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Optimal dividend policies with transaction costs for a class of jump-diffusion processes

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Abstract This paper addresses the problem of finding an optimal dividend policy for a class of jump-diffusion processes. The jump component is a compound Poisson process with negative jumps, and the drift and diffusion components are assumed to satisfy some regularity and growth restrictions. With each dividend payment there is associated a fixed and a proportional cost, meaning that if ξ is paid out by the company, the shareholders receive $k\xi - K$, where k and K are positive. The aim is to maximize expected discounted dividends until ruin. It is proved that when the jumps belong to a certain class of light tailed distributions, the optimal policy is a simple lump sum policy, that is when assets are equal to or larger than an upper barrier $\bar{u}^*$, they are immediately reduced to a lower barrier $\underline{u}^*$ through a dividend payment. The case with $K = 0$ is also investigated briefly, and the optimal policy is shown to be a reflecting barrier policy for the same light tailed class. Methods to numerically verify whether a simple lump sum barrier strategy is optimal for any jump distribution are provided at the end of the paper, and some numerical examples are given.

Keywords Optimal dividends · Jump-diffusion models · Impulse control · Barrier strategy · Singular control · Numerical solution

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1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a probability space satisfying the usual conditions, i.e. the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous and P-complete. Assume that the uncontrolled surplus process follows the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t - dY_t, \quad X_0 = x, \tag{1.1}$$

where W is a Brownian motion and Y is a compound Poisson process, i.e.

$$Y_t = \sum_{i=1}^{N_t} S_i,$$

where N is a Poisson process with intensity λ, independent of the i.i.d. positive $\{S_i\}$. We will let S be generic for the S_i, and F be the distribution function of S. A natural interpretation is that X is a model of an insurance business, where Y represents claims and the other terms represent incomes and various business fluctuations. This interpretation is further developed in Example 3.5 below. To avoid lengthy explanations, we will refer to the S_i as claims.

Assume that the company pays dividends to its shareholders, but at a fixed transaction cost $K > 0$ and a tax rate $1 - k < 1$ so that $k > 0$. We will allow $k > 1$, opening up for other interpretations than that $1 - k$ is a tax rate. This means that if $\xi > 0$ is the amount the capital is reduced by due to a dividend payment, the net amount of money the shareholders receive is $k\xi - K$. It can be argued that taxes are paid on dividends after costs, so an alternative would be to use $k(\xi - K) = k\xi - kK$, but clearly this is just a reparametrization. Furthermore, different investors may have different tax rates, so $1 - k$ should be interpreted as an average tax rate.

Since every dividend payment results in a fixed transaction cost, the company should not pay out dividends continuously but only at discrete time epochs. Therefore, a strategy can be described by

$$\pi = (\tau_1^\pi, \tau_2^\pi, \ldots, \tau_n^\pi, \ldots; \xi_1^\pi, \xi_2^\pi, \ldots, \xi_n^\pi, \ldots),$$

where τ_n^π and ξ_n^π denote the times and amounts paid. Thus, when applying the strategy π, the resulting surplus process X_t^π is given by

$$X_t^\pi = x + \int_0^t \mu(X_s^\pi)ds + \int_0^t \sigma(X_s^\pi)dW_s - Y_t - \sum_{n=1}^{\infty} 1_{\{\tau_n^\pi < t\}}\xi_n^\pi. \tag{1.2}$$

Note that X^π is left continuous at the dividend payments, so that $\xi_n^\pi = X_{\tau_n^\pi}^\pi - X_{\tau_n^\pi+}^\pi$.

Definition 1.1. *A strategy π is said to be admissible if*

(i) $0 \leq \tau_1^\pi$ *and for* $n \geq 1$, $\tau_{n+1}^\pi > \tau_n^\pi$ *on* $\{\tau_n^\pi < \infty\}$.
(ii) τ_n^π *is a stopping time with respect to* $\{\mathcal{F}_t\}_{t\geq 0}$, $n = 1, 2. \ldots$
(iii) ξ_n^π *is measurable with respect to* $\mathcal{F}_{\tau_n^\pi+}$, $n = 1, 2. \ldots$

(iv) $\tau_n^\pi \to \infty$ *a.s. as* $n \to \infty$.
(v) $0 < \xi_n^\pi \le X_{\tau_n}^\pi$.

We denote the set of all admissible strategies by Π.

Another natural admissibility condition is that net money received should be positive, that is $k\xi - K > 0$. However, as we are looking for optimal policies, and a policy that allows $k\xi - K \le 0$ can never be optimal, it can be dropped as a condition.

With each admissible strategy π we define the corresponding ruin time as

$$\tau^\pi = \inf\{t \ge 0 : \ X_t^\pi < 0\}, \tag{1.3}$$

and the performance function $V_\pi(x)$ as

$$V_\pi(x) = E_x\left[\sum_{n=1}^{\infty} e^{-r\tau_n^\pi}(k\xi_n^\pi - K)1_{\{\tau_n^\pi \le \tau^\pi\}}\right], \tag{1.4}$$

where by P_x we mean the probability measure conditioned on $X_0 = x$. $V_\pi(x)$ represents the expected total discounted dividends received by the shareholders until ruin when the initial reserve is x.

The optimal return function is defined as

$$V^*(x) = \sup_{\pi \in \Pi} V_\pi(x) \tag{1.5}$$

and the optimal strategy, if it exists, by π^*. Then $V_{\pi^*}(x) = V^*(x)$. In the control theoretic language, this is an impulse control problem.

Definition 1.1 A lump sum dividend barrier strategy $\pi = \pi_{\bar{u},\underline{u}}$ with parameters $\underline{u} < \bar{u}$, satisfies for $X_0^\pi < \bar{u}$,

$$\tau_1^\pi = \inf\{t > 0 : X_t^\pi = \bar{u}\}, \qquad \xi_1^\pi = \bar{u} - \underline{u},$$

and for every $n \ge 2$,

$$\tau_n^\pi = \inf\{t > \tau_{n-1}^\pi : X_t^\pi = \bar{u}\}, \qquad \xi_n^\pi = \bar{u} - \underline{u}.$$

When $X_0^\pi \ge \bar{u}$,

$$\tau_1^\pi = 0, \qquad \xi_1^\pi = X_0^\pi - \underline{u},$$

and for every $n \ge 2$, τ_n^π is defined as above.

With a given lump sum dividend barrier strategy $\pi_{\bar{u},\underline{u}}$, the corresponding value function is denoted by $V_{\bar{u},\underline{u}}(x)$.

A lump sum dividend strategy $\pi_{\bar{u},\underline{u}}$ is sometimes called a $(\underline{u}, \bar{u})$ strategy.

Since some results in this paper, like Theorem 2.3, can be of interest of their own, we will look for as weak assumptions as possible. The following list of partially inclusive assumptions will therefore be referred to frequently.

A1a. μ and σ are continuous on $[0,\infty)$.
A1b. μ and σ are continuously differentiable on $(0,\infty)$.
A1c. μ and σ are twice continuously differentiable on $(0,\infty)$.
A1d. μ and σ are globally Lipschitz continuous on $[0,\infty)$.
A2a. The distribution function F is continuous.
A2b. The distribution function F has a continuous density f.
A2c. The distribution function F has a continuously differentiable density f.
A2d. The distribution function F has a continuous density f, and there is an $x_f \geq 0$ so that $f(x)$ is decreasing for $x > x_f$.
A3a. μ is continuously differentiable and there is an $\alpha > 0$ so that $\mu'_M \leq r+\lambda-\alpha$, where r is the discounting rate from (1.4) and $\mu'_M = \sup_{x>0} \mu'(x)$.
A3b. μ is continuously differentiable and $\mu'_M \leq r$.
A3c. μ is continuously differentiable and there is an $\alpha > 0$ and an $x_r \geq 0$ so that $\sup_{x \geq x_r} \mu'(x) \leq r - \alpha$.
A3d. μ is continuously differentiable and there is an $\alpha > 0$ so that $\mu'_M \leq r - \alpha$
A4. μ is concave on $[0,\infty)$.
A5. $\sigma^2(x) > 0$ on $[0,\infty)$.
A6. $|\sigma(x)| \leq C(1+x)$ for all $x \geq 0$ and some $C > 0$.
A7. There are nonnegative constants M_1 and M_2 so that

$$\frac{|\mu(x)| + r + \lambda}{\sigma^2(x)} \leq M_1 + M_2 x \quad \text{on} \quad [0,\infty).$$

Note that A6 follows from A1d.

It is argued in [21] that a proper comparison is between $\mu'(x)$, the rate of growth, and r, the discounting factor. It is easy to prove that if for some x_0 and $\delta > 0$, $\mu'(x) > r + \delta$ for all $x > x_0$, then $V^*(x) = \infty$ and there is no optimal policy.

The optimal dividend problem for the classical Lundberg process

$$X_t = x + pt - Y_t, \tag{1.6}$$

where Y is as in (1.1), has a long history when there are no transaction costs. It was proved by Gerber back in 1969 that the optimal strategy can be quite complicated, but for some choices of the claim distribution F, notably the exponential distribution, a simple barrier strategy is optimal [13]. By this is meant that whenever assets hit a barrier u^*, dividends are paid at a rate p until a claim occurs. If initial assets are higher than u^*, they are immediately reduced to u^* through a dividend payment. In general, the optimal dividend strategy is a so-called band strategy, meaning that there are several barriers u_i^*, and whenever assets hit one barrrier, dividends are paid continuously at the rate p until the next claim. If initial assets are higher than the highest barrier, they are reduced to that barrier through a dividend payment.

The methods used by Gerber are somewhat obsolete today, and in their paper Azcue and Muler [4] extended and improved the results from Gerbers paper using very different methods. See also the book [23]. In the same spirit as Azcue and Muler, Albrecher and Thonhauser in [1] allowed for assets to earn interests, and again it was proved that the optimal strategy is a band strategy, but in the case of exponential claims it is a simple barrier strategy as before.

Recently there has been a considerable interest in this problem when X is a Lévy process with spectrally negative jumps, i.e. μ and σ in (1.1) are constants and Y is a nondecreasing pure jump process with stationary, independent increments [3], [18], [16]. In [16] it was proved that if the Lévy measure of Y has a log convex density, then the optimal strategy is a barrier strategy. In particular, when $\sigma > 0$ this means that the dividend process is a singular process, a fact that is well known from the theory of optimal control of ordinary diffusion processes [25]. In [9] a special case of this result was proved when Y is a compound Poisson process with exponential jumps. Extensions and variations of the Lévy problem can be found in [19] and [17].

The introduction of a proportional cost k does not alter any of the above findings in a fundamental way, but if a positive fixed cost K is added, it is a different story. In this case the lump sum barrier strategy corresponds to the simple barrier strategy. Loeffen [19] made use of the results in [16] to prove optimality of a simple lump sum barrier strategy when X is a spectrally negative Lévy process with a log convex jump density. This was also proved in [5] for the simple model (1.6) with exponentially distributed claims, and in [9] where a Brownian motion is added to (1.6), but still with exponentially distributed claims. Loeffen [19] also gives an example where he shows numerically that a simple lump sum dividend strategy cannot be optimal.

Another paper that is related to the present paper is [2], where Y in (1.1) is replaced by the geometric term

$$Y_t = \sum_{i=1}^{N_t} X_{\tau_i -} S_i \quad \text{and} \quad F(1) = 1.$$

Here the τ_i are the times of jump of N. Under assumptions rather different from ours, simple barrier strategies are proved to be optimal in the no-fixed cost case, and simple lump sum dividend strategies in the fixed cost case.

There are several papers that study the fixed cost dividend problem (1.1) when there are no jumps, going back to [15] where X is a linear Brownian motion with drift. The closest to the present paper is [21], where optimality of the simple lump sum barrier strategy is proved. In [5] the basic assumption A3b used in [21] was relaxed, and it was proved that a simple lump sum barrier strategy is no longer always optimal. These exceptional cases are further studied in [7], where it is shown that the optimal strategy sometimes becomes what is called a two-level lump sum dividend strategy.

Further variations of the fixed cost dividend problem for the model (1.1) without jumps can be found in [8] where dividend payouts are subject to certain solvency constraints, and in [22] where reinvestment of capital is allowed after it goes below zero. In both cases, under the same assumptions on the diffusion part of (1.1) as in [21], simple lump sum strategies turned out to be optimal.

Finally we should mention the papers [24] and [11] which are devoted to smoothness properties of the optimal value function for a very general multivariate jump-diffusion process. Their objective, in a setup somewhat different from ours, is to minimize expected discounted costs for some rather general cost functions. In [24] viscosity solution properties are proved, and that is improved to classical solutions in [11].

There is an obvious practical advantage with the lump sum dividend barrier strategy compared to a simple barrier strategy. Paying dividends continuously is rather unfeasable, and one would have to resort to some kind of lump sum payments anyway. So the optimal solution with a fixed positive K is in some sense more attractive.

The aim of this paper is to analyze the dividend problem for the jump-diffusion (1.1) subject to various assumptions. We will be looking for sufficient conditions for a lump sum dividend strategy to be optimal. An, admittedly small, class of distributions, that together with some other rather weak assumptions guarantees that the optimal solution is a lump sum dividend barrier strategy, is found. As could be expected, this class includes the exponential distribution, but not only that. However, in order to belong to the class, it is necessary that the density exists, is decreasing and is light-tailed. For completeness, we have also included the case when $K = 0$. Then, under the same assumptions that yield an optimal solution when $K > 0$, it is proved that the optimal solution is a barrier strategy. At the end of the paper numerical methods to check whether simple lump sum barrier strategies are optimal for any claim distribution, are introduced. Numerical examples showing the usefulness of such methods are provided.

In order to present and prove the optimality results in Section 3 and beyond, it is necessary to make a thorough analysis of a certain boundary value problem associated with the optimality problem. Section 2 is therefore dedicated to this issue.

2 Some results for the associated integro-differential equation

In this section we will study the solution and its properties of the boundary value problem

$$\begin{aligned} Lg(x) &= 0, \quad x > 0, \\ g(0) &= 0, \\ g'(0) &= 1, \end{aligned} \tag{2.1}$$

where L is the integro-differential operator

$$Lg(x) = \frac{1}{2}\sigma^2(x)g''(x) + \mu(x)g'(x) - (r+\lambda)g(x) + \lambda \int_0^x g(x-z)dF(z). \tag{2.2}$$

A twice continuously differentiable solution of (2.1) will henceforth be called a canonical solution. We will see in the next section that a canonical solution plays a crucial role in the solution of the optimization problem of this paper.

The results of this section may be of independent interest, for example in generalizing the results of Section 3 and beyond. An example is how the results in [21] are generalized in [6]. We have therefore tried to keep the assumptions at a minimum. All proofs are of technical nature, so they are given in Section 6. Although there exists several proofs for the existence and smoothness of integral-differential equations, we have not found any that covers Theorem 2.3. Lemma 3.1 in [1] covers the case with no diffusion and linear $\mu(x)$. Theorem 5 in [12] is related, but it deals with ruin theory. Another example is Theorem 2.1 in [10], and they refer to Theorem 5 in [14] for a similar proof. As mentioned in the introduction, a very general existence and smoothness result can be found in [11]. It may well be possible to adapt that proof to our setting, but that would only be worthwhile if their assumption A5 can be relaxed, since it excludes much of Example 3.5 which is maybe the most important application of the theory.

Definition 2.1 For given $\beta > 0$ and $\zeta \geq 0$ we will denote by $L^{\infty}_{\beta,\zeta}\left([0,\infty), R^n\right)$ the space of Borel measurable functions

$$\mathbf{u} = (u_1, \ldots, u_n) : [0,\infty) \to R^n$$

such that

$$\sup_{x\geq 0} \frac{|\mathbf{u}(x)|}{\exp\left(\beta x + \zeta x^2\right)} < \infty.$$

Here

$$|\mathbf{u}(x)| = \max_{1\leq i\leq n} |u_i(x)|\,.$$

With $C([0,\infty), R^n)$ the space of continuous functions, we set

$$C_{\beta,\zeta}\left([0,\infty), R^n\right) = C\left([0,\infty), R^n\right) \cap L^{\infty}_{\beta,\zeta}\left([0,\infty), R^n\right).$$

Furthermore, $C^k\left([0,\infty), R^n\right)$ is the space of all k-times continuously differentiable functions and $C^k_{\beta,\zeta}\left([0,\infty), R^n\right)$ is the subspace so that the k'th derivative belongs to $C_{\beta,\zeta}\left([0,\infty), R^n\right)$.

From the definition it is clear that $L^{\infty}_{\beta,\zeta}\left([0,\infty), R^n\right) \subset L^{\infty}_{\tilde{\beta},\tilde{\zeta}}\left([0,\infty), R^n\right)$ whenever $\tilde{\zeta} > \zeta$ or $\tilde{\zeta} = \zeta$ and $\tilde{\beta} \geq \beta$. The same kind of inclusion obviously holds for $C_{\beta,\zeta}\left([0,\infty), R^n\right)$ and $C^k_{\beta,\zeta}\left([0,\infty), R^n\right)$.

Lemma 2.2 *The space* $L^{\infty}_{\beta,\zeta}\left([0,\infty), R^n\right)$ *with norm*

$$\|\mathbf{u}\|^{\infty}_{\beta,\zeta} = \sup_{x\geq 0} \frac{|\mathbf{u}(x)|}{\exp\left(\beta x + \zeta x^2\right)}$$

is a Banach space. Furthermore, the space $C_{\beta,\zeta}\left([0,\infty), R^n\right)$ *is closed in* $L^{\infty}_{\beta,\zeta}\left([0,\infty), R^n\right)$.

Theorem 2.3 *Assume A1a, A5 and A7. Then the boundary value problem (2.1) has a unique solution in* $C^2_{\beta,\zeta}\left([0,\infty), R\right)$ *for some* $\beta > 0$ *and* $\zeta \geq 0$.

Theorem 2.3 gives sufficient, but not necessary conditions for a canonical solution to exist. The assumption A7 can probably be relaxed, so in order to have results as general as possible, in the following we will just assume that a canonical solution exists.

Theorem 2.4 *Let g be a canonical solution and assume A1a and A5. Then g is strongly increasing on $[0,\infty)$. Moreover, assume there exists positive constants c_1, c_2, c_3 with $c_3 < c_1$ so that for all $x \geq 0$,*

$$(c_1 - c_3)\sigma^2(x) + 2(c_2 + c_3 x)\mu(x) < 2\frac{r}{c_1}(c_2 + c_3 x)^2. \tag{2.3}$$

Then

$$\lim_{x\to\infty} g'(x) = \infty.$$

In particular (2.3) can be satisfied if additionally A3c and A6 are satisfied.

Define

$$x^* = \inf\{x \geq 0 : g''(x) = 0\}.$$

By this definition, g is strictly concave on $(0, x^*)$. Clearly, if $x^* = \infty$ then g is strictly concave.

Theorem 2.5 *Let g be a canonical solution.*

a) If A5 holds then $x^* = 0$ if and only if $\mu(0) \leq 0$.
If in addition A1b and A2a hold and $\mu(0) = 0$ and $\mu'(0) < r + \lambda$, then $g''(0) = 0$ and $g'''(0) > 0$.

b) Assume A5 and that $\mu(0) > 0$. Also assume that there is an $x_0 > 0$ so that

$$\frac{\mu(x_0)}{x_0} = r. \tag{2.4}$$

Then $x^* \leq x_0$.

Clearly $\mu(0) > 0$ and A3c imply (2.4), and in this case since $\mu(x) \leq a + (r - \alpha)x$ for some nonnegative a,

$$x^* \leq \frac{a}{\alpha}.$$

Definition 2.6 A function h defined on $[0,\infty)$ is strictly concave-convex if there is an $x_h \geq 0$ so that h is strictly concave on $x < x_h$ and strictly convex on $x > x_h$.

If $x_h = 0$, h is strictly convex, but for simplicity we include that case in the definition of concave-convex. If h is twice continuously differentiable, a strictly concave-convex function has at most one point x where $h''(x) = 0$. If h is three times continuously differentiable, a concave-convex function has at most one point x where $h''(x) = 0$ and $h'''(x) > 0$.

In [25] it was shown that if $\lambda = 0$, i.e. no jumps, then under conditions similar to those here, the canonical solution is either strictly concave-convex or strictly concave. This was used in [21] to give a solution of the control problem for this case. Inspired by these results, we will look for sufficient conditions to insure concave-convexity for the more general jump-diffusion studied here. Unfortunately, this is not an easy task, and we have only been able to come up with some rather strong conditions. To present the results, define

$$Ag(x) = \int_0^x g(x-z)dF(z). \tag{2.5}$$

If A2a holds, an integration by parts shows that

$$(Ag)'(x) = \int_0^x g'(x-z)dF(z) = -\int_0^x g'(z)dF(x-z), \tag{2.6}$$

and if A2b holds,

$$(Ag)''(x) = f(x) + \int_0^x g''(x-z)f(z)dz. \tag{2.7}$$

Lemma 2.7 *Let g be a canonical solution. Assume A1c, A2b, A3a and A5. Also assume that*

$$\lambda(Ag)''(x) + \mu''(x)g'(x) < 0, \tag{2.8}$$

whenever

$$(Ag)'(x) = \left(\frac{\lambda + r - \mu'(x)}{\lambda}\right) g'(x). \tag{2.9}$$

Then g is strictly concave-convex. Moreover, for every $x > x^*$,

$$(Ag)'(x) < \left(\frac{\lambda + r - \mu'(x)}{\lambda}\right) g'(x), \tag{2.10}$$

i.e. if x_0 satisfies (2.8) and (2.9) then $x_0 \leq x^*$.

Unfortunately the assumption (2.8) and (2.9) in Lemma 2.7 is not easy to verify, so something that is more easily verifiable is needed. Assume that the density f is continuously differentiable and consider the condition,

$$-f'(x) > c(x)f(0)f(x), \quad x \geq 0, \tag{2.11}$$

where

$$c(x) = \frac{\lambda}{\lambda + r - \mu'(x)}$$

and it is implicitly assumed that $f(0)$ is finite.

Theorem 2.8 *Let g be a canonical solution of (2.1). Assume A1c, A2c, A3a, A4 and A5. Furthermore, assume that (2.11) holds. Then the canonical solution g is strictly concave-convex.*

Since Theorem 2.8 gives us the result we want, it is of interest to examine a bit closer the class of distribution functions that satisfy (2.11). Clearly, (2.11) and A3a imply that f is strongly decreasing. Furthermore, integrating (2.11) from 0 to infinity and using that $\liminf_{x\to\infty} f(x) = 0$, gives

$$f(0) > f(0)\int_0^\infty c(x)f(x)dx = f(0)E[c(S)].$$

Therefore, it is necessary that $E[c(S)] < 1$.

We can write (2.11) as $\frac{d}{dx}\log f(x) < -c(x)f(0)$, and integrating this yields

$$f(x) < f(0)e^{-f(0)\int_0^x c(y)dy}.$$

By A3a, $c(x) \geq \frac{\lambda}{\alpha} > 0$ for all x and so f must be light tailed.

It is trivial to verify that the exponential distribution satisfies (2.11) provided $c(x) < 1$, i.e. provided A3d holds. The question is whether there are any other distributions that satisfy this inequality. Here are a couple of examples.

Example 2.9 Assume that $\mu'(x) = r - \alpha$ for some $\alpha > 0$ so that $c(x) = c = \frac{\lambda}{\lambda+\alpha}$. Let f be the exponential mixture

$$f(x) = a\beta_1 e^{-\beta_1 x} + (1-a)\beta_2 e^{-\beta_2 x}, \quad x \geq 0,$$

for $0 < a < 1$. Without loss of generality we can assume that $\beta_1 < \beta_2$. Then (2.11) is equivalent to $h(x) > 0$ for all $x \geq 0$, where

$$\begin{aligned} h(x) &= e^{\beta_1 x}(-f'(x) - cf(0)f(x)) \\ &= a\beta_1^2 + (1-a)\beta_2^2 e^{-(\beta_2-\beta_1)x} - c(a\beta_1 + (1-a)\beta_2)\left(a\beta_1 + (1-a)\beta_2 e^{-(\beta_2-\beta_1)x}\right). \end{aligned}$$

Since

$$h'(x) = -(1-a)\beta_2(\beta_2-\beta_1)e^{-(\beta_2-\beta_1)x}(\beta_2 - c(a\beta_1 + (1-a)\beta_2)) < 0,$$

this is satisfied if and only if $\lim_{x\to\infty} h(x) = a\beta_1^2 - c(a\beta_1 + (1-a)\beta_2)a\beta_1 \geq 0$. Easy calculations show that this is equivalent to

$$\frac{\beta_2}{\beta_1} \leq 1 + \frac{1-c}{c}\frac{1}{1-a} = 1 + \frac{\alpha}{(1-a)\lambda}.$$

Example 2.10 Assume again that $\mu'(x) = r - \alpha$ for some $\alpha > 0$ so that $c(x) = c = \frac{\lambda}{\lambda+\alpha}$. Let f be the truncated normal distribution

$$f(x) = \frac{e^{-\frac{1}{2\sigma^2}(x+\gamma)^2}}{\int_0^\infty e^{-\frac{1}{2\sigma^2}(y+\gamma)^2}dy} = \frac{\frac{1}{\sigma}e^{-\frac{1}{2\sigma^2}(x+\gamma)^2}}{H\left(\frac{\gamma}{\sigma}\right)}, \quad x \geq 0, \tag{2.12}$$

for $\gamma > 0$. Here

$$H(u) = \int_u^\infty e^{-\frac{1}{2}y^2}dy.$$

Then (2.11) is equivalent to

$$\frac{1}{\sigma}(x+\gamma) > c\frac{e^{-\frac{1}{2}\left(\frac{\gamma}{\sigma}\right)^2}}{H\left(\frac{\gamma}{\sigma}\right)}.$$

Since the left side is increasing in x, this is equivalent to

$$\frac{\gamma}{\sigma}e^{\frac{1}{2}\left(\frac{\gamma}{\sigma}\right)^2}H\left(\frac{\gamma}{\sigma}\right) > c.$$

Let

$$v(u) = ue^{\frac{1}{2}u^2}H(u).$$

Then $v(0) = 0$, and L'hôpital's rule easily shows that $\lim_{u\to\infty} v(u) = 1 > c$. Therefore, if we can show that v is strongly increasing in u, (2.11) is satisfied if and only if

$$\frac{\gamma}{\sigma} \geq u_0,$$

where u_0 is the unique solution of $v(u) = c$. To show that v is strongly increasing, differentiation gives

$$v'(u) = (1+u^2)e^{\frac{1}{2}u^2}H(u) - u.$$

An integration by parts gives that for $u > 0$,

$$H(u) > u^2\int_u^\infty \frac{1}{y^2}e^{-\frac{1}{2}y^2}dy = u^2\left(\frac{1}{u}e^{-\frac{1}{2}u^2} - H(u)\right),$$

from which we get that $(1+u^2)H(u) > ue^{-\frac{1}{2}u^2}$, and so $v'(u) > 0$. A numerical calculation with $\lambda = 1$ and $\alpha = 0.02$ shows that $u_0 = 6.936$

Remark 2.11 As mentioned in the introduction, in [19] it is shown that for the Lévy model the result of Theorem 2.8 holds if the condition (2.11) is replaced by the condition that $\log f$ is convex. This is a more attractive condition, one reason is that it includes several heavy tailed distributions like the Pareto distribution

$$F(x) = 1 - \frac{\theta^\kappa}{(\theta+x)^\kappa}, \quad x > 0, \tag{2.13}$$

for positive θ and κ. It also includes the heavy tailed Weibull distribution. On the other hand, the log-convexity assumption of f does not include (2.12) since the density in that example is not log-convex.

We conjecture that Theorem 2.8 holds also when f is log-convex. However, the proofs given in [16] and [19] rely on the Lévy strucure, so a different proof is needed.

3 The optimal solution

In this section we will assume that g is the unique canonical solution that satisfies (2.1). Then any function v that satisfies $v(0) = 0$ and $Lv(x) = 0$ is of the form

$$v(x) = cg(x), \tag{3.1}$$

for some constant c. This fact will be utilized in our quest for an optimal solution. Again proofs are of technical nature, and are therefore given in Section 6.

Consider the following set of problems with unknown V, $\bar{u}^*$ and $\underline{u}^*$.

B1:
$$V(0) = 0 \text{ and } LV(x) = 0, \qquad 0 < x < \bar{u}^*,$$
$$V(x) = V(\bar{u}^*) + k(x - \bar{u}^*), \qquad x > \bar{u}^*.$$

B2:
$$V(\bar{u}^*) = V(\underline{u}^*) + k(\bar{u}^* - \underline{u}^*) - K,$$
$$V'(\bar{u}^*) = k,$$
$$V'(\underline{u}^*) = k.$$

B3:
$$V(\bar{u}^*) = k\bar{u}^* - K,$$
$$V'(\bar{u}^*) = k,$$
$$V'(x) < k, \qquad 0 \le x \le \bar{u}^*.$$

From this and (3.1) we see that $V(x)$ can be written as

$$V(x) = \begin{cases} c^* g(x), & x \le \bar{u}^*, \\ V(\underline{u}^*) + k(x - \underline{u}^*) - K, & x > \bar{u}^*. \end{cases} \tag{3.2}$$

Here

$$c^* = \frac{k}{g'(\bar{u}^*)} = \frac{k(\bar{u}^* - \underline{u}^*) - K}{g(\bar{u}^*) - g(\underline{u}^*)}, \tag{3.3}$$

where in case B3, $\underline{u}^* = 0$. Also, if g is concave-convex then clearly $\underline{u}^* < x^* < \bar{u}^*$.

Theorem 3.1 *Assume that the canonical solution g is strictly concave-convex. Then we have:*

a) If B1+B2 or B1+B3 have a solution, this solution is unique.
b) If in addition $\lim_{x\to\infty} g'(x) = \infty$, then either B1+B2 or B1+B3 will have a solution.

It follows from the proof of Theorem 3.1(b) that it is B1+B2 that have a solution if and only if

$$\int_0^{\bar{u}} \left(1 - \frac{g'(x)}{g'(\bar{u})}\right) dx = \bar{u} - g(\bar{u}) > \frac{K}{k},$$

where $\bar{u}$ is the unique value that satisfies $g'(\bar{u}) = g'(0) = 1$.

If B1+B2 or B1+B3 have a solution, then

$$LV(x) = k\mu(x) - (r+\lambda)(V(\bar{u}^*) + k(x - \bar{u}^*)) + \lambda AV(x), \quad x > \bar{u}^*.$$

Therefore, if the canonical solution g is concave-convex, the fact that $V'(\bar{u}^*) = k$, that V and V' are continuous and that $LV(\bar{u}^*-) = 0$ gives

$$LV(\bar{u}^*+) = -\sigma^2(\bar{u}^*)V''(\bar{u}^*-) \leq 0. \tag{3.4}$$

Theorem 3.2 *Assume that the canonical solution g is a strictly concave-convex. Also assume A1d and A2a. Then we have:*

(i) Assume that either B1+B2 or B1+B3 have a solution, and that

$$LV(x) \leq 0, \quad x > \bar{u}^*. \tag{3.5}$$

Then $V^*(x) = V(x) = V_{\bar{u}^*,\underline{u}^*}(x)$ for all $x \geq 0$, where in case B1+B3, $\underline{u}^* = 0$. Thus the lump sum dividend barrier strategy $\pi^* = \pi_{\bar{u}^*,\underline{u}^*}$ is an optimal strategy. In particular (3.5) is satisfied if

$$(LV)'(x) = \lambda(AV)'(x) - k(r + \lambda - \mu'(x)) \leq 0, \quad x > \bar{u}^*. \tag{3.6}$$

(ii) If neither B1+B2 nor B1+B3 have a solution, then there do not exist an optimal strategy, but

$$V^*(x) = \lim_{\bar{u}\to\infty} V_{\bar{u},0}(x),$$

and this limit exists and is finite for every $x \geq 0$. In terms of the canonical solution,

$$V^*(x) = \frac{k}{g'_\infty} g(x),$$

where $g'_\infty = \lim_{\bar{u}\to\infty} g'(\bar{u})$.

Furthermore, case (i) occurs if $g'_\infty = \infty$. If g is concave, i.e. $x^* = \infty$, then case (ii) occurs.

Assumption A1d was made to guarantee that the stochastic differential equation (1.1) has a unique strong solution. It could be replaced by A1a and any other condition that guarantees a unique strong solution.

Remark 3.3 It was demonstrated in Example 2 in [19] that concave-convexity of g is not a necessary condition for a simple lump sum dividend barrier strategy to be optimal.

The next theorem gives sufficient, verifiable conditions for optimality.

Theorem 3.4 *Assume A1c, A1d, A2c, A3d, A4, A5, A7 and (2.11). Then either B1+B2 or B1+B3 have a solution, and an optimal policy exists. This optimal policy is given in Theorem 3.2(i).*

Example 3.5 Assume that income from the basic insurance business evolves as

$$P_t = pt + \sigma_P W_{P,t} - Y_t,$$

where $\sigma_P^2 > 0$. Also assume that assets earn return according to

$$R_t = (r-\alpha)t + \sigma_R dW_{R,t},$$

where $\alpha > 0$. Here W_P and W_R are standard Brownian motions with correlation ρ. The constant α can be seen as a cost due to inefficient investments, or as an equity premium since $r - (r-\alpha) = \alpha$.

Total assets without dividend payments are then

$$dX_t = dP_t + X_{t-} dR_t, \quad X_0 = x.$$

Combining the two Brownian motions, this can be written as (1.1), where

$$\mu(x) = p + (r-\alpha)x, \quad \sigma^2(x) = \sigma_P^2 + 2\rho\sigma_P\sigma_R x + \sigma_R^2 x^2.$$

In order for assumption A5 to hold it is necessary and sufficient that $\sigma_P^2 > 0$. If in addition A2c is satisfied and (2.11) holds, an optimal solution exists and is given in Theorem 3.4.

4 The case with no fixed transaction costs

In this section results for the case $K = 0$ similar to those in Section 3 will be presented. When $K = 0$ there is the added possibility that dividends may be paid continuously. The controlled process (1.2) therefore becomes

$$\begin{aligned} X_t^\pi = x &+ \int_0^t \mu(X_s^\pi)ds + \int_0^t \sigma(X_s^\pi)dW_s - Y_t \\ &- \sum_{n=1}^\infty 1_{\{\tau_n^\pi < t\}} \xi_n^\pi - D_t^{c,\pi}, \quad t \le \tau^\pi, \end{aligned} \tag{4.1}$$

where $D^{c,\pi}$ is a continuous, nondecreasing and adapted process. The performance function (1.4) becomes

$$V_\pi(x) = E_x \left[\sum_{n=1}^\infty e^{-r\tau_n^\pi} k\xi_n^\pi 1_{\{\tau_n^\pi \le \tau^\pi\}} - \int_0^{\tau^\pi} e^{-rs} k dD_s^{c,\pi} \right]. \tag{4.2}$$

Also, the optimal function V^* is defined as in (1.5).

Definition 4.1 A singular continuous dividend barrier strategy $\pi = \pi_u$ with barrier u satisfies:

- When $X_t^\pi < u$, do nothing.
- When $X_t^\pi > u$, reduce X_t^π to u by paying $X_t^\pi - u$ as a lump sum dividend.
- When $X_t^\pi = u$, pay dividends so that u is a reflecting barrier.

The corresponding value function is denoted by $V_u(x)$.

With the singular continuous dividend barrier strategy a lump sum is only paid at time 0, and only if $x > u$. After that dividends are paid continuously, but if A5 holds it is well known from the theory of singular stochastic control, see e.g. [25], that the dividend process $D^{c,\pi}$ is a singular process. This means that $D^{c,\pi}$ is continuous, nondecreasing and increasing on an uncountable set of Lebesgue measure zero. Therefore, as opposed to the lump-sum dividend strategy of Definition 1.1, from a practical point of view it is impossible to implement a singular continuous dividend policy.

Using the results from Section 2, the following theorem is proved as in [25].

Theorem 4.2 *Assume that a canonical solution exists and is strictly concave-convex. Also assume A1d, A2a and A5. Then we have:*

(i) If $x^* < \infty$ *let*

$$V(x) = \frac{k}{g'(x^*)}g(x), \qquad x \le x^*,$$
$$V(x) = V(x^*) + k(x - x^*), \qquad x > x^*.$$

If

$$LV(x) \le 0, \quad x \ge x^*, \tag{4.3}$$

then $V^*(x) = V(x) = V_{x^*}(x)$ *for all* $x \ge 0$*, so that the singular continuous dividend barrier strategy* $\pi = \pi_{x^*}$ *is optimal.*

(ii) If $x^* = \infty$ *so that* g *is concave, then there is no optimal strategy, but*

$$V^*(x) = \frac{k}{g'_\infty}g(x), \qquad x \ge 0.$$

Note that if $x^* = 0$, assets are immediately reduced to zero and ruin occurs because of A5.

Again, the assumptions of Theorem 3.4 are sufficient for an optimal solution to exist.

5 A numerical approach

Theorem 3.4 gives sufficient conditions for a lump sum barrier strategy to be optimal, but unfortunately the class of distributions that satisfy (2.11) is rather limited. However, if A1d, A2a, A3c, A5 and A6 are satisfied, it follows from Theorems 2.4, 2.5 and 3.2 that all that is needed for a lump sum dividend barrier policy to be optimal is that the canonical solution is strictly concave-convex and that (3.5) is satisfied. In principle, both these conditions can be tested numerically, but such a test will necessarily be on a finite interval, and there is no a priori guarantee that there are points beyond that interval where the assumptions are not satisfied. Therefore, it would be useful to prove theoretically that for some numerically calculable $x_P > x^*$, the conditions hold. In that case it is sufficient to use a numerical check on the interval $(0, x_P)$. Here we will take such an approach. All proofs are again given in Section 6. The first result is concerned with ultimate convexity.

Theorem 5.1 *Let g be a canonical solution.*

a) Assume A1b and A3c. Let

$$x_M = \inf\{x > \max\{x^*, x_r\} : g'(x) \geq g'(0)\}. \tag{5.1}$$

Then $g''(x) > 0$ for all $x > x_M$.

b) Assume A1b, A2b, A3c and A4. Let

$$x_L = \inf\left\{x > \max\{x^*, x_r\} : (r - \mu'(x))g'(x) > \lambda g(x^*) \max_{z \geq x - x^*} f(z)\right\} \tag{5.2}$$

If $x_L < \infty$ and $g''(x) > 0$ for all $x \in (x^*, x_L]$, then g is strictly convex on (x^*, ∞). Furthermore, (2.10) holds for all $x > x_L$.

Remark 5.2 Instead of searching for x_M in (5.1) or x_L in (5.2), an alternative is to take an arbitrary $x_A > \max\{x^*, x_r\}$ and check if the condition in (5.1) or in (5.2) holds. If that is the case, and it is numerically shown that $g''(x) > 0$ on (x^*, x_A), it follows from the definitions of x_M and x_L that $g''(x) > 0$ on (x^*, ∞).

We now turn to condition (3.5). Assume that B1+B2 or B1+B3 have a solution, and let $h(x) = LV(x)$. If we can find a numerically calculable $x_P \geq \bar{u}^*$ so that it is theoretically known that $h(x) \leq 0$ when $x \geq x_P$, then it is enough to numerically test whether $h(x) \leq 0$ on $(\bar{u}^*, x_P)$. By (3.4), this holds if $h'(x) \leq 0$ for $x \in (\bar{u}^*, x_P)$.

Theorem 5.3 *Let g be a canonical solution.*

a) Assume A1b, A2d and A4. Set

$$x_K = \inf\left\{x \geq \bar{u}^* + x_f : \lambda \int_0^{\bar{u}^*} g'(z) f(x - z) dz < (r - \mu'(x))g'(\bar{u}^*)\right\} \tag{5.3}$$

Then $(LV)'(x) \leq 0,\ x \geq x_K$. Also, $x_K < \infty$ if A3c holds.

b) Assume A1b, A2b and A4. Set

$$x_J = \inf\left\{x > \bar{u}^* : \max_{z \geq x - \bar{u}^*} f(z) < \frac{1}{\lambda}(r - \mu'(x))\frac{g'(\bar{u}^*)}{g(\bar{u}^*)}\right\}. \tag{5.4}$$

Then $(LV)'(x) \leq 0,\ x \geq x_J$. Also, $x_J < \infty$ if A2d and A3c holds.

Clearly, if $x_K = \bar{u}^*$ or $x_J = \bar{u}^*$, the condition (3.5) holds.

Remark 5.4 As in Remark 5.2 it is not necessary to calculate x_K and x_J. Again it is sufficient to pick an arbitry x_A, with $x_A > \bar{u}^* + x_f$ for x_M and $x_A > \bar{u}^*$ for x_J, and verify that the condition in (5.3) or in (5.4) holds for x_A. If that is the case, (3.5) holds provided it can be shown numerically that $LV(x) \le 0$ for $x \in (\bar{u}^*, x_A)$.

Example 5.5 In this example we will provide numerical results for the model presented in Example 3.5. We will use two different claimsize distributions, the exponential distribution with expectation β^{-1} and the Pareto distribution (2.13). For the latter, if $\kappa > 2$,

$$E[S] = \frac{\theta}{\kappa - 1} \quad \text{and} \quad E[S^2] = \frac{2\theta^2}{(\kappa-1)(\kappa-2)}.$$

The P process satisfies

$$E[P_t] = (p - \lambda E[S])t \quad \text{and} \quad \mathrm{Var}[P_t] = (\sigma_P^2 + \lambda E[S^2])t.$$

We let p, λ and $E[P_t]$ be the same for the two claimsize distributions. Then $E[S]$ will also be the same, so $\beta = (\kappa - 1)/\theta$. Furthermore, letting $\mathrm{Var}[P_t]$ be the same, and denoting the diffusion parameters by $\sigma_{P,E}^2$ and $\sigma_{P,P}^2$ respectively, gives

$$\sigma_{P,E}^2 = \sigma_{P,P}^2 + \frac{2\lambda\theta^2}{(\kappa-1)^2(\kappa-2)}.$$

For a numerical example we let $p = 1.5$, $\lambda = 1$, $\beta = 1$, $\kappa = 3$, $\theta = 2$, $\sigma_{P,E}^2 = 3$, $\sigma_{P,P}^2 = 1$ and $\rho = 0$, which make $E[P_t]$ and $\mathrm{Var}[P_t]$ the same for the two distributions. Furthermore, let $r = 0.1$, $\alpha = 0.02$, $\sigma_R = 0.2$, $k = 0.9$ and $K = 0.2$. Numerical calculations together with Remarks 5.2 and 5.4 show that the Pareto distribution satisfies the conditions of Theorem 3.2(i), and so the optimal policy is a lump sum dividend policy in both cases. In view of Remark 2.11, this comes as no surprise. The numerical solutions show that in the exponential case $(\bar{u}^*, \underline{u}^*) = (15.96, 6.32)$ so that $\bar{u}^* - \underline{u}^* = 9.65$. In the Pareto case $(\bar{u}^*, \underline{u}^*) = (12.84, 4.11)$ so that $\bar{u}^* - \underline{u}^* = 8.72$. Figure 5.1 shows the value function $V^*(x)$ for increasing x.

It is interesting to note that $\bar{u}^*$, $\underline{u}^*$ and $\bar{u}^* - \underline{u}^*$ are all higher for the exponential distribution than for the Pareto distribution, while the value function $V^*(x)$ is higher for the Pareto distribution. A possible reason for this is that the Pareto distribution yields many small claims and an occasional very large one, while the exponential distribution yields more similar claims. Therefore, not worrying too much about the occasional large claim, the Pareto distribution combined with a lower value of σ_P^2 is less affected with the possibility of ruin, thus allowing a bolder strategy and higher expected payout. If ruin occurs, in the Pareto case it will likely be with a very large deficit, but since the size of the deficit does not matter, this is an advantage for the Pareto distribution and so it can explain the higher value for this distribution.

Figures 5.2-5.9 show optimal barriers $\bar{u}^*$ and $\underline{u}^*$, optimal payout $\bar{u}^* - \underline{u}^*$ and optimal value when $x = 2$, i.e. $V^*(2)$, for the exponential and Pareto distributions. In all figures the parameters are the same as above, except of course

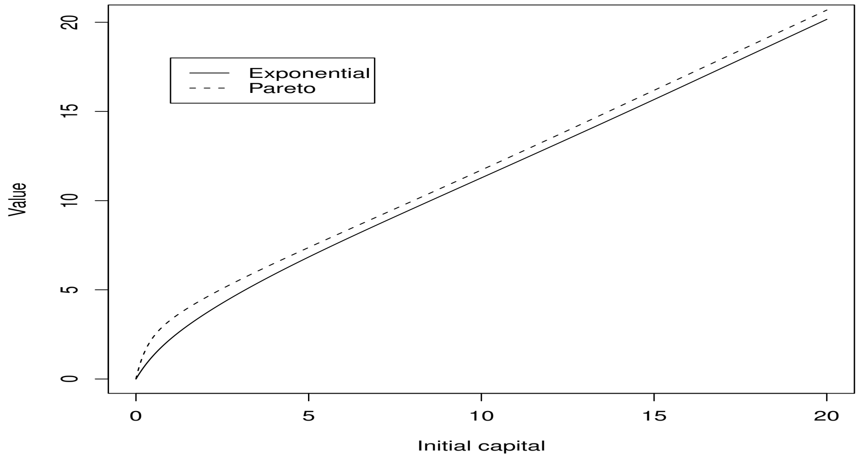

Fig. 5.1 Values of $V^*(x)$ for increasing x, using the exponential and the Pareto distributions for S. The parameters are $p = 1.5$, $\lambda = 1$, $\beta = 1$, $\kappa = 3$, $\theta = 2$, $r = 0.1$, $\alpha = 0.02$, $\sigma_R = 0.2$, $\rho = 0$, $k = 0.9$ and $K = 0.2$. The diffusion parameters are $\sigma_P^2 = 3$ in the exponential case and $\sigma_P^2 = 1$ in the Pareto case.

for the one that varies in that particular figure. Since the Pareto distribution is not covered by Theorem 3.4, a numerical test as described in Remarks 5.2 and 5.4 was used to assure that the optimal policy will always be a lump sum dividend policy. This, not surprisingly, turned out to be the case all the time. We will return to this test in Example 5.6.

Looking at the figures, the first thing to notice is that the Pareto distribution always results in a higher value of $V^*(x)$, thus supporting the argument given above. In most cases, both $\bar{u}^*$ and $\underline{u}^*$ are lower in the Pareto case, as is the payout $\bar{u}^* - \underline{u}^*$.

From Figure 5.2 we see that for $p \leq 0.63$, $\underline{u}^* = 0$ in the exponential case, and $\underline{u}^* = 0$ for $p \leq 0.46$ in the Pareto case. So when the income p is sufficiently small, it is optimal to pay everything in dividends immediately and go bankrupt. The reason is of course that the premium is too small compared to expected claims. The same optimality of immediate bankruptcy is observed in Figure 5.3 when the claim intensity λ is high.

Most plots must be said to be rather reasonable, although not apriori obvious. The main exceptions are Figures 5.2 and 5.3, where $\bar{u}^*$, $\underline{u}^*$ and $\bar{u}^* - \underline{u}^*$ all exhibit some rather unexpected patterns.

Example 5.6 In this example we again study the model of Example 5.5, but with different parameters and distribution function. Let $\sigma_P = \sigma_R = 0$, $p = 21.4$, $\lambda = 10$, $r = 0.1$, $\alpha = 0.08$, $k = 1$ and $K = 0$. Also, let the claimsizes be

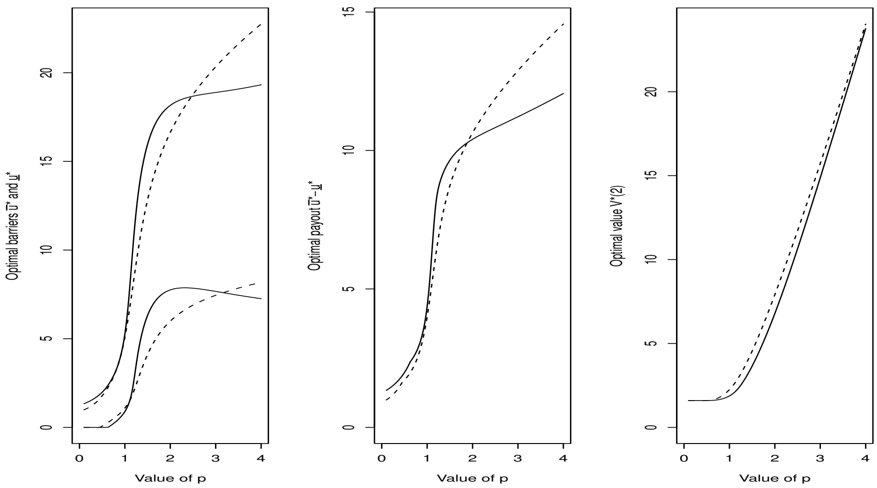

Fig. 5.2 Values for increasing p using the exponential and the Pareto distributions for S. The other parameters are as in Figure 5.1. Left panel: Values of the optimal barriers $\bar{u}^*$ and $\underline{u}^*$. Middle panel: Values of the optimal payout $\bar{u}^* - \underline{u}^*$. Right panel: The value function $V^*(2)$. Full line is exponential distribution and broken line is Pareto distribution.

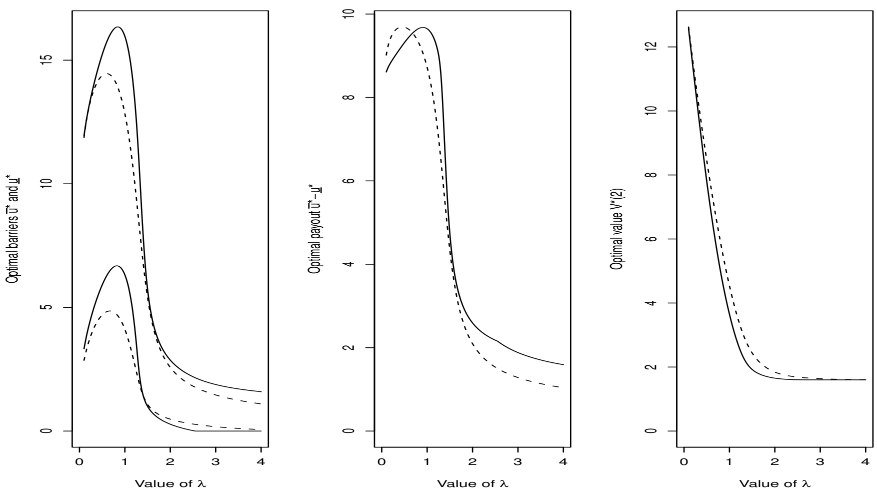

Fig. 5.3 Values for increasing λ using the exponential and the Pareto distributions for S. The other parameters are as in Figure 5.1. Panels and legends are as in Figure 5.2.

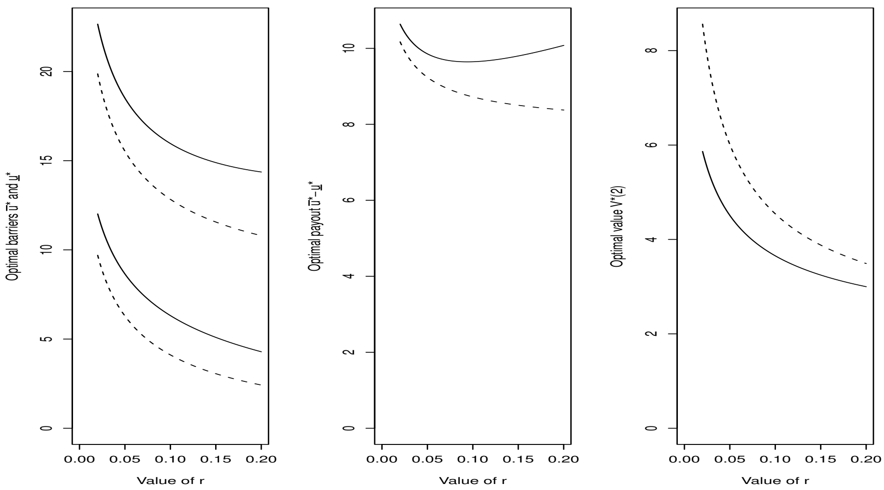

Fig. 5.4 Values for increasing r using the exponential and the Pareto distributions for S. The other parameters are as in Figure 5.1. Panels and legends are as in Figure 5.2.

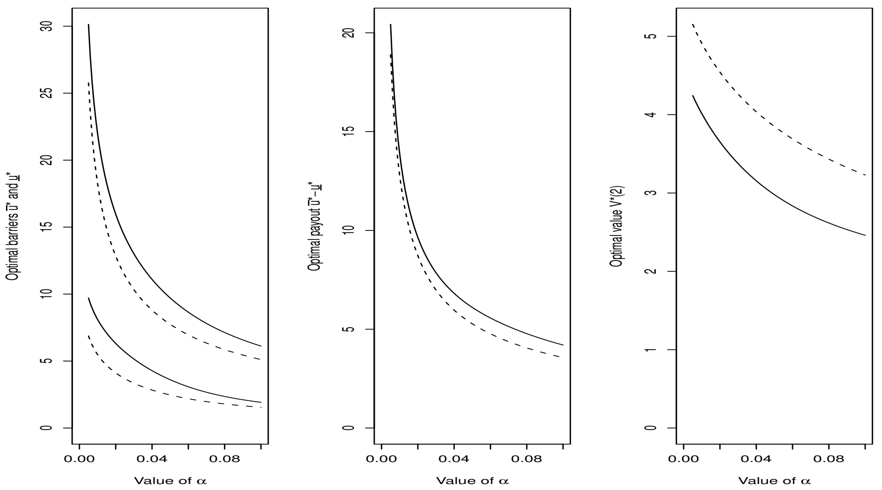

Fig. 5.5 Values for increasing α using the exponential and the Pareto distributions for S. The other parameters are as in Figure 5.1. Panels and legends are as in Figure 5.2.

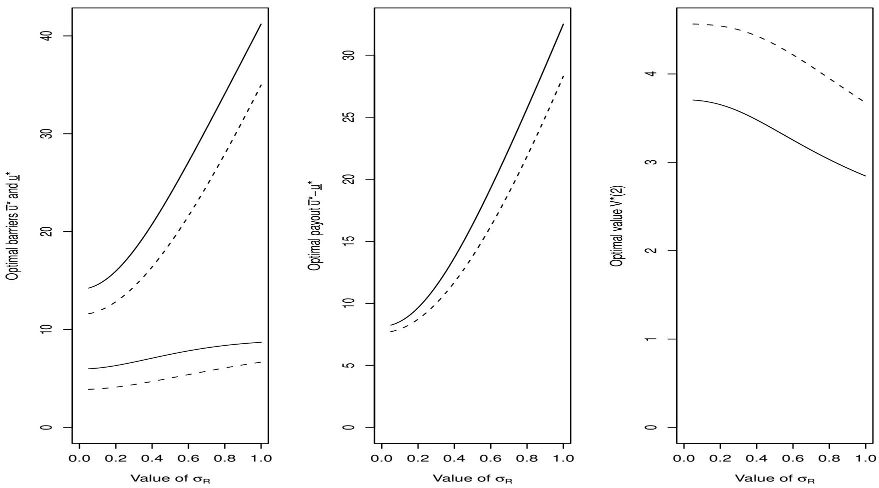

Fig. 5.6 Values for increasing σ_R using the exponential and the Pareto distributions for S. The other parameters are as in Figure 5.1. Panels and legends are as in Figure 5.2.

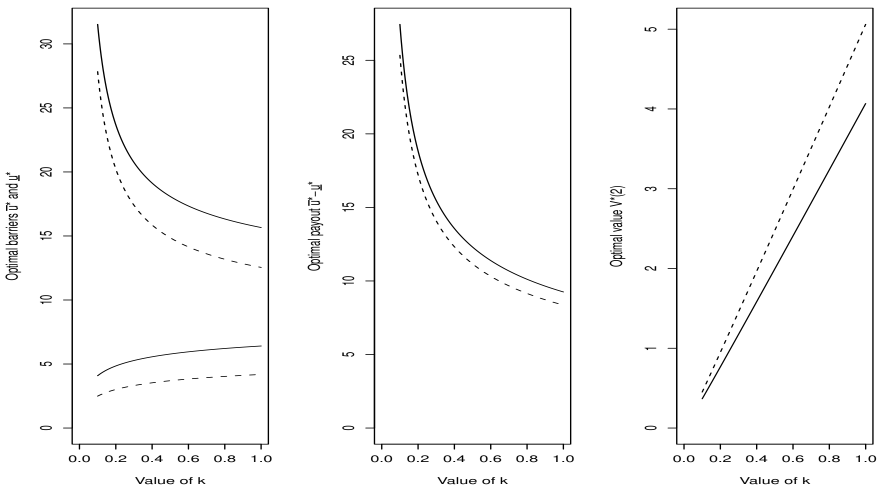

Fig. 5.7 Values for increasing k using the exponential and the Pareto distributions for S. The other parameters are as in Figure 5.1. Panels and legends are as in Figure 5.2.

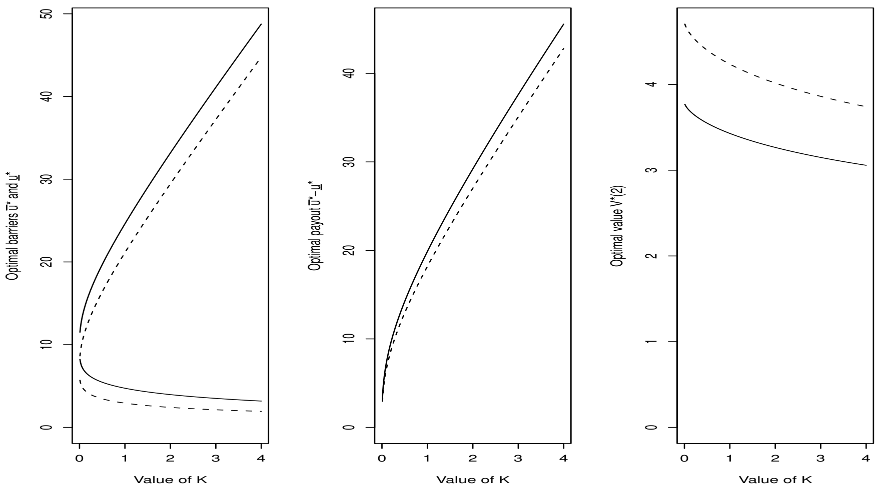

Fig. 5.8 Values for increasing K using the exponential and the Pareto distributions for S. The other parameters are as in Figure 5.1. Panels and legends are as in Figure 5.2.

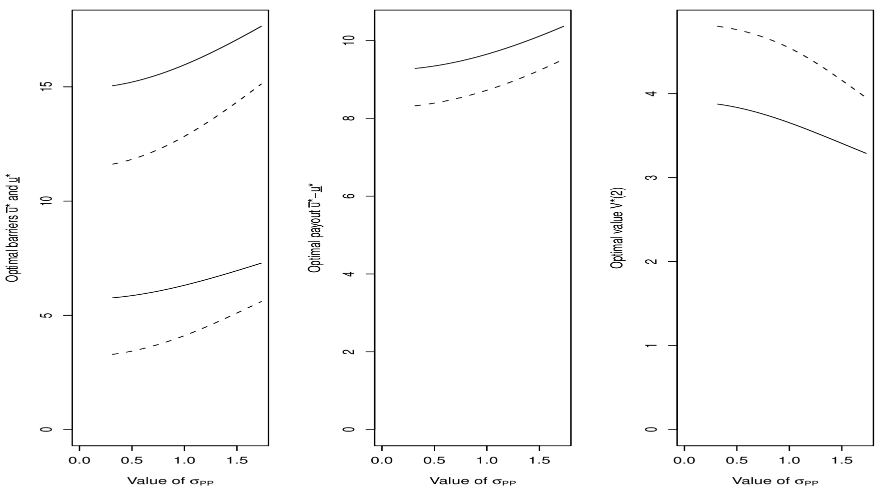

Fig. 5.9 Values for increasing σ_P using the exponential and the Pareto distributions for S. The P-process diffusion parameters are $\sigma_P^2 = \sigma_{P,E}^2 = 2 + \sigma_{P,P}^2$. The other parameters are as in Figure 5.1. Panels and legends are as in Figure 5.2.

gamma distributed with density

$$f(x) = \beta^2 x e^{-\beta x} 1_{\{x>0\}}, \tag{5.5}$$

with $\beta = 1$. Then it is proved in [1] that for this model a simple barrier strategy cannot be optimal, and the optimal band strategy is identified.

Making a few changes, let $\sigma_P = 0.5$, $\sigma_R = 0.2$ and $\rho = 0$. Although not relevant for the canonical solution, let $k = 0.9$ and $K = 0.2$. The upper left panel in Figure 5.10 shows $g''(x)$ for $x \in (0.064, 50)$. Since there are three roots $x_1 = 0.069$, $x_2 = 1.73$ and $x_3 = 12.66$, we cannot expect a simple lump sum dividend barrier strategy to be optimal, although we cannot rule that out as is shown in [19]. The upper right panel shows $LV(x)$ for $x \in (\bar{u}^*, 50)$, and since the condition in (5.4) turned out to be satisfied for $x = 50$, it follows from Remark 5.4 that (3.5) is satisfied.

Making yet another change, let $\sigma_P = 4$ and as before $\sigma_R = 0.2$. From the lower left panel we have (maybe a bit difficult to see) that there is only one root $x^* = 14.5$. Furthermore, since the condition in (5.2) turned out to be satisfied for $x = 50$, it follows from Remark 5.2 that g is strictly concave-convex. Thus the added diffusion smoothed out the non concave-convexity in the original model. Also, the condition in (5.4) was satisfied for $x = 50$, and so by Remark 5.4 and the lower right panel in Figure 5.10, (3.5) is satisfied. Therefore by Theorem 3.2, the optimal strategy is a simple lump sum dividend strategy.

6 Proofs

Proof of Lemma 2.2 It is straightforward to show that $\|\cdot\|_{\beta,\zeta}^{\infty}$ is a norm on $L_{\beta,\zeta}^{\infty}([0,\infty), R^n)$. To prove completeness, let $\{\mathbf{u}_k\}$ be a Cauchy sequence in $L_{\beta,\zeta}^{\infty}([0,\infty), R^n)$, and for each $x \geq 0$ let

$$\mathbf{u}(x) = \limsup_{k\to\infty} \mathbf{u}_k(x),$$

where the lim sup is componentwise. Choose N_1 large enough so that for $k, l \geq N_1$, $\|\mathbf{u}_k - \mathbf{u}_l\|_{\beta,\zeta}^{\infty} < 1$. Then for every $k \geq N_1$,

$$\|\mathbf{u}_k\|_{\beta,\zeta}^{\infty} \leq \|\mathbf{u}_k - \mathbf{u}_{N_1}\|_{\beta,\zeta}^{\infty} + \|\mathbf{u}_{N_1}\|_{\beta,\zeta}^{\infty} < 1 + \|\mathbf{u}_{N_1}\|_{\beta,\zeta}^{\infty} < \infty,$$

and from this it follows that $\mathbf{u} \in L_{\beta,\zeta}^{\infty}([0,\infty), R^n)$. To show that $\mathbf{u}_k$ converges towards $\mathbf{u}$, for any given $\varepsilon > 0$ choose N_ε so that for any $k, l \geq N_\varepsilon$, $\|\mathbf{u}_k - \mathbf{u}_l\|_{\beta,\zeta}^{\infty} < \frac{\varepsilon}{2}$. Also, for each $x \geq 0$ choose $m_j(x) \geq N_\varepsilon$ large enough so that $|u_{m_j(x),j}(x) - u_j(x)| < \frac{\varepsilon}{2}$. Then for the j'th component,

$$\frac{|u_{k,j}(x) - u_j(x)|}{\exp(\beta x + \zeta x^2)} \leq \frac{|u_{k,j}(x) - u_{m_j(x),j}(x)|}{\exp(\beta x + \zeta x^2)} + \frac{|u_{m_j(x),j}(x) - u_j(x)|}{\exp(\beta x + \zeta x^2)} < \varepsilon.$$

Taking supremum over x and then maximum over j gives that $\|\mathbf{u}_k - \mathbf{u}\|_{\beta,\zeta}^{\infty} < \varepsilon$, and completeness follows.

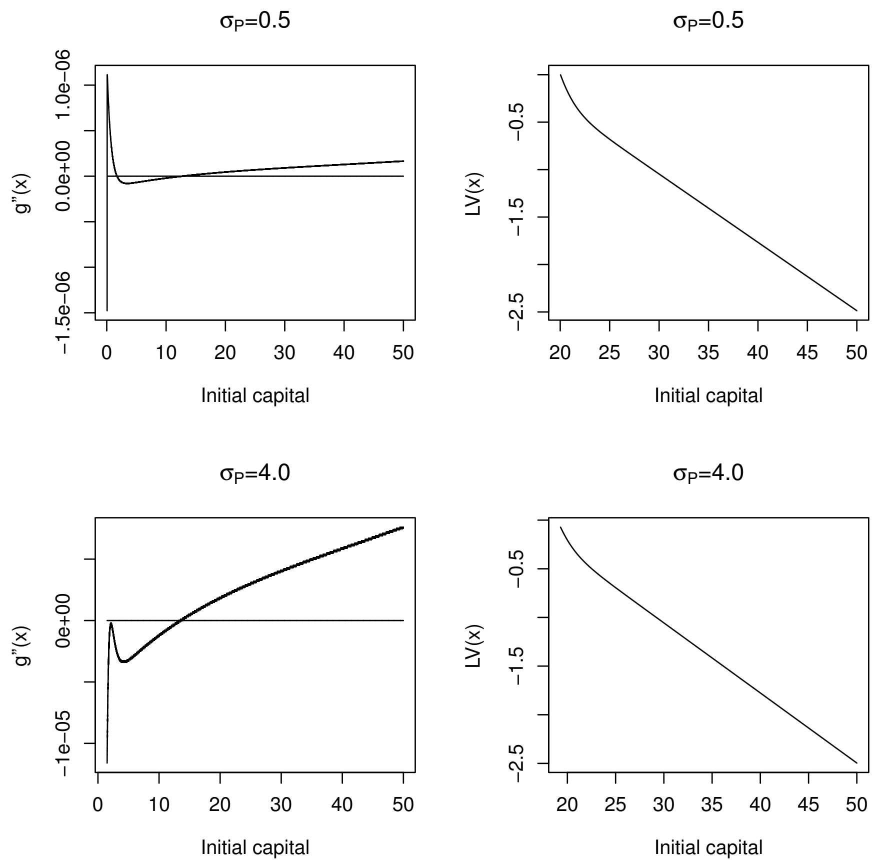

Fig. 5.10 Value of $g''(x)$ (left panels) and $LV(x)$ (right panels). The upper panels are for $\sigma_P = 0.5$, while the lower are $\sigma_P = 4$. The density (5.5) was used for the distribution of S. The other parameters are $p = 21.4$, $\lambda = 10$, $r = 0.1$, $\sigma_R = 0.2$, $\alpha = 0.08$, $k = 1$ and $K = 0$.

It remains to prove that $C_{\beta,\zeta}\left([0,\infty), R^n\right)$ is closed in $L^\infty_{\beta,\zeta}\left([0,\infty), R^n\right)$. Assume that the $\mathbf{u}_k \in C_{\beta,\zeta}\left([0,\infty), R^n\right)$ converge towards $\mathbf{u}$ in the $\|\cdot\|^\infty_{\beta,\zeta}$ norm, but that $\mathbf{u}$ is not continuous at a point x_0. With $b > x_0$ we get,

$$\begin{aligned}\sup_{0\le x\le b} |\mathbf{u}_k(x) - \mathbf{u}(x)| &\le \exp(\beta b + \zeta b^2) \sup_{0\le x\le b} \frac{|\mathbf{u}_k(x) - \mathbf{u}(x)|}{\exp(\beta x + \zeta x^2)} \\ &\le \exp(\beta b + \zeta b^2)\, \|\mathbf{u}_k - \mathbf{u}\|^\infty_{\beta,\zeta}\,.\end{aligned}$$

Hence convergence in the $\|\cdot\|^\infty_{\beta,\zeta}$ norm implies convergence in the standard sup norm on $[0, b]$. But it is well known that $C\left([0, b], R^n\right)$ is complete, hence $\mathbf{u}$ must be continuous on $[0, b]$, a contradiction. This proves the lemma.

Let

$$C_{\beta,\zeta,0}\left([0,\infty),R^n\right)=\{\mathbf{u}\in C_{\beta,\zeta}\left([0,\infty),R^n\right):u_1(0)=0\},$$

and similarly

$$C^k_{\beta,\zeta,0}\left([0,\infty),R^n\right)=\{\mathbf{u}\in C^k_{\beta,\zeta}\left([0,\infty),R^n\right):u_1(0)=0\}.$$

Then it follows trivially from Lemma 2.2 that $C_{\beta,\zeta,0}\left([0,\infty),R^n\right)$ is a Banach space with the $\|\cdot\|^\infty_{\beta,\zeta}$ norm.

Let the operator A be as in (2.5) and define

$$G_1u(x)=\int_0^x u(z)dz.$$

Assume A5 and set

$$G_2\mathbf{u}(x)=\int_0^x\frac{2}{\sigma^2(z)}\left((r+\lambda)u_1(z)-\mu(z)u_2(z)-\lambda Au_1(z)\right)dz,\quad x\geq 0.$$

Finally, let $\mathbf{Gu}(x)=(G_1u_2(x),G_2\mathbf{u}(x))$.

Lemma 6.1 *Let $u\in C_{\beta,\zeta,0}\left([0,\infty),R\right)$. Then $Au\in C_{\beta,\zeta,0}\left([0,\infty),R\right)$ and $G_1u\in C_{\beta,\zeta,0}\left([0,\infty),R\right)\cap C^1_{\beta,\zeta,0}\left([0,\infty),R\right)$. Furthermore,*

$$\|Au\|^\infty_{\beta,\zeta}\leq\|u\|^\infty_{\beta,\zeta}\tag{6.1}$$

and

$$\|G_1u\|^\infty_{\beta,\zeta}\leq\frac{1}{\beta}\|u\|^\infty_{\beta,\zeta}.\tag{6.2}$$

Proof That G_1u is continuously differentiable is obvious. Furthermore,

$$Au(x+h)-Au(x)=\int_0^x(u(x+h-z)-u(x-z))dF(z)+\int_x^{x+h}u(x+h-z)dF(z).$$

The first term goes to zero as h goes to zero because of continuity of u, and the second term goes to zero since $u(0)=0$. Also by monotonicity of F,

$$\begin{aligned}|Au(x)|&\leq\int_0^x|u(z)||dF(x-z)|\\&\leq\int_0^x e^{\beta(x-z)+\zeta(x^2-z^2)}|u(z)||dF(x-z)|\\&\leq e^{\beta x+\zeta x^2}\|u\|^\infty_{\beta,\zeta}.\end{aligned}$$

Therefore,

$$\frac{|Au(x)|}{\exp(\beta x+\zeta x^2)}\leq\|u\|^\infty_{\beta,\zeta}.$$

Taking supremum over x gives (6.1). Next

$$
\begin{aligned}
|G_1 u(x)| &\le \int_0^x e^{\beta z+\zeta z^2} \frac{|u(z)|}{\exp(\beta z+\zeta z^2)} dz \\
&\le e^{\beta x+\zeta x^2} \|u\|_{\beta,\zeta}^{\infty} \int_0^x e^{-\beta(x-z)} dz \\
&\le \frac{1}{\beta} e^{\beta x+\zeta x^2} \|u\|_{\beta,\zeta}^{\infty} .
\end{aligned}
$$

The rest of the proof of (6.2) is now the same as above.

Lemma 6.2 *Given the assumptions of Theorem 2.3, let* $\mathbf{u} \in C_{\beta,\zeta,0}\left([0,\infty), R^2\right)$ *with* $\zeta > 0$ *if* $M_2 > 0$ *in A7. Then* $G_2\mathbf{u} \in C_{\beta,\zeta,0}\left([0,\infty), R\right)$ *and*

$$
\|G_2\mathbf{u}\|_{\beta,\zeta}^{\infty} < 4 \max\left\{\frac{M_1}{\beta}, \frac{M_2}{\zeta}\right\} \|\mathbf{u}\|_{\beta,\zeta}^{\infty} , \tag{6.3}
$$

with $M_2/\zeta = 0$ *if* $M_2 = \zeta = 0$*. Moreover, for any* $\tilde{\beta} > \beta$, $G_2\mathbf{u} \in C^1_{\tilde{\beta},\zeta,0}\left([0,\infty), R\right)$.

Proof Clearly $G_2\mathbf{u}$ is continuously differentiable with $G_2\mathbf{u}(0) = 0$. Also by assumptions and (6.1),

$$
\begin{aligned}
|(G_2\mathbf{u})'(x)| &\le \frac{2}{\sigma^2(x)} \left(|\mu(x)||u_2(x)| + (r+\lambda)|u_1(x)| + \lambda |Au_1(x)|\right) \\
&\le 4(M_1 + M_2 x) e^{\beta x+\zeta x^2} \|\mathbf{u}\|_{\beta,\zeta}^{\infty} . \qquad (6.4)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|G_2\mathbf{u}(x)| &\le 4 \|\mathbf{u}\|_{\beta,\zeta}^{\infty} \int_0^x e^{\beta z+\zeta z^2} (M_1 + M_2 z) dz \\
&\le 4 e^{\beta x+\zeta x^2} \|\mathbf{u}\|_{\beta,\zeta}^{\infty} \int_0^x e^{-(x-z)(\beta+\zeta x)} (M_1 + M_2 z) dz \\
&\le 4 e^{\beta x+\zeta x^2} \|\mathbf{u}\|_{\beta,\zeta}^{\infty} \frac{M_1 + M_2 x}{\beta + \zeta x} \\
&\le 4 e^{\beta x+\zeta x^2} \|\mathbf{u}\|_{\beta,\zeta}^{\infty} \max\left\{\frac{M_1}{\beta}, \frac{M_2}{\zeta}\right\} .
\end{aligned}
$$

The result (6.3) now follows as before. From (6.4) we get

$$
\frac{|(G_2\mathbf{u})'(x)|}{\exp(\tilde{\beta} x + \zeta x^2)} \le 4(M_1 + M_2 x) e^{-(\tilde{\beta}-\beta)x} \|\mathbf{u}\|_{\beta,\zeta}^{\infty} ,
$$

which shows that $G_2\mathbf{u} \in C^1_{\tilde{\beta},\zeta}\left([0,\infty), R\right)$.

Lemmas 6.1 and 6.2 now give:

Lemma 6.3 *Under the assumptions of Theorem 2.3, for* $\mathbf{u} \in C_{\beta,\zeta,0}\left([0,\infty), R^2\right)$ *and* $\tilde{\beta} > \beta$,

$$\mathbf{Gu} \in C_{\beta,\zeta,0}\left([0,\infty), R^2\right) \cap C^1_{\tilde{\beta},\zeta}\left([0,\infty), R^2\right).$$

Furthermore,

$$\|\mathbf{Gu}\|^{\infty}_{\beta,\zeta} \leq c_G \|\mathbf{u}\|^{\infty}_{\beta,\zeta},$$

where

$$c_G = \max\left\{\frac{1}{\beta}, \frac{4M_1}{\beta}, \frac{4M_2}{\zeta}\right\}.$$

Proof of Theorem 2.3 For $\mathbf{u} \in C_{\beta_0,\zeta,0}\left([0,\infty), R^2\right)$ define

$$\mathbf{Hu}(x) = (0,1) + \mathbf{Gu}(x).$$

Choose β_0 and ζ in Lemma 6.3 (with β_0 for β) large enough so that $c_G < 1$. Then $\mathbf{H}$ is a contraction operator on $C_{\beta_0,\zeta,0}\left([0,\infty), R^2\right)$, and since $C_{\beta_0,\zeta,0}\left([0,\infty), R^2\right)$ is complete, there is a $\mathbf{v} \in C_{\beta_0,\zeta,0}\left([0,\infty), R^2\right)$ so that $\mathbf{Hv} = \mathbf{v}$. Furthermore, by Lemmas 6.1 and 6.2, for $\beta > \beta_0$,

$$\mathbf{v} = \mathbf{Hv} \in C^1_{\beta,\zeta,0}\left([0,\infty), R^2\right).$$

Let $g = v_1$. Then $g' = v_1' = (G_1 v_2)' = v_2$ and so since $g' = v_2$ is continuously differentiable,

$$\begin{aligned} g''(x) &= v_2'(x) \\ &= (G_2\mathbf{v})'(x) \\ &= \frac{2}{\sigma^2(x)}(-\mu(x)v_2(x) + (r+\lambda)v_1(x) - \lambda A v_1(x)) \\ &= \frac{2}{\sigma^2(x)}(-\mu(x)g'(x) + (r+\lambda)g(x) - \lambda A g(x)). \end{aligned}$$

Rearranging this last equation yields $Lg(x) = 0$. Also $g'(0) = v_2(0) = 1 + (G\mathbf{v})_2(0) = 1$, hence g solves (2.1).

Conversely, it can be shown that if $h \in C^2_{\beta,\zeta,0}([0,\infty), R)$ for some $\beta > 0$ and $\zeta \geq 0$ and h solves (2.1), then $\mathbf{w} = (h, h') \in C^1_{\beta,\zeta,0}([0,\infty), R^2)$ and satisfies $\mathbf{Hw} = \mathbf{w}$. Since $\mathbf{H}$ has a unique fixed point it follows that $h = w_1 = v_1 = g$.

In the remaining proofs we shall use the more convenient notation $A_g(x) = Ag(x)$, and similarly $A'_g(x) = (Ag)'(x)$.

Proof of Theorem 2.4 We start by proving that g is strongly increasing. The equation $Lg = 0$ gives

$$g''(x) = \frac{2}{\sigma^2(x)}(-\mu(x)g'(x) + (r+\lambda)g(x) - \lambda A_g(x)). \tag{6.5}$$

Let $x_0 = \inf\{x > 0 : g'(x) = 0\}$. Then g is strongly increasing on $(0, x_0)$, and so

$$\begin{aligned} g(x_0) - A_g(x_0) &= g(x_0) - \int_0^{x_0} g(x_0 - z)dF(z) \\ &\geq g(x_0)(1 - F(x_0)) \geq 0. \end{aligned} \tag{6.6}$$

Therefore, if $x_0 < \infty$,

$$g''(x_0) \geq \frac{2}{\sigma^2(x_0)} r g(x_0) > 0. \tag{6.7}$$

But since $g'(0) = 1 > 0$, it follows from the definition of x_0 that $g''(x_0) \leq 0$, a contradiction. Hence $x_0 = \infty$, and g' is strongly increasing.

Assume that (2.3) holds and let

$$H(x) = g'(x) - \frac{c_1}{c_2 + c_3 x} g(x),$$

so that in particular $H(0) = 1$. Let $x_1 = \inf\{x : H(x) = 0\}$. If $x_1 < \infty$ then $H'(x_1) \leq 0$, and we will show that this leads to a contradiction. So assume $x_1 < \infty$. Then

$$g'(x_1) = \frac{c_1}{c_2 + c_3 x_1} g(x_1)$$

and by (6.5) and (6.6),

$$\begin{aligned} g''(x_1) &\geq \frac{2}{\sigma^2(x_1)} (r g(x_1) - \mu(x_1) g'(x_1)) \\ &= \frac{2}{\sigma^2(x_1)} \left(r - \frac{c_1}{c_2 + c_3 x_1} \mu(x_1) \right) g(x_1). \end{aligned} \tag{6.8}$$

Therefore,

$$\begin{aligned} H'(x_1) &= g''(x_1) - \frac{c_1}{c_2 + c_3 x_1} g'(x_1) + \frac{c_1 c_3}{(c_2 + c_3 x_1)^2} g(x_1) \\ &\geq \left(\frac{2}{\sigma^2(x_1)} \left(r - \frac{c_1}{c_2 + c_3 x_1} \mu(x_1) \right) - \frac{c_1(c_1 - c_3)}{(c_2 + c_3 x_1)^2} \right) g(x_1) > 0, \end{aligned}$$

where the last inequality follows from (2.3). Hence $x_1 = \infty$ and so H is positive. It is easy to verify that the equation

$$g'(x) - \frac{c_1}{c_2 + c_3 x} g(x) = H(x)$$

has the solution

$$g(x) = g(1) \left(1 + \frac{c_3}{c_2} x \right)^{\frac{c_1}{c_3}} + \left(1 + \frac{c_3}{c_2} x \right)^{\frac{c_1}{c_3}} \int_1^x \left(1 + \frac{c_3}{c_2} y \right)^{-\frac{c_1}{c_3}} H(y) dy.$$

Taking the derivative yields that $g'(x) \to \infty$ as $x \to \infty$.

Now assume that A3c and A6 also hold. We will show that (2.3) can be satisfied. Let $\varepsilon < \alpha$ be positive. We will show that for $0 < \varepsilon < \alpha$ we can choose positive c_1, c_2 and c_3 with $c_3 < c_1$ so that

$$\mu(x) < \frac{r-\varepsilon}{c_1}(c_2 + c_3 x), \tag{6.9}$$

and

$$(c_1 - c_3)\sigma^2(x) < \frac{2\varepsilon}{c_1}(c_2 + c_3 x)^2. \tag{6.10}$$

Together, (6.9) and (6.10) prove the claim. To prove (6.9), by A3c there is a constant a so that $\mu(x) < a + (r-\alpha)x$ for all $x \geq 0$. Therefore,

$$\begin{aligned}\frac{r-\varepsilon}{c_1}(c_2 + c_3 x) - \mu(x) &> \frac{r-\varepsilon}{c_1}(c_2 + c_3 x) - (a + (r-\alpha)x) \\ &= \left((r-\varepsilon)\frac{c_2}{c_1} - a\right) + \left((r-\varepsilon)\frac{c_3}{c_1} - (r-\alpha)\right)x.\end{aligned}$$

This is positive for c_2 sufficiently large and c_3 so close to c_1 that $\frac{c_3}{c_1} \geq \frac{r-\alpha}{r-\varepsilon}$. The condition (6.10) is equivalent to

$$\frac{\sigma^2(x)}{2(c_2 + c_3 x)^2} < \frac{\varepsilon}{c_1(c_1 - c_3)}.$$

Using the growth constriction A6 and choosing c_3 sufficiently close to c_1, this can be satisfied and so the theorem is proved.

Proof of Theorem 2.5 By (6.5),

$$g''(0) = -\frac{2}{\sigma^2(0)}\mu(0),$$

hence $\mu(0) \leq 0$ is equivalent to $x^* = 0$. Assume that $\mu(0) = 0$ so that $g''(0) = 0$ as well. Taking the derivative in $Lg(x) = 0$ gives with $\tau(x) = \sigma^2(x)$,

$$\begin{aligned}g'''(x) = \frac{2}{\tau(x)}\Bigg(&-(\mu(x) + \frac{1}{2}\tau'(x))g''(x) \\ &+(r + \lambda - \mu'(x))g'(x) - \lambda A'_g(x)\Bigg),\end{aligned} \tag{6.11}$$

since by (2.6), A_g is continuously differentiable. By (2.6), $A'_g(0) = 0$, so therefore

$$g'''(0) = \frac{2}{\sigma^2(0)}(r + \lambda - \mu'(0)) > 0.$$

To prove part b, assume that $x^* = \infty$, meaning that g is strictly concave. Therefore we must have that for $x > 0$, $g(x) > xg'(x)$. This gives

$$rg(x_0) - \mu(x_0)g'(x_0) > \left(r - \frac{\mu(x_0)}{x_0}\right)x_0 g'(x_0) = 0,$$

so by (6.8), $g''(x_0) > 0$, a contradiction. Hence $x^* < \infty$.

Proof of Lemma 2.7 From (6.11) we have

$$g'''(x) = \frac{2}{\tau(x)} \left(-(\mu(x) + \frac{1}{2}\tau'(x))g''(x) + H(x) \right), \tag{6.12}$$

where

$$H(x) = (r + \lambda - \mu'(x))g'(x) - \lambda A'_g(x).$$

Note that (2.9) just says that $H(x) = 0$.

We start by proving that $g''(x) > 0$ for $x \in (x^*, x^* + \delta)$ for some positive δ. Assume the contrary. Then by definition of x^*, $g'''(x^*) = 0$ as well, and so by (6.12), $H(x^*) = 0$. Straightforward differentiation gives

$$g^{(4)}(x) = \frac{2}{\tau(x)} \Big(-(\mu(x) + \tau'(x))g'''(x) + \big(r + \lambda - 2\mu'(x) - \frac{1}{2}\tau''(x)\big)g''(x) \\ -\mu''(x)g'(x) - \lambda A''_g(x) \Big).$$

Therefore,

$$g^{(4)}(x^*) = -\frac{2}{\tau(x^*)}(\mu''(x^*)g'(x^*) + \lambda A''_g(x^*)) > 0$$

by assumption. But since $g''(x^*) = g'''(x^*) = 0$, we get

$$g''(x^* + u) = \int_{x^*}^{x^*+u} \int_{x^*}^{y} g^{(4)}(x)dxdy,$$

and the result follows.

We will now show that either $H(x^*) > 0$ or $H(x^*) = 0$ and $H'(x^*) > 0$. If $x^* = 0$ then $H(x^*) = r + \lambda - \mu'(0) > 0$ by assumption. Assume $x^* > 0$. Again by definition of x^*, $g'''(x^*) \geq 0$, and so by (6.12) $H(x^*) \geq 0$. Assume $H(x^*) = 0$. Since

$$H'(x) = (r + \lambda - \mu'(x))g''(x) - (\mu''(x)g'(x) + \lambda A''_g(x)) \tag{6.13}$$

and $g''(x^*) = 0$, it follows from the assumption that $H'(x^*) > 0$.

From the above results we can define

$$x_1 = \min\{x > x^* : g''(x) = 0\},$$
$$x_H = \min\{x > x^* : H(x) = 0\}.$$

Then it follows from the above:

1. $g''(x) > 0$ on (x^*, x_1).
2. $g'''(x_1) \leq 0$.
3. $H(x) > 0$ on (x^*, x_H).
4. $H'(x_H) \leq 0$.

We will prove that $x_1 = \infty$. Consider the three possibilities:

(i) $x_H < x_1$. Then $g''(x_H) \geq 0$ by item 1 and since $H(x_H) = 0$, it follows by assumption that $\mu''(x_H)g'(x_H) + \lambda A''_g(x_H) < 0$. Therefore by (6.13), $H'(x_H) > 0$, which contradicts item 4 above.
(ii) $x_H = x_1 < \infty$. Here we can use the same arguments to arrive at a contradiction.
(iii) $x_H > x_1$. But then by item 3, $H(x_1) > 0$ and so by (6.12), $g'''(x_1) > 0$ as well, which contradicts item 2.

From this it follows that $x_H = x_1 = \infty$ is the only possibility. But the inequality (2.10) just says that $x_H = \infty$, and so that this inequality is proved as well.

Proof of Theorem 2.8 We use the same notation as in the proof of Lemma 2.7. It follows easily from the assumptions that

$$\begin{aligned} A'_g(x) &= f(0)g(x) + \int_0^x g(x-z)f'(z)dz, \\ A''_g(x) &= f(0)g'(x) + \int_0^x g'(x-z)f'(z)dz. \end{aligned}$$

Assume that for some $x_0 > 0$, $\lambda A'_g(x_0) = (\lambda + r - \mu'(x_0))g'(x_0)$. Then

$$\begin{aligned} \lambda A''_g(x_0) + \mu''(x_0)g'(x_0) &\leq \lambda A''_g(x_0) \\ &= \lambda \left(f(0)g'(x_0) + \int_0^{x_0} g'(x_0 - z)f'(z)dz \right) \\ &< \lambda f(0) \left(g'(x_0) - \int_0^{x_0} g'(x_0 - z)c(z)f(z)dz \right) \\ &\leq \lambda f(0) \left(g'(x_0) - c(x_0)A'_g(x_0) \right) = 0. \end{aligned}$$

The result now follows from Lemma 2.7.

Proof of Theorem 3.1 For (a), assume that $(V_i, \bar{u}_i^*, \underline{u}_i^*),\ i = 1, 2$ are two solutions of B1+B2, and assume without loss of generality that $\bar{u}_1^* < \bar{u}_2^*$. Since $\underline{u}_i^* < x^*,\ i = 1, 2$, and $g'(\underline{u}_i^*) = g'(\bar{u}_i^*)$, it is necessary that $\underline{u}_1^* > \underline{u}_2^*$. But from

$$k(\bar{u}_1^* - \underline{u}_1^*) - (V_1(\bar{u}_1^*) - V_1(\underline{u}_1^*)) = k(\bar{u}_2^* - \underline{u}_2^*) - (V_2(\bar{u}_2^*) - V_2(\underline{u}_2^*))$$

and (3.2) and (3.3), we get

$$\int_{\underline{u}_1^*}^{\bar{u}_1^*} \left(1 - \frac{g'(x)}{g'(\bar{u}_1^*)}\right) dx = \int_{\underline{u}_2^*}^{\bar{u}_2^*} \left(1 - \frac{g'(x)}{g'(\bar{u}_2^*)}\right) dx.$$

However, $g'(\bar{u}_1^*) < g'(\bar{u}_2^*)$ and $g'(x) < g'(\bar{u}_i^*),\ \underline{u}_i^* < x < \bar{u}_i^*$ and so

$$\int_{\underline{u}_2^*}^{\bar{u}_2^*} \left(1 - \frac{g'(x)}{g'(\bar{u}_2^*)}\right) dx > \int_{\underline{u}_1^*}^{\bar{u}_1^*} \left(1 - \frac{g'(x)}{g'(\bar{u}_2^*)}\right) dx > \int_{\underline{u}_1^*}^{\bar{u}_1^*} \left(1 - \frac{g'(x)}{g'(\bar{u}_1^*)}\right) dx,$$

a contradiction. The proof that B1+B2 and B1+B3 cannot both have a solution is similar, as is the proof that B1+B3 cannot have two different solutions.

For (b), the assumption $\lim_{x\to\infty} g'(x) = \infty$ implies that for each $\underline{u} \in [0, x^*)$ there is a unique $\bar{u} = \bar{u}(\underline{u}) \in (x^*, \infty)$ so that $g'(\bar{u}) = g'(\underline{u})$. By smoothness of g' and strict concave-convexity, this $\bar{u}(\underline{u})$ is continuous in $\underline{u}$. Therefore, if

$$\int_0^{\bar{u}(0)} \left(1 - \frac{g'(x)}{g'(\bar{u}(0))}\right) dx \geq \frac{K}{k}, \tag{6.14}$$

there is a unique pair $(\underline{u}^*, \bar{u}^*)$ so that $g'(\underline{u}^*) = g'(\bar{u}^*)$ and

$$\int_{\underline{u}^*}^{\bar{u}^*} \left(1 - \frac{g'(x)}{g'(\bar{u}^*)}\right) dx = \frac{K}{k}.$$

Then,

$$V(x) = \begin{cases} \dfrac{k}{g'(\bar{u}^*)} g(x), & x \leq \bar{u}^*, \\ V(\bar{u}^*) + k(x - \bar{u}^*), & x > \bar{u}^*, \end{cases}$$

satisfies B1+B2.

If (6.14) does not hold we can find a unique $\bar{u}^*$ so that

$$\int_0^{\bar{u}^*} \left(1 - \frac{g'(x)}{g'(\bar{u}^*)}\right) dx = \frac{K}{k}.$$

Then $V(x)$ defined as above satisfies B1+B3.

Proof of Theorem 3.2 This is proved almost exactly as Theorem 2.1 in [21], and we drop the details.

Proof of Theorem 3.4 By Theorems 2.3, 2.4, 2.5, 2.8 and 3.2 it only remains to verify (3.5), and for this it is sufficient to prove (3.6). Let $h(x) = LV(x)$. Then by (3.6),

$$\begin{aligned} h'(\bar{u}^*) &= \lambda A_V'(\bar{u}^*) - k(r + \lambda - \mu'(\bar{u}^*)) \\ &= k \left(\frac{\lambda}{g'(\bar{u}^*)} A_g'(\bar{u}^*) - k(r + \lambda - \mu'(\bar{u}^*)) \right) < 0 \end{aligned}$$

by (2.10). Let $x_0 = \inf\{x > \bar{u}^* : h'(x) > 0\}$. If we can prove that $x_0 = \infty$ we are done. So assume that $x_0 < \infty$ which implies that $\lambda A_V'(x_0) = k(r+\lambda-\mu'(x_0))$. Also by definition of x_0, $h''(x_0) \geq 0$, but a direct calculation gives as in the proof of Theorem 2.8,

$$\begin{aligned} h''(x_0) &= \lambda f(0) V'(x_0) + \lambda \int_0^{x_0} V'(x_0 - z) f'(z) dz \\ &< \lambda k f(0) - \frac{\lambda^2}{r + \lambda - \mu'(x_0)} f(0) \int_0^{x_0} V'(x_0 - z) f(z) dz \\ &= \lambda f(0) \left(k - \frac{\lambda}{r + \lambda - \mu'(x_0)} A_V'(x_0) \right) = 0, \end{aligned}$$

a contradiction.

Proof of Theorem 5.1 By definition of x_M, $g''(x_M) \geq 0$. If $g''(x_M) > 0$ there is a $\delta > 0$ so that

$$g''(x) > 0 \quad \text{on} \quad (x_M, x_M + \delta). \tag{6.15}$$

Assume that $g''(x_M) = 0$. By definition of x_M, $g'(x) \leq g'(x_M)$ for $x \in (0, x_M)$, so that by (2.6), $g'(x_M) \geq A'_g(x_M)$. Therefore, by (6.11),

$$\begin{aligned} g'''(x_M) &= \frac{2}{\sigma^2(x_M)} \big(r + \lambda - \mu'(x_M))g'(x_M) - \lambda A'_g(x_M)\big) \\ &\geq \frac{2\alpha}{\sigma^2(x_M)} g'(x_M) > 0, \end{aligned}$$

and so (6.15) holds in this case as well. Let $x_0 = \inf\{x > x_M : g''(x) = 0\}$. Then if $x_0 < \infty$, $g'''(x_0) \leq 0$. But since $g'(x)$ is increasing on (x_M, x_0), the same calculations as above yield that $g'''(x_0) > 0$, a contradiction. This ends the proof of (a). To prove (b), let $x_0 = \inf\{x > x^* : g''(x) = 0\}$. By assumption, $x_0 > x_L$. Assume that $x_0 < \infty$. Then $g'''(x_0) \leq 0$. Also by assumption, $\max_{z \in [x^*, x_0]} g'(z) = g'(x_0)$, and hence a calculation using (6.11) yields

$$\begin{aligned} g'''(x_0) &= \frac{2}{\sigma^2(x_0)} \big((r + \lambda - \mu'(x_0))g'(x_0) - \lambda A'_g(x_0)\big) \\ &= \frac{2}{\sigma^2(x_0)} \Bigg((r - \mu'(x_0))g'(x_0) - \lambda \int_0^{x^*} g'(z) f(x_0 - z) dz \\ &\quad + \lambda \left(g'(x_0) - \int_{x^*}^{x_0} g'(z) f(x_0 - z) dz\right)\Bigg) \\ &> \frac{2}{\sigma^2(x_0)} \left((r - \mu'(x_0))g'(x_0) - \lambda g(x^*) \max_{z \geq x_0 - x^*} f(z)\right) \\ &\geq \frac{2}{\sigma^2(x_0)} \left((r - \mu'(x_L))g'(x_L) - \lambda g(x^*) \max_{z \geq x_L - x^*} f(z)\right) = 0, \end{aligned}$$

a contradiction. Hence $x_0 = \infty$. From this, using that $Lg(x) = 0$, we also get

$$(r + \lambda - \mu'(x))g'(x) - \lambda A'_g(x) > 0, \quad x \geq x_L. \tag{6.16}$$

Proof of Theorem 5.3 Assume that $x_K < \infty$ and let $h(x) = LV(x)$. Simple calculations using (3.6) and (2.6) yield for $x > \bar{u}^* + x_f$.

$$\begin{aligned} h'(x) &= \lambda \int_0^{\bar{u}^*} V'(z) f(x - z) dz - k\lambda \bar{F}(x - \bar{u}^*) - k(r - \mu'(x)) & (6.17) \\ &\leq \lambda \int_0^{\bar{u}^*} V'(z) f(x - z) dz - k(r - \mu'(x)). & (6.18) \end{aligned}$$

By definition of x_K, the right side is zero at x_K, and since $f(x-z)$ and $\mu'(x)$ are all decreasing in x when $x > \bar{u}^* + x_f$, the result follows. That $x_K < \infty$ if A3c holds is trivial. Part (b) is proved similarly, since the above gives for $x > \bar{u}^*$,

$$\begin{aligned} h'(x) &= \lambda \int_0^{\bar{u}^*} V'(z) f(x-z) dz - k\lambda \bar{F}(x - \bar{u}^*) - k(r - \mu'(x)) \\ &\leq \lambda \int_0^{\bar{u}^*} V'(z) f(x-z) dz - k(r - \mu'(x)) \\ &\leq \lambda V(\bar{u}^*) \max_{x-\bar{u}^* \leq z \leq x} f(z) - k(r - \mu'(x)) \\ &\leq \frac{k}{g'(\bar{u}^*)} \left(\lambda g(\bar{u}^*) \max_{z \geq x - \bar{u}^*} f(z) - k(r - \mu'(x)) g'(\bar{u}^*) \right). \end{aligned}$$

By definition, the right side is zero at x_J and is decreasing in x.

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