Detecting Periodic Elements in Higher Topological Hochschild Homology

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Contents

Preface

Introduction

Algebraic K-theory of a ring captures several important properties of the ring. The zeroth group K_0 is concerned with the projective modules over the ring, while the first group K_1 is related to the general linear group over the ring. By methods of Quillen and Waldhausen, these groups can be extended to a family of groups K_i for each natural number i. Although the K-theory of a ring has a very natural definition, it's almost impossible to compute it directly, so people have sought approximations that are easier to compute.

One approximation is Hochschild homology, another is cyclic homology, and there exists a map from K-theory to Hochschild homology, called the Dennis trace map, that factors through negative cyclic homology. Hochschild homology and cyclic homology are possible to calculate due to their algebraic nature, and by results of Goodwillie in [Goo86], rational relative K-theory is isomorphic to rational relative cyclic homology.

It's possible to generalize the definition of K-theory to the category of ring spectra, and K-theory of rings then becomes a special case by associating to each ring R, the Eilenberg Mac Lane spectrum HR of the ring. One can then hope to mimic the construction of Hochschild homology and cyclic homology in the category of ring spectra, and in the unpublished article [Bök86a], Bökstedt was able to define THH, the topological Hochschild homology, of some special spectra. In a modern framework with highly structured ring spectra, topological Hochschild homology of a commutative ring spectrum R can be defined as the tensor $S^1 \otimes R$, see [MSV97]. For a space X we will write $\Lambda_X R$ for the spectrum defined in Section 4.6 in [BCD10], which is non-equivariantly equivalent to the tensor $X \otimes R$. Martin Stolz analyzed the categorical constructions of the functor $\Lambda_X R$ in his PhD thesis [Sto11].

The cyclic group C_n with n elements act on S^1 through multiplication with the n-th roots of unity, and this induces an action of C_n on THH. Topological cyclic homology TC of a spectrum, was invented by Bökstedt, Hsiang and Madsen in [BHM93], and is defined as a limit over certain maps between the C_n fixed points of THH, where n varies over the natural numbers. Similarly to the non-topological versions, there is a map from K-theory to THH, which factors through TC, and by a result in [DGM13], the map from K-theory to TC is an equivalence in the nilpotent relative case.

Bökstedt calculated THH of the Eilenberg Mac Lane spectra $H\mathbb{F}_p$ and $H\mathbb{Z}$, in

[Bök86b], and building on these calculations Bökstedt and Madsen in [BM94] with the help of Tsalidis [Tsa94], was able to calculate $TC(\mathbb{Z})^\wedge_p$, the topological cyclic homology of the integers completed at a prime p, for all odd primes p. Later Rognes did the case $p = 2$ in [Rog99]. Hesseholt in [Hes97] and Hesselholt and Madsen in [HM97a, HM03] have calculated TC completed at a prime p for free associative $\mathbb{F}$-algebras, perfect fields of characteristic $p > 0$, truncated polynomial rings of perfect fields of characteristic $p > 0$, and certain local fields, more specifically complete discrete valuation fields of characteristic zero with perfect residue field k of characteristic $p > 2$.

Several people have put a lot of effort into computing the homotopy groups of topological Hochschild homology of various ring spectra. Some examples are calculating the mod p homotopy groups of THH of the Adams summand ℓ in [MS93], the mod v_1 homotopy groups of THH of connective complex K-theory in [Aus05] and the integral homotopy groups of $THH(\ell)$ and the 2-local homotopy groups of $THH(ko)$ in [AHL10].

Related to the fixed points of THH is the now proven Segal conjecture. One version says that for a cyclic group C_p of prime order p, the canonical map $THH(S^0)^{C_p} \to THH(S^0)^{hC_p}$, from the fixed points to the homotopy fixed points, where S^0 is the equivariant sphere spectrum, is a p-adic equivalence. In [LNR11] the authors prove similarly that $THH(MU)^{C_p} \to THH(MU)^{hC_p}$ and $THH(BP)^{C_p} \to THH(BP)^{hC_p}$ are p-adic equivalences, where MU is the complex cobordism spectrum, and BP is the Brown-Peterson spectrum, at the prime p. Another calculation in [HM97b] of similar flavour, is that for a perfect field k of characteristic p, the map $THH(k)^{C_{p^n}} \to THH(k)^{hC_{p^n}}$ induces an equivalence of connective covers.

Let C_2 act on $\Lambda_{S^2}H\mathbb{F}_2$ via the free action on S^2 given by the antipodal map. In Chapter 2 we make the following calculation: There are ring isomorphisms

$$\begin{aligned}\pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{C_2}) &\cong P_{\mathbb{Z}/4}(\alpha) \otimes_{\mathbb{Z}/4} E_{\mathbb{Z}/4}(\beta)/(2\alpha, 2\beta, \alpha^2, \alpha\beta)\\ \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2}) &\cong P_{\mathbb{Z}/4}(t, \alpha) \otimes_{\mathbb{Z}/4} E_{\mathbb{Z}/4}(\beta)/(2t, 2\alpha, 2\beta, \alpha^2, \alpha\beta)\end{aligned}$$

where $|t| = -2$, $|\alpha| = 2$ and $|\beta| = 3$, and the homomorphism

$$\Gamma_*\colon \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{C_2}) \to \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2})$$

is given by mapping α to α and β to β. Since $t\beta$ is not in the image of Γ_*, it is not an isomorphism in non-negative degrees.

In Chapter 3 we calculate the homotopy groups of iterated topological Hochschild homology of $H\mathbb{F}_p$, which is isomorphic to $\pi_*(\Lambda_{T^n}H\mathbb{F}_p)$, where T^n is the n-torus. We do these calculations for $n \leq p$ when $p \geq 5$ and $n \leq 2$ when $p = 3$. These groups are as expected, in the sense that the spectral sequence calculating them collapses at the E^2-term and $\pi_*(\Lambda_{T^n}H\mathbb{F}_p)$, is abstractly isomorphic as an $\mathbb{F}_p$-algebra to the E^∞-page as an algebra. Here abstractly isomorphic means that the $\mathbb{F}_p$-algebra isomorphism between E^∞ and $\pi_*(\Lambda_{T^n}H\mathbb{F}_p)$ is not necessarily given by the canonical isomorphism between E^∞ and the associated graded complex of $\pi_*(\Lambda_{T^n}H\mathbb{F}_p)$, coming from the filtration giving rise to the spectral sequence. There is a natural map of spectra $\omega : S^1_+ \wedge \Lambda_{T^{n-1}} H\mathbb{F}_p \to \Lambda_{T^n} H\mathbb{F}_p$, which is important when calculating the homotopy fixed points, and we attain explicit formulas for the induced map in homotopy.

After the proof of the periodicity theorem in [HS98], periodic phenomena play a prominent role in stable homotopy theory. The chromatic viewpoint on stable homotopy theory, is an organizing principle that let us see only information with particular periodicity properties. In [CDD11], the authors construct higher topological cyclic homology of a ring spectrum R, as a limit of fixed points of $\Lambda_{T^n} R$. It is hoped that higher topological cyclic homology increases the chromatic type of a spectrum.

Fix a prime p and let $k(n)$ be the n-th connective Morava K-theory. One version of periodicity as defined in Section 6 in [BDR04] is that of telescopic complexity of a spectrum X, and this is related to the chromatic type of a spectrum. If a spectrum X has telescopic complexity n, then the map $k(n)_*(\Sigma^{2p^n-2} X) \to k(n)_*(X)$ induced by multiplication of v_n is an isomorphism in high degrees.

There is an obvious action of T^{n+1} on $\Lambda_{T^{n+1}} H\mathbb{F}_p$, and it is expected that the homotopy fixed points $(\Lambda_{T^{n+1}} H\mathbb{F}_p)^{hT^{n+1}}$ has telescopic complexity n. In the last section of Chapter 3 we show that in the range were we have calculated $\pi_*((\Lambda_{T^{n+1}} H\mathbb{F}_p)^{hT^{n+1}})$ the self map

$$k(n)_*(\Sigma^{2p^n-2}(\Lambda_{T^{n+1}} H\mathbb{F}_p)^{hT^{n+1}}) \to k(n)_*((\Lambda_{T^{n+1}} H\mathbb{F}_p)^{hT^{n+1}})$$

induced by multiplication of v_n maps 1 to something non-zero, supporting the conjecture that $(\Lambda_{T^{n+1}} H\mathbb{F}_p)^{hT^{n+1}}$ has telescopic complexity n.

The calculation of $\pi_*(\Lambda_{T^n} H\mathbb{F}_p)$ should be possible to generalize to a calculation of the mod p homotopy groups $V(0)_*(\Lambda_{T^n} H\mathbb{Z})$ and the mod v_1 homotopy groups $V(1)_*(\Lambda_{T^n} \ell)$ in some range for n depending on p.

Organization

In Chapter 1 we give a short introduction to the Loday functor with some associated results. After that we introduce the bar spectral sequence, and prove some results about spectral sequences that we need later. In the last two sections we define the isotropy separation diagram of an equivariant spectrum, and some spectral sequences associated with it.

Chapter 2 begins in Section 2.1 by identifying the first possible non-zero differential in the Tate spectral sequence for an equivariant S^1 or S^3 spectrum. Continuing in Section 2.2 we find a family of non-zero differentials in $V(0)_*(\Lambda_{S^n} H\mathbb{F}_2)$ for all $n \geq 1$. We finish the chapter by calculating the homotopy groups of the Tate fixed points, homotopy fixed points, geometric fixed points, and actual fixed points of $\Lambda_{S^2} H\mathbb{F}_2$, and identify the homomorphism $\pi_*((\Lambda_{S^2} H\mathbb{F}_2)^{C_2}) \to \pi_*((\Lambda_{S^2} H\mathbb{F}_2)^{hC_2})$.

Chapter 3 is the main part of this thesis, both in length, difficulty and technicality. The first section introduces multifold Hopf algebras, which is a way to encode the connection between the Hopf algebra structures coming from the different circles in

$\Lambda_{T^n} H\mathbb{F}_p$. In Section 3.2 we prove that the structure of a multifold Hopf algebra puts restriction on the possible coalgebra structures that can appear in $\pi_*(\Lambda_{T^n} H\mathbb{F}_p)$. In Section 3.3 we explicitly calculate $\pi_*(\Lambda_{S^n} H\mathbb{F}_p)$ for $n \leq 2p$, and state several technical lemmas that are needed in Section 3.4, where we explicitly calculate $\pi_*(\Lambda_{T^n} H\mathbb{F}_p)$ for $n \leq p$ when $p \geq 5$ and $n \leq 2$ when $p = 3$. The calculation is spread over several lemmas, and consists of showing that a bar spectral sequence collapses on the E^2-page, and then find a suitable $\mathbb{F}_p$-algebra basis for $\pi_*(\Lambda_{T^n} H\mathbb{F}_p)$ that allows us to identify the algebra structure. Section 3.5 shows that there is an element in the second column of the homotopy fixed points spectral sequence that is a cycle and not a boundary, and represents v_n in $k(n)(\Lambda_{T^{n+1}} H\mathbb{F}_p)$.

The appendix contains the definition of a Hopf algebra, and the bar complex. In addition we define a spectral sequence, state some convergence theorems and define an algebra and coalgebra spectral sequence. After that, we define the Bökstedt spectral sequence and continuous homology of a Tate spectrum, two constructions that are needed in some proofs, but doesn't play a very prominent role in the thesis.

Notation and Convention

We let $\subsetneq$ denote strict inclusion and $\subseteq$ denote inclusion when equality is allowed. We let $\mathbb{N}$ denote the natural numbers including 0, and $\mathbb{N}_+$ denote the strictly positive natural numbers. Given $n \geq 1$ we let $\mathbf{n}$ denote the set $\{1, \ldots, n\}$ of natural numbers. Given a set S and an element $s \in S$ we will often write $S \setminus s$ for $S \setminus \{s\}$ to make the formulas more readable.

Given an element x in a (bi)graded module M, we let $|x|$ denote the (bi)degree of x. Given a graded module M we let M_n denote the part in degree n, and let $M_{\leq n}$ denote the module $\bigoplus_{i \leq n} M_n$, and similarly for other inequalities $<, >$ and $\geq$.

Let R be a commutative ring, let x and y be of even and odd degree, respectively. We let $P_R(x)$ be the polynomial ring over R and let $E_R(y)$ be the exterior algebra over y. When R is clear from the setup we often leave it out of the notation and write $P_p(x) = P(x)/(x^p)$ for the truncated polynomial ring. Furthermore, we let $\Gamma(x)$ be the divided power algebra over R, which as an R-module is generated by the elements $\gamma_i(x)$ in degree $i|x|$ for $i \geq 0$, with R-algebra structure given by $\gamma_i(x)\gamma_j(x) = \binom{i+j}{j}\gamma_{i+j}(x)$, and R-coalgebra structure given by $\psi(\gamma_k(x) = \sum_{i+j=k} \gamma_i(x) \otimes \gamma_j(x)$.

Homology is always with $\mathbb{F}_p$ coefficients, where p is a prime which is clear from the setting. The differentials in a spectral sequence is only given up to multiplication with a unit.

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Chapter 1

Preliminaries

In this chapter we define the Loday functor and state the properties we need from orthogonal ring spectra. After that we introduce the spectral sequences that are used throughout the thesis. See the appendix for the definition and convergence properties of a spectral sequence.

1.1 The Loday Functor

We will work in the category of orthogonal spectra, but since our goal is to calculate homotopy groups of certain spectra, we could have chosen another model. See [MM02] and [MMSS01] for details. In [MMSS01] they prove that the category of orthogonal commutative ring spectra is enriched over topological spaces, and is tensored and cotensored.

Given a simplicial set X and an commutative ring spectrum R we define the Loday functor $\Lambda_X R$ as in the beginning of Section 4.6 in [BCD10]. When X is a topological space, we write $\Lambda_X R$ for $\Lambda_{\sin(X)} R$, where $\sin(X)$ is the singular set of X.

Proposition 1.1.1. *The Loday functor has the following properties.*

1. *If R is a cofibrant commutative ring spectrum then there is a natural equivalence $\Lambda_X R \simeq X \otimes R$.*

2. *A weak equivalence $X \to Y$ of simplicial sets induces a weak equivalence $\Lambda_X R \to \Lambda_Y R$.*

3. *Given a cofibration $L \to X$ and a map $L \to K$ between simplicial sets there is an equivalence $\Lambda_{X \coprod_L K} R \simeq \Lambda_X R \wedge_{\Lambda_L R} \Lambda_K R$.*

Proof. The first and second part follows from Corollary 4.4.5 and Lemma 4.6.1 in [BCD10], respectively. The last part follows from the equivalence $\Lambda_X R \simeq X \otimes R$ and the fact that tensor commutes with colimits. □

Definition 1.1.2. *Let X be simplicial sets, and let R a commutative ring spectrum. The inclusion $\{x\} \to X$ induces a map $\Lambda_{\{x\}}R \to \Lambda_X R$, and these maps assemble to a natural map*

$$\omega_X : X_+ \wedge R \cong \bigvee_{x\in X} R \cong \bigvee_{x\in X} \Lambda_{\{x\}}R \to \Lambda_X R.$$

Let Y be a simplicial set. Composing $\omega_X : X_+ \wedge \Lambda_Y R \to \Lambda_{X\times Y}R$ with the map induced by the map $X \times Y \to X \times Y/(X \vee Y) \cong X \wedge Y$ yields a natural map

$$\widehat{\omega}_X : X_+ \wedge \Lambda_Y R \to \Lambda_{X\wedge Y}R.$$

The map ω_X was first constructed in Section 5 of [MSV97]. Given a simplicial set X the cofiber sequence $X_+ \to S^0 \to \Sigma X$ induces a stable splitting $X_+ \simeq S^0 \vee X$.

Definition 1.1.3. *Composing the maps ω_{S^1} and $\widehat{\omega}_{S^1}$ with the stable splitting $S^1_+ \simeq S^1 \vee S^0$, induce maps in homotopy*

$$\pi_*(S^1 \wedge R) \cong H_*(S^1) \otimes_{\mathbb{F}_p} \pi_*(R) \to \pi_*(\Lambda_{S^1}R)$$
$$\pi_*(S^1 \wedge \Lambda_Y R) \cong H_*(S^1) \otimes_{\mathbb{F}_p} \pi_*(\Lambda_Y R) \to \pi_*(\Lambda_{S^1\wedge Y}R).$$

Given $z \in \pi_*(R)$ and $y \in \pi_*(\Lambda_Y R)$ we write $\sigma(z)$ and $\widehat{\sigma}(y)$ for the image of $[S^1] \otimes z$ and $[S^1] \otimes y$ under the respective maps, where $[S^1]$ is a chosen generator of $H_1(S^1)$.

The following statement was proven in Proposition 5.10 in [AR05] for homology, but the same proof works for homotopy.

Proposition 1.1.4. *Let R be a commutative ring spectrum. Then $\sigma : \pi_*(R) \to \pi_*(\Lambda_{S^1}R)$ is a graded derivation, i.e.,*

$$\sigma(xy) = \sigma(x)y + (-1)^{|x|}x\sigma(y)$$

for $x, y \in \pi_* R$. From this it follows that the composite $\sigma : \pi_*(\Lambda_{S^1}R) \to \pi_*(\Lambda_{S^1\times S^1}R) \to \pi_*(\Lambda_{S^1}R)$ where the last map is induced by the multiplication in S^1 is also a derivation.

Proposition 1.1.5. *Let $n \geq 1$ and let R be a commutative ring spectrum, and assume that $\pi_*(\Lambda_{S^n}R)$ is flat as a $\pi_*(R)$-module. Then $\pi_*(\Lambda_{S^n}R)$ is an $\pi_*(R)$-Hopf algebra with unit and counit induced by choosing a base point in S^n and collapsing S^n to a point, respectively. The multiplication and coproduct is induced by the fold map $\nabla : S^n \vee S^n \to S^n$ and the pinch map $\psi : S^n \to S^n \vee S^n$, respectively, and the conjugation map is induced by the reflection map $-\mathrm{id} : S^n \to S^n$.*

Proof. We have $\Lambda_{S^n\vee S^n}R \simeq \Lambda_{S^n}R \wedge_R \Lambda_{S^n}R$ and since $\pi_*(\Lambda_{S^n}R)$ is flat as a $\pi_*(R)$-module, $\pi_*(\Lambda_{S^n}R \wedge_R \Lambda_{S^n}R) \cong \pi_*(\Lambda_{S^n}R) \wedge_{\pi_*(R)} \pi_*(\Lambda_{S^n}R)$ by Corollary 1.2.2. That the various diagrams in the definition of a $\pi_*(R)$-Hopf algebra commutes, now follows from commutativity of the corresponding diagrams on the level of simplicial sets. □

Proposition 1.1.6. *Let R be a commutative ring spectrum, and assume that $\pi_*(\Lambda_{S^1}R)$ is flat as a $\pi_*(R)$-module. Given z in $\pi_*(R)$, then $\sigma(z)$ is primitive in the the $\pi_*(R)$-Hopf algebra $\pi_*(\Lambda_{S^1}R)$.*

Proof. The diagram

$$\begin{tikzcd} S^1_+ \wedge R \arrow[r, "\omega"] \arrow[d, "\psi_+\wedge \mathrm{id}"] & \Lambda_{S^1}R \arrow[d, "\Lambda_\psi R"] \\ (S^1 \vee S^1)_+ \wedge R \arrow[r, "\omega"] & \Lambda_{S^1\vee S^1}R \end{tikzcd}$$

commutes. Hence, $\psi(\sigma(z)) = \sigma(z) \otimes 1 + 1 \otimes \sigma(z)$. □

1.2 The Bar Spectral Sequence

In this section we introduce the bar spectral sequence which is the most important tool in our calculations.

Let X_* be a simplicial spectrum and define the simplicial abelian group $\pi_t(X_*)$ to be $\pi_t(X_q)$ in degree q with face and degeneracy homomorphisms induced by the face and degeneracy maps in X_*. Write $|X_*|$ for the realization of the simplicial spectrum X_*. See Chapter X in [EKMM97] for more details.

The spectral sequence below is well known for spaces, and appears for spectra in Theorem X.2.9 in [EKMM97].

Proposition 1.2.1. *Let X_* be a simplicial spectrum, and assume that $\mathrm{sk}_s(X_*) \to \mathrm{sk}_{s+1}(X_*)$ is a cofibration for all $s \geq 0$. There is a strongly convergent spectral sequence*

$$E^2_{s,t}(X_*) = H_s(\pi_t(X_*)) \Rightarrow \pi_{s+t}(X_*).$$

Let R be a simplicial ring spectrum.

If X_* is a simplicial R-algebra, then $E^2_{s,t}(X_*)$ is an $\pi_*(R)$-algebra spectral sequence.

Proof. The skeleton filtration $\mathrm{sk}_0 X_* \subseteq \mathrm{sk}_1 X_* \subseteq \mathrm{sk}_2 X_* \subseteq \dots$ of X_* gives rise to an unrolled exact couple

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & \dots \\ E^1_0 & E^1_1 \arrow[ul] & E^1_2 \arrow[ul] & \end{tikzcd}$$

where $A_{s,t} = \pi_{s+t}(\mathrm{sk}_s X_*)$ and $E^1_{s,t} = \pi_{s+t}(\mathrm{sk}_s X_* / \mathrm{sk}_{s-1} X_*)$ when $s \geq 0$ and 0 otherwise. That the d^1-differential is the differential in the chain complex associated to $\pi_t(X_*)$ follows from a diagram chase as in Theorem 11.14 in [May72].

This spectral sequence is concentrated in the right half plane. By Theorem A.3.6, the associated spectral sequence converges strongly to the colimit $\mathrm{colim}_s A_s = \pi_*(X_*)$ since the limit $\lim_s A_s = 0$. We have the usual filtration $F_0 \subseteq F_1 \subseteq F_s \subseteq \dots$ of the colimit $\mathrm{colim}_s A_s$ as constructed in Section A.3.

Recall the definition of an algebra spectral sequence in Definition A.3.7. Given $a \in \pi_t(\mathrm{sk}_s X_*)$ and $b \in \pi_v(\mathrm{sk}_u X_*)$ represented by maps of simplicial sets $S^t \to \mathrm{sk}_s X_*$ and $S^v \to \mathrm{sk}_u X_*$, the product $ab \in \pi_{t+v}(\mathrm{sk}_{s+u} X_*)$ is represented by the composition $S^t \wedge S^u \to \mathrm{sk}_s X_* \wedge_R \mathrm{sk}_u X_* \to \mathrm{sk}_{s+u} X_* \wedge_R \wedge X_* \to$, where the first map is the smash product, the second is the inclusion, and the last map is the product map in X_*.

If X_* is a simplicial R-algebra, the product thus respects the filtration, i.e., $\phi(F_{s,t} \otimes F_{u,v}) \subseteq F_{s+u,t+v}$. Using the crossproduct in homology we get a product

$$E^2_{s,t}(X_*) \otimes E^2_{u,v}(X_*) \to E^2_{s+t,u+v}(X_* \wedge_R X_*) \to E^2_{s+t,u+v}(X_*)$$

where the last homomorphism is the standard shuffle product of simplicial modules. Thus the product satisfies the Leibniz rule, and we define the rest of the products as the homology of the product on the E^2-page. It coincide with the induced product on the associated graded complex coming from the filtration of $\mathrm{colim}\, A_s$ since both products have the same geometric origin from a map of simplicial spectra.

□

We are interested in the special case when R is a commutative ring spectrum, M a cofibrant right R-module, N is a left R module and $B(M, R, N)$ is the bar construction. I.e., $B(M, R, N)$ is the simplicial spectrum which in degree q is equal to $M \wedge R^{\wedge q} \wedge N$, and where the face and degeneracy maps are induced by the same formulas as in the algebra case using the unit map and multiplication map. By Lemma 4.1.9 in [Shi07] there is an equivalence $|B(M, R, N)| \simeq |M \wedge_R N|$.

Corollary 1.2.2. *Let R be a bounded below ring spectrum, M a right R-module and N a left R-module. Then there is a strongly convergent spectral sequence*

$$E^2_{s,t} = \mathrm{Tor}_s^{\pi_* R}(\pi_* M, \pi_* N)_t \Rightarrow \pi_{s+t}(M \wedge^L_R N).$$

Remark 1.2.3. If $\pi_*(X_*)$ is flat as an $\pi_*(R)$-module, this corollary yields an isomorphism $\pi_*(X_* \wedge_R X_*) \cong \pi_*(X_*) \otimes_{\pi_*(R)} \pi_*(X_*)$.

If X_* is a simplicial R-coalgebra, i.e., there is a coproduct map $\psi : X_* \to X_* \wedge_R X_*$ with a counit map $X_* \to R$ making the obvious diagrams commute up to homotopy, and $\pi_*(X_*)$ is flat as an $\pi_*(R)$-module, then $\pi_*(X_*)$ is an $\pi_*(R)$-coalgebra with coproduct induced by ψ followed by the isomorphism $\pi_*(X_* \wedge_R X_*) \cong \pi_*(X_*) \otimes_{\pi_*(R)} \pi_*(X_*)$.

Corollary 1.2.4. *Assume that X_* is a simplicial R-coalgebra, and assume that the map $\mathrm{sk}_s(X_*) \to \mathrm{sk}_{s+1}(X_*)$ is a cofibration for all $s \geq 0$. If each term $E^r(X_*)$ for $r \geq 1$ is flat over $\pi_*(R)$ then $E^2(X_*)$ is an $\pi_*(R)$-coalgebra spectral sequence. If in addition, $\pi_*(X_*)$ is flat as an $\pi_*(R)$-module, then the spectral sequence converges to $\pi_*(X_*)$ as an $\pi_*(R)$-coalgebra.*

Proof. Recall the definition of a coalgebra spectral sequence in Definition A.3.8. Let $\overline{\mathrm{sk}}_n(X_* \wedge_R X_*)$ be the colimit of the diagram consisting of the spectra $\mathrm{sk}_i(X_*) \wedge_R \mathrm{sk}_j(X_*)$ with $i + j \leq n$, and with one map $\mathrm{sk}_i(X_*) \wedge_R \mathrm{sk}_j(X_*) \to \mathrm{sk}_{i'}(X_*) \wedge_R \mathrm{sk}_{j'}(X_*)$ when

$i \le i'$ and $j \le j'$ with $i'+j' \le n$, induced by the inclusion of the skeletons. The natural map $\overline{\mathrm{sk}}_n(X_* \wedge_R X_*) \to \overline{\mathrm{sk}}_{n+1}(X_* \wedge_R X_*)$ is a cofibration since it can be constructed as a pushout of cofibrations, by adding the extra spectra in the diagram for $\overline{\mathrm{sk}}_{n+1}(X_* \wedge_R X_*)$ one by one.

This yields a sequence of cofibrations

$$\overline{\mathrm{sk}}_0(X_* \wedge_R X_*) \to \overline{\mathrm{sk}}_1(X_* \wedge_R X_*) \to \overline{\mathrm{sk}}_2(X_* \wedge_R X_*) \to \dots$$

with colimit equal to $X_* \wedge_R X_*$. We let $A_{s,t} = \pi_{s+t}(\overline{\mathrm{sk}}_s(X_* \wedge_R X_*))$ and $\overline{E}^1_{s,t} = \pi_*(\overline{\mathrm{sk}}_s(X_* \wedge_R X_*)/\overline{\mathrm{sk}}_{s-1}(X_* \wedge_R X_*))$ and the chain complex $\overline{E}^1$ is equal to the total complex of $E^1(X_*) \otimes_{\pi_*(R)} E^1(X_*)$, since $\overline{\mathrm{sk}}_s(X_* \wedge_R X_*)/\overline{\mathrm{sk}}_{s-1}(X_* \wedge_R X_*)$ is the wedge of $\mathrm{sk}_i(X_*) \wedge_R \mathrm{sk}_j(X_*)$ with $i+j=n$ divided by the images of lower dimensional skeletons.

This corresponding spectral sequence converges strongly

$$\overline{E}^1(X_* \wedge_R X_*) \Rightarrow \pi_*(X_* \wedge_R X_*),$$

with and since each term $E^1(X_*)$ for $r \ge 1$ is flat over $\pi_*(R)$, the Künneth isomorphism induces an isomorphism $\overline{E}^r(X_* \wedge_R X_*) \cong E^r(X_*) \otimes_{\pi_*(R)} E^r(X_*)$.

From Proposition 1.2.1 we have a spectral sequence

$$E^1(X_* \wedge_R X_*) \Rightarrow \pi_*(X_* \wedge_R X_*)$$

coming from the skeleton filtration of $X_* \wedge_R X_*$.

There is map from the filtration $\overline{\mathrm{sk}}_i(X_* \wedge_R X_*)$ to the skeleton filtration $\mathrm{sk}_i(X_* \wedge_R X_*)$ induced by the natural maps $\mathrm{sk}_i(X_*) \wedge_R \mathrm{sk}_j(X_*) \to \mathrm{sk}_{i+j}(X_* \wedge_R X_*)$. It induces the shuffle map from $\overline{E}^1(X_* \wedge_R X_*)$ to $E^1(X_* \wedge_R X_*)$, which is a chain equivalence with inverse given by the Alexander Whitney map.

The composition

$$E^r_{s,t}(X_*) \to E^r_{s,t}(X_* \wedge_R X_*) \xrightarrow{\cong} \bigoplus_{u+x=s, v+y=t} E^r_{u,v}(X_*) \otimes_{\pi_*(R)} E^r_{x,y}(X_*)$$

where the first map is induced by the map $X_* \to X_* \wedge_R X_*$, and the second map is induced by the Alexander Whitney map defines a $\pi_*(R)$-coalgebra structure on $E^r(X_*)$ satisfying the the assumption of an R-coalgebra spectral sequence in Definition A.3.8.

If in addition $\pi_*(X_*)$ is flat as an $\pi_*(R)$-module, then $\pi_*(X_*)$ is an $\pi_*(R)$-coalgebra as observed in Remark 1.2.3. Let

$$F_0 \subseteq \dots \subseteq F_{s-1} \subseteq F_s \subseteq F_{s+1} \subseteq \dots \subseteq \pi_*(X_*)$$

be the filtration associated with the skeleton filtration of X_*, let

$$G_0 \subseteq \dots \subseteq G_{s-1} \subseteq G_s \subseteq G_{s+1} \subseteq \dots \subseteq \pi_*(X_* \wedge_R X_*)$$

be the filtration associated with the skeleton filtration of $X_* \wedge_R X_*$, and let

$$\overline{F}_0 \subseteq \dots \subseteq \overline{F}_{s-1} \subseteq \overline{F}_s \subseteq \overline{F}_{s+1} \subseteq \dots \subseteq \pi_*(X_* \wedge_R X_*)$$

be the filtration associated with the filtration $\overline{\mathrm{sk}}(X_* \wedge_R X_*)$. Since the spectral sequence $E^2(X_* \wedge_R X_*)$ is isomorphic to $\overline{E}^2(X_* \wedge_R X_*)$, and they both converge strongly to $\pi_*(X_* \wedge_R X_*)$ we have a commutative square

$$\begin{tikzcd}
\bigoplus_{s\geq 0} G_s/G_{s-1} \arrow[r, "\cong"] \arrow[d, "\cong"] & \bigoplus_{s\geq 0} E_s^\infty(X_* \wedge_R X_*)/E_{s-1}^\infty(X_* \wedge_R X_*) \arrow[d, "\cong"] \\
\bigoplus_{s\geq 0} H_s/H_{s-1} \arrow[r, "\cong"] & \bigoplus_{s\geq 0} \overline{E}_s^\infty(X_* \wedge_R X_*)/\overline{E}_{s-1}^\infty(X_* \wedge_R X_*)
\end{tikzcd}$$

so $G_s = \overline{F_s}$ for all $s \geq 0$. Since the coproduct map $\psi : X_* \to X_* \wedge_R X_*$ preserves the skeleton filtration, this implies that on homotopy groups $\psi(F_s) \subseteq G_s = H_s$. Now $H_s/H_{s-1} \cong \bigoplus_s F_s/F_{s-1}$, so the spectral sequence converges to $\pi_*(X_*)$ as an $\pi_*(R)$-coalgebra. $\square$

In particular, for $B(R, \Lambda_X R, R) \simeq \Lambda_{S^1 \wedge X} R$ we have the following proposition.

Proposition 1.2.5. *Let R be a commutative ring spectrum and let X be a simplicial set. The operator*

$$\widehat{\sigma} : \pi_*(\Lambda_X R) \to \pi_*(\Lambda_{S^1 \wedge X} R)$$

takes z to the class of $[z]$ in

$$E^2_{s,t} = \mathrm{Tor}^{\pi_*(\Lambda_X R)}(\pi_*(R), \pi_*(R)) \Rightarrow \pi_{s+t}(\Lambda_{S^1 \wedge X} R),$$

where $[z]$ is in the reduced bar complex $B(\pi_*(R), \pi_*(\Lambda_X R), \pi_*(R))$.

Proof. Using the minimal simplicial model for S^1 we get a simplicial spectrum $S^1_+ \wedge \Lambda_X R$ which in simplicial degree q is equal to $(S^1_q)_+ \wedge \Lambda_X R \cong (\Lambda_X R)^{\vee q}$, the q-fold wedge of $\Lambda_X R$. In the E^2-term of the spectral sequence in Proposition 1.2.1 associated with this simplicial spectrum, the element $[S^1] \otimes z$ is represented by $1 \oplus z$ in $E^1_{1,*} \cong \pi_*(\Lambda_X R \vee \Lambda_X R) \cong \pi_*(\Lambda_X R) \oplus \pi_*(\Lambda_X R)$, where the second factor corresponds to the non-degenerate simplex in S^1_1.

Similarly, there is a simplicial model for the spectrum $\Lambda_{S^1 \wedge X} R$, which in simplicial degree q is equal to $\Lambda_{S^1_q \wedge X} R \cong \Lambda_{\bigvee_q X} R \cong (\Lambda_X R)^{\wedge_R q-1}$, the $(q-1)$-fold smash product over R. The map $\widehat{\omega} : S^1_+ \wedge \Lambda_X R \to \Lambda_{S^1 \times X} R$ is given on these simplicial models in degree q by the natural map

$$(\Lambda_X R)^{\vee q} \to (\Lambda_X R)^{\wedge q} \to (\Lambda_X R)^{\wedge_R q-1}$$

where the first map is induced by the inclusion into the various smash factors using the unit maps, and the second map is induced by the map $\Lambda_X R \to \Lambda_{\{\mathrm{pt}\}} R$ on the factor indexed by the degenerate simplex. The element $\widehat{\sigma}(z)$ in the spectral sequence from Proposition 1.2.1 associated with this simplicial spectrum, is thus represented by the element z in $E^1_{1,*} \cong \pi_*(\Lambda_X R)$.

Now we have to compare this last spectral sequence, with the spectral sequence coming from the bar complex $B(R, \Lambda_X R, R)$. In simplicial degree q, $B(R, \Lambda_X R, R)$ is equal to $R \wedge \Lambda_X R^{\wedge q-1} \wedge R \cong \Lambda_{S^0 \amalg (\coprod_q X)} R$. The equivalence between $B(R, \Lambda_X R, R)$ and the model above is induced by the map $S^0 \amalg \coprod_q X \to \bigvee_q X$ identifying S^0 and the basepoints in X to the base point in $\bigvee_q X$. The element $\widehat{\sigma}(z)$ is thus represented by the class of $[z]$ in

$$E^2_{s,t} = \mathrm{Tor}^{\pi_*(\Lambda_X R)}(\pi_*(R), \pi_*(R)) \Rightarrow \pi_{s+t}(\Lambda_{S^1 \wedge X} R),$$

where $[z]$ is in the reduced bar complex $B(\pi_*(R), \pi_*(\Lambda_X R), \pi_*(R))$. □

1.3 Hopf Algebra Spectral Sequences

This section contains some results about calculations in spectral sequences with a Hopf algebra structure. The first result is well known, and will be a cornerstone in reducing the number of potential non-zero differentials in the bar spectral sequence and the Bökstedt spectral sequence.

Proposition 1.3.1. *Let E^2 be a first quadrant connected R-Hopf algebra spectral sequence. The shortest non-zero differentials in E^2 of lowest total degree, if there are any, are generated by differentials from an indecomposable element in E^2 to a primitive element in E^2.*

Proof. If there are no d^i-differentials for $i < r$, then $E^r = E^2$ is still an R-Hopf algebra spectral sequence. Let z be an element in E^r of lowest total degree with $d^r(z) \neq 0$. If z can be decomposed as $z = xy$, with both x and y in positive degrees, then by the Leibniz rule $d^r(xy) = d^r(x)y \pm xd^r(y)$, so if $d^r(xy) \neq 0$, then $d^r(x)$ or $d^r(y)$ must be non-zero, contradicting the minimality of the degree of z.

We have $\psi(z) = 1 \otimes z + z \otimes 1 + \sum z' \otimes z''$ for some elements z' and z'' of lower degree than z. Now,

$$\psi(d^r(z)) = d^r(\psi(z)) = 1 \otimes d^r(z) + d^r(z) \otimes 1 + \sum d^r(z') \otimes z'' \pm z' \otimes d^r(z'').$$

If $d^r(z)$ is not primitive we must have that $d^r(z')$ or $d^r(z'')$ are not zero, contradicting the minimality of the degree of z.

Thus the shortest differential in lowest total degree is from an indecomposable element to a primitive element. □

The next proposition shows that in certain circumstances the coalgebra structure of the abutment in a spectral sequence is determined by the algebra structure of the dual spectral sequence. We will use it to calculate the $\mathbb{F}_p$-Hopf algebra structure of $\pi_*(\Lambda_{S^n} H\mathbb{F}_p)$.

Proposition 1.3.2. *Let R be a field, and let*

$$\begin{tikzcd}
0 \arrow[r] & A_0 \arrow[r, "i"] \arrow[d, "="] & A_1 \arrow[r, "i"] \arrow[d, "j"] & A_2 \arrow[r] \arrow[d, "j"] & \cdots \\
& E^1_0 & E^1_1 \arrow[ul, "k"] & E^1_2 \arrow[ul, "k"] &
\end{tikzcd}$$

be an unrolled exact couple of connected cocommutative R-coalgebras which are finite in each degree. The unrolled exact couple gives rise to a spectral sequence E^2 converging strongly to $\operatorname{colim}_s A_s$ *by Theorem A.3.6.*

Assume that in each degree t the map $A_{s,t} \to A_{s+1,t}$ eventually stabilizes, i.e., is the identify for all $s \geq u$ for some u depending on t. Assume the E^2-term of the spectral sequence is isomorphic, as an R-coalgebra, to a tensor product of exterior algebras and divided power algebras, and there are no differentials in the spectral sequence, i.e., $E^2 = E^\infty$. Then there are no coproduct coextensions in the abutment. Hence, $\operatorname{colim}_s A_S \cong E^\infty$ *as an R-coalgebra.*

Proof. The colimit $\operatorname{colim}_s A_S$ of R-coalgebras is constructed in the underlying category of R-modules. Applying $D(-) = \hom_R(-, R)$ to the unrolled exact couple in the proposition yields an unrolled exact couple $\ldots \to \overline{A}_{-2} \to \overline{A}_{-1} \to \overline{A}_0$ of commutative R-coalgebras with $\overline{A}_{-s} = D(A_s)$. By Theorem A.3.6 the associated spectral sequence converges strongly to $\lim_s \overline{A}_s = D(\operatorname{colim}_s A_s)$ since it is a spectral sequence with exiting differentials. Since R is a field, cohomology is the dual of homology, so $\overline{E}^r_{-s,-t} = D(E^r_{s,t})$.

Now, since $E^2 \cong \bigotimes_I E(x_i) \otimes \bigotimes_J \Gamma(y_j)$, we have $\overline{E}^2 \cong \bigotimes_I E(x_i^*) \otimes \bigotimes_J P(y_j^*)$, where x_i^* is the dual of x_i, and $(y_j^*)^k$ is the dual of $\gamma_k(y_j)$. Since there are no differentials in E^2, there are no differentials in $\overline{E}^2$, so $\overline{E}^2 = \overline{E}^\infty$. Since $\operatorname{colim} A_s$ is cocommutative, $\lim_s \overline{A}_s$ is commutative, and hence $(x_i^*)^2 = 0$ in the abutment $\lim_s \overline{A}_s$ since x_i^* is in odd degree. Furthermore, y_j^* is not nilpotent, so there is an algebra isomorphism $\lim_s \overline{A}_s \cong \overline{E}^\infty$. Since the maps $A_s \to A_{s+1}$ eventually stabilizes, $D(\lim_s A_s) \cong \operatorname{colim}_s A_s$, so we can dualize again, and get that there is an R-coalgebra isomorphism $\operatorname{colim}_s A_s \cong E^\infty$. $\square$

The final two lemmas are one standard homological calculation, and one easy homological calculation that are used to identify the E^p-term of the Bökstedt spectral sequence.

Lemma 1.3.3. *Let*

$$E^2 = A \otimes_R \Gamma_R(x_0, x_1, \ldots) \otimes E_R(y_1, y_2, \ldots)$$

be a connected R-algebra spectral sequence with x_i and y_i in filtration 1 and R a field. Assume there are differentials

$$d^{p-1}(\gamma_{p+k}(x_i)) = \gamma_k(x_i) y_{i+1},$$

for all $k, i \geq 0$. Then

$$E^p \cong A \otimes P_R(x_0, x_1, \ldots)/(x_0^p, x_1^p, \ldots).$$

Proof. Consider the R algebra $\Gamma_R(x_i) \otimes E_R(y_{i+1})$ with differentials $d^{p-1}(\gamma_{p+k}(x_i)) = \gamma_k(x_i)y_{i+1}$. The cycles are $\gamma_k(x_i)$ for $k \leq p-1$ and $\gamma_k(x_i)y_{i+1}$ for all k, but this last family are also boundaries, so the homology is $P_R(x_i)/(x_i^p)$. The lemma now follows from the Künneth isomorphism, since R is a field. □

In the next lemma we have a family of differentials $d^{p-1}(\gamma_{p+k}(x_i))$ given by certain formulas, and then another family of differentials $d^{p-1}(\gamma_{p+k}(z))$ with image in the module generated by the images of all the differentials in the first family. The lemma states how we can construct new cycles such that we are not bothered by the last family of differentials.

Lemma 1.3.4. *Let*

$$E^2 = A \otimes \Gamma_R(x_0, x_1, \ldots) \otimes E_R(y_1, y_2, \ldots) \otimes \Gamma_R(z)$$

be a connected R-algebra spectral sequence. Assume there are differentials

$$\begin{aligned} d^{p-1}(\gamma_{p+k}(x_i)) &= \gamma_k(x_i)y_{i+1} \\ d^{p-1}(\gamma_{p+k}(z)) &= \gamma_k(z) \cdot \sum_{l \in \mathbb{N}} r_l d^{p-1}(\gamma_p(x_l)), \end{aligned}$$

where r_l are elements in R.

Then there are cycles

$$\gamma_{p^k}(z') = \sum_{j=0}^{p^{k-1}} \big((-1)^j \gamma_{p^k - pj}(z) \sum_{\alpha \in \mathbb{N}^{\mathbb{N}}, |\alpha| = j} \prod_{i \in \mathbb{N}} r_i^{\alpha_i} \gamma_{p\alpha_i}(x_i)\big), \tag{1.3.5}$$

where $|\alpha| = \sum_{k \in \mathbb{N}} \alpha_i$, and the convention is that $0^0 = 1$, $\gamma_0(x) = 1$, and $\gamma_i(x) = 0$ when $i < 0$.

Furthermore, this formula induces an R-algebra isomorphism

$$A \otimes \Gamma_R(x_0, x_1 \ldots) \otimes E_R(y_1, y_2, \ldots) \otimes \Gamma_R(z') \cong A \otimes \Gamma_R(x_0, x_1 \ldots) \otimes E_R(y_1, y_2, \ldots) \otimes \Gamma_R(z).$$

Proof. First we show that the elements $\gamma_{p^k}(z')$ are cycles. By the Leibniz rule

$$\begin{aligned} d^{p-1}(\gamma_{p^k}(z')) &= d^{p-1}\Big(\sum_{j=0}^{p^{k-1}} \big((-1)^j \gamma_{p^k - pj}(z) \sum_{\alpha \in \mathbb{N}^{\mathbb{N}}, |\alpha| = j} \prod_{i \in \mathbb{N}} r_i^{\alpha_i} \gamma_{p\alpha_i}(x_i)\big)\Big) \\ &= \sum_{j=0}^{p^{k-1}} \big((-1)^j d^{p-1}(\gamma_{p^k - pj}(z)) \sum_{\alpha \in \mathbb{N}^{\mathbb{N}}, |\alpha| = j} \prod_{i \in \mathbb{N}} r_i^{\alpha_i} \gamma_{p\alpha_i}(x_i)\big) \\ &+ \sum_{j=0}^{p^{k-1}} \Big((-1)^j \gamma_{p^k - pj}(z) \sum_{\alpha \in \mathbb{N}^{\mathbb{N}}, |\alpha| = j} d^{p-1}\Big(\prod_{i \in \mathbb{N}} r_i^{\alpha_i} \gamma_{p\alpha_i}(x_i)\Big)\Big). \end{aligned}$$

Using the formula for $d^{p-1}(\gamma_{p^k-pj}(z)$ and the Leibniz rule once more,

$$\begin{aligned}
d^{p-1}(\gamma_{p^k}(z')) = &\sum_{j=0}^{p^{k-1}} \Big((-1)^j \gamma_{p^k-p(j+1)}(z)\big(\sum_{l\in\mathbb{N}} r_l d^{p-1}(\gamma_p(x_l))\big) \sum_{\alpha\in\mathbb{N}^{\mathbb{N}},|\alpha|=j} \prod_{i\in\mathbb{N}} r_i^{\alpha_i}\gamma_{p\alpha_i}(x_i)\Big) \\
&+ \sum_{j=0}^{p^{k-1}} \big((-1)^j \gamma_{p^k-pj}(z) \sum_{\alpha\in\mathbb{N}^{\mathbb{N}},|\alpha|=j} \sum_{l\in\mathbb{N}} r_l^{\alpha_l}\gamma_{p(\alpha_l-1)}(x_l) d^{p-1}(\gamma_p(x_l)) \prod_{l\neq i\in\mathbb{N}} r_i^{\alpha_i}\gamma_{p\alpha_i}(x_i)\big),
\end{aligned} \tag{1.3.6}$$

and there are no extra signs here since all the factors in the expression of $\gamma_{p^k}(z)$ are in even degrees.

In the first sum in equation 1.3.6 observe that

$$\begin{aligned}
&\big(\sum_{l\in\mathbb{N}} r_l d^{p-1}(\gamma_p(x_l))\big) \sum_{\alpha\in\mathbb{N}^{\mathbb{N}},|\alpha|=j} \prod_{i\in\mathbb{N}} r_i^{\alpha_i}\gamma_{p\alpha_i}(x_i) \\
&\qquad = \sum_{l\in\mathbb{N}} \sum_{\alpha\in\mathbb{N}^{\mathbb{N}},|\alpha|=j} r_l^{\alpha_l+1} d^{p-1}(\gamma_p(x_l))\gamma_{p\alpha_l}(x_l) \prod_{l\neq i\in\mathbb{N},} r_i^{\alpha_i}\gamma_{p\alpha_i}(x_i) \\
&\qquad = \sum_{l\in\mathbb{N}} \sum_{\alpha\in\mathbb{N}^{\mathbb{N}},|\alpha|=j+1} r_l^{\alpha_l} d^{p-1}(\gamma_p(x_l))\gamma_{p(\alpha_l-1)}(x_l) \prod_{l\neq i\in\mathbb{N}} r_i^{\alpha_i}\gamma_{p\alpha_i}(x_i)
\end{aligned}$$

Substituting this expression into equation 1.3.6 and increasing the summation index in the first sum with one, the differential is given by

$$\begin{aligned}
&d^{p-1}(\gamma_{p^k}(z')) = \\
&\sum_{j=1}^{p^{k-1}+1} \big((-1)^{j-1} \gamma_{p^k-pj}(z) \sum_{\alpha\in\mathbb{N}^{\mathbb{N}},|\alpha|=j} \sum_{l\in\mathbb{N}} r_l^{\alpha_l}\gamma_{p(\alpha_l-1)}(x_l) d^{p-1}(\gamma_p(x_l)) \prod_{l\neq i\in\mathbb{N}} r_i^{\alpha_i}\gamma_{p\alpha_i}(x_i)\big) \\
&\quad + \sum_{j=0}^{p^{k-1}} \big((-1)^j \gamma_{p^k-pj}(z) \sum_{\alpha\in\mathbb{N}^{\mathbb{N}},|\alpha|=j} \sum_{l\in\mathbb{N}} r_l^{\alpha_l}\gamma_{p(\alpha_l-1)}(x_l) d^{p-1}(\gamma_p(x_l)) \prod_{l\neq i\in\mathbb{N}} r_i^{\alpha_i}\gamma_{p\alpha_i}(x_i)\big).
\end{aligned}$$

The $j = p^{k-1}+1$ summand in the first sum is zero because $\gamma_{p^k-(p^{k-1}+1)p}(z) = \gamma_{-p}(z) = 0$. Similarly, the $j = 0$ summand in the last sum is zero because $0 = j = |\alpha|$ implies that $\alpha_l = 0$ for all l, and hence $\gamma_{p(\alpha_l-1)}(x_l) = \gamma_{-p}(x_l) = 0$.

The rest of the summands cancel pairwise, due to the factors $(-1)^{j-1}$ and $(-1)^j$. Thus $d^{p-1}(\gamma_{p^k}(z')) = 0$.

That $(\gamma_{p^k}(z'))^p = 0$ is clear by the Frobenius formula, since every summand in the expression for $\gamma_{p^k}(z')$ contain a factor from a divided power algebra.

The composite

$$\Gamma_R(z) \xrightarrow{\gamma_{p^k}(z)\mapsto\gamma_{p^k}(z')} \Gamma_R(x_0,x_1,\ldots)\otimes E_R(y_1,y_2,\ldots)\otimes\Gamma_R(z) \xrightarrow{\mathrm{pr}_{\Gamma_R(z)}} \Gamma_R(z)$$

equals the identity. Hence, the map induced by equation 1.3.5 induces an R-algebra isomorphism

$$A\otimes\Gamma_R(x_0, x_1\ldots)\otimes E_R(y_1, y_2,\ldots)\otimes\Gamma_R(z') \cong A\otimes\Gamma_R(x_0, x_1\ldots)\otimes E_R(y_1, y_2,\ldots)\otimes\Gamma_R(z).$$

□

1.4 The Isotropy Separation Diagram

Everything in this section about finite groups can be found in Section 4 in [LNR12] or in Part 1 of [GM95]. Recall from Section II.2 in [MM02] what it means for an equivariant orthogonal G-spectrum X to be indexed on various universes. We let i be the inclusion of the trivial G-universe into a complete G-universe, let i^* be the forgetful functor from G-spectra indexed on a complete universe, to a G-spectra indexed on the trivial G-universe and let i_* be the left adjoint of i^*. See Section V.1 in [MM02] for more details.

Let EG be a free, contractible G-CW complex. The collapse map from EG to a point gives a homotopy cofiber sequence

$$EG_+ \to S^0 \to \widetilde{EG} \tag{1.4.1}$$

of based $G-CW$ complexes where $\widetilde{EG}$ is the unreduced suspension of EG_+ with one of the cone points as a base point.

Definition 1.4.2. *Let X be an orthogonal G-spectrum and define the spectra*

$$\begin{aligned} X_{hG} &= (EG_+ \wedge i^*X)/G && \textit{(homotopy orbit)}\\ X^{hG} &= F(EG_+, X)^G && \textit{(homotopy fixed points)}\\ X^{tG} &= [\widetilde{EG} \wedge F(EG_+, X)]^G && \textit{(Tate spectrum).} \end{aligned}$$

We have a commutative diagram

$$\begin{tikzcd} {[EG_+ \wedge X]^G} \arrow[r] \arrow[d, "\simeq"] & X^G \arrow[r] \arrow[d, "\Gamma"] & {[\widetilde{EG} \wedge X]^G} \arrow[d, "\hat{\Gamma}"] \\ {[EG_+ \wedge F(EG_+, X)]^G} \arrow[r] & F(EG_+, X)^G \arrow[r] & {[\widetilde{EG} \wedge F(EG_+, X)]^G,} \end{tikzcd}$$

with horizontal cofiber sequences coming from 1.4.1, where $F(Y, X)$ is the mapping spectrum from Y to X and the vertical map is induced by the map $X \cong F(S^0, X) \to F(EG_+, X)$ given by collapsing EG to a point.

The left map is an equivalence by Proposition IV.6.7 in [MM02]. We also have the Adams equivalence, Equation VI 4.6 in [MM02],

$$\tau : (\Sigma^{\mathrm{ad}\,G} EG_+ \wedge i^*X)/G \xrightarrow{\simeq} [i_*(EG_+ \wedge i^*X)]^G$$

where $\operatorname{ad} G$ denotes the adjoint representation of G. When G is finite, the adjoint representation is trivial, so the diagram above can be rewritten for any genuine G-spectrum X, as the *isotropy separation* diagram

$$\begin{tikzcd} X_{hG} \arrow[r] \arrow[d, equal] & X^G \arrow[r] \arrow[d, "\Gamma"] & [\widetilde{EG} \wedge X]^G \arrow[d, "\hat{\Gamma}"] \\ X_{hG} \arrow[r] & X^{hG} \arrow[r, "R^h"] & X^{tG}. \end{tikzcd}$$

Our goal is to compute the homotopy groups of X^G and to do this we introduce some spectral sequences, first introduced in [GM95], that converges to the homotopy groups of the bottom row of the isotropy separation diagram.

1.5 Tate Spectral Sequence and Homotopy Fixed Points Spectral Sequence

In this section we will define three spectral sequences that calculates the homotopy groups of the three spectra in the lower half of the isotropy separation diagram. These were originally constructed in [GM95].

To define one of these spectral sequences we need the notion of a complete resolution and Tate cohomology of a finite group G with coefficients in a G-module M. This can be found in Chapter IV in [Bro82].

A complete resolution for G is an acyclic complex $P = (P_i)_{i\in\mathbb{Z}}$ of projective $\mathbb{F}_p$-modules together with a surjective homomorphism $\epsilon : P_0 \to \mathbb{Z}$ such that $P = (P_i)_{i\in\mathbb{N}}$ is an ordinary resolution of $\mathbb{F}$ with augmentation ϵ. From the definition there is a monomorphism $\eta : \mathbb{F}_p \to P_{-1}$ such that the diagram

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r, "d"] & P_0 \arrow[r, "d"] \arrow[d, "\epsilon"] & P_{-1} \arrow[r, "d"] & \cdots \\ & & \mathbb{F}_p \arrow[ur, "\eta"] & & \end{tikzcd}$$

commutes.

Definition 1.5.1. *Given an $\mathbb{F}_p$-module M and a complete resolution (P_*, d_*) of G, the* Tate cohomology groups *of M are defined by*

$$\hat{H}^n(G; M) = H^n(\operatorname{Hom}_{\mathbb{F}_p}(P_*, M)),$$

and the Tate homology groups *are defined by*

$$\hat{H}_n(G; M) = H_n(P_* \otimes_{\mathbb{F}_p} M).$$

These groups are independent of the chosen resolution of G, and there is an isomorphism

$$\hat{H}^n(G; M) \cong \hat{H}_{-n-1}(G; M).$$

We will not distinguish between a G-space and its suspension spectrum. Let G be a finite group, $\tilde{E}_n$ be the n-skeleton of $\widetilde{EG}$ for $n \geq 0$, while $\tilde{E}_{-n} = D(\tilde{E}_n) = F(\tilde{E}_n, S^0)$ is its function dual. Splicing the skeleton filtration of $\widetilde{EG}$ with its function dual gives the finite terms in the *Greenlees filtration* of G:

$$D(\widetilde{EG}) \to \ldots \to \tilde{E}_{-1} \to \tilde{E}_0 = S^0 \to \tilde{E}_1 \to \ldots \to \widetilde{EG}. \tag{1.5.2}$$

The successive cofibers of 1.5.2 are

$$\tilde{E}_n/\tilde{E}_{n-1} = G_+ \wedge (\vee S^n). \tag{1.5.3}$$

So, applying homology to the filtration yields a spectral sequence $E^1_{s,t} = H_{s+t}(\tilde{E}_s/\tilde{E}_{s-1})$ that is concentrated on the horizontal axis. Since both $\widetilde{EG} \simeq \operatorname{hocolim}_n \tilde{E}_n$ and $D(\widetilde{EG}) \simeq \operatorname{holim}_n \tilde{E}_n$ are non-equivariantly contractible, this spectral sequence collapses at the E^2-term, giving us a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_2(\tilde{E}_2/\tilde{E}_1) \arrow[r, "d"] & H_2(\tilde{E}_1/\tilde{E}_0) \arrow[r, "d"] \arrow[d] & H_2(\tilde{E}_0/\tilde{E}_{-1}) \arrow[r, "d"] & \cdots \\
& & H_0(S) \arrow[ur, hook] & &
\end{tikzcd}$$

of finitely generated free $\mathbb{F}_p[G]$-modules. Letting $P_n = H_{n+1}(\tilde{E}_{n+1}/\tilde{E}_n)$ yields a complete resolution (P_*, d_*) of $\mathbb{F}_p = H_0(S^0)$.

We are also interested in the non-finite, groups of units in $\mathbb{C}$ and $\mathbb{H}$. Let K be one of the fields $\mathbb{C}$ or $\mathbb{H}$ and let $k = \dim K$. Let $G = S(K)$ be the group of units in K and let S^K be the one point compactification of K thought of as the unreduced suspension of $S(K)$. Given a G-spectrum X, the *Greenlees filtration* of $\widetilde{EG} = S^{K^\infty}$ is defined as

$$\tilde{E}_{ks} = \tilde{E}_{ks+1} = \ldots = \tilde{E}_{k(s+1)-1} = S^{sK}, \tag{1.5.4}$$

with maps $i_{ks} : \tilde{E}_{ks-1} \to \tilde{E}_{ks}$ equal to the natural inclusion $S^{(s-1)K} \to S^{sK}$.

We now construct the Tate spectral sequence. Let G be a finite group, $S(\mathbb{C})$ or $S(\mathbb{H})$, and let X be an orthogonal G-spectrum. Smashing the cofiber sequence $\tilde{E}_{s-1} \to \tilde{E}_s \to \tilde{E}_s/\tilde{E}_{s-1}$, coming from the Greenless filtration, with $F(EG_+, X)$ and taking G-fixed points, yields the cofiber sequence

$$[\tilde{E}_{s-1} \wedge F(EG_+, X)]^G \to [\tilde{E}_s \wedge F(EG_+, X)]^G \to [\tilde{E}_s/\tilde{E}_{s-1} \wedge F(EG_+, X)]^G.$$

Theorem IV 2.11 in [MM02] and Corollary II 1.8 in [LMSM86] combines to give a non-equivariant equivalence $i_* i^*(X) \to X$, and the map collapsing EG to a point gives a non-equivariant equivalence $F(S^0, X) \to F(EG_+, X)$. Hence we get an equivalence

$$[\tilde{E}_s/\tilde{E}_{s-1} \wedge F(EG_+, X)]^G \simeq [\tilde{E}_s/\tilde{E}_{s-1} \wedge X]^G \simeq [\tilde{E}_s/\tilde{E}_{s-1} \wedge i_* i^*(X)]^G$$

Since $\tilde{E}_s/\tilde{E}_{s-1}$ is G-free, the Adams equivalence, equation VI 4.6 in [MM02] gives an equivalence

$$[\tilde{E}_s/\tilde{E}_{s-1} \wedge i_* i^*(X)]^G \simeq (\Sigma^{\operatorname{ad} G} \tilde{E}_s/\tilde{E}_{s-1} \wedge i^*(X))/G.$$

The Greenless filtration yields a filtration

$$* \longrightarrow \cdots \longrightarrow [\tilde{E}_{s-1} \wedge F(EG_+, X)]^G \longrightarrow [\tilde{E}_s \wedge F(EG_+, X)]^G \longrightarrow \cdots \longrightarrow X^{tG},$$

where the identification of the homotopy (co)limits follows from Lemma 4.4 in [LNR12].

Applying homotopy to this sequence gives rise to an unrolled exact couple of graded groups

$$\begin{tikzcd}
\dots \arrow[r] & A_{s-1} \arrow[r] & A_s \arrow[r] \arrow[d] & A_{s+1} \arrow[r] \arrow[d] & \dots \\
& & \hat{E}^1_{s-1} \arrow[ul, dashed] & \hat{E}^1_s \arrow[ul, dashed] &
\end{tikzcd}$$

where $A_{s,t} = \pi_{s+t}([\tilde{E}_{s-1} \wedge F(EG_+, X)]^G)$ and $\hat{E}^1_{s,t} = \pi_{s+t}(\Sigma^{ad(G)} \tilde{E}_s/\tilde{E}_{s-1} \wedge i^*(X))/G)$. The dotted line is a degree 1 homomorphism.

More general versions of the spectral sequences in the next proposition can be found in Theorem 10.3 in [GM95]. Furthermore, Theorem 10.5 and 10.6 in [GM95] proves the claim about the multiplicative property of the spectral sequences involved.

Proposition 1.5.5. *Let G be a finite group, and let X be a G spectrum. Assume that X is bounded below and with finite homotopy groups in each degree. Let M be the sphere spectrum or $V(0)$ the mod p Moore spectrum. Then there are strongly convergent spectral sequences*

$$\begin{aligned}
\hat{E}^2_{s,t} &\cong \hat{H}^{-s}(G; M_t(X)) \Rightarrow M_{s+t}(X^{tG}) && \textit{Tate spectral sequence} \\
E^2_{s,t} &\cong H^{-s}(G; M_t(X)) \Rightarrow M_{s+t}(X^{hG}) && \textit{homotopy fixed point spectral sequence} \\
E^2_{s,t} &\cong H_s(G; M_t(X)) \Rightarrow M_{s+t}(X_{hG}) && \textit{homotopy orbit spectral sequence,}
\end{aligned}$$

where the first come from the Greenlees filtration, and the second and third come from the skeleton filtration of EG.

If X is a G-ring spectrum, the first two are $M_0(X)$-algebra spectral sequences.

The restriction map $R^h : X^{hG} \to X^{tG}$ induces the standard homomorphism

$$\hat{H}^{-s}(G; M_t(X)) \to H^{-s}(G; M_t(X))$$

which is an isomorphism for $s \leq -1$ (see Section 4 in [Bro82]).

Proof. We only do the argument for the Tate spectral sequence. We do the proof for homotopy groups, but all the arguments work equally well for $V(0)$, since it is a finite spectrum.

When G is finite, the adjoint representation is trivial, so equation 1.5.3 lets us rewrite the $\hat{E}^1$ term as

$$\hat{E}^1_{s,t} \cong H_s(\tilde{E}_s/\tilde{E}_{s-1}) \otimes_{\mathbb{F}_pG} \pi_t(X) = P_{s-1} \otimes_{\mathbb{F}_pG} \pi_t(X),$$

with the d^1 differential being induced by the differential in the complete resolution (P_*, d_*). Thus,

$$\hat{E}^2_{s,t} \cong \hat{H}_{s-1}(G; \pi_t(X)) \cong \hat{H}^{-s}(G; \pi_t(X)),$$

where $\hat{H}$ is Tate-(co)homology.

To show that this spectral sequence converges conditionally we must show that $\lim A_s = 0$ and $\operatorname{Rlim}_s A_s = 0$. We have, by Theorem IX.3.1 in [BK72], an exact sequence

$$0 \longrightarrow \operatorname{Rlim}_s A_{s,*+1} \longrightarrow \pi_*(\operatorname{holim}_s[\tilde{E}_s \wedge F(EG_+, X)]^G) \longrightarrow \lim_s A_{s,*} \longrightarrow 0\,.$$

This exact sequence also holds for $V(0)_*$ since it is a finite spectrum.

Since $\operatorname{holim}_s[\tilde{E}_{s-1} \wedge F(EG_+, X)]^G$ is contractible $\pi_*(\operatorname{holim}_s[\tilde{E}_{s-1} \wedge F(EG_+, X)]^G) = 0$ so both $\operatorname{Rlim}_s A_s$ and $\lim_s A_s$ are zero. The Tate spectral sequence is concentrated in the upper half plane, so by Theorem A.3.5 the spectral sequence converges strongly when $\operatorname{Rlim}_r E^r = 0$, which is the case since $\hat{E}^2$ is finite in each bi-degree. That the Tate spectral sequence is an algebra spectral sequence when X is a G-ring spectrum, follows from Proposition 4.3.5 in [HM97b] or from [GM95].

For the homotopy fixed point spectral sequence, homotopy orbit spectral sequence see Theorem 10.3-10.6 in [GM95], and the claim about the restriction homomorphism R^h, is proven in Section 2 in [BM94]. □

By Lemma 2.12 in [BM94], the spectral sequence we get from the skeleton filtration of EG is isomorphic to the spectral sequence we get from the negative part of the Greenlees filtration

$$\ldots \to [\tilde{E}_{-1} \wedge F(EG_+, X)]^G \to [\tilde{E}_0 \wedge F(EG_+, X)]^G = F(EG_+, X)^G = X^{hG}.$$

The next proposition also appears in [GM95] as Theorem 14.2 and 14.9 in combination with Theorem 10.3

Proposition 1.5.6. *Let K be one of the fields $\mathbb{C}$ or $\mathbb{H}$, let G be the group $S(K)$ and let $k = \dim K$.*

Assume that X is a G-spectrum that is bounded below and with finite homotopy groups in each degree. The Greenlees filtration induces a strongly convergent Tate spectral sequence

$$\hat{E}^2_{s,t} \cong P(t, t^{-1}) \otimes \pi_t(X) \Rightarrow \pi_{s+t}(X^{tG}).$$

where $|t| = (-k, 0)$.

Proof. We have to identify the E^2-term. The cofiber $\tilde{E}_1/\tilde{E}_0$ may be identified with ΣG_+. In general, for $k = \dim K$, $i_{ks} = id_{S^{(s-1)K}} \wedge i_k$ so the cofiber $\tilde{E}_{ks}/\tilde{E}_{ks-1}$ may be identified with $\Sigma G_+ \wedge S^{(s-1)K}$. Thus, the action map induces an isomorphism

$$\Sigma^{\mathrm{ad}(G)} G_+ \wedge_G i^*(S^{(s-1)K} \wedge X) \cong |\Sigma^k S^{(s-1)K} \wedge X| = \Sigma^{ks}|X|.$$

Here $|X|$ denotes the underlying spectra of the G-spectra $i^*(X)$. Hence the $\hat{E}^2$-term is equal to $P(u, u^{-1}) \otimes \pi_*(X)$. To show that the spectral sequence converges strongly, we can use the same argument as in the finite case. □

Now we want to construct the homotopy fixed points spectral sequence for the group T^n, the n-fold torus. We use the setup in [BR05]. Let the unit sphere $S(\mathbb{C}^\infty)$ be our model for ES^1 with the S^1-action given by the coordinatewise action. The space $S(\mathbb{C}^\infty)$ is equipped with a free S^1-CW structure with one free S^1-cell in each even degree, and the $2k$-skeleton is the odd sphere $S(\mathbb{C}^{2k+1})$. The $2k$-skeleton is attained from the $2k-2$-skeleton $S(\mathbb{C}^{2k-1})$ by attaching a cell $S^1 \times D^{2k}$ via the T^n-action map

$$S^1 \times \partial D^{2k} \to S(\mathbb{C}^k).$$

We use the product $S(\mathbb{C}^\infty)^n$ as a model for ET^n with the product T^n-CW structure. Thus the $2k$ cells in $S(\mathbb{C}^\infty)^n$ are $T^n \times D^{2k_1} \times \ldots \times D^{2k_n}$, where $k_1 + \ldots + k_n = k$. In particular there are $2n$ number of 2-cells in ET^n, and they are attached by the T^n-equivariant extension of the inclusion $S^1 \to T^n$ of the i-the circle.

We now get a T^n-equivariant filtration

$$\emptyset \subseteq E_0T^n \subseteq \ldots \subseteq E_{2k-2}T^n \subseteq E_{2k}T^n \subseteq \ldots$$

with colimit ET^n, and T^n-equivariant cofiber sequences

$$E_{2k-2}T^n \to E_{2k}T^n \to T^n_+ \wedge (\vee S^{2k})$$

where the wedge sum runs over all $2k$-cells in ET^n. Here T^n acts trivially on the space $(\vee S^{2k})$.

Proposition 1.5.7. *Let X be a bounded below T^n-spectrum with finite homotopy groups in each degree. The skeleton filtration of ET^n induces a strongly convergent homotopy fixed point spectral sequence*

$$E^2_{s,t} \cong P(t_1, \ldots, t_n) \otimes \pi_t(X) \Rightarrow \pi_{s+t}(X^{tG}).$$

where $|t_i| = (-2, 0)$. Let M be any homology theory. When restricted to the 2-skeleton of ET^n there is a strongly convergent spectral sequence

$$E^2_{s,t} \cong \mathbb{Z}\{1, t_1, \ldots, t_n\} \otimes M_t(X) \Rightarrow M_{s+t}(F(E_2T^n_+, X)^{T^n}).$$

When $n = 1$ the restriction map R^h induces the inclusion map $P(t) \otimes \pi_t(X) \to P(t, t^{-1}) \otimes \pi_t(X)$ on spectral sequences from the homotopy fixed points spectral sequence, to the Tate spectral sequence.

Proof. Applying $F(-,X)^{T^n}$ to the skeleton filtration of ET^n_+ yields a filtration

$$\ldots \to F(E_kT^n_+,X)^{T^n} \to F(E_{k-1}T^n_+,X)^{T^n} \to \ldots \to F(E_0T^n_+,X)^{T^n} \to \{\mathrm{pt}\},$$

with the homotopy fixed point spectrum as its limit

$$X^{hT^n} = \lim_k F(E_kT^n_+,X)^{T^n}.$$

From the cofiber sequences

$$E_{k-1}T^n \to E_kT^n \to E_kT^n/E_{k-1}T^n$$

we get cofiber sequences of spectra

$$F(E_kT^n/E_{k-1}T^n,X)^{T^n} \to F(E_kT^n_+,X)^{T^n} \to F(E_{k-1}T^n_+,X)^{T^n}.$$

When k is odd the last map is an equality and $F(E_kT^n/E_{k-1}T^n,X)^{T^n}$ is contractible. When k is even $E_kT^n/E_{k-1}T^n \cong T^n \wedge \vee S^k$, and since T^n acts freely on $T^n_+ \wedge (\vee S^k)$, there are equivalences of spectra $F(T^n_+ \wedge (\vee S^k),X)^{T^n} \simeq F(\vee S^k, X) \simeq \vee \Sigma^{-k} X$.

We define an unrolled exact sequence

$$\begin{tikzcd}
\ldots \arrow[r] & A_{2s-1} \arrow[r, "="] & A_{2s} \arrow[r] \arrow[dl, dashed] & A_{2s+1} \arrow[r, "="] \arrow[dl, dashed] & \ldots \arrow[r, "="] & A_0 \\
& E^1_{2s} \arrow[u] & E^1_{2s} \arrow[u] & & & E^1_0 \arrow[u]
\end{tikzcd}$$

by $A_{s,t} = \pi_{s+t}(F(E_{-s}T^n_+,X)^{T^n})$ and $E^1_{s,t} = \pi_{s+t}(F(E_{-s}T^n/E_{-s+1}T^n,X)^{T^n})$The dotted arrow is a degree 1 homomorphism, and when s is odd or $s > 1$ then $E^1_{s,t} = 0$, and when s is even and non-positive, then $E^1_{s,t} = \pi_{s+t}(F(E_{-s}T^n/E_{-s+1}T^n,X)^{T^n}) \cong \pi_{s+t}(\vee\Sigma^s X) \cong \pi_t(\vee X)$.

This spectral sequence converges conditionally to the limit $\lim_s A_s = \pi_*(X^{hT^n})$ since $\operatorname{colim}_s A_s = 0$. By Theorem A.3.5 it converges strongly since $\mathrm{Rlim}_r E^r = 0$.

By Theorem 9.8 in [GM95] There is an isomorphism $E^2 \cong H^{-*}(BT^n, \pi_*(X)) \cong P(t_1,\ldots t_n) \bigotimes \pi_*(X)$, where the last isomorphism follows from the fact that the action of T^n on $\pi_*(X)$ is trivial.

That the spectral sequence for the 2-skeleton is as desribed is clear since ET^n only contain even degree cells. Convergence is not a problem since it is concentrated in two columns. $\square$

Chapter 2

Homotopy Groups of C_2-fixed Points of $\Lambda_{S^2} H\mathbb{F}_2$

In this chapter we will use the isotropy separation diagram for the spectrum $\Lambda_{S^2} H\mathbb{F}_2$ to calculate the homotopy groups $\pi_*((\Lambda_{S^2} H\mathbb{F}_2)^{C_2})$, where the non-trivial action of C_2, the cyclic group with two elements, is induced by the antipodal map on S^2. In Section 2.2 we construct a family of non-zero differentials in the Tate spectral sequence calculating $V(0)_*((\Lambda_{S^2} H\mathbb{F}_2)^{tC_2})$. In Section 2.3 we use the bar spectral sequence to calculate $\pi_*(\Lambda_{S^2} H\mathbb{F}_2)$, and from this we calculate the homotopy groups of $\Lambda_{\mathbb{R}P^2} H\mathbb{F}_2$, which is one of the spectra in the isotropy separation diagram. In the last section this enables us to determine all the entries, except for the actual fixed points, in the isotropy separation diagram, and the various maps connecting them, and in turn this determines $\pi_*((\Lambda_{S^2} H\mathbb{F}_2)^{C_2})$.

2.1 A Differential in the S^1 and S^3 Tate Spectral Sequence

In this section we will show that given a G-spectrum X, where G is S^1 or S^3, the first possible non-zero differential in the Tate spectral sequence converging to $\pi_*(X^{tG})$ is given by the action of G. The argument is a generalization to the field of quaternions $\mathbb{H}$ of an argument given for the complex plane $\mathbb{C}$ in [Hes96]. Originally we needed the result in this section to calculate $\pi_*((\Lambda_{S^2} H\mathbb{F}_2)^{tC_2})$, but we have changed that calculation. We choose to include it anyway, since it has independent interest.

Let K be one of the fields $\mathbb{C}$ or $\mathbb{H}$ and let $k = \dim K$. Let $G = S(K)$ be the group of units in K and let S^K be the one point compactification of K thought of as the unreduced suspension of $S(K)$. These spaces fits into a cofiber sequence $G_+ \to S^0 \to S^K$. For dimension reasons the last map is zero in homotopy, so the induced long exact sequence of stable homotopy groups splits into short exact sequences. From the equivalence $S^K \simeq \Sigma G$ this gives a splitting

$$\pi_* G_+ \cong \pi_* G \oplus \pi_* S^0. \tag{2.1.1}$$

Let η and ν be the respective Hopf maps $S^{2k-1} \to S^k$. Let $[G]$ and η_K in $\pi_{k-1}G_+$ denote the elements which reduces to $(\mathrm{id}, 0)$, and $(0, \eta)$ or $(0, \nu)$, respectively. Recall that $\operatorname{ad} G$ is the adjoint representation of G.

Proposition 2.1.2. *In the Tate spectral sequence*

$$\hat{E}^2 \cong P(u, u^{-1}) \otimes \pi_*(X) \Rightarrow \pi_*(X^{tG}),$$

where $|u| = (0, k)$, the first potentially nonzero differential $d^k_{ks,t}; \hat{E}^k_{ks,t} \to \hat{E}^k_{k(s-1),t+k-1}$, is given by the composite

$$\pi_t X \xrightarrow{[G]+n\eta_K} \pi_{t+k-1}(G_+ \wedge i^* X) \cong \pi_{t+2k-2}(\Sigma^{\operatorname{ad} G} G_+ \wedge i^* X) \xrightarrow{\mu} \pi_{t+2k-2}\, \Sigma^{\operatorname{ad} G} X \cong \pi_{t+k-1} X,$$

where the first map is exterior multiplication and the second map μ is induced by the diagonal action of G_+ on $\Sigma^{\operatorname{ad} G}$ and X.

When $G = S^1$, the adjoint representation is trivial since S^1 is abelian. When X is an $H\mathbb{F}_p$-module, it is equivalent as a spectrum to a wedge of suspensions of $H\mathbb{F}_p$, so η_K acts trivially on X, since it acts trivially on each $H\mathbb{F}_p$ summand for dimension reasons. Thus when X is an $H\mathbb{F}_p$ module and $G = S^1$, the differential is just exterior multiplication by $[S^1]$ followed by the action map on X. This is the case we are interested in.

Proof. The identification of the $\hat{E}^2$-term is done in Proposition 1.5.6 as follows: In the Greenlees filtration of $\widetilde{EG}$ in 1.5.4, the cofiber of $i_{ks} : \tilde{E}_{ks-1} \to \tilde{E}_{ks}$ is $\Sigma G_+ \wedge S^{(s-1)K}$, and we have the Adams equivalence

$$\sigma : \Sigma\Sigma^{\operatorname{ad} G} G_+ \wedge_G i^*(S^{(s-1)K} \wedge X) \to i^*(\Sigma G_+ \wedge S^{(s-1)K} \wedge X)^G,$$

and the action map of G gives an isomorphism

$$\Sigma^{\operatorname{ad}(G)} G_+ \wedge_G i^*(S^{(s-1)K} \wedge X) \cong |\Sigma^k S^{(s-1)K} \wedge X| = \Sigma^{ks}|X|.$$

After desuspending once these maps are the vertical maps in the diagram

$$\begin{tikzcd}
{[G_+ \wedge i_* i^*(S^{(s-1)K} \wedge X)]^G} \arrow[r, "\partial"] & {[i_* i^*(S^{(s-1)K} \wedge X)]^G} \arrow[r, "j"] & {[\Sigma G_+ \wedge i_* i^*(S^{(s-2)K} \wedge X)]^G} \\
\Sigma^{\operatorname{ad} G} G_+ \wedge_G S^{(s-1)K} \wedge i^* X \arrow[u, "\simeq"] \arrow[r, "\partial"] \arrow[dd, bend right=60, "\tilde{\mu}", "\cong"'] & (\Sigma^{\operatorname{ad} G} S^{(s-1)K} \wedge i^* X)/G \arrow[r, "j"] & \Sigma\Sigma^{\operatorname{ad} G} G_+ \wedge_G S^{(s-2)K} \wedge i^* X \arrow[u, "\simeq"] \arrow[dd, bend left=60, "\tilde{\mu}"', "\cong"] \\
\Sigma^{\operatorname{ad} G} G_+ \wedge S^{(s-1)K} \wedge i^* X \arrow[u, "\mathrm{pr}"] \arrow[r, "\partial"] & \Sigma^{\operatorname{ad} G} S^{(s-1)K} \wedge i^* X \arrow[r, "j"] & \Sigma\Sigma^{\operatorname{ad} G} G_+ \wedge S^{(s-2)K} \wedge i^* X \arrow[u, "\mathrm{pr}"'] \\
{|\Sigma^{\operatorname{ad} G} S^{(s-1)K} \wedge X|} \arrow[u, "\iota"] & & {\Sigma|\Sigma^{\operatorname{ad} G} S^{(s-2)K} \wedge X|.} \arrow[u, "\iota"']
\end{tikzcd}$$

The map ι is induced by the unit map $S^0 \to G_+$, and ∂ and j comes from the cofiber sequence induced by the Greenlees filtration of $\widetilde{EG}$.

After applying homotopy π_{ks+t} to this diagram, the left hand side is equal to $E^2_{ks,t}$, the right hand side is $E^2_{k(s-1),t+k-1}$, and the differential $d^k_{ks,t}$ is the composition across the top of this diagram, which due to the equivalences is equal to the composition of maps from the lower left hand corner to the lower right hand corner. The composition $\partial\iota$ is equal to the identity and since the homomorphism j is defined by the cofiber sequence giving the splitting in equation 2.1.1, it represents exterior multiplication with $[G]$.

Hence, it suffices to look at the composition $\tilde{\mu} \circ \mathrm{pr}$. If one ignores suspension, this composition is equal to

$$\begin{aligned} &\Sigma^{\mathrm{ad}\,G} G_+ \wedge S^{(s-2)K} \wedge i^*X \xrightarrow{\xi_{s-2}\wedge \mathrm{id}} \Sigma^{\mathrm{ad}\,G} S^{(s-2)K} \wedge G_+ \wedge i^*X \xrightarrow{\Delta\wedge\mathrm{id}} \\ &\Sigma^{\mathrm{ad}\,G} G_+ \wedge G_+ \wedge S^{(s-2)K} \wedge i^*X \xrightarrow{\mathrm{id}\,\tau\wedge\mathrm{id}} \Sigma^{\mathrm{ad}\,G} G_+ \wedge S^{(s-2)K} \wedge G_+ \wedge i^*X \xrightarrow{\mu\wedge\mathrm{id}\wedge\mu} \\ &\Sigma^{\mathrm{ad}\,G} S^{(s-2)K} \wedge i^*X, \end{aligned}$$

where τ interchanges G_+ and $S^{(s-2)K}$, Δ is the diagonal map, and ξ_{s-2} is defined as the composition

$$\xi_{s-2} : G_+ \wedge S^{(s-2)K} \xrightarrow{\Delta\wedge\mathrm{id}} G_+ \wedge G_+ \wedge S^{(s-2)K} \xrightarrow{\mathrm{id}\wedge\mu} G_+ \wedge S^{(s-2)K}.$$

We will prove that under the isomorphism in the splitting 2.1.1

$$\xi_{s-2} = \begin{pmatrix} 1 & 0 \\ (s-2)\eta & 1 \end{pmatrix} \qquad\qquad \xi_{s-2} = \begin{pmatrix} 1 & 0 \\ (s-2)\nu & 1 \end{pmatrix}$$

when G is equal to S^1 and S^3 respectively, and the matrix multiplies from the right. Every entry except the one in the lower left corner is clear from the expression of ξ_{s-2}

To understand ξ_{s-2} it suffices to consider the case $s = 3$ since the case $s > 3$ follows by composition and $s < 3$ follows by smashing with S^{NK} for large N as seen by the following argument: For $n > 1$, ξ_n is equal to the n-th iterated composition of $\xi_1 \wedge S^{(n-1)K}$ with shuffle maps inserted so that G_+ acts once on every S^K factor. When $n \leq 0$ we choose N such that $N + n > 1$ and get, when ignoring the required shuffle maps, that $(\xi_N \wedge S^{nK}) \circ (\xi_n \wedge S^{NK}) = \xi_{N+n}$. Now the matrix for ξ_n is clear since the matrices for ξ_N and ξ_{N+n} are known.

We will now identify the lower left entry in the matrix for ξ_1.

Let $R \subseteq K$ be the ray from the origin given by the non-negative part of the first coordinate axis in K, and let the the intersection of $\{0\} \times R$ and $S(K \times K)$ be the basepoint in $S(K \times K)$. The map j that defines the splitting in 2.1.1 fits into the commutative diagram

$$\begin{tikzcd}
 & & \Sigma G_+ \wedge S^K \\
S^{K\times K} \arrow[r, "\simeq"] \arrow[rru, bend left, "j"] \arrow[d, equal] & \Sigma S(K\times K) \arrow[r, "\Sigma\,\mathrm{pr}"] \arrow[d, equal] & \Sigma S(K\times K)/(0\times S(K)) \arrow[u, "\Sigma f"'] \arrow[d, equal] \\
(K\times K)^c \arrow[r, "\simeq"] & ((K\times K)/(0\times R))^c \arrow[r, "\mathrm{pr}"] & ((K\times K)/(0\times K))^c,
\end{tikzcd} \tag{2.1.3}$$

where the superscript c indicates compactification, pr is the projection map, and f is the weak equivalence given by $(z, w) \mapsto (\frac{z}{|z|}, \frac{w}{|z|})$, with $z = 0$ being mapped to the basepoint.

The lower left entry of ξ_1 is the composite of the left, bottom and right map in the following commutative diagram

$$\begin{tikzcd}
S(K\times K) \arrow[rrr, "\eta_K"] \arrow[d, "f\circ\mathrm{pr}"] & & & S^K \\
G_+\wedge S^K \arrow[rrru, bend left, "\mu"] \arrow[r, "\Delta\wedge\mathrm{id}"] & G_+\wedge G_+\wedge S^K \arrow[rr, "\mathrm{id}\wedge\mu"] & & G_+\wedge S^K, \arrow[u, "\mathrm{pr}"']
\end{tikzcd}$$

where the top map is identified as the respective Hopf map η or ν by diagram 2.1.3. □

2.2 A Differential in the Spectral Sequence Calculating $V(0)_*((\Lambda_{S^n}H\mathbb{F}_2)^{tC_2})$

In this section we use the map $\omega : S^n_+ \wedge H\mathbb{F}_2 \to \Lambda_{S^n}H\mathbb{F}_2$ to find a family of non-zero differential in the Tate spectral sequence calculating $V(0)_*((\Lambda_{S^n}H\mathbb{F}_2)^{tC_2})$, where the C_2-action is given by the antipodal action on S^n.

Proposition 2.2.1. *For $n \geq 1$, there is an $\mathbb{F}_2$-module isomorphism*

$$\pi_*(\Lambda_{S^n}H\mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2\{z_n\} \oplus A,$$

where $|z_n| = n+1$ and A is some $\mathbb{F}_2$-module which is zero in degree less than $n+2$.

When $n \geq 2$, z_n is equal to $\widehat{\sigma}(z_{n-1})$.

Proof. Use induction on n. By proposition A.4.6, $\pi_*(\Lambda_{S^1}H\mathbb{F}_2) \cong P(\mu)$ with $|\mu| = 2$. Assume we have proved it for $m \leq n$. The bar spectral sequence in Corollary 1.2.2 coming from applying the functor $\Lambda_- H\mathbb{F}_2$ to the pushout

$$\begin{tikzcd}
S^n \arrow[r] \arrow[d] & D^{n+1} \arrow[d] \\
D^{n+1} \arrow[r] & S^{n+1},
\end{tikzcd}$$

begins

$$E^1(S^{n+1}) = B(\mathbb{F}_2, \pi_*(\Lambda_{S^n}H\mathbb{F}_2)\mathbb{F}_2).$$

Furthermore, $E^1_{0,*} \cong \mathbb{F}_2$, and when $s > 0$ we have

$$E^1_{s,t} \cong \begin{cases} \mathbb{F}_2 & t = 0, n \\ 0 & 0 < t < n. \end{cases}$$

From this we can read off $\pi_*(\Lambda_{S^n}H\mathbb{F}_2)$ in degree less than or equal to $n+1$. That $\pi_{n+1}(\Lambda_{S^n}H\mathbb{F}_2)$ is generated by the image of $\widehat{\sigma}$ follows from Proposition 1.2.5. □

Proposition 2.2.2. *Let $n \geq 1$. There are Tate spectral sequences*

$$\hat{E}^2(n) = P(u, u^{-1}) \otimes E(x_n) \Rightarrow \pi_*(S^n_+ \wedge H\mathbb{F}_2)^{tC_2}$$
$$\hat{E}^2(V(0), n) = P(u, u^{-1}) \otimes E(\overline{\tau}_0) \otimes E(x_n) \Rightarrow V(0)_*(S^n_+ \wedge H\mathbb{F}_2)^{tC_2},$$

where $|u| = (-1, 0)$, $|x_n| = (0, n)$ and $|\overline{\tau}_0| = (0, 1)$. The differentials are given by

$$d^{n+1}(u^i) = u^{i+n+1}x_n \quad \text{and} \quad d^{n+1}(\overline{\tau}_0 u^i) = \overline{\tau}_0 u^{i+n+1}x_n$$

for all $i \in \mathbb{Z}$.

Proof. We identify the E^2-terms using Proposition 1.5.5, and the fact that the C_2 action on $\overline{\tau}_0$ is trivial. Hence the Tate cohomology is as above. The spectral sequences are not multiplicative spectral sequences, since $S^n_+ \wedge H\mathbb{F}_2$ is not an equivariant ring spectrum.

By Proposition 2.4 and Theorem 5.6 in [GM95], $\pi_*((S^n_+ \wedge H\mathbb{F}_2)^{tC_2}) = 0$, since the action of C_2 on S^n and thus on $S^n_+ \wedge H\mathbb{F}_2$, is free. Since the spectral sequence $\hat{E}^2(n)$ is concentrated in vertical degree 0 and $n-1$, the above pattern of differentials is the only one possible in this case.

We prove the proposition for $\hat{E}^2(V(0), n)$ by induction.

The inclusion $E^2(1) \to E^2(V(0), 1)$ determines the differentials $d^2(u^i) = u^{i+2}x_1$ in $E^2(V(0), 1)$. Since $V(0)_*((S^1_+ \wedge H\mathbb{F}_2)^{tC_2} = 0$, the other differentials must be as stated in the proposition.

For dimension reasons $E^2(V(0), n) = E^{n-1}(V(0), n)$. If we have proved the case $n-1$, then $E^2(V(0), n-1) = E^{n-1}(V(0), n-1)$, and the map $E^{n-1}(V(0), n-1) \to E^{n-1}(V(0), n)$ is an isomorphism in vertical degree 0 and 1, and zero elsewhere. Hence, $d^{n-1}(\overline{\tau}_0 u^i) = 0$ in $E^{n-1}(V(0), n)$, since $d^{n-1}(\overline{\tau}_0 u^i) = \overline{\tau}_0 u^{i+n}x_{n-1}$ in $E^{n-1}(V(0), n-1)$, and $\overline{\tau}_0 u^{i+n}x_{n-1}$ is mapped to zero in $E^{n-1}(V(0), n)$. From the inclusion $E^2(n) \to E^2(V(0), n)$ there are differentials $d^{n+1}(u^i) = u^{i+n+1}x_n$ in $E^2(V(0), n)$, and since $V(0)_*((S^n_+ \wedge H\mathbb{F}_2)^{tC_2}) = 0$ the other differentials must be as stated in the proposition. □

Corollary 2.2.3. *Let $n \geq 1$. The Tate spectral sequence*

$$E^2 = P(u, u^{-1}) \otimes E(\overline{\tau}_0) \otimes \pi_*(\Lambda_{S^n}H\mathbb{F}_2) \Rightarrow V(0)_*(\Lambda_{S^n}H\mathbb{F}_2)^{tC_2}$$

has differentials generated by $d^{n+1}(\overline{\tau}_0) = u^{n+1}z_n$ where z_n is the non-zero element in $V(0)_{n+1}(\Lambda_{S^n}H\mathbb{F}_2) \cong \mathbb{F}_2$.

Proof. The natural map

$$\omega : S^n_+ \wedge H\mathbb{F}_2 \to \Lambda_{S^n}H\mathbb{F}_2$$

is an equivariant map, so it induces a map of Tate spectral sequences. It follows from Corollary A.4.7, see Theorem 5.2 in [HM97b], that μ in $V(0)_2(\Lambda_{S^1}H\mathbb{F}_2)$ is the image of $[S^1]\otimes\bar{\xi}_1$ under the map $\omega_* : V(0)_*(S^1_+ \wedge H\mathbb{F}_2) \cong H_*(S^1_+)\otimes V(0)_*(H\mathbb{F}_2) \to V(0)_*(\Lambda_{S^1}H\mathbb{F}_2)$. There is a commutative diagram

$$\begin{tikzcd}
(S^1_+)^{\wedge n} \wedge H\mathbb{F}_2 \arrow[rrr] \arrow[d, "\omega"] & & & S^n_+ \wedge H\mathbb{F}_2 \arrow[d, "\omega"] \\
(S^1_+)^{\wedge n-1} \wedge \Lambda_{S^1}H\mathbb{F}_2 \arrow[r, "\widehat{\omega}"] & (S^1_+)^{\wedge n-2} \wedge \Lambda_{S^2}H\mathbb{F}_2 \arrow[r, "\widehat{\omega}"] & \dots \arrow[r, "\widehat{\omega}"] & \Lambda_{S^n}H\mathbb{F}_2
\end{tikzcd}$$

where the top horizontal map is induced by the quotient map $T^n \to T^n/T^n_{n-1} \cong S^n$. By Proposition 2.2.1, $z_n \in V(0)_*(\Lambda_{S^n}H\mathbb{F}_2)$ is the image of $[S^1]^{\otimes n}\otimes\bar{\xi}_1$ under the composition of the left and bottom maps in this diagram. Hence, z_n is the image of $[S^n]\otimes\bar{\xi}_1$ under the map $\omega_* : V(0)_*(S^n_+ \wedge H\mathbb{F}_2) \to V(0)_*(\Lambda_{S^n}H\mathbb{F}_2)$.

On the level of Tate spectral sequences ω thus induces a map which is an inclusion except in vertical degree $n-1$, so we can read off the differentials in the corollary from the differentials in Proposition 2.2.2. $\square$

The above statements can also be made for odd primes and odd dimensional spheres with some adjustments.

2.3 Calculating the Homotopy Groups of $\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2$

In this section we will use the bar spectral sequence to calculate $\pi_*(\Lambda_{S^2}H\mathbb{F}_2)$ and use this to show that there is an equivalence $\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2 \simeq \Lambda_{S^1}H\mathbb{F}_2 \wedge_{H\mathbb{F}_2} \Lambda_{S^2}H\mathbb{F}_2$.

Proposition 2.3.1. *There is an isomorphism of $\mathbb{F}_2$-Hopf algebras*

$$\pi_*(\Lambda_{S^2}H\mathbb{F}_2) \cong E(\beta),$$

where $|\beta| = 3$.

Proof. From Corollary A.4.7 there is an isomorphism $\pi_*(\Lambda_{S^1}H\mathbb{F}_2) \cong P(\mu)$ where $|\mu| = 2$. The pushout

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \\
D^2 \arrow[r] & S^2
\end{tikzcd}$$

yields by Corollary 1.2.2 a bar spectral sequence

$$E^2 = \mathrm{Tor}^{\pi_*(\Lambda_{S^1}H\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2) \cong \mathrm{Tor}^{P(\mu)}(\mathbb{F}_2,\mathbb{F}_2) \cong E(\beta) \Rightarrow \pi_*(\Lambda_{S^2}H\mathbb{F}_2),$$

where the identification of the E^2-term follows from Proposition A.2.10. Now, there are no room for any differentials or (co)multiplicative (co)extensions, since the spectral sequence is concentrated in bidegrees $(0, 0)$ and $(1, 2)$. $\square$

To calculate the homotopy groups of $\Lambda_{\mathbb{R}P^2} H\mathbb{F}_2$ we need to know which $P(\mu)$-module structure the attaching map $S^1 \xrightarrow{\cdot 2} S^1$ in the standard CW-structure of $\mathbb{R}P^2$ induces on $\pi_*(\Lambda_{S^1} H\mathbb{F}_2)$. Recall that by Proposition 1.1.5, $\pi_*(\Lambda_{S^n} H\mathbb{F}_2)$ is an $H\mathbb{F}_2$-Hopf algebra, since $\mathbb{F}_2$ is a field.

Lemma 2.3.2. *For $n \geq 1$, the degree two map from S^n to S^n induces the map $\epsilon \circ \eta$, on $\pi_*(\Lambda_{S^n} H\mathbb{F}_2)$.*

Proof. By calculation $\pi_*(\Lambda_{S^1} H\mathbb{F}_2)$ is cocommutative as an $\mathbb{F}_2$-coalgebra, and in general when $n \geq 2$ we have that $\pi_*(\Lambda_{S^n} H\mathbb{F}_2)$ is cocommutative as an $\mathbb{F}_2$-coalgebra since the pinch map on S^n is homotopy cocommutative.

An example of a degree two map from S^n to S^n is the composite

$$S^n \xrightarrow{\psi} S^n \vee S^n \xrightarrow{-\mathrm{id} \vee \mathrm{id}} S^n \vee S^n \xrightarrow{\nabla} S^n ,$$

where ψ is the pinch map, $-\mathrm{id}$ is the reflection map, and ∇ is the fold map. Applying the functor $\Lambda_{-} H\mathbb{F}_2$ yields

$$\pi_*(\Lambda_{S^n} H\mathbb{F}_2) \xrightarrow{\phi \circ (\chi \otimes \mathrm{id}) \circ \psi} \pi_*(\Lambda_{S^n} H\mathbb{F}_2),$$

and since this is a Hopf algebra, this composite is equal to $\epsilon \circ \eta$. $\square$

Using the two previous results we can deduce the following proposition.

Proposition 2.3.3. *There is an equivalence*

$$\Lambda_{\mathbb{R}P^2} H\mathbb{F}_2 \simeq \Lambda_{S^1} H\mathbb{F}_2 \wedge \Lambda_{S^2} H\mathbb{F}_2.$$

Proof. Let R be a commutative simplicial ring. Theorem 4.5 in [Sch99] gives a Quillen equivalence between simplicial R-algebras and algebras over the Eilenberg Maclane spectrum HR. In the paragraph following Theorem 5.2 in [HM97b] they show that $\Lambda_{S^1} H\mathbb{F}_2$ is equivalent to H of the free $\mathbb{F}_2$-algebra generated by an element in degree 2. By Lemma 2.3.2 the attaching map $S^1 \to S^1$ for $\mathbb{R}P^2$, yields a map $\Lambda_{S^1} H\mathbb{F}_2 \to \Lambda_{S^1} H\mathbb{F}_2$ which is zero in homotopy. Therefore the attaching map factors stably through $H\mathbb{F}_2$, so

$$\Lambda_{\mathbb{R}P^2} H\mathbb{F}_2 \simeq \Lambda_{S^1} H\mathbb{F}_2 \wedge_{H\mathbb{F}_2} \Lambda_{S^2} H\mathbb{F}_2.$$

$\square$

2.4 Calculating the Homotopy Groups of $(\Lambda_{S^2}H\mathbb{F}_2)^{C_2}$

In this section we will find a differential in the Tate spectral sequence calculating $\pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2})$, and use this to identify the homotopy groups of the spectra in the isotropy separation diagram associated with $\Lambda_{S^2}H\mathbb{F}_2$, and the maps between them. In this section tensor products and algebras are over $\mathbb{F}_2$, unless otherwise specified.

We will consider the following Tate spectral sequence and homotopy fixed points spectral sequences:

$$\hat{E}^2 = \hat{H}^*(C_2; \pi_*(\Lambda_{S^2}H\mathbb{F}_2)) \cong P(u, u^{-1}) \otimes E(\beta) \Rightarrow \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2})$$
$$E^2 = H^{-*}(C_2; \pi_*(\Lambda_{S^2}H\mathbb{F}_2)) \cong P(u) \otimes E(\beta) \Rightarrow \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2})$$
$$\overline{E}^2 = H^{-*}(C_2; V(0)_*(\Lambda_{S^2}H\mathbb{F}_2)) \cong P(u) \otimes E(\beta) \otimes E(\overline{\tau}_0) \Rightarrow V(0)_*((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2}),$$

where $|u| = (-1, 0)$, $|\overline{\tau}_0| = (0, 1)$ and $|\beta| = (0, 3)$.

Lemma 2.4.1. *There is an isomorphism*

$$\pi_0((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2}) \cong \mathbb{Z}/4.$$

Proof. For dimension reasons $E^2_{s,t} \cong E^2_{s,t}$ when $s \geq 3$, in the homotopy fixed points spectral sequence. Hence, there is an isomorphism $\pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2}) \cong \mathbb{F}_2$ when $* = 1, 2, 3$, and $\pi_0((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2})$ has order 4.

We use $S(\mathbb{R}^\infty)$ as a model for EC_2, with the antipodal action. The space $S(\mathbb{R}^\infty)$ is equipped with a free C_2-CW structure with one free C_2-cell in each degree, and the k-skeleton is the k-sphere. Restricting the skeleton filtration to the 3-skeleton, yields a filtration

$$\{\text{pt}\} \longrightarrow S^0_+ \longrightarrow S^1_+ \longrightarrow S^2_+ \longrightarrow S^3_+$$

which gives rise to two spectral sequences

$$'E^2 = P(u)/(u^4) \otimes E(\beta) \Rightarrow \pi_*(F(S^3_+, \Lambda_{S^2}H\mathbb{F}_2)^{C_2})$$
$$'\overline{E}^2 = P(u)/(u^4) \otimes E(\beta) \otimes E(\overline{\tau}_0) \Rightarrow V(0)_*(F(S^3_+, \Lambda_{S^2}H\mathbb{F}_2)^{C_2}),$$

which are equal to $E^2_{s,t}$ and $\overline{E}^2_{s,t}$, restricted to the columns $-3 \leq s \leq 0$. For dimension reasons, there are no differentials in $'E^2$.

The inclusion $S^3_+ \to (EC_2)_+$ induces a map of spectral sequence

$$\overline{E}^2 = P(u) \otimes E(\beta) \otimes E(\overline{\tau}_0) \to' \overline{E}^2 = P(u)/(u^4) \otimes E(\beta) \otimes E(\overline{\tau}_0),$$

which is the quotient map. By Corollary 2.2.3, there is a differential $d^3(\overline{\tau}_0) = u^3\beta$ in the first of these spectral sequences, and thus also in the latter. Hence, the element represented by $u^3\beta$ in $'E^2$ is zero in $'\overline{E}^2$, so $\pi_0(F(S^3_+, \Lambda_{S^2}H\mathbb{F}_2)^{C_2}) \cong \mathbb{Z}/4$. Since the inclusion $S^3_+ \to (EC_2)_+$ induces a ring map $\pi_*(F(S^3_+, \Lambda_{S^2}H\mathbb{F}_2)^{C_2}) \to \pi_0((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2})$, the unit in $\pi_0((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2})$ has order at least four. $\square$

We are now ready to do the following calculation.

Proposition 2.4.2. *The nonzero differentials in* $\hat{\mathrm{E}}^r$ *are generated by* $d^4u = u^5\beta$. *Hence, there are ring isomorphisms*

$$\pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2}) \cong P(t,t^{-1}) \otimes E(\beta)$$
$$\pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2}) \cong P_{\mathbb{Z}/4}(t,\alpha) \otimes_{\mathbb{Z}/4} E_{\mathbb{Z}/4}(\beta)/(2t, 2\alpha, 2\beta, \alpha^2, \alpha\beta)$$

with $|t| = -2$, $|\alpha| = 2$ *and* $|\beta| = 3$. *Furthermore,* $\pi_*((\Lambda_{S^2}H\mathbb{F}_2)_{hC_2}) \cong \mathbb{F}_2$ *for all* $* \geq 0$, *and the restriction homomorphism* $R^h : \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2}) \to \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2})$ *is given by mapping* t *to* t, β *to* β *and* α *to zero.*

Proof. Since $\pi_0(\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2) \cong \mathbb{F}_2$, the ring map $\pi_0(\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2) \to \pi_0((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2})$ proves that the unit in $\pi_0(\Lambda_{S^2}H\mathbb{F}_2)^{tC_2}$ has order at most 2. By Lemma 2.4.1 there is an isomorphism $\pi_0((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2}) \cong \mathbb{Z}/4$, and since $R^h : \pi_0((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2}) \to \pi_0((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2})$ is a ring map, $\pi_0((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2}) \cong \mathbb{F}_2$. Hence, there are non-zero differentials in $\hat{\mathrm{E}}^r$, and since this is a multiplicative spectral sequence, the only possible non-zero differentials are those generated by $d^4u = u^5\beta$.

These differentials also gives rise to non-zero differentials in the homotopy fixed point spectral sequence and the homotopy orbit spectral sequence, giving us the module structures in the proposition.

The element t is represented by u^2. There are no room for any additive or multiplicative extensions, except for $\pi_0((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2}) \cong \mathbb{Z}/4$. □

The proof that $\hat{\Gamma}_*$ is injective, hinges on proving that $\hat{\Gamma}_*$ of some element is nonzero in continuous homology. So before we prove this we state two lemmas needed in the proof. Consider the two C_2-spectra $\Lambda_{C_2 \times S^2}H\mathbb{F}_2$ and $\Lambda_{S^2 \vee S^2}H\mathbb{F}_2$, where C_2 acts on the C_2-factor in the first spectrum and by interchanging the two wedge factors in the second spectrum.

Lemma 2.4.3. *There are* A_**-isomorphisms*

$$H_*(\Lambda_{C_2 \times S^2}H\mathbb{F}_2) \cong A_* \otimes A_* \otimes E(z_1, z_2)$$
$$H_*(\Lambda_{S^2 \vee S^2}H\mathbb{F}_2) \cong A_* \otimes E(z_1, z_2),$$

where $|z_1| = |z_2| = 3$, *and the map induced by identifying the subspace* $C_2 \subseteq C_2 \times S^2$ *to a point, is given by multiplication of the* A_**-factors.*

Proof. Since $\Lambda_{S^2}H\mathbb{F}_2$ is an $H\mathbb{F}_2$-module $H_*(\Lambda_{S^2}H\mathbb{F}_2) \cong A_* \otimes E(z)$ where z is the image of $\beta \in \pi_3(\Lambda_{S^2}H\mathbb{F}_2)$ under the Hurewicz homomorphism. By flatness we have

$$\begin{aligned} H_*(\Lambda_{S^2 \vee S^2}H\mathbb{F}_2) &\cong H_*(\Lambda_{S^2}H\mathbb{F}_2) \otimes_{H_*(\mathbb{F}_2)} H_*(\Lambda_{S^2}H\mathbb{F}_2) \\ &\cong (A_* \otimes E(z_1)) \otimes_{A_*} (A_* \otimes E(z_2)) \cong A_* \otimes E(z_1, z_2). \end{aligned}$$

Similarly,

$$H_*(\Lambda_{C_2 \times S^2} H\mathbb{F}_2) \cong H_*(\Lambda_{S^2} H\mathbb{F}_2) \otimes H_*(\Lambda_{S^2} H\mathbb{F}_2) \cong A_* \otimes A_* \otimes E(z_1, z_2).$$

That the homomorphism is multiplication of the A_*-factors follows from the commutative diagram

$$\begin{tikzcd}
\Lambda_{C_2} H\mathbb{F}_2 \arrow[r] \arrow[d] & \Lambda_{\{\mathrm{pt}\}} H\mathbb{F}_2 \arrow[d] \\
\Lambda_{C_2 \times S^2} H\mathbb{F}_2 \arrow[r] & \Lambda_{S^2 \vee S^2} H\mathbb{F}_2.
\end{tikzcd}$$

□

Lemma 2.4.4. *The $\hat{E}^2$-page in the Tate spectral sequence that calculates the continuous homology $H^c_*((\Lambda_{S^2 \vee S^2} H\mathbb{F}_2)^{tC_2})$ is equal to $P(u, u^{-1}) \otimes A_* \otimes E(z_1 z_2)$, where $|u| = (-1, 0)$ and $|z_1 z_2| = (0, 6)$. For all i, the elements $u^i \otimes z_1 z_2$ survives to the $\hat{E}^\infty$ page.*

Proof. Observe that since the C_2-action on $\Lambda_{S^2 \vee S^2} H\mathbb{F}_2$ is given by interchanging the two wedge factors, it interchanges z_1 and z_2 in homology. A complete free $\mathbb{F}_2[C_2]$ resolution of $\mathbb{F}_2$ is given by

$$\dots \xrightarrow{1+t} \mathbb{F}_2[C_2] \xrightarrow{1+t} \mathbb{F}_2[C_2] \xrightarrow{1+t} \mathbb{F}_2[C_2] \xrightarrow{1+t} \dots$$

where t is a generator of C_2. Thus the $\hat{E}^2$-page is equal to $P(u, u^{-1}) \otimes A_* \otimes E(z_1 z_2)$.

By Proposition A.5.3 the Tate-spectral sequence calculating $H^c_*((\Lambda_{C_2 \times S^2} H\mathbb{F}_2)^{tC_2})$ collapses at the $\hat{E}^2$-page, and is given by

$$\hat{E}^2 = \hat{H}^*(C_2; (A_* \otimes E(z))^{\otimes 2}) \cong P(u, u^{-1}) \otimes \mathbb{F}_2(x^{\otimes 2}),$$

where x runs through a $\mathbb{F}_2$ basis of $A_* \otimes E(z)$. By Lemma 2.4.3 the element $u^i \otimes z_1 z_2$ is the image of the infinite cycle $u^i \otimes z^2$, and is thus an infinite cycle itself.

If $u^i \otimes z_1 z_2$ is a boundary it must be the image of a differential which have $P(u, u^{-1}) \otimes A_*$ as a source. But this is impossible since the Tate spectral sequence computing $H^c_*((\Lambda_{\{\mathrm{pt}\}} H\mathbb{F}_2)^{tC_2})$, have $\hat{E}^2$-page equal to $P(u, u^{-1}) \otimes A_*$, and there is a splitting

$$P(u, u^{-1}) \otimes A_* \to P(u, u^{-1}) \otimes A_* \otimes E(z_1 z_2) \to P(u, u^{-1}) \otimes A_*$$

induced by the C_2-map $\{\mathrm{pt}\} \to S^1 \vee S^1 \to \{\mathrm{pt}\}$.

□

Given a space X with a free C_2 action, Lemma 5.2.5 in [BCD10] yields an equivalence

$$[\widetilde{EC_2} \wedge \Lambda_X H\mathbb{F}_2]^{C_2} \simeq \Lambda_{X/C_2} H\mathbb{F}_2.$$

Hence $\hat{\Gamma}$ is a map from $\Lambda_{\mathbb{R}P^2} H\mathbb{F}_2$ to $(\Lambda_{S^2} H\mathbb{F}_2)^{tC_2}$, and we have the following proposition.

Proposition 2.4.5. *The homomorphism*

$$\hat{\Gamma}_* : \pi_3(\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2) \to \pi_3((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2})$$

is an isomorphism.

Proof. The pinch map $S^2 \to S^2 \vee S^2$ induces a homomorphism $\pi_3((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2}) \to \pi_3((\Lambda_{S^2\vee S^2}H\mathbb{F}_2)^{tC_2})$, where the C_2-action on $S^2 \vee S^2$ interchanges the two wedge factors. We will show that β the generator of $\pi_3(\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2)$ survives to $\pi_3((\Lambda_{S^2\vee S^2}H\mathbb{F}_2)^{tC_2})$, under the composition of Γ_* followed by the homomorphism induced by the pinch map on S^2, and is thus non-zero in $\pi_3((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2})$.

There is a commutative cube of C_2-spaces and C_2-equivariant maps

$$\begin{tikzcd}
C_2 \times S^1 \arrow[rr] \arrow[dr] \arrow[dd] & & S^1 \arrow[dr] \arrow[dd] & \\
& C_2 \times D^2 \arrow[rr] \arrow[dd] & & \{\mathrm{pt}\} \arrow[dd] \\
C_2 \times D^2 \arrow[rr] \arrow[dr] & & S^2 \arrow[dr] & \\
& C_2 \times S^2 \arrow[rr] & & S^2 \vee S^2,
\end{tikzcd} \tag{2.4.6}$$

where C_2 acts on the C_2-factor on the left face, with the antipodal action on S^1 and S^2, and by interchanging the two wedge-factors in $S^2 \vee S^2$. The maps in the left face are inclusions, and the top map on the back face is the identity on $\{e\} \times S^1$, where e is the identity element in C_2. The rest of the maps are defined by requiring the left, front, right and back face to be pushouts of C_2-spaces.

Taking C_2-orbits in the back face of the cube 2.4.6 and then applying $\Lambda_{-}H\mathbb{F}_2$, yields a pushout diagram

$$\begin{tikzcd}
\Lambda_{S^1}H\mathbb{F}_2 \arrow[r] \arrow[d] & \Lambda_{S^1}H\mathbb{F}_2 \arrow[d] \\
\Lambda_{D^2}H\mathbb{F}_2 \arrow[r] & \Lambda_{\mathbb{R}P^2}H\mathbb{F}_2.
\end{tikzcd} \tag{2.4.7}$$

By Theorem 5.13 in [LNR12] there is a 2-adic equivalence of spectra

$$(\Lambda_{C_2\times X}H\mathbb{F}_2)^{tC_2} \simeq \Lambda_X H\mathbb{F}_2,$$

when $X = S^1, S^2, D^2$. Applying the functor $(\Lambda_{-}H\mathbb{F}_2)^{tC_2}$ to the top and left edge of the front face of diagram 2.4.6, and then taking the pushout, thus yields a pushout diagram

$$\begin{tikzcd}
\Lambda_{D^2}H\mathbb{F}_2 \arrow[r] \arrow[d] & (\Lambda_{\{\mathrm{pt}\}}H\mathbb{F}_2)^{tC_2} \arrow[d] \\
\Lambda_{S^2}H\mathbb{F}_2 \arrow[r] & (\Lambda_{S^2}H\mathbb{F}_2) \wedge_{H\mathbb{F}_2} (\Lambda_{\{\mathrm{pt}\}}H\mathbb{F}_2)^{tC_2}.
\end{tikzcd} \tag{2.4.8}$$

The homomorphism $\hat{\Gamma}_*$ together with the cube 2.4.6 and the universal property of pushouts induces a sequence of maps

$$\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2 \xrightarrow{f} (\Lambda_{S^2}H\mathbb{F}_2) \wedge_{H\mathbb{F}_2} (\Lambda_{\{\mathrm{pt}\}}H\mathbb{F}_2)^{tC_2} \xrightarrow{g} (\Lambda_{S^2\vee S^2}H\mathbb{F}_2)^{tC_2},$$

and we will first show that $f_*(\beta) \neq 0$.

The Tate spectral sequence computing $(\Lambda_{\{\mathrm{pt}\}}H\mathbb{F}_2)^{tC_2}$ is concentrated on the x-axis and hence $\pi_*(\Lambda_{\{\mathrm{pt}\}}H\mathbb{F}_2)^{tC_2} \cong P(u, u^{-1})$ where $|u| = -1$. By flatness

$$\pi_*((\Lambda_{S^2}H\mathbb{F}_2) \wedge_{H\mathbb{F}_2} (\Lambda_{\{\mathrm{pt}\}}H\mathbb{F}_2)^{tC_2}) \cong E(\beta) \otimes P(u, u^{-1}),$$

so if $f_*(\beta) \neq 0$, then $f_*(\beta) = \beta$.

To show that $f_*(\beta)$ is non-zero we will look at the map f induces on bar spectral sequences. From the 2-adic equivalence of ring spectra $(\Lambda_{C_2 \times X}H\mathbb{F}_2)^{tC_2} \simeq \Lambda_X H\mathbb{F}_2$, we get that applying the functor $(\Lambda_{-}H\mathbb{F}_2)^{tC_2}$ to the left face of the cube 2.4.6, yields a pushout diagram of ring spectra. Composing this pushout diagram with the pushout diagram 2.4.8 thus yields a pushout diagram

$$\begin{tikzcd} \Lambda_{S^1}H\mathbb{F}_2 \arrow[r] \arrow[d] & (\Lambda_{\{\mathrm{pt}\}}H\mathbb{F}_2)^{tC_2} \arrow[d] \\ \Lambda_{D^2}H\mathbb{F}_2 \arrow[r] & (\Lambda_{S^2}H\mathbb{F}_2) \wedge_{H\mathbb{F}_2} (\Lambda_{\{\mathrm{pt}\}}H\mathbb{F}_2)^{tC_2}. \end{tikzcd} \tag{2.4.9}$$

By Corollary 1.2.2 there is a bar spectral sequence associated with this pushout diagram. Since the top map in this diagram factors through $\Lambda_{D^2}H\mathbb{F}_2 \simeq H\mathbb{F}_2$, the $P(\mu)$-module structure on $P(u, u^{-1})$ is the trivial one, so the E^2-page is isomorphic to

$$E^2 = \mathrm{Tor}^{P(\mu)}(P(u, u^{-1}), \mathbb{F}_2) \cong P(u, u^{-1}) \otimes E(\beta') \Rightarrow \pi_*((\Lambda_{S^2}H\mathbb{F}_2) \wedge_{H\mathbb{F}_2} (\Lambda_{\{\mathrm{pt}\}}H\mathbb{F}_2)^{tC_2}).$$

From the previous calculation of the abutment $\pi_*((\Lambda_{S^2}H\mathbb{F}_2) \wedge_{H\mathbb{F}_2} (\Lambda_{\{\mathrm{pt}\}}H\mathbb{F}_2)^{tC_2})$, we know that the spectral sequence must collapse, so $E^2 \cong E^\infty$.

The pushout diagram 2.4.7 gives rise to a bar spectral sequence

$$E^2(\mathbb{R}P^2) \cong \mathrm{Tor}^{P(\mu)}(P(\mu), \mathbb{F}_p) \Rightarrow \pi_*(\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2),$$

By Lemma 2.3.2 the degree two map on S^2 induces the trivial module structure on $P(\mu)$. Hence,

$$E^2(\mathbb{R}P^2) = P(\mu) \otimes \mathrm{Tor}^{P(\mu)}(\mathbb{F}_p, \mathbb{F}_p) \cong P(\mu) \otimes E(\beta').$$

By Proposition 2.3.1, there are no differentials in $E^2(\mathbb{R}P^\infty)$, so $E^2(\mathbb{R}P^2) \cong E^\infty(\mathbb{R}P^2)$, and $\beta \in \pi_3(\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2)$ is represented by β' in the spectral sequence. Now, on the E^∞-pages the homomorphism

$$f_*\colon E^\infty(\mathbb{R}P^2) \cong E(\beta') \otimes P(\mu) \to E^\infty \cong E(\beta') \otimes P(u, u^{-1})$$

maps β' to β'. Thus, $f_*(\beta)$ is non-zero, so it must be equal to β.

It is left to prove that $g_*(\beta) \neq 0$. This is equivalent to showing that the image of β is non-zero under the map

$$\Lambda_{S^2}H\mathbb{F}_2 \simeq (\Lambda_{C_2 \times S^2}H\mathbb{F}_2)^{tC_2} \longrightarrow (\Lambda_{S^2 \vee S^2}H\mathbb{F}_2)^{tC_2},$$

coming from the lower map on the front face of the cube 2.4.6. To achieve this we consider the diagram

$$\begin{tikzcd}
\pi_3(\Lambda_{S^2}H\mathbb{F}_2) \arrow[r, "\cong"] \arrow[d, "h"] & \pi_3(\Lambda_{C_2\times S^2}H\mathbb{F}_2)^{tC_2} \arrow[r] \arrow[d, "h"] & \pi_3(\Lambda_{S^2\vee S^2}H\mathbb{F}_2)^{tC_2} \arrow[d, "h"] \\
H_3(\Lambda_{C_2\times S^2}H\mathbb{F}_2)^{tC_2} \arrow[r, "\epsilon_*"] & H_3^c(\Lambda_{C_2\times S^2}H\mathbb{F}_2)^{tC_2} \arrow[r] & H_3^c(\Lambda_{S^2\vee S^2}H\mathbb{F}_2)^{tC_2},
\end{tikzcd}$$

where h is induced by the Hurewicz homomorphism. Now,

$$h : \pi_*(\Lambda_{C_2\times S^2}H\mathbb{F}_2)^{tC_2} \to H_*(\Lambda_{C_2\times S^2}H\mathbb{F}_2)^{tC_2} \cong A_* \otimes \pi_3(\Lambda_{C_2\times S^2}H\mathbb{F}_2)^{tC_2}$$

maps β to $z = 1 \otimes \beta$. If we show that the image of z is nonzero in $H_3^c(\Lambda_{S^2\vee S^2}H\mathbb{F}_2)^{tC_2}$, we know that the image of β is nonzero in $\pi_3(\Lambda_{S^2\vee S^2}H\mathbb{F}_2)^{tC_2}$, finishing the proof.

By Proposition A.5.3 the Tate spectral sequence calculating $H_*^c(\Lambda_{C_2\times S^2}H\mathbb{F}_2)^{tC_2}$ is equal to

$$\hat{E}^2 = P(u, u^{-1}) \otimes \mathbb{F}_2\{x^2\} \Rightarrow H_*^c(\Lambda_{C_2\times S^2}H\mathbb{F}_2)^{tC_2},$$

where x runs over a basis for the elements in $H_*^c(\Lambda_{S^2}H\mathbb{F}_2)$. Furthermore, this spectral sequence collapses on the E^2-page and $\epsilon_*(z)$ is represented by $u^3 \otimes z^2$.

Finally, by Lemma 2.4.4 the element $u^3 \otimes z^2$ survives to the element $u^3 \otimes z_1 z_2$ on the $\hat{E}^\infty$-page of the spectral sequence computing $H_*^c((\Lambda_{S^2\vee S^2}H\mathbb{F}_2)^{tC_2})$. Thus $g_*(\beta)$ is non-zero and hence $\hat{\Gamma}_*(\beta)$ is non-zero and thus equal to $\beta \in \pi_3((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2})$. □

Together with results by Hesselholt and Madsen in [HM97b] this lemma enables us to calculate $\hat{\Gamma}_*$.

Proposition 2.4.10. *The map*

$$\hat{\Gamma}_* : \pi_*(\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2) \cong P(\mu) \otimes E(\beta) \to \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2}) \cong P(t, t^{-1}) \otimes E(\beta)$$

is given by mapping μ to t^{-1}, and β to β.

Proof. The inclusion $S^1 \to S^2$ gives a commutative diagram

$$\begin{tikzcd}
\pi_2(\Lambda_{S^1}H\mathbb{F}_2) \arrow[r] \arrow[d, "\hat{\Gamma}_*"] & \pi_2(\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2) \arrow[d, "\hat{\Gamma}_*"] \\
\pi_2((\Lambda_{S^1}H\mathbb{F}_2)^{tC_2}) \arrow[r] & \pi_2((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2}).
\end{tikzcd}$$

In Lemma 5.4 in [HM97b] they prove that $\pi_2((\Lambda_{S^1}H\mathbb{F}_2)^{tC_2}) \cong P(t, t^{-1})$, where t is represented by u^2 in the Tate spectral sequence. (They actually state the lemma for odd primes, but everything works out for $p = 2$ with the obvious renaming of elements). The left map is then proved to be an isomorphism in Proposition 5.3 in [HM97b]. By Proposition 2.3.3 the top map is an isomorphism and the bottom map is an isomorphism if you look at the Tate spectral sequences, since both groups are represented by the infinite cycle u^{-2} in their respective Tate spectral sequences. Hence, the right map is also an isomorphism so μ is mapped to t^{-1}. That β is mapped to β follows from Proposition 2.4.5. □

Finally, we are able to calculate $\pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{C_2})$ using the isotropy separation diagram. Recall that there is a ring isomorphism

$$\pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2}) \cong P_{\mathbb{Z}/4}(t,\alpha) \otimes_{\mathbb{Z}/4} E_{\mathbb{Z}/4}(\beta)/(2t, 2\alpha, 2\beta, \alpha^2, \alpha\beta).$$

Theorem 2.4.11. *There is a ring isomorphism*

$$\pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{C_2}) \cong P_{\mathbb{Z}/4}(\alpha) \otimes_{\mathbb{Z}/4} E_{\mathbb{Z}/4}(\beta)/(2\alpha, 2\beta, \alpha^2, \alpha\beta)$$

where $|\alpha| = 2$ *and* $|\beta| = 3$*, and the homomorphism*

$$\Gamma_* \colon \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{C_2}) \to \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2})$$

is given by mapping α *to* α *and* β *to* β*.*

Proof. From the isotropy separation diagram we get a commutative diagram

$$\begin{tikzcd}
\pi_*((\Lambda_{S^2}H\mathbb{F}_2)_{hC_2}) \arrow[r] \arrow[d, equal] & \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{C_2}) \arrow[r] \arrow[d, "\Gamma_*"] & \pi_*((\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2) \arrow[r] \arrow[d, "\hat{\Gamma}_*"] & \pi_{*-1}((\Lambda_{S^2}H\mathbb{F}_2)_{hC_2}) \arrow[d, equal] \\
\pi_*((\Lambda_{S^2}H\mathbb{F}_2)_{hC_2}) \arrow[r] & \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2}) \arrow[r] & \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2}) \arrow[r] & \pi_{*-1}((\Lambda_{S^2}H\mathbb{F}_2)_{hC_2}),
\end{tikzcd}$$

where the horizontal lines are parts of two long exact sequences.

From Proposition 2.4.10 the homomorphism $\hat{\Gamma}_* : \pi_*(\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2) \to \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{tC_2})$ is an isomorphism when $* \geq 0$ and $* \neq 1$. By the five lemma we thus get that $\pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{C_2}) \cong \pi_*((\Lambda_{S^2}H\mathbb{F}_2)^{hC_2})$ when $* \geq= 2$. Since $\pi_1(\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2) = 0$ and $\pi_0((\Lambda_{S^2}H\mathbb{F}_2)_{hC_2})$ and $\pi_0(\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2)$ are isomorphic to $\mathbb{F}_2$, we know that the order of $\pi_0((\Lambda_{S^2}H\mathbb{F}_2)^{C_2})$ is four. Now, Γ_* is a ring homomorphism, so this implies that $\pi_0((\Lambda_{S^2}H\mathbb{F}_2)^{C_2}) \cong \mathbb{Z}_4$

Examining the long exact sequence in the top row we get $\pi_1((\Lambda_{S^2}H\mathbb{F}_2)^{C_2}) = \pi_1(\Lambda_{\mathbb{R}P^2}H\mathbb{F}_2) = 0$. □

Chapter 3

Homotopy Groups of $\Lambda_{T^n} H\mathbb{F}_p$ and Periodic Elements

In this chapter we calculate $\pi_*(\Lambda_{T^n} H\mathbb{F}_p)$ when $p \geq 5$ and $1 \leq n \leq p$, and $p = 3$ and $1 \leq n \leq 2$. These calculations take a lot of effort, and every section but the last revolves around it. The argument is based on the bar spectral sequence, and is heavily dependent on the Hopf algebra structures of $\pi_*(\Lambda_{T^n} H\mathbb{F}_p)$. We have one Hopf algebra structure for each circle factor, and the first two sections concerns the interplay between these Hopf algebra structures. In Section 3.3 we calculate $\pi_*(\Lambda_{S^n} H\mathbb{F}_p)$ for $n \leq 2p$, before we calculate $\pi_*(\Lambda_{T^n} H\mathbb{F}_p)$ in Section 3.4.

In the last section we use this to show that v_n is non-zero in $k(n)_*(\Lambda_{T^{n+1}} H\mathbb{F}_p)$, where $k(n)$ is the n-th connective Morava K-theory.

It's recommended to skip the first two sections on your first read, and rather go back to it when you need it.

3.1 Multifold Hopf Algebras

The homotopy groups of the spectrum $\Lambda_{T^n} H\mathbb{F}_p$ will have several Hopf algebra structures coming from the various circles. These structures will be interlinked, and this section sets up an algebraic framework for this interlinked structure. Our main goal is to be able to state Proposition 3.2.5 which is a crucial ingredient in the calculation of the multiplicative structure of $\pi_*(\Lambda_{T^n} H\mathbb{F}_p)$.

Multifold Hopf algebras have more structure than we show below. In particular, it would be interesting to have a good description of the module of elements that are primitive in all the Hopf algebra structures simultaneously. In that regard a generalization of the very special case in Lemma 3.3.12 would be welcomed.

Let CRings be the category of commutative rings. In this section we will assume that all our Hopf algebras are connected and commutative.

First we construct a category of Hopf algebras, and show that it as all small colimits. Objects in this category are ordinary Hopf algebras, but we need the morphisms to define a multifold Hopf algebra.

Definition 3.1.1. *We define the category of Hopf algebras to have objects pairs of commutative rings (A, R) where A is given the structure of a commutative connected R-Hopf algebra. A morphism from (A, R) to (B, S) consists of two maps $f\colon A \to B$ and $g\colon R \to S$ such that f is a map of R-algebras and S-coalgebras, where the R-algebra structure on B and the S-coalgebra structure on A are induced by g.*

Proposition 3.1.2. *The category of Hopf algebras has all small colimits, and the colimit $\operatorname{colim}_J(A_j, R_j)$ is equal to the pair $(\operatorname{colim}_J A_j, \operatorname{colim}_J R_j)$, of colimits in the category of commutative rings.*

Proof. The ring $\operatorname{colim}_J A_j$ is a $\operatorname{colim}_J R_j$-algebra. Since colimits commute, there is an isomorphism

$$\beta : \operatorname*{colim}_J(A_j \otimes_{R_j} A_j) \cong \operatorname*{colim}_J A_j \otimes_{\operatorname{colim}_J R_j} \operatorname*{colim}_J A_j,$$

and we define the counit and coproduct in $\operatorname{colim}_J(A_j, R_j)$ to be equal to $\operatorname{colim}_J(\epsilon_j)$ and $\beta \circ \operatorname{colim}_J(\psi_j)$, respectively. That the required diagrams in the definition of a $(\operatorname{colim}_J R_j)$- Hopf algebra commute, follows by functoriality of the colimits.

The only thing left to prove is that given a Hopf algebra (A, R) which is a cone over (A_j, R_j) with $j \in J$, the homomorphism (f, g) from $(\operatorname{colim}_J A_j, \operatorname{colim}_J R_j)$ to (A, R) induced by the universal property of colimit of commutative rings, is actually a homomorphism of Hopf algebras.

By the universal property of colimits of commutative rings, there is unique map $h : \operatorname{colim}_J(A_j \otimes_{R_j} \otimes A_j) \to A \otimes_R A$, since $A \otimes_R A$ is a cone over $A_j \otimes_{R_j} \otimes A_j$ with $j \in J$. From the fact that the coproduct and counit are algebra homomorphisms, and by functoriality of colimits, there are commutative diagrams

$$\begin{tikzcd}
\operatorname{colim}_J A_j \arrow[r, "f"] \arrow[d, "\operatorname{colim}\psi_j"] & A \arrow[d, "\psi_A"] \\
\operatorname{colim}_J(A_j \otimes_{R_j} A_j) \arrow[r, "h"] \arrow[d, "\beta"] & A \otimes_R A \\
(\operatorname{colim}_J A_j) \otimes_{\operatorname{colim}_J R_j} (\operatorname{colim}_J A_j) \arrow[ur, "f\otimes f"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{colim}_J A_j \arrow[r, "f"] \arrow[d, "\operatorname{colim}\epsilon_j"] & A \arrow[d, "\epsilon_A"] \\
\operatorname{colim}_J R_j \arrow[r, "g"] & R.
\end{tikzcd}$$

Thus, (f, g) is a homomorphism of Hopf algebras. $\square$

Our multifold Hopf algebras will be functors from the following category. Let S be a finite set a and define $V(S)$ to be the category with objects subsets of S and morphisms from U to V given by $U \cap V$, where composition is intersection. Let $[2]^S$ be the category with objects subsets of S and morphisms inclusions of sets. The category $V(S)$ is isomorphic to the category of spans in $[2]^S$.

The next definition is only a stepping stone towards the final definition of an S-fold Hopf algebra in 3.1.13.

Definition 3.1.3. *Let S be a finite set. A pre S-fold Hopf algebra A is a functor $A : V(S) \to \mathrm{CRings}$, such that:*

For every $v \in V \subseteq S$, the pair $(A(V), A(V \setminus v))$ is equipped with the structure of a Hopf algebra with unit and counit induced by the inclusion $V \setminus v \to V$, such that with this structure the composite

$$V(S) \times S \xrightarrow{\Delta \times S} V(S) \times V(S) \times S \xrightarrow{V(S) \times (- \setminus -)} V(S) \times V(S) \xrightarrow{A \times A} \mathrm{CRings} \times \mathrm{CRings},$$

becomes a functor to the category of Hopf algebras. Here Δ is the diagonal functor and the functor $(- \setminus -)$ takes (U, u) to $U \setminus u$.

We write $A(V) = A_V$, and let ψ^v_V, ϕ^v_V, η^v_V and ϵ^v_V denote the various structure maps in $(A(V), A(V \setminus v))$.

Definition 3.1.4. *A map from a pre S-fold Hopf algebra A to a pre S-fold Hopf algebra B is a natural transformation from A to B such that for every $v \in V \subseteq S$ the induced map from $(A_V, A_{V \setminus v})$ to $(B_V, B_{V \setminus v})$ is a map of Hopf algebras.*

The example we have in mind is the functor $\pi_*(\Lambda_{T^-} H\mathbb{F}_p) : A : V(S) \to \mathrm{CRings}$ that maps $U \subseteq S$ to $\pi_*(\Lambda_{T^U} H\mathbb{F}_p)$. We show in Proposition 3.4.2 that this is a (pre) S-fold Hopf algebra, where the different Hopf algebra structures, comes from the different circles.

Let $T(S)$ be the full subcategory of $[2]^S \times [2]^S$ with objects pairs (U, V) with $U \cap V = \emptyset$. There is an inclusion $[2]^S \to V(S)$ given by sending a morphism $U \subseteq V$ to the morphism U from U to V.

Example 3.1.5. Let $S = \{u, v\}$. Then, the category $V(S)$ is equal to

$$\begin{tikzcd}
\emptyset \arrow[r, shift right] \arrow[d, shift left] & \{v\} \arrow[l, shift right] \arrow[d, shift right] \\
\{u\} \arrow[u, shift left] \arrow[r, shift left] & \{u, v\}, \arrow[l, shift left] \arrow[u, shift right]
\end{tikzcd}$$

and the inclusion $[2]^S \to V(S)$ is given by picking all the inner arrows going away from $\emptyset$.

We will now define some commutative rings A^U_V that will be the source and targets for iterated coproducts in a pre S-fold Hopf algebra. These commutative rings are constructed from functors from $[2]^S$ to the category of commutative rings. Using the inclusion $[2]^S \to V(S)$ this construction applies to any pre S-fold Hopf algebra A.

Definition 3.1.6. *Let S be a finite set and A a functor $A : [2]^S \to \mathrm{CRings}$. Given finite sets $U \subseteq V \subseteq S$ we define the functor $F^U_{A,V}$ to be the composite*

$$T(U) \xrightarrow{- \cup -} [2]^U \xrightarrow{- \cup (V \setminus U)} [2]^S \xrightarrow{A} \mathrm{CRings},$$

and define A^U_V to be the colimit of the functor $F^U_{A,V}$.

Example 3.1.7. Let $U = \{u, v\} \subseteq V$. The source category $T(U)$ of $F^U_{A,V}$ is the diagram on the left, and the image of $F^U_{A,V}$ in commutative rings is the diagram on the right:

$$\begin{tikzcd}
\{u,v\},\emptyset & \{u\},\emptyset \arrow[l] \arrow[r] & \{u\},\{v\} \\
\{v\},\emptyset \arrow[u] \arrow[d] & \emptyset,\emptyset \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \emptyset,\{v\} \arrow[u] \arrow[d] \\
\{v\},\{u\} & \emptyset,\{u\} \arrow[l] \arrow[r] & \emptyset,\{u,v\}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_V & A_{V\setminus v} \arrow[l] \arrow[r] & A_V \\
A_{V\setminus u} \arrow[u] \arrow[d] & A_{V\setminus\{u,v\}} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & A_{V\setminus u} \arrow[u] \arrow[d] \\
A_V & A_{V\setminus v} \arrow[l] \arrow[r] & A_V .
\end{tikzcd}$$

In our example, $\pi_*(\Lambda_{T^-} H\mathbb{F}_p)^U_V \cong \pi_*(\Lambda_{T^{V\setminus U}\times(S^1\vee S^1)^{\times U}} H\mathbb{F}_p)$.

We will now describe a helpful way to think about the rings A^U_V.

Definition 3.1.8. *The power set, $P(U)$ of U, can be thought of as a discrete category, with objects the subsets of U. There is a functor G from $P(U)$ to $T(U)$ given by mapping $W \subseteq U$ to the pair $(U \setminus W, W)$. The composite $F^U_{A,V} \circ G$ is the constant functor A_V, so this induces a surjective map on colimits from $A_V^{\otimes P(U)}$ to A^U_V.*

An element in A^U_V can thus be represented by an element in $A_V^{\otimes P(U)}$, and we write the image of these representatives as cubes with an element of A_V in each corner, indexed by the subset of U.

Example 3.1.9. Let $U = \{u, v\} \subseteq V$. An element of A^U_V is represented by a sum of cubes

$$\begin{bmatrix} x_\emptyset & x_{\{v\}} \\ x_{\{u\}} & x_{\{u,v\}} \end{bmatrix},$$

where all the x's are elements of A_V. The four entries in the cube correspond to the four corners in the right diagram in Example 3.1.7, and the subscripts are given by the second set in the four corners in the left diagram. Multiplication is done component wise, and we have the following identifications

$$\begin{bmatrix} a_u x_\emptyset & x_{\{v\}} \\ x_{\{u\}} & x_{\{u,v\}} \end{bmatrix} = \begin{bmatrix} x_\emptyset & a_u x_{\{v\}} \\ x_{\{u\}} & x_{\{u,v\}} \end{bmatrix} \qquad \begin{bmatrix} x_\emptyset & x_{\{v\}} \\ a_u x_{\{u\}} & x_{\{u,v\}} \end{bmatrix} = \begin{bmatrix} x_\emptyset & x_{\{v\}} \\ x_{\{u\}} & a_u x_{\{u,v\}} \end{bmatrix}$$
$$\begin{bmatrix} a_v x_\emptyset & x_{\{v\}} \\ x_{\{u\}} & x_{\{u,v\}} \end{bmatrix} = \begin{bmatrix} x_\emptyset & x_{\{v\}} \\ a_v x_{\{u\}} & x_{\{u,v\}} \end{bmatrix} \qquad \begin{bmatrix} x_\emptyset & a_v x_{\{v\}} \\ x_{\{u\}} & x_{\{u,v\}} \end{bmatrix} = \begin{bmatrix} x_\emptyset & x_{\{v\}} \\ x_{\{u\}} & a_v x_{\{u,v\}} \end{bmatrix}$$

when a_u is an element in $A_{V\setminus\{u\}} \subseteq A_V$, and a_v is an element in $A_{V\setminus\{v\}} \subseteq A_V$. Observe that if a is an element in $A_{V\setminus\{u,v\}}$ we can move the element between all four corners of the cube.

Observe that the colimits of the columns in the right diagram in Example 3.1.7 are

$$A_V^{\{u\}} \longleftarrow A^u_{V\setminus\{v\}} \longrightarrow A_V^{\{u\}} .$$

Given a map of diagrams

$$\begin{tikzcd} A_V \arrow[d, "\psi_V"] & A_{V\setminus v} \arrow[l] \arrow[r] \arrow[d, "\psi_{V\setminus v}"] & A_V \arrow[d, "\psi_V"] \\ A_V^{\{u\}} & A_{V\setminus v}^{\{u\}} \arrow[l] \arrow[r] & A_V^{\{u\}}, \end{tikzcd}$$

we will write the map on the colimits of the horizontal direction as

$$\begin{bmatrix}\psi_V & \psi_V\end{bmatrix} : A_V^{\{v\}} \to A_V^{\{u,v\}}.$$

Lemma 3.1.10. *For $U \subseteq V$ and $v \in V \setminus U$, the universal property of colimits induces an isomorphism*

$$A_V^U \otimes_{A_{V\setminus v}^U} A_V^U \cong A_V^{U\cup v}$$

of commutative rings.

Proof. Both sides are the colimit of the functor $F_{A,V}^{U\cup v}$. On the left hand side the colimit is evaluated in two steps, evaluating the v-th direction in the diagram $T(U \cup v)$ last.

More explicitly, the middle term $A_{V\setminus v}^U$ is the colimit of the functor $F_{A,V}^{U\cup v}$ precomposed with the inclusion $T(U) \to T(U\cup v)$. The two outer terms A_V^U are the colimit of the functor $F_{A,V}^{U\cup v}$ precomposed with the two maps $T(U) \to T(U \cup v)$, given by adding v to the first and second set in $T(U)$, respectively. $\square$

Given a pre S-fold Hopf algebra A, there are some related multifold Hopf algebras. You can think of a pre S-fold Hopf algebra as an S-cube, with corners indexed by the subset of S, of commutative rings with extra structure. The first part of the next proposition says that every face is a pre multifold Hopf algebra in a natural way.

Proposition 3.1.11. *If U and W are subsets of S, the composite*

$$V(W) \xrightarrow{-\cup U} V(S) \xrightarrow{A} \mathrm{CRings}$$

is a pre W-fold Hopf algebra. If U is a subset of S the functor

$$A^U : V(S \setminus U) \to \mathrm{CRings}$$

given by $A^U(V) = A_{V\cup U}^U$ is a pre $S \setminus U$-fold Hopf algebra.

Proof. The first case is clear by definition. In the second case, for every $U \subseteq V \subseteq S$ and $v \in V \setminus U$, we need to give a Hopf algebra structure to the pair $\left(A_V^U, A_{V\setminus v}^U\right)$ satisfying the definition of a pre $S \setminus U$-fold Hopf algebra.

We claim there is a pushout diagram

$$\begin{tikzcd} \left(A_{V\setminus u}^U, A_{(V\setminus u)\setminus v}^U\right) \arrow[r] \arrow[d] & \left(A_V^U, A_{V\setminus v}^U\right) \arrow[d] \\ \left(A_V^U, A_{V\setminus v}^U\right) \arrow[r] & \left(A_V^{U\cup u}, A_{V\setminus v}^{U\cup u}\right) \end{tikzcd}$$

of Hopf algebras.

The identification of the pushout follows from Lemma 3.1.10. The case $U = \emptyset$ follows from the definition of a pre S-fold Hopf algebra. The rest are by induction on the number of elements in U.

The universal property of pushouts guarantees that these Hopf algebras combines to a functor satisfying the definition of a pre $S \setminus U$-fold Hopf algebra. $\square$

Composing the various coproducts in a pre S-fold Hopf algebra, gives rise to several homomorphisms that we now introduce. An S-fold Hopf algebra is a pre S-fold Hopf algebra where these various homomorphisms agree.

Definition 3.1.12. *Let A be a pre S-fold Hopf algebra. Given a pair of sets $U \subseteq V \subseteq S$ with $v \in V \setminus U$ we define*

$$\psi_V^{U,v} \colon A_V^U \to A_V^U \otimes_{A_{V\setminus v}^U} A_V^U \cong A_V^{U\cup v}$$

to be the composition of the coproduct in the Hopf algebra $\left(A_V^U, A_{V\setminus v}^U\right)$ with the isomorphism from Lemma 3.1.10

Given a sequence of distinct elements $u_1, u_2, \ldots, u_k \in V \subseteq S$, we define

$$\psi_V^{u_1,\ldots,u_k} : A_V \to A_V^{\{u_1,\ldots,u_k\}}$$

by the recursive formula

$$\psi_V^{u_i,\ldots,u_k} = \psi_V^{\{u_{i+1},\ldots,u_k\},u_i} \circ \psi_V^{u_{i+1},\ldots,u_k}.$$

Similarly, we define

$$\widetilde{\psi}_V^{u_1,\ldots,u_k} \colon A_V \to A_V^{\{u_1,\ldots,u_k\}}$$

using the reduced coproducts $\widetilde{\psi}_V^{\{u_{i+1},\ldots,u_k\},u_i}$.

Definition 3.1.13. *Let S be a finite set. An S-fold Hopf algebra A is a pre S-fold Hopf algebra A with the additional requirement that for every sequence $u_1, u_2, \ldots, u_k$ of distinct elements in $V \subseteq S$, and all permutations α of k,*

$$\psi_V^{u_{\alpha(1)},\ldots,u_{\alpha(k)}} = \psi_V^{u_1,\ldots,u_k} : A_V \to A_V^{\{u_1,\ldots,u_k\}}.$$

We denote this map

$$\psi_V^U \colon A_V \to A_V^U,$$

where $U = \{u_1, \ldots, u_k\}$.

In the case of $\Lambda_{T^-} H\mathbb{F}_p$, where the different coproducts come from the pinch map on the different circles, the extra requirement in the definition of an S-fold Hopf algebra, amounts to the fact that all the ways to go from $\Lambda_{T^V} H\mathbb{F}_p$ to $\Lambda_{(S^1\vee S^1)^{\times V}} H\mathbb{F}_p$ by pinching each circle once, are equal up to homotopy.

Definition 3.1.14. *A map from an S-fold Hopf algebra A to an S-fold Hopf algebra B is a map of pre S-fold Hopf algebras.*

It shouldn't come as a surprise that when the composition of the coproducts agree, the composition of the reduced coproducts agree. More precisely:

Proposition 3.1.15. *Let A be an S-fold Hopf algebra, and let $u_1, u_2, \dots, u_k$ be a sequence of distinct elements in $V \subseteq S$. Then*

$$\widetilde{\psi}_V^{u_{\alpha(1)},\dots,u_{\alpha(k)}} = \widetilde{\psi}_V^{u_1,\dots,u_k}.$$

for all permutations α of k.

Proof. By Proposition 3.1.11, it suffices to check the claim for transpositions, since $A^{\{u_i,\dots,u_k\}}$ is an $S \setminus \{u_i, \dots, u_k\}$-fold Hopf algebra for every $i \leq k$. Let $i, j \in V \subseteq S$. In cube notation this amounts to showing that

$$\begin{aligned}\widetilde{\psi}_V^{i,j} = \widetilde{\psi}_V^{\{j\},i} \circ \widetilde{\psi}_V^{j} &= \Big(\begin{bmatrix}\psi_V^i & \psi_V^i\end{bmatrix} - \begin{bmatrix}\mathrm{id} & \mathrm{id}\\ 1 & 1\end{bmatrix} - \begin{bmatrix}1 & 1\\ \mathrm{id} & \mathrm{id}\end{bmatrix} \Big) \circ \Big(\begin{bmatrix}\psi_V^j\end{bmatrix} - \begin{bmatrix}\mathrm{id} & 1\end{bmatrix} - \begin{bmatrix}1 & \mathrm{id}\end{bmatrix} \Big) \\ &= \Big(\begin{bmatrix}\psi_V^j\\ \psi_V^j\end{bmatrix} - \begin{bmatrix}\mathrm{id} & 1\\ \mathrm{id} & 1\end{bmatrix} - \begin{bmatrix}1 & \mathrm{id}\\ 1 & \mathrm{id}\end{bmatrix} \Big) \circ \Big(\begin{bmatrix}\psi_V^i\end{bmatrix} - \begin{bmatrix}\mathrm{id}\\ 1\end{bmatrix} - \begin{bmatrix}1\\ \mathrm{id}\end{bmatrix} \Big) = \widetilde{\psi}_V^{\{i\},j} \circ \widetilde{\psi}_V^{i} = \widetilde{\psi}_V^{j,i},\end{aligned}$$

where the horizontal direction of the cube is the j-th direction, the vertical direction is the i-th direction, $\begin{bmatrix}\psi_V^i & \psi_V^i\end{bmatrix} = \psi_V^{\{j\},i}$, and $\begin{bmatrix}\psi_V^j\\ \psi_V^j\end{bmatrix} = \psi_V^{\{i\},j}$.

Expanding the first two parentheses in the above expression we get

$$\begin{aligned}\widetilde{\psi}_V^{i,j} = \begin{bmatrix}\psi_V^i & \psi_V^i\end{bmatrix} \circ \begin{bmatrix}\psi_V^j\end{bmatrix} - \Big(\begin{bmatrix}\mathrm{id} & \mathrm{id}\\ 1 & 1\end{bmatrix} + \begin{bmatrix}1 & 1\\ \mathrm{id} & \mathrm{id}\end{bmatrix} \Big) \circ \begin{bmatrix}\psi_V^j\end{bmatrix} - \begin{bmatrix}\psi_V^i & \psi_V^i\end{bmatrix} \circ \big(\begin{bmatrix}\mathrm{id} & 1\end{bmatrix} + \begin{bmatrix}1 & \mathrm{id}\end{bmatrix} \big) \\ + \begin{bmatrix}\mathrm{id} & 1\\ 1 & 1\end{bmatrix} + \begin{bmatrix}1 & \mathrm{id}\\ 1 & 1\end{bmatrix} + \begin{bmatrix}1 & 1\\ \mathrm{id} & 1\end{bmatrix} + \begin{bmatrix}1 & 1\\ 1 & \mathrm{id}\end{bmatrix}.\end{aligned}$$

Since A is an S-fold Hopf algebra we have

$$\psi_V^{i,j} = \begin{bmatrix}\psi_V^i & \psi_V^i\end{bmatrix} \circ \begin{bmatrix}\psi_V^j\end{bmatrix} = \begin{bmatrix}\psi_V^j\\ \psi_V^j\end{bmatrix} \circ \begin{bmatrix}\psi_V^i\end{bmatrix} = \psi_V^{j,i}.$$

By naturality we have that

$$\begin{bmatrix}\psi_V^i & \psi_V^i\end{bmatrix} \circ \big(\begin{bmatrix}\mathrm{id} & 1\end{bmatrix} + \begin{bmatrix}1 & \mathrm{id}\end{bmatrix} \big) = \Big(\begin{bmatrix}\mathrm{id} & 1\\ \mathrm{id} & 1\end{bmatrix} + \begin{bmatrix}1 & \mathrm{id}\\ 1 & \mathrm{id}\end{bmatrix} \Big) \circ \begin{bmatrix}\psi_V^i\end{bmatrix},$$

and similarly for j, thus finishing the proof. □

We end this section by constructing some special S-fold Hopf algebras. In the case we are interested in, $\pi_*(\Lambda_{T^-} H\mathbb{F}_p)$, the simplest version correspond to the functor $\pi_*(\Lambda_{T_k^-} H\mathbb{F}_p)$.

Definition 3.1.16. *Let S be a finite set. We define a subcategory Δ of $V(S)$ to be* saturated *if it has the property that when $W \in \Delta$ then $V(W) \subseteq \Delta$.*

Definition 3.1.17. *Let S be a finite set and let Δ be a saturated subcategory of $V(S)$. Define a* partial S-fold Hopf algebra $A\colon \Delta \to \mathrm{CRings}$, *to be a functor A which for every $W \in \Delta$, is a W-fold Hopf algebra when restricted to W. Let $\overline{\Delta}$ be the subcategory $\bigcup_{W\in\Delta} [2]^W \subseteq \Delta$.*

The functor

$$\overline{A} : V(S) \to \mathrm{CRings}$$

defined by $\overline{A}(W) = \mathrm{colim}_{U\subseteq W, U\in\overline{\Delta}} A(U)$ has the structure of an S-fold Hopf algebra and we denote it the extension of A to S.

Given an S-fold Hopf algebra A, we define the restriction of A to Δ *to be the S-fold Hopf algebra which is the extension of the functor*

$$A|_{\Delta} : \Delta \to \mathrm{CRings}\,.$$

Since the category of Hopf algebras has all small colimits, and these are given as pair of colimits in commutative rings, it is clear that the extension of A is an S-fold Hopf algebra. All the properties of an S-fold Hopf algebra follows from functoriality of the colimit.

Definition 3.1.18. *Let S be a finite set, and let m be a positive even integer. Let $\Delta \subseteq V(S)$ be the full subcategory containing all sets with at most one element. Let $A : \Delta \to \mathrm{CRings}$ be the functor given by $A(\emptyset) = R$ and $A(\{s\}) = P_R(\mu_s)$, with $|\mu_s| = m$.*

We define $P_R(\mu_-)$, the polynomial S-fold Hopf algebra over R in degree m, *to be the extension of the functor A to all of S.*

When $m = 2$, the functor $\pi_*(\Lambda_{T_1^-} H\mathbb{F}_p)$ is isomorphic to $P(\mu_-)$.

Note that for $U = \{u_1, \ldots, u_k\}$ there is an isomorphism $P_R(\mu_U) \cong P_R(\mu_{u_1}, \ldots, \mu_{u_k})$, and for $u \in U$ the element μ_u is primitive in the Hopf algebra $(P_R(\mu_U), P_R(\mu_{U\setminus u}))$.

3.2 Coproduct in a Multifold Hopf Algebra

In this section we will state a proposition that we need when we calculate the multiplicative structure of $\pi_*(\Lambda_{T^n} H\mathbb{F}_p)$.

First we give Lucas' theorem about binomial coefficients, see Lemma 3C.6 in [Hat02] for a proof.

Proposition 3.2.1. *If p is a prime, then $\binom{n}{k} = \prod_i \binom{n_i}{k_i} \mod p$ where $n = \sum_i n_i p^i$ and $k = \sum_i k_i p^i$ with $0 \le n_i < p$ and $0 \le k_i < p$ are the p-adic representations of n and k.*

The convention is that $\binom{n}{k} = 0$ if $n < k$, and $\binom{n}{0} = 1$ for all $n \geq 0$.

Given an integer n divisible by p, we write $\frac{n}{p}$ for the image of $\frac{n}{p}$ under the ring map $\mathbb{Z} \to \mathbb{F}_p$. In the polynomial $\mathbb{F}_p$-Hopf algebra $P_{\mathbb{F}_p}(\mu)$, we write $\frac{\widetilde{\psi}(\mu^{p^i})}{p}$ for the image of $\frac{\widetilde{\psi}(\mu^{p^i})}{p}$ under the ring map $P_{\mathbb{Z}}(\mu_n) \to P_{\mathbb{F}_p}(\mu_n)$ given by mapping μ_n to μ_n. This is well defined since $\binom{p^i}{k}$ is divisible by p for all i and k with $0 < k < p^i$.

Lemma 3.2.2. *Let M be an $\mathbb{F}_p$-module, and let n be a natural number greater than 2. Let $\{r_{k,n-k}\}_{0<k<n}$ be a set of elements in M which satisfy the relations $\binom{a+b}{b} r_{a+b,c} = \binom{b+c}{b} r_{a,b+c}$ for all $a+b+c = n$ and $0 < a, c < n$. Then the following relations hold:*

1. *If $n = p^{m+1}$ for some $m \geq 0$, then*
$$r_{k,n-k} = \frac{\binom{n}{k}}{p} r_{p^m,(p-1)p^m}$$
for all $0 < k < n$.

2. *If $n = p^{m_1} + p^{m_2}$ with $m_1 < m_2$ and $k \neq p^{m_1}, p^{m_2}$, then $r_{k,n-k} = 0$.*

3. *If $n \neq p^{m+1}, p^{m_1} + p^{m_2}$ with $m_1 < m_2$, then*
$$r_{k,n-k} = \binom{n}{k} n_m^{-1} r_{p^m,n-p^m}$$
for all $0 < k < n$, where $n = n_0 + n_1 p^1 + \ldots + n_m p^m$ with $0 \leq n_i < p$ and $n_m \neq 0$ is the p-adic representation of n.

The only case which is not covered by the lemma is $n = p^{m_1} + p^{m_2}$ with $m_1 \neq m_2$, when the relations in the lemma doesn't give any relation between $r_{p^{m_1},p^{m_2}}$ and $r_{p^{m_2},p^{m_1}}$.

Proof. Given a set $\{r_{k,n-k}\}_{0<k<n}$ of elements in an abelian group, let $\sim$ be the equivalence relation generated by $\binom{a+b}{b} r_{a+b,c} \sim \binom{b+c}{b} r_{a,b+c}$. Let $\mathbb{F}_p\{r_{1,n-1}, \ldots, r_{n-1,1}\}$ be the free $\mathbb{F}_p$-module on the set $\{r_{1,n-1}, \ldots, r_{n-1,1}\}$. Since M is an $\mathbb{F}_p$-module, there is a homomorphism
$$\mathbb{F}_p\{r_{1,n-1}, \ldots, r_{n-1,1}\}/\!\sim\; \to M$$
defined by mapping $r_{k,n-k}$ to $r_{k,n-k}$. Hence it suffices to prove the lemma for the module $M = \mathbb{F}_p\{r_{1,n-1}, \ldots, r_{n-1,1}\}/\!\sim$.

Let
$$k = k_0 + k_1 p^1 + \ldots + k_j p^j$$
with $0 \leq k_i < p$ and $k_j \neq 0$, be the p-adic representations of k. Similarly, let
$$n = n_0 + n_1 p^1 + \ldots + n_m p^m$$
with $0 \leq n_i < p$ and $n_m \neq 0$ be the p-adic representation of n, except that when n is a power of p we express it as $n = p^{m+1}$.

The proof consists of two part. First we prove that unless both k and $n-k$ are powers of p, there is a sequence of equations expressing $r_{k,n-k}$ as a multiple of $r_{p^m,n-p^m}$. The second part is to identify the factor in this equation in terms of $\binom{n}{k}$.

We will now use the relations $\binom{a+b}{b} r_{a+b,c} = \binom{b+c}{b} r_{a,b+c}$ to express $r_{k,n-k}$ as a multiple of $r_{p^m,n-p^m}$. By Proposition 3.2.1 $\binom{k}{k-p^j} = \binom{k}{p^j} = k_j$, giving us the equation

$$r_{k,n-k} = \frac{\binom{n-p^j}{k-p^j}}{\binom{k}{k-p^j}} r_{p^j,n-p^j}.$$

If $j = m$ we are done. Otherwise, if $n > p^j + p^m$, the m-th coefficient in the p-adic expansion of $n - p^j$ is at least 1. Hence $\binom{n-p^j}{p^m} \neq 0$, and we have two equations

$$r_{p^j,n-p^j} = \frac{\binom{p^j+p^m}{p^m}}{\binom{n-p^j}{p^m}} r_{p^j+p^m,n-p^j-p^m} \qquad r_{p^j+p^m,n-p^j-p^m} = \frac{\binom{n-p^m}{p^j}}{\binom{p^j+p^m}{p^j}} r_{p^m,n-p^m}.$$

If $n < p^j + p^m$ there is an $i < j$ such that $n_i \neq 0$ and the i-th coefficient in the p-adic expansion of $n - p^j$ is n_i. Hence $\binom{n-p^j}{p^i} = n_i$ and $\binom{n-p^i}{p^m} = n_m$ and the four equations below move these powers of p back and forth

$$r_{p^j,n-p^j} = \frac{\binom{p^j+p^i}{p^i}}{\binom{n-p^j}{p^i}} r_{p^j+p^i,n-p^j-p^i} \qquad r_{p^j+p^i,n-p^j-p^i} = \frac{\binom{n-p^i}{p^j}}{\binom{p^j+p^i}{p^j}} r_{p^i,n-p^i}$$

$$r_{p^i,n-p^i} = \frac{\binom{p^i+p^m}{p^m}}{\binom{n-p^i}{p^m}} r_{p^i+p^m,n-p^m-p^i} \qquad r_{p^i+p^m,n-p^i-p^m} = \frac{\binom{n-p^m}{p^i}}{\binom{p^i+p^m}{p^i}} r_{p^m,n-p^m}.$$

Combining three or five equations, respectively, we get when $(k,n) \neq (p^j, p^j + p^m)$ with $j < m$, the equation

$$r_{k,n-k} = u r_{p^m,n-p^m}$$

where u is some element in $\mathbb{F}_p$.

To determine u we will take a detour through $\mathbb{Z}_{(p)}$, the integers localized at p. In the $\mathbb{Q}$-module $\mathbb{Q}\{r_{1,n-1}, \ldots, r_{n-1,1}\}/\sim$ we let $r_{1,n-1} = nr$. The formula $\binom{k}{1} r_{k,n-k} = \binom{n-k+1}{1} r_{k-1,n-k+1}$ and induction, give the equality

$$r_{k,n-k} = \frac{n-k+1}{k} r_{k-1,n-k+1} = \frac{n-k+1}{k} \binom{n}{k-1} r = \binom{n}{k} r$$

in $\mathbb{Q}\{r_{1,n-1}, \ldots, r_{n-1,1}\}/\sim$.

Thus, if $n = p^{m+1}$, then $r_{k,n-k} = \frac{\binom{p^{m+1}}{k}}{\binom{p^{m+1}}{p^m}} r_{p^m,(p-1)p^m} = \frac{\frac{\binom{p^{m+1}}{k}}{p}}{\binom{p^{m+1}-1}{p^m-1}} r_{p^m,(p-1)p^m}$, and when n is not a power of p, $r_{k,n-k} = \frac{\binom{n}{k}}{\binom{n}{p^m}} r_{p^m,n-p^m} = \binom{n}{k} n_m^{-1} r_{p^m,n-p^m}$.

By Proposition 3.2.1, $\binom{p^{m+1}}{k}$ is divisible by p for every k, but neither $\binom{p^{m+1}-1}{p^m-1}$ nor n_m are divisible by p. Hence these relations exists in $\mathbb{Z}_{(p)}\{r_{1,n-1},\dots,r_{n-1,1}\}/\!\sim\; \subseteq \mathbb{Q}\{r_{1,n-1},\dots,r_{n-1,1}\}/\!\sim$.

By the universal property of localization we get a map

$$f : \mathbb{Z}_{(p)}\{r_{1,n-1},\dots,r_{n-1,1}\}/\!\sim\; \to \mathbb{F}_p\{r_{1,n-1},\dots,r_{n-1,1}\}/\!\sim$$

by mapping $r_{k,n-k}$ to $r_{k,n-k}$.

By Proposition 3.2.1, $\binom{p^{m+1}-1}{p^m-1} = 1 \mod p$ and $\binom{n}{n_m p^m} = 1 \mod p$. So when n is not a power of p, $f\left(\frac{\binom{n}{k}}{\binom{n}{n_m p^m}}\right) = \binom{n}{k} = u$ proving part 3. In particular when $k \neq p^j, p^m$ the binomial coefficients $\binom{p^j+p^m}{k}$ are equal to 0, proving part 2 of the lemma.

When $n = p^{m+1}$, then $f\left(\frac{\frac{\binom{p^{m+1}}{k}}{p}}{\binom{p^{m+1}-1}{p^m-1}}\right) = \frac{\binom{p^{m+1}}{k}}{p} = u$ proving part 1. $\square$

Definition 3.2.3. *Let A and B be R-algebras. An R-algebra homomorphism from A to B in degree less than or equal to q, is an R-module homomorphism $f : A \to B$ which induces an R-algebra homomorphism on the quotients $A/A_{>q} \to B/B_{>q}$. We define the similar notion for coalgebras and Hopf algebras.*

First we state a similar proposition about ordinary Hopf algebras. Let $\mathbb{P}$ denote the set of integers $\{p^0, p^1, p^2 \dots\} \subseteq \mathbb{N}$.

Proposition 3.2.4. *Let R be an $\mathbb{F}_p$-algebra and let A be an R-Hopf algebra such that:*

1. *There is a sub R-Hopf Algebra $P_R(\mu) \subseteq A$.*
2. *As an R-algebra in degree less than or equal to q, this is part of a splitting $P_R(\mu) \subseteq A \stackrel{\mathrm{pr}}{\to} P_R(\mu)$.*
3. *In degree less than or equal to $q-1$ this is a splitting as an R-Hopf algebra, i.e., the following diagram commutes*

$$\begin{tikzcd} A \arrow[r, "\mathrm{pr}"] \arrow[d, "\psi_A"] & P_R(\mu) \arrow[d, "\psi_{P_R(\mu)}"] \\ A \otimes A \arrow[r, "\mathrm{pr}\otimes\mathrm{pr}"] & P_R(\mu) \otimes P_R(\mu) \end{tikzcd}$$

in degree less than or equal to $q-1$.

Let x be an element in degree q in $\ker(\mathrm{pr})$. *Then there exists elements $r_n \in R$ for $n \in \mathbb{N}_+$, and $t_{(n_1<n_2)} \in R$ for pairs $(n_1 < n_2) \in \mathbb{P} \times \mathbb{P}$, such that the coproduct satisfy*

$$(\mathrm{pr}_{P_R(\mu)} \otimes \mathrm{pr}_{P_R(\mu)}) \circ \psi(x) = \sum_{n\in\mathbb{N}_+} r_n \widetilde{\psi}(\mu^n) + \sum_{n\in\mathbb{P}} r_n \frac{\widetilde{\psi}(\mu^n)}{p} + \sum_{(n_1<n_2)\in\mathbb{P}\times\mathbb{P}} t_{(n_1<n_2)} \mu^{n_1} \otimes \mu^{n_2}.$$

Recall that $\widetilde{\psi}(\mu^n) = \sum_{k=1}^{n-1} \binom{n}{k} \mu^k \otimes \mu^{n-k}$ and that when n is a power of p, $\binom{n}{k}$ is divisible by p for all $0 < k < n$. Hence, $\frac{\widetilde{\psi}(\mu^n)}{p}$ is well defined.

Observe, that since $\widetilde{\psi}(\mu^{p^i}) = 0$ for all $i \geq 1$, the first sum is independent of the values of r_{p^i}. An example where this proposition applies is the dual Steenrod algebra A_* with $P(\bar{\xi}_1) \subseteq A_*$. Then $\widetilde{\psi}(\bar{\xi}_2) = \bar{\xi}_1 \otimes \bar{\xi}_1^p$ so $t_{1,p} = 1$.

Proof. In general $(\mathrm{pr}_{P_R(\mu)} \otimes \mathrm{pr}_{P_R(\mu)}) \circ \psi(x) = \sum_{n \in \mathbb{N}} \sum_{a+b=n} r_{a,b} \mu^a \otimes \mu^b$, for some $r_{a,b} \in R$. Since $(\epsilon \otimes \mathrm{id})\psi = (\mathrm{id} \otimes \epsilon)\psi = \mathrm{id}$ and $x \in \ker(\mathrm{pr})$, we must have that $r_{0,n} = r_{n,0} = 0$ for all n.

In degree less than or equal to q, there is a factorization of $\widetilde{\psi}$ as

$$A_{\leq q} \xrightarrow{\widetilde{\psi}} \sum_{\substack{k+l=q \\ k,l>0}} A_{\leq k} \otimes A_{\leq l} \subseteq (A \otimes A)_{\leq q}.$$

Tensoring the diagram from assumption 3 in the proposition with $A_{\leq l}$, gives a commutative diagram

$$\begin{tikzcd} \sum\limits_{\substack{k+l=q \\ k,l>0}} A_{\leq k} \otimes A_{\leq l} \arrow[r, "\sum \widetilde{\psi} \otimes \mathrm{id}"] \arrow[d, "\sum \mathrm{pr} \otimes \mathrm{pr}"] & \sum\limits_{\substack{k+l=q \\ k,l>0}} \sum\limits_{\substack{i+j=k \\ i,j>0}} A_{\leq i} \otimes A_{\leq j} \otimes A_{\leq l} \arrow[d, "\sum \mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr}"] \\ \sum\limits_{\substack{k+l=q \\ k,l>0}} P_R(\mu)_{\leq k} \otimes P_R(\mu)_{\leq l} \arrow[r, "\sum \widetilde{\psi} \otimes \mathrm{id}"] & \sum\limits_{\substack{k+l=q \\ k,l>0}} \sum\limits_{\substack{i+j=k \\ i,j>0}} P_R(\mu)_{\leq i} \otimes P_R(\mu)_{\leq j} \otimes P_R(\mu)_{\leq l}. \end{tikzcd}$$

There is also a similar diagram for $\mathrm{id} \otimes \widetilde{\psi}$. Hence we have

$$(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr})(\widetilde{\psi} \otimes \mathrm{id})\widetilde{\psi}(x) = \sum_{n \in \mathbb{N}} \sum_{d+c=n} r_{d,c} \sum_{a+b=d} \binom{d}{b} \mu^a \otimes \mu^b \otimes \mu^c,$$

and

$$(\mathrm{pr} \otimes \mathrm{pr} \otimes \mathrm{pr})(\mathrm{id} \otimes \widetilde{\psi})\widetilde{\psi}(x) = \sum_{n \in \mathbb{N}} \sum_{a+d=n} r_{a,d} \sum_{b+c=d} \binom{d}{b} \mu^a \otimes \mu^b \otimes \mu^c.$$

From coassociativity of $\widetilde{\psi}$ we know that the coefficients in front of $\mu^a \otimes \mu^b \otimes \mu^c$ in the two expressions above must be equal. Hence there are relations $\binom{a+b}{b} r_{a+b,c} = \binom{b+c}{b} r_{a,b+c}$, for all $a, c \geq 1$ and $b \geq 0$.

Given such relations, if $n = p^{m+1}$, then by Lemma 3.2.2 $r_{k,n-k} = \frac{\binom{n}{k}}{p} r_{p^m,(p-1)p^m}$, and we let $r_n = r_{p^m,(p-1)p^m}$. If $n = p^{m_1} + p^{m_2}$ with $m_1 < m_2$, then $r_{k,n-k} = 0$ when $k \neq p^{m_1}, p^{m_2}$. We let $r_n = r_{p^{m_2},p^{m_1}}$ and $t_{(p^{m_1} < p^{m_2})} = r_{p^{m_1},p^{m_2}} - r_{p^{m_2},p^{m_1}}$. Otherwise, let $n = n_0 + n_1 p^1 + \ldots + n_m p^m$, with $0 \leq n_i < p$ and $n_m \neq 0$, be the p-adic representation of n. Then $r_{k,n-k} = \binom{n}{k} n_m^{-1} r_{p^m,n-p^m}$, and we let $r_n = n_m^{-1} r_{p^m,n-p^m}$. $\square$

The next Proposition is similar to the previous proposition, but involves S-fold Hopf algebras. Although they are similar, when S contains exactly one element the next proposition doesn't specialize to the previous proposition, since in assumption 2 in the next proposition, $\widetilde{A} = R$ giving an impossible splitting $P_R(\mu) \to R \to P_R(\mu)$.

Given a finite set $U = \{u_1, \ldots, u_k\}$ we write $P_R(\mu_U)$ for the polynomial ring $P_R(\mu_{u_1}, \ldots, \mu_{u_k})$, and given an element $m \in \mathbb{N}^V$ where $U \subseteq V$, we let μ_U^m in $P_R(\mu_U)$ denote the product $\mu_{u_1}^{m_{u_1}} \cdots \mu_{u_k}^{m_{u_k}}$.

Proposition 3.2.5. *Let A be an S-fold Hopf algebra such that:*

1. *$R = A_\emptyset$ is an $\mathbb{F}_p$-algebra.*
2. *There is a splitting of S-fold Hopf algebras $P_R(\mu_-) \xrightarrow{f} \widetilde{A} \xrightarrow{\mathrm{pr}} P_R(\mu_-)$, where $\widetilde{A}$ is the restriction of A, as in Definition 3.1.17, to the full subcategory of $V(S)$ not containing S.*
3. *In degree less than or equal to q, the map pr can be extended to A_S, i.e., in degree less than or equal to q, there is an R-algebra homomorphism $\mathrm{pr} : A_S \to P_R(\mu_S)$ (see Definition 3.2.3) such that the following diagram commutes*

$$\begin{tikzcd} \widetilde{A}_S \arrow[r] \arrow[d, "\mathrm{pr}_S"] & A_S \arrow[dl, "\mathrm{pr}"] \\ P_R(\mu_S), & \end{tikzcd}$$

in degree less than or equal to q.

4. *For all $s \in S$, the map $\mathrm{pr}\colon \big(A_S, A_{S\setminus s}\big) \to \big(P_R(\mu_S), P_R(\mu_{S\setminus s})\big)$ is a map of Hopf algebras in degree less than or equal to $q-1$.*

Let x be an element in $\bigcap_{s\in S} \ker(\epsilon_S^s : A_S \to A_{S\setminus s}) \subseteq A_S$ of degree q. If $x \in \ker(\mathrm{pr})$ and $s \in S$, then there exist elements $r_b \in R$ for $b \in \mathbb{N}_+^{\times S}$ such that for every $s \in S$,

$$\begin{aligned} \big[\mathrm{pr} \quad \mathrm{pr}\big] \circ \psi_S^s(x) = \sum_{b\in\mathbb{N}_+^{\times S}} r_b \mu_{S\setminus s}^b \widetilde{\psi}^s(\mu_s^{b_s}) + \sum_{b\in\mathbb{P}^{\times S}} r_{b,s}\mu_{S\setminus s}^b \frac{\widetilde{\psi}^s(\mu_s^{b_s})}{p} \\ + \sum_{b\in\mathbb{P}^{S\setminus s}} \sum_{c_1<c_2\in\mathbb{P}\times\mathbb{P}} t_{b,c_1<c_2,s}\mu_{S\setminus s}^b \big[\mu_s^{c_1} \quad \mu_s^{c_2}\big], \end{aligned} \tag{3.2.6}$$

where b_s is the s-th component of b, and $r_{b,s}$ and $t_{b,c_1<c_2,s}$ are elements in R.

An important observation is that in the first sum, the coefficients r_b are independent of the element s. The $\mathbb{P}^{\times S}$ part in the first sum is zero since $\widetilde{\psi}^s(\mu_s^{p^i}) = 0$ for all $i \geq 0$.

Proof. In this proof we will compare $\widetilde{\psi}^{i,k}(x)$ with $\widetilde{\psi}^{k,i}(x)$ for all pair of elements $i \neq k$ in S, where the definition of $\widetilde{\psi}_S^{k,i}$ is found in Definition 3.1.13.

For every element $i \in S$ the ring A_S is an $A_{S\setminus i}$-Hopf algebra and $A_{S\setminus i}$ is an $\mathbb{F}_p$-algebra since $R = A_\emptyset = \mathbb{F}_p$. Assumption 2 and the unit $\eta_S^i : A_{S\setminus i} \to A_S$ induces an inclusion $P_{A_{S\setminus i}}(\mu_i) \cong P_R(\mu_S) \otimes_{P_R(\mu_{S\setminus i})} A_{S\setminus i} \to \widetilde{A}_S \to A_S$ so assumption 1 in Proposition 3.2.4 is satisfied for the Hopf algebra $(A_S, A_{S\setminus i})$. The splitting in assumption 2 in Proposition 3.2.4 comes from the homomorphism $A_S \to P_R(\mu_S) \otimes_{P_R(\mu_{S\setminus i})} A_{S\setminus i} \cong P_{A_{S\setminus i}}(\mu_i)$ induced by ϵ_S^i and the splitting in assumption 2. From assumption 4 this splitting induces a map of Hopf algebras

$$\big(A_S, A_{S\setminus i}\big) \to \big(P_R(\mu_S) \otimes_{P_R(\mu_{S\setminus i})} A_{S\setminus i}, P_R(\mu_{S\setminus i}) \otimes_{P_R(\mu_{S\setminus i})} A_{S\setminus i}\big) \cong \big(P_{A_{S\setminus i}}(\mu_i), A_{S\setminus i}\big),$$

satisfying assumption 3 in Proposition 3.2.4.

By Proposition 3.2.4, there exist elements $r_{b,i}$ and $t_{b,c_1<c_2,i}$ in R such that

$$\begin{bmatrix}\mathrm{pr}\\ \mathrm{pr}\end{bmatrix} \circ \psi_S^i(x) = \sum_{b\in\mathbb{N}^S} r_{b,i}\mu_{S\setminus i}^b \widetilde{\psi}(\mu_i^{b_i}) + \sum_{b\in\mathbb{N}^{S\setminus i} | b_i \in \mathbb{P}} r_{b,i}\mu_{S\setminus i}^b \frac{\widetilde{\psi}(\mu_i^{b_i})}{p} + \sum_{b\in\mathbb{N}^S} \sum_{c_1<c_2\in\mathbb{P}\times\mathbb{P}} t_{b,c_1<c_2,i}\mu_{S\setminus i}^b \begin{bmatrix}\mu_i^{c_1}\\ \mu_i^{c_2}\end{bmatrix}. \quad (3.2.7)$$

Observe that if $b_i = 1$, we can choose $r_{b,i}$ arbitrary.

We will now show that if $b_i \geq 2$ and $b_k = 0$ for some $k \neq i$, then $r_{b,i} = 0$. The counits ϵ_S^k and $\epsilon_{S\setminus i}^k$ induce a map of Hopf algebras $(A_S, A_{S\setminus i}) \to (A_{S\setminus k}, A_{S\setminus\{i,k\}})$. Since x is in $\bigcap_{s\in S}\ker(\epsilon_S^s\colon A_S \to A_{S\setminus s})$, we have $\psi_{S\setminus k}^i \circ \epsilon_S^k(x) = 0$. If $r_{b,i} \neq 0$, then $\epsilon_S^k \otimes \epsilon_S^k(\psi_S^i(x)) \neq 0$ so the commutative diagram

$$\begin{tikzcd} A_S \arrow[r, "\psi_S^i"] \arrow[d, "\epsilon_S^k"] & A_S \otimes_{A_{S\setminus i}} A_S \arrow[d, "\epsilon_S^k\otimes\epsilon_S^k"] \\ A_{S\setminus k} \arrow[r, "\psi_{S\setminus k}^i"] & A_{S\setminus k} \otimes_{A_{S\setminus\{i,k\}}} A_{S\setminus k} \end{tikzcd}$$

gives a contradiction. Thus $r_{b,i} = 0$.

From assumption 4 in the proposition, we get a commutative diagram

$$\begin{tikzcd} \ker(\epsilon_S^i) \arrow[r, "\widetilde{\psi}_S^i"] \arrow[d, "\widetilde{\psi}_S^i"] & A_S^{\{i\}} \arrow[r, "\widetilde{\psi}_S^{\{i\},k}"] & A_S^{\{i,k\}} \arrow[d, "\mathrm{pr}"] \\ A_S^{\{i\}} \arrow[r, "\mathrm{pr}"] & P_R(\mu_S)^{\{i\}} \arrow[r, "\widetilde{\psi}_S^{\{i\},k}"] & P_R(\mu_S)^{\{i,k\}}, \end{tikzcd}$$

in degree less than or equal to q, where the composition of the two morphisms on the top is the definition of $\widetilde{\psi}_S^{k,i}$. The diagram commute in degree less than or equal to q, and not just $q-1$, since we use the reduced coproduct.

From this diagram we have the formula

$$\begin{aligned}\begin{bmatrix}\mathrm{pr} & \mathrm{pr}\\ \mathrm{pr} & \mathrm{pr}\end{bmatrix} \circ \widetilde{\psi}_S^{k,i}(x) = &\sum_{b\in\mathbb{N}_+^S}\sum_{0<a_i<b_i}\sum_{0<a_k<b_k} r_{b,i}\binom{b_k}{a_k}\binom{b_i}{a_i}\mu^b_{S\setminus\{i,k\}}\begin{bmatrix}\mu_i^{a_i}\mu_k^{a_k} & \mu_k^{b_k-a_k}\\ \mu_i^{b_i-a_i} & 1\end{bmatrix}\\ &+\sum_{b\in\mathbb{N}_+^S|b_i\in\mathbb{P}}\sum_{0<a_i<b_i}\sum_{0<a_k<b_k} r_{b,i}\binom{b_k}{a_k}\frac{\binom{b_i}{a_i}}{p}\mu^b_{S\setminus\{i,k\}}\begin{bmatrix}\mu_i^{a_i}\mu_k^{a_k} & \mu_k^{b_k-a_k}\\ \mu_i^{b_i-a_i} & 1\end{bmatrix}\\ &+\sum_{b\in\mathbb{N}_+^{S\setminus i}}\sum_{c_1<c_2\in\mathbb{P}\times\mathbb{P}}\sum_{0<a_k<b_k} t_{b,c_1<c_2,i}\binom{b_k}{a_k}\mu^b_{S\setminus\{i,k\}}\begin{bmatrix}\mu_i^{c_1}\mu_k^{a_k} & \mu_k^{b_k-a_k}\\ \mu_i^{c_2} & 1\end{bmatrix}.\end{aligned}$$

The three lines correspond to the three summands in equation 3.2.7.

Since A is an S-fold Hopf algebra, $\widetilde{\psi}_S^{k,i} = \widetilde{\psi}_S^{i,k}$ so

$$\begin{bmatrix}\mathrm{pr} & \mathrm{pr}\\ \mathrm{pr} & \mathrm{pr}\end{bmatrix} \circ \widetilde{\psi}_S^{k,i}(x) = \begin{bmatrix}\mathrm{pr} & \mathrm{pr}\\ \mathrm{pr} & \mathrm{pr}\end{bmatrix} \circ \widetilde{\psi}_S^{i,k}(x).$$

In this equation we will now compare the coefficient in front of $\mu^b_{S\setminus\{i,k\}}\begin{bmatrix}\mu_i^{a_i}\mu_k^{a_k} & \mu_k^{b_k-a_k}\\ \mu_i^{b_i-a_i} & 1\end{bmatrix}$ for $b\in\mathbb{N}_+^S$, with $0<a_j<b_j$.

We will say that an integer $b_i \geq 2$ is type 1 if b_i is equal to a power of the prime p, type 2 if b_i is equal to a sum of two distinct powers of p, and type 3 otherwise.

Case 1, both b_i and b_k are type 3:

We get the equation

$$\binom{b_k}{a_k}\binom{b_i}{a_i}r_{b,i} = \binom{b_i}{a_i}\binom{b_k}{a_k}r_{b,k}.$$

Since neither b_i nor b_k are of type 1, there exists integers $0<a_i<b_i$ and $0<a_k<b_k$ such that $\binom{b_i}{a_i}\neq 0$ and $\binom{b_k}{a_k}\neq 0$. Thus $r_{b,i}=r_{b,k}$.

Case 2, b_i is type 2 and b_k is type 3:

Let $b_i = p^j + p^l$ with $j<l$. When $a_i = p^j$ we get the equation

$$\binom{b_k}{a_k}\binom{b_i}{p^j}r_{b,i} + \binom{b_k}{a_k}t_{b,p^j<p^l,i} = \binom{b_i}{p^j}\binom{b_k}{a_k}r_{b,k},$$

and when $a_i = p^l$ we get the equation

$$\binom{b_k}{a_k}\binom{b_i}{p^l}r_{b,i} = \binom{b_i}{p^l}\binom{b_k}{a_k}r_{b,k}.$$

From Proposition 3.2.1 we know that $\binom{b_i}{p^j} = \binom{b_i}{p^l} = 1$. Since b_k is not of type 1, there exists an a_k such that $\binom{b_k}{a_k}\neq 0$. The last equation thus gives $r_{b,i}=r_{b,k}$, and the second equation becomes $r_{b,i}+t_{b,p^j<p^l,i} = r_{b,k}$, so $t_{b,p^j<p^l,i}$ must be equal to 0.

Case 3, b_i is type 1 and b_k is type 3:
We get the equation

$$\binom{b_k}{a_k}\frac{\binom{b_i}{a_i}}{p} r_{b,i} = \binom{b_i}{a_i}\binom{b_k}{a_k} r_{b,k}.$$

Since b_i is equal to a power of p, $\binom{b_i}{a_i} = 0$ for all $0 < a_i < b_i$. Hence the right hand side of the equation is always 0. Since b_k is not of type 1, there exists an a_k such that $\binom{b_k}{a_k} \neq 0$, and if $b_i = p^{m+1}$, we have $\frac{\binom{b_i}{p^m}}{p} = 1$. Thus $r_{b,i} = 0$.
Case 4, both b_i and b_k are type 2:
Very similar to case 2. Let $b_i = p^{j_i} + p^{l_i}$ and $b_k = p^{j_k} + p^{l_k}$ with $j_i < l_i$ and $j_k < l_k$. When $a_i = p^{j_i}$ and $a_k = p^{l_k}$ we get the equation

$$\binom{b_k}{p^{l_k}}\binom{b_i}{p^{j_i}} r_{b,i} + \binom{b_k}{p^{l_k}} t_{b,p^{j_i}<p^{l_i},i} = \binom{b_i}{p^{j_i}}\binom{b_k}{p^{l_k}} r_{b,k},$$

and when $a_i = p^{l_i}$ and $a_k = p^{l_k}$we get the equation

$$\binom{b_k}{p^{l_k}}\binom{b_i}{p^{l_i}} r_{b,i} = \binom{b_i}{p^{l_i}}\binom{b_k}{p^{l_k}} r_{b,k}.$$

From Proposition 3.2.1 we know that $\binom{b_i}{p^{j_i}} = \binom{b_i}{p^{l_i}} = \binom{b_k}{p^{j_k}} = \binom{b_k}{p^{l_k}} = 1$. The last equation thus gives that $r_{b,i} = r_{b,k}$, and so $t_{b,p^{j_i}<p^{l_i},i}$ must be equal to 0.
Case 5, b_i is type 2 and b_k is type 1:
Let $b_i = p^j + p^l$ with $j < l$. When $a_i = p^j$ we get the equation

$$\binom{b_k}{a_k}\binom{b_i}{p^j} r_{b,i} + \binom{b_k}{a_k} t_{b,p^j<p^l,i} = \binom{b_i}{p^j}\frac{\binom{b_k}{a_k}}{p} r_{b,k},$$

and when $a_i = p^l$ we get the equation

$$\binom{b_k}{a_k}\binom{b_i}{p^l} r_{b,i} = \binom{b_i}{p^l}\frac{\binom{b_k}{a_k}}{p} r_{b,k}.$$

From Proposition 3.2.1 we know that $\binom{b_i}{p^j} = \binom{b_i}{p^l} = 1$. Since $b_k = p^{m+1}$ for some m, $\binom{b_k}{a_k} = 0$ for all $0 < a_k < b_k$, but $\frac{\binom{b_k}{p^m}}{p} = 1$. In the last equation the left hand side is always equal to zero, and hence $r_{b,k} = 0$. The first equation doesn't give any information about $r_{b,i}$ and $t_{b,p^j<p^l,i}$.
Case 6, both b_i and b_k are type 1:
We get the equation

$$\binom{b_k}{a_k}\frac{\binom{b_i}{a_i}}{p} r_{b,i} = \binom{b_i}{a_i}\frac{\binom{b_k}{a_k}}{p} r_{b,k}.$$

Both sides are 0 for all $0 < a_i < b_i$ and $0 < a_k < b_k$, so we don't get any information about $r_{b,i}$ nor $r_{b,k}$.

From these six cases we will now deduce equation 3.2.6 in the proposition.

Consider an S-tuple $b \in \mathbb{N}_+^S$. These fall in five classes:

1. All b_i's are equal to 1.
2. At least one b_i is of type 3.
3. No b_i is of type 3, but at least two are of type 2.
4. Exactly one b_i is of type 2 and the rest are of type 1 or equal to 1.
5. All b_i-s of type 1 or equal to 1.

We will now consider these cases one by one.

1. We can choose $r_{b,i}$ arbitrary, since they don't affect the sum so we let $r_b = 0$.
2. Case 2 shows that for all $b_k = p^j + p^l$ of type 2, $t_{b,p^j<p^l,k} = 0$ and $r_{b,k} = r_{b,i}$. From case 3, $r_{b,k} = 0$ for all k with b_k of type 1. Finally, case 1 says that $r_{b,k} = r_{b,i}$ for all b_k of type 3, so we let $r_b = r_{b,i}$. This correspond to the first sum in equation 3.2.6.
3. Assume $b_i = p^{j_i} < p^{l_i}$ and $b_k = p^{j_k} < p^{l_k}$ are of type 2. Then using case 4 twice, we get that $t_{b,p^j<p^l,k} = t_{b,p^j<p^l,k} = 0$ and $r_{b,k} = r_{b,i}$. If b_j is of type 1, case 5 shows that $r_{b,j} = 0$. We choose $r_b = r_{b,i}$, and this also correspond to the first sum in equation 3.2.6.
4. Assume b_i is of type 2. By case 5, for all b_k of type 1 $r_{b,k} = 0$, but nothing can be said about $r_{b,i}$ nor $t_{b,p^j<p^l,i}$. We choose $r_b = r_{b,i}$, and this correspond to one summand in the first sum and one summand in the last sum in equation 3.2.6.
5. This correspond to the middle sum in equation 3.2.6.

□

3.3 Calculating the Homotopy Groups of $\Lambda_{S^n} H\mathbb{F}_p$

In this section we will calculate $\pi_*(\Lambda_{S^n} H\mathbb{F}_p)$, when $n \leq 2p$ and p is odd. First we describe an $\mathbb{F}_p$-Hopf algebra B_n, and then we show that $L(S^n) \cong B_n$. In the end of this section we state several lemmas, which we need in the next section, about the degree of certain elements in B_n. Given a space X, we write $L(X)$ for the graded ring $\pi_*(\Lambda_X H\mathbb{F}_p)$.

Definition 3.3.1. *Given the letters μ, ϱ, ϱ^k and φ^k for $k \geq 0$. Define an* admissible word *to be a word such that*

1. *It ends with the letter μ.*

2. *The letter μ is immediately preceded by ϱ.*

3. *The letter ϱ is immediately preceded by ϱ^k.*

4. *The letters ϱ^k and φ^k are immediately preceded by ϱ or φ^l for some $l \geq 0$.*

We define a monic word *to be admissible word that begins with one of the letters $\varrho, \varrho^0, \varphi^0$ or μ.*

We define the degree of μ to be 2, and recursively define the degree of an admissible word by the rules

$$
\begin{aligned}
|\varrho x| &= 1 + |x| \\
|\varrho^k x| &= p^k(1 + |x|) \\
|\varphi^k x| &= p^k(2 + p|x|).
\end{aligned}
$$

An example of an admissible word of length 6 is $\varrho\varphi^m\varphi^l\varrho^k\varrho\mu$.

Lemma 3.3.2. *The following statements hold:*

1. *An admissible word of length at least 3 always ends with the letter combination $\varrho^k\varrho\mu$*

2. *There is at most $\frac{n-1}{2}$ occurrences of the letter ϱ in an admissible word of even degree of length n.*

3. *Every admissible word of length n has degree at least $n+1$*

4. *IAll admissible words of odd degree begin with the letter ϱ.*

5. *Given $0 \leq k < p$. A monic word of degree $2k$ modulo $2p$ is either equal to $(\varrho^0\varrho)^{k-1}\mu$, or starts with the letter combination $(\varrho^0\varrho)^{k-1}\varphi^0$ or $(\varrho^0\varrho)^k$. A monic word of degree $2k+1$ modulo $2p$ is either equal to $\varrho(\varrho^0\varrho)^{k-1}\mu$, or starts with the letter combination $\varrho(\varrho^0\varrho)^{k-1}\varphi^0$ or $\varrho(\varrho^0\varrho)^k$.*

Proof. All but the last statement is obvious. The last statement follows from the observation that the degree of a word starting with φ^l or $\varrho^l\varrho$ is 0 modulo $2p$, when $l \geq 1$, and the degree of a word starting with φ^0 is 2 modulo $2p$. □

Definition 3.3.3. *We define B_1 to be the polynomial $\mathbb{F}_p$-Hopf algebra $P(\mu)$, with $|\mu| = 2$. Given $n \geq 2$, we define the $\mathbb{F}_p$-Hopf algebra B_n to be equal to the tensor product of exterior algebras on all monic words of length n of odd degree, and divided power algebras on all monic words of length n of even degree.*

For example, the monic words of length 4 are $\varrho\varrho^k\varrho\mu$ and $\varphi^0\varrho^k\varrho\mu$. Hence, $B_4 = \bigotimes_{k\geq 0}(E(\varrho\varrho^k\varrho\mu) \otimes \Gamma(\varphi\varrho^k\varrho\mu))$.

Proposition 3.3.4. *When $n \geq 2$ there is an isomorphism of $\mathbb{F}_p$-Hopf algebras*

$$B_n \cong \mathrm{Tor}^{B_{n-1}}(\mathbb{F}_p, \mathbb{F}_p).$$

Proof. By Lemma 3.3.2, the odd degree monic words are those starting with ϱ, while the even degree monic words are those starting with ϱ^0, φ^0 or μ. From Proposition A.2.10, we get that $B_2 = E(\varrho\mu) \cong \mathrm{Tor}^{P(\mu)}(\mathbb{F}_p, \mathbb{F}_p)$. When $n \geq 3$,

$$B_{n-1} = \bigotimes_{i \in I} E(y_i) \otimes \bigotimes_{j \in J} \Gamma(z_j) \cong \bigotimes_{i \in I} E(y_i) \otimes \bigotimes_{j \in J} \bigotimes_{k \geq 0} P_p(\gamma_{p^k}(z_j)),$$

where y_i runs over all admissible words of length $n-1$, starting with ϱ and z_j runs over all admissible words of length $n-1$ starting with ϱ^0 or φ^0. The isomorphism is only an isomorphism of $\mathbb{F}_p$-algebras. By Proposition A.2.10, and the Künneth isomorphism we have an isomorphism of $\mathbb{F}_p$-Hopf algebras

$$\mathrm{Tor}^{B_{n-1}}(\mathbb{F}_p, \mathbb{F}_p) \cong \bigotimes_{i \in I} \Gamma(\sigma y_i) \otimes \bigotimes_{j \in J} \bigotimes_{k \geq 0} E(\sigma\gamma_{p^k}(z_j)) \otimes \Gamma(\varphi\gamma_{p^k}(z_j)),$$

where $|\sigma x| = 1 + |x|$ and $|\varphi x| = 2 + p|x|$.

Now, there is an homomorphism of $\mathbb{F}_p$-Hopf algebras $\mathrm{Tor}^{B_{n-1}}(\mathbb{F}_p, \mathbb{F}_p) \to B_n$ given by mapping the element σy_i to the monic word $\varrho^0 y_i$, and if $z_j = \varrho^0 z_j'$ we map $\sigma\gamma_{p^k}(z_j)$ to the monic word $\varrho\varrho^k z_j'$ and $\varphi\gamma_{p^k}(z_j)$ to the monic word $\varphi^0\varrho^k z_j'$, while if $z_j = \varphi^0 z_j'$ we map $\sigma\gamma_{p^k}(z_j)$ to the monic word $\varrho\varphi^k z_j'$ and $\varphi\gamma_{p^k}(z_j)$ to the monic word $\varphi^0\varphi^k z_j'$.

The monic words of odd degree of length n is equal to the set of words ϱx, where x runs over all admissible words of length $n-1$ starting with ϱ^k or φ^k. Similarly, the monic words of even degree of length n is equal to the set of words $\varrho^0 x$ and $\varphi^0 z$ where x runs over all admissible words that starts with ϱ and z runs over all admissible words that starts with ϱ^k or φ^k for $k \geq 0$. Hence, the homomorphism above is an $\mathbb{F}_p$-Hopf algebra isomorphism $\mathrm{Tor}^{B_{n-1}}(\mathbb{F}_p, \mathbb{F}_p) \cong B_n$. □

Before we calculate $L(S^n)$, we state a technical lemma which is needed in the proof. Given a graded module A, we will write A_i for the part in degree i. Recall that $P(B_n)$ is the submodule of primitive elements.

Lemma 3.3.5. *If $2 \leq n \leq 2p$, then $P(B_n)_{2pi-1} = P(B_n)_{2pi} = 0$ for all $i \geq 2$.*

Proof. In a divided power algebra $\Gamma(x)$, the only primitive element is $\gamma_1(x)$, so, by Proposition A.1.8 and 3.3.4, the primitive elements in B_n are linear combinations of monic words of length n.

We will show that the shortest monic word in degree 0 modulo $2p$ and of degree greater than $2p$, has length $2p + 2$.

By part 5 of Lemma 3.3.2, a monic word of degree 0 modulo $2p$ must either be equal to $(\varrho^0\varrho)^{p-1}\mu$, or start with the letter combination $(\varrho^0\varrho)^{p-1}\varphi^0$ or $(\varrho^0\varrho)^p$.

The word $(\varrho^0\varrho)^{p-1}\mu$ has degree $2p$, so the shortest monic word in degree 0 modulo $2p$ of degree greater than $2p$, is thus $(\varrho^0\varrho)^{p-1}\varphi^0\varrho^k\varrho\mu_0$, for $k \geq 1$, and it has length $2p+2$.

By a similar argument, we get that the shortest monic word in degree -1 modulo $2p$ of degree greater than $2p$, is $\varrho(\varrho^0\varrho)^{p-2}\varphi^0\varrho^k\varrho\mu_0$, for $k \geq 1$, and it has length $2p+1$. □

Applying the functor $\Lambda_{-} H\mathbb{F}_p$ to the cofiber sequence

$$S^{n-1} \longrightarrow D^n \longrightarrow S^n$$

gives rise to a bar spectral sequence

$$E^2(S^n) = \mathrm{Tor}^{L(S^{n-1})}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow L(S^n),$$

by Corollary 1.2.2.

The pinch map ψ induces vertical maps of cofiber sequences

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "\psi"] & D^n \arrow[r] \arrow[d] & S^n \arrow[d, "\psi"] \\ S^{n-1} \vee S^{n-1} \arrow[r] & D^n \vee D^n \arrow[r] & S^n \vee S^n, \end{tikzcd}$$

and this in combination with the reflection map on S^n, gives a map of simplicial spectra

$$B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, H\mathbb{F}_p) \to B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, H\mathbb{F}_p)$$

that endows this spectral sequence with a $\mathbb{F}_p$-Hopf algebra structure as explained in Corollary 1.2.4. Flatness is no problem, since $\mathbb{F}_p$ is a field.

Theorem 3.3.6. *When $n \leq 2p$, there are no differentials in the spectral sequence $E^2(S^n)$, and there is an $\mathbb{F}_p$-Hopf algebra isomorphism*

$$\pi_*(\Lambda_{S^n} H\mathbb{F}_p) \cong B_n.$$

Proof. The proof is by induction on n. Corollary A.4.7 gives us that $\pi_*(\Lambda_{S^1} H\mathbb{F}_p) \cong P(\mu) = B_1$.

Assume we have proved the theorem for $n-1$. The bar spectral sequence then becomes

$$E^2(S^n) = \mathrm{Tor}^{B_{n-1}}(\mathbb{F}_p, \mathbb{F}_p) \cong B_n \Rightarrow \pi_*(\Lambda_{S^n} H\mathbb{F}_p).$$

By Proposition 1.3.1, the shortest differential in lowest total degree goes from an indecomposable element to a primitive element. We have $E^2(S^n)_{0,*} \cong \mathbb{F}_p$, so the indecomposable elements in B_n that can support differentials, are generated by $\varrho^k w$ and $\varphi^k w$, with $k \geq 1$, where w is some admissible word of length $n-1$. By part 3 in Lemma 3.3.2 these elements are all in degrees greater than or equal to $4p$, and equal to 0 modulo $2p$ since $k \geq 1$. Thus if z is an indecomposable element, $d^r(z)$ is in degree -1

modulo $2p$, greater than or equal to $4p-1$. By 3.3.5 there are no primitive elements in these degrees when $n \leq 2p$, so there are no differentials in the spectral sequence. Hence, $E^2(S^n) = E^\infty(S^n)$.

To solve the multiplicative extensions we must determine $(\varrho^k w)^p$ and $(\varphi^k w)^p$ for all $k \geq 0$, and w an admissible word of length $n-1$.

Assume z is one of the generators $\varrho^k w$ or $\varphi^k w$ of lowest total degree with $z^p \neq 0$. Then by Frobenius

$$\begin{aligned}\psi(z^p) = \psi(z)^p = (1 \otimes z + z \otimes 1 + \sum z' \otimes z'')^p = 1 \otimes z^p + z^p \otimes 1 + \sum (z')^p \otimes (z'')^p \\ = 1 \otimes z^p + z^p \otimes 1,\end{aligned}$$

so z^p must be a primitive element in degree 0 modulo $2p$. By Proposition 3.3.5, this is impossible when $n \leq 2p$ and $|z^p| \geq 4p$, so there are no multiplicative extensions.

When $n \geq 2$ the pinch map $\psi : S^n \to S^n \vee S^n$ is homotopy cocommutative, i.e. the following diagram commutes

$$\begin{tikzcd} & S^n \vee S^n \arrow[d, "\tau"] \\ S^n \arrow[ur, "\psi"] \arrow[dr, "\psi"'] & \\ & S^n \vee S^n, \end{tikzcd}$$

where τ interchanges the two factors. Cocommutativity is shown by suspending a homotopy between the identity and antipodal map on S^1, picking one of the endpoints of the suspension as the basepoint in S^n, and identifying the suspension of two antipodal points on S^1 to a point, to define ψ.

From this is it follows that $L(S^n)$ is cocommutative as an $\mathbb{F}_p$-coalgebra when $n \geq 2$. Since $E^2(S^n)$ is a tensor product of exterior algebras and divided power algebras, Proposition 1.3.2 says that there are no coproduct coextensions. Thus $L(S^n) \cong E^\infty(S^n) = E^2(S^n) \cong B_n$ as an $\mathbb{F}_p$-Hopf algebra. $\square$

We finish this section by proving five technical statements about the degrees of certain admissible words. They are used in later sections in arguments about differentials and multiplicative extensions in spectral sequences. The first two lemmas can obviously be generalized to all n, but we only need them for $n \leq p$, so we keep their formulations as simple as possible.

Lemma 3.3.7. *Let $n \leq 2p-2$, and let x be an admissible word in B_n of even degree. Let l be the number of occurrences of the letter ϱ in the word x. The sum of the coefficients in the p-adic expansion of the number $\frac{|x|}{2}$ is equal to $n-l$.*

Proof. The proof is by induction on n. It is true for $n=1$ since $l=0$ and $|\mu|=2$. Assume it is true for all $1 \leq m \leq n-1$. An admissible word x in B_n of even degree is, by part 4 in Lemma 3.3.2, either equal to $\varphi^k y$ or $\varrho^k \varrho z$ for some $k \geq 0$, where y and z are admissible words in B_{n-1} and B_{n-2}, respectively.

First, $\frac{|\varphi^k y|}{2} = p^k(1+p\frac{|y|}{2})$, so if the sum of the coefficients in the p-adic expansion of $\frac{|y|}{2}$ is $n-1-l$, where l is the number of occurences of ϱ in y, the sum of the coefficients in the p-adic expansion of $\frac{|\varphi^k y|}{2}$ is $n-l$

Second, $\frac{|\varrho^k \varrho z|}{2} = p^k(1+\frac{|z|}{2})$, so if the sum of the coefficients in the p-adic expansion of $\frac{|z|}{2}$ is $n-2-(l-1) = n-1-l$, where $l-1$ is the number of occurences of ϱ in z, then the sum of the coefficients in the p-adic expansion of $\frac{|\varrho^k \varrho z|}{2}$ is $n-l$, unless there was carrying involved in the addition $1+\frac{|z|}{2}$.

There is only carrying involved if the degree of z is equal to -2 modulo $2p$, and by part 5 in Lemma 3.3.2 this implies that z is equal to $(\varrho^0\varrho)^{p-2}\mu$, or starts with $(\varrho^0\varrho)^{p-1}$ or $(\varrho^0\varrho)^{p-2}\varphi^0$. In these cases $\varrho^0\varrho z$ has length at least $2p-1$, so there is no carrying involved when $n \leq 2p-2$. □

Lemma 3.3.8. *Let $Q(B_n)$ be the module of indecomposable elements in B_n. If $2 \leq n \leq 2p$, then $Q(B_n)_{2pi-1} = 0$ for all i and $\bigoplus_{i\geq 1} Q(B_n)_{2pi}$ is equal to the module generated by all non-monic admissible words of length n.*

Proof. The module of indecomposable elements is generated by all admissible words of length n. All non-monic words are in degree 0 modulo $2p$. All monic words are primitive, so by 3.3.5 they are not in degree -1 or 0 modulo $2p$ when $2 \leq n \leq 2p$. □

Lemma 3.3.9. *The sum of the coefficients in the p-adic expansion of the number $\frac{|\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\ldots\mu_n^{p^{j_n}}|}{2}$, where $j_i \geq 0$ and $|\mu_i| = 2$ for $1 \leq i \leq n$, is equal to n when $0 < n < p$ and n or $n-p+1$ when $p \leq n < 2p$.*

Proof. If less than p of the numbers j_i are equal, we get the case n, and if at least p of the numbers j_i are equal, we get the case $n-p+1$. □

Corollary 3.3.10. *Let x be an admissible word in B_n of even degree.*

If $1 \leq n \leq p$, then the degree of x is not equal to the degree of $\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\ldots\mu_n^{p^{j_n}}$, where $j_i \geq 0$ for $1 \leq i \leq n$.

If $p \geq 5$, $1 \leq n \leq p$ and $1 \leq s \leq n$, then the degree of x is not equal to the degree of $(\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\ldots\mu_n^{p^{j_n}})\mu_s^{p^{j_{n+1}}}$, where $j_i \geq 0$ for $1 \leq i \leq n+1$.

Proof. By Lemma 3.3.7 the sum of the coefficients in the p-adic expansion of $\frac{|x|}{2}$ is equal to $n-l$ where l is the number of occurences of the letter ϱ in x. Part 2 in Lemma 3.3.2 says that $1 \leq l \leq \frac{n-1}{2}$, so $\frac{n+1}{2} \leq n-l \leq n-1$. By Lemma 3.3.9 the sum of the coefficients in the p-adic expansion of $\frac{|\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\ldots\mu_n^{p^{j_n}}|}{2}$ is equal to n when $0 < n < p$ and n or 1 when $n = p$. Now, $n-l \leq n-1 < n < n+1$ and when $n = p$ then $1 < \frac{n+1}{2} = \frac{p+1}{2} \leq n-l$, proving the first claim.

The sum of the coefficients in the p-adic expansion of $\frac{|(\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\ldots\mu_n^{p^{j_n}})\mu_i^{p^{j_{n+1}}}|}{2}$ is equal to $n+1$ when $0 < n < p-1$ and $n+1$ or $n-p+2$ when $p-1 \leq n \leq p$. When $n = p-1 \geq 4$ then $1 < \frac{n+1}{2} = \frac{p}{2} \leq n-l$ and when $n = p \geq 5$ then $2 < \frac{n+1}{2} = \frac{p+1}{2} \leq n-l$, proving the second claim. □

Definition 3.3.11. *Given a finite ordered set $S = \{s_1 < \ldots < s_n\}$ we define an S-labeled admissible word to be an admissible word of length n, where the first letter is labeled with s_n, the second with s_{n-1}, and so forth. We define B_S to be the $\mathbb{F}_p$-Hopf algebra that is a tensor product of exterior algebras on all S-labeled admissible monic words of odd degree and divided power algebras on all S-labeled admissible monic words of even degree. We let $B_\emptyset = \mathbb{F}_p$ be generated by the empty word in degree zero.*

Forgetting the labels on the letters induces an $\mathbb{F}_p$-Hopf algebra isomorphism between B_S and B_n. An example of an S-labeled word of length 3 is $\varrho^k_{s_3}\varrho_{s_2}\mu_{s_1}$.

Lemma 3.3.12. *Let $n \le p$, and let $P \subseteq \bigotimes_{U \subsetneq \mathbf{n}} B_U$ be the $\mathbb{F}_p$-submodule generated by all products $z_{U_1} \cdots z_{U_k}$, where $U_1, \ldots, U_k$ is a partition of $\mathbf{n}$, and, z_{U_i} is a primitive element in B_{U_i}, for every i. Then $P_{2pi-1} = 0$ for every $i \ge 2$, and the module $\bigoplus_{i \ge 2} P_{2pi}$ is contained in the module generated by all the elements $\mu_1^{p^{j_1}} \mu_2^{p^{j_2}} \ldots \mu_n^{p^{j_n}}$, where $j_i \ge 0$ for $1 \le i \le n$.*

Proof. In a divided power algebra $\Gamma(x)$, the only primitive element is $\gamma_1(x)$, and in a polynomial algebra $P(x)$ the primitive elements are generated by x^{p^j}. By Proposition A.1.8, the primitive elements in B_{U_i} are thus linear combinations of monic words w_i of length $|U_i|$ when $|U_i| > 1$, and $\mu_{U_i}^{p^{j_i}}$ when $|U_i| = 1$. Assume without loss of generality that z is a product of such elements.

Observe that the degree of a word starting with φ^k, $\varrho^k\varrho$ or μ^{p^k} is 0 modulo $2p$ when $k \ge 1$. Thus multiplication with one of these words will not change the degree of the product modulo $2p$. The degree of $\varphi^0 x$ and μ is 2 modulo $2p$, and finally the degree of $\varrho^0\varrho x$ is $2 + |x|$ modulo $2p$.

Except for the products $\mu_1^{p^{j_1}} \ldots \mu_n^{p^{j_n}}$, the smallest n where the degree of z is 0 modulo $2p$ is thus $n = p + 2$ where z may be equal to $\mu_1 \cdots \mu_{p-2} \cdot \mu_{p-1}^{p^k} \cdot \varrho^0_{p+2}\varrho_{p+1}\mu_p$. Similarly, the smallest n where the degree of z can be -1 modulo $2p$ is $n = p + 1$, where z might be equal to $\mu_1 \cdots \mu_{p-2} \cdot \mu_{p-1}^{p^k} \cdot \varrho_{p+1}\mu_p$. □

3.4 Calculating the Homotopy Groups of $\Lambda_{T^n} H\mathbb{F}_p$

In this section we will calculate the homotopy groups $\pi_*(\Lambda_{T^n} H\mathbb{F}_p)$ for $n \le p$. We will use the bar spectral sequence, and the multifold Hopf algebra structure of $\pi_*(\Lambda_{T^n} H\mathbb{F}_p)$ to make the calculation.

Fix a basepoint on the circle S^1. Let $\mathcal{I}$ be the category with objects finite sets of natural numbers, and morphisms inclusions.

We define the functor $T : \mathcal{I} \to Top$ by $T(\emptyset) = \{\mathrm{pt}\}$, and when $U \ne \emptyset$, $T(U) = T^U$, the U-fold torus. On morphisms it takes an inclusion $U \subseteq V$ to the inclusion $\mathrm{in}_U^V \colon T^U \to T^V$, where we use the basepoint in the factors not in U.

Given a finite subcategory $\Delta \subseteq \mathcal{I}$, we define

$$T^\Delta = \operatorname*{colim}_{U \in \Delta} T^U.$$

Given a finite set U we define $\Delta|_U$, the *restriction* of Δ to U, to be the full subcategory of Δ with objects $\{V \cap U | V \in \Delta\}$. The *dimension* of Δ, is the the maximal cardinality of the sets in Δ.

If U has cardinality k, there is a quotient map

$$g^U : T^U \to T^U/T^U_{k-1} \cong S^U,$$

where S^U is the U-sphere, and if $U \subseteq V$, there is a projection map

$$\mathrm{pr}^V_U \colon T^V \to T^U.$$

Given a map of spaces $f : X \to Y$ we will, when there are no room for confusion, write f for both the induced maps $\Lambda_f H\mathbb{F}_p \colon \Lambda_X H\mathbb{F}_p \to \Lambda_Y H\mathbb{F}_p$ and $L(f) : L(X) \to L(Y)$.

Proposition 3.4.1. *For each u in $U \in \mathcal{I}$, if $L(T^U)$ is flat as an $L(T^{U\setminus u})$-module, the ring $L(T^U)$ is a commutative $L(T^{U\setminus u})$-Hopf Algebra where:*

1. *Multiplication is induced by the fold map $T^U \amalg_{T^{U\setminus u}} T^U \cong T^{U\setminus u} \times (S^1 \vee S^1) \to T^U$.*
2. *Coproduct is induced by the pinch map $S^1 \to S^1 \vee S^1$ on the u-th circle in T^U.*
3. *The unit map is induced by choosing a basepoint in the u-th circle in T^U.*
4. *The counit map is induced by collapsing the u-th circle in T^U to a point.*

Proof. Since $\Lambda_{T^U} H\mathbb{F}_p \cong \Lambda_{S^1}\Lambda_{T^{U\setminus u}} H\mathbb{F}_p$ this follows from Proposition 1.1.5. □

Recall the definition of the category $\mathcal{I}$ in Section 3.1. Given a finite set $S \in \mathcal{I}$ and a full subcategory Δ of $V(S)$ we also write Δ for the full subcategory of $\mathcal{I}$ containing all the sets in Δ.

Proposition 3.4.2. *Let W be an object in $\mathcal{I}$, and let Δ be a saturated subcategory of $V(W)$, see Definition 3.1.16. Define the functor $L(T^-) : \Delta \to$ CRings on objects by $L(T^-)(U) = L(T^U)$ and on a map $U : V \to W$ by $\mathrm{in}^U_W \circ \mathrm{pr}^V_U$. The functor $L(T^-)$ is a partial W-fold Hopf algebra, when equipped with the Hopf algebra structures in Proposition 3.4.1, and we let $L(T^\Delta)$ denote its extension to W. Thus, $L(T^\Delta)(U) = L(T^{\Delta|_U})$.*

Furthermore, the map g^W induces a map of Hopf algebras

$$\big(L(T^W), L(T^{W\setminus j})\big) \to \big(L(S^{|W|}), \mathbb{F}_p\big).$$

Proof. It suffices to show that $L(T^-)$ is a W-fold Hopf algebra when $\Delta = V(W)$. Given $U \subseteq V \subseteq W$ and $v \in V$ we get two homomorphisms of Hopf algebras $\big(L(T^U), L(T^{U\setminus v})\big) \to \big(L(T^V), L(T^{V\setminus v})\big) \to \big(L(T^U), L(T^{U\setminus v})\big)$ induced by the inclusion $U \setminus v \to V \setminus v$. Hence, $L(T^-)$ is a pre W-fold Hopf algebra.

Given a set $U \subseteq V \subseteq W$, the ring $L(T^V)^U$, as defined in Definition 3.1.6, is isomorphic to $L(T^{V\setminus U} \times (S^1 \vee S^1)^U)$, since $L(-)$ commutes with colimits, and the colimit of the composite

$$T(U) \xrightarrow{-\cup-} [2]^U \xrightarrow{-\cup(V\setminus U)} [2]^S \xrightarrow{T^-} Top.$$

is $T^{V\setminus U} \times (S^1 \vee S^1)^U$, where the definitions are as in Definition 3.1.6.

That $L(T^-)$ is a W-fold Hopf algebra, follows from the geometric origin of the coproducts ψ_V^i for $i \in V \subseteq W$. Given a sequence $u_1, u_2, \ldots, u_k$ of distinct elements in $V \subseteq W$, let $U = \{u_1, u_2, \ldots, u_k\}$. The map

$$\psi_V^{u_1,u_2,\ldots,u_k} : L(T^V) \to L(T^V)^U \cong L(T^{V\setminus U} \times (S^1 \vee S^1)^U),$$

defined in Definition 3.1.12, is induced by the pinch map on every circle in $T^U \subseteq T^V$. Hence, it is independent of the order of the elements u_i in $\psi_V^{u_1,u_2,\ldots,u_k}$. $\square$

Ultimately we are interested in $L(T^{\mathbf{n}})$, so we only do the next constructions for the finite sets $\mathbf{n}$. We will now construct a family of bar spectral sequences that will be the backbone in our calculations of $L(T^{\mathbf{n}})$.

Give the circle S^1 the minimal CW-structure, and give the U-fold torus T^U the product CW-structure.

The attaching maps in the CW-structures yield cofiber sequences

$$S^{n-1} \xrightarrow{f^{\mathbf{n}}} T^{\mathbf{n}}_{n-1} \longrightarrow T^{\mathbf{n}}.$$

giving an equivalence of commutative $H\mathbb{F}_p$-algebra spectra

$$B(\Lambda_{D^n} H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, \Lambda_{T^{\mathbf{n}}_{n-1}} H\mathbb{F}_p) \simeq \Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p.$$

By Corollary 1.2.2, there is an $\mathbb{F}_p$-algebra bar spectral sequences

$$E^2(T^{\mathbf{n}}) = \mathrm{Tor}^{L(S^{n-1})}(L(T^{\mathbf{n}}_{n-1}), \mathbb{F}_p) \Rightarrow L(T^{\mathbf{n}}).$$

For each $i \in \mathbf{n}$, the pinch of the i-th circle in $T^{\mathbf{n}}$ induces a map of cofiber sequences

$$\begin{tikzcd}
S^{n-1} \arrow[r, "f^{\mathbf{n}}"] \arrow[d] & T^{\mathbf{n}}_{n-1} \arrow[r] \arrow[d] & T^{\mathbf{n}} \arrow[d] \\
S^{n-1} \vee S^{n-1} \arrow[r] & T^{\mathbf{n}}_{n-1} \amalg_{T^{\mathbf{n}\setminus i}} T^{\mathbf{n}}_{n-1} \arrow[r] & T^{\mathbf{n}} \amalg_{T^{\mathbf{n}\setminus i}} T^{\mathbf{n}},
\end{tikzcd}$$

inducing a map of simplicial spectra

$$\begin{aligned}
&B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, \Lambda_{T^{\mathbf{n}}_{n-1}} H\mathbb{F}_p) \\
&\quad \to B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p \wedge_{H\mathbb{F}_p} \Lambda_{S^{n-1}} H\mathbb{F}_p, \Lambda_{T^{\mathbf{n}}_{n-1}} H\mathbb{F}_p \wedge_{\Lambda_{T^{\mathbf{n}\setminus i}} H\mathbb{F}_p} \Lambda_{T^{\mathbf{n}}_{n-1}} H\mathbb{F}_p) \\
&\quad \simeq B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, \Lambda_{T^{\mathbf{n}}_{n-1}} H\mathbb{F}_p) \wedge_{\Lambda_{T^{\mathbf{n}\setminus i}} H\mathbb{F}_p} B(H\mathbb{F}_p, \Lambda_{S^{n-1}} H\mathbb{F}_p, \Lambda_{T^{\mathbf{n}}_{n-1}} H\mathbb{F}_p).
\end{aligned}$$

Hence by Corollary 1.2.4, if $E^r(T^{\mathbf{n}})$ is flat as an $L(T^{\mathbf{n}\setminus i})$-module then $E^2(T^{\mathbf{n}})$ is a spectral sequence of $L(T^{\mathbf{n}\setminus i})$-Hopf algebras, and if $L(T^{\mathbf{n}})$ is flat as an $L(T^{\mathbf{n}\setminus i})$-module, then $L(T^{\mathbf{n}})$ is an $L(T^{\mathbf{n}\setminus i})$-Hopf algebra and the spectral sequence converges to $L(T^{\mathbf{n}})$ as an $L(T^{\mathbf{n}\setminus i})$-Hopf algebra.

In light of part 3 in the next theorem, we will abuse notation and write $B_U \subseteq L(T^V)$ for the injective homomorphism $B_U \to \bigotimes_{W\subseteq V} B_V \overset{\alpha}{\cong} L(T^V)$, when $U \subseteq V$.

Theorem 3.4.3. *Given $1 \le k \le p$ when $p \ge 5$ and $1 \le k \le 2$ when $p = 3$, let Δ a finite subcategory of $\mathcal{I}$ of dimension at most k and let $V \subseteq W$ be two non-empty sets in $\mathcal{I}$ of cardinality at most k.*

1. *The map $L(f^{\mathbf{k}}) : L(S^{k-1}) \to L(T^{\mathbf{k}}_{k-1})$ factors through $\mathbb{F}_p$.*

2. *When $k \ge 2$, the spectral sequence $E^2(T^{\mathbf{k}})$ collapses on the E^2-term.*

3. *There is a natural $\mathbb{F}_p$-algebra isomorphism*

$$\alpha : L(T^V) \cong \bigotimes_{U\subseteq V} B_U,$$

where B_U is described in Definition 3.3.11. These isomorphisms induce an $\mathbb{F}_p$-algebra isomorphism

$$L(T^\Delta) \cong \operatorname*{colim}_{U\in\Delta} L(T^U) \cong \bigotimes_{U\in\Delta} B_U.$$

4. *Assume $|V| \ge 2$ and let v be the greatest integer in V.*

The operator

$$\sigma : L(T^{V\setminus v}) \to L(T^V)$$

is determined by the fact that σ is a derivation and that $\sigma(z) = \varrho_v z$ and $\sigma(z) = \varrho_v^0 z$ when $\emptyset \ne U \subseteq V \setminus v$ and z is an U-labeled admissible word in $B_U \subseteq L(T^{V\setminus v})$ of even and odd degree, respectively.

In particular, for any $z \in L(T^{V\setminus v})$, $\sigma(z)$ is in the kernel of the homomorphism

$$\mathrm{pr} : L(T^V) \overset{\alpha}{\cong} \bigotimes_{U\subseteq V} B_U \to \bigotimes_{i\in V} B_{\{i\}}.$$

5. *There is a commutative diagram*

$$\begin{tikzcd} L(T^W) \arrow[r, "\alpha"] \arrow[d, "\mathrm{pr}^W_V"] & \bigotimes_{U\subseteq W} B_U \arrow[d, "\mathrm{pr}"] \\ L(T^V) \arrow[r, "\alpha"] & \bigotimes_{U\subseteq V} B_U. \end{tikzcd}$$

6. *There is a commutative diagram*

$$\begin{tikzcd} L(T^V) \arrow[r, "\alpha"] \arrow[d, "g^V"] & \bigotimes_{U \subseteq V} B_U \arrow[d, "\mathrm{pr}"] \\ L(S^V) \arrow[r, "\cong"] & B_V, \end{tikzcd}$$

where the bottom isomorphism is the one from Theorem 3.3.6, together with the canonical isomorphism $B_{|V|} \cong B_V$ given by labeling the words in $B_{|V|}$.

Part 3 and 5 are equivalent to α being a natural transformation of functors from the category $V(\mathbf{k})$ to the category of $\mathbb{F}_p$-algebras.

The range $k \leq p$ comes from all the lemmas in Section 3.3 concerning the degree of primitive elements. It is possible that this range could be improved by getting better control of the degree of the primitive elements.

When $k = p = 3$ part 1 and 2 of the theorem still holds, but we are not able to determine the multiplicative structure of $L(T^{\mathbf{3}}) \cong E^\infty(T^{\mathbf{3}}) \cong L(T^{\mathbf{3}}_2) \otimes B_3$. This is because the degree of $\gamma_{p^{k+1}}(\varrho^0 \varrho \mu) \in \Gamma(\varrho^0 \varrho \mu) = B_3$ equals the degree of $\mu_1^{p^k + p^{k+1}} \mu_2^{p^k} \mu_3^{p^k}$. Thus, we can't use Proposition 3.2.5 to show that $(\gamma_{p^{k+1}}(\varrho^0 \varrho \mu))^p$ is a simultaneously primitive element.

The idea to look at the simultaneously primitive elements to show that the spectral sequence collapses on the E^2-term originated from a note by John Rognes, where he showed that the spectral sequence $E^2(T^{\mathbf{3}})$ collapses on the E^2-term.

Remark 3.4.4. It should be possible to prove a similar result for $V(0)_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{Z})$. The difference would be the degree of the elements in the rings, and thus the degree of the simultaneously primitive elements. The arguments in Section 3.3 would thus have to be adjusted for these new elements, and possibly you would want to work modulo $2p^2$ instead of modulo $2p$.

Before we prove this theorem, we give a very short sketch of the proof. We use the Bökstedt spectral sequence to identify the E^2-term $E^2(T^{\mathbf{n}})$ and to show that all d^2-differentials are zero. The S-fold Hopf algebra structure on $L(T^{\mathbf{n}})$ will help us prove that there are no other non-zero differentials, and hence $E^2(T^{\mathbf{n}}) \cong E^\infty(T^{\mathbf{n}})$.

From $E^\infty(T^{\mathbf{n}})$ we get a set of $\mathbb{F}_p$-algebra generators for $L(T^{\mathbf{n}})$. In several steps we exchange this set of generators with other sets. The elements in these new sets have extra properties, and using these extra properties we are able to prove the various statements in the theorem. In particular, we need the S-fold Hopf algebra structure and Proposition 3.2.5 to get hold of the multiplicative structure to prove part 3.

We begin the proof by stating two lemmas with corresponding corollaries. The first lemma concerns the Bökstedt spectral sequence calculating $H_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)$.

Lemma 3.4.5. *Given n, assume that Theorem 3.4.3 holds for $1 \leq k \leq n-1$, then:*

1. *The Bökstedt spectral sequence calculating* $H_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)$ *has* E^2*-term*

$$\overline{E}^2(T^{\mathbf{n}}) \cong A_* \otimes L(T^{\mathbf{n-1}}) \otimes \bigotimes_{\emptyset \neq U \subseteq \mathbf{n-1}} B_{U \cup \{n\}} \otimes E(\sigma_n \overline{\xi}_1, \sigma_n \overline{\xi}_2, \ldots) \otimes \Gamma(\sigma_n \overline{\tau}_0, \sigma_n \overline{\tau}_2, \ldots),$$

2. *There are no differentials* d^r *when* $r < p-1$, *so* $\overline{E}^2(T^{\mathbf{n}}) = \overline{E}^{p-1}(T^{\mathbf{n}})$.

3. *There are differentials*

$$d^{p-1}(\gamma_{p^l}(\sigma_n \overline{\tau}_i)) = \sigma_n \overline{\xi}_{i+1} \cdot \gamma_{p^l - l}(\sigma_n \overline{\tau}_i).$$

If, in addition, given $m \geq 0$ *the homomorphism* $f^{\mathbf{n}} \colon L(S^{n-1}) \to L(T^{\mathbf{n}}_{n-1})$ *factors through* $\mathbb{F}_p$ *in degree less than or equal to* $2pm - 1$ *and the spectral sequence* $E^2(T^{\mathbf{n}})$ *collapses in total degree less than or equal to* $2pm - 1$ *(that is* $E^2(T^{\mathbf{n}}) = E^\infty(T^{\mathbf{n}})$ *in these degrees) then:*

4. *The only other possible non-zero differentials in* $\overline{E}^{p-1}(T^{\mathbf{n}})$ *starting in total degree less than or equal to* $2p(m+1) - 1$, *are*

$$d^{p-1}(\gamma_{p^l}(\sigma_n x)) = \gamma_{p^l - p}(\sigma_n x) \sum_i r_{x,i} d^{p-1}(\gamma_p(\sigma_n \overline{\tau}_i)),$$

where x *is a generator in* $L(T^{\mathbf{n-1}})$ *of odd degree and* $r_{x,i} \in L(T^{\mathbf{n-1}}) \subset \overline{E}^{p-1}_{0,*}(T^{\mathbf{n}})$.

5. *Let* $B'_U \subsetneq \overline{E}^2(T^{\mathbf{n}})$ *be the algebra, isomorphic to* B_U, *that has the same generators as* B_U, *except that we exchange the generators* $\gamma_{p^l}(\sigma_n x)$ *in degree less than* $2p(m+1)$ *with the infinite cycles*

$$\gamma_{p^l}((\sigma_n x)') = \sum_{j=0}^{p^{l-1}} \big((-1)^j \gamma_{p^l - pj}((\sigma_n x)') \sum_{\alpha \in \mathbb{N}^{\mathbb{N}}, |\alpha| = j} \prod_{i \in \mathbb{N}} r_i^{\alpha_i} \gamma_{p\alpha_i}(\sigma_n \overline{\tau}_i)\big),$$

where $|\alpha| = \sum_{i \in \mathbb{N}} \alpha_i$, *and the convention is that* $0^0 = 1$, $\gamma_0(x) = 1$, *and* $\gamma_i(x) = 0$ *when* $i < 0$.

When $s + t \leq 2p(m+1) - 2$ *we get an isomorphism*

$$\overline{E}^\infty_{s,t}(T^{\mathbf{n}}) \cong A_* \otimes L(T^{\mathbf{n-1}}) \otimes \bigotimes_{\emptyset \neq U \subseteq \mathbf{n-1}} B'_{U \cup \{n\}} \otimes P_p(\sigma_n \overline{\tau}_0, \sigma_n \overline{\tau}_1, \ldots).$$

Proof. By Proposition A.4.1 and the Künneth isomorphism there are isomorphisms of $H_*(\Lambda_{T^{\mathbf{n-1}}} H\mathbb{F}_p) \cong A_* \otimes L(T^{\mathbf{n-1}})$-Hopf algebras

$$\begin{aligned}
\overline{E}^2(T^{\mathbf{n}}) = HH_*(H_*(\Lambda_{T^{\mathbf{n-1}}} H\mathbb{F}_p)) &\cong H_*(\Lambda_{T^{\mathbf{n-1}}} H\mathbb{F}_p) \otimes \mathrm{Tor}^{A_* \otimes L(T^{\mathbf{n-1}})}(\mathbb{F}_p, \mathbb{F}_p) \\
&\cong A_* \otimes L(T^{\mathbf{n-1}}) \otimes \bigotimes_{U \subseteq \mathbf{n-1}} \mathrm{Tor}^{B_U}(\mathbb{F}_p, \mathbb{F}_p) \otimes \mathrm{Tor}^{A_*}(\mathbb{F}_p, \mathbb{F}_p) \\
&\cong A_* \otimes L(T^{\mathbf{n-1}}) \otimes \bigotimes_{\emptyset \neq U \subseteq \mathbf{n-1}} B_{U \cup \{n\}} \otimes E(\sigma_n \overline{\xi}_1, \sigma_n \overline{\xi}_2, \ldots) \otimes \Gamma(\sigma_n \overline{\tau}_0, \sigma_n \overline{\tau}_2, \ldots),
\end{aligned}$$

where the empty set is left out in the tensor product in the last line, since $\mathrm{Tor}^{B_\emptyset}(\mathbb{F}_p, \mathbb{F}_p)$ is isomorphic to $\mathbb{F}_p$.

Proof of 2: The Bökstedt spectral sequence $\overline{E}^2(T^{\mathbf{n}})$ is an $A_* \otimes L(T^{\mathbf{n-1}})$-Hopf-algebra spectral sequence. By Proposition 1.3.1, the shortest differential is therefore from an indecomposable element to a primitive element. By Proposition A.1.8 the primitive elements are linear combinations of the monic words in $\bigotimes_{\emptyset \neq U \subseteq \mathbf{n-1}} B_{U \cup \{n\}}$, and the elements $\sigma_n \overline{\xi}_{i+1}$ and $\gamma_1(\sigma_n \overline{\tau}_i)$ for $i \geq 0$. The primitive elements are thus in filtration 1 and 2. The indecomposable elements are linear combinations of the $\mathbb{F}_p$-algebra generators in $\bigotimes_{\emptyset \neq U \subseteq \mathbf{n-1}} B_{U \cup \{n\}} \otimes E(\sigma_n \overline{\xi}_1, \sigma_n \overline{\xi}_2, \ldots) \otimes \Gamma(\sigma_n \overline{\tau}_0, \sigma_n \overline{\tau}_1, \ldots)$, given by the admissible words in $\bigotimes_{\emptyset \neq U \subseteq \mathbf{n-1}} B_{U \cup \{n\}}$ together with the elements $\sigma_n \overline{\xi}_j$ and $\gamma_{p^k}(\sigma_n \overline{\tau}_j)$, and they are in filtration $1, 2$ and pi for $i > 0$. The indecomposable elements in filtration p are generated by $\gamma_p(\sigma_n x)$ for a generator x in $A_* \otimes L(T^{\mathbf{n-1}})$ of odd degree. By Theorem A.4.5, these elements survive to $\overline{E}^{p-1}(T^{\mathbf{n}})$, so $\overline{E}^2(T^{\mathbf{n}}) = \overline{E}^{p-1}(T^{\mathbf{n}})$.

Proof of 3: Theorem A.4.5 also gives us the differentials

$$d^{p-1}(\gamma_{p+k}(\sigma_n \overline{\tau}_i)) = u_i \sigma_n \overline{\xi}_{i+1} \cdot \gamma_k(\sigma_n \overline{\tau}_i), \tag{3.4.6}$$

where u_i are units in $\mathbb{F}_p$.

Proof of 4: When $m = 0$, there is nothing to prove, since all elements in filtration p and higher are in degree at least $2p$. Since $\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p$ is an $H\mathbb{F}_p$-module it is an Eilenberg Mac Lane spectrum, so the Hurewicz homomorphism induces an isomorphism between the $\mathbb{F}_p$-modules $A_* \otimes L(T^{\mathbf{n}})$ and $H_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)$.

From the assumption that $f^{\mathbf{n}}$ factors through $\mathbb{F}_p$ in degree less than or equal to $2pm - 1$ and that $E^2(T^{\mathbf{n}})_{<2pm} \cong E^\infty(T^{\mathbf{n}})_{<2pm}$, we know the dimension of $H_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)$ as an $\mathbb{F}_p$-module in degree less than $2pm$. We will show that if there are other differentials in the spectral sequence $\overline{E}^2(T^{\mathbf{n}})$ than those in part 3 and 4 of the lemma, the dimension of $\overline{E}^\infty(T^{\mathbf{n}})$ is smaller than the abutment of the spectral sequence, which is equal to $H_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)$, thus giving us a contradiction.

Assume the only d^{p-1}-differentials in the Bökstedt spectral sequence $\overline{E}^2(T^{\mathbf{n}})$ are those generated by 3.4.6. Lemma 1.3.3 yields an isomorphism

$$\overline{E}^p(T^{\mathbf{n}}) \cong A_* \otimes L(T^{\mathbf{n-1}}) \otimes \bigotimes_{\emptyset \neq U \subseteq \mathbf{n-1}} B_{U \cup \{n\}} \otimes P_p(\sigma_n \overline{\tau}_0, \sigma_n \overline{\tau}_1, \ldots).$$

Proposition 3.3.4 together with the assumption that $f^{\mathbf{n}}$ factors through $\mathbb{F}_p$ in degree less than $2pm - 1$ and that $E^2(T^{\mathbf{n}})_{<2pm} = E^\infty(T^{\mathbf{n}})_{<2pm}$, gives us an $\mathbb{F}_p$-module isomorphism

$$\begin{aligned} L(T^{\mathbf{n}})_{<2pm} &\cong E^\infty(T^{\mathbf{n}})_{<2pm} = E^2(T^{\mathbf{n}})_{<2pm} = (L(T^{\mathbf{n}}_{n-1}) \otimes B_{\mathbf{n}})_{<2pm} \\ &\cong \Big(\bigotimes_{U \subseteq \mathbf{n}} B_U \Big)_{<2pm} \cong \Big(\bigotimes_{U \subseteq \mathbf{n-1}} B_U \otimes \bigotimes_{\emptyset \neq U \subseteq \mathbf{n-1}} B_{U \cup \{n\}} \otimes B_{\{n\}} \Big)_{<2pm} \\ &\cong \Big(L(T^{\mathbf{n-1}}) \otimes \bigotimes_{\emptyset \neq U \subseteq \mathbf{n-1}} B_{U \cup \{n\}} \otimes B_{\{n\}} \Big)_{<2pm}. \end{aligned}$$

Now, there is an $\mathbb{F}_p$-module isomorphism from $P_p(\sigma_n\overline{\tau}_0, \sigma_n\overline{\tau}_1, \ldots)$ to $B_{\{n\}}$ given by mapping $\sigma_n\overline{\tau}_i$ to $\mu_n^{p^i}$, and this isomorphism yields an $\mathbb{F}_p$-module isomorphism $\overline{E}^p(T^{\mathbf{n}})_{<2pm} \cong (A_* \otimes L(T^{\mathbf{n}}))_{<2pm} \cong H_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)_{<2pm}$.

Assume there is a d^{p-1}-differential with image in $\overline{E}^{p-1}(T^{\mathbf{n}})_{<2pm}$, which doesn't have image in the ideal $(\sigma_n\overline{\xi}_1, \sigma_n\overline{\xi}_2, \ldots) \subseteq \overline{E}^{p-1}(T^{\mathbf{n}})$, which is the ideal generated by the images of all the differentials in equation 3.4.6. Then, in the degree of the target of this differential, the dimension of the $\mathbb{F}_p$-module $\overline{E}^{\infty}(T^{\mathbf{n}})_{<2pm}$ would be smaller than the dimension of $H_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)_{<2pm} \cong \overline{E}^p(T^{\mathbf{n}})_{<2pm}$, giving us a contradiction.

To find all possible d^{p-1}-differentials with target in $(\sigma_n\overline{\xi}_1, \sigma_n\overline{\xi}_2, \ldots)$ it suffices to look at differentials from indecomposable elements. Possible non-zero d^{p-1}-differentials with image in $\overline{E}^{p-1}_{<2pm}$ are thus generated by $d^{p-1}(\gamma_{p^k}(\varrho_n^0 x))$ and $d^{p-1}(\gamma_{p^k}(\phi_n x))$ where x is an U-admissible word in $B_U \subseteq L(T^{\mathbf{n-1}})$ for some $\emptyset \neq U \subset \mathbf{n-1}$ of odd degree at most $2mp^{1-k} - 1$ and even degree at most $\frac{2mp^{1-k}-2}{p}$, respectively, and $k \geq 1$. From the calculation

$$\psi(d^{p-1}(\gamma_{p^k}(\phi_n x))) = d^{p-1}(\psi(\gamma_{p^k}(\phi_n x))) = d^{p-1}\Big(\sum_{i+j=p^k} \gamma_i(\phi_n x) \otimes \gamma_j(\phi_n x)\Big),$$

we see by induction on k, that $d^{p-1}(\gamma_{p^k}(\phi_n x))$ must be primitive. Thus it is zero, since when $k \geq 1$, it is in filtration greater than or equal to $p+1$, while the primitive elements are in filtration 1 and 2.

For the elements $\gamma_{p^k}(\varrho_n^0 x)$, Theorem A.4.5 yields the formula

$$d^{p-1}(\gamma_{p+k}(\varrho_n^0 x)) = (\sigma\beta Q^{\frac{|x|+1}{2}} x) \cdot \gamma_k(\varrho_n^0 x),$$

so $\gamma_{p+k}(\varrho_n^0 x)$ is a cycle if and only if $\gamma_p(\varrho_n^0 x)$ is a cycle.

In $\overline{E}^{p-1}_{1,*}(T^{\mathbf{n}})$, the ideal generated by the elements $\sigma_n\overline{\xi}_1, \sigma_n\overline{\xi}_2, \ldots$ is equal to $A_* \otimes L(T^{\mathbf{n-1}})\{\sigma_n\overline{\xi}_1, \sigma_n\overline{\xi}_2, \ldots\}$. Thus, if $d^{p-1}(\gamma_p(\varrho_n^0 x))$ is non-zero, $\sigma_n\beta Q^{\frac{|x|+1}{2}} x$ must be an element in $A_* \otimes L(T^{\mathbf{n-1}})\{\varrho_n\overline{\xi}_1, \varrho_n\overline{\xi}_2, \ldots\}$. Since differentials from a A_*-comodule primitive has target an A_*-comodule primitive, $\sigma_n\beta Q^{\frac{|x|+1}{2}} x$ must actually be an element in $L(T^{\mathbf{n-1}})\{\sigma_n\overline{\xi}_1, \sigma_n\overline{\xi}_2, \ldots\}$. Hence,

$$\sigma_n\beta Q^{\frac{|x|+1}{2}} x = \sum_i r_{x,i} d^{p-1}(\gamma_p(\sigma_n\overline{\tau}_i)),$$

where $r_{x,i}$ are elements in $L(T^{\mathbf{n-1}})$.

Proof of part 5: By Lemma 1.3.4, the elements $\gamma_{p^l}((\sigma_n x)')$ in part 5 are cycles, and $\overline{E}^{p-1}$ is isomorphic as an algebra to

$$\overline{E}^{p-1}(T^{\mathbf{n}}) \cong A_* \otimes L(T^{\mathbf{n-1}}) \otimes \bigotimes_{\emptyset \neq U \subseteq \mathbf{n-1}} B'_U \otimes E(\sigma_n\overline{\xi}_1, \sigma_n\overline{\xi}_2, \ldots) \otimes \Gamma(\sigma_n\overline{\tau}_0, \sigma_n\overline{\tau}_2, \ldots).$$

In total degree less than or equal to $2p(m+1)-1$, all elements in $\bigotimes_{\emptyset \neq U \subseteq \mathbf{n-1}} B'_U$ are cycles. Thus, when $s+t \leq 2p(m+1)-2$, the only differentials are those in part

3, so by Lemma 1.3.3 there is an isomorphism

$$\overline{E}^p(T^{\mathbf{n}}) \cong A_* \otimes L(T^{\mathbf{n-1}}) \otimes \bigotimes_{\emptyset \neq U \subseteq \mathbf{n-1}} B'_U \otimes P_p(\sigma_n \overline{\tau}_0, \sigma_n \overline{\tau}_1, \dots),$$

in total degree less than $2p(m+1)-2$.

All the algebra generators in filtration greater than 2 are in total degree zero modulo $2p$. All generators in total degree less than or equal to $2pm$ must be cycles, because otherwise, in the degree of the target of this non-zero differential, the dimension of the $\mathbb{F}_p$-module $\overline{E}^\infty(T^{\mathbf{n}})_{<2pm}$ will be smaller than the dimension of $H_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)_{<2pm} \cong \overline{E}^p(T^{\mathbf{n}})_{<2pm}$. Thus there are no more differentials with source in total degree less than or equal to $2p(m+1)$, so $\overline{E}^p(T^{\mathbf{n}})_{\leq 2p(m+1)-2} \cong \overline{E}^\infty(T^{\mathbf{n}})_{\leq 2p(m+1)-2}$. □

We only need this lemma to prove the following corollary, which we need to identify the E^2-term $E^2(T^{\mathbf{n}})$ and show that there are no d^2-differentials.

Corollary 3.4.7. *Given n, assume Theorem 3.4.3 holds when $1 \leq k \leq n-1$. Given $m \geq 0$, if $f^{\mathbf{n}}\colon L(S^{n-1}) \to L(T^{\mathbf{n}}_{n-1})$ factors through $\mathbb{F}_p$ in degree less than or equal to $2pm-1$ and the spectral sequence $E^2(T^{\mathbf{n}})$ collapses in total degree less than or equal to $2pm-1$ (that is $E^2(T^{\mathbf{n}}) = E^\infty(T^{\mathbf{n}})$ in these degrees), then:*

1. *The map $f^{\mathbf{n}}\colon L(S^{n-1}) \to L(T^{\mathbf{n}}_{n-1})$ factors through $\mathbb{F}_p$ in degree less than or equal to $2p(m+1)-2$.*
2. *The spectral sequence $E^2(T^{\mathbf{n}})$ collapses in total degree less than or equal to $2p(m+1)-2$.*

Proof. From the Proposition we know that as an $\mathbb{F}_p$-module

$$H_*(\Lambda_{T^n} H\mathbb{F}_p)_{\leq 2p(m+1)-2} \cong (A_* \otimes \bigotimes_{U \subseteq \mathbf{n}} B'_U))_{\leq 2p(m+1)-2}.$$

Since $\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p$ is a generalized Eilenberg Mac Lane spectrum, the Hurewicz homomorphism induces an isomorphism between the $\mathbb{F}_p$-modules $A_* \otimes L(T^{\mathbf{n}}) \cong A_* \otimes E^\infty(T^{\mathbf{n}})$ and $H_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)$.

The E^1-terms of the bar spectral sequence $E^1(T^{\mathbf{n}})$ is the two-sided bar complex

$$E^1_{s,*}(T^{\mathbf{n}}) = B(L(T^{\mathbf{n}}_{n-1}), B_{n-1}, \mathbb{F}_p) \cong L(T^{\mathbf{n}}_{n-1}) \otimes_{f^{\mathbf{n}}} B_{n-1}^{\otimes s} \otimes \mathbb{F}_p.$$

and the differential $d^1 : E^1_{s,t}(T^{\mathbf{n}}) \to E^1_{s-1,t}(T^{\mathbf{n}})$ is given by

$$d^1(a \otimes b_1 \otimes \cdots \otimes b_{s+1}) = af^{\mathbf{n}}(b_1) \otimes b_2 \otimes \cdots \otimes b_{s+1} + \sum (-1)^i a \otimes b_1 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_{s+1}.$$

If $f^{\mathbf{n}}$ factors through $\mathbb{F}_p$ in degree less than l, then

$$E^2(T^{\mathbf{n}}) = \mathrm{Tor}^{L(S^{n-1})}(L(T^{\mathbf{n}}_{n-1}), \mathbb{F}_p) \cong L(T^{\mathbf{n}}_{n-1}) \otimes \mathrm{Tor}^{L(S^{n-1})}(\mathbb{F}_p, \mathbb{F}_p) = L(T^{\mathbf{n}}_{n-1}) \otimes B_{\mathbf{n}}.$$

in bidegree (s, t) with $t < l$. Furthermore,

$$E^2_{0,l}(T^{\mathbf{n}}) \cong L(T^{\mathbf{n}}_{n-1})/\operatorname{im}(f^{\mathbf{n}})$$

If $f^{\mathbf{n}}$ doesn't factor through $\mathbb{F}_p$ in degree $l \leq 2p(m+1)-2$ then the dimension of $A_* \otimes E^2(T^{\mathbf{n}})$ in total degree l is smaller than the dimension of $H_*(\Lambda_{T^n} H\mathbb{F}_p)$ in degree l, giving us a contradiction.

Thus $f^{\mathbf{n}}$ factors through $\mathbb{F}_p$ in degree less than or equal to $2p(m+1)-2$.

By a similar argument, if there are any non-zero d^r-differentials in $E^2(T^{\mathbf{n}})$. starting in total degree less than or equal to $2p(m+1)-1$, the dimension of $A_* \otimes E^2(T^{\mathbf{n}})$ in the degree of the image of this differential, will be smaller than the dimension of $H_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)$ in this degree.

Thus the spectral sequence $E^2(T^{\mathbf{n}})$ collapses in total degree less than or equal to $2p(m+1)-2$. ☐

This lemma is about which elements in $L(T^{\mathbf{n}}_{n-1})$ are simultaneously primitive in all n Hopf algebra structures. For example $\mu_1\mu_2^p\mu_3^{p^2}$ is simultaneously primitive in $L(T^{\mathbf{3}})$ since it's a product of elements that are primitive in the different circles. We only gain control over the degree of the elements, but that is sufficient for our needs. It's probably a very special case of a more general statement about simultaneously primitive elements in an S-fold Hopf algebra, but a more general statement has eluded us.

Lemma 3.4.8. *Assume Theorem 3.4.3 holds for $1 \leq k \leq n-1$. Let S be an object in $\mathcal{I}$ and let Δ be a saturated subcategory (see Definition 3.1.16) of $V(S)$ with dimension at most $n-1$.*

Let $V \in \mathcal{I}$ and define $\mathbb{N}_V \subseteq \mathbb{N}$ to be the set of degrees of monic words in B_V when $|V| \geq 2$, and the set $\{2p^i\}_{i\geq 0}$, the set of degrees of $\mu_v^{p^i}$ when $V = \{v\}$. Let $\mathbb{N}_\Delta \subseteq \mathbb{N}$ be the set

$$\mathbb{N}_\Delta = \Big\{ \sum_{U_i \in \{U_1,\dots,U_j\}} r_{U_i} \,\Big|\, U_1, \dots U_j, \text{ is a partition of } S \text{ with } U_i \in \Delta \text{ and } r_{U_i} \in \mathbb{N}_{U_i} \Big\}.$$

If $z \in L(T^\Delta)$ is S-fold primitive, then $|z| \in \mathbb{N}_\Delta$.

Proof. We prove it by induction on the number of sets in Δ. If $S \backslash \big(\bigcup_{U\in\Delta} U\big) = W \neq \emptyset$, there are no S-fold primitive elements in $L(T^\Delta)$, since $L(T^{\Delta|_{S\backslash j}}) = L(T^\Delta)$ for any $j \in W$, so $\psi^j_S = \mathrm{id}\colon L(T^\Delta) \to L(T^\Delta)$. If $S = \{s\}$ and $\Delta = S$ the lemma holds since $L(T^\Delta) = B_{\{s\}} = P(\mu_s)$, and the primitive elements are generated by $\mu_s^{p^i}$ for $i \geq 0$.

Let $V \in \Delta$ be a maximal set in Δ, i.e., if $V \subseteq W \in \Delta$ then $V = W$. Let $\widehat{\Delta}$ be the full subcategory of Δ not containing V.

Let $z^V_0, z^V_1, \dots$ be the monomials in $B_V \subseteq L(T^V) \cong \bigotimes_{U\subseteq V} B_U$ ordered so that $|z^V_i| \leq |z^V_{i+1}|$ for all $i \geq 0$. Note that $z^V_0 = 1$.

When $z \neq 0$ we can write z uniquely as

$$z = z_l^V x_l^{\widehat{V}} + z_{l-1}^V x_{l-1}^{\widehat{V}} + \ldots + z_0^V x_0^{\widehat{V}}, \tag{3.4.9}$$

where $x_i^{\widehat{V}}$ are elements in $L(T^{\widehat{\Delta}}) \cong \bigotimes_{U \in \Delta, U \neq V} B_U$, and $x_l^{\widehat{V}} \neq 0$. This is possible since $L(T^{\Delta}) \cong L(T^{\widehat{\Delta}}) \otimes B_V$. If $l = 0$, then $z \in L(T^{\widehat{\Delta}})$ and we are done by the induction hypothesis.

Otherwise, given $j \in V$, assume $x_l^{\widehat{V}} \notin L(T^{\Delta|_{S \setminus j}}) \subseteq L(T^{\Delta})$. Then

$$\begin{aligned}
\psi_S^j(z) &= \psi_S^j(z_l^V)\psi_S^j(x_l^{\widehat{V}}) + \psi_S^j(z_{l-1}^V)\psi_S^j(x_{l-1}^{\widehat{V}}) + \ldots + \psi_S^j(z_0^V)\psi_S^j(x_0^{\widehat{V}}) \\
&= \big(1 \otimes z_l^V + z_l^V \otimes 1 + \sum (z_l^V)' \otimes (z_l^V)''\big)\big(1 \otimes x_l^{\widehat{V}} + x_l^{\widehat{V}} \otimes 1 + \sum (x_l^{\widehat{V}})' \otimes (x_l^{\widehat{V}})''\big) \\
&\quad + \ldots \\
&= 1 \otimes z_l^V x_l^{\widehat{V}} + z_l^V x_l^{\widehat{V}} \otimes 1 + z_l^V \otimes x_l^{\widehat{V}} + x_l^{\widehat{V}} \otimes z_l^V + \ldots
\end{aligned}$$

Now, $\psi_S^j : L(T^{\widehat{\Delta}}) \to L(T^{\widehat{\Delta}}) \otimes_{L(T^{\widehat{\Delta}|_{S \setminus j}})} L(T^{\widehat{\Delta}})$, so the expression on the last line can not be equal to $z \otimes 1 + 1 \otimes z$ due to the summands $z_l^V \otimes x_l^{\widehat{V}}$ and $x_l^{\widehat{V}} \otimes z_l^V$ and the fact that $z_l^V, \ldots, z_0^V$ is part of a basis and z_l^V is of highest degree. Hence we get a contradiction and $x_l^{\widehat{V}} \in L(T^{\Delta|_{S \setminus j}}) \subseteq L(T^{\Delta})$. Doing this for all j gives us that $x_l^{\widehat{V}} \in L(T^{\Delta|_{S \setminus V}}) \subseteq L(T^{\Delta})$.

For $U \in \Delta$, the projection maps $\mathrm{pr}_{U \setminus V}^U : T^U \to T^{U \setminus V}$ combines into a map

$$\mathrm{pr} : T^{\widehat{\Delta}} :\to T^{\Delta|_{S \setminus V}}.$$

Since this map collapses $T_{|V|-1}^V$ to a point, the map $g^V : T^V \to S^{|V|}$ together with pr induces a map

$$\mathrm{pr} : T^{\Delta} \to S^{|V|} \vee T^{\Delta|_{S \setminus V}}.$$

For $j \in V$ the pinch map ψ^j on the j-th circle induces a commutative diagram

$$\begin{tikzcd}
T^{\Delta} \arrow[rr, "\mathrm{pr}"] \arrow[ddd, "\psi^j"] & & S^V \vee T^{\Delta|_{S\setminus V}} \arrow[d, "\psi \vee \mathrm{id}"] \\
& & S^V \vee S^V \vee T^{\Delta|_{S\setminus V}} \\
& & \\
T^{\Delta} \amalg_{T^{\Delta|_{S\setminus j}}} T^{\Delta} \arrow[rr, "\mathrm{pr} \amalg \mathrm{pr}"] & & (S^V \vee T^{\Delta|_{S\setminus V}}) \amalg_{T^{\Delta|_{S\setminus V}}} (S^V \vee T^{\Delta|_{S\setminus V}}). \arrow[uu, "\cong"]
\end{tikzcd}$$

Similarly, for $j \in S \setminus V$ the pinch map ψ^j on the j-th circle induces a commutative diagram

$$\begin{tikzcd}
T^{\Delta} \arrow[rr, "\mathrm{pr}"] \arrow[dd, "\psi^j"] & & S^V \vee T^{\Delta|_{S\setminus V}} \arrow[d, "\mathrm{id}\vee\psi^j"] \\
& & S^V \vee (T^{\Delta|_{S\setminus V}} \amalg_{T^{\Delta|_{(S\setminus V)\setminus j}}} T^{\Delta|_{S\setminus V}}) \\
T^{\Delta} \amalg_{T^{\Delta|_{S\setminus j}}} T^{\Delta} \arrow[rr, "\mathrm{pr}\amalg\mathrm{pr}"] & & (S^V \vee T^{\Delta|_{S\setminus V}}) \amalg_{S^V\vee T^{\Delta|_{(S\setminus V)\setminus j}}} (S^V \vee T^{\Delta|_{S\setminus V}}). \arrow[u, "\cong"']
\end{tikzcd}$$

Applying the functor $L(-)$ to these two diagrams yields for $j \in V$ a commutative diagram

$$\begin{tikzcd}
L(T^{\Delta}) \arrow[r, "\mathrm{pr}"] \arrow[d, "\psi^j_S"] & B_V \otimes L(T^{\Delta|_{S\setminus V}}) \arrow[d, "\psi_{B_V}\otimes\mathrm{id}"] \\
L(T^{\Delta}) \otimes_{L(T^{\Delta|_{S\setminus j}})} L(T^{\Delta}) \arrow[r, "\mathrm{pr}\otimes\mathrm{pr}"] & B_V \otimes B_V \otimes L(T^{\Delta|_{S\setminus V}}),
\end{tikzcd} \tag{3.4.10}$$

and for $j \in S \setminus V$ a commutative diagram

$$\begin{tikzcd}
L(T^{\Delta}) \arrow[r, "\mathrm{pr}"] \arrow[d, "\psi^j_S"] & B_V \otimes L(T^{\Delta|_{S\setminus V}}) \arrow[d, "\mathrm{id}\otimes\psi^j_{S\setminus V}"] \\
L(T^{\Delta}) \otimes_{L(T^{\Delta|_{S\setminus j}})} L(T^{\Delta}) \arrow[r, "\mathrm{pr}\otimes\mathrm{pr}"] & B_V \otimes (L(T^{\Delta|_{S\setminus V}}) \otimes_{L(T^{\Delta|_{(S\setminus V)\setminus j}})} L(T^{\Delta|_{S\setminus V}})).
\end{tikzcd} \tag{3.4.11}$$

We have proved that $x_l^{\widehat{V}} \in L(T^{\Delta|_{S\setminus V}})$, so

$$\mathrm{pr}(z) = z_l^V x_l^{\widehat{V}} + z_{l-1}^V \mathrm{pr}(x_{l-1}^{\widehat{V}}) + \ldots + z_0^V \mathrm{pr}(x_0^{\widehat{V}}),$$

is non-zero since $z_l^V, \ldots, z_0^V$ is part of a basis. From Diagram 3.4.10 we know that $\mathrm{pr}(z)$ must be primitive in the $L(T^{\Delta|_{S\setminus V}})$-Hopf algebra $B_V \otimes L(T^{\Delta|_{S\setminus V}})$, where the Hopf algebra structure is induced by the $\mathbb{F}_p$-Hopf algebra structure on $B_V \cong B_{|V|} \cong L(S^V)$. By Proposition A.1.8, this implies that if $\mathrm{pr}(x_i^{\widehat{V}})$ is non-zero then z_i^V is a V-labeled monic word when $|V| \geq 2$ or an element $\mu_v^{p^m}$ for some m when $V = \{v\}$. It follows from Diagram 3.4.11 that when $\mathrm{pr}(x_i^{\widehat{V}}) \neq 0$ it is $S \setminus V$-fold primitive. By induction the Lemma holds for $\mathrm{pr}(x_i^{\widehat{V}})$, finishing the proof. □

Corollary 3.4.12. *Given $n \leq p$, assume Theorem 3.4.3 holds for $1 \leq k < n$. Let y be an $\mathbf{n}$-fold primitive element in $L(T^{\mathbf{n}}_{n-1})$. If x is an admissible word of length n and degree 0 modulo $2p$, , then $|x| - 1 \neq |y|$. If z is an admissible word of length n and of even degree, then $|z^p| \neq |y|$.*

Proof. When $2 \leq n \leq p$, Lemma 3.3.8 says the admissible words of length n and degree 0 modulo $2p$ are those that start with φ^i or ϱ^i for $i \geq 1$. Hence $x = \varrho^i x'$ or

$x = \varphi^i x'$ for x' some admissible word of length $n-1$. The element $\varrho^i x'$ is in degree greater than or equal to $4p$. By Lemma 3.4.8 and 3.3.12, there are no $\mathbf{n}$-fold primitive elements in $L(T^{\mathbf{n}}_{n-1})$ in dimension $2pm-1$ for $m \geq 2$, and hence $|x|-1 \neq |y|$.

If z is an admissible word of length n and of even degree Lemma 3.3.8 says the admissible words of length n and degree 0 modulo $2p$ are those that start with φ^i or ϱ^i for $i \geq 0$. Hence $z = \varrho^i z'$ or $z = \varphi^i z'$ for z' some admissible word of length $n-1$. So $|z^p| = |\varrho^{i+1} z'|$ or $|z^p| = |\varphi^{i+1} z'|$. By Lemma 3.4.8 and 3.3.12 the degree of the $\mathbf{n}$-fold primitive elements in $L(T^{\mathbf{n}}_{n-1})$ is equal to the degree of a product $\mu_1^{p^{j_1}} \dots \mu_n^{p^{j_n}}$. By Corollary 3.3.10 neither $\varrho^{i+1} z'$ nor $\varphi^{i+1} z'$ is in the same degree as one of the products $\mu_1^{p^{j_1}} \dots \mu_n^{p^{j_n}}$, and hence $|z^p| \neq |y|$. □

Proof of Theorem 3.4.3. The proof is by induction. Given n, with $1 \leq n \leq p$ when $p \geq 5$ and $1 \leq n \leq 2$ when $p = 3$, assume the theorem holds for all k with $1 \leq k < n$. The only place in the proof where there is a difference between $p = 3$ and $p \geq 5$ is when we invoke Corollary 3.3.10 in the proof of Lemma 3.4.13.

When $n = 2$, the theorem holds since $L(T^U_1) \cong P(\mu_U)$.

Proof of part 1 and 2 : We prove it by induction on the degree of elements in part 1 and total degree in part 2. Given m, assume that part 1 and 2 holds in degree less than or equal to $2pm-1$. This is trivially true when $m = 0$.

By Corollary 3.4.7, part 1 and 2 holds in (total) degree less than or equal to $2p(m+1)-2$. We must thus show that they hold in degree $2p(m+1)-1$.

The attaching map $f^{\mathbf{n}} : L(S^{n-1}) \to L(T^{\mathbf{n}}_{n-1})$ is determined by what it does on the set of algebra generators in $L(S^{n-1})$ given by the monic words of length $n-1$, and by Lemma 3.3.8 there are no such element in degree -1 modulo $2p$, and hence $f^{\mathbf{n}}$ factors through $\mathbb{F}_p$ in degree less than or equal to $2p(m+1)-1$. So, in vertical degree less than or equal to $2p(m+1)-1$ Proposition A.2.10 together with the Künneth isomorphism yields an $L(T^{\mathbf{n}}_{n-1})$-module isomorphism

$$E^2(T^{\mathbf{n}}) = \mathrm{Tor}^{L(S^{n-1})}(L(T^{\mathbf{n}}_{n-1}), \mathbb{F}_p) \cong L(T^{\mathbf{n}}_{n-1}) \otimes \mathrm{Tor}^{L(S^{n-1})}(\mathbb{F}_p, \mathbb{F}_p) \cong L(T^{\mathbf{n}}_{n-1}) \otimes B_n.$$

.

It remains to show that there are no d^r-differentials in $E^2(T^{\mathbf{n}})$ starting in total degree $2p(m+1)$. For every i in $\mathbf{n}$, $E^2(T^{\mathbf{n}})$ is an $L(T^{\mathbf{n}\setminus i})$-Hopf algebra spectral sequence, since $E^2(T^{\mathbf{n}})$ is flat over $L(T^{\mathbf{n}\setminus i})$. The Hopf algebra structure on $E^2(T^{\mathbf{n}})$ is the tensor product of the $L(T^{\mathbf{n}\setminus i})$-Hopf algebra structures on $L(T^{\mathbf{n}}_{n-1})$ and the $\mathbb{F}_p$ Hopf algebra structure on B_n. Thus, by Proposition 1.3.1, a shortest non-zero differential in lowest total degree, must go to a primitive element in the $L(T^{\mathbf{n}\setminus i})$-Hopf algebra structure. Hence, if a shortest non-zero differential starts in total degree $2p(m+1)$, there must elements in degree $2p(m+1)-1$ that are primitive in the $L(T^{\mathbf{n}\setminus i})$-Hopf algebra structure for all $i \in \mathbf{n}$.

The $L(T^{\mathbf{n}\setminus i})$-primitive elements in $L(T^{\mathbf{n}}_{n-1}) \otimes B_n$ are by Proposition A.1.8 linear combinations of primitive elements in $L(T^{\mathbf{n}}_{n-i})$ and B_n. By proposition A.1.8 the module of $L(T^{\mathbf{n}\setminus i})$-primitive elements in B_n is $L(T^{\mathbf{n}\setminus i})\{x_j\}$, where x_j runs over the monic words in B_n. The intersection $\bigcap_{i\in\mathbf{n}} L(T^{\mathbf{n}\setminus i})\{x_j\}$ is equal to $\mathbb{F}_p\{x_j\}$ since $\bigcap_{i\in\mathbf{n}} L(T^{\mathbf{n}\setminus i}) = \mathbb{F}_p$. Thus, the module of elements in $B_n \subseteq E^2(T^{\mathbf{n}})$ that are primitive in the $L(T^{\mathbf{n}\setminus i})$-Hopf algebra structure for every $i \in \mathbf{n}$ is $\mathbb{F}_p\{x_j\} \subseteq B_n$, which is isomorphic to the module of $\mathbb{F}_p$-primitive elements in B_n, under the projection map $E^2(T^{\mathbf{n}}) \to B_n$. By Proposition 3.3.5 there are no $\mathbb{F}_p$-primitive elements in B_n in degree -1 modulo $2p$ when $n \leq 2p$. Hence, there are no differentials starting in total degree $2p(m+1)$ that have target in filtration 1 or higher.

It remains to show that there are no differentials starting in total degree $2p(m+1)$ that have target in filtration 0. This is only possible if there are $\mathbf{n}$-fold primitive elements in $L(T^{\mathbf{n}}_{n-1})$ in the target of the differential. If z is an indecomposable element in B_n in degree $2p(m+1)$, Corollary 3.4.12 says there are no $\mathbf{n}$-fold primitive elements in $L(T^{\mathbf{n}}_{n-1})$ in degree $2p(m+1) - 1$ when $n \leq p$.

Hence, there are no differentials in $E^2(T^{\mathbf{n}})$ when $n \leq p$, so $E^2(T^{\mathbf{n}})$ collapses on the E^2-term. Since $E^2(T^{\mathbf{n}}) \cong L(T^{\mathbf{n}})$, $L(T^{\mathbf{n}})$ is flat as an $L(T^{\mathbf{n}\setminus i})$-module, so the spectral sequence converges to $L(T^{\mathbf{n}})$ as an $L(T^{\mathbf{n}\setminus i})$-Hopf algebra.

Proof of part 3- 6: We will only show the theorem for the set $V = \mathbf{n}$.

1. Let $\mathcal{G}^{\text{odd}}_{1,0}$ and $\mathcal{G}^{\text{even}}_{1,0}$ and be all admissible words of length n starting with ϱ or ϱ^0, respectively.
2. Let $\mathcal{G}_{2,-1}$ be all admissible words of length n that starts with φ^i or ϱ^{i+1} for $i \geq 0$.
3. Define $\mathcal{G}_{1,0} = \mathcal{G}^{\text{odd}}_{1,0} \cup \mathcal{G}^{\text{even}}_{1,0}$.

The set $\mathcal{G}_{2,-1}$ only contain even degree elements. These three sets generate B_n as an $\mathbb{F}_p$-algebra.

We can also think of $\mathcal{G}^{\text{odd}}_{1,0}$ and $\mathcal{G}^{\text{even}}_{1,0}$ as sets of elements in $E^2_{1,*}(T^{\mathbf{n}})$ of odd and even degree, respectively, while $\mathcal{G}_{2,-1}$ is a set of elements in $E^2_{s,*}(T^{\mathbf{n}})$ with $s \geq 2$, and together they generate $E^2(T^{\mathbf{n}})$ as an $L(T^{\mathbf{n}}_{n-1})$-algebra.

Given an element $z \in B_n$, we let $\overline{z}$ denote the corresponding element in $L(T^{\mathbf{n}})$ under the $\mathbb{F}_p$-module isomorphism $L(T^{\mathbf{n}}) \cong E^2(T^{\mathbf{n}}) \cong L(T^{\mathbf{n}}_{n-1}) \otimes B_n$. We let $\overline{\mathcal{G}}^{\text{odd}}_{1,0}$, $\overline{\mathcal{G}}^{\text{even}}_{1,0}$ and $\overline{\mathcal{G}}_{2,-1}$ be the set of elements in $L(T^{\mathbf{n}})$ corresponding under this isomorphism, to $\mathcal{G}^{\text{odd}}_{1,0}$, $\mathcal{G}^{\text{even}}_{1,0}$ and $\mathcal{G}_{2,-1}$, respectively. Let $E(\overline{\mathcal{G}}^{\text{odd}}_{1,0})$ and $P_p(\overline{\mathcal{G}}^{\text{even}}_{1,0} \cup \overline{\mathcal{G}}_{2,-1})$ be the exterior algebra and truncated polynomial algebra on the respective sets. Note that there is an $\mathbb{F}_p$-algebra isomorphism $B_n \cong E(\overline{\mathcal{G}}^{\text{odd}}_{1,0}) \otimes P_p(\overline{\mathcal{G}}^{\text{even}}_{1,0} \cup \overline{\mathcal{G}}_{2,-1})$ given by the bijections $\mathcal{G}_{1,0} \cong \overline{\mathcal{G}}_{1,0}$ and $\mathcal{G}_{2,-1} \cong \overline{\mathcal{G}}_{2,-1}$.

Given two sets $\overline{\mathcal{G}}_{1,i}$, and $\overline{\mathcal{G}}_{2,j}$ of elements in $L(T^{\mathbf{n}})$ where all elements in $\overline{\mathcal{G}}_{2,j}$ are of even degree, we define an $L(T^{\mathbf{n}}_{n-1})$-module homomorphism

$$\alpha_{i,j} : L(T^{\mathbf{n}}_{n-1}) \otimes E(\overline{\mathcal{G}}^{\text{odd}}_{1,i}) \otimes P_p(\overline{\mathcal{G}}^{\text{even}}_{1,i} \cup \overline{\mathcal{G}}_{2,j}) \to L(T^{\mathbf{n}})$$

by mapping the monomial $ax_1 \ldots x_m$, where $a \in L(T^{\mathbf{n}}_{n-1})$ and $x_l \in \overline{\mathcal{G}}_{1,i} \cup \overline{\mathcal{G}}_{2,j}$ to the corresponding element $ax_1 \ldots x_m$ in $L(T^{\mathbf{n}}_{n-1})$. This is not necessarily an algebra homomorphism since $\alpha_{i,j}(x)^p$ might not be zero for $x \in \overline{\mathcal{G}}^{\text{even}}_{1,i} \cup \overline{\mathcal{G}}_{2,j}$.

Given admissible words $x_1, \dots, x_m$ in B_n, then the product $\overline{x_1} \dots \overline{x_m}$ is equal to $\overline{x_1 \dots x_m}$ modulo everything in filtration lower than the filtration of $x_1 \dots x_m$. Thus, $\alpha_{0,-1}$ is an isomorphism of $L(T^{\mathbf{n}}_{n-1})$-modules.

We will prove part 3-6 of the theorem using the following lemma, which we prove after finishing the proof of the theorem.

Lemma 3.4.13. *Assume everything in this proof up to this point. Let $\overline{\mathcal{G}}_{1,1}$ be the set of elements $\sigma(z_{\mathbf{n-1}})$ in $L(T^{\mathbf{n}})$, where $z_{\mathbf{n-1}}$ runs over all admissible words in $B_{\mathbf{n-1}} \subseteq L(T^{\mathbf{n-1}})$. For every $l \geq 0$, there exist sets $\overline{\mathcal{G}}_{2,l}$ of elements in $L(T^{\mathbf{n}})$ with bijections $\beta_{1,1} : \overline{\mathcal{G}}_{1,0} \to \overline{\mathcal{G}}_{1,1}$ and $\beta_{2,l} : \overline{\mathcal{G}}_{2,l-1} \to \overline{\mathcal{G}}_{2,l}$ such that:*

1. *For any $x \in \overline{\mathcal{G}}_{i,j-1}$ the element x is equal to $\beta_{i,j-1}(x)$ in $L(T^{\mathbf{n}})$ modulo the ideal generated by the non-units in $L(T^{\mathbf{n}}_{n-1})$.*
2. *Given $\sigma(z_{\mathbf{n-1}}) \in \overline{\mathcal{G}}_{1,1}$, if $z \in B_{n-1}$ is the unlabeled version of the admissible word $z_{\mathbf{n-1}}$, then $g^{\mathbf{n}}(\sigma(z_{\mathbf{n-1}})) \in L(S^n) \cong B_n$ is equal to ϱz or $\varrho^0 z$, when $\sigma(z_{\mathbf{n-1}})$ is of odd or even degree, respectively.*
3. *For every $l \geq 0$ and $V \subsetneq \mathbf{n}$, the elements in $\overline{\mathcal{G}}_{1,1}$ and $\overline{\mathcal{G}}_{2,l}$ are mapped to zero under the homomorphism $\mathrm{pr}^{\mathbf{n}}_V \colon L(T^{\mathbf{n}}) \to L(T^V)$.*
4. *When $l \geq -1$, the homomorphism $\alpha_{1,l}$ is an $L(T^{\mathbf{n}}_{n-1})$-algebra isomorphism in degree less than pl.*
5. *When $l \geq 1$, the bijection $\beta_{2,l}$ is the identity on all elements not in degree $l-1$.*
6. *Composing the isomorphism $\alpha_{1,l}^{-1}$ with the projection homomorphism*

$$L(T^{\mathbf{n}}_{n-1}) \otimes E(\overline{\mathcal{G}}^{odd}_{1,1}) \otimes P_p(\overline{\mathcal{G}}^{even}_{1,1} \cup \overline{\mathcal{G}}_{2,l}) \to L(T^{\mathbf{n}}_{n-1}) \overset{\alpha}{\cong} \bigotimes_{U \subsetneq \mathbf{n}} B_U \to \bigotimes_{i \in \mathbf{n}} B_{\{i\}} \cong P(\mu_{\mathbf{n}}),$$

yields an $\mathbb{F}_p$-algebra homomorphism $\mathrm{pr}_{P(\mu_{\mathbf{n}})} \colon L(T^{\mathbf{n}}) \to P(\mu_{\mathbf{n}})$ in degree less than pl, and for every $i \in \mathbf{n}$, this homomorphism induces a homomorphism of Hopf algebras $\big(L(T^{\mathbf{n}}), L(T^{\mathbf{n}\setminus i})\big) \to \big(P(\mu_{\mathbf{n}}), P(\mu_{\mathbf{n}\setminus i})\big)$ in degree less than l.

Note that the homomorphism $\mathrm{pr}_{P(\mu_{\mathbf{n}})}$ doesn't come from a map of spaces, but is purely an algebraic construction. The bijections $\beta_{2,l}$ will be identities when l is not equal to 0 or 2 modulo $2p$, since $\overline{\mathcal{G}}_{2,l}$ only contain elements in degree 0 and 2 modulo $2p$. Furthermore, the bijections $\beta_{1,1}$ and $\beta_{2,l}$ are used for book keeping, and there is no particular relation between them and the homomorphism $\alpha_{1,l}$.

Now we continue the proof of Theorem 3.4.3. We define $\overline{\mathcal{G}}_{2,\infty}$ to be equal to $\overline{\mathcal{G}}_{2,l+1}$ in degree less than or equal to l, and by part 5 in Lemma 3.4.13 $\overline{\mathcal{G}}_{2,\infty}$ is well defined. Define $\beta : \overline{\mathcal{G}}_{1,0} \cup \overline{\mathcal{G}}_{2,-1} \cong \overline{\mathcal{G}}_{1,1} \cup \overline{\mathcal{G}}_{2,\infty}$ to be the bijection given by all the $\beta_{i,j}$'s. Recall that when we constructed $\overline{\mathcal{G}}_{1,0}$ and $\overline{\mathcal{G}}_{2,-1}$, we also showed that there was an $\mathbb{F}_p$-algebra isomorphism $E(\overline{\mathcal{G}}^{\mathrm{odd}}_{1,0}) \otimes P_p(\overline{\mathcal{G}}^{\mathrm{even}}_{1,0} \cup \overline{\mathcal{G}}_{2,-1}) \cong B_n$. Together with β this induces an $\mathbb{F}_p$-algebra isomorphism $E(\overline{\mathcal{G}}^{\mathrm{odd}}_{1,1}) \otimes P_p(\overline{\mathcal{G}}^{\mathrm{even}}_{1,1} \cup \overline{\mathcal{G}}_{2,\infty}) \cong B_n$.

Now, the map $g^{\mathbf{n}} : \Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p \to \Lambda_{S^n} H\mathbb{F}_p$ induces the projection homomorphism on spectral sequences $E^\infty(T^{\mathbf{n}}) = E^2(T^{\mathbf{n}}) \cong L(T^{\mathbf{n}}_{n-1}) \otimes B_n \to B_n \cong E^2(S^n) = E^\infty(S^n)$. Thus, the homomorphism $g^{\mathbf{n}} \colon L(T^{\mathbf{n}}) \to L(S^n)$ is surjective and maps $L(T^{\mathbf{n}}_{n-1}) \subseteq L(T^{\mathbf{n}})$ to $\mathbb{F}_p$. Since B_n is finite in each degree, and we observed above that there is an $\mathbb{F}_p$-algebra isomorphism $E(\overline{\mathcal{G}}^{\text{odd}}_{1,1}) \otimes P_p(\overline{\mathcal{G}}^{\text{even}}_{1,1} \cup \overline{\mathcal{G}}_{2,\infty}) \cong B_n$, there exists another $\mathbb{F}_p$-algebra isomorphism $\delta : E(\overline{\mathcal{G}}^{\text{odd}}_{1,1}) \otimes P_p(\overline{\mathcal{G}}^{\text{even}}_{1,1} \cup \overline{\mathcal{G}}_{2,\infty}) \cong B_n$ such that the following diagram of $\mathbb{F}_p$-modules commutes

$$\begin{tikzcd}
L(T^{\mathbf{n}}) \arrow[rr, "g^{\mathbf{n}}"] & & L(S^n) \arrow[d, "\cong"] \\
L(T^{\mathbf{n}}_{n-1}) \otimes E(\overline{\mathcal{G}}^{\text{odd}}_{1,1}) \otimes P_p(\overline{\mathcal{G}}^{\text{even}}_{1,1} \cup \overline{\mathcal{G}}_{2,\infty}) \arrow[u, "\alpha_{1,\infty}"] \arrow[r, "\mathrm{pr}"] & E(\overline{\mathcal{G}}^{\text{odd}}_{1,1}) \otimes P_p(\overline{\mathcal{G}}^{\text{even}}_{1,1} \cup \overline{\mathcal{G}}_{2,\infty}) \arrow[r, "\delta"] & B_n,
\end{tikzcd}$$

where the rightmost isomorphism is the one in Theorem 3.3.6.

By part 4 in Lemma 3.4.13 the homomorphism $\alpha_{1,\infty}$ is an algebra isomorphism. We define the $\mathbb{F}_p$-algebra isomorphism α in part 3 of the theorem to be the composite of $\mathbb{F}_p$-algebra isomorphisms

$$\begin{aligned}
\alpha : L(T^{\mathbf{n}}) \xleftarrow{\alpha_{1,\infty}} L(T^{\mathbf{n}}_{n-1}) \otimes E(\overline{\mathcal{G}}^{\text{odd}}_{1,1}) \otimes P_p(\overline{\mathcal{G}}^{\text{even}}_{1,1} \cup \overline{\mathcal{G}}_{2,\infty}) &\xrightarrow{\mathrm{id} \otimes \delta} L(T^{\mathbf{n}}_{n-1}) \otimes B_n \\
&\xrightarrow{\mathrm{id} \otimes \zeta} L(T^{\mathbf{n}}_{n-1}) \otimes B_{\mathbf{n}} \xrightarrow{\alpha \otimes \mathrm{id}} \bigotimes_{U \subseteq \mathbf{n}} B_U,
\end{aligned}$$

where ζ is the isomorphism given by labeling the words in B_n, and the last isomorphism comes from the induction hypothesis.

Part 6 of the theorem is satisfied for $V = \mathbf{n}$, since α was deliberately constructed to satisfy it.

To prove part 4 of the theorem it suffices, by the induction hypothesis, to show that under the isomorphism α, $\sigma(z_{\mathbf{n-1}}) = \varrho_n z_{\mathbf{n-1}}$ and $\sigma(z_{\mathbf{n-1}}) = \varrho_n^0 z_{\mathbf{n-1}}$ for any $\mathbf{n}$-labeled admissible word $z_{\mathbf{n-1}}$ in $B_{\mathbf{n-1}} \subseteq L(T^{\mathbf{n-1}})$. Let z be the underlying unlabeled admissible word of $z_{\mathbf{n-1}}$. As we see from the construction of $\overline{\mathcal{G}}_{1,1}$, the element $\alpha_{1,\infty}^{-1}(\sigma(z_{\mathbf{n-1}}))$ is equal to $\sigma(z_{\mathbf{n-1}})$ in $E(\overline{\mathcal{G}}^{\text{odd}}_{1,1}) \otimes P_p(\overline{\mathcal{G}}^{\text{even}}_{1,1})$. By part 2 of Lemma 3.4.13, $\delta\,\mathrm{pr}(\alpha_{1,\infty}^{-1}(\sigma(z_{\mathbf{n-1}})))$ is thus equal to ϱz or $\varrho^0 z$, in B_n, when $\sigma(z_{\mathbf{n-1}})$ is of odd or even degree, respectively. Hence, $\alpha(\sigma(z_{\mathbf{n-1}}))$ is equal to $\varrho_n z_{\mathbf{n-1}}$ or $\varrho_n^0 z_{\mathbf{n-1}}$ in $B_{\mathbf{n}} \subseteq L(T^{\mathbf{n}})$, respectively. This proves part 4 of the theorem

By the induction hypothesis and part 5 in Theorem 3.4.3, the following diagram

$$\begin{tikzcd}
L(T^{\mathbf{n}}_{n-1}) \arrow[r, "\alpha"] \arrow[d, "\mathrm{pr}^{\mathbf{n}}_V"] & \bigotimes_{U \subsetneq \mathbf{n}} B_U \arrow[d, "\mathrm{pr}"] \\
L(T^V) \arrow[r, "\alpha"] & \bigotimes_{U \subseteq V} B_U.
\end{tikzcd}$$

commutes for any set $V \subsetneq \mathbf{n}$. From part 3 of Lemma 3.4.13 we have that the non-units in $B_{\mathbf{n}} \subseteq L(T^{\mathbf{n}})$ is mapped to zero under the homomorphism $\mathrm{pr}^{\mathbf{n}}_V$. Hence, part 5 of the theorem holds for $V = \mathbf{n}$. $\square$

Proof of Lemma 3.4.13. After we have proven part 1 for the bijection $\beta_{1,1}$, we know that given $x \in \overline{\mathcal{G}}_{1,0}$, then $\beta_{1,1}(x)$ is equal to x in $L(T^{\mathbf{n}})$ modulo the ideal generated by the non-units in $L(T^{\mathbf{n}}_{n-1})$. Since $\alpha_{0,-1}$ is an $L(T^{\mathbf{n}}_{n-1})$-module isomorphism, this implies that $\alpha_{1,-1}^{-1} \circ \alpha_{0,-1}$ is an $L(T^{\mathbf{n}}_{n-1})$-module isomorphism, and hence $\alpha_{1,-1}$ is an $L(T^{\mathbf{n}}_{n-1})$-module isomorphism.

It follows by induction, that if we prove part 1 for all $l \leq m$, then $\alpha_{1,m}$ is an $L(T^{\mathbf{n}}_{n-1})$-module isomorphism. Hence, to prove part 4 it suffices to prove that $\alpha_{1,l}$ is an algebra homomorphism on the various quotients.

First we prove part 1-3 for $\overline{\mathcal{G}}_{1,1}$.

Recall that the set $\overline{\mathcal{G}}_{1,0}$ consists of the elements $\overline{\varrho z}$ and $\overline{\varrho^0 z}$ where z runs over all admissible words of length $n-1$ of even and odd degree, respectively. The bijection $\beta_{1,1}$ is given by mapping $\overline{\varrho z}$ and $\overline{\varrho^0 z}$ to $\sigma(z_{\mathbf{n-1}})$, where $z_{\mathbf{n-1}} \in B_{\mathbf{n-1}}$ is the labeled version of the word z. By part 6 of Theorem 3.4.3, $g^{\mathbf{n-1}}(z_{\mathbf{n-1}}) = z \in L(S^n) \cong B_n$.

By Proposition 1.2.5 and the commutative diagram

$$\begin{tikzcd}
S^1_+ \wedge \Lambda_{T^{\mathbf{n-1}}} H\mathbb{F}_p \arrow[r, "\omega"] \arrow[d, "S^1_+ \wedge g^{\mathbf{n-1}}"] & \Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p \arrow[d, "g^{\mathbf{n}}"] \\
S^1_+ \wedge \Lambda_{s^{n-1}} H\mathbb{F}_p \arrow[r, "\widehat{\omega}"] & \Lambda_{S^n} H\mathbb{F}_p,
\end{tikzcd}$$

the element $g^{\mathbf{n}}(\sigma(z_{\mathbf{n-1}}))$ is equal to $\varrho g^{\mathbf{n-1}}(z_{\mathbf{n-1}}) = \varrho z$. This proves part 2 of the lemma.

The map $g^{\mathbf{n}}$ induces the projection map from $E^\infty(T^{\mathbf{n}}) = E^2(T^{\mathbf{n}}) \cong L(T^{\mathbf{n}}_{n-1}) \otimes B_n$ to $E^\infty(S^n) = E^2(S^n) \cong B_n$. Hence, $g^{\mathbf{n}}(\overline{\varrho z}) = \varrho z \in L(S^n)$, since $\overline{\varrho z}$ is represented by ϱz in $E^\infty_{1,*}(T^{\mathbf{n}})$, and there is nothing in positive degree in filtration 0 or lower in $E^2(S^n)$.

Hence, $\sigma(z_{\mathbf{n-1}})$ is equal to $\overline{\varrho g^{\mathbf{n-1}}(z)}$ in $L(T^{\mathbf{n}})$ modulo the ideal generated by the non-units in $L(T^{\mathbf{n}}_{n-1})$, and we have proved part 1 for the set $\overline{\mathcal{G}}_{1,1}$.

Observe that the diagrams

$$\begin{tikzcd}
S^1_+ \wedge \Lambda_{T^{\mathbf{n-1}}} H\mathbb{F}_p \arrow[r, "\omega"] \arrow[d, "S^1_+ \wedge \mathrm{pr}^{\mathbf{n-1}}_{\mathbf{n-1}\setminus i}"] & \Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p \arrow[d, "\mathrm{pr}^{\mathbf{n}}_{\mathbf{n}\setminus i}"] \\
S^1_+ \wedge \Lambda_{T^{\mathbf{n-1}\setminus i}} H\mathbb{F}_p \arrow[r, "\omega"] & \Lambda_{T^{\mathbf{n}\setminus i}} H\mathbb{F}_p
\end{tikzcd}
\qquad
\begin{tikzcd}
S^1_+ \wedge \Lambda_{T^{\mathbf{n-1}}} H\mathbb{F}_p \arrow[r, "\omega"] \arrow[d, "\mathrm{pr}_+ \wedge \mathrm{id}"] & \Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p \arrow[d, "\mathrm{pr}^{\mathbf{n}}_{\mathbf{n-1}}"] \\
S^0 \wedge \Lambda_{T^{\mathbf{n-1}}} H\mathbb{F}_p \arrow[r, "\cong"] & \Lambda_{T^{\mathbf{n-1}}} H\mathbb{F}_p,
\end{tikzcd}$$

commute for all $i \in \mathbf{n-1}$. Let y be an element in $L(T^{\mathbf{n-1}})$. By part 5 in Theorem 3.4.3, $\mathrm{pr}^{\mathbf{n-1}}_{\mathbf{n-1}\setminus i}(y) = 0$. Let $\sigma(z)$ be an element in $\overline{\mathcal{G}}_{1,1}$. From the left diagram, we conclude that $\mathrm{pr}^{\mathbf{n}}_{\mathbf{n}\setminus i}(\sigma(z))$ is zero when $i \neq n$. From the right diagram we conclude that $\mathrm{pr}^{\mathbf{n}}_{\mathbf{n-1}}(\sigma(z))$ is zero, since $H_1(S^0) = 0$. This proves part 3 of the lemma for the set $\overline{\mathcal{G}}_{1,1}$.

Second, we construct the set $\overline{\mathcal{G}}_{2,0}$ and prove part 1 and 3 for the set $\overline{\mathcal{G}}_{2,0}$.

Given $x \in L(T^{\mathbf{n}})$, we define

$$\widehat{x} = \sum_{U \subseteq \mathbf{n}} (-1)^{n-|U|} \,\mathrm{in}^{\mathbf{n}}_U \,\mathrm{pr}^{\mathbf{n}}_U(x).$$

Now, let $\overline{\mathcal{G}}_{2,0}$ be the set of elements $\widehat{x}$ where x runs over the elements in $\overline{\mathcal{G}}_{2,-1}$ and the bijection $\beta_{2,0}$ is given by sending x to $\widehat{x}$. For every $U \subsetneq \mathbf{n}$ we have $\mathrm{in}_U^{\mathbf{n}} \mathrm{pr}_U^{\mathbf{n}}(x) \in L(T^{\mathbf{n}}_{n-1})$, so $\widehat{x}$ is equal to x in $L(T^{\mathbf{n}})$ modulo the ideal generated by the non-units in $L(T^{\mathbf{n}}_{n-1})$. This proves part 1 for the set $\overline{\mathcal{G}}_{2,0}$.

The diagram

$$\begin{tikzcd} T^{\mathbf{n}} \arrow[r, "\mathrm{pr}_U^{\mathbf{n}}"] \arrow[d, "\mathrm{pr}^{\mathbf{n}}_{U\cap V}"] & T^U \arrow[r, "\mathrm{in}_U^{\mathbf{n}}"] & T^{\mathbf{n}} \arrow[d, "\mathrm{pr}_V^{\mathbf{n}}"] \\ T^{U\cap V} \arrow[rr, "\mathrm{in}^V_{U\cap V}"] & & T^V \end{tikzcd}$$

commutes. Hence, if $V \subsetneq \mathbf{n}$, then

$$\mathrm{pr}_V^{\mathbf{n}}(\widehat{x}) = \sum_{U\subseteq \mathbf{n}} (-1)^{n-|U|} \mathrm{pr}_V^{\mathbf{n}} \mathrm{in}_U^{\mathbf{n}} \mathrm{pr}_U^{\mathbf{n}}(x) = \sum_{S\subseteq V} \sum_{W\subseteq \mathbf{n}\setminus V} (-1)^{n-|S|-|W|} \mathrm{in}_S^V \mathrm{pr}_S^{\mathbf{n}}(x) = 0,$$

since $\sum_{W\subseteq \mathbf{n}\setminus V}(-1)^{n-|S|-|W|} = \sum_{i=0}^{n-|V|}(-1)^{n-|S|-i}\binom{n-|V|}{i} = 0$. This proves part 3 for the set $\overline{\mathcal{G}}_{2,0}$.

Third, we will prove the lemma by induction on l. When $l = 0$, part 5 and 6 are empty statements. The first three parts was proven previously in the proof, while the isomorphism in part 4 is just an isomorphism of $L(T^{\mathbf{n}}_{n-1})$-modules.

Now, assume that we have proven the lemma for $0 \leq l \leq m$. We will construct the set $\overline{\mathcal{G}}_{2,m+1}$, and prove that the lemma holds for $l = m+1$. By Proposition 3.4.2, $L(T^-)$ is an $\mathbf{n}$-fold Hopf algebra. By part 3 in Theorem 3.4.3, every element in $\bigotimes_{U\subsetneq \mathbf{n}, |U|\neq 1} \subseteq L(T^{\mathbf{n}}_{n-1})$ is nilpotent. Hence, there is a splitting of $\mathbf{n}$-fold Hopf algebras $P_{\mathbb{F}_p}(\mu_-) \to L(T^-_{n-1}) \to P_{\mathbb{F}_p}(\mu_-)$, since no element in $P_{\mathbb{F}_p}(\mu_{\mathbf{n}})$ is nilpotent. Since $L(T^-_{n-1})$ is the restriction, see Definition 3.1.17, of $L(T^-)$ to the full subcategory of $V(\mathbf{n})$ not containing $\mathbf{n}$, assumption 2 in Proposition 3.2.5 is thus satisfied for the $\mathbf{n}$-fold Hopf algebra $L(T^-)$.

By part 4 and 6 in the lemma, assumption 3 and 4 in Proposition 3.2.5 holds for the $\mathbf{n}$-fold Hopf algebra $L(T^-)$ when $q = m$.

For $x \in \overline{\mathcal{G}}_{2,m}$, the degree of x is equal to the degree of an element in $\mathcal{G}_{2,-1}$ which is equal to the degree of an admissible word in $B_n \subsetneq E^2(T^{\mathbf{n}})$ of even degree. By Corollary 3.3.10, x is thus not in the same degree as any of the elements $\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\cdots\mu_n^{p^{j_n}}$ or $(\mu_1^{p^{j_1}}\mu_2^{p^{j_2}}\cdots\mu_n^{p^{j_n}})\mu_s^{p^{j_{n+1}}}$, where $1 \leq s \leq n$ and $j_i \in \mathbb{N}$ for all $1 \leq i \leq n+1$.

From part 3 in the lemma we have $x \in \bigcap_{i\in\mathbf{n}} \ker(\epsilon^i_{\mathbf{n}} : L(T^{\mathbf{n}}) \to L(T^{\mathbf{n}\setminus i}))$. By part 6, if x is in degree m, then $x \in \ker(\mathrm{pr}_{P(\mu_{\mathbf{n}})})$. Hence, Proposition 3.2.5 gives us that

$$(\mathrm{pr}_{P(\mu_{\mathbf{n}})} \otimes_{P(\mu_{\mathbf{n}\setminus i})} \mathrm{pr}_{P(\mu_{\mathbf{n}})}) \circ \psi^i_{\mathbf{n}}(x) = \sum_{b\in\mathbb{N}_+^{\mathbf{n}}} r_{b,x}\mu^b_{\mathbf{n}\setminus i}\widetilde{\psi}^i(\mu_i^{b_i})$$

for some $r_{b,x}$ in $\mathbb{F}_p$, and we define

$$\widetilde{x} = x - \sum_{b\in\mathbb{N}_+^{\mathbf{n}}} r_{b,x}\mu^b_{\mathbf{n}}.$$

Obviously, $\widetilde{x}$ is equal to x modulo the ideal generated by the non-units in $L(T^{\mathbf{n}}_{n-1})$. We define the set $\overline{\mathcal{G}}_{2,m+1}$ to be the set that contains all elements in $\overline{\mathcal{G}}_{2,m}$ in degree not equal to m, together with the elements $\widetilde{x}$, where x runs over all the elements in degree m in $\overline{\mathcal{G}}_{2,m}$. This construction gives a bijection $\beta_{2,m+1} : \overline{\mathcal{G}}_{2,m} \cong \overline{\mathcal{G}}_{2,m+1}$ satisfying part 1 and 5.

In the construction of $\widetilde{x}$ we only sum over positive integers so $\mathrm{pr}^{\mathbf{n}}_V(\widetilde{x}) = \mathrm{pr}^{\mathbf{n}}_V(x) = 0$ for all $V \subsetneq \mathbf{n}$. Thus, part 3 holds for the set $\overline{\mathcal{G}}_{2,m+1}$.

We will now prove part 4 and 6 for $l = m+1$. Since, $\beta_{2,m+1}$ is the identity in degree less than m, it suffices to prove this for elements in degree m. That the homomorphism pr in part 6 is an $\mathbb{F}_p$-algebra homomorphism in degree less than $p(m+1)$, follows from proving part 4 for $l = m+1$.

If x is an element in $\overline{\mathcal{G}}_{2,m+1}$ of degree m, then by construction

$$(\mathrm{pr}_{P(\mu_{\mathbf{n}})} \otimes_{P(\mu_{\mathbf{n}\setminus i})} \mathrm{pr}_{P(\mu_{\mathbf{n}})}) \circ \psi^i_{\mathbf{n}}(x) = 0. \tag{3.4.14}$$

Let $\sigma(z) \in \overline{\mathcal{G}}_{1,1}$ be an element in degree m. By Proposition 1.1.4, $\sigma : \pi_*(\Lambda_{T^{\mathbf{n-1}}} H\mathbb{F}_p) \to \pi_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)$ is a derivation. Hence, if $\psi^i_{\mathbf{n-1}}(z) = 1 \otimes z + z \otimes 1 + \sum z'_i \otimes z''_i$ for $i \in \mathbf{n-1}$, then

$$\psi^i_{\mathbf{n}}(\sigma(z)) = \sigma(\psi^i_{\mathbf{n}}(z)) = 1 \otimes \sigma(z) + \sigma(z) \otimes 1 + \sum \sigma(z'_i) \otimes z''_i \pm z'_i \otimes \sigma(z''_i).$$

Recall that by Proposition 3.3.4, the $\mathbb{F}_p$algebra $L(T^{\mathbf{n-1}}) \cong \bigotimes_{U \subseteq \mathbf{n-1}} B_U$ is generated by all U-labeled admissible words for $U \subseteq \mathbf{n-1}$. Since σ is a derivation and z is of degree $m-1$, to show that $\sigma(z'_i)$ and $\sigma(z''_i)$ are in the kernel of the projection homomorphism $\mathrm{pr}_{P(\mu_{\mathbf{n}})}$, it suffices to show that $\sigma(x_U)$ is in the kernel of $\mathrm{pr}_{P(\mu_{\mathbf{n}})}$ for all U-labeled admissible words x_U of dimension less than $m-1$, where $U \subseteq \mathbf{n-1}$.

Now, by part 4 of Theorem 3.4.3, if $U \neq \mathbf{n-1}$, then $\sigma(x_U)$ is in the kernel of $\mathrm{pr}_{P(\mu_{\mathbf{n}})}$. Otherwise, $\sigma(x_{\mathbf{n-1}})$ is an element in $\overline{\mathcal{G}}_{1,1}$ by the construction of $\overline{\mathcal{G}}_{1,1}$. By the induction hypothesis and part 6 of Lemma 3.4.13, $\sigma(x_{\mathbf{n-1}})$ is thus in the kernel of $\mathrm{pr}_{P(\mu_{\mathbf{n}})}$. Hence, $\sigma(z'_i)$ and $\sigma(z''_i)$ are in the kernel of $\mathrm{pr}_{P(\mu_{\mathbf{n}})}$, so

$$(\mathrm{pr}_{P(\mu_{\mathbf{n}})} \otimes_{P(\mu_{\mathbf{n}\setminus i})} \mathrm{pr}_{P(\mu_{\mathbf{n}})}) \circ \psi^i_{\mathbf{n}}(\sigma(z)) = 0 \tag{3.4.15}$$

when $i \in \mathbf{n-1}$. This equation also holds for $i = n$, since, by Proposition 1.1.6 $\sigma(z)$ is primitive as an element in the $L(T^{\mathbf{n-1}})$-Hopf algebra $L(T^{\mathbf{n}})$.

By equation 3.4.14 and 3.4.15, the homomorphism $\mathrm{pr}_{P(\mu_{\mathbf{n}})}$ induces a map of Hopf algebras $\big(L(T^{\mathbf{n}}), L(T^{\mathbf{n}\setminus i})\big) \to \big(P(\mu_{\mathbf{n}}), P(\mu_{\mathbf{n}\setminus i})\big)$ in degree less than $m+1$, and hence we have proved part 6 for $l = m+1$.

To prove part 4 for $m+1$, we must show that when $y \in \overline{\mathcal{G}}_{1,1} \cup \overline{\mathcal{G}}_{2,m+1}$ and $|y| = m$, then $y^p = 0$ and $y^2 = 0$ when y is of even or odd degree, respectively. The claim is obvious when y is of odd degree, since the ring is graded commutative.

In degree less than m the only elements in $L(T^{\mathbf{n}})$ that are non-zero when raised to the power of p are the elements in the subring $P(\mu_{\mathbf{n}}) \subseteq L(T^{\mathbf{n}})$, as we see from the homomorphism $\alpha_{1,m}$. Thus, by Frobenius and part 6 of Lemma 3.4.13 for $l = m+1$,

$$\psi^i_{\mathbf{n}}(y^p) = \psi^i_{\mathbf{n}}(y)^p = (1 \otimes y + y \otimes 1 + \sum y' \otimes y'')^p = 1 \otimes y^p + y^p \otimes 1,$$

in all $L(T^{\mathbf{n}\setminus i})$-Hopf algebra structures. Hence, y^p is primitive in the $L(T^{\mathbf{n}\setminus i})$-Hopf algebra structure for every $i \in \mathbf{n}$.

Let y^p be represented by $y_0 + \ldots + y_s \in E^\infty(T^{\mathbf{n}})$, where $y_i \in E^\infty_{i,*}(T^{\mathbf{n}})$ and $y_s \neq 0$. Then, since y^p is primitive in the $L(T^{\mathbf{n}\setminus i})$-Hopf algebra structure, y_s must be primitive in the $L(T^{\mathbf{n}\setminus i})$-Hopf algebra $E^\infty(T^{\mathbf{n}})$ for every $i \in \mathbf{n}$. Otherwise, $\psi^i_{\mathbf{n}}(y^p)$ would not be equal to $y_s \otimes 1 + 1 \otimes y_s$ in filtration s.

The $L(T^{\mathbf{n}\setminus i})$-primitive elements in $L(T^{\mathbf{n}}_{n-1}) \otimes B_n$ are by Proposition A.1.8 linear combinations of primitive elements in $L(T^{\mathbf{n}}_{n-i})$ and B_n. By proposition A.1.8 The module of $L(T^{\mathbf{n}\setminus i})$-primitive elements in B_n is $L(T^{\mathbf{n}\setminus i})\{x_j\}$, where x_j runs over the monic words in B_n. The intersection $\bigcap_{i\in\mathbf{n}} L(T^{\mathbf{n}\setminus i})\{x_j\}$ is equal to $\mathbb{F}_p\{x_j\}$ since $\bigcap_{i\in\mathbf{n}} L(T^{\mathbf{n}\setminus i}) = \mathbb{F}_p$. Thus, the module of elements in $B_n \subseteq E^2(T^{\mathbf{n}})$ that are primitive in the $L(T^{\mathbf{n}\setminus i})$-Hopf algebra structure for every $i \in \mathbf{n}$ is $\mathbb{F}_p\{x_j\} \subseteq B_n$, which is isomorphic to the module of $\mathbb{F}_p$-primitive elements in B_n, under the projection map $E^2(T^{\mathbf{n}}) \to B_n$

The degree of y^p is zero modulo $2p$ and the degree of z is at least four, so by Lemma 3.3.5 there are no $\mathbb{F}_p$-primitive elements in B_n in degree 0 modulo $2p$.

Hence, y^p must be equal to an $\mathbf{n}$-fold primitive element in $L(T^{\mathbf{n}}_{n-1})$. By Corollary 3.4.12, the degree of y^p is not equal to the degree of any $\mathbf{n}$-fold primitive element in $L(T^{\mathbf{n}}_{n-1})$ when $n \leq p$. Thus $y^p = 0$, so $\alpha_{1,m+1}$ is an algebra isomorphism in degree less than $p(m+1)$ proving part 4 when $l = m+1$. □

3.5 Periodic Elements

The connective n-th Morava K-theory $k(n)$ is a ring spectrum with coefficient ring $k(n)_* = P_{\mathbb{F}_p}(v_n)$ where $|v_n| = 2p^n - 2$. The unit map of the ring spectrum $\Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p$ induces a homomorphism $P_{\mathbb{F}_p}(v_m) \to k(m)_*(\Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p)$ and we denote the image of v_m with v_m. In this section we show that the class of $t_1\mu_1^{p^{n-1}} + t_2\mu_2^{p^{n-1}} + \ldots + t_n\mu_n^{p^{n-1}}$ in the homotopy fixed points spectral sequence calculating $k(n-1)_*(F(E_2T^n_+, \Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p)^{T^n})$ is not hit by any differential, and that this implies that $v_{n-1} \in k(n-1)_*((\Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p)^{hT^n})$ is non-zero.

See [JW75] for the following details about Morava K-theory. We have $H_*(k(n)) = \overline{A}_*$, where $\overline{A}_*$ is the dual Steenrod algebra A_*, without the generator $\overline{\tau}_n$. Multiplication by v_n yields a cofiber sequence

$$\Sigma^{2p^n-2}k(n) \to k(n) \to H\mathbb{F}_p$$

which in homology decomposes into short exact sequences

$$0 \to \overline{A}_* \to A_* \to \Sigma^{2p^n-1}\overline{A}_* \to 0.$$

Since $\Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p$ is an $H\mathbb{F}_p$-module spectrum, we have $k(m)_*(\Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p) \cong k(m)_*(H\mathbb{F}_p) \otimes L(T^{\mathbf{n}})$.

By Proposition 1.5.7, there is a homotopy fixed point spectral sequence

$$E^2 = H^{-*}(T^n, H_*(\Lambda_{T^n}H\mathbb{F}_p)) \cong P(t_1 \dots t_n) \otimes H_*(\Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p) \Rightarrow \pi_*((H\mathbb{F}_p \wedge \Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p)^{hT^n})$$

where $|t_i| = (-2, 0)$. Similarly as in Section A.5, the right hand side is called the continuous homology of $(\Lambda_{T^n}H\mathbb{F}_p)^{hT^n}$ and denoted with $H^c_*((\Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p)^{hT^n})$.

Give $ES^1 = S(\mathbb{C}^\infty)$ the free S^1-CW structure given by the odd spheres filtration, and use $S(\mathbb{C}^\infty)^n$ as a model for the free contractible T^n-CW complex ET^n. Let E_kT^n denote the k-skeleton of ET^n.

By Proposition 1.5.7 the filtration $E_0T^n_+ \to E_2T^n_+$ yields a spectral sequence

$$E^2(M, n) = M_*(\Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p)\{1, t_1, \dots t_n\} \Rightarrow M_*(F(E_2T^n_+, \Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p)^{T^n})$$

when M is $H\mathbb{F}_p$ or $k(m)$.

First we show that in our case it suffices to look at the first two columns in the homotopy fixed points spectral sequence, to determine whether v_{n-1} is non-zero in the homotopy fixed points.

Proposition 3.5.1. *Assume x in $E^2_{-2,2p^{n-1}}(k(n-1), n)$ survives to $E^3(k(n-1), n)$. If $d^2(\overline{\tau}_{n-1}) = x$ in $E^2(H\mathbb{F}_p, n)$, then $x = uv_{n-1}$ for some unit u.*

Proof. The cofiber sequence

$$\Sigma^{2p^{n-1}-2}k(n-1) \wedge \Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p \xrightarrow{v_{n-1}\wedge\Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p} k(n-1) \wedge \Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p \xrightarrow{i} H\mathbb{F}_p \wedge \Lambda_{T^n}H\mathbb{F}_p$$

preserves the filtration used to construct the spectral sequences, so it descends to a map of spectral sequences.

Now $E^\infty_{0,2p^{n-1}-2}(k(n-1), n) \cong E^\infty_{0,2p^{n-1}-2}(H\mathbb{F}_p, n)$, and, since $E^2_{-2,2p^{n-1}}(k(n-1), n)$ maps injectively to $E^2_{-2,2p^{n-1}}(H\mathbb{F}_p, n)$, the class of $d^2(\overline{\tau}_{n-1}) = x$ generates the kernel of

$$k(n)_{2p^{n-1}-2}(F(E_2T^n_+, \Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p)^{T^n}) \to H_{2p^{n-1}-2}(F(E_2T^n_+, \Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p)^{T^n}).$$

The difference between $E^2_{0,2p^{n-1}-1}(k(n-1), n)$ and $E^2_{0,2p^{n-1}-1}(H\mathbb{F}_p, n)$ is $\overline{\tau}_{n-1}$, so since $d^2(\overline{\tau}_{n-1}) = x$ and x is not hit by any differential in $E^2(k(n-1), n)$, the map i is an isomorphism on $E^2_{0,2p^{n-1}-1}$. Thus the edge homomorphism maps $E^2_{0,2p^{n-1}-1}(H\mathbb{F}_p, n)$ to zero in $E^2_{0,0}(k(n-1), n)$, so it must map $1 \in E^2_{0,0}(k(n-1), n)$ to kernel of i, which is generated by x.

□

Definition 3.5.2. *Let M be a homology theory. The homomorphism*

$$(\omega_{T^{\mathbf{n}}})_* : M_*(T^{\mathbf{n}}_+ \wedge \Lambda_{T^{\mathbf{n}}}H\mathbb{F}_p) \to M_*(\Lambda_{T^{\mathbf{n}} \times T^{\mathbf{n}}}H\mathbb{F}_p)$$

in Definition 1.1.3, together with the stable splitting $T^n_+ \simeq T^n \vee S^0$ and the multiplication map in the group $T^{\mathbf{n}}$, induces a homomorphism

$$H_*(T^{\mathbf{n}}) \otimes M_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p) \to M_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p).$$

Given $j \in \mathbf{n}$ and $x \in M_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)$ we write $\sigma_j(x)$ for the image of $[S^1_j] \otimes x$ under this map, where $[S^1_j] \in H_*(T^{\mathbf{n}})$ is the image of a fundamental class $[S^1] \in H_*(S^1)$ under the inclusion of the j-th circle.

Let $p \geq 5$ and $1 \leq n \leq p$ or $p = 3$ and $1 \leq n \leq 2$, and define $\mathfrak{p}$ to be the kernel of the projection homomorphism

$$H_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p) \cong A_* \otimes L(T^{\mathbf{n}}) \cong A_* \otimes \bigotimes_{U \subseteq \mathbf{n}} B_U \to \bigotimes_{i \in \mathbf{n}} B_{\{i\}},$$

where the isomorphism $\alpha : L(T^{\mathbf{n}}) \cong \bigotimes_{U \subseteq \mathbf{n}} B_U$ comes from Theorem 3.4.3.

We will now show that the image of the d^2-differential in the homotopy fixed points spectral sequence is usually contained in the ideal $\mathfrak{p}$.

Proposition 3.5.3. *Let $p \geq 5$ and $1 \leq n \leq p$ or $p = 3$ and $1 \leq n \leq 2$. If x is in the subring $P(\bar{\xi}_1, \bar{\xi}_2, \ldots) \otimes L(T^{\mathbf{n}}) \subseteq A_* \otimes L(T^{\mathbf{n}}) \cong H_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)$, then $\sigma_j(x)$ is in $\mathfrak{p}$ for all $j \in \mathbf{n}$.*

Proof. Since σ_j is a derivation by Proposition 1.1.4, it suffices to check the claim for the set of $\mathbb{F}_p$-algebra generators in $P(\bar{\xi}_1, \bar{\xi}_2, \ldots) \otimes L(T^{\mathbf{n}}) \cong P(\bar{\xi}_1, \bar{\xi}_2, \ldots) \otimes \bigotimes_{U \subseteq \mathbf{n}} B_U$ consisting of $\bar{\xi}_i$ for $i \geq 1$ together with all U-labeled admissible words, where $U \subseteq \mathbf{n}$.

By Proposition A.4.4, the element $\sigma(\bar{\xi}_i)$ for $i \geq 1$ is represented by $\sigma\bar{\xi}_i$ in filtration 1 in the Bökstedt spectral sequence calculating $H_*(\Lambda_{S^1} H\mathbb{F}_p)$. By Proposition A.4.6 the element $\sigma\bar{\xi}_i$ is a boundary in the Bökstedt spectral sequence, and hence $\sigma(\bar{\xi}_i) \in A_*$. It must thus be equal to zero since it is the image of $[S^1] \otimes \bar{\xi}_i$, and $[S^1]$ is mapped to zero on the left hand side in the commutative diagram

$$\begin{array}{ccc} H_*(S^1_+) \otimes H_*(H\mathbb{F}_p) & \xrightarrow{\omega_*} & H_*(\Lambda_{S^1} H\mathbb{F}_p) \\ \downarrow{\scriptstyle \mathrm{pr} \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{pr} \otimes \mathrm{id}} \\ H_*(S^0) \otimes H_*(H\mathbb{F}_p) & \xrightarrow{\omega_*} & H_*(\Lambda_{\{\mathrm{pt}\}} H\mathbb{F}_p). \end{array}$$

Hence $\sigma_j(\bar{\xi}_i) = 0$ for all $i \geq 1$ and $1 \leq j \leq n$.

We prove the proposition by induction on the degree m of an element x in $L(T^{\mathbf{n}}) \cong \bigotimes_{U \subset \mathbf{n}} B_U$. When $m = 0$, there is nothing to check since σ_j is trivial on units.

Assume the proposition holds for all elements in degree less than m. If m is even, the proposition holds because $\sigma_j(x)$ is then of odd degree, and $\bigotimes_{i \in \mathbf{n}} B_{\{i\}}$ is concentrated in even degrees. Assume m is odd, and that x is a U-labeled admissible word of degree m for some $U \subseteq \mathbf{n}$. By Proposition 3.3.2, x is thus equal to $\varrho_k y$, where k is the greatest element in U and y is a $U \setminus k$-labeled admissible word of even degree.

By part 4 of Theorem 3.4.3, x is equal to $\sigma_k(y)$ where we think of y as being an element in $B_{U\backslash k} \subseteq L(T^{U\backslash k}) \subseteq L(T^{\mathbf{k-1}})$.

If $j > k$, the element $\sigma_j(x) = \sigma_j(\sigma_k(y))$ is in $\mathfrak{p}$ by part 4 of Theorem 3.4.3.

The element $\sigma_j(\sigma_k(y))$ is equal to the image of $[S^1_j] \cdot [S^1_k] \otimes y$, where $[S^1_j] \cdot [S^1_k]$ is the product in $H_*(T^{\mathbf{n}})$. When $k = j$, $\sigma_j(\sigma_k(y))$ is thus zero since $[S^1_j]^2 = 0$.

When $j < k$, we have $\sigma_j(\sigma_k(y)) = \pm\sigma_k(\sigma_j(y))$ since the ring $H_*(T^{\mathbf{n}})$ is graded commutative. Now, $\sigma_j(y)$ is in $L(T^{\mathbf{k-1}})$, so by part 4 in Theorem 3.4.3, the element $\sigma_k(\sigma_j(y))$ is in $\mathfrak{p}$. Hence, $\sigma_j(\sigma_k(y))$ is in $\mathfrak{p}$. □

Proposition 3.5.4. *The differential in $E^2(H\mathbb{F}_p, n)$ is given by*

$$d^2(x) = t_1\sigma_1(x) + \ldots + t_n\sigma_n(x),$$

for $x \in E^2_{0,*}(H\mathbb{F}_p, n)$.

Thus, if $p \geq 5$ and $1 \leq n \leq p$ or $p = 3$ and $1 \leq n \leq 2$, and x is in the subring $P(\bar{\xi}_1, \bar{\xi}_2, \ldots) \otimes L(T^{\mathbf{n}}) \subseteq H_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p) \cong E^2_{0,*}(H\mathbb{F}_p, n)$, then $d^2(x)$ is in $\mathfrak{p}\{t_1, \ldots, t_n\}$.

Proof. There is a surjective homomorphism from the spectral sequence

$$E^2 = H^{-*}(T^n, H_*(\Lambda_{T^n} H\mathbb{F}_p)) \cong P(t_1 \ldots t_n) \otimes H_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p) \Rightarrow \pi_*((H\mathbb{F}_p \wedge \Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)^{hT^n}),$$

to $E^2(H\mathbb{F}_p, n)$. Inclusion of fixed points induces the projection homomorphism from E^2 to

$$'E^2 = H^{-*}(S^1, H_*(\Lambda_{T^n} H\mathbb{F}_p)) \cong P(t_i) \otimes H_*(\Lambda_{T^{\mathbf{n}}}) \Rightarrow \pi_*((H\mathbb{F}_p \wedge \Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)^{hS^1}))$$

where S^1 acts on the i-th circle in T^n. Now $'E^2$ maps injectively to the Tate spectral sequence, so by Proposition 2.1.2, the d^2-differential in $'E^2$ is induced by the operator σ_j.

The formula for the differential in $E^2(H\mathbb{F}_p, n)$ is thus

$$d^2(x) = t_1\sigma_1(x) + \ldots + t_n\sigma_n(x),$$

and the second claim now follows by Proposition 3.5.3. □

We will now show that the element $t_1\mu_1^{p^{n-1}} + t_2\mu_2^{p^{n-1}} + \ldots + t_n\mu_n^{p^{n-1}}$ in $E^2(H\mathbb{F}_p, n)$ is not hit by any differential in the homotopy fixed points spectral sequence. The idea of the proof is that by the previous propositions only $\overline{\tau}_i$ can hit an element in $P(\mu_1, \ldots \mu_n)\{t_1, \ldots t_n\}$. For dimension reasons this can only happen when $i \leq n-2$, but since we have one fewer variable $\overline{\tau}_i$ than μ_j, these will not add up correctly.

Proposition 3.5.5. *Let $p \geq 5$ and $1 \leq n \leq p$ or $p = 3$ and $1 \leq n \leq 2$. The element $t_1\mu_1^{p^{n-1}} + t_2\mu_2^{p^{n-1}} + \ldots + t_n\mu_n^{p^{n-1}}$ in*

$$E^2(k(n-1), n) \cong k(n-1)_*(\Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)\{1, t_1, \ldots, t_n\} \Rightarrow k(n-1)_*(F(E_2T^n_+, \Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)^{T^n})$$

is not hit by any differential, is obviously a cycle, and thus represents a non-zero element in $k(n-1)_*(F(E_2T^n_+, \Lambda_{T^{\mathbf{n}}} H\mathbb{F}_p)^{T^n})$.

Proof. Since $k(n-1)_*(\Lambda_{T^\mathbf{n}} H\mathbb{F}_p) \subseteq H_*(\Lambda_{T^\mathbf{n}} H\mathbb{F}_p)$, the differentials in $E^2(k(n-1), n)$ are determined by the differentials in $E^2(H\mathbb{F}_p, n)$. By Proposition A.4.6 and A.4.4, $\sigma_i(\overline{\tau}_j) = \mu_i^{p^j}$, so Proposition 3.5.4 yields $d^2(\overline{\tau}_j) = \sum_{i=1}^n t_i \mu_i^{p^j}$. Assume z is an element in $k(n-1)_*(\Lambda_{T^\mathbf{n}} H\mathbb{F}_p)$ with differential $d^2(z) = \sum_{i=1}^n t_i \mu_i^{p^{n-1}}$. It can be written, not necessarily uniquely, as

$$z = \overline{\tau}_0 z_0 \dots \overline{\tau}_{n-2} z_{n-2} + z',$$

where z' is in $P(\overline{\xi}_1, \overline{\xi}_2, \dots) \otimes L(T^\mathbf{n})$. By Proposition 3.5.4, $d^2(z')$ is in $\mathfrak{p}\{t_1, \dots, t_n\}$, so we must have

$$d^2(\overline{\tau}_0) z_0 + \dots + d^2(\overline{\tau}_{n-2}) z_{n-2} = \sum_{j=0}^{n-2} (t_1 \mu_1^{p^j} + \dots + t_n \mu_n^{p^j}) z_j = \sum_{i=1}^{n} t_i \mu_i^{p^{n-1}} + y, \quad (3.5.6)$$

for some y in $\mathfrak{p}\{t_1, \dots, t_n\}$.

Write the elements z_i in the monomial basis in $A_* \otimes L(T^\mathbf{n}) \cong A_* \otimes \bigotimes_{U \subseteq \mathbf{n}} B_U$. For equation 3.5.6 to hold, at least one of the z_i-s must have a non-zero coefficient in front of $\mu_1^{p^{n-1}-p^i}$. We let $k_1 \geq 0$ be the greatest integer i such that this coefficient is non-zero.

Let $k_2 < k_1$ be the greatest integer where the coefficient in front of $\mu_1^{p^{n-1}-p^{k_1}} \mu_2^{p^{k_1}-p^{k_2}}$ in z_{k_2} is non-zero. Such an integer must exist, because the coefficient in front of $t_2 \mu_2^{p^{k_1}} \mu_1^{p^{n-1}-p^{k_1}}$ on the left hand side in equation 3.5.6 would otherwise be non-zero due to the contribution from $d^2(\overline{\tau}_{k_1}) z_{k_1}$.

Continuing in this way we get that, since there are n variables t_i, there must be a sequence of integers $k_1 > \dots > k_n$ such that the coefficient in front of the monomial $\mu_1^{p^{n-1}-p^{k_1}} \mu_2^{p^{k_1}-p^{k_2}} \cdots \mu_n^{p^{k_n}-p^{k_n}}$ in z_{k_n} is non-zero. But this is impossible since there are only $n-1$ number of variables z_i.

We thus get a contradiction, so there is no element z in $k(n-1)_*(\Lambda_{T^\mathbf{n}} H\mathbb{F}_p)$ with differential $d^2(z) = \sum_{i=1}^n t_i \mu_i^{p^{n-1}}$. □

Theorem 3.5.7. *Let $p \geq 5$ and $1 \leq n \leq p$ or $p = 3$ and $1 \leq n \leq 2$. Then v_{n-1} in $k(n-1)_*((\Lambda_{T^\mathbf{n}} H\mathbb{F}_p)^{hT^n})$ is non-zero. Equivalently, the homomorphism*

$$k(n-1)_*(\Sigma^{2p^{n-1}-2} F(E_2 T^n_+, \Lambda_{T^\mathbf{n}} H\mathbb{F}_p)^{T^n}) \xrightarrow{v_{n-1}} k(n-1)_*(F(E_2 T^n_+, \Lambda_{T^\mathbf{n}} H\mathbb{F}_p)^{T^n})$$

maps 1 to something non-zero.

Proof. The unit map $S^0 \to (\Lambda_{T^\mathbf{n}} H\mathbb{F}_p)^{hT^n}$ and the inclusion $E_2 T^n \to ET^n$ induces the vertical homomorphisms in the commutative diagram

$$\begin{tikzcd}
k(n-1)_*(\Sigma^{2p^{n-1}-2} S^0) \arrow[r, "v_{n-1}"] \arrow[d] & k(n-1)_*(S^0) \arrow[d] \\
k(n-1)_*(\Sigma^{2p^{n-1}-2} (\Lambda_{T^\mathbf{n}} H\mathbb{F}_p)^{hT^n}) \arrow[r, "v_{n-1}"] \arrow[d] & k(n-1)_*((\Lambda_{T^\mathbf{n}} H\mathbb{F}_p)^{hT^n}) \arrow[d] \\
k(n-1)_*(\Sigma^{2p^{n-1}-2} F(E_2 T^n_+, \Lambda_{T^\mathbf{n}} H\mathbb{F}_p)^{T^n}) \arrow[r, "v_{n-1}"] & k(n-1)_*(F(E_2 T^n_+, \Lambda_{T^\mathbf{n}} H\mathbb{F}_p)^{T^n}).
\end{tikzcd}$$

By Proposition 3.5.5 and 3.5.1 the homomorphism v_{n-1} maps 1 in the lower left hand corner to the non-zero element represented by the cycle $t_1\mu_1^{p^{n-1}} + t_2\mu_2^{p^{n-1}} + \ldots + t_n\mu_n^{p^{n-1}}$ in the lower right hand corner. Hence, the image of v_{n-1} must be non-zero in the middle group on the right hand side of the diagram. $\square$

Appendix A

Tools for Calculation

A.1 Hopf Algebras

We will now recall the definition, and some basic properties, of a Hopf Algebra. See [MM65] and Chapter 20 in [MP12] for more details. The spectral sequences we encounter later will often have extra structure coming from a Hopf algebra, and this will aide us in our calculations.

Our ground ring will be graded, so there will be some small differences between our treatment and the classical treatments of Hopf algebras. Our constructions could be made more general, but we restrict the attention to the cases we are interested in.

We will work with graded objects, and all our objects will be non-negatively graded. Let R be a fixed graded commutative field, i.e., a graded commutative ring such that every graded R-module is free, and write $\otimes$ for $\otimes_R$.

Definition A.1.1. *An R-algebra is a graded R-module A together with morphisms of graded R-modules $\phi : A \otimes A \to A$, called the* multiplication*, and $\eta : R \to A$, called the* unit *such that the following diagrams are commutative*

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes \phi"] \arrow[d, "\phi \otimes \mathrm{id}"] & A \otimes A \arrow[d, "\phi"] \\ A \otimes A \arrow[r, "\phi"] & A \end{tikzcd} \qquad \begin{tikzcd} R \otimes A \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\phi"] & A \otimes R \arrow[l, "\mathrm{id} \otimes \eta"'] \arrow[dl, "\cong"] \\ & A. & \end{tikzcd}$$

A morphism of R-algebras $f : A \to B$ is a morphisms of graded R-modules, such that the following diagrams commute

$$\begin{tikzcd} A \otimes A \arrow[r, "\phi_A"] \arrow[d, "f \otimes f"] & A \arrow[d, "f"] \\ B \otimes B \arrow[r, "\phi_B"] & B \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "f"] \\ R \arrow[ur, "\eta_A"] \arrow[r, "\eta_B"'] & B. \end{tikzcd}$$

Given two R-modules A and B we define the *twist map* $\tau_{A,B} : A \otimes B \to B \otimes A$ by the formula $\tau(a \otimes b) = (-1)^{kl} b \otimes a$ where a is an element in A of degree k and b is an element in B of degree l

Given two R-algebras A and B, then $A \otimes B$ is an R-algebra with multiplication the composition

$$A \otimes B \otimes A \otimes B \xrightarrow{A \otimes \tau \otimes B} A \otimes A \otimes B \otimes B \xrightarrow{\phi_A \otimes \phi_B} A \otimes B$$

and unit

$$R \cong R \otimes R \xrightarrow{\eta_A \otimes \eta_B} A \otimes B\ .$$

An R-algebra A is said to be *commutative* if the diagram

$$\begin{tikzcd} A \otimes A \arrow[d, "\tau"] \arrow[dr, "\phi"] & \\ A \otimes A \arrow[r, "\phi"'] & A \end{tikzcd}$$

commutes.

An R-algebra A is said to be *connected* if $\eta\colon\ : R_0 \to A_0$ is an isomorphism, and is said to be *augmented* if there is an R-algebra map $\epsilon\colon A \to R$ such that $\epsilon\eta = \mathrm{id}$. Given an augmentation ϵ, the kernel $\ker(\epsilon)$ is denoted $I(A)$ and is called *the augmentation ideal.* The splitting $\epsilon\eta = \mathrm{id}$ induces an isomorphism $A \cong R \oplus I(A)$.

Definition A.1.2. *An R-coalgebra is a graded R-module A together with morphisms of graded R-modules $\psi : A \to A \otimes A$, called the* coproduct*, and $\epsilon : A \to R$, called the* counit *such that the following diagrams are commutative*

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\psi"] & A \otimes A \arrow[d, "\psi \otimes \mathrm{id}"] \\ A \otimes A \arrow[r, "\mathrm{id} \otimes \psi"] & A \otimes A \otimes A \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "\cong"'] \arrow[d, "\psi"] \arrow[dr, "\cong"] & \\ R \otimes A & A \otimes A \arrow[l, "\epsilon \otimes \mathrm{id}"'] \arrow[r, "\mathrm{id} \otimes \epsilon"] & A \otimes R \end{tikzcd}$$

A morphism of R-coalgebras $f : A \to B$ is a morphisms of graded R-modules, such that the following diagrams commute

$$\begin{tikzcd} A \arrow[r, "\psi_A"] \arrow[d, "f"] & A \otimes A \arrow[d, "f \otimes f"] \\ B \arrow[r, "\psi_B"] & B \otimes B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "f"] \arrow[dr, "\epsilon_A"] & \\ B. \arrow[r, "\epsilon_B"'] & R \end{tikzcd}$$

Given two R-coalgebras A and B, then $A \otimes B$ is an R-coalgebra with coproduct the composition

$$A \otimes B \xrightarrow{\psi_A \otimes \psi_B} A \otimes A \otimes B \otimes B \xrightarrow{A \otimes \tau \otimes B} A \otimes B \otimes A \otimes B$$

and counit

$$A \otimes B \xrightarrow{\epsilon_A \otimes \epsilon_B} R \otimes R \cong R\,.$$

An R-coalgebra A is said to be *cocommutative* if the diagram

$$\begin{tikzcd} & A \otimes A \arrow[d, "\tau"] \\ A \arrow[ur, "\psi"] \arrow[r, "\psi"'] & A \otimes A. \end{tikzcd}$$

commutes.

An R-coalgebra A is said to be *connected* if $\epsilon\colon\ : A_0 \to R_0$ is an isomorphism., and is said to be *unital* if there is an R-coalgebra map $\eta\colon R \to A$ such that $\epsilon\eta = \mathrm{id}$. Given such a map, the cokernel $\mathrm{Coker}(\eta)$ is denoted $J(A)$. The splitting $\epsilon\eta = \mathrm{id}$ induces an isomorphism $A \cong R \oplus J(A)$

Definition A.1.3. *Let A be an augmented R-algebra. We define the R-module $Q(A)$ of indecomposable elements in A by the exact sequence*

$$I(A) \otimes I(A) \xrightarrow{\phi} I(A) \longrightarrow Q(A) \longrightarrow 0.$$

Let A be a unital R-coalgebra. We define the R-module $P(A)$ of primitive elements in A by the exact sequence

$$0 \longrightarrow P(A) \longrightarrow J(A) \xrightarrow{\psi} J(A) \otimes J(A).$$

Let A be a unital R-coalgebra and let $I(A) = \ker \epsilon$. We say that an element x in $I(A)$ is primitive if its image in $J(A)$ lies in $P(A)$.

Definition A.1.4. *Given an R-coalgebra A with coproduct ψ, we define the reduced coproduct $\widetilde{\psi} : A \to A \otimes A$ to be equal to $\widetilde{\psi} = \psi - \mathrm{id} \otimes 1 - 1 \otimes \mathrm{id}$.*

Lemma A.1.5. *If A is a unital R-coalgebra, the primitive elements in $I(A)$ is equal to the kernel of $\widetilde{\psi}$. I.e., if $x \in I(A)$ is primitive, then*

$$\psi(x) = x \otimes 1 + 1 \otimes x.$$

Proof. This is clear from the definitions. $\square$

Definition A.1.6. *An R-Hopf algebra is a graded R-module A together with morphisms of graded R-modules*

$$\phi : A \otimes A \to A \qquad \psi : A \to A \otimes A$$
$$\eta : R \to A \qquad \epsilon : A \to R$$

such that the morphisms ϕ, η and ϵ makes A into an augmented R-algebra, ψ, ϵ and η makes A into a unital R-coalgebra, and the following diagram commutes

$$\begin{tikzcd} A \otimes A \arrow[r, "\phi"] \arrow[d, "\psi\otimes\psi"] & A \arrow[r, "\psi"] & A \otimes A \\ A \otimes A \otimes A \otimes A \arrow[rr, "\mathrm{id}\otimes\tau_{A,A}\otimes\mathrm{id}"] & & A \otimes A \otimes A \otimes A. \arrow[u, "\phi\otimes\phi"] \end{tikzcd}$$

Commutativity of the last diagram is equivalent to ψ being a morphism of R-algebras, or ϕ being a morphism of R-coalgebras.

We say that an R-Hopf algebra is *connected* if A is connected as an R-algebra, or equivalently as an Rcoalgebra. We say A is *commutative* if A is commutative as an R-algebra, and cocommutative if A is cocommutative as an R-coalgebra.

Proposition A.1.7. *If A is a connected commutative R-Hopf algebra, there is an R-module map $\chi : A \to A$ called the* conjugation *such that* $\chi^2 = \mathrm{id}$ *and the following diagram commutes*

$$\begin{tikzcd} A \arrow[r, "\epsilon"] \arrow[d, "\psi"] & R \arrow[r, "\eta"] & A \\ A \otimes A \arrow[rr, "\mathrm{id}\otimes\chi"] & & A \otimes A. \arrow[u, "\phi"] \end{tikzcd}$$

Proof. We have $A \cong I(A) \oplus R$. Let x be an element in $I(A)$ of degree q. Then $\psi(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$, and since A is connected, $0 < |x''| < q$. We inductively define χ by the formula $\chi(x) = -x - \sum x'\chi(x'')$. That χ^2 is the identity follows from Section 8 in [MM65]. The generalization to a graded ground ring R is straightforward. $\square$

Proposition A.1.8. *Let A and B be unital R-coalgebras. Then there is a split short exact sequence of R-modules*

$$0 \longrightarrow P(A) \xrightarrow{P(A\otimes\eta_B)} P(A \otimes B) \xrightarrow{P(\epsilon_A\otimes B)} P(B) \longrightarrow 0,$$

where the splitting is given by $P(\eta_A \otimes B)$.

Proof. When R is concentrated in degree zero, a more general statement is given in Proposition 3.12 in [MM65], and the proof below is an adaption of the proof of this proposition.

Let $i_A = A \otimes \eta_B$ and $\mathrm{pr}_B = \epsilon_A \otimes B$.

Exactness of the short exact sequence in the proposition is clear except at the middle term. That $P(\mathrm{pr}_B) \circ P(i_A) = 0$ is clear since $\mathrm{pr}_B \circ i_A = \eta_B \circ \epsilon_A$

The only thing left to prove is that $\ker(P(\mathrm{pr}_B)) \subseteq \mathrm{im}(P(i_A))$. Observe that there is an exact sequence of R-modules

$$0 \longrightarrow A \xrightarrow{i_A} A \otimes B \xrightarrow{f} A \otimes B \otimes B$$

where f is given by $f = \mathrm{id}_{A\otimes B} \otimes 1 - (\mathrm{id}_{A\otimes B} \otimes \mathrm{pr}_B) \circ \psi_{A\otimes B}$. Exactness is clear except for $\ker(f) \subseteq \mathrm{im}(i_A)$. The composite

$$A \otimes B \xrightarrow{f} A \otimes B \otimes B \xrightarrow{\mathrm{id}_A \otimes \epsilon_B \otimes \mathrm{id}_A} A \otimes R \otimes B \cong A \otimes B$$

is equal to $\mathrm{pr}_A \otimes 1 - \mathrm{id}$, and the kernel of this map is equal to $\mathrm{im}(i_A)$, hence $\ker(f) \subseteq \mathrm{im}(i_A)$.

If $x \in P(A \otimes B)$ satisfy $P(\mathrm{pr}_B)(x) = 0$, then $\mathrm{pr}_B(x) = 0$ so $f(x) = 0$. Hence $x \in A \cap P(A \otimes B)$, so x is an element in $P(A)$. Thus $\ker(P(\mathrm{pr}_B)) \subseteq \mathrm{im}(P(i_A))$. □

A.2 The Bar Complex

In this section we let k be a field. Everything in this section can be found in Chapter VIII and X in [ML95].

Definition A.2.1. *A simplicial k-module M is a family of k-modules M_n, $n \geq 0$ together with k-module homomorphisms*

$$\begin{aligned} &d_i : M_n \to M_{n-1}, i = 0, \ldots, n, && \textit{called } \text{face maps } \textit{and} \\ &s_j : M_n \to M_{n+1}, j = 0, \ldots, n, && \textit{called } \text{degeneracy maps,} \end{aligned}$$

satisfying the simplicial identities. See Section VIII.5. in [ML95] for more details.

Associated to every simplicial k-module M we have an associated chain complex (M_*, d), called the Moore complex, with differential

$$d = \sum_{i=0}^{n} (-1)^i d_i.$$

Let M be a simplicial k-module and denote with DM_n the submodule of M_n generated by the degenerate simplices, i.e., $DM_n = s_0 M_{n-1} + \ldots S_{n-1} M_{n-1}$. The relations between the face and degeneracy maps show that DM_* is a subcomplex of M_*. We call the complex M_*/DM_* the normalized complex of M_*.

Proposition A.2.2. *The canonical projection $M_* \to M_*/DM_*$ is a quasi-isomorphism.*

Proof. See Theorem VIII.6.1 in [ML95]. □

Given two simplicial k-modules M and N their product $M \times N$ is defined component wise, i.e $(M \times N)_n = M_n \otimes N_n$, $d_n = d_n \otimes d_n$ and $s_n = s_n \otimes s_n$.

Before we define the shuffle product, we must give the definition of a shuffle. Let m and n be two non-negative integers. An (m, n)-shuffle (μ, ν) is a partition of the set $\{0, \ldots, m + n - 1\}$, into two disjoint subsets $\mu_1 < \ldots \mu_m$ and $\nu_1 < \ldots, \nu_n$ of m and n integers, respectively. The sign of this permutation is denoted with $\mathrm{sgn}(\mu, \nu)$.

Theorem A.2.3. *There is a natural map of chain complexes $sh : M_* \otimes N_* \to (M \times N)_*$, called the* shuffle map, *given by*

$$sh(a \otimes b) = \sum_{(\mu,\nu)} \operatorname{sgn}(\mu,\nu)(s_{\nu_n} \dots s_{\nu_1}(a) \otimes s_{\mu_m} \dots s_{\mu_1}(b),$$

where $a \in M_m$, $b \in N_n$ and the sum runs over all $(m+n)$ shuffles. The map is associative, graded commutative and a chain equivalence.

Proof. See Theorem VII.8.8 in [ML95]. □

This map induces a map on normalized chain complexes as well.

Corollary A.2.4. *The shuffle map induces a chain transformation on the normalized chain complexes*

$$sh : DN_* \otimes DM_* \to (DN \times DM)_*.$$

Proof. See Corollary VII.8.9 in [ML95]. □

If M is a simplicial k-algebra then composing the shuffle product with the algebra product gives M_* a k-algebra structure.

Write $\tilde{d}$ for the "last" face map in a simplicial module. That is if $a \in N_n$ then $\tilde{d}a = d_n a$.

Theorem A.2.5. *There is a natural map of chain complexes $f : (M \times N)_* \to M_* \otimes N_*$, called the* Alexander-Whitney map, *given by*

$$f(a \times b) = \sum_{i=0}^{n} \tilde{d}^{n-i} a \otimes d_0^i b, \quad a \in N_n, b \in M_n.$$

Proof. See Theorem VII.8.5 in [ML95]. □

Corollary A.2.6. *The Alexander-Whitney map induces a chain transformation on the normalized chain complexes*

$$f : (DN \times DM)_* \to DN_* \otimes DM_*.$$

Proof. See Corollary VII.8.6 in [ML95]. □

We will now elaborate on a particular simplicial module. Let R be a k-algebra let M be a left R-module and N a right R-module, and define the simplicial k-module A by $A_n = M \otimes R^{\otimes n} \otimes N$. The face and degeneracies are given by

$$\begin{aligned} d_i(a_0 \dots, a_{n+1}) &= (a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}) && \text{for } i = 0, \dots, n-1, \\ s_j(a_0 \dots, a_{n+1}) &= (a_0, \dots, a_j, 1, a_{j+1}, \dots, a_{n+1}) && \text{for } j = 0, \dots, n, \end{aligned}$$

where $a_0 \in M$ and $a_{n+1} \in N$.

The (normalized) two-sided Bar complex is defined to be the associated chain complex of the normalization of this simplicial module, and is denoted with $B(M,R,N)$. It is a standard fact that $B(M,R,N)_n = M \otimes (R/k)^{\otimes n} \otimes N$, where R/k is the cokernel of the augmentation $k \to R$. An element in $B(M,R,N)_n$ is written $l[a_1|\dots|a_n]r$ with $l \in N$, $a_i \in R$ and $r \in M$, and these elements are normalized in the sense that $l[a_1|\dots|a_n]r = 0$ when any one $a_i \in k$.

Proposition A.2.7. *The chain complex $B(R,R,M)$, respectively $B(M,R,R)$, is a free resolution, of left, respectively right, modules, of N.*

Corollary A.2.8. *There is an isomorphism*

$$\mathrm{Tor}^R(M,N) \cong H_*(B(M,R,N))$$

There is a standard coproduct in the bar complex, and by the next proposition this coproduct is "unique".

Proposition A.2.9. *Let R be a k-coalgebra. The composition of $B(k,\psi_R,k)$ and the Alexander-Whitney map, induces the standard coproduct on the bar complex $\psi : B(k,R,k) \to B(k,R,k) \otimes_R B(k,R,k)$, which is given by*

$$[a_1|\dots|a_n] \mapsto \sum_{i=0}^{n} [a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_n].$$

Since we work over the field k, it descends to a coproduct in homology via the Künneth isomorphism.

Proof. See Corollary 7.12 in [McC01]. □

Proposition A.2.10. *Let x and y be of even and odd degree, respectively. For all primes p there are isomorphisms of $\mathbb{F}_p$-Hopf algebras*

$$E(\sigma x) \cong \mathrm{Tor}^{P(x)}(\mathbb{F}_p,\mathbb{F}_p) \cong H_*(B(\mathbb{F}_p,P(x),\mathbb{F}_p))$$
$$\Gamma(\sigma y) \cong \mathrm{Tor}^{E(y)}(\mathbb{F}_p,\mathbb{F}_p) \cong H_*(B(\mathbb{F}_p,E(y),\mathbb{F}_p)),$$

and when p is odd there is an isomorphism of $\mathbb{F}_p$-Hopf algebra

$$E(\sigma x) \otimes \Gamma(\varphi x) \cong \mathrm{Tor}^{P_p(x)}(\mathbb{F}_p,\mathbb{F}_p) \cong H_*(B(\mathbb{F}_p,P_p(x),\mathbb{F}_p)),$$

given by sending σx to the class of $[x]$, $\gamma_1(\sigma y)$ to the class of $[y]$ and $\gamma_1(\varphi x)$ to the class of $[x^{p-1}|x]$ in the bar complex.

Proof. The differentials in the bar complexes $B(\mathbb{F}_p,P(x),\mathbb{F}_p)$ and $B(\mathbb{F}_p,E(y),\mathbb{F}_p)$ are all zero, giving us the $\mathbb{F}_p$-module structure in the first two cases.

There is a free resolution

$$\dots \longrightarrow x^{p+1}P_p(x) \longrightarrow x^{p+1}P_p(x) \longrightarrow x^pP_p(x) \longrightarrow xP_p(x) \longrightarrow P_p(x)$$

of $\mathbb{F}_p$ giving us the $\mathbb{F}_p$-module structure of $\mathrm{Tor}^{P_p(x)}(\mathbb{F}_p, \mathbb{F}_p)$.

In the first case, there is only one possible $\mathbb{F}_p$-Hopf algebra structure.

In the second case we have to check that the homomorphism respects the multiplicative structure. Let $y^{[i]} = [y|\dots|y]$ (i-fold product). Then the product on the right hand side is given by

$$\begin{aligned} sh(y^{[n]} \otimes y^{[m]}) &= \sum_{(\mu,\nu)} \mathrm{sgn}(\mu,\nu)(s_{\nu_m}\dots s_{\nu_1}(y^{[n]}) \otimes s_{\mu_m}\dots s_{\mu_1}(y^{[m]})) \\ &= \sum_{(\mu,\nu)} (\mathrm{sgn}(\mu,\nu))^2 (y^{[n+m]}) = \binom{n+m}{n} y^{[n+m]} \end{aligned}$$

where the extra $\mathrm{sgn}(\mu,\nu)$ come from the graded product in $E(y)^{\otimes n+m}$ when we shuffle the non-degenerate factors in $s_{\mu_m}\dots s_{\mu_1}(y^{[m]})$ past the non-degenerate factors in $s_{\nu_m}\dots s_{\nu_1}(y^{[n]})$.

The coalgebra structure follows from Proposition A.2.9.

For the $\mathbb{F}_p$-Hopf algebra structure of $\mathrm{Tor}^{P_p(x)}(\mathbb{F}_p, \mathbb{F}_p)$, we refer to Proposition 7.24 in [McC01]. □

A.3 Spectral Sequences

The construction of a spectral sequence and the convergence properties of a spectral sequence are from Boardmans paper [Boa99]. The algebra and coalgebra properties of spectral sequences are from [McC01].

Definition A.3.1. *An* unrolled exact couple *is a diagram of graded abelian groups and homomorphisms of the form*

$$\begin{tikzcd} \dots \arrow[r, "i"] & A_{s-1} \arrow[r, "i"] & A_s \arrow[r, "i"] \arrow[d, "j"] & A_{s+1} \arrow[r] \arrow[d, "j"] & \dots \\ & & E^1_s \arrow[ul, "k"] & E^1_s \arrow[ul, "k"] & \end{tikzcd}$$

where j or k is a homomorphism of degree 1*, while the other two are of degree* 0*, and where each triangle $A_{s+1} \to A_s \to E_s \to A_{s+1}$ is a long exact sequence.*

Note that we use a different indexing than Boardman.

An unrolled exact couple as above gives rise to a spectral sequence $\{E^r, d^r\}$. That is a sequence of differential bigraded abelian groups $E^r_{s,t}$ for $r \geq 1$ with differential $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ such that $E^{r+1} \cong H_*(E^r, d^r)$.

When we draw our spectral sequence on a grid in the plane, we will put the group $E^r_{s,t}$ in the (s,t)-coordinate. Thus our differentials will go up and to the left.

We will say that a spectral sequence is concentrated in a half plane or a quadrant, if $E^2_{s,t}$ is zero outside of the half plane or quadrant, respectively.

We denote the s-th grade the *horizontal grade* and the t-th grade the *vertical grade.*

For readers who are unfamiliar with the construction of a spectral sequence from an unrolled exact couple, we point to Boardmans paper [Boa99].

Definition A.3.2. *Given an unrolled exact couple as in A.3.1 we*

1. *Filter the colimit* $\operatorname{colim}_s A_s$ *by the subgroups* $F_s A_\infty = \operatorname{im}[A_s \to \operatorname{colim}_s A_s]$*, i.e., there is a sequence of inclusions*

$$\ldots \subseteq F_{s-1} \subseteq F_s \subseteq F_{s+1} \subseteq \ldots \subseteq \operatorname*{colim}_s A_s.$$

2. *Filter the limit* $\lim_s A_s$ *by the subgroups* $F_s A_{-\infty} = \ker[A^\infty \to \lim_s A_s]$*, i.e., there is a sequence of inclusions*

$$\ldots \subseteq F_{s-1} \subseteq F_s \subseteq F_{s+1} \subseteq \ldots \subseteq \lim_s A_s.$$

The homomorphisms $A_s \to \operatorname{colim}_s A_s$ *and* $\lim_s A_s \to A^s$ *are the canonical ones coming from the colimit and limit construction, respectively.*

Given a filtration $\ldots \subseteq F_{s-1} \subseteq F_s \subseteq F_{s+1} \subseteq \ldots \subseteq H$ of the group H, we write $\operatorname{Gr} H$ for the associated graded complex

$$\operatorname{Gr} H = \bigoplus_s F_s/F_{s-1}.$$

We will write $\operatorname{Rlim} A_s$ for the the derived limit of the sequence A_s, see [Boa99] for more details.

Definition A.3.3. *Given an unrolled exact couple we say that the associated spectral sequence* (E^r, d^r) converges strongly *to* H *where* $H = \operatorname{colim}_s A_s$ *or* $H = \lim_s A_s$ *if*

1. *There is an isomorphism* $\operatorname{colim}_s F_s \cong H$*.*

2. *There are isomorphisms* $E_s^\infty \cong F_s/F_{s-1}$*.*

3. *We have* $\lim_s F_s = \operatorname{Rlim}_s F_s = 0$*.*

When a spectral sequence converges strongly, we have by the second property above an isomorphism

$$\operatorname{Gr} H \cong \bigoplus_s E_s^\infty.$$

We will now give two theorems that suffices to prove strong convergence in the cases we are interested in. Before we do that we need the definition of conditionally convergence which together with some extra properties will guarantee strong convergence.

Definition A.3.4. *Given an unrolled exact couple, we say the resulting spectral sequence* converges conditionally to the colimit $\operatorname{colim}_s A_s$ *if* $\lim_s A_s = \operatorname{Rlim}_s A_s = 0$. *We say that the spectral sequence* converges conditionally to the limit $\lim_s A_s$ *if* $\operatorname{colim}_s A_s = 0$.

The next two theorems are Theorem 6.1 and 7.1 in [Boa99], rephrased in the language of half plane spectral sequences.

Theorem A.3.5. *Given an unrolled exact couple, suppose the resulting spectral sequence is concentrated in the right half plane, or the lower half plane.*

1. *If* $\lim_s A_s = 0$*, the spectral sequence converges strongly to the colimit* $\operatorname{colim}_s A_s$
2. *If* $\operatorname{colim}_s A_s = 0$*, the spectral sequence converges strongly to the limit* $\lim_s A_s$.

Theorem A.3.6. *Given an unrolled exact couple, suppose the resulting spectral sequence is concentrated in the left half plane or the upper half plane, and that it converges conditionally to the colimit* $\operatorname{colim}_s A_s$ *or the limit* $\lim_s A_s$. *If* $\operatorname{Rlim}_r E^r = 0$*, the spectral sequence converges strongly.*

Note that $\operatorname{Rlim}_r E^r = 0$ if $E^r_{s,t}$ is finite for all s and t, which will always be the case in our applications.

We finish this section by defining what it means for a spectral sequence to have an algebra and coalgebra structure. This extra structure will be crucial for our calculations. See Section 2 of [McC01] for more details.

A differential bigraded R-algebra $\{E_{*,*}, d\}$ is a bigraded R-module with a product structure $\phi : E_{s,t} \otimes_R E_{u,v} \to E_{s+u,t+v}$ such that d is a derivation, i.e., satisfies the Leibniz rule

$$d(xy) = d(x)y + (-1)^{s+t} x d(y)$$

when $x \in E_{s,t}$ and $y \in E_{u,v}$, and such that it satisfies the usual associativity and unit conditions.

Definition A.3.7. *Assume we have an unrolled exact couple of graded R-algebras, with a spectral sequence (E^r, d^r) converging strongly to H where $H = \operatorname{colim}_s A_s$ or $H = \lim_s A_s$, such that the product ϕ on H satisfy*

$$\phi(F_{s,t} \otimes_R F_{u,v}) \subseteq F_{s+u,t+v}.$$

We say that the spectral sequence is an R-algebra spectral sequence if:

1. *For every $r \geq 1$, $\{E^r_{*,*}, d^r\}$ is a differential bigraded R-algebra.*
2. *The homomorphism ϕ^{r+1} is given as the composite*

$$E^{r+1}_{s,t} \otimes_R E^{r+1}_{u,v} \cong H_*(E^r_{s,t}) \otimes_R H_*(E^r_{u,v}) \to H_*(E^r_{s,t} \otimes_R E^r_{u,v}) \xrightarrow{H_*(\phi^r)} H_*(E^r_{s+u,t+v}) \cong E^{r+1}_{s+u,t+v}$$

where the unlabeled homomorphism is the cross product in homology.

3. *The induced pairing on E^∞ makes the following diagram commute*

$$\begin{tikzcd} \mathrm{Gr}_s H_t \otimes \mathrm{Gr}_u H_v \arrow[r, "\phi"] \arrow[d, "\cong"] & \mathrm{Gr}_{s+u} H_{t+v} \arrow[d, "\cong"] \\ E^\infty_{s,t} \otimes E^\infty_{s,t} \arrow[r, "\phi^\infty"] & E^\infty_{s+u,t+v} \end{tikzcd}$$

where the vertical isomorphisms comes from strong convergence.

Dually, a differential bigraded R-coalgebra $\{E_{*,*}, d\}$ is a bigraded R-module with a coproduct structure

$$\psi : E_{s,t} \to \bigoplus_{\substack{u+x=s \\ v+y=t}} E_{u,v} \otimes_R E_{x,y}$$

such that if $\psi(x) = \sum x' \otimes x''$ then $\psi^r(d(x)) = \sum d(x') \otimes x'' + (-1)^{|x'|} x' \otimes d(x'')$ where $|x'| = u + v$ is the total degree of $x' \in E_{u,v}$, and such that it satisfies the usual coassociativity and counit conditions.

Definition A.3.8. *Assume we have an unrolled exact couple of graded R-coalgebras, with a spectral sequence (E^r, d^r) converging strongly to H where $H = \mathrm{colim}_s A_s$ or $H = \lim_s A_s$. We say that the spectral sequence is an R-coalgebras spectral sequence if:*

1. *For every $r \geq 1$, $\{E^r, d^r\}$ is a differential bigraded R-coalgebra.*
2. *The R-module $E^r_{s,t}$ is flat*
3. *The homomorphism ψ^{r+1} is given as the composite*

$$\begin{aligned} E^{r+1}_{s,t} \cong H_*(E^r_{s,t}) \overset{H_*(\psi^r)}{\to} H_*(&\bigoplus_{\substack{u+x=s \\ v+y=t}} E^r_{u,v} \otimes_R E^r_{x,y}) \\ &\cong \bigoplus_{\substack{u+x=s \\ v+y=t}} H_*(E^r_{u,v}) \otimes_R H_*(E^r_{x,y}) \cong \bigoplus_{\substack{u+x=s \\ v+y=t}} E^{r+1}_{u,v} \otimes_R E^{r+1}_{x,y} \end{aligned}$$

where the second to last isomorphism is the Künneth isomorphism, which exists since $E^r_{s,t}$ is flat as an R-module.

If H is an Rcoalgebra we say that an R-coalgebra spectral sequence converges to H as an R-coalgebra if the coproduct ψ on H satisfy

$$\psi(F_{s,t}) \subseteq \bigoplus_{\substack{u+x=s \\ v+y=t}} F_{u,v} \otimes_R F_{x,y}.$$

and the induced pairing on E^∞ makes the following diagram commute

$$\begin{tikzcd}
\operatorname{Gr}_s H_t \arrow[r, "\psi"] \arrow[d, "\cong"] & \bigoplus_{\substack{u+x=s\\ v+y=t}} \operatorname{Gr}_u H_v \otimes_R \operatorname{Gr}_x H_y \arrow[d, "\cong"] \\
E^\infty_{s,t} \arrow[r, "\psi^\infty"] & \bigoplus_{\substack{u+x=s\\ v+y=t}} E^\infty_{u,v} \oplus E^\infty_{x,y}
\end{tikzcd}$$

where the vertical isomorphisms comes from strong convergence.

Observe that if the definition only holds for E^r when $r \leq r_0$, the coalgebra structure still gives valuable information about the differentials in this range.

A.4 Bökstedt Spectral Sequence

In this section we define the Bökstedt spectral sequence, and give some results about it.

Let R be a graded commutative ring and let A be a augmented R-algebra. See [Lod98] for the definition of the A-Hopf algebra $HH_*(A)$, the Hochschild homology of A.

All the information we need about Hochschild homology can be found in the following proposition, which is similar to Proposition 2.1 in [MS93].

Proposition A.4.1. *Let A be a commutative augmented $\mathbb{F}_p$-algebra. There is an isomorphism of A-Hopf algebras*

$$HH_*(A) \cong A \otimes \operatorname{Tor}^A(\mathbb{F}_p, \mathbb{F}_p).$$

Proof. There is an isomorphism $HH_*(A) \cong \operatorname{Tor}^{A\otimes A^{\mathrm{op}}}(\mathbb{F}_p, \mathbb{F}_p)$ by Proposition 1.1.13 in [Lod98], and by Theorem X.6.1 in [CE56] there is an isomorphism $\operatorname{Tor}^{A\otimes A^{\mathrm{op}}}(\mathbb{F}_p, \mathbb{F}_p) \cong \operatorname{Tor}^A(A', \mathbb{F}_p)$ where A' is A with the trivial A-module structure. Hence, $\operatorname{Tor}^A(A', \mathbb{F}_p) \cong A \otimes \operatorname{Tor}^A(\mathbb{F}_p, \mathbb{F}_p)$. □

There is a Bökstedt spectral sequence first introduced in [Bök86b].

Proposition A.4.2. *Let R be commutative ring spectrum. There is a strongly convergent spectral sequence*

$$E^2_{s,t} = HH_s(H_t(R)) \Rightarrow H_{s+t}(\Lambda_{S^1} R).$$

An overview of this spectral sequence can be found in [AR05], and their Theorem 4.5 states:

Theorem A.4.3. *Let R be a commutative ring spectrum.*

1. *If $H_*(\Lambda_{S^1}R)$ is flat over $H_*(R)$, then $H_*(\Lambda_{S^1}R)$ is an A_*-comodule $H_*(R)$-Hopf algebra.*
2. *If each term E^r for $r \geq 2$ in the Bökstedt spectral sequence calculating $H_*(\Lambda_{S^1}R)$ is flat over $H_*(R)$, then E^r is an A_*-comodule $H_*(R)$-Hopf algebra spectral sequence. In particular, the differentials d^r respect the coproduct ψ.*

From Proposition 4.9 in [AR05] we have

Proposition A.4.4. *Let R be a commutative ring spectrum. Given x in $H_*(R)$ the element $\sigma(x)$ in $H_{*+1}(\Lambda_{S^1}R)$ is represented by σx in $HH_1(H_*(R))$.*

A helpful tool for calculations is Theorem 1 in [Hun96], which is a generalization of an argument by Bökstedt in [Bök86b].

Theorem A.4.5. *Suppose $x \in H_n(R)$ with n odd and positive. Then in the Bökstedt spectral sequence*

$$HH_*(H_*(R)) \Rightarrow H_*(\Lambda_{S^1}R)$$

the element $\gamma_{p^k}(\sigma x)$ lives to E^{p-1} and

$$d^{p-1}(\gamma_{p^k}(\sigma x)) = \sigma(\beta Q^{\frac{n+1}{2}}x)\gamma_{p^k-p}(\sigma x).$$

Given a prime p, let A_* be the dual Steenrod algebra, see [Mil58] for details. When p is odd $A_* = P(\bar{\xi}_1, \bar{\xi}_2, \ldots) \otimes E(\bar{\tau}_0, \bar{\tau}_1, \ldots)$ where $|\bar{\xi}_i| = 2p^i - 2$ and $|\bar{\tau}_i| = 2p^i - 1$, and when p is even $A_* = P(\bar{\xi}_1, \bar{\xi}_2, \ldots)$ where $|\bar{\xi}_i| = 2^i - 1$. Bökstedt proved the following in [Bök86b]. See Theorem 5.2 in [HM97b] for a published account.

When p is odd the Bökstedt spectral sequence calculating

Proposition A.4.6. *When p is odd, the Bökstedt spectral sequence calculating the A_*-comodule $H_*(\Lambda_{S^1}H\mathbb{F}_p)$ begins*

$$E^2 = HH_*(A_*) \cong A_* \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \ldots) \otimes \Gamma(\sigma\bar{\tau}_0, \sigma\bar{\tau}_1, \ldots),$$

and the only non-zero differential is given by Theorem A.4.5, so the E^∞ page is equal to

$$E^\infty \cong A_* \otimes P_p(\sigma\bar{\tau}_0, \sigma\bar{\tau}_1, \ldots).$$

There is an isomorphism of A_*-comodules

$$H_*(\Lambda_{S^1}H\mathbb{F}_p) \cong A_* \otimes P(\sigma\bar{\tau}_0),$$

where $(\sigma\bar{\tau}_0)^{p^i}$ is represented by $\sigma\bar{\tau}_i$ on E^∞.

Similarly, when $p = 2$ there is an isomorphism of A_*-comodules

$$H_*(\Lambda_{S^1}H\mathbb{F}_2) \cong A_* \otimes P(\sigma\bar{\xi}_1).$$

We can use this to calculate the homotopy groups.

Corollary A.4.7. *For any prime p, there is an isomorphism of $\mathbb{F}_p$-algebras*

$$\pi_*(\Lambda_{S^1}H\mathbb{F}_p) \cong P(\mu).$$

A.5 Continuous Homology of Tate Spectra

In this section we define continuous homology of a Tate spectrum, and state some results about the corresponding homological Tate spectral sequence. See [LNR12] for more details. We will need continuous homology in Section 2.4 to show that a homomorphism is not zero.

When G is a finite group, the Greenlees filtration of G gives rise to a filtration

$$X^{tG} \to \cdots \to [\widetilde{EG}/\tilde{E}_{n-1} \wedge F(EG_+, X)]^G \to [\widetilde{EG}/\tilde{E}_n \wedge F(EG_+, X)]^G \to \cdots \to *$$

where the identification of the homotopy (co)limit follows from Lemma 4.4 in [LNR12].

The next definition is Definition 4.7 in [LNR12].

Definition A.5.1. *Let G be a finite group and X an orthogonal G-spectrum whose underlying non-equivariant spectrum is bounded below and of finite type over $\mathbb{F}_p$. By the* continuous homology *of X^{tG} we mean the complete A_*-comodule*

$$H_*^c(X^{tG}) = \lim_{n \to -\infty} H_*([\widetilde{EG}/\tilde{E}_{n-1} \wedge F(EG_+, X)]^G).$$

The following proposition is part of Proposition 4.15 in [LNR12]

Proposition A.5.2. *Let G be a finite group and X a G-spectrum. Assume that X is bounded below and of finite type over $\mathbb{F}_p$. Then the homological Tate spectral sequence*

$$\hat{E}^2_{s,t}(X) = \hat{H}^{-s}(G; H_t(X)) \Rightarrow H^c_{s+t}(X^{tG})$$

converges strongly to the continuous homology of X^{tG} as a complete A_*-comodule.

When $X = B \wedge B$ for some spectrum B, with the action of C_2 being permutation of the two factors, there is more to say about this spectral sequence. The examples we are interested in are are $\Lambda_{C_2 \times X} H\mathbb{F}_p \simeq (\Lambda_X H\mathbb{F}_p) \wedge (\Lambda_X H\mathbb{F}_p)$ for some space X. Below we state the homological version of Proposition 5.14 in [LNR12].

Proposition A.5.3. *Let B be a bounded below spectrum of finite type over $\mathbb{F}_2$. The homological Tate spectral sequence*

$$\hat{E}^2 = \hat{H}^{-s}(C_2; H_*(B)^{\otimes 2}) \Rightarrow H_*^c((B \wedge B)^{tC_2})$$

collapses at the $\hat{E}^2$-term. Hence the $\hat{E}^2 = \hat{E}^\infty$-term is given by

$$\hat{E}^\infty = P(u, u^{-1}) \otimes \mathbb{F}_2\{\alpha\}$$

where α runs through an $\mathbb{F}_p$-basis for $H_*(B)$.

The map $\epsilon_* : H_*((B \wedge B)^{tC_2}) \to H_*^c((B \wedge B)^{tC_2})$ maps an A_*-comodule primitive element $z \in H_n((B \wedge B)^{tC_2})$ to the element represented by $u^n \otimes z^2$ in the Tate spectral sequence.

Proof. The identification of the spectral sequence, and the fact that it collapses is part of Proposition 5.14 in [LNR12]. By Corollary 2.9 in [LNR12] continuous homology is the dual of continuous cohomology. We can now read of the formula for ϵ_* from Proposition 5.12, Formula 3.8 and Proposition 5.14 in [LNR12]. □

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