Annals of Global Analysis and Geometry (2022) 61:759–776
https://doi.org/10.1007/s10455-022-09827-3

# A horizontal Chern–Gauss–Bonnet formula on totally geodesic foliations

**Fabrice Baudoin[1] · Erlend Grong[2] · Gianmarco Vega-Molino[1]**

Received: 8 July 2021 / Accepted: 8 January 2022 / Published online: 1 February 2022


**Abstract**
Under suitable conditions, we show that the Euler characteristic of a foliated Riemannian manifold can be computed only from curvature invariants which are transverse to the leaves. Our proof uses the hypoelliptic sub-Laplacian on forms recently introduced by two of the authors in Baudoin and Grong (Ann Glob Anal Geom 56(2):403–428, 2019).

## 1 Introduction

The goal of the paper is to prove the following result:

**Theorem 1.1** *Let $\mathbb{M}$ be a smooth, connected, oriented and $n + m$ dimensional compact manifold. We assume that $\mathbb{M}$ is equipped with a Riemannian foliation $\mathcal{F}$ with bundle-like metric $g$ and totally geodesic $m$-dimensional leaves. We also assume that the horizontal distribution $\mathcal{H} = \mathcal{F}^{\perp}$ is bracket-generating and that there exists $\varepsilon > 0$ such that*

$$(\nabla_v J)_w = -\frac{1}{2\varepsilon}[J_v, J_w] \tag{1.1}$$

*for any $v, w \in T_x\mathbb{M}$, $x \in \mathbb{M}$, where $\nabla$ is the Bott connection of the foliation and $J$ is the tensor defined in (2.2). Denoting $\chi(\mathbb{M})$ the Euler characteristic of $\mathbb{M}$:*

- *If $n$ or $m$ is odd, then $\chi(\mathbb{M}) = 0$;*

F. Baudoin: Author supported in part by the NSF Grant DMS 1901315. E. Grong: Author supported by grant from the Trond Mohn Foundation—Grant TMS2021STG02 (GeoProCo).

✉ Erlend Grong
erlend.grong@uib.no

Fabrice Baudoin
fabrice.baudoin@uconn.edu

Gianmarco Vega-Molino
gianmarco.molino@uconn.edu

1 Department of Mathematics, University of Connecticut, Storrs, CT, USA

2 Department of Mathematics, University of Bergen, P.O. Box 7803, 5020 Bergen, Norway

- *If n and m are both even, then*

$$\chi(\mathbb{M}) = \int_{\mathbb{M}} \hat{\omega}^{\varepsilon}_{\mathcal{H}} \wedge \left[ \det \left( \frac{\mathscr{T}}{\sinh(\mathscr{T})} \right)^{1/2} \right]_m .$$

Notations are further explained in Sect. 4, but we point out that a remarkable feature of that result is that the density $\hat{\omega}^{\varepsilon}_{\mathcal{H}} \wedge \left[ \det \left( \frac{\mathscr{T}}{\sinh(\mathscr{T})} \right)^{1/2} \right]_m$ essentially only depends on *horizontal curvature quantities*. Therefore, the theorem illustrates further the fact already observed in [4] that topological properties of $\mathbb{M}$ might be obtained from horizontal curvature invariants only provided that the bracket-generating condition of the horizontal distribution is satisfied; thus, in essence, the theorem is a sub-Riemannian result. We also note that the condition (1.1) is satisfied in a large class of examples including the H-type foliations introduced in [5], see Example 2.4.

The proof of Theorem 1.1 is based on the study of the heat semigroup generated by the hypoelliptic sub-Laplacian on forms recently introduced in [4]. The heat equation approach to Chern–Gauss-Bonnet type formulas (or index formulas) that we are using is of course not new: It was suggested by Atiyah–Bott [1] and McKean-Singer [16] and first carried out by Patodi [18] and Gilkey [12] and is by now classical, see the book [9]. However, a difficulty in our setting is that the sub-Laplacian on forms we consider is only hypoelliptic but not elliptic. To carry out the required small-time asymptotics analysis to obtain the horizontal Chern–Gauss–Bonnet formula, we will make use of the probabilistic Brownian Chen series parametrix method first introduced in [3] and which is easy to adapt to hypoelliptic situations, see [2].

The paper is organized as follows. In Sect. 2, we introduce the horizontal Laplacian on forms $\Delta_{\mathcal{H},\varepsilon}$ and prove that it is a self-adjoint operator if and only if the condition (1.1) is satisfied. In Sect. 3, we prove a McKean–Singer type formula for $\Delta_{\mathcal{H},\varepsilon}$, namely that for every $t > 0$,

$$\mathbf{Str}(e^{t\Delta_{\mathcal{H},\varepsilon}}) = \chi(\mathbb{M}).$$

Finally, in Sect. 4 we study the small-time asymptotics of $\mathbf{Str}(e^{t\Delta_{\mathcal{H},\varepsilon}})$ and conclude the proof of Theorem 1.1.

## 2 Preliminaries

In this section, we first recall the framework and notations of Baudoin and Grong [4] and the references therein to which we refer for further details. We then prove a necessary and sufficient condition for the form horizontal Laplacian of a totally geodesic foliation to be a symmetric operator.

### 2.1 Totally geodesic foliations

Let $(\mathbb{M}, g)$ be a smooth, oriented, connected, compact Riemannian manifold with dimension $n + m$. We assume that $\mathbb{M}$ is equipped with a foliation $\mathcal{F}$ with $m$-dimensional leaves. The distribution $\mathcal{V}$ formed by vectors tangent to the leaves is referred to as the set of *vertical directions* (or *vertical subbundle*). Define *the horizontal subbundle* $\mathcal{H} = \mathcal{V}^{\perp}$ as its orthogonal complement. We will always assume in this paper that the horizontal distribution $\mathcal{H}$ is everywhere bracket-generating. The foliation is called *Riemannian* and *totally geodesic* if for any $X \in \Gamma(\mathcal{H})$, $Z \in \Gamma(\mathcal{V})$, the respective conditions are satisfied,

$$(\mathcal{L}_Z g)(X, X) = 0, \quad (\mathcal{L}_X g)(Z, Z) = 0.$$

Equivalently, we can describe these conditions using *the Bott connection*. Write $\pi_{\mathcal{H}}$ and $\pi_{\mathcal{V}}$ for the respective orthogonal projections to $\mathcal{H}$ and $\mathcal{V}$. Let $\nabla^g$ be the Levi–Civita connection of $g$. Introduce a new connection $\nabla$ on $T\mathbb{M}$ according to the rules,

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}}(\nabla^g_X Y) & \text{for any } X, Y \in \Gamma(\mathcal{H}), \\ \pi_{\mathcal{H}}([X, Y]) & \text{for any } X \in \Gamma(\mathcal{V}), Y \in \Gamma(\mathcal{H}), \\ \pi_{\mathcal{V}}([X, Y]) & \text{for any } X \in \Gamma(\mathcal{H}), Y \in \Gamma(\mathcal{V}), \\ \pi_{\mathcal{V}}(\nabla^g_X Y) & \text{for any } X, Y \in \Gamma(\mathcal{V}). \end{cases} \tag{2.1}$$

We observe that $\nabla$ preserves $\mathcal{H}$ and $\mathcal{V}$ under parallel transport. The foliation $\mathcal{F}$ is then both Riemannian and totally geodesic if and only if $\nabla g = 0$. For the rest of the paper, we will assume that $\nabla$ is indeed compatible with the metric $g$. The torsion $T$ of $\nabla$ is given by

$$T(X, Y) = -\pi_{\mathcal{V}}[\pi_{\mathcal{H}} X, \pi_{\mathcal{H}} Y].$$

Define a corresponding endomorphism valued one-form $Z \mapsto J_Z$ by

$$\langle J_Z X, Y\rangle_g = \langle Z, T(X, Y)\rangle_g, \qquad X, Y, Z \in \Gamma(T\mathbb{M}). \tag{2.2}$$

Let $g_{\mathcal{H}}$ and $g_{\mathcal{V}}$ be the respective restrictions of $g$ to $\mathcal{H}$ and $\mathcal{V}$. We then define *the canonical variation* $g$ by $g_\varepsilon = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}$, $\varepsilon > 0$, and make the following observations:

(i) If $(\mathbb{M}, \mathcal{F}, g)$ is a Riemannian, totally geodesic foliation, then so is $(\mathbb{M}, \mathcal{F}, g_\varepsilon)$.
(ii) Although the Levi-Civita connection $\nabla^{g_\varepsilon}$ of $g_\varepsilon$ is different from the connection $\nabla^g$ of $g$, replacing $\nabla^g$ with $\nabla^{g_\varepsilon}$ in formula (2.1) will lead to exactly the same connection. In other words, when defining the Bott connection $\nabla$, we obtain the same connection for any metric $g_\varepsilon$ in the family of canonical variations.
(iii) For any fixed $\varepsilon > 0$, define a connection

$$\hat{\nabla}^\varepsilon_X Y = \nabla_X Y + \frac{1}{\varepsilon} J_X Y. \tag{2.3}$$

This connection preserves $\mathcal{H}$ and $\mathcal{V}$ under parallel transport and is compatible with $g_{\varepsilon'}$ for any $\varepsilon' > 0$. Furthermore, its torsion

$$\hat{T}^\varepsilon(X, Y) = T(X, Y) + \frac{1}{\varepsilon} J_X Y - \frac{1}{\varepsilon} J_Y X,$$

is skew-symmetric with respect to $g_\varepsilon$. Hence, if we consider its adjoint connection

$$\nabla^\varepsilon_X Y = \hat{\nabla}^\varepsilon_X Y - \hat{T}^\varepsilon(X, Y) = \nabla_X Y - T(X, Y) + \frac{1}{\varepsilon} J_Y X, \tag{2.4}$$

it will also be compatible with $g_\varepsilon$. However, $\mathcal{H}$ and $\mathcal{V}$ are not parallel with respect to $\nabla^\varepsilon$.

### 2.2 Horizontal Laplacian on forms

For the totally geodesic Riemannian foliation $(\mathbb{M}, \mathcal{F}, g)$, define its horizontal Laplacian on functions $f \in C^\infty(\mathbb{M})$ by

$$\Delta_{\mathcal{H}} f = \operatorname{tr}_{\mathcal{H}} \nabla_\times df(\times). \tag{2.5}$$

We note that since $\mathcal{H}$ is assumed to be bracket-generating, from Hörmander's theorem, $\Delta_{\mathcal{H}}$ is a subelliptic operator. We also note that since $g_{\mathcal{H}}$ and the Bott connection are independent of $\varepsilon > 0$, the horizontal Laplacian is as well; that is, the choice of any metric $g_\varepsilon$ in the canonical variation family will not change $g_{\mathcal{H}}$, the Bott connection, or the horizontal Laplacian.

Consider now the totally geodesic Riemannian foliation $(\mathbb{M}, \mathcal{F}, g_\varepsilon)$ for some fixed $\varepsilon > 0$. We want to extend the horizontal Laplacian on functions (2.5) to a differential operator on forms $\Delta_{\mathcal{H},\varepsilon}$ satisfying the following requirements:

(I) $\Delta_{\mathcal{H},\varepsilon} f = \Delta_{\mathcal{H}} f$ for any smooth function $f$;

(II) The operator $\Delta_{\mathcal{H},\varepsilon}$ is of Weitzenböck type, i.e., $\Delta_{\mathcal{H},\varepsilon} = L_{\mathcal{H},\varepsilon} + \mathscr{R}_\varepsilon$ where $\mathscr{R}_\varepsilon$ is a zero-order differential operator and

$$L_{\mathcal{H},\varepsilon} = \operatorname{tr}_{\mathcal{H}} \tilde{\nabla}^2_{\times,\times}, \tag{2.6}$$

is the connection horizontal Laplacian of some connection $\tilde{\nabla}$ compatible with $g_\varepsilon$;

(III) If $d$ is the exterior differential, then

$$[\Delta_{\mathcal{H},\varepsilon}, d] = 0.$$

Given these requirements, there is an essentially unique extension of $\Delta_{\mathcal{H}}$ to forms, see [4,15] for details. We call $\Delta_{\mathcal{H},\varepsilon}$ the $\varepsilon$-horizontal Laplacian on forms. This operator can described as follows.

**Proposition 2.1** (Horizontal Laplacian on forms, see [4]) *Consider the $\varepsilon$-horizontal divergence operator defined by*

$$\delta_{\mathcal{H},\varepsilon}\eta = -\operatorname{tr}_{\mathcal{H}}(\nabla^\varepsilon_\times \eta)(\times, \cdot).$$

*The operator*

$$\Delta_{\mathcal{H},\varepsilon} = -\delta_{\mathcal{H},\varepsilon} d - d\delta_{\mathcal{H},\varepsilon}$$

*is called the $\varepsilon$-horizontal Laplacian on forms, and it satisfies the requirements* (I), (II), (III). *In particular, this operator has Weitzenböck decomposition $\Delta_{\mathcal{H},\varepsilon} = L_{\mathcal{H},\varepsilon} + \mathscr{R}_\varepsilon$ where $L_{\mathcal{H},\varepsilon}$ is defined as in* (2.6) *relative to $\nabla^\varepsilon$.*

We can describe the zero order operator $\mathscr{R}_\varepsilon$ can be made explicit, see [4]. For later use, we will prefer to write the operators using Fermion calculus, see Appendix A.1. Let $X_1, \dots, X_n$ and $Z_1, \dots, Z_m$ be local orthonormal bases of, respectively, $\mathcal{H}$ and $\mathcal{V}$. Define $a_i = \iota_{X_i}$ and $b_r = \iota_{Z_r}$ for the corresponding annihilation operators, with the dual operators $a_i^* = X_i^* \wedge$ and $b_r^* = Z_r^* \wedge$ acting by wedge products. The dual are here relative to the $L^2$ inner product with respect to the fixed metric $g$. Relative to the curvature tensor $\hat{R}^\varepsilon$ of $\hat{\nabla}^\varepsilon$, write

$$\hat{R}^{\varepsilon,l}_{ijk} = \langle \hat{R}^\varepsilon(X_i, X_j)X_k, X_l\rangle_g, \tag{2.7}$$

and use similar notation for other tensors with indices $i, j, k, l$ denoting evaluations with respect to the basis of $\mathcal{H}$, indices $r, s$ with respect to the basis of $\mathcal{V}$. We emphasize that these indices are always defined relative to the fixed metric $g$. Then, $\mathscr{R}_\varepsilon$ is given by

$$\begin{aligned}\mathscr{R}_\varepsilon = &\sum_{i,j,k=1}^{n} \hat{R}^{\varepsilon,i}_{ijk} a_k^* a_j + \sum_{i,k=1}^{n}\sum_{r=1}^{m} \hat{R}^{\varepsilon,i}_{irk} a_k^* b_r + \frac{1}{2}\sum_{i,j,k,l=1}^{n} \hat{R}^{\varepsilon,l}_{ijk} a_k^* a_l^* a_j a_i \\ &+ \sum_{i,j,k=1}^{n}\sum_{r=1}^{m} \hat{R}^{\varepsilon,l}_{irk} a_k^* a_l^* b_r a_i + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r,s=1}^{m} \hat{R}^{\varepsilon,j}_{rsi} a_i^* a_j^* b_r b_s.\end{aligned} \tag{2.8}$$

We want to give a formula for this operator that shows the dependence of $\varepsilon$ explicitly. Let $T$ and $R$ be the curvature of the Bott connection $\nabla$ and use indices after semi-colons to denote covariant derivatives with respect to this connection. Using Lemma A.2, Appendix, we can write

$$\begin{aligned}\mathscr{R}_\varepsilon = &\sum_{i,j,k=1}^{n}\left(R^k_{kji} + \frac{1}{\varepsilon}\sum_{r=1}^{m} T^r_{ik}T^r_{jk}\right)a_i^* a_j - \sum_{i,j=1}^{n}\sum_{r=1}^{m} T^r_{ij;i}a_j^* b_r \\ &+ \frac{1}{2}\sum_{i,j,k,l=1}^{n}\left(R^j_{kli} + \frac{1}{\varepsilon}\sum_{r=1}^{m} T^r_{kl}T^r_{ij}\right)a_i^* a_j^* a_l a_k + \sum_{i,j,k=1}^{n}\sum_{r=1}^{m}\frac{1}{\varepsilon}T^r_{ij;k}a_i^* a_j^* b_r a_k \\ &+ \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r,s=1}^{m}\left(\frac{2}{\varepsilon}T^s_{ij;r} + \frac{1}{\varepsilon^2}\sum_{k=1}^{n}(T^r_{kj}T^s_{ik} - T^s_{kj}T^r_{ik})\right)a_i^* a_j^* b_s b_r. \end{aligned} \tag{2.9}$$

### 2.3 Symmetry of the horizontal Laplacian

Consider the exterior algebra

$$\Omega = \Omega(\mathbb{M}) = \bigoplus_{k=0}^{\dim \mathbb{M}} \Omega^k,$$

with the $L^2$-inner product from $g_\varepsilon$. When restricted to elements in $\Omega^0 \oplus \Omega^1$, the operator $\Delta_{\mathcal{H},\varepsilon}$ is symmetric if and only if $\mathcal{H}$ satisfies the Yang–Mills condition, i.e., if

$$\sum_{i=1}^{n} T^r_{ij;i} = 0, \qquad \text{for any} \quad j = 1,\ldots,n,\ r = 1,\ldots,m.$$

see [6]. Considering all forms, we have the following result.

**Proposition 2.2** *The operator $\Delta_{\mathcal{H},\varepsilon}$ is symmetric with respect to the $L^2$-inner product of $g_\varepsilon$ if and only if*

$$(\nabla_v J)_w = -\frac{1}{2\varepsilon}[J_v, J_w], \tag{2.10}$$

*for any $v, w \in T_x M$, $x \in M$. In particular, $\nabla_v J = 0$ for any $v \in \mathcal{H}$.*

We note that under the above condition, the expression of $\mathscr{R}_\varepsilon$ reduces to

$$\mathscr{R}_\varepsilon = \sum_{i,j,k=1}^{n}\left(R^k_{kji} + \frac{1}{\varepsilon}\sum_{r=1}^{m} T^r_{ik}T^r_{jk}\right)a_i^* a_j + \frac{1}{2}\sum_{i,j,k,l=1}^{n}\left(R^j_{kli} + \frac{1}{\varepsilon}\sum_{r=1}^{m} T^r_{kl}T^r_{ij}\right)a_i^* a_j^* a_l a_k. \tag{2.11}$$

***Proof*** $L_{\mathcal{H},\varepsilon}$ is symmetric by Grong and Thalmaier [15, Lemma A.1], so we only need to determine when $\mathscr{R}_\varepsilon$ is symmetric. We choose a local bases $X_1,\ldots,X_n$ and $Z_1,\ldots,Z_m$ of, respectively, $\mathcal{H}$ and $\mathcal{V}$. We consider the representation of $\mathscr{R}_\varepsilon$ as in (2.9). Then, for $\mathscr{R}_\varepsilon$ to be symmetric, we must have

$$
\begin{aligned}
0 &= \langle \mathscr{R}_\varepsilon X_k^* \wedge Z_r^*, X_i^* \wedge X^j \rangle_\varepsilon - \langle \mathscr{R}_\varepsilon X_i^* \wedge X_j, X_k^* \wedge Z_r^* \rangle_\varepsilon = \frac{1}{\varepsilon} T_{ij;k}^r, \\
0 &= \langle \mathscr{R}_\varepsilon Z_r^* \wedge Z_s^*, X_i^* \wedge X^j \rangle_\varepsilon - \langle \mathscr{R}_\varepsilon X_i^* \wedge X_j, Z_r^* \wedge Z_s^* \rangle_\varepsilon \\
&= \frac{2}{\varepsilon} T_{ij;r}^s + \frac{1}{\varepsilon^2} \sum_{k=1}^n (T_{kj}^r T_{ik}^s - T_{kj}^s T_{ik}^r).
\end{aligned}
$$

These equations are clearly equivalent to (2.10). If these hold, then $\mathscr{R}_\varepsilon$ reduces to the expression (2.11), which is symmetric by Lemma A.3 (i). □

***Remark 2.3*** If we assume that $m = 1$ (i.e., the leaves are one-dimensional), then it is immediate from the previous result that the following are equivalent:

(i) $\Delta_{\mathcal{H},\varepsilon}$ is symmetric for some $\varepsilon > 0$.
(ii) $\Delta_{\mathcal{H},\varepsilon}$ is symmetric for all $\varepsilon > 0$.
(iii) $\nabla J = 0$.

Recall that the statement $\nabla J = 0$ is equivalent to $\nabla T = 0$. For $m > 1$, the above statement remains true if we replace (i) by the following assumption

(i') $\Delta_{\mathcal{H},\varepsilon}$ is symmetric at least two values $\varepsilon > 0$ and $\varepsilon' > 0$.

***Example 2.4*** (*H-type foliations*) Following definitions given in [5], we say that a foliated Riemannian manifold $(\mathbb{M}, \mathcal{F}, g)$ is of *H-type* if for every $Z \in \Gamma(\mathcal{V})$, we have $J_Z^2 = -\|Z\|_{\mathcal{V}}^2 \pi_{\mathcal{H}}$. Expand the definition of $J$ from taking values from $\mathcal{V}$ to its Clifford algebra $\mathbf{Cl}(\mathcal{V})$ by the rule $J_1 = \pi_{\mathcal{H}}$ and iteratively $J_{u \cdot v} = J_u J_v$, $u, v \in \mathbf{Cl}(\mathcal{V})$. We then further say that the foliation is of horizontally parallel Clifford type if $\nabla_X J = 0$ for any horizontal vector fields $X \in \Gamma(\mathcal{H})$ and while for $u, v \in \mathcal{V}$.

$$
(\nabla_u J)_v \in J_{\mathbf{Cl}(\mathcal{V})}.
$$

It then turns out that for some $\kappa \in \mathbb{R}$,

$$
(\nabla_u J)_v = -\kappa J_{u \cdot v + \langle u, v \rangle} = -\frac{\kappa}{2} [J_u, J_v].
$$

The number $\kappa$ determines the Ricci curvature of $\nabla$, see [5, Theorem 3.16]. We see that if we have an H-type Riemannian foliation $(\mathbb{M}, \mathcal{F}, g)$ of horizontally parallel Clifford type, then $\Delta_{\mathcal{H},\varepsilon}$ is symmetric with respect to $g_\varepsilon$ for $\varepsilon = \frac{1}{\kappa}$.

Finally, to conclude the section we point out the following result. For the definition of the Carnot–Carathéodory metric $d_{cc}$ of the sub-Riemannian manifold $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ and the tangent cone of a metric space, see, e.g., [13].

**Corollary 2.5** *Assume that $\Delta_{\mathcal{H},\varepsilon}$ is symmetric on forms for some fixed $\varepsilon > 0$. Then, the following holds:*

(a) *The horizontal bundle $\mathcal{H}$ has step 2, that is $\mathcal{H} + [\mathcal{H}, \mathcal{H}] = T\mathbb{M}$. In particular, the torsion $T$ of the Bott connection $\nabla$ will be surjective on $\mathcal{V}$.*
(b) *The tangent cones of the metric space $(\mathbb{M}, d_{cc})$ at any pair of points $x, y \in \mathbb{M}$ are isometric.*

***Proof*** (a) Recall that if $\Delta_{\mathcal{H},\varepsilon}$ is symmetric on forms for some $\varepsilon > 0$, then in particular $\nabla_v J = 0$ for any $v \in \mathcal{H}$. We can rewrite it as $\nabla_v T = 0$ for any $v \in \mathcal{H}$ since $\nabla$ is compatible with $g$. Define $\mathcal{H}^2 = \mathcal{H} + [\mathcal{H}, \mathcal{H}]$ and let $X_1, X_2, X_3 \in \Gamma(\mathcal{H})$ be arbitrary. We first see that

$$T(X_2, X_3) = \nabla_{X_2} X_3 - \nabla_{X_3} X_2 - [X_2, X_3] = 0 \mod \mathcal{H}^2,$$

since $\nabla$ preserves $\mathcal{H}$. Furthermore, by the definition of the Bott connection

$$\begin{aligned}[X_1, [X_2, X_3]] &= -[X_1, T(X_2, X_3)] \mod \mathcal{H}^2 = -\nabla_{X_1} T(X_2, X_3) \mod \mathcal{H}^2 \\ &= -T(\nabla_{X_1} X_2, X_3) - T(X_2, \nabla_{X_1} X_3) \mod \mathcal{H}^2 = 0 \mod \mathcal{H}^2.\end{aligned}$$

It follows that $\mathcal{H}$ only generates $\mathcal{H}^2$. As we assumed that $\mathcal{H}$ is bracket generating, we have $\mathcal{H}^2 = TM$.

(b) Since both $\mathcal{H}$ and $\mathcal{H}^2 = \mathcal{H} + [\mathcal{H}, \mathcal{H}] = TM$ have constant rank, it follows by Mitchell [17] and Bellaïche [8] that the tangent cone at a point $x$ is a Carnot group $G_x$. Its Lie algebra $\mathfrak{g}_x$ is given by

$$\mathfrak{g}_x = \mathfrak{g}_{x,1} \oplus \mathfrak{g}_{x,2} = \mathcal{H}_x \oplus T_x M/\mathcal{H}_x,$$

where $TM/\mathcal{H}_x$ is the center, and for $X_x, Y_x \in \mathcal{H}_x = \mathfrak{g}_{x,1}$ the Lie bracket is defined as

$$[\![ X_x, Y_x ]\!] = [X, Y]|_x \mod \mathcal{H}_x.$$

where $X, Y$ are any pair of vector fields extending this vectors. The Carnot group $G_x$ is then the corresponding simply connected Lie group of $\mathfrak{g}_x$ with the sub-Riemannian structure given by left translation of $\mathfrak{g}_x = \mathcal{H}_x$ and its inner product.
If identify $\mathfrak{g}_x = \mathcal{H}_x \oplus T_x M/\mathcal{H}_x$ with $T_x M = \mathcal{H}_x \oplus \mathcal{V}_x$ through the map $v \mod \mathcal{H}_x \mapsto \pi_{\mathcal{V}_x}(v)$, $v \in T_x M$, then the Lie bracket becomes,

$$[\![ v, w ]\!] = -T(v, w), \qquad v, w \in T_x M.$$

Let now $y$ be any other point and let $\gamma : [0, 1] \to \mathbb{M}$ be any horizontal curve from $x$ to $y$, which exists form our assumption that $\mathcal{H}$ satisfies the bracket-generating condition. Then, $\nabla_{\dot{\gamma}(t)} T = 0$ for any $t \in [0, 1]$, so if we write

$$/\!\!/_{\gamma,t} = /\!\!/_t : T_x\mathbb{M} \to T_{\gamma(t)}\mathbb{M},$$

for the parallel transport map along $\gamma$, then this satisfies

$$/\!\!/_t\, T(u, v) = T(/\!\!/_t\, u, /\!\!/_t\, v), \qquad v, w \in T_x\mathbb{M}.$$

As a consequence, $/\!\!/_1 : \mathfrak{g}_x = T_x\mathbb{M} \to \mathfrak{g}_y = T_y\mathbb{M}$ is a Lie algebra isomorphism, which can be integrated to a Lie group isomorphism from $G_x$ to $G_y$. Since the parallel transport $/\!\!/_1$ also maps $\mathcal{H}_x$ onto $\mathcal{H}_y$ isometrically, the induced map on Carnot groups is in fact a sub-Riemannian isometry.

□

## 3 Horizontal McKean–Singer theorem

We work on a totally geodesic foliation $(\mathbb{M}, \mathcal{F}, g)$ and assume that there is some $0 < \varepsilon < +\infty$ such that horizontal Laplacian $\Delta_{\mathcal{H},\varepsilon}$, is symmetric. From Proposition 2.2, this assumption is equivalent to the fact that

$$(\nabla_v J)_w = -\frac{1}{2\varepsilon}[J_v, J_w].$$

Since $\Delta_{\mathcal{H},\varepsilon}$ commutes with $d$ on smooth forms and is symmetric, it also commutes on smooth forms with the coderivative $\delta_\varepsilon$, and thus, it also commutes with the Hodge–de Rham operator $\Delta_\varepsilon := -d\delta_\varepsilon - \delta_\varepsilon d$ on smooth forms. From Hodge theorem, the operator $\Delta_\varepsilon$ is elliptic with a compact resolvent and the space of $L^2$-forms can be decomposed as $\oplus_{k=0}^{+\infty} E_{\lambda_k}$ where the $E_{\lambda_k}$'s are the eigenspaces of $\Delta_\varepsilon$. Those eigenspaces only contain smooth forms, therefore $\Delta_{\mathcal{H},\varepsilon}(E_{\lambda_k}) \subset E_{\lambda_k}$. This implies that $\Delta_{\mathcal{H},\varepsilon}$ is essentially self-adjoint and generates the semigroup:

$$e^{t\Delta_{\mathcal{H},\varepsilon}} = \oplus_{k=0}^{+\infty} e^{t\Delta_{\mathcal{H},\varepsilon}|E_{\lambda_k}} \tag{3.1}$$

By hypoellipticity (see [4, Lemma 4.9]), this semigroup has a smooth kernel $p_{\mathcal{H},\varepsilon}(t,x,y)$ and is a bounded trace class operator in $L^2_\mu(\wedge^\cdot \mathbb{M}, g_\varepsilon)$. Let us denote by $E_0^+(\Delta_{\mathcal{H},\varepsilon})$ (resp. $E_0^-(\Delta_{\mathcal{H},\varepsilon})$) the space of harmonic even forms for $\Delta_{\mathcal{H},\varepsilon}$ (resp. the space of harmonic odd forms for $\Delta_{\mathcal{H},\varepsilon}$).

The goal of the section is to prove the following theorem, which is an analogue for our horizontal Laplacian of the classical McKean–Singer formula found in [16] :

**Theorem 3.1** (Horizontal McKean-Singer formula) *For every $t > 0$,*

$$\begin{aligned}\mathbf{Str}(e^{t\Delta_{\mathcal{H},\varepsilon}}) :&= \int_{\mathbb{M}} \mathbf{Tr}(p^+_{\mathcal{H},\varepsilon}(t,x,x))d\mu(x) - \int_{\mathbb{M}} \mathbf{Tr}(p^-_{\mathcal{H},\varepsilon}(t,x,x))d\mu(x)\\ &= \dim E_0^+(\Delta_{\mathcal{H},\varepsilon}) - \dim E_0^-(\Delta_{\mathcal{H},\varepsilon})\\ &= \chi(\mathbb{M})\end{aligned}$$

*where $\chi(\mathbb{M})$ is the Euler characteristic of $\mathbb{M}$.*

We turn to the proof of Theorem 3.1. We denote by

$$\mathbf{D}_\varepsilon = d + \delta_\varepsilon$$

the Dirac operator of the metric $g_\varepsilon$. Observe that $\mathbf{D}_\varepsilon$ commutes with $\Delta_{\mathcal{H},\varepsilon}$ since both $d$ and $\delta_\varepsilon$ commute with it. The main idea to prove Theorem 3.1 is to introduce a deformation of $\Delta_{\mathcal{H},\varepsilon}$ as follows:

$$\square_{\varepsilon,\theta} = (1-\theta)\Delta_{\mathcal{H},\varepsilon} - \theta\mathbf{D}_\varepsilon^2, \quad \theta \in [0,1].$$

A first lemma is the following:

**Lemma 3.2** *Let $\lambda$ be a nonzero eigenvalue of $\square_{\varepsilon,\theta}$. Then, $\mathbf{D}_\varepsilon : E_\lambda^+(\square_{\varepsilon,\theta}) \to E_\lambda^-(\square_{\varepsilon,\theta})$ is an isomorphism. Therefore, $\dim E_\lambda^+(\square_{\varepsilon,\theta}) = \dim E_\lambda^-(\square_{\varepsilon,\theta})$.*

***Proof*** Let $\lambda$ be a nonzero eigenvalue of $\square_{\varepsilon,\theta}$. The corresponding eigenspace $E_\lambda(\square_{\varepsilon,\theta})$ is finite-dimensional since $e^{t\square_{\varepsilon,\theta}}$ is a compact operator for $t > 0$. Moreover, since $\mathbf{D}_\varepsilon$ commutes with $\square_{\varepsilon,\theta}$, $\mathbf{D}_\varepsilon : E_\lambda^+(\square_{\varepsilon,\theta}) \to E_\lambda^-(\square_{\varepsilon,\theta})$ is well defined. Let now $\alpha \in E_\lambda^+(\square_{\varepsilon,\theta})$ such that $\mathbf{D}_\varepsilon\alpha = 0$. One has then

$$d\alpha = -\delta_\varepsilon\alpha.$$

This implies that

$$\|d\alpha\|^2_{L^2(\wedge^\cdot\mathbb{M},g_\varepsilon)} = -\langle d\alpha, \delta_\varepsilon\alpha\rangle_{L^2(\wedge^\cdot\mathbb{M},g_\varepsilon)} = 0,$$

so $d\alpha = 0$. Similarly, one has $\|\delta_\varepsilon \alpha\|^2_{L^2(\wedge^\cdot \mathbb{M}, g_\varepsilon)} = 0$, so $\delta_\varepsilon \alpha = 0$. Therefore,

$$\alpha = \frac{1-\theta}{\lambda} \Delta_{\mathcal{H},\varepsilon} \alpha = -\frac{1-\theta}{\lambda} (d\delta_{\mathcal{H},\varepsilon} + \delta_{\mathcal{H},\varepsilon} d)\alpha = -\frac{1-\theta}{\lambda} d\delta_{\mathcal{H},\varepsilon}\alpha.$$

One deduces

$$\|\alpha\|^2_{L^2(\wedge^\cdot \mathbb{M}, g_\varepsilon)} = -\frac{1-\theta}{\lambda} \langle \alpha, d\delta_{\mathcal{H},\varepsilon}\alpha \rangle_{L^2(\wedge^\cdot \mathbb{M}, g_\varepsilon)} = -\frac{1-\theta}{\lambda} \langle \delta_\varepsilon \alpha, \delta_{\mathcal{H},\varepsilon}\alpha \rangle_{L^2(\wedge^\cdot \mathbb{M}, g_\varepsilon)} = 0.$$

As a consequence, $\mathbf{D}_\varepsilon : E^+_\lambda(\Box_{\varepsilon,\theta}) \to E^-_\lambda(\Box_{\varepsilon,\theta})$ is injective. Let us now prove that it is surjective. Let $\alpha \in E^-_\lambda(\Box_{\varepsilon,\theta})$ which is orthogonal to the space $\mathbf{D}_\varepsilon E^+_\lambda(\Box_{\varepsilon,\theta})$. For every $\omega \in E^+_\lambda(\Box_{\varepsilon,\theta})$, one has

$$0 = \langle \alpha, \mathbf{D}_\varepsilon \omega \rangle_{L^2(\wedge^\cdot \mathbb{M}, g_\varepsilon)} = \langle \mathbf{D}_\varepsilon \alpha, \omega \rangle_{L^2(\wedge^\cdot \mathbb{M}, g_\varepsilon)}.$$

Thus, $\mathbf{D}_\varepsilon \alpha = 0$ and from the first part of the proof, we deduce that $\alpha = 0$. We conclude that $\mathbf{D}_\varepsilon : E^+_\lambda(\Box_{\varepsilon,\theta}) \to E^-_\lambda(\Box_{\varepsilon,\theta})$ is indeed an isomorphism. □

A second lemma is the following:

**Lemma 3.3** *For every $t > 0$, the map $\theta \to \mathbf{Str}(e^{t\Box_{\varepsilon,\theta}})$ is continuous on* $[0, 1]$.

***Proof*** Let $q_{\varepsilon,\theta}(t, x, y)$ be the heat kernel of $\Box_{\varepsilon,\theta} = (1-\theta)\Delta_{\mathcal{H},\varepsilon} - \theta \mathbf{D}^2_\varepsilon$, $p_{\mathcal{H},\varepsilon}(t, x, y)$ be the heat kernel of $\Delta_{\mathcal{H},\varepsilon}$ and $p_\varepsilon(t, x, y)$ be the heat kernel of $-\mathbf{D}^2_\varepsilon$. Since $-\mathbf{D}^2_\varepsilon$ and $\Delta_{\mathcal{H},\varepsilon}$ commute, we have

$$e^{t\Box_{\varepsilon,\theta}} = e^{t(1-\theta)\Delta_{\mathcal{H},\varepsilon}} e^{-t\theta \mathbf{D}^2_\varepsilon}.$$

Therefore:

$$q_{\varepsilon,\theta}(t, x, y) = \int_{\mathbb{M}} p_{\mathcal{H},\varepsilon}(t(1-\theta), x, z) p_\varepsilon(t\theta, z, y) dz$$

and the result easily follows since

$$\mathbf{Str}(e^{t\Box_{\varepsilon,\theta}}) = \int_{\mathbb{M}} q_{\varepsilon,\theta}(t, x, x) dx.$$

□

We are now ready for the proof of Theorem 3.1.

***Proof*** From the first lemma:

$$\begin{aligned} &\mathbf{Str}(e^{t\Box_{\varepsilon,\theta}}) \\ &\quad = \dim E^+_0(\Box_{\varepsilon,\theta}) - \dim E^-_0(\Box_{\varepsilon,\theta}) + \sum_{\lambda \neq 0} (\dim E^+_\lambda(\Box_{\varepsilon,\theta}) - \dim E^-_\lambda(\Box_{\varepsilon,\theta})) e^{\lambda t} \\ &\quad = \dim E^+_0(\Box_{\varepsilon,\theta}) - \dim E^-_0(\Box_{\varepsilon,\theta}). \end{aligned}$$

Therefore, $\mathbf{Str}(e^{t\Box_{\varepsilon,\theta}}) \in \mathbb{Z}$. From the second lemma, $\theta \to \mathbf{Str}(e^{t\Box_{\varepsilon,\theta}})$ is continuous, thus constant. We deduce

$$\mathbf{Str}(e^{t\Box_{\varepsilon,0}}) = \mathbf{Str}(e^{t\Box_{\varepsilon,1}}).$$

Since $\Box_{\varepsilon^*,1} = -\mathbf{D}^2_\varepsilon$ is the Hodge–de Rham Laplacian of the Riemannian manifold $(\mathbb{M}, g_\varepsilon)$, from the usual Riemannian Hodge theory (see [16]), we have

$$\mathbf{Str}(e^{t\Box_{\varepsilon,1}}) = \chi(\mathbb{M}),$$

4 which concludes the proof. □

***Remark 3.4*** (*Dependence on the symmetry condition*) It would obviously be beneficial to prove the above statement without the assumption of symmetry on $\Delta_{\mathcal{H},\varepsilon}$. A semigroup approach to non-symmetric horizontal Laplacians has been used, see [15, Appendix A]. In the above proof, however, we really rely on the fact that $\Delta_{\mathcal{H},\varepsilon}$ commutes with the codifferential $\delta_\varepsilon$, and with the Laplace–Beltrami operator $-\mathbf{D}_\varepsilon^2$. We can no longer use these properties if we remove the symmetry assumption.

## 4 Horizontal Chern–Gauss–Bonnet formula

As before, we consider the horizontal Laplacian

$$\Delta_{\mathcal{H},\varepsilon} = -d\delta_{\mathcal{H},\varepsilon} - \delta_{\mathcal{H},\varepsilon} d,$$

and assume that it is symmetric for a fixed $\varepsilon$. As seen earlier, $\Delta_{\mathcal{H},\varepsilon}$ satisfies the Weitzenböck identity

$$\Delta_{\mathcal{H},\varepsilon} = L_{\mathcal{H},\varepsilon} - \mathscr{R}_\varepsilon = -(\nabla_{\mathcal{H}}^\varepsilon)^* \nabla_{\mathcal{H}}^\varepsilon - \mathscr{R}_\varepsilon. \tag{4.1}$$

where the later equality follows from [15, Lemma 2.1]. The goal of the section is to compute the pointwise limit

$$\lim_{t\to 0} \mathbf{Str}\, (p_{\mathcal{H},\varepsilon}(t,x,x))$$

and deduce from it our horizontal Chern–Gauss–Bonnet formula. The computation of that limit will be based on the probabilist method of Brownian Chen series (see [3,7]) which has the advantage of being easily adapted to subelliptic operators like $\Delta_{\mathcal{H},\varepsilon}$, see [2]. For convenience and to introduce notation, we include in Appendix A.2 the main elements of that theory.

A first step to implement the method in [2] is to study the small-time heat kernel asymptotics of a diffusion tangent to the scalar horizontal Laplacian $\Delta_{\mathcal{H}}$ . Since we assume that $\Delta_{\mathcal{H},\varepsilon}$ is symmetric, from Corollary 2.5 one has $T\mathbb{M} = \mathcal{H} + [\mathcal{H},\mathcal{H}]$, and thus the tangent diffusion will take its values in a two-step Carnot group [the so-called tangent cone, see Corollary 2.5(b)] for which an explicit formula for the heat kernel is known (see [10,11]). In a local horizontal frame $\{X_1,\dots,X_n\}$ around $x_0$ write

$$V_t(x_0) = \sum_{i=1}^n \sqrt{2} X_i(x_0) B_t^i + \sum_{1\le i<j\le n} \pi_{\mathcal{V}}([X_i,X_j](x_0)) \int_0^t B_s^i dB_s^j - B_s^j dB_s^i,$$

where $(B_t)_{t\ge 0}$ is a Brownian motion in $\mathbb{R}^n$. We note that $V_t(x_0)$ can be written in a basis free way as

$$\sqrt{2} B_t(x_0) - \int_0^t T(B_s(x_0), dB_s(x_0))$$

where $B_t(x_0) = \sum_{i=1}^n X_i(x_0) B_t^i$ is a standard Brownian motion in $\mathcal{H}_{x_0}$.

**Lemma 4.1** *Let $x_0 \in \mathbb{M}$. For $t > 0$, let $d_t(x_0)$ be the density at 0 of the $T_{x_0}\mathbb{M}$ valued random variable $V_t(x_0)$. Then, when $t \to 0$,*

$$d_t(x_0) \sim \frac{2^m}{(4\pi t)^{\frac{n}{2}+m}} \int_{\mathcal{V}_{x_0}} \det\left( \frac{\sqrt{J_z^* J_z}}{\sinh \sqrt{J_z^* J_z}} \right)^{1/2} dz.$$

***Proof*** The process $(V_t(x_0))_{t\geq 0}$ is the horizontal Brownian motion in the tangent cone $G_{x_0}$ which is a 2-step Carnot group when it is identified with $T_{x_0}\mathbb{M}$ using the group exponential map. The heat kernel of the horizontal Laplacian is known explicitly in 2-step Carnot groups (see [10,11]) which yields the small-time asymptotics. □

***Remark 4.2*** We note that $d_t(x_0)$ is independent of $x_0$ because of Corollary 2.5(b).

In the sequel, we will use the notation $\mathcal{F}_I$ (defined with respect to the connection $D = \nabla^{\varepsilon}$) and $\Lambda_I(B)_t$, as introduced and discussed in Appendix A.2.

**Corollary 4.3** *It will hold that as* $t \to 0$

$$\mathbf{Str}(p_{\mathcal{H},\varepsilon}(t,x_0,x_0)) \sim d_t(x_0)\mathbb{E}\left(\mathbf{Str}\left(\exp\left(\sum_{I,d(I)\leq n+2m}\Lambda_I(B)_t\mathcal{F}_I\right)(x_0)\right)\middle| B_1 = 0\right)$$

*where* $d_t(x_0)$ *is the density at 0 of* $V_t(x)$*, as in Lemma 4.1.*

***Proof*** Since $\mathcal{H}$ is two-step bracket generating, the homogeneous dimension is $Q = \dim\mathcal{H} + 2\dim\mathcal{V} = n+2m$. Taking $N = n+2m$ in Theorem A.1, and applying similar arguments as in the proof of Proposition 4.2 in [3], the corollary follows by recognizing that for $|I| > 2$, $X_I$ is a linear combination of $X_i, [X_j, X_k]$ so that when $t \to 0$ the density at 0 of

$$\sum_{I,d(I)\leq n+2m}\Lambda_I(B)_t X_I$$

is equivalent to $d_t(x_0)$ from the previous lemma. □

Applying the previous results, we are now able to compute $\lim_{t\to 0}\mathbf{Str}(p_{\mathcal{H},\varepsilon}(t,x_0,x_0))$. Choose local orthonormal bases $X_1,\ldots,X_n$ and $Z_1,\ldots,Z_m$ of, respectively, $\mathcal{H}$ and $\mathcal{V}$.

**Lemma 4.4** *The integral*

$$\mathcal{J} = \mathcal{J}(x_0) = \frac{2^m}{(2\pi)^{\frac{n}{2}+m}}\int_{\mathcal{V}_{x_0}}\det\left(\frac{\sqrt{J_z^*J_z}}{\sinh\sqrt{J_z^*J_z}}\right)^{1/2}dz,$$

*is a constant, so independent of the point* $x_0 \in \mathbb{M}$ *chosen. Furthermore, it holds that*

$$\lim_{t\to 0}\mathbf{Str}(p_{\mathcal{H},\varepsilon}(t,x_0,x_0)) = \begin{cases}\frac{\mathcal{J}}{(\frac{n}{2}+m)!}\mathbb{E}\left(\mathbf{Str}\left[A_{x_0}^{\frac{n}{2}+m}\right]\middle| B_1 = 0\right), & \text{if } n \text{ is even}\\ 0, & \text{if } n \text{ is odd.}\end{cases}$$

*where the random variable* $A_{x_0}$ *is given by*

$$A_{x_0} = -\frac{1}{2}\sum_{i,j,k,l=1}^{n}\left(R_{kli}^{j} + \frac{1}{\varepsilon}\sum_{r=1}^{m}T_{kl}^{r}T_{ij}^{r}\right)a_i^*a_j^*a_l a_k$$
$$, \sum_{1\leq i<j\leq n}\sum_{r,s=1}^{m}T_{ij;r}^{s}b_r^*b_s\int_0^1 B_t^i dB_t^j - B_t^j dB_t^i. \tag{4.2}$$

***Proof*** First, observe that

$$\mathcal{J}(x_0) = (2t)^{\frac{n}{2}+m} d_t(x_0),$$

and so the independence of $\mathcal{J}(x_0)$ from $x_0$ follows from Corollary 2.5(b) as in Remark 4.2.

Consider the expansion

$$\mathbf{Str}\left[\exp\left(\sum_{I,d(I)\leq n+2m} \Lambda_I(B)_t \mathcal{F}_I\right)(x_0)\right] = \sum_{k\geq 0} \frac{1}{k!}\mathbf{Str}\left[\left(\sum_{I,d(I)\leq n+2m} \Lambda_I(B)_t \mathcal{F}_I\right)^k (x_0)\right].$$

From the Weitzenböck identity (4.1), we have for $i, j \in \{1, \ldots, n+m\}$ that

$$\mathcal{F}_0 = -\mathscr{R}_\varepsilon, \quad \mathcal{F}_i = 0, \quad \mathcal{F}_{(i,j)} = \hat{R}^\varepsilon(Y_i, Y_j)$$

where $\{Y_1, \ldots, Y_{n+m}\}$ form a local orthonormal frame and the $\{c_i, c_i^*\}_{i=1}^{n+m}$ form the associated Fermion calculus of $T\mathbb{M}$. Equation (2.11) allows us to write

$$\mathscr{R}_\varepsilon = \sum_{i,j,k=1}^{n} \langle \hat{R}^\varepsilon(X_i, X_k)X_j, X_i\rangle_g a_k^* a_i + \sum_{i,j,k,l} \langle \hat{R}^\varepsilon(X_i, X_j)X_k, X_l\rangle_g a_i^* a_j^* a_l a_k$$

where $\{a_i, a_i^*\}$ form the Fermion calculus for $\mathcal{H}$.

Recalling equation (A.1) in the appendix, we see that the supertrace will vanish for any term that is not of full degree; from our expressions for $\mathcal{F}_I$, it is thus clear that for $k < \frac{n}{2}+m$

$$\mathbf{Str}\left[\left(\sum_{I,d(I)\leq n+2m} \Lambda_I(B)_t \mathcal{F}_I\right)^k (x_0)\right] = 0.$$

Let us assume that $n$ is even. Applying the scaling property of Brownian motion, when $t \to 0$ the term $k = \frac{n}{2}+m$ will be dominant. More precisely,

$$\begin{aligned}&\mathbb{E}\left(\mathbf{Str}\left[\exp\left(\textstyle\sum_{I,d(I)\leq n+2m} \Lambda_I(B)_t\mathcal{F}_I\right)(x_0)\right]\Big| B_1 = 0\right)\\ &= \frac{1}{(\frac{n}{2}+m)!}\mathbb{E}\left(\mathbf{Str}\left[\left(\textstyle\sum_{I,d(I)\leq n+2m} \Lambda_I(B)_t\mathcal{F}_I\right)^{\frac{n}{2}+m}(x_0)\right]\Big| B_1 = 0\right) + O\left(t^{\frac{n}{2}+m+\frac{1}{2}}\right). \quad (4.3)\end{aligned}$$

Then, we have,

$$\begin{aligned}&\mathbb{E}\left(\mathbf{Str}\left[\left(\textstyle\sum_{I,d(I)\leq n+2m} \Lambda_I(B)_t\mathcal{F}_I\right)^{\frac{n}{2}+m}(x_0)\right]\Big| B_1{=}0\right)\\ &=\mathbb{E}\left(\mathbf{Str}\left[\left(-t\mathscr{R}_\varepsilon(x_0)+\textstyle\sum_{1\leq i<j\leq n}\sum_{r,s=1}^{s} \hat{R}_{iir}^{\varepsilon,s} b_r^* b_s \int_0^t B_u^i dB_u^j - B_u^j dB_u^i\right)^{\frac{n}{2}+m}\right]\Big| B_1{=}0\right)+O\left(t^{\frac{n}{2}+m+\frac{1}{2}}\right) \quad (4.4)\end{aligned}$$

We can further simplify this expression using that by Lemma A.2, Appendix, we know that $\hat{R}_{ijr}^{\varepsilon,s} = R_{ijr}^s = T_{ij;r}^s$. We also use (2.11) and the fact that only the last term in $\mathscr{R}_\varepsilon$ contributes to the supertrace. Combining Lemma 4.1, Corollary 4.3, and Eqs. (4.3) and (4.4), we apply the scaling property of Brownian motion again to find

$$\begin{aligned}\mathbf{Str}(p_{\mathcal{H},\varepsilon}(t, x_0, x_0)) = \frac{\mathcal{J}}{(\frac{n}{2}+m)!}\mathbb{E}\left(\mathbf{Str}\left[A_{x_0}^{\frac{n}{2}+m}\right]\Big|\right.\\ \left. B_1 = 0\right) + O\left(t^{\frac{1}{2}}\right).\end{aligned}$$

If $n$ is odd, we get by similar arguments that

$$\mathbf{Str}(p_{\mathcal{H},\varepsilon}(t, x_0, x_0)) = O\left(t^{\frac{1}{2}}\right).$$

completing the proof. □

In what follows, we will introduce the tensor $\mathscr{T}$ by

$$\mathscr{T}(Y_1, Y_2) = \hat{R}^{\varepsilon}(\pi_{\mathcal{H}} Y_1, Y_2)\pi_{\mathcal{V}} = \pi_{\mathcal{V}} \hat{R}^{\varepsilon}(\pi_{\mathcal{H}} Y_1, Y_2).$$

We observe that for any $X_1, X_2 \in \Gamma(\mathcal{H})$ and $Z \in \mathcal{V}$,

$$\mathscr{T}(X_1, X_2)Z = (\nabla_Z T)(X_1, X_2) = \frac{1}{2\varepsilon}\left(T(J_Z X_1, X_2) + T(X_1, J_Z X_2)\right),$$

where the latter equality follows from the symmetry condition of $\Delta_{\mathcal{H},\varepsilon}$.

***Example 4.5*** (H-type foliation) We again consider the case of the of H-type foliations as in Example 2.4. We recall that in this case, we have that $\Delta_{\mathcal{H},\varepsilon}$ for $\varepsilon = \frac{1}{\kappa}$. Let $x \in \mathbb{M}$ be a fixed point and let $\mathbf{Cl}(\mathcal{V}_x)$ be the Clifford algebra of the vertical space. We remark that in this case, for any $u, v \in \mathcal{H}_x$ with $v \in (\mathrm{span}_{\zeta \in \mathbf{Cl}(\mathcal{V}_x)} J_\zeta u)^{\perp}$, we have $\mathscr{T}(u, v) = 0$. On the other hand, if $v = J_\zeta u$, then for any $z \in \mathcal{V}_x$,

$$\mathscr{T}(u, J_\zeta u)z = \kappa \pi_{\mathcal{V}_x}(z \cdot \zeta^{\mathrm{odd}}),$$

where $\zeta^{\mathrm{odd}}$ is the odd part of $\zeta$ and $\pi_{\mathcal{V}_x} \mathrm{Cl}(\mathcal{V}_x) \to \mathcal{V}_x$ is the projection to the first-order part.

We can use the above definition and the previous lemma to prove the following.

**Proposition 4.6** *Assume that $n$ or $m$ is odd, then*

$$\lim_{t\to 0} \mathbf{Str}\,(p_{\mathcal{H},\varepsilon}(t, x, x))\,dx = 0$$

*Assume that both $n$ and $m$ are even, then*

$$\lim_{t\to 0} \mathbf{Str}\,(p_{\mathcal{H},\varepsilon}(t, x, x))\,dx = \hat{\omega}^{\varepsilon}_{\mathcal{H}} \wedge \left[\det\left(\frac{\mathscr{T}}{\sinh(\mathscr{T})}\right)^{1/2}\right]_m$$

*where $[\cdot]_m$ denotes the $m$-form part and $\hat{\omega}^{\varepsilon}_{\mathcal{H}}$ is the horizontal Euler form, locally defined as*

$$\hat{\omega}^{\varepsilon}_{\mathcal{H}} = \frac{(-1)^{n/2} m!}{2^{n/2}\left(\frac{n}{2} + m\right)!} \mathcal{J} \sum_{\sigma,\tau \in \mathfrak{S}_n} \epsilon(\sigma)\epsilon(\tau) \prod_{i=1}^{n-1} \hat{R}^{\varepsilon,\tau(i+1)}_{\sigma(i)\sigma(i+1)\tau(i)} dx_{\mathcal{H}},$$

*In the above formula, $\mathfrak{S}_n$ is the set of the permutations of the indices $\{1, ..., n\}$, $\epsilon$ the signature of a permutation, $\hat{R}^{\varepsilon,l}_{ijk}$ is as in (2.7) and $dx_{\mathcal{H}}$ the $n$-form $X_1^* \wedge \cdots \wedge X_n^*$.*

***Proof*** We first assume that both $n$ and $m$ are even. It remains to compute $\mathbb{E}\left(\mathbf{Str}\left[A_{x_0}^{\frac{n}{2}+m}\right]\middle| B_1 = 0\right)$. Looking at (4.2), we have

$$\mathbb{E}\left(\mathbf{Str}\left[A_{x_0}^{\frac{n}{2}+m}\right]\middle| B_1=0\right)$$
$$= \mathbf{Str}\left[\left(-\sum_{i,j,k,l}\langle \hat{R}^{\varepsilon}(X_i,X_j)X_k,X_l\rangle_g a_i^* a_j^* a_l a_k\right)^{n/2} \mathbb{E}\left[\left(\sum_{1\le i<j\le n} \mathscr{T}(X_i,X_j)(x_0)\int_0^1 B_s^i dB_s^j - B_s^j dB_s^i\right)^m \middle| B_1=0\right]\right]$$

The term $\left(\sum_{i,j,k,l}\langle \hat{R}^{\varepsilon}(X_i, X_j)X_k, X_l\rangle_g a_i^* a_j^* a_l a_k\right)^{n/2}$ is then analyzed as in the proof of Proposition 5.6 in [7] (see also Lemma 2.35 in [19]) and up to constant yields the horizontal Euler form $\hat{\omega}^{\varepsilon}_{\mathcal{H}}$. On the other hand, using again the formula for the supertrace, the term

$$\mathbb{E}\left[\left(\sum_{1\le i<j\le n} \mathscr{T}(X_i, X_j)(x_0)\int_0^1 B_s^i dB_s^j - B_s^j dB_s^i\right)^m \middle| B_1 = 0\right]$$

can be replaced with

$$m!\,\mathbb{E}\left[\exp\left(\sum_{1\le i<j\le n}\mathscr{T}(X_i,X_j)(x_0)\int_0^1 B_s^i dB_s^j - B_s^j dB_s^i\right)\middle| B_1=0\right]$$

and is analyzed using the Lévy area formula as in the proof of Theorem 4.3 in [3]: it yields the top degree Fermionic piece of $\det\left(\frac{\mathscr{T}}{\sinh(\mathscr{T})}\right)^{1/2}(x_0)\in \mathbf{End}\left(\wedge\mathcal{V}_{x_0}^*\right)$ (Fermionic calculus is done here on $\mathcal{V}_{x_0}$).

If $n$ is even and $m$ is odd, a similar analysis shows that

$$\mathbb{E}\left(\mathbf{Str}\left[A_{x_0}^{\frac{n}{2}+m}\right]\middle| B_1=0\right)=0.$$

□

Combining Theorem 3.1 and Proposition 4.6 finally yields our main theorem:

**Theorem 4.7** *Assume that both n and m are even, then*

$$\chi(\mathbb{M})=\int_{\mathbb{M}}\hat{\omega}_{\mathcal{H}}^{\varepsilon}\wedge\left[\det\left(\frac{\mathscr{T}}{\sinh\mathscr{T}}\right)^{1/2}\right]_m.$$

*Assume that n or m is odd, then* $\chi(\mathbb{M})=0$.

As a corollary, since $\nabla J=0$ implies $\mathscr{T}=0$, we obtain the following result:

**Corollary 4.8** *Assume that* $\nabla J=0$, *then* $\chi(\mathbb{M})=0$.

**Funding** Open access funding provided by University of Bergen (incl Haukeland University Hospital).



# A Appendices

## A.1 Fermion calculus and supertraces

In this section, we recall some basic elements of Fermion calculus, see section 2.2.2 in [19] for more details. Let $V$ be a $d$-dimensional Euclidean vector space. We denote $V^*$ its dual and $\wedge V^*=\bigoplus_{k\ge 0}\wedge^k V^*$, its exterior algebra. If $u\in V^*$, we denote $a_u^*$ the map $\wedge V^*\to\wedge V^*$, such that $a_u^*(\omega)=u\wedge\omega$. The dual map is denoted $a_u$. Let now $\theta_1$,..., $\theta_d$ be an orthonormal basis of $V^*$. We denote $a_i=a_{\theta_i}$. If $I$ and $J$ are two words with $1\le i_1<\cdots<i_k\le d$ and $1\le j_1<\cdots<j_l\le d$, we denote

$$A_{IJ}=a_{i_1}^*\cdots a_{i_k}^* a_{j_1}\cdots a_{j_l}.$$

The family of all the possible $A_{IJ}$ forms a basis of the $2^{2d}$-dimensional vector space $\mathbf{End}\,(\wedge V^*)$.

If $A \in \mathbf{End}\,(\wedge V^*)$, the supertrace $\mathbf{Str}(A)$ is the difference of the trace of $A$ on even forms minus the trace of $A$ on odd forms. If $A = \sum_{I,J} c_{IJ} A_{IJ}$, then we have

$$\mathbf{Str}(A) = (-1)^{\frac{d(d-1)}{2}} c_{\{1,\dots,d\}\{1,\dots,d\}}. \tag{A.1}$$

In this paper, $c_{\{1,\dots,d\}\{1,\dots,d\}}$ will be called the top degree Fermionic piece of $A$ and

$$[A]_d := (-1)^{\frac{d(d-1)}{2}} c_{\{1,\dots,d\}\{1,\dots,d\}} \theta_1 \wedge \cdots \wedge \theta_d$$

the $d$-form part of $A$.

### A.2 The Brownian Chen series parametrix method

For the sake of completeness and to introduce some notations used in the paper, we reproduce here the essential ideas from [2,3,7] to which we refer for further details. Let $\mathcal{E}$ be a finite-dimensional vector bundle over a compact manifold $\mathbb{M}$ equipped with a connection $D$ and consider a second-order differential operator $\mathcal{L} = D_0 + \sum_{i=1}^d D_i^2$ with $D_i = \mathcal{F}_i + D_{X_i}$ for some smooth vector fields $X_i$ and potentials $\mathcal{F}_i$ on $\mathcal{E}$. It is known that the differential equation

$$\frac{\partial \Phi}{\partial t} = \mathcal{L}\Phi, \quad \Phi(0, x) = f(x)$$

has solution

$$\Phi(t, x) = (e^{t\mathcal{L}} f)(x) = P_t f(x).$$

At strongly regular points $x_0 \in \mathbb{M}$, it is furthermore true that $P_t$ admits a smooth heat kernel

$$\begin{aligned} p_t(x_0, \cdot) \colon \mathbb{R}_{>0} &\to \Gamma(\mathbb{M}, \mathrm{Hom}(\mathcal{E})) \\ t &\mapsto p_t(x_0, \cdot) \end{aligned}$$

which is to say

$$(P_t f)(x_0) := (e^{t\mathcal{L}} f)(x_0) = \int_{\mathbb{M}} p_t(x_0, y) f(y)\, dy.$$

We have a method of approximation for the heat kernel in this setting.

**Theorem A.1** *Let $N \geq 1$ and define $(P_t^N f)(x) = \mathbb{E}(\Psi(1, x))$ where $\Psi(\tau, x)$ solves the random differential equation*

$$\frac{\partial \Psi}{\psi \tau} = \sum_{I\colon d(I) \leq N} \Lambda_I(B)_t (D_I \Psi)(\tau, x), \quad \Psi(0, x) = f(x). \tag{A.2}$$

*where $I = (i_1, \dots, i_k) \in \{0, \dots, d\}^k$ is a word, $D_I = [D_{i_1}, [\dots, [D_{i_{k-1}}, D_{i_k}] \dots]]$, $d(I) = n(I) + k$ with $n(I)$ the number of 0's in $I$, and the random coefficients are defined by*

$$\Lambda_I(B)_t = 2^{d(I)/2} \sum_{\sigma \in \mathfrak{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 \binom{k-1}{e(\sigma)}} \int_{\Delta^k[0,t]} \circ dB^{\sigma^{-1}(I)}$$

*where $(B_t)_{t \geq 0}$ is a standard Brownian motion in $\mathbb{R}^d$. Then,*

- *For $k \geq 0$, define the norm*

$$\|f\|_k = \sup_{0\leq l\leq k} \sup_{0\leq i_1,\dots,i_k} \sup_{x\in\mathbb{M}} \|D_{i_1}\cdots D_{i_l} f(x)\|.$$

*It will hold that for any $k \geq 0$*

$$\|P_t f - P_t^N f\|_k = O\left(t^{\frac{N+1}{2}}\right), \quad t \to 0$$

- *$P_t^N$ admits a smooth kernel $p_t^N$ such that for $N \geq 2$*

$$p_t(x_0, x_0) = p_t^N(x_0, x_0) + O\left(t^{\frac{N+1-Q}{2}}\right), \qquad t \to 0$$

*where $Q$ is the homogeneous dimension at $x_0$.*

- *Write $\mathcal{F}_I = D_I - D_{X_I}$. For $N \geq 2$, it holds as $t \to 0$ that*

$$p_t^N(x_0, x_0)$$
$$= d_t^N(x_0)\mathbb{E}\left(\exp\left(\sum_{I,d(I)\leq N} \Lambda_I(B)_t \mathcal{F}_I\right)(x_0)\middle| \sum_{I,d(I)\leq N} \Lambda_I(B)_t X_I(x_0) = 0\right) + O\left(t^{\frac{N+1-Q}{2}}\right)$$

*where $d_t^N(x)$ is the density at 0 of the random variable $\sum_{I,d(I)\leq N} \Lambda_I(B)_t X_I(x)$.*

We refer to Baudoin [2] and Baudoin [7, Section 5.1] for the proofs and further details, but we remark that roughly the theorem says that in small time we can approximate the heat kernel of $\mathcal{L}$ by the kernel associated with solutions of Eq. (A.2), for which we will be able to say much more.

### A.3 Curvature of the connection $\hat{\nabla}^\varepsilon$

We want to give details on writing the curvatures of $\hat{\nabla}^\varepsilon$ in terms of the Bott connection $\nabla$.

**Lemma A.2** *Relative to the notation of (2.7) we have the following identities. Recall that $i, j, k, l$ denotes vector fields from a basis of $\mathcal{H}$, while indices $r, s$ denotes such elements from a basis of $\mathcal{V}$*

(i) $R_{ijk}^l = R_{kli}^j$, $R_{r_1 s_1 r_1}^{s_2} = R_{r_2 s_2 r_1}^{s_1}$,
(ii) $R_{ijr}^s = T_{ij;r}^s$, $R_{irk}^l = 0$, $R_{is_1 r_2}^{s_2} = 0$,
(iii) $T_{ij;r}^r = 0$. *Equivalently $(\nabla_Z J)_Z = 0$ for any vector field $Z$ with values in $\mathcal{V}$.*
(iv) $\hat{R}_{ijk}^{\varepsilon,l} = R_{ijk}^l + \frac{1}{\varepsilon}\sum_{s=1}^m T_{ij}^s T_{kl}^s$.
(v) $\hat{R}_{irk}^{\varepsilon,l} = \frac{1}{\varepsilon} T_{kl;i}^s$.
(vi) $\hat{R}_{rsk}^{\varepsilon,l} = \frac{2}{\varepsilon} T_{kl;r}^s + \frac{1}{\varepsilon^2}\sum_{i=1}^n (T_{il}^r T_{ki}^s - T_{il}^s T_{ki}^r)$

***Proof*** From (2.3), we observe that

$$\hat{R}^\varepsilon(X, Y)Z = R(X, Y)Z + \frac{1}{\varepsilon}(\nabla_X J)_Y Z - \frac{1}{\varepsilon}(\nabla_Y J)_X Z + \frac{1}{\varepsilon} J_{T(X,Y)} Z + \frac{1}{\varepsilon^2}[J_X, J_Y]Z. \tag{A.3}$$

We will also use the first Bianchi identity for connections with torsion

$$\circlearrowright R(X, Y)Z = \circlearrowright (\nabla_X T)(X, Y) + \circlearrowright T(T(X, Y), Z),$$

where $\circlearrowright$ denotes the cyclic sum. We furthermore observe the following identities.

(i) Since $\langle T(Y_1, Y_2), Y_3\rangle$ and $T(T(Y_1, Y_2), Y_3)$ vanishes if $Y_1, Y_2, Y_3$ are either all vertical or all horizontal,

$$\langle R(X_1, X_2)X_3, X_4\rangle_g = \langle R(X_3, X_4)X_1, X_2\rangle_g,$$
$$\langle R(Z_1, Z_2)Z_3, Z_4\rangle_g = \langle R(Z_3, Z_4)Z_1, Z_2\rangle_g,$$

for any $X_i \in \Gamma(\mathcal{H})$, $Z_i \in \Gamma(\mathcal{V})$, $i = 1, 2, 3, 4$.

(ii) From Grong [14, Appendix A], we know that for $X_1, X_2 \in \Gamma(\mathcal{H})$, $Z_1, Z_2 \in \Gamma(\mathcal{V})$,

$$R(X_1, X_2)Z_1 = (\nabla_{Z_1}T)(X_1, X_2), \quad R(X_1, Z_1)X_2 = 0 \quad R(X_1, Z_1)Z_2 = 0.$$

(iii) Since $\nabla$ is compatible with the metric then $(\nabla_Z J)_Z = 0$ for any $Z \in \Gamma(\mathcal{V})$, as for any $X_1, X_2 \in \Gamma(\mathcal{H})$,

$$\begin{aligned} 0 = \langle Z, R(X_1, X_2)Z\rangle_g &= \langle Z, \circlearrowright R(X_1, X_2)Z\rangle_g \\ &= \langle Z, (\nabla_Z T)(X_1, X_2)\rangle_g = \langle X_2, (\nabla_Z J)_Z X_1\rangle_g. \end{aligned}$$

(iv) We observe first that from (A.3), for any $X_1, X_2, X_3, X_4 \in \Gamma(\mathcal{H})$

$$\begin{aligned} \langle \hat{R}^\varepsilon(X_1, X_2)X_3, X_4\rangle_g &= \langle R(X_1, X_2)X_3, X_4\rangle_g + \frac{1}{\varepsilon}\langle J_{T(X_1,X_2)}X_3, X_4\rangle_g \\ &\overset{\text{(i)}}{=} \langle R(X_3, X_4)X_1, X_2\rangle_g + \frac{1}{\varepsilon}\langle T(X_1, X_2), T(X_3, X_4)\rangle_g. \end{aligned}$$

(v) Next, for any $X_1, X_2 \in \Gamma(\mathcal{H})$, $Z \in \Gamma(\mathcal{V})$,

$$\hat{R}^\varepsilon(X_1, Z)X_2 \overset{\text{(ii)}}{=} \frac{1}{\varepsilon}(\nabla_{X_1} J)_Z X_2.$$

(vi) For the final property observe that

$$R(Z_1, Z_2)X_1 \overset{\text{(ii)}}{=} \circlearrowright R(Z_1, Z_2)X_1 = 0.$$

Hence,

$$\begin{aligned} \hat{R}^\varepsilon(Z_1, Z_2)X_1 &= \frac{1}{\varepsilon}(\nabla_{Z_1} J)_{Z_2}X_1 - \frac{1}{\varepsilon}(\nabla_{Z_2} J)_{Z_1}X_1 + \frac{1}{\varepsilon^2}[J_{Z_1}, J_{Z_2}]X_1 \\ &\overset{\text{(iii)}}{=} \frac{2}{\varepsilon}(\nabla_{Z_1} J)_{Z_2}X_1 + \frac{1}{\varepsilon^2}[J_{Z_1}, J_{Z_2}]X_1. \end{aligned}$$

□

## References


1. Atiyah, M.F., Bott, R.: A Lefschetz fixed point formula for elliptic complexes. I. Ann. Math. **2**(86), 374–407 (1967)
2. Baudoin, F.: A Bismut type theorem for subelliptic heat semigroups. C. R. Math. Acad. Sci. Paris **344**(12), 765–768 (2007)
3. Baudoin, F.: Brownian Chen series and Atiyah–Singer theorem. J. Funct. Anal. **254**(2), 301–317 (2008)
4. Baudoin, F., Grong, E.: Transverse Weitzenböck formulas and de Rham cohomology of totally geodesic foliations. Ann. Glob. Anal. Geom. **56**(2), 403–428 (2019)
5. Baudoin, F., Grong, E., Molino, G., Rizzi, L.: H-type foliations. arXiv preprint arXiv:1812.02563 (2018)
6. Baudoin, F., Kim, B., Wang, J.: Transverse Weitzenböck formulas and curvature dimension inequalities on Riemannian foliations with totally geodesic leaves. Commun. Anal. Geom. **24**(5), 913–937 (2016)
7. Baudoin, F.: Stochastic Taylor expansions and heat kernel asymptotics. ESAIM:PS **16**, 453–478 (2012)
8. Bellaïche, A.: The tangent space in sub-Riemannian geometry. In: Sub-Riemannian geometry, Volume 144 of Progr. Math., pp. 1–78. Birkhäuser, Basel (1996)

9. Berline, N., Getzler, E., Vergne, M.: Heat kernels and Dirac operators. Grundlehren Text Editions. Springer, Berlin (2004). Corrected reprint of the 1992 original
10. Cygan, J.: Heat kernels for class 2 nilpotent groups. Studia Math. **64**(3), 227–238 (1979)
11. Garofalo, N., Tralli, G.: Heat kernels for a class of hybrid evolution equations. arXiv e-print arXiv:2007.10862 (2020)
12. Gilkey, P.B.: Curvature and the eigenvalues of the Laplacian for elliptic complexes. Adv. Math. **10**, 344–382 (1973)
13. Gromov. M.: Carnot-Carathéodory spaces seen from within. In: Sub-Riemannian geometry, Volume 144 of *Progr. Math.*, pp. 79–323. Birkhäuser, Basel (1996)
14. Grong, E.: Affine connections and curvature in sub-Riemannian geometry. arXiv e-prints arXiv:2001.03817 (2020)
15. Grong, E., Thalmaier, A.: Stochastic completeness and gradient representations for sub-Riemannian manifolds. Potential Anal. **51**(2), 219–254 (2019)
16. McKean, H.P., Jr., Singer, I.M.: Curvature and the eigenvalues of the Laplacian. J. Differ. Geom. **1**(1), 43–69 (1967)
17. Mitchell, J.: On Carnot–Carathéodory metrics. J. Differ. Geom. **21**(1), 35–45 (1985)
18. Patodi, V.K.: An analytic proof of Riemann–Roch–Hirzebruch theorem for Kaehler manifolds. J. Differ. Geom. **5**, 251–283 (1971)
19. Rosenberg, S.: The Laplacian on a Riemannian manifold. Volume 31 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge (1997). An introduction to analysis on manifolds