \frac{1}{2}\left(\frac{\Omega_0}{6}\zeta^3 - \frac{\Omega_0}{2}z^2\zeta - \frac{2}{3}\Omega_0 z^3 - \frac{Q}{3}\zeta + z^2 \frac{Q}{\zeta}\right)\left(2Q\frac{\zeta'^2}{\zeta^3} - \zeta''(\frac{Q}{\zeta^2} + \frac{1}{2}\Omega_0)\right).$$

The main result of paper C is that, when increasing the vorticity effects, there will be a dip in the pressure and the maximum pressure is no longer found under the wave crest. Moreover, the wave-length may decrease or increase. We define a
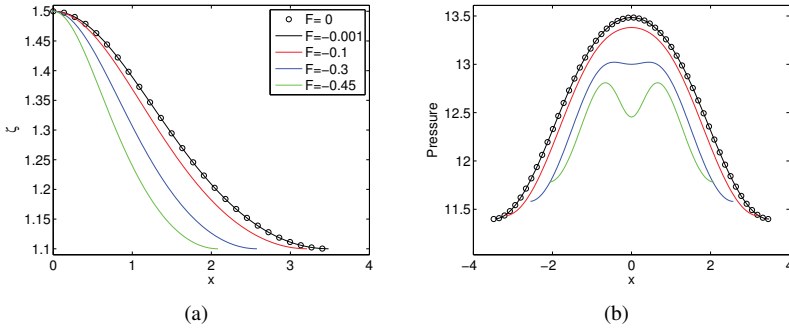

(a)   (b)

Figure 4.3: In panel (a) the solution of (4.6) is found for different negative values of F, where the corresponding pressure profiles are shown in panel (b). The ratio of the wave amplitude to the undisturbed water depth is $\alpha = 0.1771$. For the case where $F = 0$, the exact cnoidal-wave solution shown by circles is used.

non-dimensional number F, which represents the ratio of the maximum mass flux due to the vorticity against the constant mass flux Q, as

$$F = \frac{\Omega_0 M^2}{2Q},$$

where M is the maximum wave height. If m represents the wave trough, then the parameter α can be defined as $\alpha = (M - m)/(2h_0)$. Consider for example that $\Omega_0 < 0$. Therefore, as the absolute value of F increases, the wave-length decreases and the pressure starts introducing a dip, this can be seen in the example shown in figure 4.3. Figure 4.3 also shows that when $|F| \to 0$, the numerical solution approaches the exact cnoidal-wave solution ($F = 0$).

Further work

It should be possible to extend the formulation of mechanical balance laws to KdV equations for rotational waves [21, 76] which arise in the study of wave-current interactions [67], and to higher-order equations for surface waves [36, 41, 60, 61, 63]. Another possible extension would be in the study of model equations for internal waves [65].

The KdV equation also appears in this situation [28, 60], but it would also be interesting to look at higher-order models such as the extended KdV equation [43, 44, 49, 55] or fully nonlinear models [22]. Internal waves are also often described by non-local models [4, 23, 47, 55, 66], and a parallel development should also be possible here. As this study is devoted to one-dimensional models, it should possible to make an extension to the two-dimensional case [20, 31, 32].

It would also be interesting to see if the techniques in this thesis apply to the equations used in [37] and [60].

Bibliography

[1] A. Ali and H. Kalisch, *Energy balance for undular bores*, Compt. Rend. Mec. **338** (2010), 67–70.

[2] A. Ali and H. Kalisch, *Mechanical Balance Laws for Boussinesq Models of Surface Water Waves*, to appear in J. Nonlinear Sc.

[3] G. B. Airy, *Tides and Waves*, Encyclopedia Metropolitan, London (1842), article 192.

[4] T. B. Benjamin, *Internal waves of permanent form in fluids of great depth*, J. Fluid Mechanics **29** (1967), 559–592.

[5] T. B. Benjamin, *The solitary wave on a stream with an arbitrary distribution of vorticity*, J. Fluid Mechanics **12** (1962), 97–116.

[6] T. B. Benjamin and M. J. Lighthill, *On cnoidal waves and bores*, Proc. Roy. Soc. London Ser. **A 224** (1954), 448–460.

[7] T. B. Benjamin, J. B. Bona and J. J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Philos. Trans. Roy. Soc. London, 1972, **A 272**, 47–78.

[8] A. M. Binnie and J. C. Orkney, *Experiments on the flow of water from a reservoir through an open horizontal channel II*, Proc. Roy. Soc. London Ser. **A 230** (1955), 237-246.

[9] M. Bjørkavåg and H. Kalisch, *Wave breaking in Boussinesq models for undular bores*, Phys. Lett. A **375** (2011), 1570–1578.

[10] J. L. Bona and M. Chen, *A Boussinesq system for two-way propagation of nonlinear dispersive waves*, Physica D **116** (1998), 191–224.

[11] J. L. Bona, M. Chen and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I: Derivation and linear theory*, J. Nonlinear Sci. Vol. **12**, 283-318 (2002).

[12] J. L. Bona, M. Chen and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II: The nonlinear theory* Nonlinearity **17** (2004), 925–952.

[13] J. L. Bona, W. G. Pritchard and L. R. Scott, *An evaluation of a model equation for water waves*, Phil. Trans. Roy. Soc. London **A 302** (1981), 457–510.

[14] J. L. Bona and R. Smith, *A model for the two-way propagation of water waves in a channel*, Math. Proc. Cambridge Phil. Soc. **79** (1976), 167–182.

[15] J. Boussinesq, *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond*, J. Math. Pures Appl. **17** (1872), 55–108.

[16] J. Boussinesq, *Théorie de l'intumescence liquide appele onde solitaire ou de translation se propageant dans un canal rectangulaire* , Comptes Rendus Acad. Sci (Paris), **72** (1871) 755–759.

[17] T. J. Bridges, *Spatial Hamiltonian structure, energy flux and the water-wave problem*, Proc. Roy. Soc. London Ser. A **439** (1992), 297-315.

[18] H. Chanson, *Current knowledge in hydraulic jumps and related phenomena. A survey of experimental results*, Eur. J. Mech. B/Fluids **28** (2009), 191–210.

[19] H. Chanson, *Undular Tidal Bores: Basic Theory and Free-surface Characteristics*, J. Hydraulic Engineering, ASCE, Vol. 136, No. 11 (2010), 940–944.

[20] M. Chen, *Numerical investigation of a two-dimensional Boussinesq system*, Discrete Continuous Dyn. Syst. **23** (2009), 1169–1190.

[21] W. Choi, *Strongly nonlinear long gravity waves in uniform shear flows*, Phys. Rev. E **68** (2003), 026305.

[22] W. Choi and R. Camassa, *Fully nonlinear internal waves in a two-fluid system.* J. Fluid Mech. **396** (1999), 1–36.

[23] W. Choi and R. Camassa, *Weakly nonlinear internal waves in a two-fluid system.* J. Fluid Mech. **313** (1996), 83–103.

[24] A. Constantin, M. Ehrnström and E. Wahlén. *Symmetry of steady periodic gravity water waves with vorticity.* Duke Math. J. **140** (2007), 591–603.

[25] A. Constantin and W. Strauss. *Exact steady periodic water waves with vorticity.* Comm. Pure Appl. Math. **57** (2004), 481–527.

[26] A. Constantin and W. Strauss. *Pressure beneath a Stokes wave.* Comm. Pure Appl. Math. **63** (2010), no. 4, 533–557.

[27] W. Craig and M. D. Groves, *Hamiltonian long-wave approximations to the water-wave problem*, Wave Motion **19** (1994), 367–389.

[28] W. Craig, P. Guyenne and H. Kalisch, *Hamiltonian long-wave expansions for free surfaces and interfaces*, Comm. Pure Appl. Math. **58** (2005), 1587–1641.

[29] A. D. D. Craik, *The origins of water wave theory*, Annu. Rev. Fluid Mech. **36** (2004), 1–28.

[30] R. G. Dean and R. A. Dalrymple, *Water wave mechanics for engineers and scientists* (World Scientific, Singapore, 1991).

[31] V. A. Dougalis, D. E. Mitsotakis and J.-C. Saut, *On some Boussinesq systems in two space dimensions: theory and numerical analysis*, M2AN Math. Model. Numer. Anal. **41** (2007), 825–854.

[32] V. A. Dougalis, D. E. Mitsotakis and J.-C. Saut, *On initial-boundary value problems for a Boussinesq system of BBM-BBM type in a plane domain*, Discrete Cont. Dyn. Syst. **23** (2009), 1191–1204.

[33] P. G. Drazin and R. S. Johnson, *Solitons: an introduction* (Cambridge University Press, 1989).

[34] D. Dutykh and F. Dias, *Energy of tsunami waves generated by bottom motion*, Proc. R. Soc. **A 465** (2009), 725–744.

[35] M. Ehrnström and G. Villari, *Linear water waves with vorticity: rotational features and particle paths*, J. Differential Equations **244** (2008), 1888–1909,

[36] G. A. El, R. H. J. Grimshaw and N. F. Smyth, *Asymptotic description of solitary wave trains in fully nonlinear shallow-water theory*, Physica D, **237**, (2008) 2423-2435.

[37] G. A. El, R. H. J. Grimshaw and N. F. Smyth, *Unsteady undular bores in fully nonlinear shallow-water theory*. Phys. Fluids, **18**, (2006), 027104.

[38] W. A. B. Evans and M. J. Ford, *An Exact Integral Equation for solitary waves (with New Numerical Results for Some 'Internal' properties)*, Proc. R. Soc. **452** (1945), 373–390.

[39] H. Favre, *Ondes de Translation* (Dunod, Paris, 1935).

[40] A. S. Fokas and B. Pelloni, *Boundary value problems for Boussinesq type systems*, Math. Phys. Anal. Geom. **8** (2005), 59–96.

[41] A. E. Green and P. M. Naghdi, *A derivation of equations for wave propagation in water of variable depth*, J. Fluid Mech. **78**, (1976), 237–246.

[42] R. Grimshaw and N. Joshi *Weakly nonlocal solitary waves in a singularly perturbed Korteweg-de Vries equation* SIAM J. Appl. Math. **55** (1995) , 124–135.

[43] R. Grimshaw, E. Pelinovsky and O. Poloukhina, *Higher-order Korteweg-de Vries models for internal solitary waves in a stratified shear flow with a free surface*, Nonlinear Processes in Geophysics **9** (2002), 221–235.

[44] R. Grimshaw, E. Pelinovsky, T. Talipova, and A. Kurkin, *Simulation of the transformation of internal solitary waves on oceanic shelves.* J. Physical Oceanography **34** (2004), 2774–2791.

[45] R. S. Johnson, *A Modern Introduction to the Mathematical Theory of Water Waves* (Cambridge University Press, 1997).

[46] H. Kalisch, *A uniqueness result for periodic traveling waves in water of finite depth.* Nonlinear Anal. **58** (2004), 779–785.

[47] H. Kalisch, *Derivation and comparison of model equations for interfacial capillary-gravity waves in deep water*, Math. Comput. Simulation **74**, (2007) 168-178.

[48] H. Kalisch and J. L. Bona, *Models for internal waves in deep water*, Disc. Cont. Dyn. Sys. **6** (2000) 1-19.

[49] H. Kalisch and N. T. Nguyen, *On the stability of internal waves*, J. Phys. A, **43** (2010), 495205.

[50] Y. Kang and J. M. Vanden-Broeck, *Gravity-capillary waves in the presence of constant vorticity.* Eur. J. Mech. B Fluids **19** (2000), no. 2, 253–268.

[51] D. J. Kaup, *A higher-order wave equation and the method for solving it*, Prog. Theor. Phys. **54** (1975), 396–408.

[52] G. H. Keulegan and G. W. Patterson, *Mathematical theory of irrotational translation waves*, Nat. Bur. Standards J. Res. **24** (1940), 47–101.

[53] J. Ko and W. Strauss, *Effect of vorticity on steady water waves*, J. Fluid Mech. **608** (2008), 197–215.

[54] J. Ko and W. Strauss, *Large-amplitude steady rotational water waves*, Eur. J. Mech. B Fluids **27** (2008), 96–109.

[55] C. G. Koop and G. Butler, *An investigation of internal solitary waves in a two-fluid system*, J. Fluid Mech. **112** (1981), 225–251.

[56] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular channel and on a new type of long stationary wave*, Philos. Mag. **39** (1895), 422–443.

[57] P. K. Kundu and I. M. Cohen, *Fluid Mechanics*, third edition, (Academic Press, New York, 2008).

[58] C. Koch and H. Chanson, *Unsteady turbulence characteristics in an undular bore* in River Flow 2006, (Taylor & Francis Group, London, 2006).

[59] D. Lannes and P. Bonneton, *Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation*, Phys. Fluids **21** (2009), 016601.

[60] C.-Y. Lee and R.C. Beardsley, *The generation of long nonlinear internal waves in a weakly stratified shear flow.* J. Geophys. Res. **79** (1974) 453–462.

[61] O. Le Métayer, S. Gavrilyuk and S. Hank, *A numerical scheme for the Green-Naghdi model*, J. Comput. Phys., **229** (2010), 2034–2045.

[62] R. Lemoine, *Sur les ondes positives de translation dans les canaux et sur le ressaut ondule de faible amplitude*, Jl. La Houille Blanche (1948) 183–185.

[63] P. A. Madsen and H. A. Schäffer, *Higher-order Boussinesq-type equations for surface gravity waves: derivation and analysis*, Phil. Trans. Roy. Soc. Lond. Ser. A **356** (1998), 3123–3184.

[64] P. Milewski, *Resonant interactions between vortical flows and water waves. II. Shallow water*, Stud. Appl. Math. **94** , no. 3, (1995) 225–256.

[65] H. Y. Nguyen and F. Dias, *A Boussinesq system for two-way propagation of interfacial waves*, Phys. D **237** (2008), 2365–2389.

[66] H. Ono, *Algebraic Solitary Waves in Stratified Fluids*, J. Phys. Soc. of Japan **39** (1975), 1082–1091.

[67] D. H. Peregrine, *Interaction of water waves and currents*, Advances in Applied Mechanics **16** (1976) , Academic Press, 9–117.

[68] D. H. Peregrine, *Calculations of the development of an undular bore*, J. Fluid Mech. **25** (1966), 321–330.

[69] Lord Rayleigh, *On the theory of long waves and bores*, Proc. R. Soc. Lond. **A 90** (1914), 324–328.

[70] Lord Rayleigh, *On waves*, Phil. Mag. (5), **1** (1876), 257–279.

[71] J. J. Stoker, *Water Waves: The Mathematical Theory with Applications*, Pure and Applied Mathematics, Vol. IV. (Interscience Publishers, New York, 1957).

[72] G. G. Stokes, *On the theory of oscillatory waves*, Trans. Camp. Phil. Soc. **8** (1847), 441–455.

[73] B. Sturtevant, *Implications of experiments on the weak undular bore*, Phys. Fluids **6** (1965), 1052–1055.

[74] A. F. Teles da Silva and D. H. Peregrine, *Steep, steady surface waves on water of finite depth with constant vorticity*, J. Fluid Mech. **195** (1988), 281–302.

[75] J.-M. Vanden-Broeck *Free-surface flow over an obstruction in a channel*, Phys. Fluids **30**, 2315 (1987); doi:10.1063/1.866121 (3 pages)

[76] E. Wahlén, *Hamiltonian long-wave approximations of water waves with constant vorticity*, Phys. Lett. A **372** (2008), 2597–2602.

[77] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).