Tensor Induction As Left Kan Extension

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1 Introduction

A function between sets can be extended by many different ways! If A,B and C are sets and A is non-empty, B $\hookrightarrow$ C, then a function f : B$\rightarrow$ A, can be extended as$f' : C \rightarrow A$, by many different ways. But there is not a canonical or unique way. Besides, if A, B and C are even groups or Rings or Modules, f can be extended as many different functions. But it is not same in Category theory, if we have a functor T : M $\rightarrow$ A, and M is subcategory of C and all colimits and limits exist in A, there is ways to find two canonical extension functors from M to functors L, R : C $\rightarrow$ A. These extensions functors are called Left Kan Extension functor L and Right Kan Extension R. I am going to study here in my thesis the category which is all colimits exist and the Left Kan Extension between Category of R-Modules (Mod_R). I start with the category R[fin], its objects are finite sets and its hom sets are R-modules. R[fin] is full subcategory of Mod_R and the Left Kan Extension of T along the inclusion functor will be found later in the chapter 2.

In the processes of constructing the left kan extension L, some tools are necessary to use. I have found the co-equalizer, co-product, bi-product diagrams in the category Mod_R as my tools. After I define our functor L as co-equalizer digram, the universal property of co-equalizer diagram gives beautifully the unique natural transformation between two functors T and L along the full and faithful functor M to C. which is necessary to prove L is the left kan extension.

Tensor product ($\otimes$) is though as another parallel functor with L in here. Tensor is bilinear as defining property but it is not a linear. As of Kan extension properties, another parallel functor is not an additive functor, tensor is not linear nor additive, we need to make a long proof to find the unique natural transformation between functors L and $\otimes$ by using universal property of co-equalizers. I could manage to prove that tensor product has a quality to use as a parallel functor of the left kan extension.

In the last part of chapter 2, the natural transformation γ between L and $\otimes$ is proved as a unique isomorphism. It becomes L $\cong \otimes$ and it shows that Tensor product is a kind of left kan extension.

In chapter (3), I introduce two category C_G and B^{OP}, the category of the transitive G-set of finite group G and Category of finite G-sets. I construct these two categories with the maps between objects are composing three kinds of maps, the induction, restriction and transferring. I am going to use three kinds of functions when I need the finite g-sets to move between G's subgroups. Then I prove that C_G is full subcategory of B^{OP}. Being C_G is full subcategory and the left Kan extension properties construct the left induction which is a functor category. This left induction functor category gives the connection between tensoring and the Grothendicks group representation.

End of chapter three I introduce the tensor induction with our categories B_H^{OP}, B_K^{OP}, B_H and B_K. If we defining the $Tens_H^K$ to get well adjustment between the two Modules categories $Mod_{R(B_H)}$ and $Mod_{R(B_K)}$, It works and we get the commute diagram with $Tens_H^K$ as the left kan extension.

2 Tensor product

2.1 Tensor Product is not a additive functor

Definition 2.1 (Tensor products of Rmodules, $\bigotimes$)**.** *: Tensor product is bilinear maps. For any two Rmodules M and N, there exist a pair (T, g), Rmodules T and Rmordules morphism g : M $\times$N $\to$ T, with the following property: Given any module P and bilinear f : M $\times$N $\to$ P, there exists a unique morphism $f' : T \to P$ such that $f = f' \circ g$. Every R-bilinear map on M $\times$ N factors through T. Moreover, (T, g) and (T', g') are two pairs with this property, then there exists unique isomorphism $j : T \to T' suchthatj \circ g = g'$.*

The modules T constructed above is called the tensor product of M an N, and is denoted by $M \otimes_R N$. It is generated as an Rmodule by the products $x \otimes_R y$. The elements $x_i \otimes_R y_j$ generate $M \otimes_R N$ if $x_i and y_j$ are families of generators of M and N.

The tensor product is not an additive functor.

Definition 2.2 (Additive functor)**.** *A functor T from additive categories U to V with properties T(f+g)= Tf + Tg for any parallel pair of arrows f, g : $u \to u'$ in U and T send zero object to zero object of V and binary bi-product diagram in U to a bi-product diagram in V.*

Lemma 2.3. *: Tensor product is not additive functor.*

Proof. Tensor product is though as a functor as follow: $\otimes$: Hom(A,A') $\times$ Hom(B,B') $\longrightarrow$ Hom (A $\otimes$ B, $A' \otimes B'$). If we consider our categories A, A', B, B'=R, then $\otimes$: Hom(R,R) $\times$ Hom(R,R) $\longrightarrow$ Hom (R $\otimes$ R, R $\otimes$ R) is R $\times$ R $\longrightarrow$ R since Hom(R,R)= R and R $\otimes$ R= R.

Let ⊗ (a,b) = a.b and f(1)=a≠0 and g(1)=b≠0 , (a,b) ∈ R × R and (f,g) ↦ f ⊗ g, f,g are morphisms in Hom(R,R). We consider (1, 1) in R × R, (1,1)= (1,0)+ (0,1). (f ⊗ g)(1,1)= ⊗[f(1), g(1)]=⊗(a,b)= ab ≠ 0. It is bilinear. But (f ⊗ g)[(1,0)+ (0,1)]=a.0+0.b=0 and (f ⊗ g) [(1,1)] ≠ (f ⊗ g) [(0,1) + (1,0)].

Tensor product does not have the property as additive functor. So, Tensor product is not an additive functor. □

2.2 Left Kan extension

In this chapter we are going to study about the left kan extension of the following diagram:

$$\begin{tikzcd}
R[\mathrm{fin}] \times R[\mathrm{fin}] \arrow[r, "-X-"] \arrow[rr, bend left, "T"] \arrow[d, "F_R \times F_R"] & R[fin] \arrow[r, "F_R"] & \mathrm{Mod}_R \\
\mathrm{Mod}_R \times \mathrm{Mod}_R \arrow[urr, "L"] \arrow[urr, bend right, "\otimes"'] & &
\end{tikzcd}$$

Let L : $\mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Mod}_R$ be a functor together with a natural transformation η : T → $L(F_R \times F_R)$. I am going to prove that the functor ⊗ together with the natural transformation β : T → $\otimes \circ F_R \times F_R$ is a Left Kan extension of T along $F_R \times F_R$. Let $\gamma : L \to \otimes$ is natural transformation.

I am proving that the γ such that β = γK $\circ\eta$ is an unique natural transformation. That is as follow:

$$Nat(L, \otimes) \cong Nat(T, \otimes \circ F_R \times F_R)$$

$$\gamma \mapsto (\gamma F_R \times F_R \circ \eta) = \beta.$$

Definition 2.4 (Left Kan extension)**.** *Let $T : M \to A$ and $K : M \to C$ be functors. In the diagram,*

$$\begin{tikzcd}
M \arrow[r, "T"] \arrow[d, "K"] & A \\
C &
\end{tikzcd}$$

the left Kan extension of T along K is a functor $L_K T : C \to A$ together with a natural transformation $\eta : T \to L_K T\ K$ with the following properties: given any functor $S : C \to A$ together with the natural transformation $\beta : T \to SK$, there exist a unique natural transformation $\gamma : L_K T \to S$ such that $\beta = \gamma K \circ \eta$.

$$Nat(L, S) \cong Nat(T, S \circ K)$$

$$\gamma \mapsto (\gamma K \circ \eta) = \beta$$

is bijection.

We illustrate the concept at a left Kan extension in the follwing diagrams category and functors:

Given two functors T : M $\to$ Mod_R and K : M $\to$ C, then the left Kan extension L_KT = L of T along K exists and L : C $\to$ Mod_R is characterized by a universal property.

$$\begin{tikzcd} M \arrow[r, "T"] \arrow[d, "K"'] & A = \mathrm{Mod}_R \\ C \arrow[ur, "L"] \arrow[ur, bend right, "S"'] & \end{tikzcd}$$

Natural transformations $\eta :$ T $\to$ LK, $\beta :$ T $\to$ SK, and $\gamma :$ L $\to$ S with $\beta = \gamma$K $\circ\eta$ give the diagram.

$$\begin{tikzcd} T \arrow[r, "\eta"] \arrow[dr, "\beta"] & LK \arrow[d, "\gamma K"] \\ & SK \end{tikzcd}$$

Now I want to explain a notation γ**K** which I am going to use.

Definition 2.5. *γ**K**: γ is the natural transformation defined as above. γK is the morphism γ_c : L(c) $\to$ S(c) for each object c of C such that c=Km. γK_m: L(c=Km) $\to$ S(c=Km). Note that L(Km)=(LK)(m) and S(Km)=(SK)(m). The morphisms γK_m for m in M is a natural transformation from LK to SK, which we call γK. Let $\alpha : m \to m'$ be a morphism in M and the diagram*

$$\begin{tikzcd} L(c) \arrow[r, "\gamma K_m"] \arrow[d, "LK(\alpha)"] & S(c) \arrow[d, "SK(\alpha)"] \\ L(c') \arrow[r, "\gamma K_{m'}"] & S(c') \end{tikzcd}$$

commutes because γ is the unique natural transformation γ : L(c) $\to$ S(c) for all c $\in$ C. So, γK is natural too.

Lemma 2.6. *: If [L, η : T $\to$ LK] and [L', η' : T $\to$ $L'K$] are left Kan Extensions, then there exists a unique isomorphism γ : L $\to$ L' with η' = γK $\circ\eta$.*

Proof. By the definition property of left Kan extension unique natural transformations $\gamma :$ L $\to$ L' and $\gamma' : L' \to$ L. with $\eta' = \gamma$ K $\circ\eta$ and $\eta = \gamma'$ K $\circ\eta'$.

Now $\gamma' \circ \gamma : L \to$ L is a natural transformation, with $(\gamma' \circ \gamma)K \circ \eta = \gamma'$K $\circ\gamma$K $\circ\eta$ $=\gamma'$K $\circ\eta' = \eta$ as a natural transformation $\eta :$ T $\to$ LK. Also id : L $\to$ L is a natural transformation with η= $(id_L K) \circ \eta$, so by uniqueness in the defining property of Left Kan extensions we have that $\gamma' \circ \gamma = id_L$.

Similarly, $(\gamma \circ \gamma')K \circ \eta' = \gamma$K $\circ\gamma'$K $\circ\eta' = \gamma$K $\circ\eta = \eta'$ as a natural transformation $\eta' :$ T $\to L'K$. $id_{L'} : L' \to L'$ is natural transformation with $\eta' = (id_{L'}K) \circ \eta'$. Again uniqueness of natural transformation gives $\gamma \circ \gamma' = id_{L'}$.

$\gamma' \circ \gamma = id_L$ and $\gamma \circ \gamma' = id_{L'}$ give that γ and γ' are bijections and one of them is the inverse of the other.

γ is isomorphism. □

2.3 All colimit exist in Mod_R, then a left Kan extension of T along K exists.

Mod_R is a cocomplete category by the Theorem 3.13 of the reserch paper named " Limits, colimits and how to calculate them in the category of modules over a PID" by KAIRUI WANG. The theorem states that;

Theorem 2.7 (Theorem 3.13)**.** *Cocompleteness Theorem,: A category C is cocomplete if and only if the coproduct of any set of objects in C exists and the coequalizer between any two morphisms with the same source and target exists.*

Definition 2.8 (Cocomplete category)**.** *a cocomplete category is a category where colimits over diagrams F with a small source category J exist. F is an object of the categoty of functors C^J, J is a small category.*

Definition 2.9 (Coequlizer)**.**

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "u"] \arrow[dr, "h"'] & e \arrow[d, dotted, "\exists! h'"] \\ & & c \end{tikzcd}$$

Given in a category a pair of maps f and g with the same domain a and codomain b, a coequalizer of [f, g] is a pair (u,e)of a morphism u: b→ e and codomain e such that (1) uf=ug (2) if h: b→ c has hf=hg then $h = h'u$ for a unique $h' : e \to c$.

Definition 2.10 (A map of co-equlizer diagrams)**.** *A map of co-equalizer diagrams is a diagram of the form:*

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\alpha"] & b \arrow[r, "u"] \arrow[d, "\beta"] & e \arrow[d, "\gamma"] \\ a' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & b' \arrow[r, "u'"] & e' \end{tikzcd} \tag{1}$$

So that the rows are co-equalizer diagrams and

$$\beta f = f'\alpha, \beta g = g'\alpha \quad and \quad \gamma u = u'\alpha.$$

Lemma 2.11. *If in a map of co- equalizers diagrams (1) , the maps α and β are isomorphisms, then γ is an isomorphism.*

Proof.

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "u"] \arrow[dr, "h"] & e \arrow[d, dotted, "\exists! h'"] \\ & & c \end{tikzcd}$$

Given diagram, maps f and g are such that : uf=ug and if h: b→ c has hf=hg then $h = h'u$ for a unique $h' : e \to$ c.

$$\begin{tikzcd} a' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & b' \arrow[r, "u'"] \arrow[dr, "j"] & e' \arrow[d, dotted, "\exists! j'"] \\ & & c' \end{tikzcd}$$

h is the surjective map and the maps f', g' and u' are such that : $u'f' = u'g'$ and if$j : b' \to c'$ has$jf' = jg'$ then $j = j'u'$ for a unique $j' : e' \to c'$. We get the diagram below:

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\alpha"] & b \arrow[r, "u"] \arrow[d, "\beta"] \arrow[rr, bend left, "h"] & e \arrow[r, dotted, "h'"] \arrow[d, "\gamma"] & c \\ a' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & b' \arrow[r, "u'"] \arrow[rr, bend right, "j"'] & e' \arrow[r, dotted, "j'"] & c' \end{tikzcd}$$

If α and β are isomorphisms, γ must be a isomorphism because in this diagram, we know that

$$\beta f = f'\alpha, \beta g = g'\alpha \quad \text{and} \quad \gamma u = u'\alpha.$$

$h = h'u$ and $j = j'u'$, then c $\cong c'$. □

Definition 2.12 (co-product diagram)**.**

$$\begin{tikzcd} a \arrow[r, "i_1"] \arrow[dr, "f"'] & a \sqcup b \arrow[d, dotted, "\exists! \mu"] & b \arrow[l, "i_2"'] \arrow[dl, "g"] \\ & d & \end{tikzcd}$$

is a coproduct diagram. i_1 and i_2 are injectives. If there exists d, f: $a \to d$ and $g : b \to d$, then there always exists unique μ such that $f = \mu \circ i_1$ and $g = \mu \circ i_2$.

2.3.1 Prosition

Given diagram of the form,

$$\begin{tikzcd} M \arrow[r, "T"] \arrow[d, "K"] & A = Mod_R \\ C & \end{tikzcd}$$

a left Kan extension of T along K exists. The functor L : C $\to$ A and natural transformation $\eta : T \to LK$ can be constructed as follows: For c, an object of C, the value L(c) of L of c is given by the coequalizer of the diagram

$$\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) \underset{b}{\overset{a}{\rightrightarrows}} (\bigoplus_{Km \to c} Tm)$$

where the upper map a takes an element

$$x = (\alpha : m_1 \to m_0, f : Km_0 \to c, t \in Tm_1) \quad \text{of} \quad \bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1)$$

to the element $(f \circ K\alpha, t)$ of $\bigoplus_{Km \to c} Tm$, and the lower map b takes x to the element $(f, T(\alpha)(t))$of $\bigoplus_{Km \to c} Tm$. The natural transformation η : T $\to$ LK takes and element t of Tm to the element in the co-equalizer LKm represented by the element

$$[id : Km \to Km, t \in Tm] \quad \text{of} \quad \bigoplus_{Km_0 \to c} (Tm_0).$$

Proof. First we define L. Given an object c of C, let Lc be the co-equalizer described in the statement of the proposition. Given h : c $\to c'$,

$$\begin{tikzcd} \bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[d, "h\circ"] & (\bigoplus_{Km \to c} Tm) \arrow[r, "\mu"] \arrow[d, "h\circ"] & Lc \arrow[d, dotted, "\exists! Lh"] \\ \bigoplus_{Km_0 \to c'} \bigoplus_{m_1 \to m_0} (Tm_1) \arrow[r, shift left, "c"] \arrow[r, shift right, "d"'] & (\bigoplus_{Km \to c'} Tm) \arrow[r, "\theta"] & Lc' \end{tikzcd}$$

we define Lh : Lc $\to Lc'$ as follow;

an element $\quad x = (\alpha : m_1 \to m_0, f : Km_0 \to c, t \in Tm_1) \quad$ of $\displaystyle\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1)$

will be sent to x' by composing with h

$$x' = (\alpha : m_1 \to m_0, h \circ f : Km_0 \to c \to c', t \in Tm_1) \quad \text{in} \quad \bigoplus_{Km_0 \to c'} \bigoplus_{m_1 \to m_0} (Tm_1).$$

$$\text{And the map c send} \quad x' \quad \text{to} \quad (h \circ f \circ K\alpha, t) \quad \text{in} \quad \bigoplus_{Km \to c'} Tm.$$

The map a sent x to

$$(f \circ K\alpha, t) \quad \text{in} \quad (\bigoplus_{Km \to c} Tm) \quad \text{and it is sent to} \quad (h \circ f \circ K\alpha, t) \quad \text{in} \quad \bigoplus_{Km \to c'} Tm.$$

So,

$$(h \circ a)(x) = (c \circ h)(x)$$

We get the commute diagram for the upper maps a and c. For maps b and d.

$$\text{an element} \quad x = (\alpha : m_1 \to m_0, f : Km_0 \to c, t \in Tm_1) \quad \text{of} \quad \bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1)$$

will be sent to x' by composing with h

$$x' = (\alpha : m_1 \to m_0, h \circ f : Km_0 \to c \to c', t \in Tm_1) \quad \text{in} \quad \bigoplus_{Km_0 \to c'} \bigoplus_{m_1 \to m_0} (Tm_1)$$

$$\text{and the map d send} \quad x' \quad \text{to} \quad (h \circ f, T(\alpha)(t))) \quad \text{in} \quad \bigoplus_{Km \to c'} Tm$$

the map b sent x to

$$(f, T(\alpha)(t)) \quad \text{in} \quad (\bigoplus_{Km \to c} Tm) \quad \text{and it is sent to} \quad (h \circ f, T(\alpha)(t)) \quad \text{in} \quad \bigoplus_{Km \to c'} Tm.$$

$$(h \circ b)(x) = (d \circ h)(x)$$

So we get commute digram for both of the pairs of maps a and c and b and d. It gives the commuted diagram below and the defined properties of Lc gives the unique morphism Lh from Lc to Lc$'$ which gives the commute diagram as $(Lh \circ \mu)(t) = (\theta \circ h)(t)$, for all follow $t \in (\bigoplus_{Km \to c} Tm)$.

L is defined for all map h in C.

We are going to show that L is a functor L : C $\to$ Mod$_R$ and η is a natural transformation. We have proved that Lh is exist in Mod$_R$ for all h in C. In C, there exists Id_c : c $\to$ c in C. Composing with Id_c to x and get the commute diagram below and get Id_{Lc}.

$$\begin{tikzcd}
\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[d, "id"] & (\bigoplus_{Km \to c} Tm) \arrow[r] \arrow[d, "id"] & Lc \arrow[d, dotted, "\exists! L(Id_c) = Id_{Lc}"] \\
\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & (\bigoplus_{Km \to c} Tm) \arrow[r] & Lc
\end{tikzcd}$$

$Id_{Lc} = L(Id_c)$ exists .

If g: $c' \to c''$ in C, $g \circ h : c \to c''$ will induced a unique map Lg $\circ Lh : Lc \to Lc''$ as follow:

$$\begin{tikzcd}
\bigoplus_{Km_0\to c}\bigoplus_{m_1\to m_0}(Tm_1) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[d, "h\circ"] & (\bigoplus_{Km\to c} Tm) \arrow[r] \arrow[d, "h\circ"] & Lc \arrow[d, dotted, "\exists! Lh"] \\
\bigoplus_{Km_0\to c'}\bigoplus_{m_1\to m_0}(Tm_1) \arrow[r, shift left, "c"] \arrow[r, shift right, "d"'] \arrow[d, "g\circ"] & (\bigoplus_{Km\to c'} Tm) \arrow[r] \arrow[d, "g\circ"] & Lc' \arrow[d, dotted, "\exists! Lg"] \\
\bigoplus_{Km_0\to c''}\bigoplus_{m_1\to m_0}(Tm_1) \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & (\bigoplus_{Km\to c''} Tm) \arrow[r] & Lc''
\end{tikzcd}$$

$$x = (\alpha : m_1 \to m_0, f : Km_0 \to c, t \in Tm_1) \quad \text{of} \quad \bigoplus_{Km_0\to c}\bigoplus_{m_1\to m_0}(Tm_1)$$

is sent same as above by map a, b, c and d. Again, sent x' to $x''$by composing with g.

$$x'' = (\alpha : m_1 \to m_0, g \circ h \circ f : Km_0 \to c'', t \in Tm_1) \quad \text{in} \quad \bigoplus_{Km_0\to c''}\bigoplus_{m_1\to m_0}(Tm_1).$$

We get

$$(g \circ h \circ a)(x) = (u \circ g \circ h)(x) = (g \circ h \circ f \circ K\alpha, t)$$

and

$$(g \circ h \circ b)(x) = (v \circ g \circ h)(x) = (g \circ h \circ f, T(\alpha)(t))$$

and the commute diagrams with the unique map

$$Lh : Lc' \to Lc''.$$

Again, we consider map $L(g \circ h)$, we get

$$\begin{tikzcd}
\bigoplus_{Km_0\to c}\bigoplus_{m_1\to m_0}(Tm_1) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[d, "g\circ h\circ"] & (\bigoplus_{Km\to c} Tm) \arrow[r] \arrow[d, "g\circ h\circ"] & Lc \arrow[d, dotted, "\exists! L(g\circ h)"] \\
\bigoplus_{Km_0\to c''}\bigoplus_{m_1\to m_0}(Tm_1) \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & (\bigoplus_{Km\to c''} Tm) \arrow[r] & Lc''
\end{tikzcd}$$

This diagram works same way and get the same equations above,

$$(g \circ h \circ a)(x) = (u \circ g \circ h)(x) = (g \circ h \circ f \circ K\alpha, t)$$

and

$$(g \circ h \circ b)(x) = (v \circ g \circ h)(x) = (g \circ h \circ f, T(\alpha)(t))$$

So,

$$L(g \circ h) = Lg \circ Lh.$$

L is a functor.

Then we are going to show that η is natural transformation. The morphisms

$$\eta_m : Tm \to LKm$$

is such that:

$$t \mapsto L(id_{Km})(\eta_m t) = \eta_m t.$$

and for any t $\in$ TM and morphism f ,

$$(f : Km \to c, t) \mapsto (L(f \circ K_\alpha)(\eta m_1 t)).$$

Then the diagramas

$$\begin{tikzcd}
Tm_1 \arrow[r, "T\alpha"] \arrow[d, "\eta m_1"] & Tm_0 \arrow[d, "\eta m_0"] & \\
L(Km_1) \arrow[r, "LK_\alpha"] & LKm_0 \arrow[r, "Lf"] & Lc
\end{tikzcd}$$

Any element t in Tm_1 is sent by map $(LK_\alpha \circ \eta m_1)$

$$(f : Km \to c, t) \mapsto (L(f \circ K_\alpha)(\eta m_1 t))$$

t is sent by map $(T\alpha \circ \eta m_0)$

$$(f : Km \to c, t) \mapsto (L(f)\eta m_0(T(\alpha)t))$$

$$(L(f \circ K\alpha)(\eta m_1 t)) = (L(f)\eta m_0(T(\alpha)t)).$$

It makes the previous diagram commute. And η is natural.

Let S is the another functor C $\to$ Mod_R together with β : T $\to$ SK. I am going to prove that there is a unique natural transformation γ : L $\to$ S such that $\beta = \gamma$K $\circ \eta$,

$$\begin{tikzcd}
M \arrow[r, "T"] \arrow[d, "K"'] & A = Mod_R \\
C \arrow[ur, "L"] \arrow[ur, bend right, "S"'] &
\end{tikzcd}$$

The morphisms β_m : Tm $\to$ SKm induces a morphism $\bigoplus_{Km \to c} Tm \to S$c

$$[f : Km \to c, t \in Tm)] \mapsto S(f)(\beta_m t).$$

And it gives a commute diagramas ;

$$\begin{tikzcd}
Tm_1 \arrow[r, "T\alpha"] \arrow[d, "\beta m_1"] & Tm_0 \arrow[d, "\beta m_0"] & \\
S(Km_1) \arrow[r, "SK_\alpha"] & SKm_0 \arrow[r, "Sf"] & Sc
\end{tikzcd}$$

Any element t in Tm_1 is sent by map $(SK_\alpha \circ \beta m_1)$

$$(f : Km \to c, t) \mapsto (S(f \circ K_\alpha)(\beta m_1 t))$$

t is sent by map $(T\alpha \circ \beta m_0)$

$$((f : Km \to c, t)) \mapsto (S(f)\beta m_0(T(\alpha)t))$$

$$(S(f \circ K\alpha)(\beta m_1 t)) = (S(f)\beta m_0(T(\alpha)t)).$$

It gives a commute diagram and the map ϕ as follow;

$$\begin{tikzcd}
\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0}(Tm_1) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & (\bigoplus_{Km \to c} Tm) \arrow[r, "\phi"] & Sc
\end{tikzcd}$$

By universal property of coequalizer , we get a uniquely determined morphism $\gamma_c : Lc \to Sc$

$$\begin{tikzcd}
\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0}(Tm_1) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & (\bigoplus_{Km \to c} Tm) \arrow[r, "\psi"] \arrow[dr, "\phi"] & Lc \arrow[d, dotted, "\exists ! \gamma_c"] \\
& & Sc
\end{tikzcd}$$

for any modules c in C such that $\phi = \gamma_c \circ \psi$. Then we get the unique natural transformation γ : L$\to$ S, $\forall c \in C$. It holds for any free module of finite set m in M, so we get γK_m : LKm $\to$ SKm. We have defined β_m : Tm $\to$ SKm which gives ϕ in above co-equalizerby composing with f: Km $\to c$ and η_m : Tm $\to$ LKm which gives ψ in above co-equalizerby composing with f: Km $\to c$. The composite of γK_m and η_m is

$$\beta_m = \gamma K_m \circ \eta_m : Tm \to SKm, \forall m \in M.$$

We can express it as

$$\beta = \gamma K \circ \eta.$$

and the diagram is ,

$$\begin{tikzcd} T \arrow[r, "\eta"] \arrow[dr, "\beta"] & LK \arrow[d, "\gamma K"] \\ & SK \end{tikzcd}$$

So defining functor L as coequqlizer and unique natural transformation γ as above make the L is left Kan extension. $\square$

Definition 2.13. *: R[fin] is the category with finite sets as objects and the hom set in* $R[fin](X, Y)$ *is Rmodules generated by maps between two finite sets* X *and* Y*.*

$$\sum_i a_i f_i, a_i \in R, f_i \in hom(X, Y).$$

.

Definition 2.14 (Full subcategory)**.** *: We say that S is a full subcategory of C when the inclusion functor T : S → C is full. If every function* $T_{(c,c')}$ *: hom (c,c') → hom (Tc, Tc'), for all pair (c, c') of C, is surjective, T is full.*

Definition 2.15. *: Let X and Y are finite sets.* F_R *is a full embedding functor which makes a finite set to a free R modules.*

$$F_R X = \bigoplus_{x \in X} R.$$

Every map of R[fin](X, Y) is sent the map in $map(X, F_R Y)$ *as follow:*

$$R[fin](X, Y) \to map(X, F_R Y)$$

$$(\sum_i a_i f_i) \mapsto (x \mapsto \sum_i a_i f_i(x)),$$

and every map α *in* $(X_0, F_R Y)$ *will send to a map in* $Mod_R(F_R X, F_R Y)$ *as follow:*

$$map(X, F_R Y) \to Mod_R(F_R X, F_R Y).$$

$$(\alpha : X \to F_R Y) \mapsto [(\sum_i \lambda_i x_i) \mapsto \sum_i \lambda_i \alpha(x_i)].$$

In this chapter I am going to prove that R[Fin] is full subcategory of Mod_R *by using the left kan extension as co-equalizer.*

Definition 2.16 (-X-)**.** *:$R[fin] \times R[fin] \to R[fin]$, -X- is a functor which makes pair of two finite sets to a Cartesian product of two finite sets.*

$$(X, Y) \mapsto X \times Y$$

and morphisms

$$(\sum a_i f_i, \sum b_j g_j) \mapsto \sum_{i,j} a_i b_j (f_i, g_j).$$

Definition 2.17 (T and η)**.** *: T is a functor of composition of two functors $F_R \circ -\times$-,*

$$T(X, Y) = \bigoplus_{X\times Y} R,$$

with a natural transformation

$$\eta : T \to \otimes_R \circ F_R \times F_R.$$

$$\bigoplus_{X\times Y} R \to F_R X \otimes F_R Y$$

$$\sum_{(x,y)} c_{(x,y)}(x, y) \mapsto \sum_{(x,y)} c_{(x,y)}(x \otimes y).$$

Theorem 2.18. *In the diagram (1) if there is a functor $\otimes : Mod_R \times Mod_R \to Mod_R$ and an natural transformations $\beta : T \to \otimes \circ F_R \times F_R$, then there exist a unique natural isomorphism $\gamma : L \to \otimes$ such that $\beta = (\gamma F_R \times F_R \circ \eta) : T \to S$.*

$$\begin{tikzcd}
R[\text{fin}] \times R[\text{fin}] \arrow[r, "-X-"] \arrow[d, "F_R\times F_R"] \arrow[rr, bend left, "T"] & R[fin] \arrow[r, "F_R"] & \mathrm{Mod}_R \\
\mathrm{Mod}_R \times \mathrm{Mod}_R \arrow[urr, "L"] \arrow[urr, bend right, "\otimes"'] & &
\end{tikzcd} \tag{2}$$

Lemma 2.19. *There is a natural isomorphism*

$$LM \cong M$$

Proof. Let
$$\alpha : LM \to M$$
such that: we have co-equalizer diagram

$$\bigoplus_{FX_0 \to M} \bigoplus_{X_1 \to X_0} (FX_1) \underset{b}{\overset{a}{\rightrightarrows}} (\bigoplus_{FX \to M} FX) \xrightarrow{\psi} LM$$

an element $\quad x = (g : X_1 \to X_0, f : FX_0 \to M, t \in FX_1) \quad$ of $\quad \bigoplus\limits_{FX_0 \to M} \bigoplus\limits_{X_1 \to X_0} (FX_1)$

will be sent to x' by map a

$$x' = (f, F(g)(t)) \quad \text{of} \quad \bigoplus_{FX \to M} FX$$

an element $\quad x = (g : X_1 \to X_0, f : FX_0 \to M, t \in FX_1) \quad$ of $\quad \bigoplus\limits_{FX_0 \to M} \bigoplus\limits_{X_1 \to X_0} (FX_1)$

will be sent to x'' by map b

$$x'' = (f \circ Fg, t) \quad \text{of} \quad \bigoplus_{FX \to M} FX.$$

An element
$$y = (f : Km_0 \to M, t \in FX) \quad \text{of} \quad \bigoplus_{FX \to M} FX$$
will be sent to $(f(t))$ in M by map ϕ in diagram below .

$$\begin{tikzcd}
\bigoplus_{FX_0 \to M} \bigoplus_{X_1 \to X_0} (FX_1) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & (\bigoplus_{FX \to M} FX) \arrow[r, "\psi"] \arrow[dr, "\phi"] & LM \arrow[d, dotted, "\exists! \alpha"] \\
 & & M
\end{tikzcd}$$

In above diagram ψ is surjective and we get the unique α according to the universal properties of Co-equalizer. It is factor out the map ϕ such that $\phi = \alpha \circ \psi$, $(\alpha \circ \psi)(y) = f(t)$.

Case 1. If M is a free Rmodules : M= FY, Y is a finite set.

Let
$$\alpha : LM \to M$$
$$[f : FX \to M = FY, t \in FX] \mapsto f(t),$$

and

$$\beta : M \to LM$$
$$m \mapsto (id : FY \to M, m \in FY),$$

we consider

$$\alpha(\beta(m)) = \alpha(id : FY \to M = FY, m \in FY) = id(m) = m$$

$$\begin{tikzcd}
\bigoplus_{FX_0 \to M} \bigoplus_{X_1 \to X_0}(FX_1) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & (\bigoplus_{FX \to M} FX) \arrow[r, "\psi"] \arrow[dr, "f=\phi"'] & LM \arrow[d, "\exists!\alpha"'] & \\
& & (M = FY) \arrow[r, "(\alpha\circ\beta)=id"] & (M = FY) \arrow[ul, dotted, "\beta"']
\end{tikzcd}$$

and

$$\beta(\alpha(f : FX \to M = FY, t \in FX)) = \beta(f(t)) = (id : FY \to M, f(t) \in FY)$$

we know that, in LM,

$$(f : FX \to M = FY, t \in FX)) = (id : FY \to M, f(t) \in FY)$$

because t ∈ FX will be sent to f(t) ∈ FY by f and f(t) ∈ M=FY will be to itself by id_{FY}. Both elements are in the same equivalence class of LM. So, we have prove LM ≅ M for M, any finitely generated FREE module.

Case 2. If

$$M = \oplus_{x \in X} R,$$

for X is an infinite set. Let

$$\alpha_M : LM \to M$$
$$[f : FX \to M, t \in FX] \mapsto f(t).$$

Let any m $\in M, m = \sum m_x[x]$, only finitely many m_x are not zero. Let Y= [$x \in X/m_x \neq 0$]. So we can express m as m=$\sum \lambda_y[y]$.

L is left adjoint functor. Then we get the diagram below,

$$\begin{tikzcd}
LM \arrow[r, "="] \arrow[d, "\alpha_M"] & L(\oplus_{x \in X} R) \arrow[r, "\cong"] \arrow[d, "\alpha_M"] & \oplus_{x \in X} L(R) \arrow[d, "(\oplus\alpha_R)=\cong"] \\
M \arrow[r, "="] & \oplus_{x \in X} R \arrow[r, "="] & \oplus_{x \in X} R.
\end{tikzcd}$$

We have $\oplus_{x\in X}L(R)$ is isomorphic to $\oplus_{x\in X}R$. Then we get

$$L(\oplus_{x\in X}R) \cong \oplus_{x\in X}L(R) \cong \oplus_{x\in X}R$$

and an isomorphism

$$\alpha_M : L(\oplus_{x\in X}R) \to \oplus_{x\in X}R.$$

$$LM \cong M.$$

Case 3. If M is any Rmodule: we can write M as a co-equalizer of free Rmodules

$$K = ker(\oplus_{m\in M}Rm \to M)$$

$$\sum_{m\in M} a_m[m] \mapsto \sum a_m m.$$

Get a surjective map

$$\oplus_{k\in K}Rk \to K$$

We have exact sequence

$$\oplus_{k\in K}Rk \xrightarrow{\beta} \oplus_{m\in M}Rm \to M \to 0$$

Thus,

$$\begin{tikzcd} \oplus_{k\in K}Rk \arrow[r, shift left, "0"] \arrow[r, shift right, "\beta"'] & \oplus_{m\in M}Rm \arrow[r] & M \end{tikzcd} \tag{3}$$

is an coequalizer sequence. We get coequalizer sequence with L too as L is left adjont.

$$\begin{tikzcd} L(\oplus_{k\in K}Rk) \arrow[r, shift left, "L0"] \arrow[r, shift right, "L(\beta)"'] & L(\oplus_{m\in M}Rm) \arrow[r] & LM \end{tikzcd} \tag{4}$$

As the lemma 2.11, these two co-equalizers diagram 3 and 4 have the same universal property of co-equalizer. We get commute diagram as follow:

$$\begin{tikzcd} L(\oplus_{k\in K}Rk) \arrow[r, shift left, "L0"] \arrow[r, shift right, "L\beta"'] \arrow[d, "\alpha_{\oplus Rk}=\cong"] & L(\oplus_{m\in M}Rm) \arrow[r] \arrow[d, "\alpha_{\oplus Rk}=\cong"] & LM \arrow[d, "\alpha_M"] \\ \oplus_{k\in K}Rk \arrow[r, shift left, "0"] \arrow[r, shift right, "\beta"'] & \oplus_{m\in M}Rm \arrow[r] & M \end{tikzcd}$$

α_M works same as above in case2. It is an isomorphism. Therefor

$$\alpha : LM \to M$$

is isomophism for all mordules M $\in$ Mod$_R$ and

$$LM \cong M.$$

$\square$

2.4 Defining L, the left kan extension and a co-equalizer

Let functors - $\times$ -, product of sets. $\times$ (X,Y)= X$\times$Y, F_R is a functor which makes free modules of finite sets, $F_R(X \times Y) = \oplus_{X\times Y} R$ and $L : mod_R \times mod_R \to mod_R$ be a co-equalizer functor of modules. T is a functor of composition of two functors $F_R \circ -\times$-, $T(X, Y) = \oplus_{X\times Y} R$.

$$\begin{tikzcd}
R[fin] \times R[fin] \arrow[r, "-X-"] \arrow[rr, bend left, "T"] \arrow[d, "F\times F"] & R[fin] \arrow[r, "F"] & Mod_R \\
Mod_R \times Mod_R \arrow[urr, "L"] & &
\end{tikzcd} \tag{5}$$

In the diagram 5, Let L is a coequalizer such that:

$$\begin{tikzcd}
\bigoplus_{FX_1 \to M, FY_1 \to N} \bigoplus_{FX_0 \to FX_1, FY_0 \to FY_1} F(X_0 \times Y_0) \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] \\
\bigoplus_{FX \to M, FY \to N} F(X \times Y) \arrow[d, "\zeta"] \\
L(M, N)
\end{tikzcd} \tag{6}$$

Lemma 2.20. *There is a coequalizer diagram in 5 as follow:*

$$\begin{tikzcd} \bigoplus_{FX_1\to M, FY_1\to N} \bigoplus_{FX_0\to FX_1, FY_0\to FY_1} FX_0 \times FY_0 \arrow[d, shift right, "u'"'] \arrow[d, shift left, "v'"] \\ \bigoplus_{FX\to M, FY\longrightarrow N} FX \times FY \arrow[d, "\zeta'"] \\ M \times N \end{tikzcd} \tag{7}$$

Proof. We have shown in 2.19 that LM is isomorphic to M and we have the co-equalizer diagram:

$$\begin{tikzcd} \bigoplus_{FX_1\to M} \bigoplus_{X_0\to X_1}(FX_0) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & (\bigoplus_{FX\to M} FX) \arrow[r, "\psi"] \arrow[dr, "h"] & M \arrow[d, dotted, "!\xi"] \\ & & M' \end{tikzcd} \tag{8}$$

h works

$$(h \circ a)(x) = (h \circ b)(x), \forall x \in (\bigoplus_{FX_1\to M} \bigoplus_{X_0\to X_1} (FX_0)), h = \xi \circ \psi.$$

The two maps work such that : en element x in FX_0 is send to different element in FX as follow:

$$x = (\alpha : FX_1 \to M, f : FX_0 \to FX_1, t \in FX_0) \mapsto (\alpha, f(t))$$

by map a and

$$x = (\alpha : FX_1 \to M, f : FX_0 \to FX_1, t \in FX_0) \mapsto (\alpha \circ f, t)$$

by map b. But these two different elements in FX are sent to same elements f(t) of M by ψ.

There is a co-equalizer in FY too such that:

$$\begin{tikzcd} \bigoplus_{FY_1\to N} \bigoplus_{FY_0\to FY_1}(FY_0) \arrow[r, shift left, "a'"] \arrow[r, shift right, "b'"'] & (\bigoplus_{FY\to N} \to FY) \arrow[r, "\psi'"] \arrow[dr, "h'"] & N \arrow[d, dotted, "!\xi'"] \\ & & N' \end{tikzcd} \tag{9}$$

h' works

$$(h' \circ a')(y) = (h' \circ b')(y), \forall y \in (\bigoplus_{FY_1 \to N} \bigoplus_{Y_0 \to Y_1} (FY_0)), h' = \xi' \circ \psi'.$$

$$y = (\beta : FY_1 \to N, g : FY_0 \to FY_1, s \in FY_0) \mapsto (\beta, g(s))$$

by map a' and

$$y = (\beta : FY_1 \to N, g : FY_0 \to FY_1, s \in FY_0) \mapsto (\beta \circ g, s)$$

by map b' . But these two different elements in FY are sent to same elements g(s) of N by ψ'.

The co-product of co-equalizer diagrams is a co-equalizer diagram.

In our category R[fin] there are objects which co-product of its object, finite set. So these coproduct objects will be the

coproduct of 8 and 9 gives the following equation

$$\begin{tikzcd}
\bigoplus_{FX_1 \to M} \oplus_{FX_0 \to FX_1} (FX_0) \bigoplus \oplus_{FY_1 \to N} \oplus_{FY_0 \to FY_1} (FY_0) \arrow[d, shift right, "u \oplus u'"'] \arrow[d, shift left, "v \oplus v'"] \\
(\oplus_{FX \to M} FX) \oplus (\oplus_{FY \to N} FY) \arrow[d, "\psi \oplus \psi'"] \\
M \oplus N
\end{tikzcd}$$

This is equal to

$$\bigoplus_{FX_1 \to M, FY_1 \to N} \bigoplus_{FX_0 \to FX_1, FY_0 \to FY_1} (FX_0 \times FY_0) \underset{v \times v'}{\overset{u \times u'}{\rightrightarrows}} \bigoplus_{FX \to M, FY \to N} (FX \times FY) \xrightarrow{\psi \times \psi'} (M \times N)$$

Since we have co-product diagram:

$$\begin{tikzcd}
(\bigoplus_{FX \to M} FX) \arrow[r, "i_1"] \arrow[dr, "\psi"] \arrow[ddr, "h"'] & (\bigoplus_{FX \to M} FX) \bigoplus (\bigoplus_{FY \to N} FY) \arrow[d, "\psi \oplus \psi'"] & (\bigoplus_{FY \to N} FY) \arrow[l, "i_2"'] \arrow[dl, "\psi'"'] \arrow[ddl, "h'"] \\
& M \oplus N \arrow[d, dotted, "\exists ! \xi \oplus \xi'"] & \\
& M' \oplus N' &
\end{tikzcd}$$

and get unique map $\xi \oplus \xi'$ such that h= $(\xi \oplus \xi') \circ \psi$ and h'= $(\xi \oplus \xi') \circ \psi'$, we get co-equalizer diagram:

$$\begin{tikzcd}
\bigoplus_{FX_1 \to M, FY_1 \to N} \bigoplus_{FX_0 \to FX_1, FY_0 \to FY_1} (FX_0 \times FY_0) \arrow[r, shift left, "u \times u'"] \arrow[r, shift right, "v \times v'"'] & \bigoplus_{FX \to M, FY \to N} (FX \times FY) \arrow[r, "\psi \times \psi'"] \arrow[dr, "h \otimes h'"] & (M \times N) \arrow[d, dotted, "! \xi \times \xi'"] \\
& & M' \times N'
\end{tikzcd} \tag{10}$$

□

Lemma 2.21. *The bilinear map*

$$\hat{\phi} : FX \times FY \to F(X \times Y)$$

$$\hat{\phi}(\sum_i \lambda_i x_i, \sum_j \mu_j x_j) = \sum_{i,j} \lambda_i \mu_j (x_i, y_j)$$

induces a map of coequalizer diagrams and the map

$$\phi : M \times N \to L(M, N).$$

Proof. We have defined the co-equalizer diagram 6 as follow

$$\bigoplus_{FX_1 \to M, FY_1 \to N} \bigoplus_{FX_0 \to FX_1, FY_0 \to FY_1} (F(X_0 \times Y_0) \underset{b}{\overset{a}{\rightrightarrows}} \bigoplus_{FX \to M, FY \to N} F(X \times Y) \xrightarrow{\zeta} L(M \times N)$$

and I have got a co-equalizer in the lemma 2.20

$$\bigoplus_{FX_1 \to M, FY_1 \to N} \bigoplus_{FX_0 \to FX_1, FY_0 \to FY_1} (FX_0 \times FY_0) \underset{v \times v'}{\overset{u \times u'}{\rightrightarrows}} \bigoplus_{FX \to M, FY \to N} (FX \times FY) \xrightarrow{\psi \times \psi'} (M \times N)$$

From the commute diagram of the two co-equalizer diagrams, get a map ϕ as follw:

$$\begin{tikzcd}
\bigoplus_{FX_1 \to M, FY_1 \to N} \bigoplus_{FX_0 \to FX_1, FY_0 \to FY_1} (FX_0 \times FY_0) \arrow[r, shift left, "u \times u'"] \arrow[r, shift right, "v \times v'"'] \arrow[d, "\oplus \hat{\phi}"] & \bigoplus_{FX \to M, FY \to N} (FX \times FY) \arrow[r, "\psi \times \psi'"] \arrow[d, "\oplus \hat{\phi}"] & (M \times N) \arrow[d, "\phi"] \\
\bigoplus_{FX_1 \to M, FY_1 \to N} \bigoplus_{FX_0 \to FX_1, FY_0 \to FY_1} F(X_0 \times Y_0) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \bigoplus_{FX \to M, FY \to N} F(X \times Y) \arrow[r, "\zeta"] & L(M, N)
\end{tikzcd}$$

In the diagram, $\forall$ finite set X and Y,

$$\hat{\phi} : FX \times FY \to F(X \times Y)$$

$$(\sum_i \lambda_i x_i, \sum_j \mu_j x_j) \mapsto \sum_{i,j} \lambda_i \mu_j (x_i, y_j).$$

$\square$

Proposition 2.22. *ϕ is bilinear.*

Proof. ϕ inherit bilinearlity from the bilinear $\hat{\phi}$ such that: Given x_0, x_0' and y_0, choose x, x', y such that

$$\psi(x) = x_0, \psi(x') = x_0', \psi'(y) = y_0.$$

We define $\hat{\phi}$ is bilinear map, then

$$\hat{\phi}(x + x', y) = \hat{\phi}(x, y) + \hat{\phi}(x', y)$$

$$(\xi)((\hat{\phi})(x, y) + \hat{\phi}(x', y)) = (\xi)(\hat{\phi})(x, y) + (\xi)(\hat{\phi})(x', y)......(*),$$

since ξ is bilinear too. In the above commute diagram

$$(\xi)(\hat{\phi})(x, y) = (\phi)(\psi)(x), (\phi)(\psi')(y) = \phi(x_0, y_0).$$

In the (*)

$$(\xi)((\hat{\phi})(x, y) + \hat{\phi}(x', y)) = (\xi)(\hat{\phi})(x, y) + (\xi)(\hat{\phi})(x', y) = \phi(x_0, y_0) + \phi(x_0', y_0).$$

We have

$$(\xi)((\hat{\phi})(x, y) + \hat{\phi}(x', y)) = \phi((\psi(x) + \psi(x'), \psi'(y)) = \phi(x_0 + x_0, y_0)$$

$$\phi(x_0 + x_0', y_0) = \phi(x_0, y_0) + \phi(x_0', y_0).$$

ϕ is a bilear map. $\square$

Proposition 2.23. *There is a homomorphism $\bar{\phi}$: $M \otimes_R N \to$ L(M,N)*

Proof: Universal properties for defining tensor product(this is the unique natural morphism γ).

Lemma 2.24. *The maps $FX \to$ M and $FY \to$N induces a homomorphism $\bar{\psi}$: L(M,N) $\to M \otimes_R N$*

proof:

$$\begin{tikzcd} FX \otimes_R FY \arrow[r, "\theta=(\psi\otimes\psi')"] \arrow[d, "\cong"] & M \otimes_R N \\ F(X \times Y) \arrow[ur, "\psi"] & \end{tikzcd}$$

$$\begin{tikzcd} F(X \times Y) \arrow[r, "\cong"] \arrow[d, "\xi"] \arrow[dr, "\psi"] & (FX \otimes_R FY) \arrow[d, "\theta=(\psi\otimes\psi')"] \\ L(M \times N) \arrow[r, dotted, "\bar{\psi}"] & M \otimes_R N \end{tikzcd}$$

Lemma 2.25. *$\bar{\psi} \circ \bar{\phi} = id[M \otimes_R N]$*

Proof. Tensor is bi-linearity, so θ and ξ are mordules homomorphisms and conjugacy of upper horizontal maps give the identity map $\bar{\psi} \circ \bar{\phi}$.

$$\begin{tikzcd} \oplus_{FX\to M, FY \to N} FX \otimes_R FY \arrow[r, "\cong"] \arrow[d, two heads, "\theta"] & \oplus_{FX\to M, FY \to N} F(X \times Y) \arrow[r, "\cong"] \arrow[d, two heads, "\xi"] & \oplus_{FX\to M, FY \to N} FX \otimes_R FY \arrow[d, two heads, "\theta"] \\ M \otimes_R N \arrow[r, "\bar{\phi}"] & L(M, N) \arrow[r, "\bar{\psi}"] & M \otimes_R N \end{tikzcd}$$

□

Lemma 2.26. *$\bar{\phi} \circ \bar{\psi} = id[L(M\ ,\ N)]$*

Proof.

$$\begin{tikzcd} \oplus_{FX\to M, FY \to N} F(X \times Y) \arrow[r, "\cong"] \arrow[d, two heads, "\xi"] & \oplus_{FX\to M, FY \to N} FX \otimes_R FY \arrow[r, "\cong"] \arrow[d, two heads, "\theta"] & \oplus_{FX\to M, FY \to N} F(X \times Y) \arrow[d, two heads, "\xi"] \\ L(M, N) \arrow[r, "\bar{\psi}"] & M \otimes_R N \arrow[r, "\bar{\phi}"] & L(M, N) \end{tikzcd}$$

Tensor is bi-linearity, θ and ξ are mordules homomorphisms and conjugacy of upper horizontal maps give the identity composing $\bar{\phi} \circ \bar{\psi}$.

□

Lemma 2.27.
$$\bar{\psi} : L(M,N) \to M \otimes_R N$$
is a natural isomorphism

Proof :Lemma 2.26 and2.27 give that both $\bar{\phi}$ and $\bar{\psi}$ are natural isomorphism. And the unique natural morphism γ of diagram 1 is

$$\gamma = \bar{\psi} : L(M,N) \to M \otimes_R N$$

Conclusion is our two functor are isomorphic.

$$L \cong \otimes$$

Theorem 2.28. *Let $R[\text{fin}]$ be the category of finitely generated free R-modules (2.13). Let $F_R : R[\text{fin}] \to \text{Mod}_R$ be the full embedding from (**??**) and $T : R[\text{fin}] \times R[\text{fin}] \to \text{Mod}_R$ is a composing of F_R and $- \times -$, $T(X,Y) = F_R(X \times Y)$, as (2.17). Let L be the left Kan Extension of T along $F_R \times F_R$,*

$$L : \text{Mod}_R \times \text{Mod}_R \to \text{Mod}_R$$

with the natural transformation $\eta : T \to L \circ F_R \times F_R$, and the another functor $\otimes$ with the natural transformation $\beta : T \to \otimes \circ F_R \times F_R$, then there exist a unique natural isomorphism

$$\gamma : L \to \otimes$$

such that

$$\beta = (\gamma F_R \times F_R \circ \eta).$$

$$\begin{tikzcd}
R[\text{fin}] \times R[\text{fin}] \arrow[r, "-X-"] \arrow[d, hook, "F_R \times F_R"] \arrow[rr, bend left, "T"] & R[\text{fin}] \arrow[r, hook, "F_R"] & \text{Mod}_R \\
\text{Mod}_R \times \text{Mod}_R \arrow[urr, "L"] \arrow[urr, bend right, "\otimes"'] & &
\end{tikzcd}$$

3 Tensor induction

3.1 Constructing the category B^{op}

I am going to start with the category of G-sets. The category B^+ will be constructed with objects of the category of G-sets but maps are only some kinds of G-maps we need. Then I will get the B from B^+ by Grothendieck construction. It is an additive category. Then a contra-variant functor will give the category of B^{op} which I am going to study.

There are two different categories of "Mackey functors" but I use the original one defined by Dress.

Definition 3.1 (A Mackey functor). *Mackey functor is an additive functor from an additive category B^{OP} to Ab catigory Mod_R. We work with Mackey functors over a commutative ring R. A Mackey functor over R is a functor*

$$M : B^{OP} \to \mathrm{Mod}_R$$

Definition 3.2 (Additive Category). *Additive category is an Ab Category which has a zero object and a bi-product for each pair of its objects.*

Definition 3.3 (Bi-product diagram). *Bi-product diagram for the objects a, $b \in \mathcal{A}$ is a diagram*

$$\begin{tikzcd} a \arrow[r, bend left, "i_1"] & c \arrow[l, bend left, "p_1"] \arrow[r, bend right, "p_2"'] & b \arrow[l, bend right, "i_2"'] \end{tikzcd}$$

So that,

$$a \xleftarrow{p_1} c \xrightarrow{p_2} b$$

is a product diagram and

$$a \xrightarrow{i_1} c \xleftarrow{i_2} b$$

is a co-product diagram since

$$p_1 i_1 = 1_a, p_2 i_2 = 1_b \quad and \quad i_1 p_1 + i_2 p_2 = 1_c$$

Definition 3.4 (G-set). *: Let G is a finite group. A left G-set is a set and a group homomorphism*

$$f : G \times X \to X,$$

$$(g, x) \mapsto gx \in X,$$

such that the following conditions hold:

1) if g,h $\in$ G and x $\in$ X, then g.(h.x)= (gh)x,

2) if 1_G is identity element of G then 1_G.x=x.

Definition 3.5 (G-equivariant map or G-map). *If G is a group and X and Y are left G-set, a morphism of G-sets from X to Y is a map $f : X \to Y$ such that f(gx)= g f(x), for any g$\in$G and x $\in$X. Such a map is called G=equivariant map from X to Y and the set of such a map is denoted by $Hom_G(X, Y)$.*

B^+ is constructed from category of G-sets by taking all objects, G-sets, $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$..**etc** and G-maps which are able to be written by the composition of induction, transfer and restriction maps in representation ring. For example: a map f from a to b of B^+ we can describe such that

$$a \overset{f_1}{\dashleftarrow} c \xrightarrow{f_2} b$$

f_1 get from f_1', a G map from **c** to **a**. f_1 is from a to c rather than c to a. f_1 and f_2 are G equavarience maps. f_1 the map with dotted arrow in B^+, correspond to induction maps with indentity or a transfer maps in the familier makey functors like representation ring and so are called transfer. f_2 induces the restriction maps and are called restrictions. The hom set of B^+ are commutative monoids (semi group with identity).

If two maps are determin the same map in B^+, then there is an inner isomorphism of c and d as shown in diagram;

$$\begin{tikzcd}
c \arrow[r, "f_2"] \arrow[d, dotted, "f_1"'] \arrow[dr, dashed, "\cong"] & b \\
a & d \arrow[l, dotted, "g_1"] \arrow[u, "g_2"']
\end{tikzcd}$$

Composition of two maps f and g is

$$a \overset{f_1}{\dashleftarrow} c \xrightarrow{f_2} b \overset{g_1}{\dashleftarrow} e \xrightarrow{g_2} d$$

and this compositions of two maps are given by the following pullback diagram:

$$\begin{tikzcd}
h \arrow[r, "P_2"] \arrow[d, dotted, "P_1"'] & e \arrow[d, dotted, "g_1"] \\
c \arrow[r, "f_2"'] & b
\end{tikzcd}$$

$$c \overset{P_1}{\dashleftarrow} h \xrightarrow{P_2} e$$

We get the composition map a to d in B^+ is as follow:

$$a \overset{(gf)_1}{\dashleftarrow} h \xrightarrow{(gf)_2} d$$

$$g \circ f : a \to d.$$

In B^+ zero set is the initial and terminal object. Disjoin union of sets, in the B^+, get from the direct sum and direct produce of each map as follow':

$$\begin{tikzcd}
a \arrow[d] & u \arrow[l, dotted] \arrow[d] \arrow[dr] & \\
a \sqcup b & u \sqcup v \arrow[l, dotted] \arrow[r] & c \\
b \arrow[u] & v \arrow[l, dotted] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

This is a pair of maps out of $a \sqcup b$ and it is coproduct diagram in B^+ and

$$\begin{tikzcd}
{} \arrow[d, dotted] & u \arrow[r] \arrow[d, dotted] \arrow[dl, dotted] & a \arrow[d, dotted] \\
c & u \times v \arrow[l, dotted] \arrow[r] & (a \times b) \\
 & v \arrow[r] \arrow[u, dotted] \arrow[ul, dotted] & b \arrow[u, dotted]
\end{tikzcd}$$

the above is a poduct diagram in B^+

Category B is obtained from with B^+. They have same objects, finite G-sets but hom set are free abelian group. An abelian monoid, set of homomorphism of B^+ is quotient by the subgroup generated by the elements of the form

$$[f \sqcup g] - [f] - [g],$$

f and g are g-maps in of B^+ , [f] denotes the isomorphism class of f and $f \sqcup g$ is disjoint union of f and g. So, objects of B are finite G-sets same as B^+ objects and the morphisms of B are formal differences of maps inB^+. That's why hom sets in B become abelian groups. There is an obvious functor from B to it's opposite category B^{OP}.

3.2 The category C_G

G is a finite group. I am going to construct the C_G from category C(G) and C(G) is constructed from category $\mathcal{C}$. $\mathcal{C}$ is the same category $\mathcal{C}$ in the book Biset Functors for Finite Groups of Serge Bous. It is the biset category of finite groups. Objects of the $\mathcal{C}$ are finite groups and morphism from finite groups G to H are

$$Hom_{\mathcal{C}}(G, H) = B(H, G).$$

Definition 3.6. *B(H,G) B(H,G), the Grothendieck group of the category (H, G) bisets, is defined as the quotient of the free abelian group on the set of isomorphism classes of finite (H,G)-bisets by the subgroup generated by the element of the form,*

$$[X \sqcup Y] - [X] - [Y],$$

where X and Y are finite (H, G)-bisets, [X] is an isomorphism class of X and $X \sqcup Y$ is disjoint union of X and Y.

Definition 3.7. *(H,G) biset If H and G are finite groups and X is (H, G)-biset. (H,G)-biset is a left H-set and right G-set, such that*

$$\forall h \in H, \forall x \in X, \forall g \in G, (h.x).g = h.(x.g) in X.$$

In $\mathcal{C}$, The every morphism between finite groups G to H can be factored as the composition of $Ind_D^H \circ Inf_{D/C}^D \circ Iso(f) \circ Def_{B/A}^B \circ Res_B^G$. f is isomorphism from B/A to D/C. B and D are sub groups of G and H, A and C are normal subgroups of B and D.

$$Hom_{\mathcal{C}}(G, H) = B(H, G)$$

We have fundamental bisets in $C(G)$ which connected with the three types of maps we are having in the category $C(G)$. Let H is a subgroup of G.

1. G is an (H,G)-biset for the actions given by left and right multiplication in G and it is denoted by Res_H^G.

2. G is an (G,H)-biset for the actions given by left and right multiplication in G and it is denoted by Ind_H^G.

3. If f : B $\rightarrow$ D is a group isomorphism, then the set D is an (D,B)-biset, for the left action of D by multiplication, the right action of B given by taking image by f, and then multiplying on the right in D. It is denoted by **Iso(f)**.

Category $C(G)$ can be constructed from $\mathcal{C}$ by taking a fixed finite group G. $C(G)$ has objects the group G and its subgroups H, K, A, B, C, D... ect. The morphisms in $C(G)$ can be shown as composition of only three types of maps, induction map (Ind) , inner isomorphism (Iso) and restriction map(Res). Any Map from H to K can be factored as

$$Ind_D^K \circ Iso(f) \circ Res_B^H,$$

induction maps from subgroup D to K (Ind_D^K) , inner isomorphisms from B to D (Iso(f)) and restriction maps from H to B (Res_B^H). For any objects of C(G) H and K,

$$Hom_{C(G)}(H, K) \subsetneq Hom_{\mathcal{C}}(H, K)$$

$Mod_{C(G)}$ is not equivalent to the category of $Mod_{B^{OP}}$ due to the Theorem A of Mackey Functors and Bisets, Hambleton, Taylor and Williams.

Then C_G the category I aim, will be constructed from C(G) by a functor.

$$F : C(G) \rightarrow C_G.$$

The paper of Hambleton, Taylor and Williams, I mention above, gives such a functor, where C_G is full subcategory of B^{OP} with objects G/H where H is a subgroup of G.

3.3 B^{OP} and C_G

Category C_G is full subcategory of B^{OP}. All maps of C_G is in B^{OP} since maps are composition of Induction, inner Isomorphism and restriction maps. There is a functor

$$i : C_G \hookrightarrow B^{OP}.$$

Definition 3.8 ($\mathcal{A}$b)**.** *$\mathcal{A}b$ is the category which whose objects are all small (additive) abelian groups and morphisms are all homomorphisms of abelian groups.*

Definition 3.9 (Left Induction ${}^{L}Ind_{C_G}^{B^{OP}}$)**.** *The Left Induction functor is from the book named Biset Functors for Finite Groups by Serge Bouc .*

Let the functor $i : C_G \hookrightarrow B^{OP}$,

$$G/H \mapsto iH = G/H$$

In the diagram

$$\begin{tikzcd} C_G \arrow[r, "F"] \arrow[d, "i"'] & Ab \\ B^{OP} \arrow[ur, "L"'] & \end{tikzcd}$$

${}^{L}Ind_{C_G}^{B^{OP}}$ is a functor of $\mathcal{A}b$ categories . It sends from Mod_{C_G} to $Mod_{B^{OP}}$. Let A is the $\mathcal{A}$ b category.

$$Mod_{B^{OP}} = [B^{OP} \to A] \quad and$$
$$Mod_{C_G} = [C_G \to A]$$

$${}^{L}Ind_{C_G}^{B^{OP}} : Mod_{C_G} \to Mod_{B^{OP}}$$

F is a functor in The functor category $Mod_{C_G}, F : C_G \to \mathcal{A}b$.

$${}^{L}Ind_{C_G}^{\mathcal{B}^{\mathcal{OP}}}(F)(iG/H) = {}^{L} Ind_{C_G}^{\mathcal{B}^{\mathcal{OP}}}(F)(G/H)$$

${}^{L}Ind_{C_G}^{\mathcal{B}^{\mathcal{OP}}}$ works as follow,

$${}^{L}Ind_{C_G}^{\mathcal{B}^{\mathcal{OP}}}(F)(G/H) = [\bigoplus_{K \in S} Hom_{\mathcal{B}^{\mathcal{OP}}}(G/K, G/H) \bigotimes F(K)]/I.$$

S is set of representative of objects of $C(G)$, set of subgroups of G. I is the submodule generated by the elements

$$(u \circ \alpha) \otimes f - u \otimes F(\alpha)(f),$$

For any elements J and K of S, any morphism $\alpha \in Hom_{C_G}(G/J,\ G/K)$, any $f \in F(G/J)$, and any u in $\in Hom_{\mathcal{B}^{\mathcal{OP}}}(iG/K,\ iG/H)$. J, K and H are subgroups of G.

$$i^{*} \circ^{L} Ind_{C_G}^{\mathcal{B}^{\mathcal{OP}}} \quad sends \quad F(G/H) \mapsto \bigoplus_{K \in S} Hom_{\mathcal{B}^{\mathcal{OP}}}(iG/K, iG/H) \bigotimes F(G/K)/I.$$

Let f $\in F(G/H)$*, Then* $H \in S$ *and* $Hom_{\mathcal{B^{OP}}}(G/H, G/H) \bigotimes F(G/H)/I$*. So,* $f \mapsto [id_{G/H} \bigotimes f]$

Any map v *in* $Hom_{\mathcal{B^{OP}}}$*,* $v : iG/K \to iG/J$*, the map*

$$^{L}Ind_{C_G}^{\mathcal{B^{OP}}}(F)(v) :^{L} Ind_{C_G}^{\mathcal{B^{OP}}}(F)(iG/K) \to^{L} Ind_{C_G}^{\mathcal{B^{OP}}}(F)(iG/J)$$

is induced by composition on the left in $\mathcal{B^{OP}}$*.*

Theorem 3.10. *There is an equivalence of categories* $Mod_{B^{OP}}$ *to* Mod_{C_G}*.*

Proof. Let every objects of B^{OP} is finite sum of objects of C_G.

Claim 1. The functor i^* :$Mod_{B^{OP}} \to Mod_{C_G}$ is full and faithful.

Proof for claim 1,

$$Mod_{B^{OP}} = [B^{OP} \to \mathcal{A}],$$

$\mathcal{A}$ is the $\mathcal{A}$b category, and

$$Mod_{C_G} = [C_G \to \mathcal{A}]$$

Let functor i: $C_G \hookrightarrow \mathcal{B^{OP}}$. Every object H in C_G,

$$i(G/H) = G/H \in ob(\mathcal{B^{OP}}).$$

Let isomorphism f : G/B $\longrightarrow$ G/D in C_G $(D = gBg^{-1})$. i sent Iso(f) to

$$G/D \xleftarrow[f]{} G/B \xrightarrow{id} G/B$$

For Ind_D^K (D $\subset$ K) , $Ind_{C_G} : G/K \to$ G/D will be sent

$$G/K \xleftarrow[f_1]{} G/D \xrightarrow{id} G/D$$

For Res_B^H (B$\subset$ H), $Res_{C_G} : G/B \to$ G/H will be sent to

$$G/B \xleftarrow[id]{} G/B \xrightarrow{f_2} G/H$$

$$Mod_{B^{OP}} = [B^{OP} \to \mathcal{A}],$$

if we pre-compose i to any functor F' of $Mod_{B^{OP}}$, we will get the

$$i \circ F' : C_G \to \mathcal{A}$$

$$i^* : Mod_{B^{OP}} \to Mod_{C_G}.$$

If any pair of objects in $Mod_{B^{OP}}$ are exist in Mod_{C_G}, every morphism between these objects will exist in Mod_{C_G} too. So, i^* is full and faithful.

Claim 2.

$$i : C_G \hookrightarrow B^{OP}.$$

i is a full and faithful functor from C_G to B^{OP}.

proof for claim 2,

Every object of C_G are exist in B^{OP} since every object in B^{OP} is the finite sum objects of C_G. Any pair of objects of C_G are exist in B^{OP} as I showed above. So, i is a full and faithful functor from C_G to B^{OP}.

Let the functor ${}^L Ind_{C_G}^{B^{OP}} : Mod_{C_G} \to Mod_{B^{OP}}$.

$$\begin{tikzcd} C_G \arrow[r, "T"] \arrow[d, hook, "i"] & \mathcal{A} \\ B^{OP} \arrow[ur, "L"'] & \end{tikzcd}$$

Both B^{OP} and $\mathcal{A}$ are addictive categories. There exists Left Kan extension L of T along i. L is an addictive functor and pair with the natural transformation ϵ : T $\to$ Li. It is a functor of the functor category ${}^L Ind_{C_G}^{\mathcal{B}^{\mathcal{OP}}}$. I give a short name

$$i' =^L Ind_{C_G}^{B^{OP}}.$$

According to the Corollary 3, Section X.3 of Categories for the Working Mathematician by Mac Lane, if the functor **i** is full and faithful, then the universal arrow η : T $\to$ Li for Functor L along **i** is a Natural Isomorphism η :T $\cong$ Li. But I know

$$Li = i'T \quad \text{and} \quad i^* \circ i'T = i'T.$$

By the adjunction, there is a natural bijection map

$$(T \xrightarrow{\eta_T} i^* \circ i'T) \overset{bijection}{\longleftrightarrow} (i'T \longrightarrow i'T)$$

$$\text{There exists} \quad id \in [i'T \to i'T] \Longleftrightarrow id \in [T \to i^* \circ i'T]$$

Then, get $i^* \circ i' \cong id_{Mod_{C_G}}$

On the other hand, the Theorem 1 of adjunction, chapter IV.1 of Saunders Mac Lane, gives a natural map

$$(i' \circ i^* L \xrightarrow{\epsilon_L} L) \overset{bijection}{\longleftrightarrow} (i^* L \longrightarrow i^* L)$$

$$\text{There exists} \quad id \in [i^* L \to i^* L] \Longleftrightarrow id \in [i' \circ i^* L \to L]$$

$$\begin{tikzcd} i^* \circ i' \circ i^* L \arrow[r, "i^* \circ \epsilon"] & i^* L \\ i^* L \arrow[u, "i^* \circ \eta"] \arrow[ur, "id"'] & \end{tikzcd}$$

The two isomorphisms $i^* \circ \eta$ and id are give that $i^* \circ \epsilon$ is isomorphism in the naturally commute diagram. And the following proposition 3.11 gives that ϵ is isomorphism for all L of Mod_{BOP} .

$$i' \circ i^* L \xrightarrow{\epsilon} L$$

$$i' \circ i^* = Id_{Mod_{BOP}}$$

So, If every objects of B^{OP} is finite sum of objectives of C_G, then The functor i^* :$Mod_{BOP} \to Mod_{C_G}$ is equivalence of categories. □

Proposition 3.11. *For every additive functor $M : \mathcal{B}^{\mathcal{OP}} \to \mathcal{A}$, the natural map $M(a \oplus b) \to M(a) \times M(b)$ is an isomorphism.*

Proof. Inmage of disjoint union of Gsets, a $\bigsqcup b$ in $\mathcal{B}^{\mathcal{OP}}$ is M(a$\bigsqcup b$) in $\mathcal{A}$.

Claim. M(a $\bigsqcup b$) is isomorphic to M(a) × M(b).

Due to definition of Additive functor 2.2, M send the bi-product diagram to a bi-product diagram in $\mathcal{A}$.

According to the Theorem 2 of the section VIII.2, Categories for working Mathematician of Mac Lane, for any two objects a and b in an $\mathcal{A}$b category $\mathcal{A}$, $\mathcal{A}$ has bi-product of them if and only if $\mathcal{A}$ has product of them.

According to the definitions of bi-product 3.3 and co-product 2.12,

$$\begin{tikzcd} M(a) \arrow[r, "i_1"] \arrow[dr, "i_1"'] & M(a \sqcup b) \arrow[d, dotted, "\exists! \alpha"] & M(b) \arrow[l, "i_2"'] \arrow[dl, "i_2"] \\ & M(a) \times M(b) & \end{tikzcd}$$

there is the unique map between M(a ⊕ b) and M(a) × M(b) and the unique map α should be an isomorphism since

$$\alpha \circ i_1 = i_1 \quad \text{and} \quad \alpha \circ i_2 = i_2$$

$$M(a \oplus b) \cong M(a) \times M(b)$$

□

3.4 Tensor induction of representations

Let R is a commutative ring, then the tensor product M $\otimes_R$ N of two R-modules is itself an R-module (by functoriality). This allows us to iterate the tensor product construction. In particular, we can consider

$\bigotimes_{x\in X}$ M = M $\otimes_R M \otimes_R M \otimes_R$ ....$\otimes_R$ M

This construction can also be considered as a left Kan extension : F is functor for making free modules and L(M)=$\bigotimes_{x\in X} M \in Mod_R$, L is a functor of left Kan extension. Let a finite set X is fixed. In the following diagram

$$\begin{tikzcd}
R[\text{fin}] \arrow[r, "map(X,-)"] \arrow[d, hook, "F"] & R[\text{fin}] \arrow[r, "F"] & \text{Mod}_R \\
\text{Mod}_R \arrow[urr, "L"] & &
\end{tikzcd}$$

$$map(X,-) : R[\text{fin}] \to R[\text{fin}]$$

$$Y \mapsto map(X,Y).$$

The functor $map(X,-)$ sends the maps $f_i : Y \to Y' \in R[\text{fin}]$ and $a_i \in R$,

$$\sum_{i=1,..n} a_i f_i \mapsto \phi = [\sum_{\underline{i}:X\to 1,..n} (\prod_{x\in X} a_{\underline{i}(x)}) f_{\underline{i}}] \ni R[\text{fin}].$$

Map

$$f_{\underline{i}} : map(X,Y) \to map(X,Y')$$

is given by the formula $f_{\underline{i}}(k)(x) = f_{\underline{i}(x)}(k)(x)$ for k $\in$ map(X,Y) and $x \in X$. We can show the previous diagram as a commute diagram as follow too,

$$\begin{tikzcd}
Y \in R[\text{fin}] \arrow[r, "map(X,-)"] \arrow[d, "F"] & R[\text{fin}] \ni map(X,Y) \arrow[d, "F"] \\
FY \in \text{Mod}_R \arrow[r, "M\mapsto \otimes_{x\in X} M"] & \text{Mod}_R \ni F(map(X,Y)
\end{tikzcd}$$

The total number of maps in map(X,Y) is$|Y|^{|X|}$ maps and when we make the free module

$$F(map(X,Y)) \cong \bigoplus_{f\in map(X,Y)} R = R^{|y|^{|X|}}.$$

and $FY = \oplus_{y\in Y} R$. So,

$$\otimes_{x\in X} FY = \otimes_{x\in X}(\oplus_{y\in Y} R) = R^{|y|^{|X|}}.$$

$$F(map(X,Y)) \cong \otimes_{x\in X} FY.$$

We define the functor L, left induction functor,

$$LM = \bigotimes_{x\in X} M = M \otimes_R M \otimes_R M \otimes_R \ldots.. \otimes_R M$$

as a co-equalizer

$$\bigoplus_{FY_1\to M} \bigoplus_{FY_0\to FY_1} F(map(X,Y_0)) \underset{b}{\overset{a}{\rightrightarrows}} \bigoplus_{FY\to M} F(map(X,Y)) \xrightarrow{u} LM \tag{11}$$

In equation 11, the element

$$x = (\alpha : FY_1 \to M, f : FY_0 \to FY_1, t \in map(X,Y_0)) \quad \text{of} \quad \bigoplus_{FY_1\to M} \bigoplus_{FY_0\to FY_1} F(map(X,Y_0))$$

will be sent by map a to $(\alpha \circ f, t) \in (\bigoplus_{FY\to M} F(map(X,Y)))$ and it will be sent to an element $Lm \in LM$ by u. The element x will be sent by map b to $(\alpha, f(t)) \in (\bigoplus_{FY\to M} F(map(X,Y)))$ and it will be sent to the same element $Lm \in LM$ by u.

Lemma 3.12. *There is a coequalizer diagaram as follow:*

$$\bigoplus_{FY_1\to M} \bigoplus_{FY_0\to FY_1} map(X,FY_0) \underset{b'}{\overset{a'}{\rightrightarrows}} \bigoplus_{FY\to M} map(X,FY) \xrightarrow{u'} map(X,M) \tag{12}$$

In equation 12

Two parallel morphisms a' and b' send en map to two different maps of $\bigoplus_{FY\to M} map(X,FY)$ but coequalizer u' make both of them send to same maps in map(X, LM) in In equation 11.

Let the element

$$x' = (\alpha : FY_1 \to M, f : FY_0 \to FY_1, a \in map(X,FY_0)) \quad \text{of} \quad \bigoplus_{FY_1\to M} \bigoplus_{FY_0\to FY_1} map(X,FY_0)$$

send by map a' to $(\alpha\circ f, a) \in (\bigoplus_{FY\to M} map(X,FY))$ and then we get the map $(\alpha\circ f\circ a) \in map(X,M)$ by map u'.

Let x' send by map b' to the $(\alpha, f(a) \in map(X,FY))$ and get $(\alpha\circ f\circ a) \in map(X,M)$ by u'.

Proof. The proof for this lemma is the same with case of Lemma 2.20 if the the fix set X has the only two elements. If X has more than two elements we can use the induction method to prove it is right for all finite set X. I will omit this detail proof here in my thesis. □

Definition 3.13 (The tensor induction in the diagram)**.** *The formula for the tensor induction of representations. Let G and H be finite groups and let X be a left H, right G -set which is free as an H -set. F is functor of free modules. We define a functor* $map_H(X,-)$ *from* $HR[\text{fin}]$ *to* $GR[\text{fin}]$ *taking an object* Y *to* $map_H(X,Y)$*.*

$$\begin{tikzcd} HR[\text{fin}] \arrow[r, "map_H(X,-)"] \arrow[d, "F"] & G-Set \arrow[r, "F"] & R[G]-Mod \\ R[H]-Mod \arrow[urr, "Tens_H^G"'] & & \end{tikzcd}$$

It takes a H-morphism $f = \sum_{i=1,..n} a_i f_i, f_i : Y \to Y'$ *and* $a_i \in R$*, to*

$$\phi = [\sum_{I:H\backslash X \to [1,..n]} (\prod_{u \in H\backslash X} a_{I(u)}) f_{I \circ p}] \ni GR[\text{fin}],$$

where $p : X \to H\backslash X$ *is the projection and given* $J : X \to [1, ..., n]$*, the map*

$$f_J : map(X,Y) \to map(X,Y')$$

is given by the formula $f_J(k)(x) = f_{J(x)}(k(x))$ *for* $k \in map(X,Y)$ *and* $x \in X$*. It is straight forward to check that*

$$\phi(gk) = g\phi(k)$$

for every $g \in G$*. Using that H acts freely on X, f is an H-morphism we can verify that if k is a H-map, then* $\phi(k) \in F(map_H(X,Y'))$*. This means that we have a G-morphism* $\phi : map(X,Y) \to map(X,Y')$*. We define*

$$map(X,-)(f) := \phi.$$

Here in the diagram, $Tens_H^G M$ *is the tensor induction functor, R[G]-Mod is R module with an action of G. That is* $\bigotimes(M) = Tens_H^G M \in$ *R[G]- Mod. There is an isomorphism of R-modules*

$$Tens_H^G M \cong \bigotimes_{G/H} M.$$

3.5 Tensor induction with the category of B_G^{OP}

Let H and K are subgroups of G and H $\subset$ K. We construct two categories B_H^{OP} and B_K^{OP} from H and K. Then the Mackey functors give $Mod_{R(B_H)}$ and $Mod_{R(B_K)}$ and we can have $Tens_H^K$ as a functor between two categories of modules.

$$\begin{tikzcd} R(B_H^{OP}) \arrow[r, "P_H^K"] \arrow[rr, bend left, "T"] \arrow[d, "F_H"] & R(B_K^{OP}) \arrow[r, "F_K"] & Mod_{R(B_K)} \\ Mod_{R(B_H)} \arrow[urr, "Tens_H^K"'] & & \end{tikzcd} \tag{13}$$

in the diagram $Mod_{R(B_H)}$ is the category of the functors from RB_H to Mod_R, and the functor $P_H^K = map(K, -)$. Let X, X' and Y are objects of RB_H^{OP}, P_H^K takes a map in RB_H^{OP}

$$X \overset{f_1}{\dashleftarrow} c \xrightarrow{f_2} Y \tag{14}$$

where X, Y and c are H-set, to

$$map(K,X) \overset{f_1'}{\dashleftarrow} map(K,c) \xrightarrow{f_2'} map(K,Y)$$

F_H takes the map X to X' of $R(B_H^{OP})$

$$X \overset{f_1''}{\dashleftarrow} c' \xrightarrow{f_2''} X'$$

to $RB_H(X, -)$. For any Y in $ob(B^{OP})$, there is $RB_H(X, Y)$. The map (X, X') in RB_H^{OP} induces $RB_H(X', Y)$ by Yoneda embedding lemma as follow:

$$X' \overset{g_1}{\dashleftarrow} c' \xrightarrow{g_2} X$$

inRB_H,and

$$X \overset{f_1}{\dashleftarrow} c \xrightarrow{f_2} Y$$

give by composing and having pull back

$$X' \overset{h_1}{\dashleftarrow} e \xrightarrow{h_2} Y$$

$$\text{Yoneda embedding} \quad : RB_H^{OP} \to [Mod_{RB_H} = (RB_H, \mathrm{Mod}_R)]$$

Another functor F_K is working same as F_H. If we define Tensor induction $Tens_H^K$ similar as previous section, we get the functor which makes commute the diagram 13.

References

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