Square-free modules and ideals: Brill–Noether theory, polarizations, and deformations.

Dissertation for the degree of Philosophiae Doctor

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April, 2013

Preface

This thesis is organized as follows.

Part I. The first part consists of a brief introduction to combinatorial commutative algebra in general, and also a brief introduction to this thesis. In Chapter 1 we give the basic background for the theory of Stanley–Reisner rings, simplicial complexes and resolutions of such rings. We will also introduce the consept of depth of a graded S-module, and its relation to its minimal free resolution.

In Chapter 2, we introduce the notion of a polarization of a general monomial ideal. This technique reduces many questions about monomial ideals in general to Stanley–Reisner rings, where we can use combinatorial methods. Since there are many possible ways of polarizing a monomial ideal, interesting questions about polarizations itself occur. We therefore also include a brief introduction to deformation theory, which will be a tool used to study different polarizations.

Part II. The second part consists of three papers. Paper A is accepted for publication and will appear in Journal of pure and applied algebra, Volume 217, Issue 5, May 2013, Pages 803–818. Preprints of Paper B and Paper C are available on arXiv, and final versions will soon be submitted for publication.

Acknowlegdements. First of all, I want to thank my supervisor Professor Gunnar Fløystad for introducing me to this area of research. His deep insight and knowledge has produced many interesting and fruitful problems, and his guidance and advices has been of great help when writing this thesis.

I also want to thank Professor Frank O. Schreyer and the University of Saarland and Professor Tim Römer and the University of Osnabrück for their hospitality during my visits there during the fall of 2009 and the spring of 2010. I had a very good time during my visits, and discussions with the professors and PhD students there were of great value to me.

Finally, I would like to thank everybody at the department for making such a good place to work. A special thanks goes to Nils Henry, Alexander, Trine and many others for their friendship and support during my PhD period. I really enjoyed working with you all, and I have nothing but good memories from these years.

Contents

Part I

Introduction and background

Chapter 1

Introduction

In the field of commutative algebra and algebraic geometry, one of the main goals is to understand the connection between algebra and geometry. One such approach would be to describe all projective varieties up to isomorphism classes, and similarly, to describe all graded rings S/I up to isomorphism classes. This is of course too ambitious, and most studies are restricted to some smaller classes of varieties or rings. One strategy for such a classification is to set an invariant of varieties, or ideals, and thereafter study all varieties or ideals with this invariant. Usually, this invariant is the Hilbert polynomial and the Hilbert scheme is the family of varieties with a given Hilbert polynomial. Other important invariants, which are refinements of the Hilbert polynomial are the Hilbert function, the Betti numbers and the graded minimal free resolution. A classic result by Macaulay [11] says that the Hilbert function of S/I is the same as the Hilbert function of the quotient $S/\mathrm{in}_<(I)$, where $\mathrm{in}_<(I)$ is the initial ideal of I. So the problem of calculating the Hilbert polynomial of a quotient ring S/I is reduced to the case of finding the Hilbert polynomial of quotients S/J of S by a monomial ideal J.

Combinatorial commutative algebra is the research area where one uses combinatorial methods or structures to describe the algebra or algebraic properties of commutative rings or modules over a commutative ring. One of the most successful topics is the theory of Stanley–Reisner rings, where there is a one-to-one correspondence between simplicial complexes and square-free ideals. In this case there is a formula, called Hochster's formula, which is a very nice example of how the combinatorial structure can be used to compute the Betti numbers of a Stanley–Reisner ring.

In this thesis, we will study two different areas of combinatorial commutative algebra. In paper A, we study square-free modules. Square-free modules is a generalization of square-free ideals, and was introduced by Kohij Yanagawa in the article [18]. Square-free modules are always supported on a simplicial complex. We study square-free Cohen–Macaulay modules supported on a connected simplicial graph. Such modules, which are what we will call locally of rank 1, behave similarly as line bundles on curves. We investigate this relationship and prove that many results for line bundles on curves also hold for these modules on a graph.

In paper B and C, we study another topic of combinatorial commutative algebra. Namely, polarizations. In general, if I is a monomial ideal (for instance the initial ideal of a graded ideal), then we want to produce a square-free monomial ideal $\widetilde{I}$ such that I and $\widetilde{I}$ have the same numerical invariants (i.e. Betti numbers, Hilbert functions, etc.). Now $\widetilde{I}$ corresponds to a simplicial complex, and it is possible to use the techniques from the theory of Stanley–Reisner rings to produce these invariants. There is a standard way of making such an ideal, but in general this is an ideal of a very big polynomial ring, and for practical purposes, this may not be the best way of calculating the Betti numbers of I. Another approach is to find a special kind of monomial ideal with the same invariants. One such ideal is the generic initial ideal $\mathrm{gin}_<(I)$. This ideal is a Borel-fixed ideal, and there are methods for producing a minimal free resolution for such ideals, for instance by the Eliahou–Kervaire resolution (see [14]). There are also other cellular resolution of Borel-fixed ideal that are generated in one degree. This is done by Sinefakopoulos in [16] and by Nagel and Reiner in [13]. The construction of Nagel and Reiner is quite interesting, since they use a new polarization of the ideal for making the cellular resolution. In paper B, we study the different polarizations of powers of the maximal ideal. When the polynomial ring has only three variables, we show that every cellular minimal free resolution of the ideal m^d comes from such a polarization. Recently, there has been some other work on cellular resolutions of this and other monomial ideals. In the paper [7], of Dochtermann, Joswig and Sanyal, the authors uses different arrangement of tropical hyperplanes to construct minimal cellular resolutions of the ideal m^d. This interesting technique extends the work of Sinefakopoulos, and its connection to our work on different polarizations might be interesting to explore. See also [9] and [6] for more related work on cellular resolutions. Furthermore, we also study polarizations of square-free ideals and give some new results. In paper C, we also study different polarizations of powers of the maximal ideal. This time, we study the different polarizations as different points in their Hilbert scheme. We show that some polarizations are smooth points in the Hilbert scheme, and we calculate the dimension of their components.

Chapter 2

Stanley–Reisner rings and Cohen–Macaulay modules

Here we will give a brief introduction to this field of combinatorial commutative algebra and the theory of Stanley–Reisner rings. For more details, we suggest the book of Miller and Sturmfels [12], the book of Bruns and Herzog [4], the book of Stanley [17], and the book of Eisenbud [8].

2.1 Definitions and basic results

Let $[n] = \{1, 2, \ldots, n\}$. A subset $F \subseteq [n]$ is called a face.

Definition 2.1. A simplicial complex is a collection of faces Δ, such that if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

A maximal face is called a facet. A simplicial complex is completely specified by its facets.

Let $S := k[x_1, x_2, \ldots, x_n]$ be the polynomial ring over a field k.

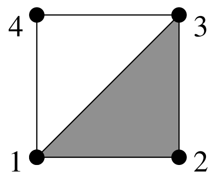
We often identify each face $F \in \Delta$ with its vector in $\{0,1\}^n$ which has entry 1 in the spots where $i \in F$ and 0 otherwise. This convention allows us to write $\mathbf{x}^F = \prod_{i \in F} x_i$.

Definition 2.2. The Stanley–Reisner ideal of the simplicial complex Δ is the square-free monomial ideal

$$I_\Delta = \left(x^F \mid F \notin \Delta\right)$$

generated by monomials corresponding to non-faces F of Δ. The Stanley–Reisner ring of Δ is the ring $k[\Delta] := S/I_\Delta$

Example 2.3. Let Δ be the simplicial complex on $[4]$ with facets $\{1,2,3\}, \{1,4\}$ and $\{3,4\}$. Then we can draw Δ as



We have that $\{1, 3, 4\}$ and $\{2, 4\}$ are the minimal non-faces, so that

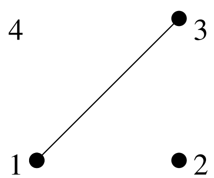
$$I_\Delta = (x_1x_3x_4, x_2x_4)\,.$$

Definition 2.4. Let Δ be a simplicial complex. Then we define the Alexander dual of Δ to be the simplicial complex

$$\Delta^* = \{F^c \mid F \notin \Delta\}.$$

If I_Δ is the Stanley–Reisner ideal of Δ, then I_{Δ^*} is called the Alexander dual of I_Δ.

Example 2.5. If Δ is as in Example 2.3 above, then Δ^* is the simplicial complex with facets given by the complements of the minimal non-faces of Δ. This is $\{2\}$ and $\{1, 3\}$, and we can draw Δ^* as



We have that $\{4\}$, $\{1, 2\}$ and $\{2, 3\}$ are the minimal non-faces of Δ^*. So we have that

$$I_{\Delta^*} = (x_4, x_1x_2, x_2x_3)\,.$$

We also observe that $I_{\Delta^*} = (x_1, x_3, x_4) \cap (x_2, x_4)$, and that $I_\Delta = (x_4) \cap (x_1, x_2) \cap (x_2, x_3)$. So the Alexander duality of square-free ideals interchanges the minimal generators and the irreducible components of the ideal.

If M a finitely generated graded S-module, then it is possible to find a graded free resolution of M:

$$0 \longleftarrow F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \longleftarrow \cdots \xleftarrow{\varphi_l} F_l \longleftarrow 0.$$

That is, a set of free modules $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$, and homogeneous maps φ such that $\operatorname{coker}(\varphi_1) \cong M$ and $\ker(\varphi_i) = \operatorname{im}(\varphi_{i+1})$. If the modules F_i are chosen to be of the smalles possible ranks, the resolution is called a minimal free resolution. In this case, the numbers β_{ij} are called the graded Betti numbers of M, and $l = \max\{i \mid F_i \neq 0\}$ is called the projective dimension of M. We write $\operatorname{pdim}(M) = l$.

Since the polynomial ring is $\mathbb{Z}^n$-graded, we may consider $\mathbb{Z}^n$-graded S-modules. If M is a $\mathbb{Z}^n$-graded S-module, then it is possible to find a $\mathbb{Z}^n$-graded minimal free resolution of M as above, with $F_i = \bigoplus_{\mathbf{a}_j} S(-\mathbf{a_j})^{\beta_{i,\mathbf{a}_j}}$, where $\mathbf{a}_j \in \mathbb{Z}^n$. The numbers $\beta_{i,\mathbf{a}_j}$ are called the $\mathbb{Z}^n$-graded (or multigraded) Betti numbers of M.

Example 2.6. If Δ is the simplicial complex from Example 2.3 above, then the Stanley–Reisner ring S/I_Δ is a $\mathbb{Z}^4$-graded S-module, and it has a minimal free resolution gives as

$$S \xleftarrow{[x_1x_3x_4\; x_2x_4]} S(-(1,0,1,1)) \oplus S(-(0,1,0,1)) \xleftarrow{\begin{bmatrix} x_2 \\ -x_1x_3 \end{bmatrix}} S(-(1,1,1,1)).$$

Since the twists occuring in a minimal free resolution of a Stanley–Reisner ring are always an element of $\{0,1\}^n$, we can identify the twists by faces $\sigma \in [n]$. For instance, we have that $\{1,0,1,1\}$ can be identified with the face $\{1,3,4\} \in [4]$.

One can use reduced homology or cohomology of simplicial complexes to compute the multigraded Betti numbers of a Stanley–Reisner ring $k[\Delta] = S/I_\Delta$. This is called Hochster's formula and it comes in two versions. Before we state the theorem, we need one more definition

Definition 2.7. If Δ is a simplicial complex, and if F is a face of Δ, then we define the link of F to be

$$\mathrm{lk}_\Delta(F) = \{G \in \Delta \mid F \cup G \in \Delta \text{ and } F \cap G = \emptyset\}.$$

Theorem 2.8 (Hochster's formula)**.** *All non-zero Betti numbers of S/I_Δ lie in square-free degrees σ (i.e. is an element of $\{0,1\}^n$), and*

$$\beta_{i,\sigma}(S/I_\Delta) = \dim_k \widetilde{H}_{i-2}(\mathrm{lk}_{\Delta^*}(\sigma^c); k) = \dim_k \widetilde{H}^{|\sigma|-i-1}(\Delta|_\sigma; k).$$

2.2 Cohen–Macaulay modules

If $S = k[x_1, x_2, \ldots, x_n]$, and if M is a finitely generated graded (or multigraded) S-module. Then the projective dimension of M is $\leq n$ by Hilbert's syzygy theorem [8, Theorem 1.13]. The projective dimension of M can be calculated by the algebraic invariant called depth. Here we present some basic definitions and results from the book of Bruns and Herzog [4].

Definition 2.9. An element $y \in S$ is called an M-regular element if $yz = 0$ for $z \in M$ implies that $z = 0$. A sequence of elements $(s_1, s_2, \ldots, s_r)$ in S is called a regular M-sequence if the following are satisfied:

(i) s_i is an $M/(s_1, \ldots, s_{i-1})M$-regular element for $i = 1, \ldots, r$ and

(ii) $M/(s_1,\dots,s_r)M \neq 0$

We will only be interested in the case where the s_i's are homogeneous elements of positive degrees. So (ii) automatically holds because of Nakayama's lemma.

Definition 2.10. Let M be a graded finitely generated S-module. Then we define

$$\operatorname{depth}(M) = \max\{r \mid (s_1,\dots,s_r) \text{ is a regular } M\text{-sequence}\},$$

where all s_i are homogeneous of positive degree.

The inequality $\operatorname{depth} M \leq \dim M$ always holds.

Definition 2.11. Let M be a finitely generated S-module. If $\operatorname{depth} M = \dim M$ then M is said to be Cohen–Macaulay.

The depth of an S-module can be read of a minimal free resolution by its projective dimension. This is given by the following theorem.

Theorem 2.12 (Auslander–Buchsbaum)**.** *Let $S = k[x_1,\dots,x_n]$ be the polynomial ring in n variables, and let M be a finitely generated graded S-module. Then*

$$\operatorname{pdim}(M) = n - \operatorname{depth}(M).$$

If $M = S/I_\Delta$ is a Stanley–Reisner ring, then we can use Hochster's formula to calculate its projective dimension, and use the Auslander Buchsbaum theorem to calculate its depth. So it is clear that it is possible to deside if S/I_Δ is Cohen–Macaulay by analyzing the simplicial complex Δ. This is done by Reiner's Criterion [4, Corollary 5.3.9].

Theorem 2.13. *Let Δ be a simplicial complex. Then the following are equivalent:*

(a) $k[\Delta]$ *is Cohen–Macaulay*

(b) $\widetilde{H}_i(\operatorname{lk} F; k) = 0$ *for all* $F \in \Delta$ *and all* $i < \dim \operatorname{lk} F$.

The Cohen–Macaulay property for rings and modules has turned out to be quite important. From the Auslander–Buchsbaum theorem, we see that Cohen–Macaulay modules has the shortest possible length, given a fixed dimension. So in some sense, we can think of them as having the simplest possible algebraic structure. In paper A, we will study square-free Cohen–Macaulay modules supported on a graph. Here the Cohen–Macaulay property is important to make an analogy to the theory of line bundles on curves. We also give another equivalent description of Theorem 2.13.

Chapter 3

Polarizations and deformations

3.1 Polarizations

Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring on n variables over a field k. If I is a monomial ideal, then we are interested in finding a graded minimal free resolution of I (or S/I). As explained in Chapter 2 above, there are techniques for doing this in the case where I is the Stanley–Reisner ideal of a simplicial complex. That is, if I is a square-free monomial ideal. The idea of a polarization of I is to produce a square-free ideal $\widetilde{I}$ in a bigger polynomial ring $\widetilde{S}$ such that the $\widetilde{I}$ has the same numerical invariants as I. For instance, if $I = (x^2, y^3)$, then we can polarize the x-variable and the y-variable to produce an ideal $\widetilde{I} = (x_1x_2, y_1y_2y_3)$ in the polynomial ring $\widetilde{S} = k[x_1, x_2, y_1, y_2, y_3]$. We can go the other way, by depolarizing the ideal $\widetilde{I}$ as follows. Let $J = (x_1 - x_2, y_1 - y_2, y_1 - y_3)$ be an ideal of $\widetilde{S}$. Then $\widetilde{S}/J \cong S$ and $\widetilde{I} \otimes_{\widetilde{S}} \widetilde{S}/J \cong I$ as an S-module. We want to be able to use techniques for Stanley–Reisner rings to find a minimal free resolution of $\widetilde{I}$ as an $\widetilde{S}$-module, and tensor it by $\widetilde{S}/J$ to produce a minimal free resolution of I as an S-module. Since tensoring is not exact, this will in general not produce a minimal free resolution. However, if we require the sequence $(x_1 - x_2, y_1 - y_2, y_1 - y_3)$ to be a regular $\widetilde{S}/\widetilde{I}$-sequence, it will.

Definition 3.1. Let I be an ideal in $S = k[x_1, \ldots, x_n]$. A polarization $\widetilde{I}$ of I is an ideal in the polynomial ring

$$\widetilde{S} := k\left[x_{11}, \ldots, x_{1r_1}, x_{21}, \ldots, x_{2r_2}, \ldots, x_{nr_n}\right]$$

such that the sequence

$$\sigma = (x_{11} - x_{12}, x_{11} - x_{13}, \ldots, x_{11} - x_{1r_1}, x_{21} - x_{22}, \ldots, x_{n1} - x_{nr_n})$$

is a regular $\widetilde{S}/\widetilde{I}$-sequence, and that $\widetilde{I} \otimes \widetilde{S}/\langle\sigma\rangle \cong I$. The homomorphism $\varphi : \widetilde{I} \longrightarrow I$ is called the depolarization of $\widetilde{I}$.

Example 3.2. The standard way of polarizing an ideal is to replace every power of a variable $x_i^{d_i}$ in a monomial by the product $x_{i1}x_{i2}\cdots x_{id_i}$. We will call this polarization of an ideal I the standard polarization of I. For instance, if $I = (x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3^2, x_3^3)$, then the standard polarization of I is the ideal

$$\widetilde{I} = (x_{11}x_{12}, x_{11}x_{21}, x_{11}x_{31}, x_{21}x_{22}, x_{21}x_{31}x_{32}, x_{31}x_{32}x_{33})\,.$$

When I is a Borel-fixed ideal generated in one degree, then in both [16] and [13], a cellular minimal free resolution of I is produced. In the article [13], then the polyhedral cell complex giving this cellular resolution is called the complex of boxes. This complex is produced by introducing a new type of polarization. Inspired by this work, we call this polarization for the box polarization. In a paper by Yanagawa [18], it is shown that such a polarization exists for all Borel-fixed ideals.

Example 3.3. This polarization uses an ordering of the varibles $x_1 < x_2 < \cdots < x_n$. In general, a monomial $m = x_{i_1}^{d_1}x_{i_2}^{d_2}\cdots x_{i_r}^{d_r}$, for $i_1 < i_2 < \cdots i_r$, is polarized to the monomial

$$m = x_{i_1,1}x_{i_1,2}\cdots x_{i_1,d_1}x_{i_2,(d_1+1)}\cdots x_{i_2,(d_1+d_2)}\cdots x_{i_r,d},$$

where $d = d_1 + d_2 + \cdots d_r$. To make things clearer, we show this with an example. Let I be the ideal $I = (x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3^2, x_3^3)$ from Example 3.2 above. More precisely, we get the ideal

$$\widetilde{I} = (x_{11}x_{12}, x_{11}x_{22}, x_{11}x_{32}, x_{21}x_{22}, x_{21}x_{32}x_{33}, x_{31}x_{32}x_{33})\,.$$

This polarization is called the box polarization of I.

One of the differences of the two polarizations is explained in the following example:

Example 3.4. Let $I = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4)$. Then the standard polarization of I is the ideal

$$I_S = (x_{11}x_{21}, x_{11}x_{31}, x_{11}x_{41}, x_{21}x_{31}, x_{21}x_{41}, x_{31}x_{41})\,,$$

while the box polarization gives us the ideal

$$I_B = (x_{11}x_{22}, x_{11}x_{32}, x_{11}x_{42}, x_{21}x_{32}, x_{21}x_{42}, x_{31}x_{42})\,.$$

So the standard polarization does not change a square-free ideal, but the box polarization does.

In paper B we study different polarizations of powers of the maximal ideal, and also in particular polarizations of square-free versions of these.

3.2 Deformations

Another observation about the standard polarization and the box polarization of the ideal I in Example 3.2 and 3.3, is that they are both ideals of the same polynomial ring, with the same graded Betti numbers. This means that they also have the same Hilbert polynomial, and that they both corresponds to points of a common Hilbert scheme. Every other polarization of I in this polynomial ring will also correspond to a point of this Hilbert scheme. In paper C we calculate the tangent spaces of the standard polarization and the box polarization of powers of the maximal ideal. This is done by the language of deformation theory which we will recall here. For more details, see [10] and [15].

The first theorem/definition is [10, Theorem 1.1]

Theorem 3.5. *Let Y be a closed subscheme of the projective space $X = \mathbb{P}^n_k$ over a field k. Then*

(a) There exists a projective scheme H, called the Hilbert scheme, parametrizing closed subschemes of X with the same Hilbert polynomial P as Y, and there exists a universal subscheme $W \subseteq X \times H$, flat over H, such that the fibers of W over closed points $h \in H$ are all closed subschemes of X with the same Hilbert polynomial P.

(b) The Zariski tangent space of H at the point $h \in H$ corresponding to the subscheme Y is given by $H^0(Y, \mathcal{N}_{Y/X})$, where $\mathcal{N}_{Y/X}$ is the normal sheaf of Y in X

Example 3.6. In our case, we want to study different polarizations of the ideal $m^d = (x_1, x_2, \ldots, x_n)^d$. Since both the standard polarization and the box polarization of m^d lie in the polynomial ring $S = k[x_{11}, \ldots, x_{nd}]$, we take $X = \mathrm{Proj}(S)$, and we take Y to be the closed subscheme $\mathrm{Proj}(S/M_d)$, where $M_d = (x_{11}, x_{21}, \ldots, x_{n1})^d$ corresponds to the trivial polarization of m^d in S. Then the standard polarization P_{nd} corresponds to the closed subscheme $\mathrm{Proj}(S/P_{nd})$ and the box polarization B_{nd} corresponds to the closed subscheme $\mathrm{Proj}(S/B_{nd})$.

We want to calculate the tangent space of the points in the Hilbert scheme corresponding to different polarizations of the ideal m^d. This can be done by finding all local deformations of the ideals. Such deformations are called first order deformations. We present the basics from [10, Chapter 1.2].

Let $D = k[t]/t^2$ denote the dual numbers. Let X be a scheme over k, and let Y be a closed subscheme of X.

Definition 3.7. A deformation of Y over D in X is a closed subscheme $Y' \subseteq X' = X \times D$, flat over D, such that $Y' \times_D k = Y$.

In our case, when $X = \mathrm{Proj}(S)$ and $Y = \mathrm{Proj}(S/I)$ for a homogeneous ideal I, we can reformulate the definition to the following.

Definition 3.8. A deformation of I over D in S is an ideal $I' \subset S' = S[t]/t^2$, with S'/I' flat over D, such that $(S'/I') \otimes_D k \cong S/I$.

The set of deformations over D can be given a module structure by the following identification stated in [10, Proposition 2.3].

Proposition 3.9. To give a deformation $I' \subset S'$ of I over D as in Definition 3.8 is equivalent to give an element $\varphi \in \mathrm{Hom}_S(I, S/I)$.

Since $\mathcal{N}_{Y/X} \cong \mathcal{H}om_X(\mathcal{I}, \mathcal{O}_Y)$, we can use the graded S-module $\mathrm{Hom}_S(I, S/I)$ to calculate the tangent space of I in the Hilbert scheme H.

Proposition 3.10. Suppose that $\operatorname{depth} \mathrm{Hom}_S(I, S/I) \geq 2$. Then

$$H^0(\mathcal{N}_{Y/X}, Y) \cong (\mathrm{Hom}_S(I, S/I))_0 \,,$$

where $(\mathrm{Hom}_S(I, S/I))_0$ denotes the k-vector space consisting of elements of degree 0.

When I is a polarization of m^d in the polynomial ring $S = k[x_{11}, \ldots, x_{nd}]$, with $d \geq 2$ and $n \geq 2$, then the depth of $\mathrm{Hom}_S(I, S/I)$ is always greater than or equal to 2. We can therefore calculate the dimension of the tangent space of the subscheme corresponding to I by finding all deformations $I' \subset S'$ over D. The next thing we are interested in, is to show if the polarization corresponds to a smooth point in the Hilbert scheme H. There are two possible methods for doing this, and both are used in paper C. The first is to find the dimension of the component of the Hilbert scheme which contains $\mathrm{Proj}(S/I)$. If this is the same as the dimension of the tangent space, then it must correspond to a smooth point. This is possible for the box polarization, since in this case, B_{nd} is the initial ideal of a determinantal ideal. For more information of determinantal ideals and Gröbner basis of such see for instance [5] and [3]. The second method is to first find all first order deformations $I' \subset S'$ over D. Then show that all such deformations are also deformations over $k[t]$. That is, to show that all local deformations lift to global deformations. We can actually calculate this since the relations between the generators of a monomial ideal or in fact a Stanley–Reisner ideal are quite simple. We will use this method for the standard polarization. We have been partly motivated by some results on deformations of Stanley–Reiser rings by Altmann and Christophersen. See for instance [1] and [2].

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Part II

Included Papers

A

Paper A

Brill-Noether theory of squarefree modules supported on a graph *

* arXiv: http://arxiv.org/abs/1001.4375