Wittgenstein on the Anthropological Grounds of Mathematics

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Abstract

The significance of Ludwig Wittgenstein's later writings on mathematics has been a subject of extensive debate within the philosophical community. This thesis undertakes a critical survey of various alternative interpretations of these writings, including conventionalist, formalist, and naturalist readings. It is shown that these interpretations in different ways underappreciate or even neglect Wittgenstein's anthropological understanding of mathematics.

According to the interpretation developed in this thesis, Wittgenstein conceived of mathematics as a family of human practices which play inextricable roles in broader social and practical contexts. This conception challenges the traditional understanding of mathematics as an independent, abstract theoretical activity. The thesis examines Wittgenstein's philosophy of mathematics by studying his remarks on rule-following, numbers, calculation, and proof, presenting a cohesive picture of elementary mathematics as fundamentally based on practice.

The thesis also explores the connection between mathematics and human life by drawing on Wittgenstein's well-known concepts of 'language game' and 'form of life'. The notion of 'formal properties of language games' is introduced, which are conditions that are more or less essential to a language game in general but do not amount to strictly necessary or sufficient conditions for the game to take place. The concept of a form of life is analyzed in terms of patterns of such properties, and in particular the formal property of 'deference', a willingness to comply with prevailing procedures, is introduced and discussed as an essential feature of mathematical language games.

The exploration of the formal property of deference, in Wittgenstein's writings, serves to elucidate the structure of open-ended, iterative rules that shape mathematical practices and provides insight into the practical, human-rooted nature of mathematics. It is argued that by clarifying this formal property, Wittgenstein offers a non-Platonist way to circumvent radical conceptual relativism, identifying the source of the normativity of mathematics with the role of calculation and geometry in practice rather than with entities or facts in an abstract realm.

The thesis includes a close examination of Wittgenstein's later use of the (dis-)analogy between chess and mathematics, focusing on the way mathematical techniques

are integrated in people's lives beyond strictly pure mathematics. Mathematical propositions are understood as both expressing and reproducing rules, with both their explicit articulation and tacit enactment in diverse practical contexts constituting a part of the same tendency of deference, together defining a mathematical form of life. This dual role draws a deep parallel with the grammatical rules of indexical language.

Finally, the thesis ties the anthropological perspective on mathematics to Wittgenstein's views on the conceptually formative function of proof, along with his critical remarks on generalization, infinity, and formalization. The work engages with the prevailing literature on these topics and aims to provide a comprehensive exploration of the nature of mathematics, its foundations, and its context within human life, offering a basis for further inquiry and debate in both philosophy of mathematics and Wittgenstein studies.

Abstrakt

Ludwig Wittgensteins senere syn på matematikk har vært gjenstand for omfattende debatt innenfor filosofien. Denne avhandlingen tar for seg en kritisk gjennomgang av forskjellige alternative tolkninger av de relevante skriftene, inkludert konvensjonalistiske, formalistiske, og naturalistiske lesninger. Det argumenteres at disse tolkningene på forskjellig vis undervurderer eller overser Wittgensteins dypt antropologiske forståelse av matematikk.

Ifølge tolkningen som er formulert i denne avhandlingen, så Wittgenstein på matematikk som en familie av praksiser som er uløselig knyttet til bredere sosiale og praktiske sammenhenger. Dette står i kontrast til den tradisjonelle oppfatningen av matematikk som en uavhengig, abstrakt teoretisk disiplin. Avhandlingen går grunding inn i Wittgensteins syn på matematikk ved å se på hans tanker om regelfølging, tall, beregning og bevis, og fremstiller et helhetlig bilde av matematikk som dypt forankret i menneskelig praksis.

Videre utforsker avhandlingen sammenhengen mellom matematikk og menneskelig liv ved å støtte seg på Wittgensteins kjente begreper om 'språkspill' og 'livsform'. Begrepet 'formelle egenskaper ved språkspill' blir introdusert. Dette er forhold som er mer eller mindre essensielle for et språkspill, sett på et generelt plan, men som ikke nødvendigvis må være til stede i et hvert tilfelle for at spillet skal finne sted. Konseptet 'livsform' blir analysert som mønstre av slike egenskaper, og egenskapen 'ettergivenhet' ("*deference*") – en vilje til å rette seg etter gjeldende prosedyrer – viser seg å være helt vesentlig for matematiske språkspill.

Undersøkelsen av ettergivenhet, og rollen til dette konseptet i Wittgensteins skrifter, bidrar til å klargjøre åpne, iterative regler som del av strukturen til matematiske praksiser, og gir dermed innsikt i matematikkens praksis. Ved å klarlegge denne formelle egenskapen, finner Wittgenstein en ikke-platonistisk måte å omgå radikal konseptuell relativisme på, gjennom å koble matematikkens normativitet til praksiser fremfor abstrakte enheter eller fakta.

Avhandlingen inkluderer en grundig analyse av Wittgensteins sammenligning mellom matematikk og spillet sjakk. Fokuset her er på rollen matematikken spiller i avveielser og beslutninger, i motsetning til sjakk, noe som fremhever matematikk som

en integrert del av menneskelige samfunn snarere enn et isolert og avskåret tegnspill. For Wittgenstein forstås matematiske utsagn som både uttrykk for og gjengivelse av regler, og både deres bruk i ren matematikk og deres anvendelse i forskjellige praktiske sammenhenger utgjør del av samme tendens til ettergivenhet som definerer en matematisk livsform. Denne tosidigheten belyses av en dyp parallell med grammatikken til indeksikalsk språk.

Til slutt binder avhandlingen sammen den menneskesentrerte forståelsen av matematikk med Wittgensteins tanker om bevis, som en konseptuelt formativ prosess, samt hans kritiske bemerkninger om generalisering, uendelighet og formalisering. Arbeidet tar for seg den eksisterende litteraturen rundt disse temaene og streber etter å gi en dypere forståelse av matematikkens natur, dens fundament og dens plass i menneskelivet. Det legges grunnlag for videre forskning og diskusjon innen matematikkens filosofi og Wittgensteinstudier.

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1 Wittgenstein's anthropological view of mathematics

1.1 Introduction

In his later period, Ludwig Wittgenstein rarely wrote in a very systematic way, for better or for worse.[1] In consequence, the relationships between his philosophical remarks can be difficult to understand. One area in which this is particularly true is in the philosophy of mathematics, which Wittgenstein considered to be his chief contribution.[2] The situation is not improved by the disorganized form of the voluminous collection of notes published in *Remarks on the Foundations of Mathematics* [RFM], much of which Wittgenstein wrote alongside material that has been published as part of *Philosophical Investigations* [PI]. As indicated by the final remark of the latter (*PI PoP* xiv §372), he regarded his thoughts on mathematics to be conceptually related to other areas of his later philosophy, even if they were never compiled as a single text.

This can make it difficult to grasp Wittgenstein's later views on mathematics, to see what they entail and how they connect. The reception of *RFM* has been characterized in part by confusions about the basic intentions of the author. Some reviewers have resorted to taking Wittgenstein's writings on mathematics out of their philosophical context, forming an opinion on the basis of select passages in isolation.[3] A prominent theme in these writings has thereby been underexplored, namely the idea that "mathematics is after all an anthropological phenomenon" (*RFM*, VII-33). This dissertation seeks to remedy this and shed light on Wittgenstein's anthropological orientation in his later writings on mathematics.

It should be kept in mind, here, that the prospect of "clarifying" something Wittgenstein said is ambiguous between the project of answering *why* he said it, on the one hand, and the task of answering *how* he meant it (or *what* he intended by it), on the other. This means that there are at least two kinds of possible clarifications to be sought here. Firstly, in terms of Wittgenstein's philosophical justifications and motivations, the aim is to see *why* he thought that mathematics is to be considered an anthropological

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[1] From here on out, unless otherwise indicated, "Wittgenstein" is used to refer to the philosopher from roughly 1929 to his death in 1951. The exegetical scope of the thesis is explained below.
[2] See Monk (1990, p. 466).
[3] Or "drive-by quotation", as Floyd (2021, p. 1) puts it. Cf. Kreisel (1958, p. 136).

phenomenon. Secondly, in terms of the intended philosophical significance of that idea, the aim is to attempt to explain *in what way* he understood mathematics to be an anthropological phenomenon.

Recognizing the distinction between "why" and "how" does not imply that it is necessarily beneficial to keep these concerns separate. Answers to the two questions are by their nature mutually explicatory and for that reason will be frequently merged throughout the thesis. Still, these two questions will occasionally be addressed separately in order to take stock of what, exactly, has and has not been established on the way towards an overall clarification of his ideas. A working hypothesis of this thesis is that the *how* might be especially relevant. The implications of conceiving mathematics anthropologically are not obvious, and must be elaborated in light of Wittgenstein's writings in general. The conclusion in Chapter 6 answers both the "why" and the "how", and Wittgenstein's reasoning is evaluated in an overall way.

The focus of the dissertation is thematic, rather than strictly exegetical or historical. As a result, it does not proceed in the order in which Wittgenstein actually developed his ideas. Since it brackets parts of his philosophy of mathematics into sections according to their topic, the goal is not to produce an exhaustive overview of Wittgenstein's writings on mathematics. Instead, the aim is to explore specifically the anthropological side of his views on this subject. To that end, Wittgenstein's writings from the 1930s through the 1940s, which are most relevant to that theme, will be emphasized. The focus is, in other words, on the so-called "later Wittgenstein", bringing in remarks from the so-called "middle period" when needed to illuminate and fill out his later remarks. This is not to deny continuities in the evolution of Wittgenstein's thinking on mathematics. Several aspects of his views discussed in this thesis can be traced back, in one way or another, already to his early period. However, confines of time and space necessitate selectivity, so the majority of the thesis will be spent on exploring the theme of the anthropological nature of mathematics in its most developed and explicit form.

1.2 The anthropological reading

Wittgenstein made several statements describing mathematics as a *human* phenomenon, in the 1940s, but the precise interpretation of these remarks is a matter of dispute. For example, after a series of reflections on the limits of mathematical practice, while questioning whether a formal system used solely in "fanciful" (patently unmathematical) ways might nevertheless constitute mathematics, Wittgenstein (*RFM*, VII-32-33 (Ms-124,115; 1944)) wrote:

> *I have asked myself: if mathematics has a purely fanciful application, isn't it still mathematics? – But the question arises: don't we call it 'mathematics' only because e.g. there are transitions, bridges from the fanciful to non-fanciful applications? That is to say: should we say that people possessed a mathematics if they used calculation, operating with signs, merely for occult purposes?* (RFM, VII-32)
> *But in that case isn't it incorrect to say: the essential thing about mathematics is that it forms concepts? – For mathematics is after all an anthropological phenomenon. Thus we can recognize [concept formation] as the essential thing about a great part of mathematics (of what is called 'mathematics') and yet say that it plays no part in other regions. This insight by itself will of course have some influence on people once they learn to see mathematics in this way. Mathematics is, then, a family; but that is not to say that we shall not mind what is incorporated into it.* (RFM, VII-33)

Such statements on the anthropological nature of mathematics will in this thesis be taken as pronouncements of the general upshot of Wittgenstein's views. The interpretation will not be restricted to that level of generality, however. His remarks on mathematics will be related to remarks on language and human activity, many of which are from *PI* (and *vice versa*), ending up with direct logical relations between Wittgenstein's views on mathematics, on the one hand, and themes in his "remarks on the natural history of human beings" (*PI* §415), on the other.

The emphasis on anthropology in this thesis is consistent with most widely accepted features of Wittgenstein's philosophy of mathematics, such as the idea that mathematics is (in a sense to be explained) a human invention, that mathematics is "grammatical", and that mathematical signs presuppose extra-mathematical uses (*RFM,* I-168, III-26, and V-2). However, such ideas are here understood as aspects, or consequences, of Wittgenstein's understanding of mathematics as a family of human practices and techniques.

Framing the reading as *anthropological* is not meant to suggest that others have completely failed to acknowledge Wittgenstein's repeated appeals to the contingencies of human life and social practices in his writings on mathematics. My anthropological reading here contrasts with what might be called "weakly anthropological" interpretations, which acknowledge Wittgenstein's identification of mathematics as an anthropological phenomenon, but first and foremost see such remarks as a call for attentiveness to the diversity of mathematical techniques. In contrast, the reading presented in this dissertation will emphasize the importance of understanding mathematics in terms of its role(s) in human life.

In contrast to weakly anthropological readings, I will argue that Wittgenstein's remarks on mathematics involve appeals to anthropological *depth* (corresponding to the "*deep* need for the convention" (*RFM*, I-74)). Mathematics cannot be divorced from the practices in which it is applied, since its use is what makes mathematics meaningful. This has implications for how we should read Wittgenstein's remarks more generally, such as the idea that a mathematical proposition 'lays down a path for us' (*RFM*, IV-8) and 'constructs a conceptual path' (*RFM*, V-42). It will be maintained that Wittgenstein was here describing how mathematics is involved in the way *other* language games, aside from pure mathematics, are set up and played.

By the end of this work, the strongly anthropological (hereafter just "anthropological") reading will be argued to be not only exegetically accurate, but also to offer an interesting view independently of its interpretational accuracy. This is partly because, judged on its own merits as an account of mathematics as an anthropological phenomenon, this reading provides clarifications of basic and ubiquitous features of mathematics. In order to develop an overview of the implications of Wittgenstein's view of mathematics, some general features of mathematics are listed below. These six features, and the way Wittgenstein's characteristically anthropological point of view accounts for them, will be revisited in the final chapter:

1. *Normativity*. Mathematical practice is normally rule-bound; the (in-)correctness of any act of calculation is, in some sense, predetermined according to how it 'should' go.
2. *Significance.* Mathematical propositions are translingual and cross-historically

understandable, and any proof is in principle surveyable and reproducible.

3. *Grounding*. If a calculation or its expression in an equation is correct, it is in principle beyond dispute and incontestable, though it is neither empirically confirmable nor disconfirmable.
4. *Coherence*. Mathematical expressions are systematically relatable and interchangeable across disparate contexts, provided certain conditions are met.
5. *Applicability*. Mathematical propositions can in many cases be applied in order to form empirical descriptions and predictions.
6. *Constancy*. A mathematical problem, if solvable, is always solvable in the same way.

These serve as basic, if not completely uncontroversial, descriptions of important aspects of mathematics. Both the relevance of, and Wittgenstein's anthropological way of accounting for, the features 1-6 will be established over the course of this thesis. The point of appealing to this list is not to suggest that Wittgenstein's clarifications of these features would independently suffice to win over any skeptical reader; they would be unlikely to persuade anyone harboring strongly Platonist preconceptions, for example. Rather, the way these six basic features are accounted for, on the anthropological reading, will itself hopefully prove illuminating both with respect to Wittgenstein's writings and with respect to mathematics. The advantages of the anthropological reading, when it comes to these features, will be summed up in the conclusion.

1.3 Organization of the dissertation

The dissertation is thematically structured and divided into 6 chapters.

Chapter 1 is this introduction. It sets out the main topic of the dissertation and the goals it seeks to achieve, then delineates the structure of the text.

Chapter 2 opens with the argument of the dissertation in a nutshell, proposing that Wittgenstein held that a background of particular forms of life are a precondition for both the meaning and the possibility of mathematics. A pivotal issue is Wittgenstein's rule-following considerations, which do I take to show that behavior is understood as adhering to a rule only as part of a given surrounding. This *contextualizes* rule-following, while undermining both absolutist and relativist accounts.

The chapter continues with a discussion of conventionalist interpretations of Wittgenstein. A distinction is made between 'moderate' and 'radical' conventionalism. The former is a broad category which, in some formulations, can justifiably be attributed to Wittgenstein. However, some limits of the concept 'convention' are pointed out. A concept of 'convention' as deriving from instrumental reasoning, while potentially relevant, is neither basic nor broad enough to capture everything Wittgenstein described in terms of 'language games', including in mathematical contexts. Following this, Dummett's full-blooded conventionalism, Kripke's rule-following skepticism, and Wright's appeal to constitutive self-ascription qualify as, and are discussed as, examples of radical conventionalism. At issue in this discussion is the view that mathematical propositions are independent conventions, such that each move in the course of a proof is a new stipulation that bears no logical relation to previous stipulations. This reading is based on quotations in which Wittgenstein discusses proofs as involving innovation and seemingly freestanding normative decisions, involving *deciding* upon a new rule.

Some of Wittgenstein's formulations on the topic of proofs are suggestive of radical conventionalism. However, I argue that radical conventionalist readings struggle to account for Wittgenstein's treatment of rule-following as engagement in practices. Proofs involve decision not in the sense of the enactment of convention, but in the sense of exhibiting ways of engaging in practices. In proofs, it is not as if new rules are arbitrarily stipulated. Rules are modified and extended, and these are themselves (part of) practices, not mere stipulations. An entire background of practices *guides us* (*RFM*, IV-30) in accepting a proof.

Chapter 3 focuses on Wittgenstein's views on counting, numbers, equations, and measuring. I argue that, for the later Wittgenstein, there is an *internal* relation between a given number system, a method of counting, and a form of measurement. These are connected as part of the same (family of) language game(s). There is no absolute distinction between the act of transitive and intransitive counting, i.e. between acts of counting objects and acts of counting *simpliciter*, which, I will argue, shows that the distinction between applied and pure mathematics is contextual. That is, the difference between applied and pure mathematics is not syntactically absolute. Sign for equations can be used both as empirical descriptions or predictions *and* as parts of a calculus, though it is important to distinguish between these two roles.

Wittgenstein drew a distinction between ordinary counting or tallying and counting in mathematics (*LFM*, XII, p. 114). For an example of the latter, counting the number of parameters in a formula is not merely a tally of objects, but a kind of prescription of a method of counting. If 3 parameters are counted in an equation, and the equation is applied, it now describes a physical system as determined by 3 different parameters. If 2 are counted, the physical system has 2 parameters, etc. Counting in mathematics is not restricted to scientific modeling, however. It informs various distinctions and procedures in ordinary language. Quantities and shapes are taken in at a glance *as* instances of mathematical paradigms, these paradigms being taught and treated as fixtures in our language games. Mathematics 'teaches how to count' (*RFM*, VII-18), shaping how things are enumerated also outside mathematics.

Chapter 3 continues by considering the later Wittgenstein's views on equations. As he highlighted in lectures and writings from the late 1930s and the 1940s, a sentence like "2 apples added to 2 apples is 4 apples" can be used strictly as part of a calculus, replaceable by "2 + 2 = 4". In that case it is not *about* apples at all. It could instead be said to be 'about numbers', but not in the same way that a description of apples is 'about apples'. The word "is", when used in calculating contexts, should for Wittgenstein not be taken as a copula or sign of identity, but as a sign for mathematical equality. This means that the two symbols flanking the "is" are given the same roles in their respective language games, such as when counting. The equation "2 + 2 = 4" tells us that counting to 2 twice is, as an action, intersubstitutable with counting to 4. Mathematical equality is therefore a grammatical notion, in Wittgenstein's broad sense, pertaining to the way expressions are used. Analogously with chess, we could place the equality sign between a physical rook-piece and the sign "R", which in algebraic chess notation denotes the rook piece. These have the same role in their respective language games.

Chapter 4 treats the topic of the function of mathematics within human life. A major theme here is the concept of 'deference', which I introduce as a way of framing what Wittgenstein writes about the role of mathematics in everyday situations. People use mathematics to determine how to complete specific tasks; they do not insist on their own experiences or their idiosyncratic results when these deviate from pure mathematical practice. In particular, when applying mathematics, people correct their calculations or geometric constructions to align with pure mathematics, not the other way around, and moreover teachers and adults instill this attitude in pupils and children whenever mathematics is taught. This should be approached as an important but contingent fact about the relation between applied and pure mathematics in the lives of people who have been brought up in broadly speaking ‘mathematical’ societies.

Chapter 4 continues by outlining the notion of ‘formal properties’ of language games. That concept is introduced as a way of clarifying the relationship between language games and forms of life, while staying consistent with Wittgenstein’s writings on family resemblance (*PI* §67) and the heterogeneity of linguistic structures (*PI* §108). The formal properties of a language game hold, *in general*, when the game is played, but these properties do not form disjointed sets, nor are they jointly necessary and sufficient for the game to take place. For instance, while football involves flat terrain, people can also play football on uneven terrain. Similarly, people can play chess with the rules for *en passant* in play despite never making that move. So, a game is identified as an *imperfectly* recurring pattern of features. As Wittgenstein (*PI* §§48-77) makes clear, concepts vary in specificity, similarly to how games differ in their level of complexity, that is, in the specificity of conditions that must hold for us to recognize them. The chapter proceeds by analyzing ‘form of life’ as a pattern of formal properties, a pattern of ways in which language games are played. Since forms of life are connected with language games while not being identical with them, Wittgenstein’s philosophy has room for the possibility that the same forms of mathematics can be used across different forms of life.

Chapter 5 builds on the preceding discussions by first distinguishing between two views which Wittgenstein opposed. First, the view of mathematics as an ethereal machine which operates according to its own independent principles; second, the view of mathematics as a repository of epistemically useful knowledge. Wittgenstein’s anthropological perspective differs from both of these perspectives, but it also goes some way towards harmonizing their respective insights. To argue for this, I bring in the concepts ‘intermediate link’ (*PI* §122), ‘direct affinity’, and ‘indirect affinity’ (*PI* §65, §76) and analyze them in terms of practices, which in turn are connected to ‘proof’.

For the later Wittgenstein, a proof is a mathematical procedure presented in the form of a reproducible picture. In order for a picture to serve as a proof, the elements of the picture must already play recognizable roles in persistent practices, which is what makes the production of the picture, i.e. the proof, a significant result.

The chapter rounds out by discussing the implications of this in light of the broader literature. On the anthropological interpretation, calculi, as a family of rule-following activities, are distinguished from other language games by the fact that they operate with pictures that serve as *paradigms* of action in those activities. In other words, the recognizable features of a calculus are products of calculation within that calculus. Even though a proof is fundamentally a procedure, the pattern left by the procedure is accepted as *the* sign of the procedure being completed. The acceptance of a proof is thus the normative establishment of a paradigm, which is to say that the procedure exhibited by the proof goes on to be repeated in mathematical practice. Rather than *contingently* getting 625 when multiplying 25 by 25, as an isolated event, the process of multiplying 25 by 25 *and thereby getting 625* is incorporated as a picture in and of the calculus as such, the practice expanding. For Wittgenstein, this makes it vital that we operate with clearly delineated pictures so as to maintain an adequate understanding of what is actually done in mathematical practice.

Chapter 6 revisits the list of six basic features of mathematics previously outlined. These features are clarified in terms of themes in Wittgenstein's philosophy that were discussed over the preceding chapters. By clarifying these features, we get a comprehensive picture of what is entailed by Wittgenstein's anthropological view of mathematics, understanding mathematics as a family of rule-based practices which are integral to human forms of life.

2 Interpreting Wittgenstein's philosophy of mathematics

> *In order to understand the nature of mathematics, one must not look out of its window, but into it from the outside.*
> (Wittgenstein, 16.11.1940)[4]

This chapter seeks to explore the significance of Wittgenstein's claim that mathematics is an anthropological phenomenon. The implications of this assertion could range from a mere acknowledgement of human involvement in mathematical practice to positing a more encompassing connection between mathematics and specific aspects of human forms of life. The goal of this chapter is to argue that Wittgenstein developed a strongly anthropological perspective, connecting mathematics to particular characteristics of human society.

Mathematics is often considered infallible and devoid of contingencies, making it seem as if nothing external could match its epistemological status or contribute to accounting for its methods. As a result, one might think that mathematics could only be understood through the use of resources internal to mathematics, such as mathematical logic, set theory, or category theory. Wittgenstein opposed this path, as evidenced by the opening quote of this chapter (Ms-123,17v-18r), likening it to looking *out* of the mathematical window rather than into it.

In both his early and later works, Wittgenstein focused on fundamental mathematical techniques, exploring the role of mathematics in human life and its relation to ordinary language, practice, and history. In the *Tractatus*, he developed a definition of cardinal numbers in terms of operations and focused on arithmetical equations. In his later works, he similarly focused on the kinds of mathematics people use in everyday practical calculations. This should be taken as an indication of the nature and goal of his investigations; Wittgenstein's main motivation appears to have been to address philosophical problems pertaining to the role of mathematics within human life. As is recorded from one of his lectures in 1939:

> *I can as a philosopher talk about mathematics because I will only deal with puzzles which arise from the words of our ordinary everyday language, such as 'proof', 'number', 'series', 'order', etc. Knowing*

[4]My translation from Wittgenstein's German original (Ms-123,17v-18r): "*Wer das Wesen der Mathematik verstehen will, muß nicht aus ihrem Fenster heraus, sondern von außen hinein schauen.*"

> *our everyday language – this is one reason why I can talk about them.* (*LFM*, I, p. 14)

The approach of drawing on special techniques within mathematics would, for Wittgenstein, come with the consequence of shrinking the scope of discussion, changing the subject matter from the overall role of mathematics in human life to that of structural relationships *within* mathematics. His philosophical investigations are not primarily concerned with technicalities, and he accordingly preferred using simple examples and illustrations. As in other areas of philosophy, his efforts were not intended to lead to new discoveries, but to remind the reader of considerations which tend to fall out of focus precisely due to their prevalence:

> *The aspects of things that are most important for us are hidden because of their simplicity and familiarity. (One is unable to notice something – because it is always before one's eyes.)* (*PI* §129)

Wittgenstein's perspective should also be sharply distinguished from reductively empiricist philosophies of mathematics. These approaches explain away the apparent phenomenon of *a priori* mathematical knowledge by appealing to empirical facts, such as an unwillingness to alter or let go of certain assertions. If everything humans take themselves to know constitutes a uniform 'web of belief' (cf. Quine & Ullian, 1970), no node in this web can be guaranteed to be correct with *a priori* certainty. One advantage of such a reductive approach is that it ensures consistency among various forms of knowledge-seeking activities and avoids ascribing to humans any super-empirical epistemic abilities. However, by reducing 'knowledge' to a single role and function, it risks doing injustice to essential differences among our practices.

The chapter commences with an exploration of core aspects of Wittgenstein's philosophy, particularly rule-following, language games, and forms of life. These concepts play a pivotal role in his understanding of human life, and they also constitute an essential part of his philosophy of mathematics. Throughout the thesis, these concepts will be examined in greater detail. In the latter half of this chapter, the anthropological interpretation will be contrasted with other exegetical approaches. In particular, it will be argued that Wittgenstein rejected conventionalist, formalist, and naturalist philosophies of mathematics, formulating a unique perspective which understands mathematics through its role in language games.

2.1 Anthropology

Wittgenstein's philosophy is characterized by a distinct anthropological dimension. This dimension is most explicitly demonstrated in his *Remarks on Frazer's Golden Bough*, written in 1931. In these remarks, Wittgenstein emphasized the indispensable role of rituals in human life, viewing rituals as actions that are not adequately explained in terms of practical function. Rituals can come in many forms, and their significance is best conveyed through comparisons, analogies, and images, rather than through causal theories.[5] For instance, a handshake may be given causal explanations. However, in Wittgenstein's view, it would be better understood as a ritual symbolizing respect or politeness, regardless of its causal origins and effects.

In general, Wittgenstein's philosophical approach is 'anthropological' in that it relates to human phenomena, focusing on how people use words and the roles these words play in their lives. Something is deemed 'anthropological' in this sense if it relies, either wholly or partially, on human interaction; without humans, it could not have existed, occurred, or made sense. For Wittgenstein, it is the latter possibility, 'would not have made sense', that is most relevant. The *meaningfulness* of concepts, including mathematical concepts, hinges on human activity. Therefore, according to Wittgenstein, seeing mathematical activity in the right light is crucial for resolving traditional philosophical problems related to mathematics.

A suitable object of comparison here is a system of currency. Currency functions as money only insofar as it is imbued with value, though its value is not necessarily a product of conscious decision-making. Analogous to language and mathematics, the 'value' of currency is linked to its usage (cf. *PI* §120, §268). However, unlike currency, the value of which can fluctuate from day to day, mathematics remains unaffected by our daily affairs. Mathematics encompasses more fundamental features of a form of life, in comparison to the use of any currency. Consequently, Wittgenstein's philosophy of

[5] See Child (2011, pp. 229-230) and Glock (1996, pp. 35-36).

mathematics appeals to ‘deeper’ anthropological considerations, features of human life that are not subject to fluctuations.

2.1.1 Forms of life

One of the building blocks of Wittgenstein's understanding of human life and his anthropological approach to philosophy is the concept of ‘form of life’ (‘*Lebensform*’), which first appears in his notes from 1936. As Peter Hacker (2015, p. 13) argues, with the phrase ‘form of life’, Wittgenstein aimed to emphasize the integration of language into “entrenched practices that are not called into doubt”. The concept thus signifies fundamental behavioral patterns, dispositions, and activities. Hacker concludes that *Homo sapiens*, as such, does not constitute a form of life in Wittgenstein’s sense. Even if one can speak of a ‘human form of life’, this should not be considered a biological notion, since forms of life include cultural practices that are historically contingent and specific to particular communities and individuals.

However, it should be noted that ‘form of life’ has etymological links to biology and was employed in the natural sciences as well as the humanities during Wittgenstein's time, as Hacker (2015, p. 2) acknowledges. Moreover, Wittgenstein equated his writings with remarks on “the natural history of human beings” (*PI* §415, cf. *PI PoP* §365). His conception of ‘natural history’ was in part cultural, as is evident from *PI* §25: “Giving orders, asking questions, telling stories, having a chat, are as much a part of our natural history as walking, eating, drinking, playing.” Here, rather than prioritizing ‘cultural’ over ‘natural’ behaviors, Wittgenstein effectively drew a parallel between them. This does not imply that the concept of ‘form of life’ is primarily either cultural or biological, but suggests that Wittgenstein appreciated its ambiguity, allowing it to straddle both culture and nature. With this in mind, the concept should be read as *multivocal*: it covers both cultural and natural characteristics of human beings. ‘Form of life’ will be discussed in greater depth in Chapter 4 (page x), where Wittgenstein’s conception of the intrinsic connection between mathematics and form of life is explored further.

One objection that could be raised against such a multivocal reading of ‘form of life’ is based on a focus on Wittgenstein’s remarks on rules and interpretation.

According to Kripke's (1982) exposition of the rule-following paradox, the phenomenon of (correctly) following a rule as such cannot be illuminated by reference to facts about humans. Empirical facts do not determine what is involved in following rules. Instead, rules are a matter of sheer intersubjective correction, a matter of "the conditions when a move […] is to be made in the 'language game'" (Kripke, 1982, p. 74). From this perspective, given that Wittgenstein thought of mathematics in terms of language games, the empirical facts studied by biologists should be seen as irrelevant to any attempt to describe the role of mathematics in a form of life.

To address this objection, Wittgenstein's view that mathematics is a matter of practice must be distinguished from the 'skeptical solution' that Kripke presents to the rule-following paradox as he reads it. Wittgenstein did not conceive of rules merely in terms of tendencies of intersubjective correction; rather, he situated rules *within* language games (*PI* §54). By adopting a non-reductive reading that acknowledges the different roles of rules in practice, it is possible to sidestep rule-following skepticism while still affirming the fundamentally human nature of mathematics. This approach meets Wright's (2001, p. 54) challenge of rationally elaborating "the constructivist imagery which is so prominent in the *Remarks on the Foundations of Mathematics*" without resorting to radical conventionalism; the response to radical conventionalism is elaborated in section 2.2.2 (p. 43). In summary, Wittgenstein's writings allow for a multivocal interpretation of 'form of life', acknowledging both natural and cultural dimensions in the meaning of the term. This multivocal interpretation not only addresses objections and counterarguments to Wittgenstein's position, but also effectively highlights the human nature of mathematics.

Language games and the normativity of meaning

The word "normativity" has several uses. The normativity of *meaning* is the idea that meaningful language accords with pre-established standards.[6] Entirely idiosyncratic uses of words result in incoherent solecisms. A similar form of normativity holds for mathematics. Calculations conform to established standards of reasoning; randomly

[6] See Glock (2019) and Glüer et al. (2022) for discussions of the normativity of meaning.

connected symbols are incomprehensible.[7] Wittgenstein's perspective on the normativity of meaning underwent significant changes from 1929 through the 1930s. As Jaakko Hintikka (1989, p. 284, cf. 1996, pp. 209-232) argues, the concept of 'language game' can be interpreted as Wittgenstein's attempt to understand the normativity of language and mathematics in light of the fact that the rules for any word or symbol extend beyond what is expressed in any single instance.

One example of the kind of question occupying Wittgenstein in the early 1930s was the following: If someone asserts $\neg p$, what makes it the case that asserting $\neg\neg p$ would be equivalent to affirming p?[8] The rule of double negation elimination is not contained in the assertion of $\neg p$ by itself. Wittgenstein rejected the assumption that individuals mentally anticipate any possible negation of $\neg p$. Instead, he drew a comparison with playing a game (*TBT*, p. 125). When playing a game, the rules need not be apparent in every move. For example, one can play a game of chess without ever applying the *en passant* rule, while that rule is in force nonetheless.

Wittgenstein thus articulated a methodological tool, 'language game' ('*Sprachspiel*'), that served as a way of understanding commitments to meaning and rule-following through the role of words and rules in recurring, recognizable practical contexts. Although Wittgenstein stated that language games are meant as mere 'objects of comparison' (*PI* §§130-131, cf. *TBT*, p. 156), and it is important to keep in mind that he was not positing the existence of a special kind of object or social structure, this was not a philosophically inconsequential move.[9] After introducing the expression "language game" in the *Investigations*, Wittgenstein clarified that the notion is meant to "bring into prominence the fact that the speaking of language is part of an activity, or a form of life" (*PI* §23). In other words, the use of this phrase was not motivated by the voluntariness and frivolity that might otherwise be associated with games. Rather,

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[7] As Wittgenstein put it in *RFM*, VII-61: "What I am saying comes to this, that mathematics is normative. But 'norm' does not mean the same thing as 'ideal.'"

[8] See *TBT*, pp. 124-126 and cf. *PI* §§556-557.

[9] Wittgenstein added a methodological comment in *TBT*, p. 156, having in mind a comparative use early on: "When I describe certain simple language-games, I don't do this so I can use them to construct gradually the process of a fully developed language – or of thinking – (Nicod, Russell), for this only results in injustices. – Rather, I present the games as games and allow them to shine their illuminating effects on particular problems."

language games highlight the role of language use in specific *forms* of human activity. They link the normativity of meaning to specific anthropological environments.[10]

Juliet Floyd (2016) and Chon Tejedor (2015) argue that there is a continuity between the notion of logical form in Wittgenstein's *Tractatus* and his later references to forms of life. A certain kind of continuity can be discerned in the idea that the normativity of meaning is illustrated by language games. If the normativity of meaning is captured by the analogy of language games, then language games are themselves at most implicitly related. Although we can describe relations between language games, doing so effectively involves positing a more encompassing language game that subsumes these interactions. Strictly speaking, there are only tacit similarities and analogies, and in that sense *formal* relations, between language games.

This has implications for how Wittgenstein conceived of the relationship between forms of life and mathematics. Although the enduring relevance of 'form' speaks to a certain kind of continuity, Wittgenstein's later approach is dynamic, in contrast to the static treatment of logical form in *TLP*. Language games change and overlap with one another in various ways (*PI* §23). Such interactions can be rendered explicit by describing the rules of overarching language games, just as these rules can be seen as formal relations between games. Rather than logical analysis, Wittgenstein's descriptive approach (*PI* §109, §122) aims at perspicuity in anthropological form, finding the normativity of meaning in the tapestry of life.

Family resemblance and open-endedness

According to a traditional, essentialist current of thought, any two uses of a word must have something in common in order for the word to convey a consistent meaning; both uses of the term must accord with an explicit standard or definition. Wittgenstein (*PI* §66-70) took aim at this view by considering the word "game" itself and reflecting on what its uses might have in common. He argued that there need not be a commonality between any two uses of the word. Instead, each use may overlap in some way with other uses, creating a network of affinities. The uses of "game" differ while being part

[10] Cf. Wilson's (2006, pp. 171-177) description of the "distributed normativity" of linguistic procedures, which he relates to Wittgenstein. However, he interprets language games as "restrictive structures" (ibid. p. 279).

of a common pattern, similar to how the members of a family differ from one another yet resemble each other in various respects.

One retort to Wittgenstein's remarks on family resemblances is that games must, at the very least, be *rule-bound*. After all, in order for there to be a game, there must be a set of rules that all its players follow. In this vein, Suits (1978, p. 34) proposes the following definition:

> *[T]o play a* game *is to engage in activity directed towards bringing about a specific state of affairs, using only means permitted by rules, where the rules prohibit more efficient in favor of less efficient means, and where the rules are accepted just because they make possible such activity.*

Though this describes important features of many kinds of games, Wittgenstein (*PI* §83-84) would reject the idea that it serves to *define* "game", insofar as it requires the rules to be readily specifiable. Examples of ludic flexibility and open-endedness show that games need not be rule-bound. Children can and do make up games on the fly without ever stipulating conditions of participation or victory and defeat. Such open-ended games *could* very well be standardized, making participation conditional on adherence to a set of explicit regulations, but a potential characteristic of a game is not the same as an actual characteristic of it.

From this it follows that rules, explicit regulations as opposed to tacit norms, do not reach the same fundamental level as language games, for Wittgenstein. Recognizable forms of human activity are a precondition for explicitly standardized rules, not the other way around. As Wittgenstein put it in *The Blue Book*, his newfound conception of language games entailed that "[t]he rule which has been taught and is subsequently applied interests us only so far as it is involved in the application. A rule, so far as it interests us, does not act at a distance" (*BBB*, p. 14). This does not mean that a rule is reducible to a specific set of applications or instantiations, but that rules are not abstract entities. The normative significance of a rule is understood by reference to the way it is generally adhered to or invoked in actual practice.[11]

[11] As Wittgenstein puts it in *OC* §139: "Our rules leave loop-holes open, and the practice has to speak for itself." Cf. Johannessen (1988) on Wittgenstein on 'practice'. Note, also, that *PI* §7 implies that 'language game' *does* serve as a metaphor for a practice, specifically highlighting meaningful, repeatable (inter-)action.

By adopting an anthropologically descriptive approach, Wittgenstein came to recognize that rules have various roles in practice. Some rules are regulatory, like traffic laws, while some are merely advisory, like recipes. Some rules characterize a kind of move in a language game, such as the rules for the iteration of arithmetic progressions. Other rules, such as *en passant*, are part of the stage-setting of a game, and are generally in force insofar as the game is played.[12] There are gradations, or border-cases, and crisscrossing similarities. Comparing different language games can be compared to tracing the lines constituting the Necker cube (*TLP* 5.5423) or the duck-rabbit (*PI PoP*, §118). The lines in these illustrations are open to being seen as part of distinct figures, different morphologies, similarly to how a rule can be taken as part of different language games and a mathematical technique often can be used in multiple ways.

Despite their diversity, language games can be clearly described. As Wittgenstein (*PI* §54) wrote, there are characteristic signs in the behavior of players whenever they make a mistake in a language game, such as when they make a slip of the tongue or confuse a move in one game for a move in another. Fundamental confusions rarely occur due to the connection of language games with its readily recognizable surroundings. We are generally able to see what rules people are following by discerning the role of the activity in which they are engaged, the 'point' of the language game that they are playing (*PI* §142; *RFM*, I, Appx. I-18-20).

To conclude this section, Wittgenstein's use of "form of life" is connected with his more frequently used concept, 'language game', through the notion of 'family resemblance'. The 'family'-metaphor is not only *metaphorically* genealogical but, as comes out from the elaboration in terms of intertwining threads (*PI* §67), literally genealogical. Language games intersect and interrelate, constituting dynamic elements of a form of life. Wittgenstein (*PI* §14, §23, §421) compared linguistic expressions to the various tools in a toolbox; a given form of life is the pattern of disparate activities in which these tools are embedded. Hence, according to the reading that will be developed further in Chapter 4 of this thesis, a parallel could be drawn between the

[12] This latter kind comprises what Searle (1969, p. 33, 1995) calls 'constitutive' rules; cf. Glüer & Pagin (1999, pp. 221-222) and Hindriks (2009). Wittgenstein (*PI* §§31, 49, 257) compared preparing a language game to setting up chess.

concept of ‘form of life’ and Durkheim’s concept of ‘organic society’ as opposed to ‘mechanical society’.[13] That is to say, a form of life is constituted by a multitude of interweaving language games which operate in different ways.

2.1.2 Mathematics as a family of techniques

Having made the concept ‘language game’ central to his philosophical approach, Wittgenstein broadened his view of mathematics. According to his later writings, mathematics is a “mixture of proof techniques” (*RFM*, III-46, III-48),[14] mathematics is “a family of activities with a family of purposes” (*RFM*, V-15, VII-33), and mathematics “forms concepts” (*RFM*, IV-29). The acknowledgment that Wittgenstein used the ‘family’-metaphor to indicate a genealogical relation among language games helps avoid the impression that he took mathematics to consist in a set of unrelated and haphazardly generated (because *autopoietic*) language games.

For example, in *RFM* IV-23, Wittgenstein described the process of recognizing and accepting a mathematical proof as follows: “‘We decide on a new language game.’ ‘We decide *spontaneously*’” (though note the use of quotation marks). Misinterpreting these statements may lead some to think that Wittgenstein grossly underestimated the consistency and structure of mathematics. They might argue that he failed to see the meaningful links between different mathematical fields as well as the relation between mathematics and its practical applications.

On the contrary, Wittgenstein (*PI* §122) aimed to create perspicuous overviews of conceptual connections, and with that came a recognition of the interconnectedness of mathematics. Any given piece of mathematics derives, historically and practically if not logically, from one or several branches of mathematics. Mathematical methods often involve transforming one concept into another, such as fractions which can interchangeably be used as decimals, percentages, or ratios. For Wittgenstein, the applicability of mathematics comes precisely from such conceptual connections. As he

[13] See Durkheim (1893). Roughly, an *organic* society is constituted by members having particular relationships with one another, such as a specific division of labor, while potentially differing in individual properties and beliefs. A *mechanical* society, by contrast, is characterized by shared properties and beliefs.

[14] In the 3rd edition of *RFM*, this is translated as “motley of techniques”. However, Wittgenstein’s original wording reads: “*Ich möchte sagen: Die Mathematik ist ein buntes Gemisch von Beweistechniken. – Und darauf beruht ihre mannigfache Anwendbarkeit & ihre Wichtigkeit,*” or, in English: “I would like to say: Mathematics is a colorful mixture of proof techniques. – And on this rests its manifold applicability & its importance.”

put it in the *Tractatus* (6.211):

> *In life it is never a mathematical proposition which we need, but we use mathematical propositions* only *in order to infer from propositions which do not belong to mathematics to others which equally do not belong to mathematics.*

In the *Tractatus*, Wittgenstein suggested that we calculate, not in order to arrive at an equation or mathematical proposition in and of itself, but in order to clarify the form of our *empirical* propositions. If you owe each of your friends $15, and you have 7 friends, the calculation 15 × 7 = 105 reminds you that you can infer that you owe your friends $105 in total. Even if that equation is explicitly written down, it is not used as a *premise* of a deductive inference, but as a guide and record of the process of inference (see Kremer, 2002, p. 293). In his later writings, Wittgenstein greatly expanded upon this understanding of the applicability of mathematics.

Stage-setting and bridging language games

Wittgenstein's 'family' metaphor (*RFM*, V-15, VII-33) likens mathematics to grammar, emphasizing that it has various roles in language games. This comparison aligns with his assertion (*RFM*, I-128) that any connection that is so 'rigid' that the one thing somehow already *is* the other, which includes the numerical relationships expressed in equations, are grammatical in nature. In this light, mathematics is seen as a formative element, intricately woven into the fabric of our language games and, by extension, our forms of life, lending credence to the anthropological perspective of Wittgenstein's philosophy of mathematics.

The idea that mathematics is analogous to grammar implies that the rules of mathematics are involved in the stage-setting of language games. That is, mathematics not only fulfils a special function within given practices, but is also employed to set up and coordinate practices. To see this, imagine that mathematical techniques were confined to individual language games. In this case, any activity involving numerals would necessitate a unique rule for every operation. For example, a game like *Monopoly*, which inherently involves arithmetic, would need explicit rules for each calculation, such as a rule stating that 2 + 3 equals 5.[15] In actuality, a game like *Monopoly presupposes* mathematical practices (cf. *RFM*, VI-32).

It could be argued that a pastime game such as *Monopoly* is relatively artificial. Games played for entertainment simply presuppose the use of more fundamental language games such as elementary arithmetic. An analogy could be drawn to how a player in a football match simultaneously abides by the laws of his or her society. While this is a valid observation, it is also true that mathematical techniques are used to modify or shift from one language game into another. Suppose someone needs to attend an event which begins at 18:00 and knows it takes 1.5 hours to reach there, while the current time is 16:30. The conclusion is to leave immediately. This decision tacitly incorporates the application of arithmetic to seamlessly move between language games of time-keeping, measuring duration, and decision-making.

In this context, arithmetic serves as a silent aid, facilitating transitions between language games without requiring explicit verbalization. Although arriving at the decision to leave immediately is an inference, it is not a *formal* inference requiring a chain of propositions in which each step is shown to follow from previous steps. If pressed or in need of further confidence, the person might apply modulo 60 arithmetic and calculate 16:30 + 1.5 = 18:00. This equation is then a *justification* for drawing the conclusion (*RFM*, I-6, I-17). That is, the equation 16:30 + 1.5 = 18:00 reminds us of what it makes sense to do given a plan for an event to start at 18:00, the current time being 16:30, and the prediction that travel takes 1.5 hours.

This underscores the stage-setting role of arithmetic in our practices, its role in structuring and facilitating language games. Without a mathematical underpinning it would be difficult to conceptualize not only artificial games like *Monopoly* but also prevalent everyday practices such as time-keeping and planning. If we were to imagine our forms of life without mathematics, we would at the very least have to imagine very different practices of measuring and keeping time, making it questionable whether we would still be describing the same forms of life. This underlines Wittgenstein's view

[15] The rules would have to specifically prescribe instances of substituting '2' and '3' for '5', which would include stating the conditions for when two numbers (here 2 and 3) are licitly 'added together'. Alternatively, it would require the contingent fact that a specific, unrelated language game (arithmetic with natural numbers) was played simultaneously with this quasi-*Monopoly*. More on this below and in section 2.2.

that mathematics is not an isolated system but an integral part of our forms of life. The practically fundamental role of mathematics is highlighted by the anthropological reading as core to his writings on mathematics.

Concepts and calculations

While taking mathematics to have a grammatical role in setting up and facilitating transitions between language games, Wittgenstein also described ‘mathematical language games’ as such (e.g., *RFM*, II-27, V-42, VI-25). This does not strictly refer to pure mathematics but includes practices involving applied mathematics. However, had the applications of mathematics been limited to stage-setting, or to facilitating transitions between language games, it would be a misnomer to speak of ‘mathematical language games’ *tout court*. Examining the way Wittgenstein used the expression “language game” when it comes to mathematical practices, such as “[s]olving a problem in applied arithmetic” listed in *PI* §23, reveals that he had a dynamic view of the relationship between pure and applied mathematics.

The signs used in pure mathematics have their roots in ordinary language, as far as Wittgenstein is concerned.[16] We use mathematical concepts in everyday discourse. For example, the sentence “the room contains eight tables, each with two chairs” involves an application of mathematics in the sense that it uses cardinal numbers and alludes to the relationship ‘x with y each’, which is equivalent to ‘$x \times y$’. The sentence can be paraphrased as “the room contains eight tables and sixteen chairs”, a transformation which is itself an application of the mathematical equation $8 \times 2 = 16$. This transformation is another type of application, enabled by the use of mathematical concepts in the original sentence. In other words, we can distinguish between at least two types of mathematical application:[17]

1. Applying a mathematical concept in an utterance (*conceptual* application)

[16] See *RFM*, V-2, which is discussed more closely in section 2.3 (p. 51). See also Fogelin (2009, pp. 90-95) on the importance of the adjectival use of numerals for Wittgenstein.

[17] The distinction drawn here is unique, and pertains to practice rather than syntax or semantics. Cf. Steiner (2002, p. 16) on mathematical applicability. It is common to distinguish between “mixed” and “pure” mathematical contexts on a syntactical or semantic basis.

2. Applying a formula to transition from one utterance to another (*mediative* application)

The statement "the room contains eight tables, each with two chairs" involves the conceptual application of mathematical vocabulary, and is local to a given language game. However, a painter might use this statement to infer "I need paint for 16 pieces of furniture", and this would be a mediative application that is not confined to a single language game; it transitions from taking an inventory to preparing a shopping list, guided by the calculation $8 \times 2 = 16$.

Wittgenstein (*PI* §23) included "[s]olving a problem in practical arithmetic" in his list of language games, meant to illustrate their multiplicity.[18] This aptly describes the inventorying of furniture described above. The painting scenario would potentially involve more than one language game, depending on how the activities are demarcated. Either way, the use of practical arithmetic involves participation in at least one form of linguistic activity, which coheres with, and underscores, Wittgenstein's overall point about the diversity of language games.

Wittgenstein frequently wrote about pure mathematics in terms of 'calculi' ('*Kalkül*' and '*Rechnungsarten*') and systems of calculation. By using these expressions he did not have in mind detached formal systems. Rather, he meant rule-governed activity in a broader sense. That is clear, for instance, from *RFM*, VII-24, where Wittgenstein argued that the characteristic surroundings of the activity of calculation are essential to what we recognize as 'mathematics', adding that the language game in which calculation occurs determines the 'meaning' of the calculation. This being so, calculi can be conceptualized as a subset of language games, which coheres with Kuusela's (2019, pp. 176-177) interpretation:

> *Given that every calculus can be understood as a game according to rules, but not every language-game as a calculus, any calculus can be characterized as a language-game, but not vice versa. In this sense the notion of a language-game is broader than that of a calculus, and*

[18] Wittgenstein focused on what Hacking (2011, p. 157) describes as "common-or-garden maths, including the arithmetic used by carpenters and shop-keepers". 'Applied mathematics' in this sense does not allude to a special field of mathematics or physics, but the use of mathematics for an extramathematical purpose.

Wittgenstein's method can be characterized as extending logic beyond calculus-based approaches.

Calculi, in a broad sense, are rule-governed activities where actions are taken in accordance with the rules inherent to that practice. Already in 1931, Wittgenstein (*WVC*, p. 171) described scheduling events through a diary as a calculus, aligning with his later view that rules are not abstract entities that "act at a distance" but require actual enactment or enforcement (*PI* §202; *BBB*, p. 14). By using a mathematical calculus, we are not merely utilizing an independently existing tool, but we are actively reproducing the rules of that calculus through our calculations.[19] Subsequent chapters will delve deeper into Wittgenstein's writings on calculi, showing how calculations allow transitions from one language game to another while still being part of language, as opposed to being 'meta-linguistic' (cf. Shanker, 1987, p. 46).

From especially his 1939 lectures onwards (e.g. *LFM*, XII, pp. 113-114, XV, p. 151), Wittgenstein emphasized that mathematical applications are *contextual* rather than strictly syntactic or semantic. Formulae which are part of pure mathematical calculi, such as '2 + 2 = 4' and 'the square root of 49 is 7', need not exclusively be used for calculation; they can alternatively serve as descriptions or predictions for empirical purposes. Whether they are used in one way or the other may not be apparent and depends on the circumstances. Philosophical confusion arises when the two uses of formulae are conflated with one another, making it seem as if a calculus describes an independent reality (*LFM*, XXVI, p. 42; cf. Conant, 1997, p. 219).

Nevertheless, the distinction between a calculus and the use of concepts *outside* that calculus (that is, outside any process of calculation) is dynamic. There is an interaction between the role of mathematics in enabling quantitative and geometric forms of description, on the one hand, and in informing various quantitative and geometric empirical descriptions, on the other. All in all, mathematics plays a role in our forms of life that includes both the facilitation of language games and the making of moves within those language games.

[19] As Wittgenstein put it, "[i]n mathematics the result itself is also a criterion for correct calculation" (*RFM*, VII-7).

Rule-following and anthropological depth

In addition to the stage-setting role and the interaction between mathematics and ordinary language, Wittgenstein's focus on rules and how they are followed also serves to highlight the fundamentally anthropological nature of mathematics. For Wittgenstein, given that rules have to be understood as part of practices, it is an important fact that there are practices which play an essential role in establishing and delineating how rules are followed. One example of that is education and training, in which instructors convey rules and concepts in paradigmatic ways. Wittgenstein consistently focused on situations of teaching and instruction in his writings on rule-following. A notable passage in this respect is *RFM*, VII-47:

> *[T]hat everything can (also) be interpreted as [rule-]following, doesn't mean that everything is following. / But how then does the teacher interpret the rule for the pupil? (For he is certainly supposed to give it a particular interpretation.) – Well, how but by means of words and training? / And if the pupil reacts to it thus and thus; he possesses the rule inwardly. / But* this *is important, namely that this reaction, which is our guarantee of understanding, presupposes as a surrounding particular circumstances, particular forms of life and speech. (As there is no such thing as a facial expression without a face.) / (This is an important movement of thought.)*

This passage stresses the importance of a 'surrounding' of particular forms of life and speech, but Wittgenstein left it unclear what this surrounding is and how it links up with training and rule-following. He continued to elaborate on this, adding in *RFM*, VII-52:

> *Following a rule is a particular language-game. How can it be described? When do we say he has understood the description? – We do this and that; if he now reacts in such-and-such a way, he understood the game.*

One could perhaps read this as Wittgenstein making the claim that, for any given rule *R*, there is a unique language game which is *constituted by* the following of *R*.[20] However, another disambiguation matches better with other pertinent remarks. Instead of the following of a given rule determining a specific language game, any instance of

[20] A less plausible interpretation is that any instance of following any rule whatsoever constitutes, or involves taking part in, the same language game each time. However, how would this language game be described or played, given that its rules can be of any kind? The scope in *RFM*, VII-52 appears to be narrow: following a rule is a particular language-game, not one language game in particular.

rule-following is linked to *some* language game, but not necessarily a language game determined by the rule in question.

On this reading, the first sentence of *RFM*, VII-52 can be paraphrased as follows: "To follow a rule requires engaging in some practical and/or linguistic activity." This reading makes sense of the subsequent sentences, as well: Whether or not a person follows or understands a rule is not apparent in the abstract, but is judged by engaging in the same *overall* activity as the would-be rule-follower. That is to say, Wittgenstein was pointing out that a person judges whether a rule has been followed, or understood, by adjudicating *within* a particular practical context. In line with this, Wittgenstein (*RFM*, VII-53) continued as follows:

> *For doesn't the technique (the possibility) of training someone else in following it belong to the following of a rule? To be sure, by means of examples. And the criterion of his understanding must be the agreement of their individual actions.*

Finding out whether a rule has been followed is associated with the rule; it belongs to the same language game as following the rule, and does not transcend it. In other words, Wittgenstein's "important movement of thought" (*RFM*, VII-47) was that to question, evaluate, and answer whether a person understands or follows a rule should itself be seen as part of the practical setting to which that rule belongs, and cannot be detached from it.

Fogelin (2009, p. 31) argues on similar grounds that the later Wittgenstein had a 'rich' understanding of rule-following, implying that rules are part of practical settings with their own inherent goals and purposes. This is due to Wittgenstein's understanding of rules as part of language games. The role of a rule in a given practice is shaped by other features of that practice, such as infrastructure and technology, motivations and attitudes, and expressions used as part of teaching and correction.[21] Hence, attributing rule-following effectively constitutes 'thick description' (cf. Ryle, 1968; Geertz, 1973). That is, attributing the following of a rule to someone is not to describe a generic phenomenon, some given category of mental or physical fact, but to describe a regulated move within a culturally recognizable context.

[21] As Floyd (2021, p. 53) writes, there is "plasticity in projecting concepts", but this does not show that our procedures "are *not* rule-governed, but rather that that notion itself requires parochial elements".

With regard to mathematics, Wittgenstein (*PI* §185) made this point with his example of teaching how to write the terms of an arithmetic progression. A teacher judging whether a pupil follows the rule of +2 is, in a sense, on the same level as the pupil. The two of them play an instructional language game, and their behavior exhibits rules within this setting. The teacher is not trying to convey an infinite amount of information, 'the entire +2 series', but to pass on a *skill* that is part of the language game. What is taught and learned is a technique.

As Wittgenstein detailed in the remarks leading up to *PI* §185, prior to learning to add 2, it is essential for the pupil to have learned the series of natural numbers, which were taught on the basis of preliminary techniques, such as writing tally marks. At each stage, if the pupil committed a 'systematic mistake', this was pointed out and corrected on the basis of previously learned techniques (*PI* §143). There is thus an unavoidable reflexivity in evaluating and correcting rule-following; such actions happen *within* the pertinent linguistic and mathematical contexts. As argued by McDowell (2009), the moves needed to steer deviant rule-followers back in the right direction are rooted in the very practices to which the rule belongs.

Wittgenstein's shift towards understanding rules as part of language games was a critical turning point, instigating the turn from the *Tractatus*' project of logical analysis and its account of meaning in terms of propositional structure. Instead, he adopted a descriptive, anthropological approach, treating the signs of mathematics as an integral part of human forms of life. This approach recognizes the possibility of mistakes and confusions in calculation, while being able to overcome the paradox of re-interpretability (which Wittgenstein brought up for this very reason in *PI* §201) by focusing on the role of the practices in which rules are grounded. As Wittgenstein (*PI* §432) put it: "Every sign *by itself* seems dead. *What* gives it life? – In use it *lives*". The significance of the signs of mathematics, as well as the function of the rules by which we reproduce them, emerge from their use within human practices.

The upcoming section proceeds by examining prominent alternative interpretations of Wittgenstein's writings on rule-following and mathematics. This will be fruitful for orientation with respect to the exegetical terrain, and will help when further exploring Wittgenstein's views in later chapters. The discussion will focus on readings that in some ways acknowledge the ‘anthropological thrust’ of his later philosophy of mathematics yet diverge in emphasis from the interpretation that has been outlined so far. In order, these readings include *conventionalism*, with its emphasis on the process of establishing mathematical necessity; *formalism*, highlighting the meaning or lack thereof of mathematical propositions; and *naturalism*, focusing on the relationship between mathematics and natural phenomena.

2.2 Conventionalism

Wittgenstein’s writings have been associated with various kinds of conventionalism about mathematics. As a broad definition, “conventionalism” here refers to any philosophical stance that takes mathematics to be comprised of conventions which could, in principle, be stipulated by *fiat*, or which historically were stipulated by *fiat*. The term thus captures a spectrum, with moderate conventionalism on the one end and radical conventionalism on the other. Moderate conventionalist readings of Wittgenstein read him as construing mathematics as a logically consistent and coherent network of conventions, while radical conventionalism posits that each convention is logically arbitrary and disconnected from other conventions.[22]

The above definition has the benefit of immediately encapsulating the difference between an anthropological perspective and conventionalism (in either form) as thus understood. On an anthropological reading of Wittgenstein’s remarks on mathematics, it need not be the case that mathematics consists of conventions that either were potentially or actually stipulated by *fiat*, that is, accepted as a matter of decision. Indeed, decisions are not necessarily foundational, given that rules are understood as forming part of *practices*. Social practices may include conventions but cannot be assumed to be reducible to them.

That being said, the word “convention” is also used more loosely as a synonym of “custom”, and there are interpretations which fall under the label of “conventionalism” that do *not* necessarily contrast with the reading offered thus far. One example is Yemima Ben-Menahem’s (2006) ‘iconoclastic conventionalism’, describing

[22] This distinction is influenced by Dummett (1959) and aligns with Baker & Hacker’s (2009, pp. 356-370) discussion. As these authors point out, conventionalist interpretations of Wittgenstein on mathematics can be traced to the Vienna Circle and its reading of the *Tractatus*, as well as to (critiques of) logical empiricism.

Wittgenstein's conventionalism as "neither explanatory nor justificatory, but, rather, *descriptive*" (ibid. p. 259). Insofar as mathematics is not regarded as strictly reducible to conventions, but is seen through a practical lens and accounted for as part of human life (*RFM*, III-65), this approach aligns with Wittgenstein's anthropological perspective. However, for the sake of argument such an approach will not be considered 'conventionalist' here.[23]

Conventionalism should also be distinguished from the claim that language and mathematics are arbitrary. As Davidson (1984, p. 265) writes, even if "what is conventional is in some sense arbitrary, what is arbitrary is not necessarily conventional". More specifically, a convention is an arbitrary *agreement*. On a conventionalist interpretation of Wittgenstein, then, any language game in which a given term is used is *governed by* conventions for the use of that term. For example, in language games featuring "bachelor", that term can be substituted for "unmarried man" as a matter of convention. Participants of the language game must follow that convention, but the convention itself is an arbitrary agreement among English speakers.

Thus, on a conventionalist reading, Wittgenstein takes mathematical formulae to have a meta-linguistic function, determining the legitimate employment of symbols. The formula '$3^2 = 9$' expresses the convention of converting any language game featuring "3^2" into one featuring "9" in the same syntactic context.[24] To apply an equation is to adhere to a convention to the effect that the language game being played is one in which a given syntactic substitution is allowed. On this reading, the proof of a mathematical formula persuades us, in one way or another, to decide on a new meta-linguistic convention. In sum, the conventions of mathematics jointly determine the rules for using mathematical symbols in our language games.

Wittgenstein never described himself as a "conventionalist". Nevertheless, an array of material has been taken as supportive of conventionalism about mathematics,

[23] Wright (1980, p. 365) describes conventionalism as the rejection of the idea that "necessary statements state a priori *facts* whose acknowledgement constitutes our recognition of necessity, and failure to acknowledge which is a kind of worldly ignorance". As Diamond (1981, pp. 354-355) points out, the early Wittgenstein would count as a 'conventionalist' in that sense. His later writings retain parts of that rejection as well: see e.g. *RFM*, VII-18.

[24] Cf. *WVC*, pp. 152-158, recorded 1931, where Wittgenstein expands '1 + 1 = 2' into the substitution rule: $f(2)$, $1 + 1 = 2 : f(1 + 1)$. However, note his comment that "when I am talking *about* equations a substitution-rule […] must mean something entirely different from the substitution-rules that the equations actually are".

from the later period included. One passage in which the term "convention" has a particularly central role (that is, given Anscombe's translation of "*Übereinkunft*") is *RFM*, I-74, written 1937-1938:

> *If the form of the group was the same, then it must have had the same aspects, the same possibilities of division. If it has different ones then it isn't the same form; perhaps it somehow made the same impression on you; but it is the* same *form only if you can divide it up in the same way." / It is as if this expressed the essence of form. – I say, however: if you talk about* essence *–, you are merely noting a convention. But here one would like to retort: there is no greater difference than that between a proposition about the depth of the essence and one about – a mere convention. But what if I reply: to the* depth *that we see in the essence there corresponds the* deep *need for the convention.*

Below, moderate conventionalism and radical conventionalism are treated in turn. The response to both forms of conventionalism will draw on the preceding reflections on normativity. Over the course of the discussion, it will become clearer that, despite passages such as *RFM*, I-74, Wittgenstein should not be considered a conventionalist in either a moderate or radical sense.

2.2.1 Moderate conventionalism

For a moderate conventionalist, mathematics consists of conventions which were not merely enacted by *fiat*. Each mathematical proposition is accepted as a matter of convention, but any new mathematical convention is adopted on the basis of processes that are themselves justified by mathematical conventions, particularly through mathematical *proofs*. Part of the plausibility of moderate conventionalism is that it yields a straightforward understanding of Wittgenstein's sentiment in *RFM,* I-74. Mathematical propositions are not 'deep' because they express 'the essence of form'. Rather, the depth of a mathematical proposition is due to its connections with other parts of mathematics. New conventions are justified on the basis of previously established conventions. This being so, giving up any *one* convention would seem to require giving up all the conventions built up around it. However, humans are *deeply* indebted to mathematics from a practical perspective, and mathematics is comprised of interconnected conventions, hence the apparent 'depth' felt with respect to any given mathematical proposition.

Commonsense realism about rules and irrational numbers

Despite its apparent plausibility, there are fundamental issues with moderate conventionalist readings of Wittgenstein's writings on rule-following and mathematics. An especially relevant thinker in this context is Hilary Putnam, who initially criticized what he saw as radical conventionalism in Wittgenstein's philosophy of mathematics (Putnam. 1979, pp. 117-119; cf. Garavaso. 2013). Later, Putnam (2001, 2002) has taken a more reconciliatory view, maintaining that Wittgenstein championed a more moderate realism with respect to rules.[25]

In this context, Putnam (2001, p. 393) considers a remark of Wittgenstein's (*RFM*, V-41) which appears to deny that, without a constructive proof, even God could know whether the sequence "7777" occurs within the decimal expansion of π. Nowadays, with the availability of computers generating trillions of digits, we might update the discussion by considering an even more intricate question, such as whether the sequence '7_1, …, $7_{7!}$' (i.e. 5040 tokens of "7" in a row) occurs within the first *quadrillion* digits of the decimal representation of π.

Putnam (2001, p. 396, 2002, pp. 438-440) argues that Wittgenstein at his best held that mathematicians *do* understand such questions. He cites *PI* §516 in support of this view. Here, however, Wittgenstein was less than definitive, writing: "Our understanding of that question [whether "7777" occurs in the decimal of π] reaches just so far, one may say, as such explanations reach." *One* way of reading this is that Wittgenstein identified the sense of questions about the extension of π with the capacity to produce the extension; it would then follow that our understanding of *any* question related to this extension can reach strictly 'as far as such explanations go', e.g. to whatever extent an algorithm can generate it. Such a radically 'verificationist' reading fails upon scrutiny, however, to be discussed in section 2.2.2 (p. 50).

Putnam (2001, pp. 396-397) reads *PI* §516 as suggesting that mathematicians can meaningfully question whether a sequence like '7_1, …, $7_{7!}$' occurs as some part of the decimal expansion of π even without having the ability to prove it. Now, there are ways of estimating the *probability* of a pattern's occurrence in a sequence prior to direct

[25] 'Realism' for Putnam is not meant in a metaphysical sense, but in "something more like the novelist's sense", involving faithfulness to the complexity of everyday reality, according to Conant (1997, p. 208).

verification, but Putnam's claim is not limited to an acknowledgment of this. Rather, he holds that such questions about π have a determinate answer *independently* of our having any way of answering them.[26]

A relevant premise of conventionalist interpretations of Wittgenstein is that language games are governed by conventions. On this view, generating a decimal representation of π is effectively to follow conventions for the 'computing π-language game' (cf. *RFM*, V-9). A moderate conventionalist way of elaborating Putnam's position, then, is to take the ways in which π is computed to be fixed by the conventions that determine that language game. That is, our updates of the language game follow overarching principles of mathematical reasoning, determining what we take to be an adequate algorithm for generating π. This is 'objective' in the sense of being independent of our familiarity with limited sequences of π. So, we can freely project the development of the expansion of π into possible scenarios in which "7777" (or '7_1, …, $7_{7!}$') occurs, or does not occur, giving substance to questions about its potential occurrence.

On this view, our position with respect to mathematics could be compared to that of a programmer who can ask and answer questions about how a program *would* run given some unreachable state, by imagining the program as being able to reach such a state. This is because the programmer has an overarching conception of the function of the program.[27] When it comes to mathematics, however, the issue with this view is that we must be able to state *which* rules are maintained as the language game changes. Since there is an internal relation between a language game and its rules, any overarching principles or patterns guiding changes in mathematical language games must be *independent* from the rules of mathematics themselves.

Wittgenstein rejected the idea that there are such independent principles or patterns dictating the ways our practices change or evolve over time. As he wrote in *RFM*, IV-45, "then by what principle is something recognized as a new proof? Or rather

[26] One way of arguing for this would be to appeal to dispositions involved in computing the decimal expansion of π; we can know how we will be disposed to behave in the future. Putnam (2001, 2002) rejects 'behavioristic' arguments in this sense; his contention is rather that that we have to *acknowledge* it as a part of the grammar of mathematical truth that such questions about irrational numbers *have* an answer, true or false (2001, p. 397).

[27] Wittgenstein's *RFM*, V-41 allusion to God being unable to 'complete' the decimal expansion of an irrational number is an analogue of such a programmer being unable to 'oversee the mathematical program', as it were.

there is certainly no 'principle' here". The closest Wittgenstein got to 'principles' in the relevant sense might be his notion of 'family resemblances'. However, as already discussed in section 2.1.1, and as will be developed further in Chapter 4 (p. 122), family resemblances should be understood in terms of relations among language games, not external patterns or principles dictating how language games *should* or *must* develop, as if it were from the outside. In other words, a family of mathematical language games might be said to exhibit a 'pattern', but patterns in this sense *emerge* from the way language games are played; they do not dictate how they are played.

Accordingly, for Wittgenstein, the proof that '$7_1, \ldots, 7_{7!}$' occurs prior to the *n*th digit in the decimal expansion of π would involve *showing* that it does occur. This is meant, not as an allusion to the *Tractatus* dichotomy of showing versus saying, but as an allusion to a *practical* demonstration actually transforming the language game: "[T]he further expansion of an irrational number is a further expansion of mathematics" (*RFM*, V-9). Although this practical point of view gave prominence to constructive proofs, Wittgenstein did not reject existence proofs as such. His stated aim was to avoid embroilment in mathematical disputes (*PI* §§124-125, cf. Marion, 1998, p. 173). Still, as Hacker (1986, pp. 164-165) argues, by saying that philosophy "leaves mathematics as it is" Wittgenstein simply meant that it is mistaken to think of philosophy as pertaining to mathematical problems, and *vice versa*, not that philosophy has no relevance to the understanding of mathematical practice. Accordingly, he also wrote that "what a mathematician is inclined to say about the objectivity and reality of mathematical facts is not a philosophy of mathematics, but something for philosophical *treatment*" (*PI* §254).

Putnam (2001, pp. 401-402) elsewhere denies that rule-following can be understood apart from practices of embodied beings. The conventionalist interpretation outlined earlier holds that rules *govern* our practices, as opposed to forming *part of* our practices. This entails that any behavior belonging to a mathematical practice (e.g. computing the digits of π) constitutes rule-following, while also allowing for the rules to *exceed* our actual practices. For Wittgenstein, however, "[t]he rule does not do work, for whatever happens according to the rule is an interpretation of the rule" (*RFM*, IV-48). Mathematical rules are elaborated through, and play concrete roles within, mathematical practice. It is this practice-oriented view of rules that explains Wittgenstein's suggestion in *PI* §516 that our understanding of π reaches *just as far as* our explanations of that number, as it is actually used, go; there is no 'rule' for an irrational number existing above and beyond practices of its computation and use, which again is not to reduce it merely to whatever decimal representation happens to be as-of-yet computed.

Having broached commonsense realism about rules and the potential tension with Wittgenstein's view that rules are grounded in practice, it is clear that it is necessary to delve deeper into the concept of 'convention' itself. The next section will examine the relation between David Lewis' influential game-theoretic account of conventions to language games, which will help further clarify Wittgenstein's relation to conventionalism about mathematics.

The concept of 'convention' and the limits of language games

In Lewis's (1969/2002) seminal study of 'convention', conventions are analyzed as regular solutions to game-theoretic coordination problems. A convention emerges when a solution to a coordination problem becomes salient due to previous solutions. For example, say that two hikers face one another on a narrow trail and have to pass each other on either the left or the right, nothing weighing one way or the other. If the hikers step to the right, then, on subsequent encounters, they can draw on precedent and step right again. Part of the reason for this decision is that they expect the other person to also draw on the same precedent, while taking the other to expect oneself to do the same. Hence, on Lewis's account, the convention obviates the need for further arbitrary decisions. To act on a convention is to act on mutual expectations, that is, shared first- and higher-order beliefs about others' (beliefs about others') behavior.

Lewis's idea of conventions as based on mutual expectations shares some surface similarities with Wittgenstein's notion of language games. Language games involve a set of shared rules and standards, consisting in contingent forms of coordination. Nevertheless, conventions in this sense *presuppose* features that Wittgenstein often builds into his characterizations of language games. For instance, Lewis (1969/2002, pp. 37-38) points out that what is recognized as 'a recurring coordination problem' depends on the use of analogy:

> *Guided by whatever analogies we notice, we tend to follow precedent by trying for a coordination equilibrium in the new problem which uniquely corresponds to the one we reached before. [...] In fact, there are innumerable alternative analogies. Were it not that we happen uniformly to notice some analogies and ignore others [...] precedents would be completely ambiguous and worthless.*

As Lewis goes on to point out, what constitutes a given, recurring coordination problem is often unambiguous in practice. However, when individuals face situations while subjectively predicting others' responses, and others' beliefs about others' responses, the fact that there is significant conformity in recognizing what constitutes a given coordination problem is a philosophically significant precondition: conformity in this sense is characteristic of language games. Although Lewis articulates a compelling account of conventions, both the initial arbitrary decision and the subsequent behavioral regularity are already taken as part of systems of possible, distinctive behavioral responses, and thus constitute *moves* in language games.[28]

Even in the simple case of pedestrian traffic, people passing one another by on a crowded path, there may be any number of tacit norms and preconceptions in play, as detailed by Goffman (1971, pp. 9-14). Tacit 'pedestrian routing practices' inform choices and potential conventions that individuals may or may not enact. This setting can be described through and with language games without appealing to conventions in the sense elaborated by Lewis (2002). This kind of setting, involving vague coordinative norms, is by no means exceptional.

Sillari (2013) has argued that Wittgenstein's remarks on rule-following align with Lewis' game-theoretic analysis of conventions, understanding rules in terms of game-theoretic equilibria. However, while the evolutionary game-theory considered by Sillari may provide a model for *describing* language games, it does not follow that it serves to *explain* either language games or their rules. Wittgenstein summed up why in a remark written in 1948: "Instinct comes first, reason second. Reasons only exist in a

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[28] Pointing out a similar presupposition of Lewis' account of conventions, Glock (2019, pp. 312-313) argues that it leaves us with an overly intellectualist concept of 'convention'. However, the intellectualist requirements of the account could be weakened while still leaving a concept that is either logically posterior or equivalent to Wittgenstein's notion of 'language game', insofar as it still presupposes some meaningful decision-making.

language game."[29] Even if conventions play an important role in language games, individuals 'strategize' or make decisions in relation to the features of already existing linguistic practices. So, to conceive of language games as based on conventions, understood game-theoretically, is to put the cart before the horse.

To exemplify, Wittgenstein's (*PI* §2, §6) builders' language game, discussed by Sillari (2013, p. 877), is not merely a signaling system in Lewis' (1969/2002, pp. 143-146) sense. The utterance of "slab!" might be conventionally linked to the act of bringing a slab, but understanding the convention still *presupposes* the framing of the builders' language game, a context in which words or symbols are expressed, slabs are used, and building is going on. Without the language game, the convention alone would not make sense. Of course, there was a moment when "slab!" was first uttered, before its use was established convention. However, the utterance could at that point not have been contingent on a belief about others believing a slab to be requested by this utterance, since that would have already required the association.

Chapter 3 (p. 105) will discuss another, sophisticated game-theoretic approach to Wittgenstein's views on rule-following, but similar considerations apply there. The limits of a game-theoretic approach to rules and language games can be linked to Wittgenstein's understanding of ritual and ceremony as expressed in his *Remarks on Frazer's Golden Bough*. Wittgenstein here criticized Frazer's methodology, taking him to misunderstand rituals by reducing magical and religious views to erroneous forms of proto-science (cf. Cahill, 2021, pp. 163-165). He wrote that there is a sense in which the human being could be called a "ceremonial animal" (Wittgenstein, 2018, p. 42; cf. Collins, 1996), emphasizing that rituals have a role in people's lives independently of their utility in terms of means-end reasoning. Even modern society is characterized by various kinds of ritual, such as greetings and goodbyes.

Wittgenstein's (*PI* §217) remarks on rule-following ultimately being a matter of "what I do" imply that he extended a similar line of thinking to rules; rule-following need not be based on strategic problem-solving, but plays distinct, *sui generis* roles in

[29] The original reads: "*Der Instinkt ist das Erste, das Raisonnement das Zweite. Gründe gibt es erst in einem Sprachspiel*" (Ms-137,66b; Wittgenstein, 1980, §689). Davidson (1984, p. 280) argues that language is a necessary element of conventions (cf. Glock, 2019, p. 317). From a Wittgensteinian perspective, conventions can be taken as part of language games regardless of whether or not they presuppose some form of language.

our language games (*PI* §54). In particular, for Wittgenstein, mathematical calculi have features which are not well understood on the model of instrumental reasoning. Counting procedures are characterized by ritual aspects, in addition to their overall utility. Things are counted in specific sequences, words and gestures are expressed in a specific order, and these aspects are inherent to what we call "counting" (*RFM*, I-4). Normally, when someone counts the apples on a tree, uttering "4" after "3" is not a strategic decision made in favor of uttering, say, "5"; it is not a *decision* at all, but an immediate, unreflective action. For Wittgenstein, people do not decide to act in these ways because doing so accords with established conventions. They simply act in these ways.

Conventions or language games

Barry Stroud (1965) formulates a reading of Wittgenstein along moderate conventionalist lines. Stroud (1965, p. 512) points out that many of Wittgenstein's imagined scenarios, including the deviant pupil adding 2 and the so-called 'wood sellers' who price logs proportionally to the area of arbitrarily tall heaps (*RFM*, I-143-152), turn on the way the specifics of the scenario are meant to relate to the rest of the world. It is unclear, for example, whether the number of logs required to construct a house is taken to correlate consistently with the fluctuating amounts of wood sold for a given price. From this, Stroud (1965, pp. 512-513) infers, firstly, that Wittgenstein's examples are often, when scrutinized, *unintelligible*. Secondly, the increasing (or dawning?) unclarity as his examples are fleshed out, when it becomes plain that the implications for social and physical reality need unpacking, is precisely Wittgenstein's point.

Thus, according to Stroud, Wittgenstein was revealing the degree to which *our* practices involving *our* concepts of counting, calculating, inferring, etc., are related to specific, contingent features of our world. This shows the limit to which he was a conventionalist.[30] Our established forms of language are contingent because the world could be different. Earth *could* have been populated by beings who do not use "same"

[30] Cf. also Moyal-Sharrock (2017) and Baghramian & Coliva (2020, pp. 113-114), who restrict or repudiate Wittgenstein's alleged conceptual relativism in a similar way.

in some of the ways that we do, perhaps (seemingly) taking themselves to continue "…, 996, 998, 1000" in the 'same' way by adding "1004". However, Stroud (1965, pp. 514-515) argues, it does not follow that we could *understand* these people or their apparently rule-governed practices. Whatever the 'logically alien' might be, it does not constitute a coherent, let alone *legitimate*, alternative for us.

While Stroud is right to stress the need to elaborate on the *context* of practices – though Wittgenstein's (*PI* §2, §5) examples are often offered precisely *as* simplifications – it seems an overstatement to state that we cannot *understand* alternative practices such as consistently going from "…, 996, 998, 1000" to "1004", or the wood sellers' business, or even imagine any way of legitimating them. In the first case, we can imagine ourselves as having developed a different notation, other than the decimal system, such that *we* would continue the sequence like that while writing out an arithmetical progression (cf. Medina, 2003, pp. 466-467). In the second case, Wittgenstein (*LFM,* XXI, p. 204) himself elaborated on the wood-sellers scenario by outlining some possible historical contexts, including that a king once told them to do it that way, adding: "They do this. And they get along all right. What more do you want?"[31]

Stroud (1965, p. 515) further argues that, from the contingency of rules, it does not follow that we are free to go as we like when adding 2 to 1000: "That we take just the step we do here is a contingent fact, but it is not the result of a decision; it is not a convention to which there are alternatives among which we could choose." Stroud here points out that voluntariness does not follow from contingency; taking the step from "1000" to "1002", when adding 2, is contingent but would normally not constitute a voluntary decision. However, we can conceive of contexts in which taking that step *would be* a convention. Even if there is no alternative for us, an indefinite number of hypothetical alternative conventions would cohere with the deviant pupil's behavior, with 'add 2 up to 1000, 4 up to 2000, 6 up to 3000, etc.' being just one.

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[31] Hersh (1997, pp. 203-204) appeals to principles of capitalism in order to rule out the legitimacy or possibility of the wood-sellers scenario. That is a *non sequitur*, since the scenario is meant to take place in a non-capitalist environment: these people "have a quite different system of payment from us" (*RFM*, I-150). Wittgenstein's question, "[a]nd is there anything to be said against simply giving the wood away?" would be answered "yes" by a businessowner in Wittgenstein's own time, but this does not undermine the hypothetical scenario.

The point Wittgenstein made with the deviant pupil in *PI* §185 is not that rule-following conforms to conventions. Appeals to conventions or the absence thereof about how to continue a sequence fail to properly address Wittgenstein's point, since the deviant pupil's confusion could remain either way. An individual who is unclear about how to follow a given rule, or how to use a concept such as '+2,' is unclear about how to *do* something in a particular situation. Appeals to conventions do not necessarily help clarify that issue, since we can be unsure about what we are doing even if our behavior coincides with established conventions, just as we can be certain or confident in cases in which there are no established conventions.

Wright (1980, p. 378) criticizes Stroud for thinking that we can understand "alternative methods of inference non-constructively; that is, without it being in principle possible to construct intelligible examples". Indeed, Stroud appears to argue that there is a sense in which our mathematical and linguistic practices are 'contingent' even if we could not conceive of any alternatives to them, which threatens to make the concept of 'contingency' empty. Moreover, there is a difference between particular counting conventions, or commercial practices like that of the wood sellers – localizable social phenomena that can be posited, or described and studied by historians and anthropologists – and 'methods of inference' in all generality. Both Stroud and Wright seem to suggest that we have a prior, independent idea of logical inference as a generic category that we can then relate to, or contrast with, the categories of conventionality and empirical fact.[32] However, part of the function of the concept of 'language game', for Wittgenstein, was precisely to scrutinize the mutual independence of such generic distinctions.

[32] As this discussion illustrates, the term "convention" in translations of Wittgenstein's writings carries with it potential for unclarity. The parenthetical remark in *PI* §355: "*Und diese Sprache beruht, wie jede andere, auf Übereinkunft*" is translated: "And this language, like any other, rests on convention" (in the 4th edition). Provided conventions are regarded as practical regularities based on some established arbitrary decision(s), this translation suggests that language rests on a decision. In light of the context of *PI* §355, a better translation would be that languages rest on social arrangements or accord. Similarly, an '*Übereinstimmung*' ('accordance') need not be the outcome of a decision. In *PI* §241, Wittgenstein clarified that he is not talking about a consensus in opinion; the point is that humans "*stimmen überein*", or accord, in their language-use. This need not be a matter of agreed-upon uniformity, but perhaps of convergent forms of life in general.

Wittgenstein used the word "*Konvention*" infrequently, mostly in the early 1930s. In the handful of later occurrences of that word, it implies artificial contingency or arbitrariness, without any indication that it served a pivotal role in his thinking. One apparent counterexample is *RFM*, I-74, where Wittgenstein talked of "the deep need for the convention", but the original German word is "*Übereinkunft*". Again, this means an arrangement which is contingent but not necessarily a mere decision (cf. Wright, 1980, pp. 104-109).

By describing language games, we can distinguish the question of whether a behavior is based on arbitrary decision or constitutes rule-following from whether the behavior is consistent with continued participation in a practice. For instance, teaching and instruction are language games which incorporate the fact that people make mistakes, make progress, rely on rules, master and creatively bend rules, and so on. The process of an individual learning to participate in a language game involves a tapestry of logical, empirical (e.g. psychological), and conventional factors. Moreover, language games are 'objects of comparison' (*PI* §§130-131); they are described by way of contrast with other language games.[33] By painting a backdrop of conditions and relations which are contingent without being necessarily based on individual decision, individual behaviors can come to stand out as inherently intelligible. We can gain an understanding of behavior through its context, precisely by not delimiting the meaning of 'context' to a setting that is strictly defined by any given condition.

Hence, while appealing to conventions risks conflating the contingency of rules with voluntariness, the analogy of language games helps differentiate these two notions. This, however, will be elaborated in subsequent chapters. First, the next section will address ways in which philosophers have responded to some of these deficiencies in moderate conventionalist readings of Wittgenstein, not by turning away from or questioning the pertinence of the concept of 'convention' itself, but by understanding it in a more radical way.

2.2.2 Radical conventionalism

Radical conventionalism posits that mathematics is based on rules enacted by *fiat*, with these rules being effectively *ad hoc* and independent of one another. Whereas moderate conventionalism allowed for bidirectional normativity in mathematical reasoning, with one convention rationally justifying further conventions, radical conventionalism treats normativity as directionless and inert. For example, a step in a proof is justified solely on its own, carrying no logical consequences in and of itself. Michael Dummett (1959)

[33] Wittgenstein's wood merchant scenario in *LFM*, XXI, p. 204, and *RFM*, I §143 is a good example of the 'relativistic' function of language games; part of their purpose is to show how our use of concepts is informed by contextual conditions (comparable to clocks in relativity-theory; cf. *RFM*, VI-28), without implying that different conditions necessarily render the concepts inapplicable. For a 'relativistic' (as opposed to absolutist *and* relativist) interpretation of the wood sellers along these lines, see Mion (2021) and Penco (2020).

interprets Wittgenstein's later work on mathematics in this way, labeling him a "full-blooded conventionalist". The acceptance of a mathematical formula, as proven or true, is the consequence of sheer decision, irrespective of our adherence to other conventions.[34] Dummett (1959, pp. 329-330) takes Wittgenstein to have come to this view on the basis of his reflections on rules:

> *[I]n order to follow [a] proof, we have to recognize various transitions as applications of the general rules of inference. Now even if these rules had been explicitly formulated at the start, and we had given our assent to them, our doing so would not in itself constitute recognition of each transition as a correct application of the rules.*

From this, Dummett argues, Wittgenstein concluded that each transition or inference is an independent decision. This reading rules out the idea that the transitions in an argument or proof are made 'automatically' once the rules which justify those steps are accepted.

While this interpretation aligns to some extent with an anthropological reading which emphasizes the fact that rules are enacted within practices, it faces similar challenges as moderate conventionalism due to its focus on conventions rather than practice. Axioms and definitions are formulations of inference rules, and they do not by their own accord determine how they are to be implemented in actual mathematical practice. This does not mean, however, that individuals are free to apply these rules in arbitrary ways and still be doing mathematics.

The way in which a proof is justified on an individual level is of interest only if it helps us see how proofs are accepted and incorporated into mathematical practice overall. While transitions in a proof can result from a conscious decision or be made without thought or hesitation in actual practice, this distinction is inessential. Wittgenstein described conscious decision-making in some instances (*RFM*, III-27) and immediate reactions in others (*RFM*, I-3, *PI* §219, §437). The key question instead revolves around what constitutes a 'step in a proof' and what it means for something to be proven. In mathematical practice, the concept 'proof' does not presuppose

[34] In their critique of Dummett's reading, Baker & Hacker (2009, pp. 366-367) focus on the idea of conventions 'making mathematical propositions *true*', something they point out that Wittgenstein never advocated. However, Dummett's (1959, p. pp. 329-330) reading centers on weakening the notion of '*correct* application of rules' and can be considered a species of rule-following skepticism. Hence, it is here classified alongside Kripke's reading.

justification but *constitutes* justification, which means that the depiction of each step in a proof as an unjustified choice is a misrepresentation of practice.

If moderate conventionalism ran into issues with securing the continuity of concepts across updates to the language game, full-blooded conventionalism faces the same issue on a smaller scale. What ties down the identity of a proof from one step to another? For Wittgenstein, if Dummett (1959, p. 341-343) is right, every step in a proof is independent of any other. In this case, it would be unclear both (1) *which* transition a mathematician makes at any given point, and (2) *whether* it constitutes a transition from one step of the proof to another, rather than a change of subject, producing some other proof entirely. Dummett's reading effectively conceives of proof as a 'game' in which every move made completely redefines the rules of the game. Inasmuch as the concept of 'prove' is one that *justifies* a result, this is incoherent.

For Wittgenstein, it is clear that we follow rules of inference when calculating. What he denied was that, when calculating, we are appealing to a preconceived model to which our rules correspond. To follow a rule is a matter of practice, not of conformity with an external ideal:

> *"But still, I must only infer what really* follows*! – Is this supposed to mean: only what follows, going by the rules of inference; or is it supposed to mean: only what follows, going by* such *rules of inference as somehow agree with some (sort of) reality?"* (*RFM*, I-8)[35]

Wittgenstein did not conceive of the steps in a proof as *normatively* voluntary. If the steps were normatively voluntary, this would not only undermine agreement in mathematical practice, but relinquish the concepts of 'proof' and 'step in a proof'. After all, there is a sense in which these concepts belong to activities involving compulsory commitments and procedures. If you accept that $x + y = z$, you are left with no choice over whether to accept that $z - y = x$. Only, the condition of having no choice is normative, not psychological. It means that solving the equation $x + y = z$ simply *is* what counts as justification for the equation $z - y = x$, and failure to acknowledge it as

[35] This quote strikes a balance, but more empathic remarks about the normatively inviolable status of mathematics can be adduced, e.g. *RFM*, III-35: "If the calculation has been done right, then this must be the result." Cf, Klenk (1976, pp. 43-44) for more examples. See also Wittgenstein's critique of intuitionism in *LFM*, XXIV, p. 237: "We might as well say that we need, not an intuition at each step, but a *decision*. – Actually there is neither. You don't make a decision: you simply do a certain thing. It is a question of a certain practice."

justificatory simply *is* what it means to commit a mathematical error.

Since the problem with the full-blooded conventionalism Dummett attributes to Wittgenstein is conceptual, disregarding or denying the internal standards of justification at play in mathematics, it can be neither attacked nor supported by citing the purported empirical fact that people conform in mathematical practice. Dummett (1959, p. 337) criticizes what he takes to be Wittgenstein's position on the basis of the consensus people overwhelmingly get when they calculate and prove theorems, taking this to demonstrate the need for a more robust account of proof. At the same time, Dummett (1959, pp. 343-344) presents Wittgenstein's appeals to conformity as attempts to justify the view that proof is arbitrary. However, denying the justificatory role of proof departs from Wittgenstein's point of view, which acknowledges that proofs and axioms are precisely what count as justificatory in mathematical practice.

Rule-following skepticism

The previous point is connected to the skeptical solution offered in Saul Kripke's (1982) exposition of Wittgenstein's remarks on rule-following. In Kripke's famous example, we are asked to answer what it is about a person that binds her (normatively) to giving a specific answer to a given arithmetic problem; specifically what about the act of setting out to calculate 68 + 57 dictates she should give the answer 125? Kripke reads Wittgenstein as answering "nothing." The argument can be divided in three: First, per stipulation, no fact about the individual is tied to the answer, since she has not calculated it before. Second, since arithmetic operations on natural numbers are unbounded while mental and physical facts are finite, no set of such facts uniquely correspond to arithmetical operations. This generalizes to the idea that any rule is congruent with any fact, on some interpretation. Third, since a rule can be interpreted to conform with any fact, no fact can determine whether a rule is followed in a given case.

The upshot is a radical form of skepticism. Any basis for talk of commitment to rules or attributions of rule-following appears to vanish. As Wittgenstein (*PI* §201) put it, "if everything can be made out to accord with the rule, then it can also be made out to conflict with it". In consequence, the very notion of 'rule-following' seems to be left without meaning. Indeed, linguistic meaning and understanding are rendered impossible. After all, who is to say what is meant by a word, considering its past use is consistent with infinite rules for its future use? When I said ‘book’ I might have meant ‘book*’: ‘a text before *t*, a cup after *t*’, with *t* being the current moment. Since Kripke’s exposition is formulated in language, this is paradoxical.

Kripke (1982, p. 55) reads Wittgenstein as offering a ‘skeptical solution’ to that paradox by reconceptualizing rule-following. To attribute the following of a rule to a person (e.g. to say that she is calculating 68 + 57) is not to *describe* him or her, it is instead to *take* the person to be a rule-follower (e.g. a calculator). This attribution is ‘correct’ (or ‘apposite’) to the extent that it accords with the judgement of a linguistic community in general, which is to say that others *do* the same (or *would* do so; Kripke allows for this modal knowledge on an individual level through the internalization of community practices). In consequence, as Kusch (2006, p. 92) puts it, the normativity of meaning – understood as the idea that mental states guide and justify the applications of signs[36] – is replaced by intersubjective normativity. Intersubjective normativity is a statistical matter. People correct each other’s use of words until they reach an equilibrium, this being a sociological fact rather than a logical or philosophical one.

In *PI* §198, Wittgenstein *does* air a paradox similar to the one Kripke (1982) expounds and Kusch (2006) elaborates. For Wittgenstein, the paradox undermines an interpretational or biographical picture of the normativity of meaning. However, he does not supplant that picture with another. Specifically, he does not appeal to arbitrary assertability conditions enforced in an amorphously defined community in the way that Kripke does. Wittgenstein’s upshot can be taken to go in a different direction entirely. Here is how he concludes *PI* §198:

> *“Then can whatever I do be brought into accord with the rule?” – Let me ask this: what has the expression of a rule – say a sign-post – got to do with my actions? What sort of connexion is there here? – Well, perhaps this one: I have been trained to react to this sign in a particular way, and now I do so react to it. / But that is only to give a causal connexion; to tell how it has come about that we now go by the sign-post; not what this going-by-the-sign really consists in. On the contrary; I have further indicated that a person goes by a sign-post*

[36] Note that this contrasts with the notion of normativity of meaning outlined in section 2.1.1; as argued, rather than mental states, the later Wittgenstein understood the normativity of meaning in terms of language games.

> *only in so far as there exists a regular use of sign-posts, a custom.*

The final sentence can be read in two different ways. On the one hand, "on the contrary" could indicate that Wittgenstein *did* take himself to have explained what going-by-the-sign really consists in, implicitly affirming that this is a mere causal connection based on training. This could be taken to cohere with Kripke's reading. On the other hand, "on the contrary" can be read as a rejection of the entire preceding sentence, with Wittgenstein denying that he was *merely* giving a causal connection. Considering that he followed up by saying that going by a sign-post requires a custom, and customs *involve* causal connections (such as the training mentioned in the previous paragraph) but are not reducible to them, the latter is more likely.

Leading up to the remark *PI* §205, Wittgenstein considered a hypothetical alternative world-history in which no human has *ever* played a game, then asked whether it is possible for someone to have nevertheless *invented* a game that was interrupted before it could be played, in this game-less world-history. He sought to expose this as an empty idea because the playing of a game is a living practice. Even the would-be 'rules' here would not count as rules unless and until they were put into practice and enforced in some way. Of course, *we* may understand the rules of merely hypothetical games, but that is because we already know actual games. The opening voice here is that of an imagined philosophical interlocutor (*PI* §205):

> *"But that is just what is remarkable about* intention*, about the mental process, that the existence of a custom, of a technique, is not necessary to it. That, for example, it is imaginable that two people should play a game of chess, or even only the beginning of a game of chess, in a world in which otherwise no games existed – and then be interrupted." / But isn't chess defined by its rules? And how are these rules present in the mind of someone who intends to play chess?*

The rules of a game are part of the way it is played; they do not define the game as if they were necessary and sufficient conditions of it. A rule, such as *en passant*, might be an implicit part of the stage setting and only come into play under certain conditions. Wittgenstein was here pointing out that established practices inform rules. As this shows, what he was after in this stretch of remarks was not to dig down to what '*must*' be involved in rule-following, or to the purely *causal* regularities that explain our (supposedly) deceptively normative notions.[37]

On the contrary, the 'rule-following paradox' shows that the search for the essential nature of rules and rule-following as such, independently of specific activities in which rules are acted upon or enforced in specific ways, is unfounded. This is why, in *PI* §198, Wittgenstein ends by referring to 'customs' (not tendencies of intersubjective correction and behavioral conformity among individuals) and, in *PI* §201, to "what is exhibited in what we call 'obeying the rule' and 'going against it' in actual cases," that is, *within* actual practices involving rules. Kripke's framing of the skeptical solution to the paradox is thus a misattribution.

As argued in chapter 2.1.1 (p. 22), for Wittgenstein, to say someone follows a rule is to see them as part of a language game. The appropriateness of this is not determined by facts in isolation, as Kripke's argument (1982) effectively highlights, but it is not arbitrary, either. The relation between the rule and the rule-follower is determined by the continuity between the context of participation and the context of evaluation or judgement; in a broad sense, the same game is played in them both. Someone is regarded as following a rule only if there is a custom of *attributing* a rule and *acting on it*, as Wittgenstein (*PI* §198) suggests, and our recognition of and participation in this custom suffices to determine the rule to "within uniqueness" (cf. Wright, 2001, p. 395). This avoids both determinism of meaning (cf. Kusch, 2006) and a reduction of rule-following to statistical norms upheld on an individual basis.

Here it may be objected that we can, for instance, comment on someone having played a card game in the past without playing that card game now. In this case it might seem like Kripke's (1982, pp. 95, 110) allusions to "taking someone into one's community" is truer to life than positing that the commentator engages in the very same game she comments upon. Kripke's (1982, p. 96) Wittgenstein would take the commentator to conform with her community's rule-following attributions. On closer scrutiny, however, it is this focus on rule-following that is overly restrictive, since a game is more inclusive than a rule. A commentator need not conform to any particular rule in order to commentate on a game. Instead, her commentating activity is inextricable from the game itself, as an institution. Similarly, to say that someone

[37] McDowell (1984, p. 340) identifies this kind of reductive reading as an attempt to 'dig beneath bedrock.'

calculated the area of a circle "correctly" because they applied the formula $A = \pi r^2$ is itself to do geometry. Describing someone in that way presupposes, and gets its meaning from, contact with the mathematical practice involving the technique that may or may not be utilized in the case described. Nevertheless, giving such descriptions does not mean that one '*would*' do the same as the person in question and conform with them in any direct sense.

Self-ascription and analogical understanding

Related reflections on rule-following lead Crispin Wright (1980, 2001) to defend a form of radical conventionalist reading. Like Kripke, Wright reads the later Wittgenstein's writings on rules as implying that rule-following requires an alternative form of explanation (Wright, 1980, p. 21-22; 2001, p. 6). Rather than intersubjective correction, the possibility of rule-following is for Wright (2001, p. 137-138) ensured by the special authority of individual self-ascriptions of meaning, intention, and decision. This 'special authority' is constitutive. That an individual who is well-functioning, not self-deceived, and sincere intends plus by "+", and thereby knows that she intends plus rather than some other operation, *makes it the case that* she is following the rule of addition. This solves the rule-following paradox, for Wright, because intention is not passive, merely conforming to an infinite number of interpretations. Rules do not determine how they are to be followed in advance; on the contrary, it is our self-ascribed meanings, intentions, and decisions that determine what it is to follow the rules that we set out to follow.[38]

The shift from perceiving humans as passive observers to active participants and conceptual inventors is well-grounded in Wittgenstein's writings. Wright (2001, pp. 79-80) decouples the question of the impossibility of private language from the question of the normativity of meaning; on the assumption that avowals of meaning are authoritative, there simply *is* no objectivity of meaning as such. Instead, Wright traces Wittgenstein's rejection of private language to his anti-Cartesian philosophy of mind, rejecting the idea that avowals of mental states such as intentions are inward

[38] Note also that, for Wright (2001, p. 141), "[t]he roots of first-personal authority for the self-ascription of these states reside not in cognitive achievement, based on cognitive privilege, but in the success of the practices informed by this cooperative interpretational scheme".

observations (Wright, 2001, pp. 291-299).

In giving primacy to avowals, however, this interpretation is in danger of obscuring the institutional nature of rules that Wittgenstein emphasized in *PI* §199. For instance, the distinction between correct and incorrect multiplication, between 25 × 25 = 625 and 25 × 25 = 624, cannot solely be attributed to the fact that well-functioning individuals take themselves to multiply by expressing the former but not the latter. To say that getting 625 when calculating 25 × 25 is 'correct' indicates the institutional role of this procedure; we "turn our back upon" it (*RFM*, IV-35). People could play a language game in which getting 625 was a deviation from the rules, possibly involving a corresponding change in their avowals, but a change in avowals would not necessarily itself constitute a different mathematical language game.

Even granting this, Wright's (2001, p. 419) interpretation that rule-following "is not a matter of learning to keep track of something whose direction is dictated […] independently of the judgements on the matter of anyone who might be regarded as competent" still stands. However, that insight should be considered part of Wittgenstein's recognition of the fact that rules are normally understood and (dis-)obeyed as part of engagement with particular practices or institutions. In other words, Wittgenstein did not dismantle some *default* concept of 'rule-following', only to leave a vacuum to be filled by an alternative theorization of rules.[39] His point was that we do not *have*, nor want, a default concept of 'rule-following'.

Reinforcing this point, and as mentioned in the previous section, Wittgenstein (*PI* §516) wrote about our understanding of the decimal expansion of π in a way that goes against a radical conventionalist reading. He indicated that there are at least two reasons to think we might understand a question about whether "777" occurs in π.

1. It is an English sentence.
2. It can be shown what it means for e.g. "415" to occur in the decimal expansion.

These "and similar" considerations are said to count as evidence in favor of people's capacity to understand the question of whether "777" occurs in π. However, if we were to assume that a radical conventionalist reading of Wittgenstein is correct, it seems that

[39] This latter point is elaborated in Finkelstein (2000). See also Diamond (1981, pp. 360-361). Indeed, Wright (2001, especially p. 395) also makes comments that can be read as pointing in this direction.

neither of these considerations would add to our understanding of that question. Firstly, the question of whether "777" occurs in the decimal of π would be decided as a matter of agreement. This makes it unclear how the fact that it is posed in English speaks in favor of its comprehension. Secondly, again on a radical conventionalist view, answering the question would require a decision for the expansion of a rule. It is unclear how an answer to a question about a prior decision (what it means for "415" to occur) could help answer a question about a decision that is not yet made. So, radical conventionalist readings are *prima facie* unable to interpret *PI* §516 in a cogent way.

Having discussed it vis-à-vis both moderate and radical conventionalism, it is worth reiterating the point of this remark from the perspective of the anthropological reading. The first thing to note is that Wittgenstein acknowledged that we can have some understanding of whether "777" occurs in π, but he added that this understanding reaches only "as far as *such explanations* reach" (*PI* §516, my emphasis). The understanding of questions about π is tied to the potential for giving explanations; it is explained, partly and in different ways, by 1 and 2.

Sentences 1 and 2 characterize the language game in which the question "does '777' occur in the decimal expansion of π?" is posed. Our limited understanding of that question comes from the fact that 1 and 2, along with other explanations, do not settle the mathematical implications of that language game. If they did, they would jointly determine whether or not "777" *does* occur in π, or a way of calculating the answer. Instead, this is left open, leaving the question *mathematically* indeterminate. So, 1 and 2 (etc.) are simply qualitatively distinct ways of outlining a possible language game with *kinship* to one in which it would be answered whether "777" occurs in π. Hence, in contrast to Putnam's (2001) reading, cf. section 2.2.1 (p. 35), the form of understanding these sentences produce is *analogical*, not mathematical.[40]

Analogies, or family resemblances, are not perfectly transitive, which implies that they cannot be projected indefinitely. Using our understanding of questions pertaining to a segment of the decimal expansion of π as evidence for our understanding

[40] Wittgenstein (*PI* §517) went on to ask: "Can't we be mistaken in thinking that we understand a question?" Some mathematical proofs lead us to "revise what counts as the domain of the imaginable". We can formulate an apparent question even if we lack mathematical understanding of it. Dawson (2016, pp. 90-92) similarly describes the (limited) understanding of the question in *PI* §516 as a matter of analogy, not mathematics *per se*.

of questions related to '*all*' of π would therefore be misconceived. It is unclear how to address whether a specific sequence appears 'somewhere' in any given non-repeating sequence of digits, unless that (apparent) question can be tied to some established mathematical method such as an estimation of probability. As far as Wittgenstein is concerned, the irrational numbers are *inherently* tied to such methods. They do not form a system (*RFM*, II-33), but are "special cases" (*RFM*, V-37).[41]

Having highlighted issues with radical conventionalism, it is worth reiterating that these stem from underestimating the anthropological nature of the normativity of meaning. Moderate conventionalism relies on the idea of conventions as conduits of normativity, while radical conventionalism redirects meaning to the level of immediate behavior, thereby ridding it of normativity altogether. Conventionalist interpretations of Wittgenstein fall short because they attempt to ascribe a fixed meaning to his concept of 'language games', or to replace that concept with 'convention'. In doing so, these readings fail to acknowledge the fact that rules play different roles in different practices, and that there is no *one* mechanism behind them.

Other accounts of rule-following face similar issues. For instance, Warren (2020, pp. 73-74) offers a dispositionalist response to radical conventionalism. He argues that we have 'composite dispositions', dispositions to act in ways that are conditional on our having already acted in other ways. Such dispositions are then taken to enable us to follow rules with indefinite complexity, such as computing π or writing out arithmetical progressions. However, whether or not it makes sense to ascribe such dispositions to individuals, the question of whether an individual conforms to a rule is ultimately anthropological, which is why that question has an ineliminable normative dimension when posed within the appropriate social setting.[42] Rule-following is not determined by dispositions in isolation, no matter how inexhaustive they might be, but depends on how the individual's actions fit into a larger context.[43] The idea of a 'larger context' here cannot be pinned down once and for all, or characterized with a single model like that

[41] Note that π, √2, and other irrational numbers are generated by following determinate rules. For instance, π/4 can be computed by starting with 1, then alternately subtracting and adding the reciprocals of the odd numbers.
[42] See Glock (2019, p. 297) and Kripke (1982, p. 37).
[43] Beyond this, Warren (2020) argues that our following conventions does not require explicit psychological representations of rules, including rules of inference. Conventions are instead rooted in linguistic behavior in a broad sense. This aspect of his thesis is in line with Wittgenstein's views on the anthropological interpretation.

of conventions, but has to be recognized as essentially varied and dynamic.

2.3 Formalism and structuralism

In addition to conventionalism, Wittgenstein has been interpreted as a formalist and/or structuralist about mathematics. According to Victor Rodych (2018, 2008, 1997), the later Wittgenstein's philosophy of mathematics constitutes a kind of finitistic formalism. This view, as Rodych sees it, is based on a syntactical conception of mathematics according to which mathematical formulae are, semantically speaking, meaningless. Rodych (1997, pp. 196-197) distinguishes between two kinds of formalism:

> *Strong Formalism (SF): A mathematical calculus is defined by its accepted or stipulated propositions (e.g., axioms) and rules of operation. Mathematics is syntactical, not semantical: the meaningfulness of propositions within a calculus is an entirely intrasystemic matter.* […] *If, however, a mathematical calculus has a semantic interpretation or an extrasystemic application, it is inessential, for a calculus is essentially a "sign-game" – its signs and propositions do not refer to or designate extramathematical objects or truths. / Weak Formalism (WF): A mathematical calculus is a formal calculus in the sense of SF, but a formal calculus is a mathematical calculus only if it has been given an extrasystemic application to a real world domain.*

Rodych (2018, 1997, p. 196) argues that Wittgenstein began his career with a variant of weak formalism, transitioning to strong formalism in his so-called 'intermediate period' in the 1930s, and later returned to an altered form of weak formalism. On this reading, Wittgenstein retained a view of mathematics as a syntactical system with signs lacking inherent meaning, but eventually added that a syntactical system which properly speaking constitutes 'mathematics' also requires applicability in the real world, beyond the system itself.

In a related vein, but with an historical approach that provides additional context to the formalist reading, Sören Stenlund (2015) aligns Wittgenstein's writings on mathematics with the tradition of 'symbolic mathematics' beginning with Franciscus

Vieta's contributions to modern algebraic notation in the 16th century.[44] As Stenlund sees it, symbolic mathematics represents a decoupling of mathematics from traditional ontological concerns over the 'nature of number'. As modern mathematics gained momentum through the development of methods and notations on purely symbolic grounds, any question of the philosophically correct 'interpretation' or 'application' of its concepts and formulae proved increasingly unnecessary.[45]

Stenlund (2015, p. 46) appears to land close to a strictly historical interpretation of this transformation, writing that "an essential feature of the symbolic point of view was the logical separation of a symbolic system from its application to some subject-matter outside pure mathematics". Stenlund (2015, p. 56) interprets Wittgenstein as a formalist in Rodych's sense, taking him to reject the notion that there is any *a priori* link between mathematics and ordinary language. He argues that Wittgenstein favored a modern perspective on mathematics as a study of self-contained symbolisms, and regarded it as a sign of progress that we no longer need to worry about inherent logical relationships between mathematics and ordinary language.

Given a focus on the formal definitions involved in modern algebra and a Hilbertian conception of geometry, the view of mathematics as an axiomatic system (the applicability of which is a contingent matter) has some intuitive appeal. However, at least when it comes to Wittgenstein's later writings, formalist readings tend to overlook what he had to say about elementary mathematics and its intricate, dynamic relation to everyday practice and ordinary language (Fogelin, 2009, p. 90). In several cases, Wittgenstein's reflections on applications of mathematics and the adjectival use of numerals ramify to an understanding of mathematics in general. The beginning of part V of *RFM* is an important stretch of remarks in this context:

> *I want to say: it is essential to mathematics that its signs are also employed in* mufti.[46] */ It is the use outside mathematics, and so the meaning of the signs, that makes the sign-game into mathematics. / Just as it is not logical inference either, for me to make a change from one formation to another (say from one arrangement of chairs to another)*

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[44] Vieta is taken to have introduced the use of letters to represent known and unknown quantities, using consonants for known quantities and vowels for unknown quantities. On this, see Cajori (1993, pp. 181-187).
[45] See Maddy (2008, pp. 16-20) on mathematics decoupling from science; cf. Pérez-Escobar & Sarikaya (2022).
[46] "*Mufti*" here means 'civilian clothes' or 'out of uniform', apparently used by Wittgenstein as slang for an informal context, i.e. ordinary and/or empirical language.

if these arrangements have not a linguistic function apart from this transformation. (RFM, V-2)

Rodych (2018, §3.5) acknowledges that Wittgenstein held that "extra-mathematical application [is] a necessary condition of a mathematical language-game". However, on a formalist reading, this 'necessary condition' appears relatively synthetic. Rather than highlighting the embedding of mathematics in a form of life, Rodych (2018, §3.5) argues that Wittgenstein distinguished between mathematics and mere sign-games in order to resolve a tension in his own views: "By demarcating mathematical language-games from non-mathematical sign-games, Wittgenstein can now claim that, 'for the time being', set theory is merely a formal sign-game."[47]

The analogy between mathematics and a game of symbols can be used to support both weak and strong formalism, according to which mathematics is a system defined by its accepted or stipulated formulae (e.g. axioms) and rules governing symbolic operations.[48] However, Wittgenstein seems to have been ambivalent about "mathematical language games" as if it were an isolated phenomenon, instead stressing intricate links between mathematics and language, as Conant (1997, p. 221) highlights. Indeed, as outlined in section 2.1.2 (p. 28), there are at least two senses in which mathematics can be applied: mediative and conceptual application. The former was highlighted in the *Tractatus*; it involves using a calculating procedure – paradigmatically arithmetic – as a guide or justification for making an inference.

In addition to mediative applications, however, there are *conceptual* applications, the use of mathematical vocabulary within nonmathematical sentences. This is what Wittgenstein referenced when talking of the use of mathematical signs "in *mufti*" (*RFM*, V-2). When a numeral is used in a sentence outside of a calculus, it helps set the stage for other language games by effectively providing an 'address' for mediative applications.[49] If we perform a calculation as a means of transitioning from one utterance to another, we rely on a grammatical system to identify the start- and end-

[47] Rodych (1997, pp. 217-219) argues that Wittgenstein retained his intermediate view that one calculus is as good as another, and so required the criterion of extra-mathematical applicability in order to maintain his critique of transfinite set theory. The former may be right, but this framing of the motivation for the criterion appears to understate the extent of his anthropological orientation. This is elaborated in Chapter 4 (p. 151).
[48] For instance, Johannes Thomae (1898) explained his 'formal arithmetic' through a comparison with chess.
[49] Or a position within a 'coordinate system'; *RFM*, VII-74. Cf. *RFM*, I-165-167.

points of this kind of transition. This grammatical system is largely facilitated by the use of mathematical vocabulary in ordinary discourse.

To illustrate, consider how both arithmetic and chronometry are involved in the distinction between planning an event in 45 minutes and in 50 minutes. When a language game features such distinctions, we can calculate to make precise moves in the game. For example, 50 – 45 = 5 can be calculated to infer "I arrived 5 minutes early". This simple example shows that language games come to *share* features with pure mathematics insofar as mathematics is used to navigate them. This interaction can be bidirectional: phrases such as "5 dollars" and "5 cents" do not have fixed meanings independently of how people calculate with them (cf. *PI* §120). This emphasis on dynamic relationships between different kinds of mathematical applications distinguishes Wittgenstein's anthropological perspective from formalism.

These considerations, which are expanded in Chapter 3 and Chapter 4, encapsulate the difference in focus between formalist or structuralist interpretations of Wittgenstein and the anthropological reading advanced in this thesis. Wittgenstein's critique of formalism neither echoed the ontological objections raised by Frege,[50] nor did it restate Hermann Weyl's question to Hilbert: "Why should we take consistency of a formal system of mathematics as a reason to believe in the truth of the pre-formal mathematics it codifies?" (Zach, 2019, §3).

Rather, Wittgenstein's critique was made on non-Platonist grammatical grounds, making it comparatively internal: The question is how a formal system of mathematics can be useful without involving the form of (empirical or ordinary) language which facilitates its use. The author of the *Tractatus* could restrict his attention to mediative applications of mathematics because he analyzed all propositions as products of a single form of operation, but the later Wittgenstein's recognition of differences among linguistic practices required a greater attention to the dynamical use of mathematical concepts. He showed *why* we should avoid drawing an absolute distinction between pure mathematics and its applications in ordinary language.

[50] This is not to deny significant connections; Wittgenstein was influenced by Frege's understanding of the applicability of arithmetic. On the latter, see Steiner (2002, pp. 17-23). See also Russell (1959, pp. 110).

2.4 Naturalism

This discussion of the applicability of mathematics leads to the final interpretive strand to be compared with the strongly anthropological reading: naturalism. Here, naturalism will mean a focus on the relationship between mathematics and natural phenomena. Both in scientific endeavors and everyday life, humans employ mathematics to describe and predict events. The union of mathematics and science has proven remarkably successful, with technological innovations attesting to its potency. Incorporating these facts into a Wittgensteinian perspective may seem challenging, as mathematical propositions do not directly reflect natural phenomena. According to Wittgenstein (*RFM*, I-167-168, Appx. II-2), they are human inventions. The question, then, is why appeals to mathematics are so practical, effective, and reliable. Addressing this issue, Wittgenstein devoted considerable effort to exploring the nature of the applicability of mathematics in *RFM*, *LFM*, and, to a lesser extent, the *Investigations*.[51]

Naturalistic rule-following

One apparently direct way of accounting for mathematical usefulness is to postulate a correspondence between mathematics and physical reality, taking concepts and/or formulae of pure mathematics to correspond to lawlike patterns of nature. This is Penelope Maddy's (1990, p. 58-61) position, who argues for 'set theoretic realism', the Platonist conception of sets as independently existent elements of the natural world.[52] In her early answer to Wittgenstein's (arguably Kripke's/Wright's) rule-following paradox, Maddy (1984, 1990, pp. 79-80) appeals to a distinction between 'natural-' and 'random collections'. Triangles and squares are natural, not random, collections, and they are thus naturally distinguished. The natural distinction between triangles and squares is taken to guarantee that what I meant by "triangle" in the past is not '*triangle**', that is, 'a triangular shape until now; a square shape henceforth':

> *It is a brute fact that triangular figures are more similar to one another than to squares and that a natural grouping corresponds to this*

[51] Clarifying the applicability of mathematics is a major theme in Wittgenstein's later writings on mathematics. See Steiner (2002, p. 13) on the importance of the question of applicability.

[52] The adjective "Aristotelian" might be more accurate, but, as Maddy (1990, p. 158) notes, any view which credits mathematical entities with independent objective existence is now commonly called "Platonism" in the philosophy of mathematics.

> *similarity. […] But our reference is to the underlying kind responsible for that perceptual similarity.* (Maddy, 1984, pp. 470-471)

This remark is, in turn, justified by an appeal to scientific explanatory power. Attributing the natural kind 'triangle' to any given triangle is consistent with scientific practice and has more explanatory power than interpreting it as a 'triangle*' or any of infinitely many other strictly congruent but arbitrary possibilities. However, this appeal to the efficacy of science works on the assumption that scientists (or perhaps, in this case, geometers) have not in fact been studying triangles*, which itself seems impossible to rule out through empirical observation.

Maddy's (1984, p. 472) response is that "[n]atural individuals and kinds correspond to objective traits, figure in true scientific explanations, while the weird kind and odd worm [e.g. triangle*] do no such thing." This is accurate if taken strictly on its own terms, but the circularity of the argument shows that it fails as a response to the apparent paradox Wittgenstein airs in *PI* §201: How do we know that "the weird kind and odd worm" are not precisely the entities that scientific explanations have been addressing if we operate on the assumption that the meaning of terms is determined by *underlying* natural kinds?

Later, Maddy (2014) paints a naturalistic picture of Wittgenstein's views on logic and mathematics. According to Wittgenstein, on Maddy's interpretation, logic and mathematics are simply two among many other forms of rule-following, which itself is a natural phenomenon explicable by 'general facts of human nature' (cf. *PI* §143, *PoP* xxi §365). The main idea is that "our ability to follow rules, our practices of rule-following rests on a number of […] very general facts about us" (Maddy, 2014, p. 68), which is taken to open for physiological and neurological research to expand and inform our concept of 'rule-following' (ibid., p. 111).

However, as Maddy (2014, p. 102, cf. Pears, 1988, pp. 451) acknowledges, Wittgenstein's own position differed, denying the role of scientific explanations in philosophy:

> *"It was correct that our considerations must not be scientific ones. […] The problems are solved, not by coming up with new discoveries, but by assembling what we have long been familiar with."* (*PI* §109; cf. *PI PoP* xii §365)

For Wittgenstein, 'what we have long been familiar with' include general facts of nature without which rule-following practices would not get off the ground.[53] However, it does not follow that such facts *explain* rules or clarify the notion of 'rule-following' in philosophically illuminating ways. Wittgenstein included natural phenomena as features of practices, part of what makes rules within those practices reliable and intelligible to us, but not as *explanations* of the practices. Moreover, the idea of basing rule-following on empirical facts seems to turn the normativity of meaning into a kind of *determinism*: there are empirical facts – involving natural phenomena, psychology, and/or biology – which of their own accord determine our concepts and our rules. Wittgenstein expressly rejected that picture without resorting to skepticism about meaning, emphasizing the possibility of *alternative* language games.[54]

Standardization and practicality

If rules and concepts in mathematics are not determined by underlying natural kinds, the question remains how they *do* emerge and relate to natural phenomena. In addressing this question, Mark Steiner (2009) emphasizes the relationship between mathematical sentences and their canonical application in the later Wittgenstein's philosophy of mathematics. In particular, Steiner attributes to Wittgenstein the idea that mathematics is based on standardized empirical regularities. According to this view, some quantitative empirical descriptions, like "combining 25 pebbles 25 times gives 625 pebbles," must have been customary within a given social context.[55] Perhaps people regularly collected and counted up 25 heaps of 25 pebbles and got 625 pebbles. Subsequently, this result was standardized. Rather than describing what tends to obtain, it became a rule for what *should* obtain: combining 25 pebbles 25 times *should* give 625

[53] For instance, Maddy (2014, p. 69) quotes *PI* §142: "The process of putting a lump of cheese on a balance and fixing the price by the turn of the scale would lose its point if it frequently happened that such lumps suddenly grew or shrank with no obvious cause." Note that Wittgenstein was not saying that the *viability* of the practice of measuring in this way is explained by natural conditions, but that a given *counterfactual* condition (the irregular behavior of cheese) would render the practice, from our point of view, pointless.

[54] In *PI PoP* xii §366, Wittgenstein wrote that imagining different facts of nature can render intelligible the formation of alternative concepts. Pears (1988, p. 455) sees in this a tension in Wittgenstein's views. Note that, even though physical causes do not determine concepts or rules, forms of life involve natural phenomena, e.g. the environment. One way of understanding Wittgenstein is that hypothetically different language games become more immersive if they are embedded in forms of life that are somehow correspondingly different.

[55] The "$25^2 = 625$" example in this context is Wittgenstein's own (see *LFM* lecture IV and *RFM*, VI-23).

pebbles.[56] Textual evidence for this quasi(-pre-)historical line of thought, in Wittgenstein, includes *PI* §206-207, *RFM*, VI-39, *LFM*, XI, p. 107 and *LFM*, XXXI, pp. 291-292.

There might be numerous reasons for a sentence to transform from an empirical description to standard of description. For example, the counted objects could have been used as building material, and the need for a more reliable construction practice could have led people to adopt a system of quantitative prescriptions. There is no metaphysical obstacle to the standardization of other results, in principle, but there are strict *empirical* restrictions, according to Steiner (2009, p. 12). The adoption of the rules depends on the viability of 'hardening' empirical propositions (*RFM*, VI-22), which presupposes reliability in our results. People must have *generally* counted 625 pebbles when counting 25 heaps of 25 pebbles.[57]

In any case, it is only through normativity that mathematics as such can emerge. The aforementioned rule becomes constitutive for an arithmetical operation, so that if someone now apparently combines 25 pebbles 25 times and obtains any result *other* than 625 pebbles, we know they have failed to perform the operation 25 × 25. To this day, equations serve as fixed standards, set apart from confirmation or disconfirmation. For example, if we begin with three coins and obtain two more, but somehow end up with four coins, we do not conclude that $3 + 2 \neq 5$; we conclude that this story skips a step, and that one coin must have disappeared.

Wittgenstein's remarks in this context should not be taken as historical in an empirical or speculative sense. Rather, they can be taken as him filling in a philosophical picture. He rejected the idea that elementary arithmetic consists in *a priori* truths, so it cannot have dawned on us through 'ratiocination'. That is, we did not simply infer all of mathematics from a set of self-evident axioms. Likewise, Wittgenstein denied that mathematical equations are isolable rules without any broader function in linguistic and social practice, so it could not have sprung into existence through arbitrary stipulation. That, apparently, left him with an explanatory gap. He instead imagined arithmetic as

[56] Steiner's (2009) interpretation is wider in scope than this might suggest, taking grammatical rules to be based on empirical regularities in general. However, as he says (ibid., p. 7), arithmetic is the clearest case.
[57] See Bangu (2018) for an account of the regularity required.

coming about through a contingent historical process, as *normal* results of customary procedures were transformed into *normative* standards.[58]

This genealogical theme plays an important role in Wittgenstein's mature philosophy of mathematics. However, the details of the story matter. These will be unpacked over the course of the thesis, but at this point Wittgenstein's remarks in this context must be situated within the overall reading outlined so far. This section opened with a general observation of the practical nature of mathematics. Taken together with the genealogical theme just outlined, it might be tempting to conclude that the pragmatic value of mathematics derives directly from the descriptive value of the standardized propositions. After all, in order for "adding 25 pebbles into 25 heaps gives 625 pebbles" to have become prescriptive, it must have described a reliable empirical regularity; hence, mathematics is useful because it is a record of reliable information.

One retort to this would be to point out that empirical regularities need not be important or useful. Even if mathematics were a record of reliable information, that would not necessarily grant it utility. However, Wittgenstein, *LFM*, XI, p. 107 provided a stronger rebuttal:

> *Mathematical truth isn't established by their all agreeing that it's true – as if they were witnesses of it.* Because *they all agree in what they do, we lay it down as a rule, and put it in the archives. Not until we do that have we got mathematics.*

Steiner (2009, p. 9) quotes this passage in support of reading Wittgenstein as conceiving of mathematical theorems as 'synthetic' because their origin involves empirical regularities. However, Wittgenstein was here saying that mathematical standardization is based, not on natural regularities which everyone can witness, but on conformity in *action*: people agree in what they *do*. They count pebbles in the same ways, and therefore tend to agree when counting 625 pebbles. So, Wittgenstein's remark is precisely a rejection of the more empiricist idea that mathematical truth is based on standardization of empirical descriptions in and of itself.

[58] See *RFM*, VI-26. In fact, Wittgenstein left it open to what extent he intended this as an actual (pre-)historical thesis, or simply meant to illustrate the transformation of empirical propositions into mathematical formulae. In any case, the employment of the *form* of historical reasoning for the purpose of clarifying how an idea relates to historical facts has deep philosophical precedents. See, for instance, Rousseau (1754/1984, p. 78): "One must not take the kind of research which we enter into as the pursuit of truths of history, but solely as hypothetical and conditional reasonings, better fitted to clarify the nature of things than to expose their actual origin."

Elsewhere, Steiner (2000, p. 335) notes that, for the later Wittgenstein, arithmetical formulae involve a systematic ambiguity, quoting *RFM*, IV-21: "The twofold character of the mathematical proposition – as *law* and as *rule*". Insofar as a formula is applied, it corresponds to a lawlike regularity; we can use it to describe and predict the actions of human calculators, including ourselves (*RFM*, IV-44). However, this point should be distinguished from the genealogical theme and the idea that the usefulness of mathematics comes from recording empirical descriptions. Rules correspond to *behavioral* regularities, not natural phenomena.

That is to say, the standardization of empirical descriptions like "there are 25 times 25 pebbles", licensing its substitution for "there are 625 pebbles", leaves us only with a quasi-mathematical technique for modifying specific sentences. Even though arithmetic can be used in this way to form shorthand descriptions, a more distinctive function of arithmetic lies in its use for *calculation*, which allows transitions from one sentence to another, completely different sentence. By its very nature, the mediative function of mathematics cannot come about through the sheer standardization of empirical descriptions. As Wittgenstein highlighted, calculation involves following rules that have an altogether different role in language than empirical descriptions. This goes to show the "limits of empiricism" (cf. *RFM*, III-71):

> *"One says that calculation is an experiment, in order to shew how it is that it can be so practical. For we do know that an experiment really does have practical value. Only one forgets that it possesses this value in virtue of a technique which is a fact of natural history, but whose rules do not play the part of propositions of natural history."* (*RFM*, VII-17)

What Wittgenstein was pointing out was that the pragmatic value that mathematics adds to our empirical descriptions, like "combining 25 pebbles 25 times gives 625 pebbles", is not directly the result of turning them into rules. As Wittgenstein put it in *RFM*, VI-30, if practically everyone already agreed to that description, then "what do we need the rule for?"[59] Rather, the unique practical value of mathematics comes, at least in part,

[59] Notably, this remark (Ms-164,87) appears to be mistranslated in the 3rd edition of *RFM*, VI-30. Translated directly, it continues:"'$25^2 = 625$' cannot therefore be the empirical proposition that people calculate like that, because $25^2 \neq 625$ would in that case not be the proposition that people get not this but another result; and could be true if people did not calculate at all" [*"$25^2 = 625$" kann darum nicht der Erfahrungssatz sein, daß die Menschen so rechnen, weil $25^2 \neq 625$ dann nicht der Satz wäre daß die Menschen nicht dieses, sondern ein*

from enabling grammatical moves across disparate contexts, such as going from "there are 25 heaps of 25 pebbles" to "there are 625 pebbles". Note that, in this example, the latter sentence does not mention heaps.

This observation is relevant because it shows that mathematics allows for important grammatical moves which empirical generalizations could not already give us, and this function ties mathematics *directly* to other social practices. This point becomes clearer when considering transitions such as going from "there are 25 heaps of 25 pebbles" to "there is building material for a 2.5-meter-long wall" or "there are 50 silver coins worth of stone" or "there are 3 days of work left". The ratio, or conversion rule, in all such transitions is adapted to the practical situation at hand, making sense only against a background of language games which are already characterized by mathematics. Nevertheless, the calculations, and hence the inferences themselves, exemplify ubiquitous kinds of uses of arithmetic. So, although empirical regularities play a role as background conditions, they do not account for the practical value of mathematics. For that, as Wittgenstein saw, a more anthropological perspective is needed.

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anderes Resultat erhalten; & auch wahr sein könnte wenn die Menschen überhaupt nicht rechneten.]. Wittgenstein here rejects the idea that '$25^2 = 625$' asserts that people calculate like that, because, if so, $25^2 \neq 625$ could be true even if people never calculated. So, contrary to what Anscombe's translation *might* be taken to suggest, Wittgenstein was *not* saying that $25^2 = 625$ 'could be true' even if people did not calculate at all.

3 Paths among concepts

> *We* make *mathematics. Just as one speaks of 'writing history' and 'making history,' mathematics can in a certain sense only be made.*
> (*WVC*, p. 34; 18 December 1929)

The present chapter takes a closer look at the anthropological and linguistic bearing of mathematics from Wittgenstein's perspective. It contains several threads of thought elaborating and drawing from his later writings on mathematical activity, these threads being related by a common philosophical purpose. The aim is to get to a point where elementary mathematical behaviors, such as counting and basic arithmetic, can be understood anthropologically, as part of everyday life and ordinary language. Hence, the chapter opens by focusing on Wittgenstein's writings on what is involved in (proto-)mathematical capacities and activities.

Although these writings delve into logical issues pertaining to mathematical capacities and activities, they encapsulate Wittgenstein's uniquely anthropological perspective, which will become clearer as the chapter progresses. The culmination of these threads of thought will be that, for the later Wittgenstein, rather than pure mathematics serving as the basis of applied mathematics, it is the training and use of numerical and geometrical concepts in everyday, extramathematical practices that lays the foundation for mathematics as such. Additionally, it will become clear that the dichotomy between pure and applied mathematics is dynamic, the line being drawn and redrawn depending on the context.[60]

Language games and rule-following

In Chapter 2 it was argued that the concept 'language game' has an essential place in the later Wittgenstein's philosophy. That concept is both methodologically and philosophically pivotal, serving as a basis from which he approached other topics, including rule-following, meaning, and forms of life. Language games are practices involving language.[61] Any language game is part of a recurring surrounding of

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[60] A similar reading of Wittgenstein's view of this dichotomy is defended by Pérez-Escobar & Sarikaya (2022), building on Dawson (2014). The implications are spelled out further in Chapter 4, section 4.3.2 (p. 146).

[61] They involve language in general, but language need not be expressed in all their instances. As an analogy, competitive games involve a victor and loser, but, in many cases, they can also end in a draw. Subsequent discussions will occasionally refer to "linguistic practices", where this qualification should be kept in mind.

associated conditions. This same surrounding provides a backdrop of conditions which inform any attributions of rule-following within the game. Part of the motivation for using *games* as a metaphor for activities involving language is that games are variegated, mutable, modular (composed of distinct, reusable elements/aspects), and in some respects open-ended. For example, think of how the rules of chess can be adapted to create a new game, such as 'speed chess'. In this case, the change in time constraints creates a new variation of the original game. A mark of continuity within a game can also end up distinguishing games. For example, card game players freely adhere to their own idiosyncratic strategies, but if someone began deciding their card moves by throwing dice, this practice might wind up as its own game variant.

Hence, the remarks on the open-endedness of games and concepts, around §§54-68 of the *Investigations*, are related to the remarks on rule-following around §198. One consequence of this connection is that there is no Archimedean point from which to get a handle on what it means to 'follow a rule'. As can be seen from Wittgenstein's '+2'-example (*PI* §185), teaching a rule calls for engaging in, and conveying, a certain *context* that is presupposed in recognizing what it means to 'do the same thing', i.e. 'play the same game', from instance to instance. This is what correcting the pupil who writes "996, 998, 1000, 1004" comes down to: showing that he/she is no longer continuing to act consistently within the same context.

However, it is not necessarily enough to point out that 1004 – 1000 = 4, in contrast to all previous terms a, for which $a_n - a_{n-1} = 2$; and $2 \neq 4$, either. The deviant pupil in the case described by Wittgenstein (*PI* §185) insists that he/she is following the same rule as before, adding 2, and he/she says that 1004 is supposed to follow 1000. This misunderstanding is recognizable as a 'systematic mistake' (*PI* §143) because that continuation (adding 2 for each 1000) *does* recognizably constitute a rule that we could follow. Nevertheless, when *we* generate the terms of the +2 series, we write a different sequence. There might not be any principle that could be cited to *convince* the pupil of the correct way of proceeding. The sheer difference between the pupil's behavior and that of others is what implies that the pupil is deviating.

The point is that the act of following a rule is not to sign up for an infinite number of commitments, or to foreclose every mistake. A rule is not inscrutable, it is evident from *what is done* under normal conditions (*PI* §186-190). Moreover, after a successful learning period, an individual's actions are not mere interpretations of the rules, but are *exemplary* of engaging in the rule-governed practice under consideration (*PI* §§201-202).[62] This implies that, for the later Wittgenstein, language games – practices involving language – are conceptually prior to rules. Hintikka (1989) reaches a similar conclusion, documenting an underemphasized upshot of the development in Wittgenstein's philosophy in the period 1929-1934.[63] According to Hintikka, Wittgenstein at this point initially entertained an analysis of rule-following in terms of 'immediate experience' but then came to realize that

> *"The only "criterion" that can help us to decide whether a rule is being followed is ultimately the entire language-game to which the rule belongs. Even though Wittgenstein himself did not emphasize the fact, this* […] *implied a highly significant further conclusion. This conclusion is that language-games are conceptually prior to their rules. In the last analysis, a language-game is not defined by means of its rules. Any one of its rules can only be understood in the context of the entire game."* (Hintikka, 1989, p. 284)

Gone was the idea that individuals, upon reading out words on a page or going through the steps of a calculation, rely on their own interpretations of rules to direct their own behavior. What took the place of internal interpretations are historical, anthropological practices involving, at least potentially, the expression of publicly accessible language. As part of such practices (and only against the background of them) do individual human beings form and understand their own intentions and normative commitments. This chapter aims towards an understanding of the implications of this shift for Wittgenstein's later philosophy of mathematics, beginning with the topic of number, before moving on to calculation and proof.

3.1 The grammar of number

Throughout his career, Wittgenstein maintained a *structural* emphasis in his conceptualization of numbers. In his middle period, he held that the properties of a

[62] The span of *PI* §§201-202 marks a shift from a concern about the interpretation of rules to a question of rule-following in practice. With the mastery of a practice, a broader range of more or less creative uses of associated vocabulary opens, and thinking in terms of 'following rules' becomes potentially restrictive.

[63] Cf. Hintikka (1996), especially chapter 15; this is not, however, meant to signal agreement with other aspects of Hintikka's reading, beyond the primacy of language games being an upshot of thinking about rule-following.

number are properties of a position in a structure (*PG,* p. 457). That is, a numeral exhibits the properties of a number by occupying a given role in a system of notation. For example, the numeral "3" exhibits the properties of 3 in the decimal system because it is introduced via 3 instances of "1", that is, 3 = 1 + 1 + 1. Similarly, the sign "| | |" exhibits properties of 3 in a system of lines provided it is formed by 3 iterations of drawing a line; we could equally imagine a system in which "| | |" was used as a simple sign to form other numbers, in which case it would exhibit properties of 1. Wittgenstein argued in favor of this syntactic conception of numbers from 1929 through the early 1930s. Even at this point, however, he did not hold that numbers *are* numerals or lines, any more than they are beads of an abacus. Instead, numbers are the roles that such signs have been given in a system of calculation. As he maintained throughout his life, it is a confusion to focus on the *content* of mathematics as opposed to what is *done* with it: "Arithmetic doesn't talk about the lines, it operates with them" (*PG*, p. 333; cf. Rodych, 2018, §2.1; 2008).

This syntactic point of view, already centered on calculation, morphed into what might be loosely called a '*ludic*' approach. Rather than producing a new philosophical position about numbers, Wittgenstein's later thinking took a methodological turn, concentrating on the role of numerals in language games. The label "structuralism" accordingly becomes less appropriate, with the metaphor of position in a system fading into the background (cf. *RFM*, VI-11, VII-10). He focused on how numbers are used, in a less algorithmic sense, in actual practices. Still, some threads of thought remained, notably the rejection of an extensional view of numbers.

That is, Wittgenstein maintained a deflationary, non-Platonistic attitude to questions about the meaning (*Bedeutung*) of numerals and the reference to numbers.[64] The sentence "*K* refers to number *k*", where *K* is a numeral, functions similarly to "*K* is the *k*th item on the list". The referent (the number or the list-item) is not an object which exists or fails to exist independently of the system or list to which both it and the numeral or list-index belongs. A number 'exists' through our ways of counting and

[64] As Floyd (2021, p. 7) puts it, "For Wittgenstein, mathematics may be said to be 'about' numbers, aspects of concepts, and so on, but only in an ordinary language sense familiar from Austin." Note that this contrasts with formalism, which (in nominalist formulations) outright denies that mathematics is 'about' anything except signs.

calculating. Therefore, we should speak of 'number systems' rather than merely 'systems of notation', although numbers are inseparable from numerals.

There are a variety of uses of numerical terms in ordinary language, as adjectives, adverbs, and singular terms. We also have various methods of numerical comparison, of which *counting* is probably the most widespread and fundamental. Other methods involve juxtaposing classes of different kinds (objects, names, events, etc.) and relating member of one class to a member of another. These latter methods, however, are limited: they can be used to determine whether classes of things have the same number of members (cf. *RFM*, III-6), but they do not determine *what* number that is.[65] The upcoming section will go into the role of numbers vis-à-vis correlation and counting, examining how Wittgenstein understood their interrelation.

Number and correlation of classes

Rather than the technique of counting being a consequence of a number system, Wittgenstein tended to describe matters the other way around: Children learn counting prior to learning mathematics, and counting plays an essential role in establishing and using our number concepts. Sequences are themselves understood in terms of activity:

> *"We learn an endless technique: that is to say, something is done for us first, and then we do it; we are told rules and we do exercises in following them; perhaps some expression like "and so on* ad inf.*" is also used, but what is in question here is not some gigantic extension."* *(*RFM*, V-19)*

A number system is linked to a way of counting. The way people count in a given number system *ties down* the use of "same number" within that system, allowing "count" to be used as a metonym for "establish equinumerosity." For example, if I count to n using the natural numbers, I can count to $n/2$ in equally many 0.5 increments, which means that $n \times 0.5 = n/2$. Consider the case of counting knives and forks on a table when there are n knives and $n \times 2$ forks. In this case, I can arrive at the number of forks by counting twice for each knife.

[65] An apparent exception to this is measurement. We can arrive at a number by measuring – juxtaposing a sample, e.g. a ruler or counterbalance, to an object or collection – but here the extent of the sample has already been counted as such. For instance, we take for granted that a given ruler really is 3 meters long. A sample or object of comparison is not a standard of measurement; it has itself already been measured and is, on that basis, used to measure other things (cf. *PI* §50, section 3.2.1 (p. 87)).

For Wittgenstein, the way we understand a proposition such as 'There are as many *F*s as there are *G*s' presupposes a method of counting the *F*s and the *G*s in the same number of steps. Here it might be objected that we can understand 'There are as many *F*s as there are *G*s' independently of whether we actually have a method of counting the *F*s and the *G*s in the same number of steps. That is a natural thought: someone might grant that the *truth* of the proposition 'There are as many *F*s as there are *G*s' requires a way of establishing the equinumerosity of *F*s and *G*s but still argue that we should be able to *understand* it even if it is false, given that the numbers of *F*s and *G*s (e.g. knives and forks) is a contingent matter. While that is true, it is also true that our understanding of correlation involves a given method of counting.

Russell and Frege sought to define numbers as equivalence classes. To this end, they drew on Hume's Principle, the idea that the number of *F*s is equal to the number of *G*s just in case there is a '1-1 correspondence' between the things that are *F* and the things that are *G*.[66] For Russell (1920/1993, pp. 18-19) a number is a set of all classes in a 1-1 correspondence with one of its members (for example, 3 is the set of all classes in a 1-1 correspondence with any given trio). Wittgenstein criticized this appeal to 1-1 correspondence on several occasions throughout his career. In 1931, he argued, first, that what would be required for the definition to work is the notion of a *possible* 1-1 correlation, since two classes can be equinumerous even if they have not *actually* been correlated. Secondly, he argued that the *possibility* of correlating *F*s and *G*s 1-1 presupposes precisely that the *F*s and *G*s are equinumerous. This being so, the strategy of appealing to 1-1 correlation does not succeed in avoiding the concept of 'number':

> *"When Frege and Russell attempt to define number through correlation, the following has to be said: A correlation only obtains if it has been* produced. […] *But if in this whole chain of reasoning the* possibility *of correlation is meant, then it presupposes precisely the concept of number. Thus there is nothing at all to be gained by the attempt to base number on correlation."* (WVC, p. 165)

He continued to make similar points with regard to correlation (*LFM*, lecture XVI).

[66] See Russell (1920/1993, pp. 15-19) and Frege (1980a, pp. 73-74). For Frege, numbers are not sets of sets. Rather, for any concept *F*, the number that belongs to *F* is the extension of the concept 'equinumerous to *F*'.

Wittgenstein here addressed the *concept* of 'number'. He should therefore be taken to say that a *concept* of 'sameness of number', not actual sameness of number, is presupposed by the possibility of correlating *F*s and *G*s. In other words, he was not talking of the number of *F*s and *G*s itself. A concept of 'sameness of number' is a sense in which two classes may or may not be equinumerous, whether or not they *are* equinumerous. One such concept would be for the same number to be reachable by tallying *F*s, then, tallying *G*s, by hand. So, say that there are, for example, 15 *F*s. The *possibility* of correlating the *F*s with the *G*s, in that sense, would not presuppose that there are exactly 15 *G*s. Rather, it would presuppose the availability of a method, a *way of counting* the *G*s that aligns with the way in which the *F*s have been counted.

Stated generally, Wittgenstein's argument is as follows: (1) If it is possible to 1-1 correlate the *F*s and the *G*s, then there is a way of counting that there are as many *F*s as there are *G*s.[67] Counting may not be physically possible, but two classes that can be 1-1 correlated in principle could also in principle be counted through in the same number of steps. (2) A way of counting presupposes the concept of 'number'. (3) Therefore, the possibility of correlating the *F*s and the *G*s presupposes the concept of 'number'. This argument does not assume that an *actual* counting procedure takes place in any process of correlation. Wittgenstein's remarks here should instead be reflective of his distinction between counting in mathematics – which includes setting up, modifying, and elaborating different *methods* of counting – and ordinary counting, that is to say, actual procedures of counting collections of things (*LFM*, XII, p. 114, *RFM*, VII-36, *RFM*, VII-18). This distinction will be explored further over the next chapters.

This implies that Wittgenstein's dissent from definitions of 'number' in terms of correlation is not as strong as it might first appear. Indeed, the possibility of 1-1 correlating *F*s and *G*s might be a useful criterion for the sameness of number of *F*s and *G*s. What he denied is that the possibility of correlation suffices to *explain* the concept of 'number' as such: "I don't want to run down Russell's definition. Although it does not do all of what it was supposed to do, it does some of it" (*LFM*, XVI, p. 156). That

[67] Cf. De Bruin (2008) and Marion & Okada (2014). I take it that Wittgenstein argued that criteria of sameness of number are relative to a number system, not merely that we have different sources of knowledge of equinumerosity. He was making a grammatical, not epistemological, point. Note also that Friedrich Waismann's (1951) treatment of the topic, and its exegetical value with respect to Wittgenstein, is not considered here.

is, Wittgenstein did not reject correlation as an explanation of 'number' for the reason that he regarded correlation as unimportant or inconsequential for the use of numbers, but for the reason that we employ different *methods* of correlating *and* counting, and it is not always clear what these methods would involve:

> *"We must always think of* number *as we think of* length *or* weight*, and of* counting *or* correlating *as we think of* weighing *or* measuring*. We say that two things have the same weight if on the balance they counterbalance each other, or counterbalance the same number of weights. If we are trying to find out whether there are the same number of people here as in the next room, one method we can adopt is one-one correlation: we could tie a string to each man here and attach it to one there. Then if there is one without attachment, . . . This is one way of finding numerical equality."* (*LFM*, XVI, p. 156)

This does not suggest a radical difference between counting and correlating; Wittgenstein likened them to each other. At the same time, he highlighted the multiplicity of methods for both counting and correlating (Frascolla, 1994, p. 46). The point of doing so is to show that it is not always clear how one would practically go about correlating two classes of things, even in principle or hypothetically. We might in some way know that two classes *could* be mapped onto one another, but, if we can conceive of no clear correlating method, we would know *that* only by relying on some *other* criterion to determine equinumerosity:

> *"At first you thought of cases where the correlation was the criterion. But if the correlation isn't possible, then it is the other way round: if they have the same number by such-and-such a criterion, then it is possible for them to be correlated."* (*LFM*, XVI, p. 158)

A practical example can be given to show that correlation is in some cases not possible, in a practical rather than logical or metaphysical sense. We are unable to correlate the leaves on the trees of a given forest with the waves in the Baltic Sea. There is no reliable *way* of relating each leaf to a unique wave in the Baltic Sea, whether by physically juxtaposing them, connecting them by string, or naming them and comparing the list of names, such that it would be clear whether any *two* attempts at (or results of) correlation confirm or refute each other.[68] More indirect empirical methods might be devised. For

[68] On a similar note, Bangu (2016, p. 239) reads Wittgenstein (in *LFM*, XVI, p. 160) as arguing "that we know *what to call a one-to-one correlation* is presupposed in Russell's definition."

instance, the number of leaves in the forest could be calculated based on an average number of leaves per square meter, and the number of waves can perhaps be estimated using physics. This would allow for a statistical comparison, but this comparison would not itself be a correlation. So, these two classes cannot be correlated.

That is admittedly an extreme example, but it highlights the issue. Büttner (2016) offers an alternative view on this, arguing to the contrary that the notion of 'correlation' should be understood widely enough so as to include counting.[69] Expanded to encompass counting, it seems unavoidable that 1-1 correlation would serve to define 'number.' There are grounds for thinking of correlation in such fundamental terms. In mathematics, equinumerosity is often *defined* via bijective functions. And, in practical contexts, it is often the case that 'count the *F*s' involves correlating (pointing at) *F*s with (voiced) numerals. Indeed, as mentioned, Wittgenstein (*LFM,* XVI, p. 156) *did* recognize counting as closely connected to correlating.

However, the similarities do not collapse the distinction. Counting is not strictly object counting; we also count to keep track of time, organize lists, maintain and update scores, and so on, and these activities can hardly be boiled down to 'correlation' or 'establishing a 1-1 correspondence.' Indeed, even object counting is not necessarily correlative. Consider the process of finger-counting by extending one's fingers one by one. This process does not necessarily involve correlating fingers with numerals. It is an important fact about finger-counting that it produces sets of increasing cardinalities, i.e. the extended fingers (cf. Wiese, 2003, pp. 136-139). Extending fingers *might* be accompanied by voicing numerals, but that is not a strictly necessary part of the process. We can just state the result at the end.[70]

Further, people often correlate things as a *consequence* of assigning numbers to them, whether by counting them as groups or as coinciding terms of series. For example, historical events can correlate because of the way we count years and days; events might share a given relation to *other* dates or events (cf. *RFM*, III-9). An economic report might go as follows: "Increased spending on Fridays correlates with …" Here, the

[69] See also Schroeder (2021, pp. 19-21).
[70] That might involve subitizing (see section 3.2.2, below), but not necessarily so. As Wiese (2007, p. 766) points out, finger-counting in various cultures follows stable conventional sequences. Going through the sequence, or even just a segment of it, can itself serve as a criterion for having counted to a certain number.

structure of the weekdays serves as a basis of the sense of the correlation. Moreover, things are often correlated through rankings, as in the case of two marathon runners both placing 5th in separate runs.

Indeed, correlation is often a product, not just of a method of counting, but of a specific sequence: the alphabet is sequenced so that "A, B, C" correlates with "1, 2, 3". From an ordinal point of view, considering numbers as terms in sequences, *correlation* can often be seen as a kind of *counting*: counting two classes in sequentially equivalent ways. From that perspective, even the syntax of English sentences pertaining to correlation exhibits a method of counting. We are familiar with sentences of the form "each *x* is paired with a *y*". Upon reading "each knife is paired with a fork", we do not infer that each fork is paired with a knife, because 'knife' sequentially *precedes* 'fork' (Everett, 2020, pp. 208-209; cf. Wiese, 2003, 2007).

Büttner (2016, p. 174) proposes a way of understanding 'correlation' which zeroes in on the idea of two classes in principle being equinumerous *when* they are correlated:

> *'The Fs can be one-one correlated to Gs' might be interpreted as 'When the Fs are one-by-one correlated to the Gs, they are co-correlated', where 'The Fs are one-by-one correlated to the Gs' means: Any F is correlated to at most one G; any G is correlated to at most one F; and there is no pair of an F and a G such that the former is not correlated to any G and the latter is not correlated to any F.*

This analysis appears to read 'the *F*s can be 1-1 correlated to *G*s' subjunctively, as a modal notion,[71] ensuring that it does not presuppose actual sameness of number. However, given that some correlations are not practically possible, it is not always clear what "when an *F* is correlated to a *G*" means. To reiterate the example given earlier, it is unclear what would be meant by the sentence "Any leaf in the nearby forest is correlated with at most one of the waves in the Baltic Sea," making it equally unclear to talk of "when" such a statement holds true.

In general, sameness or difference of number is not captured by the concept 'possible correlation', since correlation is not always possible, and to say that correlation is not possible here means that it is unclear what correlation would

[71] Cf. Wittgenstein on *possible* correlation in *LFM*, XVI, p. 157 and Bangu's (2016, p. 244) discussion.

hypothetically *involve*. We can make the stipulation to call it '*logically* possible' for *F*s and *G*s to be correlated in cases when *F*s and *G*s as a matter of practical fact cannot be correlated, but this notion would have to apply only in those cases in which the *F*s and *G*s are *actually* equinumerous.[72] There might be any number of methods by which the *F*s and *G*s could be determined to *be* equinumerous, but if we cannot correlate the *F*s and *G*s, none of these methods would amount to correlation. So, as Wittgenstein argued, treating 1-1 correlation as a *general* definition of 'number' involves circularity. There are multiple possible methods that might be called 'correlation', leaving its meaning unclear aside from the requirement that the sameness of number is preserved.

Sameness of number manifests a counting method

What, then, of Wittgenstein's positive understanding of statements of sameness of number? It was claimed that the way we understand a proposition such as 'There are as many *F*s as there are *G*s' presupposes a method of counting the *F*s and the *G*s in the same number of steps, a proposal that seemed to threaten the possibility of understanding such a proposition whenever it is false. However, whether the *F*s and the *G*s are counted in the same number of steps depends on *how* they are counted. Whether there actually happens to *be* as many *F*s as there are *G*s is a different question. So, we might still understand such a question even if its answer is negative.

And yet, asking whether there are as many *F*s as there are *G*s *does* involve a way of counting the *F*s and the *G*s in the same number of steps. After all, we have to treat the *F*s and the *G*s consistently to relate their respective quantities arithmetically, in terms of 'more,' 'fewer,' or 'as many.' It might then be asked why 'treat the *F*s and the *G*s consistently' should be conceived in terms of counting them in the same number of steps. To answer this with an example, consider the sets $A = \{a, b, c\}$ and $B = \{d, e, f, g\}$. Correlating them involves identifying members of *A* and *B*, one by one, and pairing them, which requires a given *number* of steps. That is, the game of correlating the sets involves distinguishing a given number of pairwise combinations of members from both

[72] See Goodstein (1951, p. 19; 1956). As he argues, the ability to establish a 'logical correspondence' between two classes is a consequence of, not a condition for, the two classes having an equal number of members.

sets, regardless of what might remain to be counted after the actual correlation.[73] Precisely the fact that we go through them in the same number of steps allows us to see one uncorrelated member in *B*. Of course, spatiotemporally, we can go through the sets independently, but we must align our methods of counting them.

It is possible to make a mistake when trying to correlate *A* and *B*, for example by pairing *c* with two members of *B*. This is a matter undercounting or overcounting either *A* or *B*, or both. The possibility of pairing *c* to two members of *B* is the applicability of the equation $1 + 1 = 2$ to model those pairings, and rectifying it, removing one of the pairings, is the application of $2 - 1 = 1$. It might be objected that another way of describing such a mistake is available, one which does not involve numbers; any *F* is meant to be correlated to a unique *G*, and we can visually distinguish unique *F*s and *G*s. That is true, but that simply describes a given correlation between the members of the two classes, when what is at issue is the *possibility* of correlating the two classes. The question is what *constitutes* 'a *F*' and 'a *G*', when pairing them, and the answer is whatever would be *counted as* an *F* and a *G* during the procedure.

The word "correlate" is both a telic and an atelic verb, meaning that it is used both for the process of attempting to correlate and the accomplishment of correlating two classes. If correlating *F*s and *Gs* 1-1 fails, we might use the telic verb and say "we did not 1-1 correlate *F*s and *G*s, after all". In typical cases, however, 'there are as many *F*s as there are *G*s' would be uttered when we discern an immediate link between elements of classes, so in a certain sense we see *F*s-and-*G*s,[74] or when the number system intended – how, in general, we would count members of the two classes, and how we would correlate them – is obvious. Other counting methods, though entirely possible, are tacitly ruled out. We are typically selective about which classes of objects to correlate precisely because we are after a specific correlation: in setting the table, we only count knives and forks that have actually been brought to the table.

[73] The dichotomy can thus be put in terms of *distinguishing* versus counting, as in Frascolla (1994, p. 66): "[B]y the statement that there are six permutations of a three-element set, such permutations are *distinguished*, as are the cases in grammar, not *counted* (and the same holds true for the algebraic theorem that there are two roots of a second degree equation, etc.)". In *LFM* (XII, p. 114) and *RFM* (VII-36, VII-18), Wittgenstein generally instead talked of counting in mathematics and ordinary counting (e.g. tallying for practical purposes).

[74] Consider, say, 3 pears and 3 apples in a bowl; they can immediately be perceived as equinumerous, exhibiting an obvious symmetry. Cf. *TBT*, p. 414, Marion & Okada (2014, p. 68), and section 3.2.2 below, on subitizing.

In any case, whenever we *attempt* to correlate two classes, we use a given number of steps. When trying to draw a line from each *F* to each *G*, we try to draw as many lines as there are *F*s. Obviously, people do not have to recite numerals while drawing the lines, or when they juxtapose objects or names thereof. Nevertheless, correlation is justified by appeal to a counting method, by *how many F*s and *G*s we take there to be, not the other way around.

So, in conclusion, given that a counting method is bound up with a number system, 1-1 correlation *does* tacitly presuppose a number system, no matter how natural or intuitive we might find the correlation procedure to be. The procedure of 1-1 correlation *manifests* the concept of 'number', through the employment of a counting method, and, by the same token, the possibility of 1-1 correlation cannot serve as an independent foundation for 'number'.

Wittgenstein suggested in *RFM*, III-47 that, when it comes to arithmetic, counting methods are more important than, and should not be conflated with, methods of 1-1 correlation:

> *For arithmetic, which does talk about the equality of numbers, it is indeed a matter of complete indifference how equality of number of two classes is established – but for its inferences it is not indifferent how its signs are compared with one another, and so e.g. what is the method of establishing whether the number of figures in two numerical signs is the same. / It is not the introduction of numerical signs as abbreviations that is important, but the* method *of counting.*

The idea here is that the method of counting associated with a given number system is essential to that system because it is employed to construct the numerical signs composing that system. For example, someone counting in a given setting might produce a string of numerical signs such as "1/3, 2/3, 3/3, 4/3", etc. In another context, someone might begin counting with the same first three signs, but then take a different path: "1/3, 2/3, 3/3, 1/4, 2/4, 3/4, 4/4, 1/5", etc. The first person keeps count by adding thirds, while the latter counts through each denominator.

Here, the different uses of fractions constitute different systems, which is manifested in the fact that the numerical signs are counted in different ways; different signs have been given sense, and not given sense, in the different contexts (cf. *RFM*, II-

42-52). We have to be careful not to be carried astray by the naturalness of using whole ordinal numerals to describe such sequences. We count "2/3" as the 'second' term from the left in both strings, above, but this sign is the 'two thirds' term *within* these systems; it occupies, not the 2nd, but the 2/3rd position. "4/3" has the position of 4/3 in the first system and "1/4" has the position of 1/4 in the second system; the first system has no 1/4th position and the second system has no 4/3rd position.

Wittgenstein noted that we are easily misled by the rendition of signs on a page, thinking of them as similar objects. [75] An example of that is our propensity to consider numerals in different number systems as mere letters or sounds. A numeral cannot be divorced from a method of counting, except for typographical purposes. That does not imply that Wittgenstein saw natural numbers '*as*' ordinal numbers, i.e. 'first', 'second', etc., in contrast to cardinal numbers.[76] Rather, he highlighted the variety of uses of numbers in people's lives.

Moreover, Wittgenstein (*RFM*, III-47) pointed out that a method of counting is connected with a kind of arithmetic, or, more generally, calculation. In the first number system above, but not the second, counting to n/m takes half as many steps as counting to $2n/m$. Both systems might make use of the equation '1/3 + 1/3 = 2/3', but in the first system 3/3 + 1/3 = 4/3 whereas in the second system 3/3 + 1/3 = 1/4. That latter form of calculation might seem artificial, if not simply wrong, but applications can be hypothesized. For example, imagine water pouring into a bucket which is placed inside a larger bucket, inside a larger bucket, etc. Using 'u/v' as parameters for the water level u of bucket v, the calculation 3/3 + 1/3 = 1/4 might be used as a model of the water overflowing from one bucket into the next.[77]

This is not to suggest that such different ways of operating with numbers are absolutely incommensurable, as if there could not possibly be conceived ways of

[75] See *PI* §11: "Of course, what confuses us is the uniform appearance of words when we hear them in speech, or see them written or in print. For their *use* is not that obvious. Especially when we are doing philosophy!"

[76] It is open whether to view natural numbers first and foremost as ordinals or cardinals, the latter being favored by Frege, Russell, and contemporary neo-logicists (cf. Linnebo, 2009a, pp. 63-64; 2009b). Although ordinal numbers and well-orderings (and, more colloquially, lists) play important roles in language and everyday life, this for Wittgenstein should not be taken to show that natural numbers '*are*' finite ordinals (cf. *TBT*, p. 396).

[77] Notably, associativity here would not hold, e.g. (3/3 + 1/3) + 1/4 would not equal 3/3 + (1/3 + 1/4). More familiar examples, such as modular arithmetic or complex numbers, illustrate how distinct concepts are incorporated *alongside* one another as part of different but compatible practices (see e.g. Peck, 2018, p. 6).

combining, comparing, or generalizing them. Both of these systems *could* be treated merely as different ways of using rational numbers; for the later Wittgenstein (*PI* §68), number systems are not insulated, and there are family resemblances between different kinds of number. At the same time, numbers are shaped by the kinds of activity people engage in when using them (cf. *RFM*, I, Appx. I-9-10). We *do* sometimes need a more fine-grained look at a practice of operating with numbers in order to appreciate what does, and does not, make mathematical sense in a given setting.

3.1.1 Counting as an elementary technique

Counting is taught and learned as a rote skill, comparable to the memorization and recitation of the alphabet. In particular, elementary school pupils learn the series of natural numbers and are expected to be able to continue reciting the series indefinitely. This skill is demonstrated in completing an assortment of tasks, often through basic applications (say one's age, count small collections, tell the time, etc.), involving the first dozen or so verbal numerals along with additional milestones, such as "one hundred", while learning to write the Arabic numerals for the corresponding number of digits. If a pupil deviates substantially, confusing one term for another in the series or for some unrelated expression, or shows significant effort or hesitation in attempting to continue the series, this is considered a discrepancy that has to be addressed.

Wittgenstein highlighted the uniformity that is achieved in the teaching and learning of counting, and suggested that uniformity is partly constitutive of what we call "counting":

> *"We should presumably not call it "counting" if everyone said the numbers one after the other* anyhow*; but of course it is not simply a question of a name. For what we call "counting" is an important part of our life's activities. Counting and calculating are not – e.g. – simply a pastime. Counting (and that means: counting like* this*) is a technique that is employed daily in the most various operations of our lives. And that is why we learn to count as we do: with endless practice, with merciless exactitude; that is why it is inexorably insisted that we shall all say "two" after "one", "three" after "two" and so on."* (*RFM*, I-4)

Counting is an *elementary technique*, 'elementary' both in the sense of being rudimentary and in the sense of playing a role in all kinds of meaningful activities.

Despite being rudimentary, the utility of counting stands and falls on the specifics; even just a single deviation, e.g., uttering "6, 7, 9, 10, 11" when attempting to count objects, would in most cases not be considered 'counting' but, rather, a mishap to be corrected. Any decisions made on the basis of the results of this act of would-be counting would, if the error was discovered, be deemed ill-advised.

What we describe as 'making a decision on the basis of counting' means to act in a way that depends on the specific number obtained at a given position in the sequence, and this is precisely what deviates whenever a mistake of such a kind is made. So, Wittgenstein observed, the reason we stridently demand conformity when it comes to learning to count has to do with both our reliance on counting under variegated conditions and the specificity of that reliance.

Consider the following example, similar to several of Wittgenstein's own (e.g. *PI* §2, *OC* §564, and *RFM*, I-143): Say that *A* and *B* are foresters tasked with counting trees in a given region, perhaps for logging or conservatory purposes. They take turns so that each day, after a period of counting trees, *A* reports on the progress to *B*, before *B* takes over and continues. These hypothetical foresters maintain their language game by jointly adhering to a counting method, with the report shared after each session effectively being an instruction on how to keep going in the same way. Say that both *A* and *B* had been carving lines into a tree every 10 minutes of walking at an even pace, increasing the number of lines each time. Now *C* joins and takes the third shift, counting a new numeral out loud for each tree perceived.

This might be a substantial change. For *C* to actually count in the same way as *A* and *B* would mean for there to be no problem combining the results of *C*'s counting with those of *A* and *B*. Only when their efforts flow seamlessly into one another would *C* count 'the same way' as *A* and *B* do. The different methods used by the foresters might yield equivalent answers to 'How many trees are there in the region?', but they may nevertheless constitute different language games. One way of seeing that is by looking at the number of steps in the counting procedure, each of which *might* be considered a separate move or turn in the game. For example, if the foresters conclude that there are 50,000 trees, having counted this by tracking 10-minute intervals of roughly 100 trees each, the counting procedure had 500 steps; if each tree was counted individually, this took 50,000 steps. Whether we here should consider these two *different* language games depends on our clarificatory aims and concerns (cf. Kuusela, 2019, pp. 156-159; 2008). Whether to draw a distinction between methods depends on the *context* of the activity: Averaging the trees per 10-minute interval would fit into certain *other* language games better than other methods would. Say the foresters were to be remunerated; the interval-counting might then be fed directly into a calculation of wages per hour of work.

In any case, this illustrates that correlating objects falling under two different concepts does not merely identify numbers, but joins up two language games characterized by counting. To say that there are as many *F*s as there are *G*s is to overlay counting *F*s with counting *G*s. Even if the two variables vary independently of one another, correlating them joins the methods by which they are counted. To pay the foresters a wage in proportion to the number of trees in the region is to link counting trees to remunerating foresters. For the practices to be linked in this way, the precise number of *F*s and *G*s, or trees in the forest and wages paid, is inessential. The point is that the procedures are henceforth considered and treated *as* mutually related. The act of counting what to pay someone is taken to correspond to counting their hours of work.

One technique, many methods

The above goes to show that counting is grounded in methods which also contribute to our recognition *of* given practices *as* the practices that they are. Counting is a uniform technique, in a logical sense even if not in a historical and geographical sense (cf. Everett (2017, pp. 113-118)), even though the way this technique is employed can, and often does, vary from case to case. In other words, for the later Wittgenstein (*RFM*, III-15, VI-43, *LFM*, p. 31) counting constitutes *one* technique involving *many* methods, spread across indefinitely many language games (cf. Schmidt, 2015). This suggests that counting is a form of rule-following, a given technique that can be extended indefinitely, while at the same time this form of rule-following takes on an abundance of *different* roles in people's lives.

Although these two aspects might seem contradictory, they are synthesized by considering counting as a matter of making moves in language games. Methods of counting differ from one another, the language games differ, but counting is nevertheless uniform across distinct linguistic contexts, similarly to how, for instance, a set of distinct games may all feature *turns* (cf. *RFM*, III-66-67). Different approaches to counting might be incompatible for a given purpose, but, overall, the technique of counting produces and exhibits agreement:

> *"It is a fact that different methods of counting practically always agree. / When I count the squares on a chess-board I practically always reach '64'. / If I know two series of words by heart, for example numerals and the alphabet, and I put them into one-one correspondence: a 1, b 2, c 3, etc. / at 'z' I practically always reach '26'."* (*RFM*, III-15)

People are able to use and combine different methods of counting consistently. This is connected with Wittgenstein's remark that rule-following is *blind* (*PI* §219), a metaphor that helps clearing away certain misconceptions about what counting, and numbers, involve. How these two ideas are connected is what will be explored in the present section.

The process of counting specified items, i.e. the counting *of countables* of some kind, can be called 'transitive counting'.[78] Such processes often involve visually or physically 'taking hold' of countables (*OC* §459, §§510-511), e.g. pointing to a tree while voicing a numeral or carving an increasing number of strokes into the nearest tree at regular intervals. Nevertheless, these steps can be taken without having *any* belief about, or specific thoughts pertaining to, the objects counted. We attribute mastery of complex and informative methods of counting to individuals who are able to perform the counting procedure *unthinkingly*.

While a routine technique, counting is at the same time a *custom* (cf. *PI* §199, *RFM*, VI-21), a normative regularity in a given setting. That is, to learn to count is to master a behavior in which others already engage. In the transitive case, we recognize someone as being able to count when they succeed in going through procedures of counting in the same ways as others do. For example, children are often taught to finger-count, and are expected to get a specific answer, the *same* answer as practically

[78] See Schroeder (2021, p. 76) and Benacerraf (1965, p. 49) on transitive and intransitive uses of "to count".

everyone else get when counting fingers. The procedure of counting is here taught *together with* its correct results.

So, Wittgenstein's (*RFM*, III-15) remark about counting methods agreeing with one another, quoted above, should be seen together with the remark that, "One does not learn to obey a rule by first learning the use of the word 'agreement'. Rather, one learns the meaning of 'agreement' by learning to follow a rule" (*RFM*, VII-39, cf. *PI* §224). The notion that the concept of 'agreement' is taught along with a rule is a key element of Wittgenstein's later perspective. To illustrate the idea, say that there is a rule for transposing information from one table into another. The rule says: Given a cell *C* containing some string '*x*,*y*', find the cell with coordinates *x*,*y* in another table, and enter the coordinates of *C* into that cell. So, take Table 1:

	a	*b*
c	*b,d*	*a,d*
d	*b,c*	*a,c*

By following the transposition rule and going through all the cells, we construct Table 2:

	a	*b*
c	*b,d*	*a,d*
d	*b,c*	*a,c*

Table 1 and Table 2 are not merely visually identical, but also, with respect to this transposition rule, isomorphic to one another. Structural isomorphism is a matter of rule-bound agreement; it is *one* example of agreement in Wittgenstein's (*RFM*, III-15, VII-39) sense. The tables agree with one another in the sense that they are used in the same way for the purposes of the rule. This can be seen from the fact that the rule is perfectly reversible: Table 1 can equally be constructed by beginning with Table 2 and transposing each of its cells into Table 1.

Fundamentally, a relation of agreement holds between the tables as a consequence of the fact that people can follow the transposition rule irrespective of whether they begin, and end up with, Table 1 or Table 2. Given the rule, these tables do not lead to conflict. Two people can collaborate even though one begins with Table 1 and the other with Table 2. In a similar way, the procedures of (1) reciting the alphabet from "A" to "Z" and (2) counting 26 numerals do not clash, and can be done in concert, in that sense agreeing with one another.

Agreement between methods of counting, then, is not a consequence of the constancy of physical or metaphysical laws (or chess boards having a solid construction, or letters of the alphabet never varying (cf. *RFM*, III-15)), but of similarity and compatibility in rule-bound procedures. Similarity and compatibility come together with an absence of conflict among rule-followers. So, the tendency that different results of counting agree with one another is not a happy accident, as it were, but reflects the custom-based, practice-bound nature of rule-following. We have many methods of counting, but, on the whole, these methods produce results which agree with one another. In light of this, counting can be considered *one* technique.

Coordination and correctness

Despite the fact that counting is a single technique, counting should for Wittgenstein be understood as part of an interplay between language games and the world, as part of people's efforts to coordinate and achieve procedural agreements of various kinds. Thus, the counting methods people use depend on their motivations and practical relations to the things that are counted. For example, measure words like "slice" and "part" are used to modify acts of counting and apply them appropriately in different settings. If one were to serve pie to one's guests, asking "how many 8^{ths} of pie do you want?" might help specify how, and whether, the answers/requests might be fulfilled. Alternatively, the size of each slice can be left unspecified, allowing the guests to request varying numbers in total while varying the size of each slice.

For the concept 'countable' to apply to some concept '*F*' requires that we typically get a single result across all methods used to count *F*s whenever the number of *F*s is constant. If results vary unsystematically, the concept is uncountable (*RFM*, I-37, cf. *RFM*, I-69, *RFM*, VII-61).[79] If different methods do not agree, the difference between, for instance, "there are 18 *F*s" and "there are 19 *F*s" becomes unclear. So,

[79]Accordingly, part of what it means to say that e.g. 'water' is *not* countable is that it is unclear where the line between different methods and different results of counting goes, whether to count a given body of water as '1 puddle', '5 drops', etc., due to the behavior of water. See also *OC* §558 on liquids in language games.

some concepts, such as ‘squares on a chess board’, ‘letters of the alphabet’, and ‘fingers on a healthy hand’ are made part of criteria for counting. That is, counting specific sums (i.e., 64, 26, and 5) with respect to those concepts is taken as part of what it means to know how to count. However, even if getting those sums is part of knowing how to use specific counting methods, those sums are not ‘necessary truths’. For example, different cultures have differing methods of counting on their fingers.[80]

Proficiency when it comes to intransitive counting is evaluated at the same time: reciting (vocalizing, subvocalizing, or writing down) number words in a given sequence. Here, the standard of success is sheer absence of hesitation while uttering or writing down numerals in the correct order. For Wittgenstein (*PI* §§151-153, §§179-180), it is important that we do not need to stop and reflect on our technique, as individuals, in order to properly recite a sequence. On the contrary, when counting, we attempt to ignore anything that would throw the procedure into disorder, and generally succeed in this. Irrespective of the mental details of an episode of counting, successful counting does not require going through a mental process, but involves achieving a specific form of behavioral coordination with oneself and/or others.

Ginsborg (2020, p. 9) argues that, for Wittgenstein, the notion of *correctly* going on with a given behavior, such as counting, cannot be reduced to the idea of conformity with a rule. Children learn what it means to ‘go on’ when performing rote activities, learning the ‘correct’ responses to gestures, such as looking in the right direction when someone points their finger (cf. *PI* §185). Normativity in this ‘primitive’ sense is exhibited without grasp of intentional content or explicit commitment to rules. In this view, the correctness *simpliciter* of some behaviors, in given situations, can be judged without deriving this judgment from a rule. Ginsborg sees Wittgenstein’s (*PI* §§143-155, §185) discussion of the child learning to write down natural numbers in the decimal system as describing such primitive normativity.

However, the fact that people (notably children learning to use numbers) can know the correctness of a behavior without drawing on rules does not mean that the

[80] See Bender & Beller (2012) for documentation of this diversity. Given the importance of finger counting in many cultures, different methods often come with different number systems, e.g. the use of base 5, 12, or 20. See also Wittgenstein’s discussion of the non-absolute compositeness/simplicity of a chessboard in *PI* §47.

correctness of these behaviors is independent of rules.[81] Wittgenstein (*RFM*, I-64, III-15, cf. *PI PoP* §341) noted the importance of basic agreement among humans, but he also emphasized that we understand this agreement, as such, as already constituting a (fundamental) feature of language and mathematics (*PI PoP* §§ 346-349). Seeing two behaviors as 'the same' implies surroundings in which people already interact with one another, an environment marked by the potential for procedural agreement. Despite being 'blind', individual rule-following must be understood as belonging to, or standing out in contrast with, ongoing practices featuring rules, which include various counting methods. An anthropological perspective is thus required to understand the phenomenon of counting, both when considered locally as involving specific (linguistic, cultural) features and when considered more generically as an elementary technique.

3.1.2 Numerals, reference, and abbreviation

So far it has been argued that, for Wittgenstein, understanding the phenomenon of counting requires an anthropological perspective focusing on concrete practices of using numerals. However, it might be objected that counting is ancillary to, and has to been seen in light of, the use of numerals to refer to numbers. This line of thought regards numerals mainly as *names* of numbers, taking their use in counting to be a consequence of this function. The present section will examine whether Wittgenstein held such a view, and, if not, how he responded to it. As a point of comparison to his views on numbers, it will be helpful to consider a recent argument comparing the reference to numbers via Arabic numerals, formed by combining Hindu-Arabic numeral glyphs, such as "30", to that of verbal numerals, that is, words for numbers in a local language with a phonetic component, such as "thirty". Gómez-Torrente (2019, p. 110) argues that there is a gap between the two numeral types due to how they are generated:

> *[W]hile the generation of the Arabic numerals and presumably the fixing of their referents are intuitively rule-governed, there are simply no conventional rules of any kind (explicit or implicit) for generating*

[81] Ebbs' (2021, pp. 379-380) comments point in a similar direction: "We do not describe a subject as recognising that a new step is correct unless we also describe her as understanding how to go on, and vice versa. Neither description is more fundamental than the other." Cf. also Figueiredo's (2019, p. 285) commentary on Brandom's reading of Wittgenstein on rules, which deals with a similar issue: "Wittgenstein does not hold that a certain notion of primitive correctness is implicit in [social] practices. He rather indicates that it is owing to those practices that we are indeed able to grasp what is meant by the notion 'primitive correctness'."

more verbal numerals than those conventionally existing at a given time. […] For if N is the last verbal numeral conventionally existing at a given time, then, if n is the number N refers to, the Arabic numeral for n+1 is intuitively generatable and will have n+1 as its reference, but surely will not acquire its reference by being the reference of a corresponding conventional verbal numeral.

As will be argued in the next section, Wittgenstein would agree that 'the referents of Arabic numerals', numbers, are not fixed via corresponding verbal numerals. However, it is somewhat unclear how it should be stipulated that *N* is the 'last verbal numeral conventionally existing at a given time'. After all, the verbal numerals in many languages *do* follow regular naming conventions. Consider the long and short scales, which feature numerals formed by combining increasing numeral prefixes and the suffix "-illion" for powers of one million and one thousand, respectively. When the need arises, or before, a new word of the same order (we might even say, another term in the sequence) is introduced relative to a prior word, e.g. "millicentillion" = *def.* 'thousand centillion'. If we run out of numerals, we simply add another prefix, e.g. "quinmillicentillion" = *def.* 'five millicentillion'. So, additional verbal numerals of this form can be (and are) 'generated' indefinitely, not unlike Arabic decimal numerals.

Still, Gómez-Torrente's argument approaches the issue of the reference of numerals from a relevant angle. Adopting, provisionally, the terminology of numerals 'referring to' numbers, the question is how to think of the reference of numerals in relation to the reference of complex numerical expressions involving operators.[82] For example, if the referent of "23" is the number 23, would the 'referent' of "2 + 3" be the numbers 2 and 3, or the number 5? If it is the former, and the expression refers to multiple things, then "2 + 3" is not a name, but a list or set. Arabic numerals have a special role, according to Gómez-Torrente (2019), since, for any natural number *n*, *n* + 1 is already named by a single Arabic numeral. For example, if *n* is named by "$9_1 \ldots 9_k$", i.e. *k* digits of 9, then *n* + 1 is named by the numeral "$10_1 \ldots 0_k$".

This is in contrast to verbal numerals. We can talk of large numbers by stringing together verbal numerals, like "a thousand centillion", but we do not thereby *name* the

[82] Here, 'numerical expression' means a mathematical 'term', a numeral or well-formed mathematical expression (not an equation) involving numerals, operators, and/or variables, without any unit/count noun.

large numbers, on Gómez-Torrente's view. If we want to produce verbal names for numbers, we have to define a *new* symbol, e.g. "millicentillion". This indicates a logical distinction between Arabic numerals and verbal numerals, since it suggests that only the former can be used to *directly* refer to large numbers via induction. A similar line of thinking has roots back to the early modern era. Developing the decimal cyphers in the 16th century, Simon Stevin declared that the 'unit' is "of the same material" as a 'multitude of units', namely that of number (Klein, 1936/1968, pp. 191-192).[83] He appears to have thought that, with the indefinite generatability of numerals in a symbolic notation, all numbers were determined by a given process, and they were therefore all of the same 'kind' as 1, the 'unit', being given immediately by numerals.

This highlights a distinction in the ways we introduce new Arabic numerals in contrast to verbal numerals. However, one issue with the argument just stated is that, if complex numerical expressions should not be regarded as names, then Arabic numerals should not be regarded as names, either. The two notions are linked. This can be seen from the process of attempting to prove inductively that any natural number is named by an Arabic numeral. Assume that 1 is named by the Arabic numeral "1". For any number *n*, if *N* is the numeral naming *n*, the numeral N_{+1} naming *n* + 1 can be formed by running the following algorithm:

0. Begin in the position of the rightmost digit.
1. If the digit in this position is "9", replace it with "0". Then:
   1.1. If this is the leftmost digit, go one position left and write "1". Otherwise:
      1.1.1. Go one position left and repeat step 1.
2. Otherwise, increment the digit and repeat step 1.

However, if for instance "2 + 3" is not a name, then neither is any result of this algorithm. Take the case of *n* = 1. Here, to form the name for *n* + 1, the algorithm simply tells us to increment the digit. Spelling this out, there are two possibilities for what 'incrementation' might entail.

First, "increment" might mean to add 1 to the number represented by the digit,

[83] According to Klein (1936/1968), Stevin's use of 'unit' distinguishes his concept 'number' from that of ancient thinkers, notably Aristotle and Euclid, for whom numbers were quantities composed of 'units', making 2 the first number. Cf. Aristotle (2016, 1020a14) and Euclid (2002, Book VII, Definition 1 and 2).

which is to say that "2" = *def.* '1 + 1'. This would entail that "2" is no more a name than "1 + 1" is. Unless such expressions name their sum, Arabic numerals are not names. Second, and more interestingly, "increment" might mean a function from numeral to numeral, e.g. *Increment*("1") = "2". In the context of an algorithm, a function is a set of cases with accompanying instructions. That is, the word "increment" in step 2 would then be short for another set of steps, one for each numeral substitution: if the digit is "1", replace it with "2"; if "2", with "3"; and so on. So, 'step 2' then expands into steps 2.1 to 2.8, for the digits from "1" to "8".

Recalling Wittgenstein's view discussed in section 3.1, that a number system exhibits its own method of counting, the question is how this algorithm for producing Arabic decimal numerals, and steps 2.1 to 2.8 in particular, is actually *constructed*. The steps 2.1 to 2.8 manifest knowledge of which one-digit numeral M 'refers to' m, for any $m < 9$, and which one-digit numeral M_{+1} 'refers to' $m + 1$. Only by exhibiting this knowledge are we able to form the instruction to replace the digit M by M_{+1}. In constructing steps 2.1 to 2.8, in other words, we are drawing on an understanding of how to count in the relevant number system. The attempt to define "2" via an instruction to 'increment' "1" by replacing the digit "1" with "2", *does* effectively mean that "2" = *def.* '1 + 1', implying that "2" is a name only if "1 + 1" is a name.

The numeral "2" is defined via the operation of adding in the decimal system since, in forming instructions for how to use it, we tacitly or explicitly perform the operation of adding in that system.[84] The *correctness* of using the verb "increment" in step 2, to abbreviate the steps 2.1 to 2.8, and the incorrectness of abbreviating these steps with, for example, "subtract" or "square", is a sign that our understanding of the number system is in play when we form the Arabic numerals. It might be replied that we can understand the generation of Arabic numerals in terms of a relation such as '>', rather than incrementation. However, this presupposes a recognition of numerical value, meaning that similar considerations apply.[85] Wittgenstein's (*RFM*, III-47) linking

[84] The same goes for the exception, the propagation of the carry. Even if the algorithm does not state this directly, the steps 1 and 1.1 are likewise an instruction for 'incrementation' in the case of "9".
[85] Cf. Russell (1920/1993, pp. 5-10) on Peano's axioms of arithmetic with natural numbers, which takes as primitive '0', 'number' and 'successor of *n*'. Peano's axioms ensure that every number has a successor, 0 is not the successor of any number, and that if the successors of two numbers are equal, then the original numbers are

together of a number system and counting method is therefore justified. In a certain respect, there is no perspective on the use of numbers 'from sideways on';[86] constructing a number system involves demonstrating its use. By the same token, a system of numbers cannot be detached from the language games in which they are used.

The fact that counting, calculating, and numbers have to be understood together, as aspects of number systems, does not immediately entail that we should avoid thinking of numerals as names of numbers, however. We sometimes call verbs 'names' of actions or changes, and adverbs can 'name' methods or procedures. In a similar vein, numerals might be considered 'names' of numbers, as far as Wittgenstein is concerned. That coheres with the foregoing discussion. Similarly to how "read a book" names the same action as "read the pages of a book", provided they are interchangeable, "4" names the same number as "2 + 2", provided they are interchangeable. Nevertheless, numerals are not names in the sense in which proper nouns are names; their function is not to refer to objects, whether 'abstract' or 'concrete'.[87]

The decimal system as a system of abbreviated techniques

It has been argued that number systems hang together with counting methods, but this remains a somewhat abstract notion. A more concrete question is how numbers connect with language. It can be tempting to think of numbers as primarily quantifying or describing things; we say, for example, "there are six cars on the street" and "she is 40 years old". However, as argued by Wiese (2007, p. 761), numbers also serve an essential role as part of progressions that we use to *do* things with language. Wittgenstein shows an acute awareness of this function throughout his writings, such as in his allusions to reciting the alphabet and forming lists as parallels of the use of natural numbers (e.g. *LFM*, XVII, pp. 165-166; *RFM*, III-3; *PI* §8, §148).

These comparisons are evidence that he saw the importance of progressions as an aspect of numbers, but more explicit support is given in *RFM*, I-4: "[T]he *truth* is that counting has proved to pay … but that it can't be said of the series of natural

equal. As Russell (1920/1993, p. 7-8) pointed out, these axioms are "capable of an infinite number of different interpretations" given that "every progression verifies Peano's five axioms". Cf. Benacerraf (1965, p. 51, fn. 3).

[86] Cf. McDowell (2000, p. 44).

[87] So, Wittgenstein rejected *extensionalism* about numbers (Floyd & Mühlhölzer, 2020, pp. 30- 34). As Kripke (1982, p. 76) presents it, "to say words stand for (natural) numbers *is* to say that they are used as numerals".

numbers – any more than of our language – that it is true, but: that it is usable, and, above all, it is used."[88] Another remark that takes this theme further is *RFM*, III-12, where Wittgenstein wrote that the invention of the decimal notation was not merely the invention of a system of abbreviations of signs, for instance producing "325" by removing the signs for powers of 10 in the phrase "three hundred and twenty five", but a system of *applying* signs for the purpose of abbreviation.

This somewhat difficult remark can be understood by thinking of decimal notation as involving recursively abbreviated procedures of using counting words, like "one, two, three". The decimal system of natural numbers then effectively shortens a number of procedures depending on the position of each Hindu-Arabic glyph from right to left. These procedures can be unabbreviated and distributed into a sequence of actions that reflects their order. The place values of 1, 10, 100, etc., are then in effect used to *count to* a given number 1, 10, 100, etc., time(s). For example, "325" can be unpacked with the following sign:

_ _ (1, 2, 3, 4, 5), _ (2 × 10) _, (3 × 100) _ _

Here, the "_" are empty spaces (positions) that are included to retain the form of the decimal numeral. The signs can be read as instructions: On the right, count the sequence from 1 to 5; in the middle position, count to 10 twice; on the left side, count to 100 three times. Note that the sequential order of these instructions is reversed from the syntactic order of the Arabic numeral, reflecting the order in which we actually count, which begins with 5, then 25, then 325. It is the *action* of going through these instructions that is abbreviated by the numeral "325", leaving open the *method* by which this is done. If the sequences happen to be written down, the following expression is produced (retaining the 3 instances of "100" for lack of space):

100, 100, 100, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 1, 2, 3, 4, 5

The signs with the multiplication operator for the 10 and 100 places in "325", above, "_ (2 × 10) _" and "(3 × 100) _ _", can in turn likewise be understood as abbreviating

[88] This emphasis on counting can be read in contrast to Russell (1920/1993, p. 17): "In counting, it is necessary to take the objects counted in a certain order, as first, second, third, etc., but order is not of the essence of number: it is an irrelevant addition, an unnecessary complication from the logical point of view."

several procedures. Unpacked, the former can be read as follows: In the middle position, count to 2 first, then to 2, then to 2, etc., 10 times (or count to 10, then to 10). The latter can be read as: In the left position, count to 3 first, then to 3, then to 3, etc., 100 times (or count first to 100, then to 100, then to 100). With this, the position of numerals in "325" as well as the position of numerals in its expanded form are translated into sequences of counting.

On this interpretation, Wittgenstein saw a decimal numeral as expressing an abbreviated *technique*, making the point that abbreviation in that sense is far from trivial (cf. *RFM*, III-47). The point is that a decimal numeral does not merely abbreviate a sign, but that we *use* it to abbreviate counting, similarly to how, for example, the phrase "*A* to *Z*" is used to abbreviate the action of *reciting* the alphabet.[89] Taken in isolation, a numeral such as "5" does not abbreviate anything. However, since "5" has the place value of 1 in "325", it abbreviates the procedure of expressing the sequence "1, 2, 3, 4, 5" in that context. That does not imply that the numeral here means 'count to 5', any more than "*A* to *Z*" means 'recite the alphabet'. We often use counting sequences to talk about equivalent sequences of objects or events, so we can replace the number words in our sequences with whatever (if anything) we count or measure.

It might seem as if we could shed another layer of abbreviations in the numeral so as to reach signs composed only of "1", or of "1" and "0", and their relative positions. In that case, however, what *was* a system of 10 Hindu-Arabic symbols, from "0" to "9", would be reduced to 1 or 2. More importantly, their order would no longer reflect how we actually use counting words in language, so at that point the result could no longer be taken to represent the base 10 place-value system of integers. The latter is a crucial point, for Wittgenstein (cf. *RFM*, III-2). Someone might want to generalize and say that any standard positional numeral composed of symbols $a_1, \ldots, a_n$, where $n > 1$, can, in the first instance, be expanded into the expression "$(a_1 \times 10^{n-1}) + \ldots + (a_n \times 10^0)$". For instance, the numeral "325" is defined by the formula

$$325 = (3 \times 10^2) + (2 \times 10^1) + (5 \times 10^0).$$

[89] This should not be confused with Wittgenstein (e.g. *LFM*, XVIII, p. 171) sometimes using abbreviations of the alphabet as an example of a sequence to make a related point, namely that number series are not lists.

However, this formula makes essential use of exponents, which Wittgenstein (*RFM*, III-47) saw as a substantial modification of the number system, rather than merely a convenient method of multiplication. The exponent notation gives a new method of constructing and counting signs, e.g. using the expression just shown, which opens new calculating techniques.[90] It can be useful to think of positional numeration in terms of exponentiation, but this comes at the cost of missing what is distinctive about the decimal system, the positional or sequential use of numbers. In general, Wittgenstein (*LFM*, III, p. 33) warned against thinking of definitions merely as convenient abbreviations that do not effect a change in technique.

If the decimal system constitutes a way of applying signs for the purpose of abbreviation, and number systems in general involve counting methods, then decimal numerals cannot be defined without *de facto* changing the technique and operating with a different system. Wittgenstein came to question the entire idea of a hierarchy between mathematical systems such that one is the logical basis of the other (cf. Frascolla, 1993, pp. 152-153), and this theme can be seen already in his thinking about decimal numbers as a system of techniques. He posed a rhetorical question: "If a number in the decimal system is defined in terms of 1, 2, 3, ... 9, 0, and the signs 0, 1... 9 in terms of 1, 1 + 1, (1 + 1) + 1, ... can one then use the recursive explanation of the decimal system to reach a sign of the form 1 + 1 + 1... from any number?" (*RFM*, III-13). Concretely speaking, it is not feasible to reach such a sign in general.

A unary representation of, say, 93,829,716 in terms of "1" could not serve the same role as the Arabic numeral. Although the unary numeral could signify a progression (one can read "1, 1 + 1, (1 + 1) + 1" as "1, 2, 3"), it would not immediately delimit how the corresponding Arabic numeral ("93,829,716") is to be constructed and used, meaning that it cannot adjudicate its use. Writing down such a sign would take up 10s of thousands of pages. Even with the help of computers, we would question whether the string of 1s *really* adds up to 93,829,716 long before we would question whether "93,829,716" signifies the number that we *meant* to write in those pages. So, thinking of mathematics anthropologically, unary expressions are generally unfit to stand in for their Arabic decimal counterparts. The decimal system is not grounded in a unary

[90] That final claim might seem contentious; it is elaborated in section 3.4, below, and defended in Chapter 5.

system, or any other system such as Russellian mathematical logic.[91]

This leads to the conclusion that Gómez-Torrente (2019, p. 110) is right to deny that the meaning of Arabic numerals depends on the meaning of their corresponding verbal numerals, just as the meaning of a decimal numeral does not depend on its translation into a unary system. However, equally, the decimal system does not refer to numbers in a way that antecedes or grounds the verbal reference to numbers. Insofar as we can directly translate Arabic numerals into their local/verbal counterparts and *vice versa*, without needing a proof for justification, the signs in question all name one and the same number in analogous systems.[92] Intertranslatability between numerical expressions is prevalent, especially across ordinary languages; consider "4", "four", "*quatre*", and "四". These signs, along with the 4th sign of other number systems people employ, such as "IV" taken as part of a base 10 counting system, all 'name' the number 4, i.e. four, *quatre*, 四, and IV, inasmuch as they are *used* interchangeably in language games featuring their respective number systems.[93]

Nevertheless, fundamental practical dissimilarities, such as the use of disparate quantities of unique counting words (the number systems having different bases), *do* entail a distinction in the type of number used. On this reading, Wittgenstein would agree with Kripke in the Whitehead Lectures, as described by Steiner (2011), according to whom representations of a number showing the structure of the number system are 'buck-stoppers'; such expressions are the ultimate answer to "but what number is that?" In various contexts, the Arabic numerals serve as buck-stoppers, since the numerals show the base and the positional nature of the system itself. As Steiner (2011, p. 165) explains, this grounds numbers in cultural practices:

> *Kripke notes himself that his proposal implies that the identity of the numbers is culturally dependent. A culture that calculated with* **the**

[91] This is linked to Wittgenstein's understanding of proof and induction (see Maron, 1998, p. 228). As he wrote, "something stops being a proof when it stops being a paradigm, for example Russell's logical calculus; and on the other hand any other calculus which serves as a paradigm is acceptable" (*RFM*, III-14).
[92] The Arabic numerals up to, very roughly, "1,000,000,000" ("one billion") are similar to alphabetic letters or common words in that they are interchangeable with their local verbal/phonetic counterparts, at least in English. "Numeral" is in these cases polysemous, like "word" is in general, referring to both a spoken and written sign.
[93] Cf. *TBT*, p. 397. This is not to deny more or less subtle differences; there are different counting conventions between cultures. For example, in French, "74" is "*soixante quatorze*", literally "sixty fourteen". Note, though, that "*soixante quatorze*" is still the numeral for the 74th natural number in French, and that "seventy four" could be described as the 'sixty fourteenth' numeral, even if this sounds awkward in English.

base 7 would perforce be calculating with different numbers – not just different numerals – from ours. […] *Thus, the proposition that there is no one preferred system of numbers, originally put forth by Benacerraf (1965) with reference to the Zermelo and the von Neumann numbers, is given a surprising twist, in the direction of cultural, rather than professional, pluralism.*

The evolution of Wittgenstein's views on numerals illuminates his progression towards an anthropological perspective. In the *Tractatus*, he defined numbers as 'exponents' of operations, with numerals signifying the successive application of operations (*TLP* 6.02-6.021; Frascolla, 1994, pp. 1-33; Floyd, 2001). He articulated this account using a variable ranging over all operations, Ω, along with a recursively defined construction signifying the result of a number of steps or iterations. Since he took all propositions to be constructed through the successive application of 'the general form of an operation' (*TLP* 6.01), there was neither need nor room for a plurality of number systems in order to account for mathematical applications.[94]

Wittgenstein's view changed when he realized that systems of measurement make irreducible contributions to the sense of our sentences. For example, if a given patch is red all over, it cannot also be green all over, and *vice versa*. Similarly, if a road is 5 km long, it cannot also be 6 km long, and *vice versa*, and each individual kilometer can hardly be taken as the constituent of its own separate fact. As Wittgenstein came to see, there is no general form of an operation, just as there is no ultimate method of logical analysis. The numbers we use belong to language games where 'going on in the same way' can mean something quite distinctive, but, at the same time, most of our practices overlap and flow into one another in a variety of ways.

3.2 Measurement and quantity

So far, it has been seen that Wittgenstein saw number systems as linked with different methods of counting. This was considered without explicitly discussing the role of measurement, but given that we rely on measurements to relate numbers to the world, it is natural to ask how the relation should be conceived. As just mentioned, this was a question that was instrumental in Wittgenstein's turn from absolute analyzability (cf.

[94] On the significance of the unitary nature of logic in the *Tractatus*, see White (2017, pp. 297-301).

Floyd, 2021) – positing the possibility, in all cases, of reducing a proposition to a truth-functional combination of atomic propositions comprised of names – to his more systems-oriented and, later, anthropological perspective. So, the present section explores his later views on measuring vis-à-vis number systems.

Among Wittgenstein's final writings are remarks on the grammar of knowledge as it pertains to counting and measuring. In *OC* §455 he stated: "Every language-game is based on words 'and objects' being recognized again. We learn with the same inexorability that this is a chair as that 2 × 2 = 4." Our practices often depend on the recognizability and reliable behavior of some common objects. Take the example of a sign-post; people can orient themselves with a sign-post as such only if it remains stationary in a given position. However, Wittgenstein here seems to have been making a stronger point; he was talking first and foremost about the recognizability of *words*, and the quotation marks around 'and objects' can be taken to indicate that he considered this is to be, in some sense, a redundant addition.

In these later writings, Wittgenstein stressed that, as children, we do not first learn the definition of words like 'chair'; we first use chairs and interact with them (*OC* §7, §173, §476). That is, we do not first come to recognize a *type* of object, called "chair", but physical chairs about which, around which, and connected to the presence or absence of which, we use the common noun "chair" (cf. *PI* §80). There is a symmetry here, in that the use of "chair" has an analogous *role* in language as chairs (things we call "chair") have in our lives.[95] Similarly, in chess, the use of "knight" and the use of knight pieces is coupled. It is unclear what "she moved her knight one square" means, because such a move is not allowed. It might be thought that this coupling is only due to knight pieces being rule-bound, as the rules of chess determine the grammar of the term "knight piece".[96] However, if "knight" and knight pieces are mutually regulated in chess, "chair" and chairs are mutually unregulated in people's lives.

Hence, for the later Wittgenstein, there are still formal relations between

[95] An obvious retort would be that people use "chair" to talk about chairs when no chair is physically present, in which case "chair" has a role while chairs do not. But this assumes that a 'role' in a language game is some sort of physical property or relation. We say "there is no chair nearby", not "there is no 'chair' nearby", so we can in some sense give chairs a role to play despite (and sometimes because of) their relative physical absence.
[96] Cf. Gustafsson (2020, p. 206) on the relation between the rules of a game and grammar.

language and life, but these relations are now seen as manifold and inherent in practice. This theme evolved from *Zettel* §55 (ca. 1932): "Like everything metaphysical the harmony between thought and reality is to be found in the grammar of the language." It received a more explicit development in *PI* §429: "The agreement, the harmony, between thought and reality consists in this: […] that if I want to explain the word 'red' to someone, in the sentence 'That is not red', I do so by pointing to something that is red." Finally, related ideas were expressed in *OC*, e.g. §473 (cf. *PI* §19): "Just as in writing we learn a particular basic form of letters and then vary it later, so we learn first the stability of things as the norm, which is then subject to alterations."

The stability of things Wittgenstein referenced here are the common configurations or features we take for granted as part of our language games; conditions we normally do not think about, let alone question, although we constantly presuppose them in practice. Wittgenstein (*OC* §§476, cf. *PI* §28) emphasized the normative stage-setting of our language games. When novices first learn to measure things, such as meters of length, they do not first learn the necessary and sufficient conditions of objects or units of measurement and only subsequently form beliefs about their instances. Rather, they learn, in practice, various ways of using a counting method and measuring instruments, and as part of that very process learn to recognize things that are quantifiable using those instruments.[97] Wittgenstein accordingly hypothesized scenarios in which people use numbers in alternative ways as simultaneously being contexts in which objects are measured, used, and exchanged in alternative ways, that is, in ways that we, as readers, are meant to find unfamiliar (e.g. *RFM*, I-143-144).

However, there is no need to hypothesize unfamiliar scenarios in order to illustrate formal connections between the use of numbers and our interactions with objects. Imagine a teacher counting out loud while handing a pupil first 2 apples and then 3 apples, followed by making the pupil count the resulting number of apples as 5. In this case, the pupil is being taught that 2 + 3 = 5 using apples. The teacher might also engage in something like what Steinbring (2006, p. 158) calls 'comparing showing', pointing first to one group of apples and then to another while adding them, driving home the kind of technique that is conveyed.

[97] Cf. Hanson's (1958, pp. 4-19) illustrations of the context-bound nature of scientific observation.

This exercise is not just a preliminary to teaching the equation "2 + 3 = 5", but one way in which it is taught, full stop. This kind of basic and ubiquitous exchange of objects is part and parcel of the application of that equation. Not only are the expressions "2 apples", "3 apples" and "5 apples" here used to signify natural numbers, effectively serving as elaborate versions of the signs "2", "3", and "5", but also *the groups of apples themselves* are used as equivalents of these numerals, carrying a numerical function (see *LFM*, XII, pp. 112-114 cf. *PG*, p. 308 and Schroeder, 2021, p. 192). This, however, is not to say that the teacher is informing the pupil about apples. Rather, the teacher is teaching an elementary calculation by using apples, thereby teaching a way of operating with the numbers 2, 3, and 5.

As this illustrates, one way in which the link between numbers and concepts such as 'apple' is forged is via calculations with objects falling under the concepts. Calculating with apples teaches children how to quantify apples and apply equations in relation to them; it teaches what "number of apples" *means*. This can be seen in contrast to Frege's definition of numbers as equivalence classes of concepts. Wittgenstein regarded Frege's definition as an important advance over the idea that numbers are properties of objects, but he also took it to be limited and inaccurate (*LFM*, XVII, pp. 166-169, cf. Hacker, 1999, p. 238).

We use numbers without necessarily forming *predicates* to which the numbers apply. To say that the teacher gives the pupil 2 apples and then 3 more apples is not to say that 'there are 5 objects which instantiate the property of being an apple that is given to the pupil'. Rather, the way the teacher and the pupil together *operate with* the apples demonstrates 2 + 3 = 5, and the apples are given a numerical function with respect to that demonstration. Again, though, that is not to say that 'the property of being 2' and 'the property of being 3' hold of the apples, so it remains true that numbers are not properties of objects or groups thereof.

This suggests that the notions that have been explored so far – 'system of measurement', 'method of counting', and 'number system' – are all grammatically interlinked. People can count without strictly speaking measuring, but a system of measurement is linked to a counting method, as will be discussed further in the upcoming section. For Wittgenstein, we engage in language games in which we count or measure things that play more or less circumscribed roles in our lives. If objects were to disappear, amalgamate, or duplicate at random, they would not serve as objects to be counted and, by the same token, there could be no arithmetic that usefully applied to them (*RFM*, III-75-76). That is not just a criterion for the applicability of arithmetic, but, as discussed in section 3.1.1 (p. 72), a grammatical criterion for being able to 'follow a rule' when applying the 'same' counting noun on separate instances. As will be argued in the upcoming section, measurement is distinguished from mere counting due to presupposing (in the historical sense) a paradigmatic way of counting by using a particular standard.

3.2.1 Standards and units of measurement

In the first appendix of *RFM*, Part I, Wittgenstein reflected on how measurement, and the different ways in which we measure things, contribute to the meaning of numerals. He began by highlighting the difference between the use of numbers as the results of measurement and as cardinalities, as exemplified by measuring a given rod as being *n* meters long and counting *n* soldiers in a row. He then pointed out that these different uses are combined in sentences such as "On every 1 meter there stands 1 soldier, every 2 meters 2 soldiers, and so on" (*RFM*, I, Appx. I-10). As this exemplifies, we often set up, observe, and draw on systematic quantitative relationships, making numbers serve different but related roles in our lives.

The topic Wittgenstein discussed in these remarks is how to understand such relationships. He argued that they may not be adequately captured by equations and inequations, since equations and inequations makes the different anthropological roles of numbers appear to consist merely in numerical or quantitative relations. To highlight this issue, Wittgenstein stipulated a unit of measurement called "W", used for measuring length alongside the more familiar measurement of *feet*, such that 1 W = 1 foot. However, 2 W = 4 feet, 3 W = 9 feet, and so on (*RFM*, I, Appx. I-11). He then asked whether "W" and "foot" means the same in the two sentences "This post is 1 W long" and "This post is 1 foot long". His answer was that the question is "framed wrong" and that this becomes apparent if we express identity of meaning by means of an equation, asking "does W = foot or not?" (*RFM*, I, Appx. I-13).

Wittgenstein's implicit answer is that this latter formulation, presenting a quantitative relation as a kind of relation between units, is itself misguided. It would clearly be wrong to infer that foot = W from the fact that 1 foot = 1 W. However, from Wittgenstein's description, or definition, of "W", we can see that n W = n^2 foot. So, the question we might ask is whether it follows that "W" and "foot" have the same meaning in "this post is n W long" and "this post is n^2 foot long." However, if that is the case, then let $n = 1$, and, since $1 = 1^2$, we can see that n W = n foot. By that reasoning, it could be argued that it *is* the case that "W" and "foot" have the same meaning in "this post is 1 W long" and "this post is 1 foot long".

Wittgenstein's rejoinder to that line of argument is that "[t]he sentences in which these signs occur disappear in this way of looking at it" (*RFM*, I, Appx. I-13). To see what he means here, consider that, even if "this post is n W long" says the same as "this post is n^2 foot long", syntactically, these are not sentences in which "W" and "foot" would ordinarily occur. They contain variables, requiring substitutions for n to form completed sentences.[98] We can stipulate that something is "n feet long", but such expressions are derived from concrete measurements, for example, 'the fence is 40 W long', or 'Bob is a foot taller than Alice'.

If the same number is substituted for both variables, producing e.g. "this post is 3 W long" and "this post is 3^2 foot long", the resultant sentences have the same meaning. However, the question is whether that *follows from* 'n W = n^2 foot'. Wittgenstein argues against the idea that the former is derived from the latter. Though the equation 'n W = n^2 foot' expresses the equivalence of the substitution of 'n' for variable W with the substitution of 'n^2' for variable feet, the equation does not entail a biconditional of propositions in which "W" or "foot" occur. That is, for any given numbers u and v the equation 'u W = v foot' does not entail the biconditional that 'this post is u W long' if and only if 'this post is v foot long'.

However, there is something *prima facie* implausible about this. If it is true that 'u W = v foot', then it seems we should be able to infer, for any given object, that the object is u W long if and only if it is v foot long. If Wittgenstein thinks such inferences

[98] Cf. Wittgenstein's concept of 'propositional variable' from *TLP* 3.313.

are, at least ‘framed wrong’, his objection has to be elaborated in some way. To clarify this, it might help to take a step back and look at his line of thinking in *RFM*, I, Appx. I-10-13 more broadly. In these remarks, Wittgenstein was effectively opposing a view that can be called ‘abstractionism’, after Peter Geach (1957, p. 18).[99] According to abstractionism, whenever we measure something, we attribute a general property to it on the basis of abstraction. That is, in the act of measuring, we disregard, or ‘abstract from’, innumerable individual properties of the thing measured.

On the assumption that whatever is measured with the units *A* and *B* is quantified on the basis of abstraction, if specific measurements of *A* and *B* are quantitatively related in some way, then either *A* must be abstracted from *B*, or *vice versa*. On that assumption, provided that we can measured that $u\ A = v\ B$, for some specific numbers *u* and *v*, we can *infer* a constant conversion factor between *A* and *B*. For a straightforward example, consider meters and centimeters. If we measure that something is both *u* meter(s) and $100 \times u$ centimeters long, for any number *u*, we can infer that *meter* is related to *centimeter* in such a way that ‘$\forall x \forall n$(x is n meters)’ is true if and only if ‘$\forall x \forall n$(x is $100 \times n$ centimeters)’ is true.

In contrast to the view that sentences relating different measurements are grounded in abstract relations between units, however, Wittgenstein held that such sentences are “grounded in a technique” (*RFM*, VII-1). This explains why he (*RFM*, I Appx. I-3) criticized the idea that ‘u W = v foot’ entails that ‘this post is *u* W long’ if and only if ‘this post is *v* foot long’. After all, we measure feet through the use of rulers. Wittgenstein left it open how W is supposed to be measured, but we might imagine that we measure W by changing the area of a square instrument and marking the extent of one of its sides, and so, whenever we can use the one measure to measure the result of the other, we find that n W = n^2 foot.

However, the applicability of such techniques varies from situation to situation. The issue is that, in order for ‘n W = n^2 foot’ to entail that ‘this post is *u* W long’ if and only if ‘this post is *v* foot long’, for some numbers *u* and *v*, the corresponding *inequation*

[99] Abstractionism is associated with Aristotle and, in a broader sense, Platonism, but it should be considered a broad foil. It is not attributed to any individual in particular. See also Frege’s (1980a, pp. 61-63) critique of appeals to abstraction as a strategy for defining numbers as units and as properties of objects.

'n W ≠ n^2 foot' would have to entail the opposite biconditional (cf. *RFM*, VII-1). That is, it would have to entail that 'this post is *u* W long' if and only 'this post is *not v* foot long'. However, the latter biconditional is not the *negation* of the former, but a separate biconditional altogether.

The thrust of Wittgenstein's argument here might be unclear. One reason for this may be the entrenched idea that measuring practices are based on logical relationships. According to Wittgenstein, it is the other way around. Assume that it is false that 'this post is *u* W long' if and only if 'this post is *v* foot long'. Even if that is false, it does not follow that n W ≠ n^2 foot. After all, the technique of measuring W might be inapplicable in this instance – perhaps no side of the square used to measure W can be physically aligned with the post – even if the length in feet could be measured. Applying the inequation n W ≠ n^2 foot requires not just a lack of agreement in our results of measuring W and feet, but a collective, regular agreement in getting some *other* result (*PI* §242): when measuring this post, *u* W must be specifically *un*equal to *v* foot. But no such collective, regular agreement has been established here.

Still, it could be argued that, regardless of whether we can physically measure W, if we can measure feet, we can go by 'n W = n^2 foot' in order to infer the number of W from the number of feet. Whether this makes sense would depend on the nature of the technique in which the use of "W" is grounded. In particular, it would depend on whether we defer to the same *standard* when measuring W and measuring feet, in which case (or, to that extent) 'foot' and 'W' have to be considered part of the same system of measurement. After all, regardless of whether one or the other unit is inapplicable in any given case, if both types of measurement are defined by reference to the same standard, then they in principle intertranslatable, which is the case with meters and centimeters. 'Standards' are not abstract units, either, however; Wittgenstein's understanding of their role is discussed in the upcoming section.

Standards, samples, and instruments of measurement

In *PI* §50, Wittgenstein made the claim that the standard meter rod in Paris – a given metal rod that was used to determine the meaning of 'meter' – can, paradoxically enough, neither be said to *be* a meter long nor to *not* be a meter long. In discussing this claim, Machado (2022, p. 9) provides a solid description of the later Wittgenstein's account of standards of measurement:

> *The property of being a standard is not a natural one. To be a standard is to be used in a certain way in our linguistic practices. A standard is an object of comparison. […] Therefore, an object has a measure in this system if it can be compared to the standard. But, if to be a standard of a system of measurement is to be used in a certain way in our linguistic practices, then a standard is something that lasts in time, not something located at an instant of time.*

One addendum to this description is that a standard is not just an object of comparison, but a *paradigm* that determines what it means to 'measure' within the system. We use instruments and samples for measuring things all the time, but, because instruments and samples are themselves open to correction, they do not establish or constitute standards of measurement. A person might find a ruler in a drawer and use it to measure the length of a table, then say "the table is 80 cm long". Even so, the ruler is not a standard of what constitutes '80 cm long'. It is a measuring *instrument*, and it is used on the assumption that its length has itself been measured to some degree of accuracy, so as to give reliable results.

After having measured the table, the person could now place a chair next to it, and to align them would mean that the chair is also 80 cm long. In this case, the person would be using the table as a *sample* for further measurement. For many kinds of measurement, instruments such as rulers are calibrated to produce transitive samples, so that, if *A* is measured to the length of *B*, and the length of *B* is that of *C*, then the length of *A* equals that of *C*. Still, the level of accuracy is always relative, since no two measurements will be 'absolutely' equal. Moreover, Wittgenstein *RFM*, I-5 argued that both the level and form of accuracy we demand from our units of measurement are *relative* to the general purpose for which measurements are made; a ruler made of soft rubber, shortening and lengthening in response to different temperatures, could usefully measure how things compare in their reaction to different temperatures. Wittgenstein (*RFM*, I-5) went so far as to imagine a shopkeeper using a freely stretchable ruler on clothing garments, stretching it so as to treat customers differently.

Wittgenstein's lax attitude here has been the subject of criticism. Wright argues that "our measurements, in order to deserve that name, must be roughly in agreement with our observational assessments, whereas the readings of such a soft ruler may differ wildly from our visual impression of an object's length" (Wright, 1980, p. 58). However, we *do* have forms of measurement that give readings which deviate from our other observations. For instance, current measurements of the distance from Earth to the Moon and the Sun could be said to disagree with the visual impressions people have had, in various times and places, of these distances. Wright (1980, p. 58) raises another objection that seems more apposite:

> *It is a feature of the concept of measuring that an accurately measured object will yield distinct readings at distinct times only if it changes; so much is implicit in the notion that measuring is to ascertain a property of the object measured.*

Indeed, the notion of a result of measurement and a change in measurement (e.g. the phrases 'it is *x* feet long' and 'its length has changed') are internally related. If a shopkeeper stretches a ruler at will, she does not *appeal to* the ruler in order to make measurements. In treating customers differently, the shopkeeper lets her relationship to the customers dictate how far to stretch the ruler. In effect, the shopkeeper is using *the customers* as an 'instrument of measurement', not the ruler. However, as Schroeder (2021, p. 114) notes, Wittgenstein did not unequivocally insist that he was describing measurement, as such. His point was that different practices are possible: "'But surely that isn't measuring at all!' – It is similar to our measuring and capable, in certain circumstances, of fulfilling 'practical purposes'" (*RFM*, I-5).

Whether or not Wittgenstein's biased shopkeeper should be taken to 'measure', or doing something comparable to it, his remarks highlight the contingent yet normative nature of measurement. Given that decisions are made on the basis of measurement, a sample serves as a 'carrier of normativity'. If one reliable measuring sample deviates significantly from a sample produced by another instrument, this is generally taken as a sign of the instruments having been calibrated wrongly. In that case, one of the measuring instruments is taken to be more accurate than the other, or neither is given any validity. This is adjudicated by reference to the *standard* of measurement. That is precisely what distinguishes a standard from mere samples or ordinary measuring instruments. The procedure of measuring samples using the standard, finding how the sample(s) compare against the standard, is a *paradigmatic* form of measurement. This is the grammatical, as opposed to historical, point Wittgenstein made in *PI* §50: the meter rod in Paris has a *dispositive* role, settling any controversy of whether something is a meter.[100]

Müller (2023) criticizes the literature around *PI* §50 for lack of historical accuracy, reminding us that "metrology, the art and science of measurement standards, is at the service of both society as a whole and the sciences in particular" (2023, p. 174). Standards of measurement do not apply themselves, but have to be institutionalized. Müller (2023, p. 175) discusses attributions to Wittgenstein of the view that 'object A is *x* meters long' means 'object A is *x* times as long as *that*', uttered while pointing to a material artefact (i.e. the standard meter rod). Müller points out that, in France ca. 1805, the meter was standardized in law, and multiple copies of the rod were made and compared against each other. The network of metrological practices of making and employing these copies was as important as the artifacts themselves.

Nevertheless, as Pollock (2004, p. 155) highlights, the grammar of number reflects the distinction between sample and standard. The use of "1 meter" when measuring an ordinary object differs from its use when identifying something as the standard meter. When measuring things, to borrow Wiese's (2003, p. 265) distinctions, we use "1" *instrumentally*, as a counting word. By contrast, when talking of the standard meter, if we were to say "it is 1 meter", we would be using "1" *referentially* to specify that the rod in question is *the* 1-meter standard. If the standard meter rod were found, one day, to be shorter or longer than its copies, this *could* justify withdrawing the title "1-meter standard" from it. At this point, one of its copies might be defined as the standard meter instead. The standard should thus be seen as part of a practice of *maintaining* the standard, since the initially chosen artifact can be (and *has* been) replaced.

These observations are not in conflict with Wittgenstein's overall point. A system of measurement involves a whole suite of measuring technologies of differing

[100] For the sake of the cogency of this discussion, it should be assumed that the physical meter rod is still used as the standard for the metric unit. More recently, 'meter' has been defined in terms of distance travelled by light in a vacuum over a physically specified measure of time.

levels of precision (cf. *PI* §88), but it remains true that a core of physical artifacts is given a dispositive role, and that the language game of measuring cascades outwards from this core. Measuring devices are produced on the basis of the standards, before precision tools are designed using these devices. Those tools, in turn, are used in engineering, architecture, etc. As Avital (2008, p. 323) puts it, "Wittgenstein denies that there is a more basic ethereal Platonic measuring rod ('a fact of meaning') from which the rod draws its meaning. Meanings in this respect are materialized."

Terminology such as "*n* meters" is used together with measuring devices, being applied in measurements similarly to how we apply physical samples. If we measure that a table is 2 meters long, we can write down "2 meters" to fix its role as a sample of length, and if the table is subsequently measured with a ruler, this thereby also adjudicates what was written down. If it is now found that the table is 3 meters, what was written is off by 1 meter. This connects a system of measurement with the use of a calculus, to be discussed in section 3.3.1 (p. 104).

When Wittgenstein (*PI* §50) claimed that the meter rod in Paris is the "*one* thing of which one can state neither that it is 1 meter long, nor that it is not 1 meter long",[101] he added that he was marking the role of the rod in the game of measuring with a meter ruler. A standard is *core* to a system of measurement, and measuring with rulers, as well as reporting or recording these measurements, is logically 'downstream' from the standard. Kripke (1972, pp. 54-56) has contested Wittgenstein's views, arguing that if, at time *t*, the length of rod *S* (that is, the meter rod in Paris) is defined as '1 meter', nevertheless it might not have been 1 meter in length at that time. The reason, according to Kripke, is that "1 meter" is a rigid designator, while "the length of *S* at *t*" is not. The reference to 1 meter remains fixed regardless of whether *S* at *t* is actually 1 meter or not 1 meter, which means that Wittgenstein (*PI* §50) was incorrect.

However, as Machado (2022, p. 9) writes, a standard of measurement is not a result of measurement, at some moment *t*, but a persistent object around which a system of measurement is organized. So, we have to consider our own present relation to the standard. For us to 'counterfactually' describe *S* as not being 1 meter at *t* would involve

[101] In reply, it might be asked whether it makes sense to assert or deny that e.g. a musical note is 1 meter long. Whether or not Wittgenstein's (*PI* §50) wording is too strong overall is, however, beside the point.

hypothesizing a standard which we *now* call "meter" to be compared against the object *S* at *t*. Otherwise, 'compare *S* against the standard meter' would mean to compare *S* against itself, which is incoherent, as Pollock (2004) argues. Of course, we *could* posit such an alternative standard, but doing so would be pointless since it would not constitute the standard meter that we *do* use, which is *S*.

Kripke (1972, pp. 54-56) suggests an alternative argument against Wittgenstein as well. We can measure *S* in another unit, such as inches, and then convert the unit back into meters. If we found that *S* is 39.3701 inches, then, since 1 meter = 39.3701 inches, it follows that *S* is 1 meter. Again, if this were right, it would contradict Wittgenstein's (*PI* §50) remark. This effectively raises the dilemma that was discussed in the previous section.[102] Either inches = meters, in which case *S* is the standard against which inches are measured, returning to the problem that 'compare a standard against itself' has not been given meaning, or inches ≠ meters. In the latter case, the inference from '1 meter = 39.3701 inches' and '*S* is 39.3701 inches' to '*S* is 1 meter' is unsound. In reality, inches are now based on the metric system, being defined as exactly 25.4 millimeters, so the former condition applies. Hence, defining 'meter' in terms of inches does not circumvent *S*, since inches are defined in terms of *S*.

3.2.2 Subitizing

To recapitulate, several aspects of Wittgenstein's remarks on number systems and counting have been explored. Systems of notation are internally related to number systems and methods of counting. The latter, in turn, are to be understood as language games, where each numeral (e.g. each Arabic glyph) is used in particular ways in sequences, playing a role vis-à-vis all others in its system, and that role constitutes a particular number. Language games presuppose stable environments in which objects (including signs) behave in relatively regular ways. These regularities, for instance the fact that objects do not randomly evaporate or duplicate, allow us to count and calculate with objects themselves. So, there is no gap between mathematics and its domains of empirical applicability. That is not to say that a mathematical formula *is* an empirical

[102] Steiner (2011, p. 174) describes Kripke's implicit commitment to rigidly designating 'length' as realism about magnitudes; in any case, his abstractionism causes a 'communication gap' with Wittgenstein.

proposition, but that the difference between them is not syntactic or semantic. What matters is how the sentence or formula is used: as part of calculation or for description.

At the same time, methods of counting, and consequently consensus in estimates and judgments of quantities, are not based on generalizations of our encounters with objects, but are a matter of how our language games *work*. For example, if someone is said to count a basket full of apples, the role of this basket, its contents, and the person in the language game will determine what is counted and how; the basket itself, a number of whole apples, a number of prospective apple-pieces of some size, liters of apple cider, a number of different cultivars, etc.

Some tension might be felt between that last observation and a naturalist interpretation of 'subitizing', or the immediate perception and judgement of small quantities of objects.[103] Wittgenstein (*PI* §9) himself was aware of the phenomenon, describing the "ostensive teaching of number-words […] to signify groups of objects that can be taken at a glance", adding that children "learn the first five or six elementary number-words in this way". Relatedly, he discussed the ostensive use and definition of numerals in sections *PI* §§26-30. His discussion of aspect-perception is also relevant; there are connections between the expressions "count as", "take together", "belong together", and "see as" (*PI PoP xi* §221, cf. *PI PoP xi* §162).

There are two different reasons why the phenomenon of subitizing might be taken to be in some tension with the preceding discussions. Firstly, it could be argued, subitizing is an instinctive, natural ability and proclivity for numerical judgements in humans, not learned as part of partaking in language games. So, it might appear, the phenomenon cannot be adequately explained in terms of language games. Secondly, it could be argued that the fact that humans have this natural ability implies that our numerical designations (directly for numbers within subitizing range (generally below 5), but indirectly for larger natural numbers) *do* depend on independently existing, objective quantities or cardinalities of sets.

[103] See Kaufman et al. (1949) for more on 'subitizing' as well as the origin of the term. A closely related phenomenon is the immediate perception of relative or comparative numerousness: seeing that one group of objects (or, in some cases, events, processes, etc.) is large or small, or larger, more numerous, than another. For present purposes, there is also a close link between subitizing and the phenomenon of immediately identifying geometric properties and relationships (cf. Pantsar, 2022; *PI PoP xi* §§167-175). See Stam et al (2022) for a Wittgensteinian perspective on recent research on subitizing and the acquisition of numeracy.

To answer the first objection it should be noted, as Wittgenstein (*PI* §9) did, that ostensive teaching of number words does not teach counting *per se*, which is learned at a later stage (see Starkey & Cooper, 1995, p. 401; cf. Baroody et al. 2008). Both intransitive and transitive counting, such as tallying, correlating a procedure of successively indicating objects to a process of reciting or marking down symbols, is a deliberate form of action. Meanwhile, to subitize is to think or utter e.g. "three" immediately upon perceiving a group of objects.

For Wittgenstein, as far as philosophy is concerned, what is at stake here are normative or grammatical distinctions, not empirical differences (cf. Dromm, 2003). Someone might utter numerals in immediate responses to stimuli in a way that *accords* with a rule, but that is not to say that they are *following* the rule. This distinction between mere accordance with a rule and following a rule should not be exaggerated or taken as an appeal to a psychological 'grasp'. The point here is not to draw a rigid distinction between an internal/emic perspective and an external/etic perspective (cf. Brandom. 1994, 64-65). Rather, the relevant notion is that of an individual *merely* according with a rule, who, under specific conditions, can do *part* of what a rule-follower would do, without being able to do the full range of things that rule-followers do. In the case of numbers, competency requires the ability to count and calculate in *various* situations, in *multiple* ways, not just in direct response to stimuli. In this sense, subitizing merely accords with a method of counting, and does not constitute rule-following *per se*.

Nevertheless, subitizing is relevant from a pedagogical perspective. As children first learn to count, they are faced with having to learn language games in which things are counted in various ways. It is significant that Wittgenstein (*PI* §9) noted the *ostensive* teaching of elementary number words as part of this process. He recognized that use of the word "3" is taught through clear examples of *use*, and thus in similar ways as words like "apple" or "meter". Examples of counting manifest features of the context in which people engage in language games featuring a given counting method (cf. *PI* §§71-74). An elementary school teacher presenting an image of 5 apples and counting them one by one is displaying a characteristic way of counting using natural numbers. Immediately identifying the number 5 upon seeing such an image is analogous to responding with the word "apple" on the same occasion, or, for a different example, similar to recognizing a meterstick *as* 1 meter long; these are not moves in languages of counting, but preliminary steps towards engaging in such language games.

In this respect, ostensive teaching of counting words through subitizing can be compared to the function of name-giving, as Wittgenstein described in *PI* §26. Attaching a name-tag to a thing prepares the use of a word, but, as Wittgenstein added, what is important is the activity prepared. What is being named is not immediately gathered from its ostensive explanation; sheer name-giving is followed up with particular kinds of use. Similarly, teachers judge whether pupils identify cardinalities of small sets, or use counting words in the right order, in light of calculi which are only taught later (Stam et al, 2022, pp. 6-7). This mirrors Wittgenstein's emphasis on the *method* of counting in *RFM*, III-47. His views can thus be contrasted with the idea that full-blown counting is mere *induction* from counting within subitizing range (cf. Pelland, 2020, p. 3807). Such a view understates the normative distinction between someone *only* being able to subitize and someone having learned to count.

It might be objected that, even if subitizing is seen as *preparation* for the use of a number system, the ability to subitize is nevertheless empirically continuous with full-fledged use of numbers, making subitizing continuous with counting. However, again, the physical *details* of subitizing are irrelevant from Wittgenstein's perspective. We attribute rule-following as, and about, participants in practices, but this says little about the attainment of the skills required for participation in practices.[104] Whatever the causal mechanics, precisely the ability to shift from subitizing to participation in ongoing language games of counting, going from identifying 3 apples to exercising the unrestricted technique of counting natural numbers by voicing series of counting words in various settings, is a *criterion* for knowing how to count.

This leads to the second objection aired above, which denied that counting is explicable in terms of language games on the grounds that the ability to form immediate judgements of discrete quantities implies that cardinalities inhere in the structure of the physical world. However, though subitizing does not demonstrate capacity for counting

[104] For example, there must be a cultural aspect to any attribution of rule-following, since practices vary between cultures, but that is not to say that an individual (potential) rule-follower is necessarily behaving as a result of strictly cultural (as opposed to biological) causes. However, cf. Núñez (2017) for a 'culturalist' view.

as such, it *does* exhibit another closely related ability: the ability to recognize contexts in which a given number system is used. Subitizing is thus linked to counting as a social phenomenon in much the same way that recognizing the movement of wooden pieces *as* chess moves is linked to chess as a social phenomenon. These recognitional capacities are in both cases contingent on physical characteristics of human beings, but they do not indicate that numbers, any more than chess moves, inhere in the physical structure of the world independently of human beings.

3.3 Equations and calculation

Having considered his views on numbers and judgements of quantity, the chapter will now examine Wittgenstein's understanding of arithmetic and calculation, beginning with his so-called 'middle period', roughly 1929-1936. At this point, Wittgenstein criticized the idea that there is a logical gap between mathematics and its applications. In particular, he held that arithmetic and geometry *guarantee* their own applicability (*PG*, p. 307, *PR* §111). In the remarks §§100-111 from the *Philosophical Remarks*, and in conversations with members of the Vienna Circle,[105] Wittgenstein argued against the idea that arithmetical equations are replaceable by tautologies. Some tautologies involve applications of rules for the substitution of numerical signs. Recognizing them as applications of arithmetic (or as mathematical at all) involves 'apprehending their internal multiplicity' by correlating the number of symbols on one side to the number of symbols on the other side. This being so, the tautological nature of such propositions is irrelevant; we could equally use contradictions as equations in the same way. In general, Wittgenstein rejected the vision of tautologies as forming a *logical substratum* of arithmetic, a logically consistent foundation on which to rest the concept of 'number'.

This argument shows Wittgenstein's emerging dynamic view of mathematics. As he (*PR* §109) put it, "[a]rithmetic doesn't talk about numbers, it works with numbers". He was here building on the idea, present in very general outline (and in strictly logical, rather than anthropological, terms) in the *Tractatus* (5.232) that the internal properties of a formal system are identifiable with its operations, or, more loosely, with how that system is used. He was arguing that, particularly in arithmetic

[105] See Wittgenstein (1979, pp. 34-35), transcribed by Friedrich Waismann in December 1929.

and geometry, the ways in which signs and shapes are constructed are themselves determined by calculation or geometric construction, not by external legislations *about*, or representations *of*, calculations and geometric shapes (cf. Nakano, 2020). The rules for the construction of signs are also ways those signs are used to count and calculate with, and *vice versa*: the rules are internally related to their applications.

Wittgenstein maintained the idea that there is an internal relation, a merely *apparent* distinction, between numbers and equations, on the one hand, and their respective forms of applicability, on the other, into his later period. In fact, he expanded upon it. As he wrote in *RFM*, III-4: "The *application* of the calculation must take care of itself. And that is what is correct about 'formalism'." He developed this line of thought to distinguish calculations and proofs from what he called 'experiments': The result of a calculation is an internal, *constant*, part of it. So, a calculation is a reproducible pattern which can equally be conveyed by a picture of the calculation.[106] The distinction between experiments and calculations in the context of proof is discussed in the final section of this chapter, section 3.3.2 (p. 111), and is elaborated further in Chapter 5. Before that, the present chapter will explore Wittgenstein's understanding of *equations*, and its relation to a 'calculus', beginning with some more background from his middle period before turning to his later, more anthropological understanding of calculation.

The calculus of extensions

In the late 1920s, Frank Ramsey was drawing on insights from the *Tractatus* in an attempt to ameliorate problems in Russell and Whitehead's *Principia Mathematica*. In this effort, Ramsey (1931, p. 13) sought to give credence to the 'extensional' nature of mathematics, explaining: "[I]n calling mathematics extensional we mean that it deals not with predicates but with classes, not with relations in the ordinary sense but with possible correlations." Inspired by Cantor's definition of 'similarity'/'co-cardinality', Ramsey simply took 'class' to mean any set of things, of the same logical type, and a 'relation in extension' to mean any set of ordered couples.

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[106] This line of thought, and the idea of using pictoriality as a test of the character of some sentence or symbolism, is connected to the distinction between 'mirror' and 'picture'/'painting' which Wittgenstein drew in his critique of Ramsey's theory of identity in the early 1930s (*PG*, p. 315). More on that below.

In his paper on the foundations of mathematics of 1926, posthumously published in a book in 1931, Ramsey was responding to Wittgenstein's *TLP* conception of mathematical propositions as equations (*TLP* 6.2), seeking instead to construe mathematical propositions as *identities* and to expand the notion of 'tautology' to include these identities. Ramsey hoped to use elements of Wittgenstein's early writings to ground the theory of classes, but in order to succeed in this he had to find a way of addressing the critique of Russell's definition of 'identity' and the identity of indiscernibles in *TLP* 5.5302, which had informed Wittgenstein's claim that "[t]he theory of classes is altogether superfluous in mathematics" (*TLP* 6.031).

Wittgenstein had argued in *TLP* 5.5302 that, in principle, it makes sense to describe *two* numerically distinct objects as alike in all respects; all propositional functions may have the same truth-value taking x and y as arguments without x and y being identical. An equation such as '2 + 2 = 4' signifies the intersubstitutability of "2 + 2" and "4" in propositions, helping us infer e.g. "I have 4 hats" from "I have 2 + 2 hats". What the equation '2 + 2 = 4' does *not* do, for Wittgenstein, is identify the sense of "2 + 2"-propositions (e.g. 'I have 2 +2 hats') with that of their "4"-counterparts. The latter would make mathematics depend on (the truth-value of propositions about) classes of objects (e.g. the fact that 'hats in my possession' has a certain extension), but for Wittgenstein the generality at play in mathematics is not 'accidental' (*TLP* 6.031).[107] An (arithmetical) equation is not a statement of identity, which could in principle be true or false, but is *strictly* a symbolic record to be consulted when inferring from certain non-mathematical propositions to others (*TLP* 6.211, cf. Kremer, 2002; Floyd, 2001).

However, Ramsey saw mathematics as built on logical tautology in precisely the way Wittgenstein had denied. To defend this view, Ramsey introduced the notion of 'function in extension', a function specified by arbitrary correlations of individuals with propositions. A function in extension does not require a systematic definition: for any two inputs a and b, the corresponding outputs $\phi_e a$ and $\phi_e b$ may be completely unrelated. A function in extension just maps each individual to *some* unique proposition (say, Socrates to 'Queen Anne is dead').

[107] See Demopoulos (2013, ch. 13) and Fogelin (1983) for discussion on this.

On Ramsey's proposal, then, the following is a tautology if $x = y$, and a contradiction otherwise: $\forall\phi_e(\phi_e x \equiv \phi_e y)$. That is, if x and y have the same value, then, as a matter of tautology, *all* propositions arbitrarily correlated with x are equivalent to propositions arbitrarily correlated with y (we would need a 'that'-clause preceding the functional expressions to talk of their logical equivalence). '$\forall\phi_e(\phi_e x \equiv \phi_e y)$' is a contradiction if $x \neq y$, since, among all the correlations, there will then be some proposition p correlated with x such that $\neg p$ is correlated with y. Hence, while accepting Wittgenstein's (*TLP* 5.5303) criticism of identity statements, Ramsey (1931, p. 53) took '$\forall\phi_e(\phi_e x \equiv \phi_e y)$' to serve as an alternative definition of '$x = y$' fit to satisfy the demands of the theory of classes, extensionally conceived.[108]

With this definition, equations are not tautologies in the sense of being truth-functions which are true for all the truth-possibilities of their arguments. Instead, equations are, as it were, 'underwritten' by tautology, so they might be called 'tautologous*'. Assume that $x = 2 + 2$ and $y = 4$. Now, x and y have the same value, and so it is tautologously true that all propositions 'arbitrarily correlated' with x and y are equivalent, and therefore, by definition, $x = y$. The biconditional of the propositions 'I have 2 + 2 hats' and 'I have 4 hats', and similar, becomes tautologous* as well, since '$2 + 2 = 4$' is a tautology in this expanded sense.

In response to Ramsey's 'calculus of extensions', Wittgenstein (*PG*, p. 315) wrote:

> *Ramsey's theory of identity makes the mistake that would be made by someone who said that you could use a painting as a mirror as well, even if only for a single posture. If we say this we overlook that what is essential to a mirror is precisely that you can infer from it the posture of a body in front of it, whereas in the case of the painting you have to know that the postures tally before you can construe the picture as a mirror image.*

The definition of '$x = y$' as an identity in an expanded sense involving functions in extension seems to be what Wittgenstein here meant by "Ramsey's theory of identity". Although there are disagreements over the nature of Wittgenstein's critique of this

[108] The concept 'function in extension' tracks onto an encompassing 'standard' interpretation of the range of second-order quantifiers, contrasting with more restrictive 'non-standard' interpretations. As discussed by Marion (1998, p. 48), Wittgenstein arguably favored a form of non-standard interpretation.

theory in the literature (cf. Fogelin, 1983; Marion, 1998), arguably the main thrust of his argument was directed against Ramsey's use of 'non-predicative' propositional functions. The symbols "$\phi_e a$", "$\phi_e b$", etc. were introduced as functions from individuals to propositions. Wittgenstein rebutted that they are functional expressions only insofar as "Co(rn)", "Co(al)", and "Co(lt)" are as well (*TBT*, p. 389). Ramsey was arguing that '$a = b$' can be defined via a tautology, tautologies being 'degenerate propositions' that say nothing and have no significant composition. In the context of the tautology '$Fa \vee \neg Fa$', "*Fa*" is not a genuine function and argument (*TLP* 4.466). However, if $a = b$ *=def.* $\forall\phi_e(\phi_e a \equiv \phi_e b)$ *being a tautology*, then the latter says nothing, and "$\phi_e a$" and "$\phi_e b$", in it, are not composed of function and argument, either. So, for Wittgenstein, Ramsey was effectively proposing to define "$a = a$" and "$a = b$" as inarticulate signs, contrary to what is required in e.g. "$\exists a(a = a)$" (cf. *TBT*, p. 389).[109]

Ramsey's proposed definition also did not escape the logical issues connected with conceiving of identity as a relation. If we assume that "$a = b$" makes sense, being true or false depending on whether '$\forall\phi_e(\phi_e a \equiv \phi_e b)$' is a tautology, its negation must also make sense. But, assume that '$a = b$' is true. We must still be able to express '$a \neq b$'. The latter, however, has been defined as the condition that '$\forall\phi_e(\phi_e a \equiv \phi_e b)$' is a contradiction, and this is true only if there is some proposition p correlated with a such that $\neg p$ is correlated with b. But there can be no such proposition, since it has been assumed that $a = b$. So, the condition that would make '$a \neq b$' true cannot be formulated, meaning that "$a \neq b$" lacks sense. Hence, if '$a = b$' is true, "$a \neq b$" lacks sense. But, if "$a \neq b$" lacks sense, so does its negation "$a = b$".

Such technical objections could be seen as lending support to a formalist vision of arithmetic, *contra* the logicist aims of Ramsey, who passed away before having the opportunity to respond to Wittgenstein's full criticisms. However, that would be the wrong conclusion; Wittgenstein's disagreement with logicism was less technical and more fundamental than this would suggests. He took issue with Ramsey's and Russell's

[109] Wittgenstein (*TBT*, p. 389) in fact offered several criticisms, as Ramsey's project of amending *Principia Mathematica*, particularly Russell's definition of 'identity', was in many ways opposite his own. He noted that the definitions were introduced unsystematically (that is, not via operations) which, from his perspective, would render them inadequate for dealing with sets of transfinite cardinalities, contrary to their job description.

attempts to give arithmetic a foundation by “making preparations for a possible grammar” of classes and *n*-place relations (*TBT* p. 384). He objected to this by comparing arithmetic to a game: “In arithmetic we cannot make preparations for a grammatical application” and there is no reason to want to do so: “For if arithmetic is only a game, its application too is only a game, and either the same game (in which case it takes us no further) or a different game – and in that case we were already able to play it in *pure* arithmetic” (*TBT*, p. 385). So, although comparisons between mathematics and games have long played an important role in formalist writings,[110] Wittgenstein’s *comparative* use of the analogy would serve to undermine both logicism *and* formalism.

3.3.1 Laying down a path in language

The logicist attempt to derive arithmetical equations as tautologies was, from Wittgenstein’s perspective, correct only in the negative assessment that arithmetic should *not* be understood in synthetic or empirical terms. For Wittgenstein, an equation such as “2 + 2 = 4” is not a statement about logical objects, but it is not a generalization about signs, their usage, or physical objects, either. Rather than empirical propositions, the ‘propositions’ of mathematics are *rules* (cf. e.g. *RFM*, VI-4). The questions then become what kind of rules are involved in mathematics, how we arrive at them, and how we use them. It will be argued here and in subsequent chapters that the sense of ‘rule’ Wittgenstein had in mind, at least from his lectures in 1939 onwards, is not a directive *about* anything, but a regulated action or practice.

Wittgenstein frequently used dynamic and spatial metaphors in describing equations as ‘moving between positions’, taking e.g. “2 + 2 = 4” to ‘move’ from 2 + 2 to 4 (e.g. *RFM*, I-165), or as determining a ‘path’ *on which* to move from one concept to another (e.g. *RFM*, IV-7). A mathematical formula is connected to a procedure of *calculation*, while also constituting the result of a proof. For example, we understand an equation such as $20^2 + 32^2 = 1424$ in light of a calculation which expands (dis-abbreviates; cf. section 3.1.2) the numerals in a certain way, thus proving the equation to be correct. Our understanding of it is the exercise of a skill; it is the calculating technique that shows the meaning of the equation. So, as will be explored in the

[110] As in Thomae (1898) and Hilbert (1927). Cf. Stenlund (2015, pp. 49-52).

upcoming section, by associating equations with 'movement', Wittgenstein meant that we use them as part of a calculus in order to make moves in a language game.

Equations and their applicability

Wittgenstein frequently reflected on what is at stake in simple situations involving applications of arithmetic (e.g. *RFM*, I-157, *RFM*, I-37, I-100, *PI* §466). The following is a familiar kind of example: Consider a baker storing cakes in an empty freezer, storing 3 in the morning and 2 more in the evening. Returning the next day, she opens the freezer expecting to find 5 cakes, but is surprised to find 6 inside. In response, she might question her memory or her senses. However, she would not question whether it really is the case that 3 cakes + 2 cakes = 5 cakes. As Wittgenstein highlighted (*RFM*, I-157), we do not ascribe fault to equations. Rather, we hold ourselves accountable for applying them erroneously. To talk of the "falsity" of an applied equation immediately implicates a shift in the calculation. In the given example, as soon as the baker finds the 6th cake, her 'rejection' of 3 + 2 = 5 is a rejection of calculating *in that way* on this occasion, in favor of calculating something else, or nothing. If she infers that an additional cake must unexpectedly have been added, she effectively calculates 3 + 2 + 1 = 6.

As this scenario illustrates, an equation is not a standalone representation, even if it is used for prediction. Rather, the entire system of calculations, the calculus to which the equation belongs (cf. *RFM*, III-56, III-58, III-81, III-85, VII-11, *LFM*, IV, p. 40, VIII, p. 82, cf. *PI* §81)), is employed *as a system*. Here it is useful to recall Wittgenstein's earlier comment on Ramsey's extensional understanding of mathematics, about conflating a mirror with a portrait: "[W]hat is essential to a mirror is precisely that you can infer from it the posture of a body in front of it" (*TBT*, p. 388). When we apply an equation, we utilize the calculus as a whole in order to form a mirror: we *demand* that both sides of the equals sign reflect one another.

In other words, in using mathematics to make models or predictions, we leave mathematics *as such* uncontestable. The baker does not replace "3 + 2 = 5" with "3 + 2 = 6", as would be the case if she was appealing to a standalone proposition; she remains within the calculus. This illustrates how training in calculating hangs together with basic applications. We have learned to calculate so as to conform in our *application* of equations with the rules of the calculus.[111] To apply arithmetic to a given situation means that both sides of the equals sign in some way model what is going on in the situation at hand. Given this criterion, it is always the equation *as such* that applies, or fails to apply, not just one of the sides of the equation.

This goes some way towards illustrating what Wittgenstein meant by describing mathematics as 'autonomous' and 'grammatical' (e.g. *LFM*, XXVI, pp. 248-251, *TBT*, p. 382). The former should not be taken as a statement about the role of mathematics in human life as such, since, on Wittgenstein's view, mathematics is not disconnected from other practices. Rather, both the autonomy and the grammatical status of equations can be understood, without divorcing mathematics from its applications, by comparing equations with deictic language.

The way a given equation applies is dependent on context, similarly to the meaning of deictic terms such as "here," "now," and "you" (cf. *TBT*, pp. 366-368).[112] Depending on the language game, "2 + 3 = 5" can be applied in myriad ways, modeling various situations. Abstracted from any language game, however, "2 + 3 = 5" is an autonomous grammatical form, like "I am here now". As Wittgenstein (*LFM*, XII, p. 113) came to see, merely adding counting nouns or writing the equation out in words does not by itself imply that the equation is applied. For example, "adding 3 cakes to 2 cakes gives 5 cakes" remains mathematically correct, but its applicability is no less dependent on context. Again, this can be compared to "I am here now" or "here is closer than there". Such sentences are grammatically or logically correct, but they become significant utterances only through their use in concrete situations.

Several of Wittgenstein's 1939 lectures discuss mathematical applicability, and they generally support such comparisons with deictic language. For instance, *LFM*, XXVI, p. 250:

> *If I wanted to show the reality corresponding to "30 × 30 = 900" – I'd have to show all the connexions in which this transformation occurs. –*

[111] Note, also, that we simply do not apply arithmetic in cases in which the number of objects, or groups, is completely unforeseeable and unstable. For example, bakers might calculate when it comes to cakes, but there would be little point in trying to calculate to predict quantities of breadcrumbs (cf. *RFM*, I-37).

[112] I am not identifying equations with indexical terms in some strict sense, denying broader aspects of deixis in mathematical communication and practice (cf. Goodwin & Duranti, 1992, and Barnes & Law, 1976, pp. 229-235). On the contrary, the comparison with indexical phraseology is just a way of bringing out the contextual nature of applications of mathematics, which is an important theme in Wittgenstein's post-1939 remarks.

Notice the difference between asking, "Is there a reality corresponding to '30 × 30 = 900'?" taken alone, and saying this of it as a proposition in a system. Taken by itself we shouldn't know what to do with it: it's useless. But there is all kind of use for it as a part of a calculus.

This comes almost immediately after the following:

"'300' is given its meaning by the calculus – that meaning which it has in the sentence 'There are 300 men in this college'. In the sense in which we might say 'This is a chair' gives a meaning to 'chair'" (LFM, XXVI, p. 249).

So, Wittgenstein was arguing that the applications of an equation are rooted in a calculus, but he was not suggesting that the calculus is itself free-standing and separate from its applications. Rather, the idea is that an equation applies to a *specific* variable situation, similarly to deixis. The 'reality' that corresponds to the equation '30 × 30 = 900,' in general, is all the different contexts in which the transformation from 30 × 30 to 900 occurs. In those cases, a concrete multiplication technique is actually performed, either verbally or in practice (e.g. via automated operations by electronic monetary systems,[113] to give a contemporary example). Analogously, the 'reality' that corresponds to a demonstrative term like "there" is its use in particular contexts. Its 'application,' in those cases, is to point to a concrete location or area.

What is distinctive about this view of equations is its concomitant understanding of the idea of equations being 'rules' of a calculus. Specifically, it modifies the idea that equations *license* moves in language games (cf. Hacker, 2021, pp. 182-183). Equations are not 'rules' in the sense of external *decrees*, as if they were stipulated individually. Rather, they are "surface rules related to action by interpretive procedures acquired in mathematical training," as Barnes & Law (1976, p. 235) put it. The rules are part of *calculi*, and the calculus is attached as a whole, as a form of grammar, to an entire domain of empirical applicability (cf. *LFM,* IV, p. 43). This helps illuminate the conclusion Wittgenstein draws in *LFM*, XXVI, p. 251:

This brings an entirely different sense of how a reality corresponds to mathematics. Because now, if '30 × 30 = 900' is not a proposition 'about 30', you will look for the reality corresponding to it in an

[113] Note that magnetic traces on computer disks create *representations* of monetary transactions; the quantities of money as such need not have any physical realization (cf. Searle, 2010, p. 20; Searle, 1995).

entirely different place; not in mathematics but in its application.

Again, a comparison with deictic language is in order. Consider a sentence such as "Tomorrow is Monday." In most cases, to utter that sentence would not be to say something 'about Monday.' Rather, it would be to say what day of the week is coming up next. The exception would be when teaching someone the weekdays; then, you might say "Today is Sunday," "Tomorrow is Monday," etc., in order to teach the meaning of 'Sunday,' 'Monday,' etc.[114] Analogously, we do not apply "30 × 30 = 900" in order to say something 'about numbers'.[115] We use it to say how much of something specific is obtained by multiplying two concrete quantities. Again, the exception would be when learning or teaching arithmetic.

Granted, there are also disanalogies. Deictic language is to a varying degree *referential* (Silverstein, 1976, p. 36). For Wittgenstein, the closest to a 'referent' of "3 + 2 = 5," its 'reality', is a concrete calculation, such as the act of inferring that there are 5 cakes in the freezer after having stored 3 and 2. However, again, calculations come in many forms, involving everything from writing symbols on a blackboard to forming predictions about the contents of a freezer. This openness in how any given equation is applied adds to the usefulness of the calculus as a whole; it can span multiple language games, similarly to how, for instance, the openness of "there" renders it useful for assigning relative position to all kinds of locations (cf. *PI* §88).

The more pertinent difference is that, in contrast with deictic language, we *calculate* with mathematical expressions. Each operation is a unique 'move' in a calculus. For example, if two sacks each contains 27 bananas, we can calculate that they together contain 54 bananas and by the same token that they do *not* contain 50,51, 52, 53, 55, 56, 57, etc. bananas in total. However, this should be seen as a difference in degree, rather than a difference in kind. Other forms of deictic grammar also come in

[114] See also *LFM*, XII, p. 114, concerning the sentence "2 apples + 2 apples = 4 apples": "In fact it is the same proposition when it is about numbers and when it is about apples, only it is used in an entirely different way. When it is put in the archives at Paris, it is about numbers." Also, see *RFM*, VII-2: "If you know a mathematical proposition, that's not to say you yet know anything. […] [I]f we agree, then we have only set our watches, but not yet measured any time. If you know a mathematical proposition, that's not to say you yet know anything. I.e., the mathematical proposition is only supposed to supply a framework for a description."

[115] Wittgenstein (*LFM*, XXVI, p. 250) also states that "Mathematical propositions do not treat of numbers. Whereas a proposition like 'There are three windows in this room' does treat of the number 3." As Conant (1997, pp. 218-219) suggests, his point is not about whether the calculus or its applications 'treats of numbers,' but to highlight the *distinction* between talking of numbers in calculations and in applications.

systems, such as the words for weekdays and spatial relations, even though the systematicity of these grammars is in general far less intricate than that of mathematics. For Wittgenstein, to calculate is to operate with mathematical systems of grammar.[116]

Rule-following and second order equilibria

These observations on the context-sensitivity of the 'meaning' of equations can be further elaborated by considering a recent game-theoretic response to Kripke's (1982) reading of the rule-following paradox. Drawing on an alternative to Lewis' analysis of convention developed by Vanderschraaf (2018), Matthíasson (2022) seeks to account for meaning in terms of 'basic constitutive practices', which he takes to be practices that 'provide their own stage-setting'. Key to Matthíasson's (2022, p. 12) account is that it distinguishes between intentional strategies and the actual behavior of agents. As was argued in Chapter 2, such a distinction is missing from Sillari's (2013) appeal to Lewisian conventions as an account of rule-following, in effect requiring that agents already play language games in their decision-making. If game-theoretical strategizing already involves participation in language games, as Wittgenstein conceives of the latter, then game theoretical strategizing cannot be taken to underlie language games.

Matthíasson (2022) introduces the concept of 'supergames', which are sequences of games played one after another all with the same form: featuring the same number of players who are given a fixed problem with the same payoff matrix. An 'equilibrium path' is a way for a given set of agents to solve each game in such a sequence. Further, Matthíasson (2022, p. 14) stipulates that agents are trained to have stable dispositions to use expressions in certain ways, ensuring that their solutions to addition-problems such as 68 + 57 are statistically related (most of us would answer 125). Addition-games are then generalized to include indefinitely many agents and to pose indefinitely many different addition-problems (covering any natural numbers $n + m$). Matthíasson (2022, p. 16) explains that there is a single equilibrium path that goes through *all* such supergames, a 'second order equilibrium'.

[116] There are multiple occasions on which Wittgenstein classifies mathematics as 'grammatical' in this way (e.g. *RFM*, VII-3), but one instance might be read as a deviation, *RFM*, IV-18: "It is clear that mathematics as a technique for transforming signs for the purpose of prediction has nothing to do with grammar." One way of making sense of this is simply that mathematical techniques are not part of 'grammar' narrowly conceived.

The idea is that the second order equilibrium path of addition-games "defines what it is to take part in the basic constitutive practice of using the symbol '+' and therefore also what counts as doing the same thing as in a previous case" (Matthíasson, 2022, p. 16). This is taken to provide an answer to Kripke's rule-following skeptic, since the structure of basic constitutive practices "picks out one sameness relation from a set of exemplars to future cases as correct, and it does so by selecting it as being constitutive of the practice itself. The agents do not need to have any particular action in mind for the practice to settle on a given answer as correct" (ibid.). Anyone who gave the answer "5" to "what is 68 + 57?" would not merely *not* be adding, but, given that the person has been trained on the same set of exemplars as others, would be giving an incorrect answer. After all, such a person is definitely *not* quadding, i.e. following some alternative rule. The person *was* adding before, taking part in the same basic constitutive practice as others, and when answering "5" is now verifiably no longer doing the same thing.

The appeal to actions as constitutive of meaning, in response to a Kripkean exposition of the rule-following paradox, broadly aligns with the view on rule-following outlined in Chapter 2. However, though the matter might seem purely methodological, the appeal to game-theory is a significant departure from Wittgenstein's anthropological approach. The notion of a sequence of (the same) games involves a concept of 'sameness of games', and so sequences of games cannot explain the concept of 'sameness'. There are various criteria for sameness at play in different linguistic practices. For example, differing criteria are involved when we consider the 'sameness' of two football matches, two chess matches, two interviews, two arithmetic classes in first grade, two subfields of pure mathematics, and two collaborative engineering projects. Two situations can be instances of 'the same' practice in various ways.

At the same time, positing a sequence of hypothetical games requires an independent criterion for identifying games. So, the issue here is not, as with Sillari's (2013) account, that the posited supergames presuppose that the agents already follow rules or know the meaning of words. Evolutionary biologists studying unthinking bacteria can use the formalism of game-theory to explain empirical processes, as Matthíasson (2022, pp. 25-26) points out. Similarly, a repeated game where agents are asked to solve arithmetic problems can be thought of as a situation in which agents do not need to understand their own responses.[117] The second order equilibrium path through indefinitely many such games can, without circularity, be imagined as a model of an understanding of "+", showcasing a form of use of that symbol. The problem, rather, is that in such a case we rely on a pre-theoretical understanding that this constitutes *one* use of the symbol, and so we cannot take the model to explain *that* this is one use of the symbol.

In other words, the second order equilibrium does not *explain* the 'correctness conditions' of words or symbols (such as "+"), since, in hypothesizing a sequence of games, we must already understand the words or symbols in the coordination problem in a given way. To posit a supergame in which agents solve 57 + 68 requires that 'solve', 'addition', 'number system', and 'arithmetic problem' are understood in a fixed way throughout all the games in the sequence. It presupposes this meaning-invariance *on our part*, not on the hypothetical agents' part. Given that the agents' solutions to each game, when taken in total, are meant to be constitutive of the meaning of the concepts used, that is precisely what cannot be assumed.

To illustrate the issue, notice that a game is introduced through a short description, e.g. "two agents are asked to solve 68 + 57", leaving the environment open to substantial variations. As mentioned, different sameness criteria apply in different practices. But then, since the environment of each game is left undisclosed, indefinitely many distinct supergames can be imagined, each iterating the 'same' game in different senses. Specifically, two sequences of games can require its participants to solve intensionally different problems. That is, a given agent answering "125" in two games might be giving different answers, depending on the context of the games. That is not because the agent subjectively *thinks* the questions or answers differ, but because they *do* differ. For example, "125" means one thing in the decimal system, but something else in the duodecimal system. The game-theoretic model does not prevent one sequence of games from taking place in a decimal setting while another sequence of

[117] This issue is closely related to the so-called 'frame problem' in symbolic artificial intelligence (see e.g. Haselager & van Rappard, 1998). That is, the problem of pinning down an 'action' arises even without presupposing that a program must 'understand' its own code, however the latter is conceived.

games take place in a duodecimal setting. So, indefinitely many agents converging on a given answer, within a game-theoretic model, is not constitutive of rule-following.

This might be seen as unfair, given that it is stipulated that the agents have the same dispositions; a natural response would be that we simply exclude external conditions that impact the setup of the game. However, such stipulations only go so far. We then have to localize the sequences of games to a given persistent background that is rich enough to settle what is going on, and at that point we can no longer confine ourselves to game-theory, adopting an anthropological approach. *Ceteris paribus*, most people likely read "what is 57 + 68?" as phrased in the decimal system, but even just acknowledging different number systems requires paying attention to conditions that go beyond the agents having similar dispositions.

Non-decimal number systems are used daily, even among Westerners. For example, sexagesimal (base-60) is used for timekeeping. If someone adds 68 + 57 when considering passage of time in terms of minutes, she could paraphrase it as 1:08 + 0:57, and her answer of 125 can be paraphrased as 2:05. We do not allow such paraphrases, or mixed equations such as 68 + 57 = 2:05 and 1:08 + 0:57 = 125, unless the problem involves timekeeping. Given that these kinds of operations make for meaningful distinctions, an answer to 125 to "what is 68 + 57?" can 'mean' different things depending on the situation, even in everyday settings.

So, while game-theoretic models can present an idealized, or sterilized, expression of the meaning of words and the rules of arithmetic, they do not *explain* meaning or rule-following. This is implied by Wittgenstein's turn towards understanding the following of a rule as part of (a) language game(s). Rules are seen in terms of the ways in which people proceed in recurring situations, which can be demonstrated through a few practical examples given in the right contexts. Rules are not understood in terms of *solutions* to indefinitely many problems:

> *"[W]hat the correct following of a rule consists in cannot be described more closely than by describing the learning of 'proceeding according to the rule.' And this description is an everyday one, like that of cooking and sewing, for example. It presupposes as much as these. It distinguishes one thing from another, and so it informs a human being who is ignorant of something particular. / For if you give me a*

> *description of how people are trained in following a rule and how they react correctly to the training, you will yourself employ the expression of a rule in the description and will presuppose that I understand it."* (*RFM*, VII-26)

In fact, Wittgenstein made very similar arguments to the ones given above, highlighting the necessity of bringing in the context surrounding a given instance of apparently rule-following behavior, in order to understand its implications in terms of what concepts (if any) are involved:

> *"Let us imagine a god creating a country instantaneously in the middle of the wilderness, which exists for two minutes and is an exact reproduction of a part of England, with everything that is going on there in two minutes.* […] *One of these people is doing exactly what a mathematician in England is doing, who is just doing a calculation. – Ought we to say that this two-minute-man is calculating? Could we for example not imagine a past and a continuation of these two minutes, which would make us call the processes something quite different? / And suppose that they were doing something that we were inclined to call "calculating"; perhaps because its outward appearance was similar. – But* is *it calculating; and do (say) the people who are doing it know, though we do not?"* (*RFM*, VI-34)

To conclude, while models featuring instances of rule-following behavior can be helpful, they do not *explain* rule-following. Such models abstract from the surroundings that shape the significance of the behavior of the people involved. Novices are instructed in following a given rule *R* on the basis of exemplary instances, but it does not follow that the meaning of 'following *R*' is explained by the sum total of instances of following *R*. A model exhibiting instances of *R* is only as informative as it is attentive to the particular practical distinctions that people draw in given cases, which is shaped by the role the rule has in the practices in which it is embedded.

3.3.2 Means of investigation and judgement

Having considered equations, the chapter will round out by touching on the topic of proof. Whether justifiably so or not, this is probably the area of the later Wittgenstein's thinking on mathematics that has generated the most debate, including the most detractors, as evidenced in part by the reception of *RFM*; cf. e.g. Anderson (1958) and Kreisel (1958). It thus deserves a longer discussion of its own, which will be found in

Chapter 4 and Chapter 5. At this point, his view of 'proof' will only be situated relative to the preceding reflections, with a basic sketch of how proof relates to the anthropological nature of mathematics.

As discussed in depth by Marion (2003), Wittgenstein's view of proof has some likeness to L. E. J. Brouwer's intuitionist conception, specifically in that it stresses the *active* over the *passiv*e aspect of proof, that is, stressing the *proving* over 'the proof'.[118] The meaning of a 'mathematical proposition' (a term Wittgenstein often used in a qualified manner to ward off too close associations with non-mathematical propositions) comes from its proof. The proof is not an inert object or fact, but a (series of) act(s). Where the later Wittgenstein departed from Brouwer, going in a radically different direction, is in his understanding of what the act(s) in question *are* and *how* they come to justify a given result. On the one hand, for Wittgenstein, proof is originally a kind of experiment which is turned into a picture; what is *discovered* by a proof is not the theorem, but the fact that the theorem is reached or produced in a certain way. So, a proof invents, extends, or elaborates a *technique*. On the other hand, proof is multifarious. Mathematics consists in, as Wittgenstein put it, "a colorful mixture" of proof techniques.[119]

Wittgenstein described a proof as being conceived as a kind of experiment:

> *"Proof, one might say, must originally be a kind of experiment – but is then taken simply as a picture.* […] *The proof is our model for a particular result's* being yielded*, which serves as an object of comparison (yardstick) for real changes."* (*RFM*, III-23-24)

The idea that proof is originally a kind of experiment might best be illustrated by synthetic geometry. Following a geometrical procedure can easily lead to a result that was not predicted in advance, as in the famous case of the exercise given to the slave in Plato's *Meno*. In this dialogue, the 'slave-boy' attempted to construct a square with twice the area of an initial square, and was led, with a certain surprise, to find a technique based on the diagonal of the initial square. Socrates purportedly took the

[118] As is well documented (Monk, 1990, p. 206), Wittgenstein's return to philosophy was prefigured by him attending a talk by Brouwer in 1928, in which the latter explained his intuitionism. Brouwer's account of numbers resembled the operationalist view Wittgenstein had espoused in the *Tractatus*, apart from the distinct absence of any psychological framing in Wittgenstein's writings. This difference is scrutinized further by Marion (2003, p. 110; 1995), who reads the early Wittgenstein on operations in 'phenomenological' terms.
[119] See *RFM*, III-45.

example of the recognition of the validity of the proof to be evidence of '*anamnesis*', or the recollection of inborn knowledge, but the story neatly illustrates the potentially surprising nature of a proof as revealing a technique.

Originally, a new form of mathematics is approached tentatively and used for the purposes of investigation. It is considered a possible way of answering an experiment designed to put a particular question to the test. However, rather than finding the answer to an empirical question, a 'mathematical experiment' is designed specifically to *test a technique for answering a question*. If and when the experiment has been affirmatively concluded, then, the entire 'experiment' turns into a picture that is used as a means of *judgement*. That is, it is accepted as a *way to* calculate, as a picture *against which* calculations are subsequently judged.

The technique is then no longer experimental, according to Wittgenstein. The proof may result in a constructed proposition, a theorem, but it would be misleading to think of it as an *upshot*, or as a discovery that could be divulged in various ways (*RFM*, III-28). In general, likening a theorem to an empirical proposition is misconceived. Doing so obscures the *action*, the way of forming and judging calculations that is exhibited in, and accepted due to, the proof (*LFM*, XIV, pp. 138-139). Further, Wittgenstein argued that, having obtained the picture, or the pattern which displays a proof-technique, any experimental (or empirical, temporal, etc.) feature of the way in which the experiment was initially conducted – anything inessential to the *reproduction* of the picture and its use as a model of further calculation – drops out as insignificant:

> *"There are no causal connexions in a calculation, only the connexions of the pattern. And it makes no difference to this that we work over the proof in order to accept it. That we are therefore tempted to say that it arose as the result of a psychological experiment. For the psychical course of events is not psychologically investigated when we calculate."* (*RFM*, VII-18)

The *normative* nature of proof is elaborated here. For Wittgenstein, it is essential to a proof that it is used to *judge* or *measure* subsequent (attempts at) mathematics. Nevertheless, that is not to say that the proof is accepted as if it were any arbitrarily chosen standard, or that it is accepted purely as a description of a contingent norm followed by some individuals.

To illustrate this, we distinguish visual proofs in geometry, e.g. proving the Pythagorean theorem by rearranging a 3:4:5-triangle, from illusory proofs which feature 'triangles' with slightly bent hypotenuses or other discrepancies, such as the 'missing square puzzle'. The latter seem to prove a surprising result but are invalidated by inconspicuous circumstances specific to their setup. Note that with any visual proof there will be physical discrepancies, but in the case of illusory proofs the procedure turns *crucially* on inconspicuous discrepancies. When people attempt to judge their own results against illusory proofs, they are unable to do so due to the misleading setup of those would-be 'proofs', and this is why they are not proofs.

The demand for the normativity of proof, and for the reproducibility of the conditions that allow for that normativity, is elaborated further by the following remark:

> *"The mathematical proposition determines a path, lays down a path for us. / It is no contradiction of this that it is a rule, and not simply stipulated but produced according to rules."* (Wittgenstein, *RFM*, IV-8)

The 'mathematical proposition', or theorem, lays down a path for us *in language* (*RFM*, III-29). Equipped with the phraseology of 'language games', Wittgenstein had a model of the connection between proofs and language at his disposal. As we go through the proof, we apply mathematics meditatively in retracing its path. The proof consists of a series of moves *across* language games; in retracing or reproducing the proof, we recognize one language game and transform it into another. As Wittgenstein noted, this model is not hampered by the consideration that we follow rules of inference. The proof lays down a *new* path among language games we engage in, but it is at the same time shaped by those language games.

Nor does this line of thinking rule out proofs in areas of mathematics that have, as of yet, no useful application. For Wittgenstein, mathematical proofs are like bridges between mathematical practices, and they need in this respect not be directly constrained by utility. Networks of paths lead from the peripheries of higher mathematics back to calculi with useful practical applications. That being said, Wittgenstein denied the coherence of the idea of proofs corresponding to structures that are there 'in advance', particularly criticizing the idea that proofs unearth 'mathematical truths' that would exist *regardless* of mathematical practice.[120] Indeed, he denied the very intelligibility of the idea of divorcing mathematics from practice:

> *Imagine the following queer possibility: we have always gone wrong up to now in multiplying 12 × 12. True, it is unintelligible how this can have happened, but it has happened. So everything worked out in this way is wrong! – But what does it matter? It does not matter at all! – And in that case there must be something wrong in our idea of the truth and falsity of arithmetical propositions.* (*RFM*, I-135)

Though he concluded by implying that there is something misleading about speaking of the *truth* or *falsity* of arithmetical propositions, Wittgenstein's immediate qualification that it is "unintelligible how this can have happened" already encapsulates his reasoning. What is proven is inextricably linked to the technique(s) to prove it, as a part of a calculus. To question whether 12 × 12 = 144, despite, or while disregarding, careful calculation, effectively brings into question what *you mean* by "12 × 12". To elaborate Wittgenstein's example, if we assume that some new proof, employing a novel technique, obtained a different result of 12 × 12, this technique would *shape* its result: the result would belong to a different calculus. Arithmetic as currently practiced would be left untouched, the equation 12 × 12 = 144 included.

More generally, then, Wittgenstein's point in *RFM*, I-135 was that what might seem like a mathematical discovery that something is 'awry' with a calculus cannot be the result of that calculus itself, as actually practiced up until now. After all, the calculus is constituted by its own rules and their application, making it unclear what it would mean for it to contradict itself.[121] That is not to say that there can be no reason to substitute one calculus for another. This, however, is logically similar to finding that something is awry with a game: in that case we aim to modify the game in order to fulfil *external desiderata* for what is wanted in the game.

Contradictions and games

Gerrard (1991, p. 136-137) reads the middle Wittgenstein as having dismissed any fear

[120] Note, though, that these are two distinguishable conceptions: the idea that mathematical structures can be given prior to and independently of mathematical activity, on the one hand, and the idea that mathematical proofs are discoveries of truths, on the other. The former view can be held by formalists as well as Platonists, and the latter view can be held by empiricists as well as Platonists. The distinction is elaborated in Chapter 5.

[121] Shanker (1987, pp. 251-255) stresses this point. However, despite continuity, this is one area where the *later* Wittgenstein arguably gained a more nuanced view after the mid-1930s.

of contradictions due to his belief in the autonomy of the systems of rules that comprise mathematics. At this point, Wittgenstein denied that there is any *external* standpoint from which to judge the (in-)consistency of calculi. He regarded the calculi of mathematics as autonomous, so the 'Law of Noncontradiction' could neither be assumed nor imposed from a meta-mathematical, or meta-systemic, vantage point. By contrast, the later Wittgenstein (*PI* §125) understood mathematics in terms of language games, and he described contradictions as a matter of people being 'entangled' in their own rules. On Gerrard's (1991, p. 138) interpretation, this signaled that he was finally able to take contradiction seriously; a contradiction *destroys* any possibility of playing a (mathematical) language game.

However, the later Wittgenstein continued to question the idea that contradiction poses an *inherent* problem. He suggested that the impact of a contradiction depends on the practical role of the calculus; cf. e.g. *RFM* III-80 and *RFM*, VII-35: "[I]f a contradiction were now actually found in arithmetic – that would only prove that an arithmetic with such a contradiction in it could render very good service." At the same time, though, Gerrard (1991) is right to point out that Wittgenstein, in his later period, also was cognizant of the *possibility* of contradiction causing entanglements that impact the practical use of the relevant form(s) of mathematics.

As Wittgenstein put it in *PI* §125, this question of 'entanglement' is a matter of "[t]he civic status of a contradiction, or its status in civic life"; the contradiction potentially ramifies to the uses and applications that are *expected* of the relevant form of mathematics. A contradiction could be a practical obstacle when the point of the calculus, the purpose for which it is designed, involves consistency in arriving at particular results (cf. *RFM*, III-78). However, in the case of ubiquitous and elementary forms of mathematics, as Wittgenstein suggests in *RFM*, VII-35, it is unclear that we even *could* find a method through which we 'discovered a contradiction' which would not instead curtail or cast doubts about that very method.[122]

In a critical review of *RFM*, Anderson (1958) interrogates Wittgenstein's unwillingness to regard mathematics as a game of contradiction-avoidance. He quotes him (*RFM*, V-46) as saying that 'mathematics' is not a sharply delimited concept and

[122] Cf. Wright (1980, pp. 305-307).

that mathematical propositions cannot be reduced to positions in a game, denying the idea that mathematics is a mere a sign-game. Anderson (1958, p. 455) retorts: "But surely 'mathematics' is no less sharply delimited than 'speaking English'; if the latter can be regarded as a language game, why not the former?" Note, though, that Wittgenstein did not say that mathematics cannot be compared with games, only that it is nonsense to say that mathematical propositions are *mere* positions in a game.

Although pure mathematical practice is describable as a language game, it is important to keep in mind that, for Wittgenstein, mathematical calculi, like grammar in general, are "part of an activity, or of a form of life" (*PI* §23). Calculi are precisely *not* just self-contained formal systems. Saying that mathematical propositions are *mere* positions in a game would be akin to insisting that deictic language *strictly* consists in autonomous grammatical forms, disregarding the denotative meaning given by context. In other words, such a view obscures the *raison d'être* of mathematical calculi overall: their applicability outside of pure mathematics.

Anderson (1958, p. 455) continues: "Adopting Wittgenstein's own analogy, one might as well say 'The superstitious fear and awe of chess players in the face of a checkmate'." This remark is in response to Wittgenstein's (*RFM*, I Appx. III-17) description of the fear of contradiction as 'superstitious' in mathematics. Wittgenstein's reasoning here is, firstly, that rules are in effect only insofar as they are put into practice, which renders the idea of a hidden rule, and hidden incompatibilities, confused (*RFM*, V-80). Secondly, there is a *grammatical* link between calculi and their applications; a mathematical proposition is a grammatical rule (cf. *RFM*, III-26). A contradiction thus cannot impact how a calculus *has* been applied, and does not necessarily change how it is applied in the future (*RFM*, III-81 cf. *WVC*, p. 201).

Wittgenstein's attitude to consistency in mathematics, which has puzzled Turing as well as subsequent commentators, is rooted in his understanding of the applicability of calculi. The role of mathematics in its practical applications is grammatical, not empirical or epistemic. If mathematicians arrive at contradictory formulae, this is comparable to two grammarians deriving, from grammatical principles, two conflicting rules for English. Such situations do arise, and they are resolved, for instance, by stipulating that a third rule is followed whenever conflict arises. It is not as if the process ends with a logical contradiction.[123] That is, it is not as if the grammarians see themselves forced to conclude that people behave in self-inconsistent ways, or that the very prospect of speaking English is logically contradictory.

Even if the grammarians *do* conclude that two incompatible rules are followed, say, that a given sentence both conjugates a verb and does not conjugate the verb, this would be met with a specific interpretation or resolution (such as "here, the verb is conjugated according to this rule, not according to this one" or "in this case, the two rules cancel one another out"). This shows that the so-called Principle of Explosion, which states that anything can be derived from a contradiction, is restricted to formal logic (see Schroeder, 2021, p. 194). In practice, contradictions are delimited and handled in various ways. What makes sense in mathematics, as in language, is a matter of what people actually do. On this line of reasoning, it would be wrongheaded both to see Wittgenstein as identifying contradictions with ludic constraints *and* to criticize him for failing to regard mathematics as a game of contradiction-avoidance.

[123] Cf. Matthíasson (2021 and 2013, pp. 86-89) on Wittgenstein on paraconsistency vis-à-vis Turing and Gödel.

4 Mathematics and forms of life

The present chapter will examine the relationship between mathematics and forms of life, and the case will be made that there is a conceptual relationship between them. In Chapter 2, it was argued that Wittgenstein's use of the expression "form of life" ("*Lebensform*") should be understood in a *multivocal* way, insofar as it is used about human beings. That is, rather than being restricted solely to social phenomena, or alluding strictly to humans as biological organisms, the expression should be allowed to encompass any kind of feature of a human life, be it cultural, social, economic, physical, etc. This multivocality is in line with the broad and varied historical use of the term and in intellectual sources by which Wittgenstein was, directly or indirectly, influenced. Still, more remains to be said about the meaning of 'form of life' if the notion of linking mathematics to a form of life is to have a clear significance.

The chapter proceeds as follows. First, the 'localizing function' of Wittgenstein's use of the expression "form of life" is highlighted. The expression alludes to the contingency and particularity of our linguistic and mathematical practices. These contingencies and particularities do not contradict the multivocality just mentioned, because they may involve specific physical conditions just as much as they may involve specific social (or cultural, economic, etc.) conditions. Second, the notion of 'formal properties of games', cohering with Wittgenstein's writings on family resemblance in the *Investigations*, is introduced. This notion will be used as a conceptual link between 'form of life' and 'language game'. In particular, it will be argued that a form of life can be understood as a pattern of formal properties of language games, implying that language games and forms of life shape one another. Finally, the chapter will explore the formal properties of specifically *mathematical* language games, the function of '*deference*' in linking pure mathematics to its applications, and the inextricable role these formal properties play in broadly speaking *mathematical* forms of life.

4.1 Form and forms of life

In the preface to *Philosophical Investigations*, Wittgenstein credited the stimulus of its most fruitful ideas to his frequent interlocutor Piero Sraffa. Sraffa was an Italian economist, a close friend of Antonio Gramsci, and, through his work *Production of*

Commodities by Means of Commodities, founder of the 'Neo-Ricardian' school of economic thought.[124] Monk (1990, p. 261) notes that, according to Wittgenstein, "[t]he most important thing he gained from talking to Sraffa was an 'anthropological' way of looking at philosophical problems" (cf. Sen, 2003). As suggested by Engelmann (2012), Floyd (2016), and Schroeder (2015), this influence can be seen in the later Wittgenstein's approach to mathematics, as in his hypothesizing of different possible quantitative and computational practices (e.g., the 'wood sellers' described in *RFM*, I-143-151) and the relation suggested thereby – encapsulated in Wittgenstein's concept of 'form of life' – that mathematical practice is characteristic of a culture and *vice versa*.

Danièle Moyal-Sharrock (2015) draws a distinction between the singular and plural uses of "form(s) of life" in Wittgenstein's writings. According to Moyal-Sharrock, for Wittgenstein, there is both an overall, fundamental human form of life, on the one hand, and some/many/innumerable specific forms of life, on the other (ibid. p. 27). (For the sake of discussion, "forms of life" will henceforth refer to specifically *human* forms of life.) All human beings have deep features in common, including the acquisition of language (Moyal-Sharrock, 2015, p. 30), but people(s) are at the same time also differentiated in more local ways.[125] For Moyal-Sharrock, the global/local axis (my terminology) corresponds roughly to a physical/cultural axis, with fundamental commonalities being rooted in a shared biology.

On this model, although all humans share *one* form of life, the lives of people in e.g. a 21st century Central-Asian city might contrast with life in a 3rd century South-American village to such an extent that they also constitute *two* forms of life, culturally speaking. So, Moyal-Sharrock (2015, p. 31) argues that talking of a global form of life, even if it does depend on a physical aspect, does not collapse distinctions on the cultural layer. This accords with the argument in Chapter 2 (p. 18) in response to Hacker's (2015) cultural reading; Wittgenstein acknowledged natural (*PI* §244) and shared behaviors (*PI* §206), and there are little grounds for denying that there is a physical aspect to what he here had in mind.

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[124] See Arena (2013, 2015). Arena argues that the relation is reciprocal, and that Wittgenstein's noncausal conception of 'surveyable representation' converges with Sraffa's notion of societal 'snapshots'.
[125] Moyal-Sharrock's (2015, pp. 26-27) 'bilateral reading' builds on Conway (1989) and Cavell (1996).

That being said, the apparently simple solution of classifying forms of life on a global/local axis is complicated by Wittgenstein's use of "*a* form of life" (*PI* §19) and "*this* […] form of life" (*PI PoP*, i-1), which at first glance can be read as carrying either a global or a local meaning. So, to chart the distinctions in use, and simply to get an overview of this important but sparsely used expression,[126] below are all 5 occurrences of "form(s) of life" in the *Investigations*, divided into the grammatically singular and plural:

Singular:	
? *PI* §241	agreement in language is not agreement in opinion ('*keine Übereinstimmung der Meinungen*'), but in form of life
? *PI PoP,* i-1	manifestations of hope are modifications of this complicated form of life
Plural:	
? *PI* §19	imagining a language means to imagine a form of life
PI §23	speaking language is part of an activity ('*Tätigkeit*'), or of a form of life
PI PoP, xi-345	what has to be accepted, the given is – one might say – *forms of life*

The question mark prefixing both of the singular uses and the first plural use of the phrase is meant to indicate that they could be classified in the opposite category, instead. Notably, this leaves room for questioning whether Wittgenstein ever intended *any* specifically singular use of the concept 'form of life', at least in *PI*. Must an 'agreement in form of life', which is involved in agreement in language (*PI* §241), span all of humanity? Different cultures differ with respect to their linguistic agreements, even if some (e.g., the use of (cognates of) words deriving from Proto-Indo European language) might be more widely shared than others.

[126] Although there are only 5 uses of "form(s) of life" in the *Investigations*, there are also the similar notions of 'tapestry of life' (*PI PoP,* i-2, '*Lebensteppich*') and 'weave of our lives' (*PI PoP*, xi-362, '*Band des Lebens*').

At least *in principle*, then, the 'agreement' (that is, '*Übereinstimmung*', i.e., 'convergence' or 'accordance') Wittgenstein mentioned in *PI* §241 can be culturally specific. Similarly, when Wittgenstein characterized the manifestations of a psychological state, hope, as "modifications of this complicated form of life", this could be read relatively locally. Perhaps the "this" is meant to carry some emphasis, and other cultures *might* (Wittgenstein left open) express hope, or a close relative of the concept of 'hope', in other ways.[127]

On the other hand, among the cases of the *plural* use, *PI* §19 has a question mark due to the possibility that imagining a language (which might mean something like imagining analogues or fragments of actual languages, German, Swahili, etc., or something more generic) could mean to imagine a *general* human form of life. Perhaps hypothesizing participants of *any* language game requires us to imagine the *same* kind of being each time, a human with a certain set of (biological?) characteristics. However, if that were the case, Wittgenstein should not have written "imagine *a* language", he should have written "imagine language *as such*". On the contrary, in context, the use of the indefinite article "a" ("*eine*") is crucial:

> *"It is easy to imagine a language consisting only of orders and reports in battle. – Or a language consisting only of questions and expressions for answering Yes and No – and countless other things. — And to imagine a language means to imagine a form of life."* (*PI* §19)

As argued by Haller (2014), Wittgenstein can therefore consistently be seen as discussing, at least potentially, *multiple* forms of life. What this shows is that the distinction between plural and singular uses of "form(s) of life" is of limited exegetical relevance, as is also pointed out by Boncompagni (2022, p. 10) in her study of Wittgenstein's use of the phrase.

An alternative line of inquiry instead focuses on what *function* the phrase plays in Wittgenstein's writings. In light of the questionable 'singular' status of the

[127] Granted, Wittgenstein was here distinguishing the absence of a capacity for hope in non-linguistic animals from its presence in linguistic animals (humans as such). However, similarly to the previous case, we should consider Wittgenstein's (*PI POP*, i-1) wording: "Can only those hope who can talk? Only those who have mastered the use of a language." Again, this can be read locally: *a* language (einer *Sprache*). This is supported by the illustration he gives in parentheses: "If a concept points to a characteristic of human handwriting, it has no application to beings that do not write." This could imply that such a concept would have no application to human communities which have no handwriting, only spoken language and oral traditions.

employment of "form of life" in *PI* §241 and *PI POP* i-1, it is worth noting that the function of the phrase in these passages is essentially *localizing*. That is to say, Wittgenstein was arguing that to imagine *a* language means to imagine *a* form of life, because words have meaning in *specific* activities in which those words are used, and speaking *a* language is part of *an* activity, or *a* form of life, because many of the things that people do (i.e., their activities) involve, and sometimes even require, *specific* uses of language. This observation accords with Boncompagni's (2022, p. 64) 'methodological reading', according to which "form of life" was meant to highlight that "Only within a whole world saturated with human and cultural meanings can gestures, like strokes on a canvas, express ideas with subtleties and nuances and be understood (and misunderstood)."

In other words, one can read Wittgenstein, at least in these thought-through remarks in the *Investigations*, as using "form of life" to highlight the multiplicity *of* and philosophically relevant specificities *in* the ways that humans can live their lives. The "can" here indicates that what is at stake is not a matter of resorting to empirical ethnography or data from the historical record but, rather, of paying attention to and reserving *room for* anthropological specifics. What was at issue for Wittgenstein was the recognition of *contingent possibilities*, this being a requirement for putting language and mathematics into the right context. His approach can still be considered *anthropological* since such contingencies are highlighted by hypothesizing variations in practices while paying attention to their conceptual ramifications, i.e., how people 'would' use concepts in given (real or hypothetical) contexts (cf. *PI POP*, xii-366).

The upcoming section will look more deeply at how the concept 'form of life' relates to Wittgenstein's writings on *language games*. In recent literature (e.g. Moyale-Sharrock, 2015, pp. 37-39; Garver, 1994, p. 246) language games have been seen as primarily linguistic and rooted in forms of life. Here, the case will be made that there is a bidirectional relation between a form of life and a language game, for Wittgenstein. A given form of life cannot be understood in isolation from specific relationships between linguistic practices and *vice versa*. Although this discussion might seem like a detour from Wittgenstein's philosophy of mathematical practice, it will turn out to form a crucial component of it. In sections 4.2 and 4.3, mathematics is argued to be an inextricable part of a form of life, and as such a fully anthropological phenomenon. Nevertheless, an attempt will be made to clarify how and why Wittgenstein also should be taken to hold that mathematics is applicable *across* different forms of life.

4.1.1 Family resemblances and formal properties of games

As outlined in Chapter 2, language games are meant to serve as objects of comparison, so the method of describing language games is inherently comparative (cf. *PI* §§130-133). Several examples of Wittgenstein's comparative way of appealing to games might be considered. To begin with two of them, consider *RFM*, I, Appx. III-8, where Wittgenstein wrote that a position we call 'losing' in chess may constitute winning in another game, implying by analogy that a proposition which is 'false' in one calculus might be the opposite in another calculus. Notably, this suggests that something like the 'same' position, or move, can be identified in two different language games. Second, consider this remark from near the end of *PI PoP*, xiii-369:

> *I get the idea of a memory content only through comparing psychological concepts. It is like comparing two games. (Soccer has goals, volleyball doesn't.)*

Looking past the specific subject at hand (the attribution of 'content' to memories), the stated rationale here is that features of concepts, including contrafactual possibilities, stand out to us through comparison with *other* concepts. Wittgenstein appealed to our understanding of games to elucidate why conceptual understanding is comparative in this way. We throw light on concepts by juxtaposing their uses, similarly to how games are explained by comparing how they are played. If such comparisons are taken too far, or are made without care, they can become the source of philosophical confusion, when we unwittingly transpose features of one concept (one family of language games) onto another. New patterns of behavior can amount to a shift in the game being played, unwittingly or intentionally, and this is analogous to the divergent use of an expression leading to a difference in the concept actually expressed.

Wittgenstein went on to state that the comparative way of attending to language games would be fruitfully extended to an investigation of mathematics:

> *"An investigation entirely analogous to our investigation of psychology is possible also for mathematics. It is just as little a*

mathematical *investigation as ours is a psychological one. It will* not *contain calculations, so it is not, for example, formal logic. It might deserve the name of an investigation of the 'foundations of mathematics'."* (*PI PoP*, xiv-372)

As has already been stated, something like an analogous investigation *was* pursued by Wittgenstein in the remarks collected in *RFM* and in his lectures on mathematics. However, despite its accuracy in a literal sense, Wittgenstein could be criticized for tentatively categorizing this investigation as pertaining to the 'foundations of mathematics' ('*Grundlagen*', also ramifying to the title of *RFM* and *LFM*), if only because it has led to the mistaken impression that he was trying to answer 'foundational questions' about mathematics in a way that is continuous with the way that work in this area has traditionally been conducted.[128]

An example of this impression is apparent in Kreisel (1958, p. 135), who charged that Wittgenstein's "most striking fault is that he believed that all significant philosophical problems occur at the level of elementary computations, and that he made unwarranted generalisations from this limited region of mathematics to mathematics generally". Wittgenstein's remarks were not meant to generalize from the simple to the complex, however. He highlighted the practical basis for mathematics in an effort to combat what he saw as fundamental misconceptions. Philosophical clarity *does* require a sensitivity to the variety in activities we call 'mathematical', but these activities are rooted in prevalent practices (*RFM*, VII-31-33).[129]

One could nevertheless say that Wittgenstein's approach was 'formal', as opposed to empirical. The sensitivity he took to be required pertains to *language games*, and concerns the ramifications and interweaving of concepts. For example, he approached the notion of transfinite cardinality as a modification of the *concept* of 'number'. As Wrigley (1977) argues, even though Wittgenstein did not believe that all

[128] Note, though, that Wittgenstein explicitly distinguished his approach from that tradition in *LFM*, I, p. 14: "Another idea might be that I was going to lecture on a particular branch of mathematics called 'the foundations of mathematics'. There is such a branch, dealt with in *Principia Mathematica*, etc. I am not going to lecture on that. I know nothing about it – I practically know only the first volume of *Principia Mathematica*."

[129] Wittgenstein was, of course, well aware that his approach to the philosophy of mathematics went against the grain of his time: "A mathematician is bound to be horrified by my mathematical comments, since he has always been trained to avoid indulging thoughts and doubts of the kind I develop [...] I trot out all the problems that a child learning arithmetic, etc., finds difficult, the problems that education represses without solving. I say to those repressed doubts: you are quite correct, go on asking, demand clarification!" (*PG*, p. 381-382).

conceptual problems occur on an elementary level, he was interested in core mathematical concepts and their applications, and the effectiveness and relevance of his method comes from making stark, plain comparisons.

In effect, through the method of language games, Wittgenstein was reorienting the notion of a 'formal' investigation (cf. Kuusela, 2014; 2019, p. 178; 2022). The formal nature of the method of language games can be seen in the fact the utterance "soccer has goals while volleyball does not" (*PI PoP*, xiii-369) is not merely an empirical observation, but *draws a distinction* between 'soccer' and 'volleyball'. An empirical investigation would not convince someone that, *actually*, volleyball has goals while soccer does not. For such an inversion to occur would mean that these two games would have completely changed, and that is a matter of the way these games are played *as such*, not something that could be empirically discovered.

To illustrate, say that a person, A, left civilization for a period and came back to discover that volleyball now has goals. A would conclude that the term "volleyball" has shifted in meaning; in this case, A would hold that this new game is *distinct* from volleyball and that a change in terminology would be warranted. At the same time, smaller changes could be imagined which, though they would have modified the form of the game, would not have sufficed to change *which* game was being played, that is, would not be enough to warrant the use of a new term. This spectrum from 'essential' to 'inessential' features of language games is highlighted in Wittgenstein's comments in *PI* §§558-568 (see also *RFM*, I Appx. I-18).

In the *Tractatus* (*TLP* 2.033), Wittgenstein described 'form' as the possibility of a structure, the structure of a proposition being that in virtue of which it represents a possible situation. His later *grammatical* approach (Schroeder, 2021, pp. 58-62) highlighted the irreducible variety of language games. Nevertheless, references to 'forms of language' (*PI* §111) remain central to his thinking, and, as discussed in section 3.2 (p. 87), he developed the idea of 'formal agreement' between language and reality in a pluralistic, practice-oriented direction. Together, this motivates the following stipulations, giving "form" a modified use:

1. A formal property of a game is a feature of how that game is played *in general*, not necessarily how it is played in any given instance. (It includes both typical

behaviors among players and generic properties of the game that are not contingent upon player behavior, e.g., the fact that football is played on a field, chess on a board, etc.)

2. A change in the formal properties of a game implies a more or less *essential* change, a change from it having one form to it having another (or, perhaps, no) form. (This includes cases of homonymic changes, e.g., volleyball becoming volleyball*.)[130]
3. A formal property can be brought up, within the game or outside it, to *identify* the game. The attribution of such a property can therefore serve as a partial explanation of the *meaning* of the name of the game ("football" is partly explained by football having goals, that property distinguishing it from comparable games which do not have goals).

This list is not meant to amount to a strict definition of "form" or "formal property" when it comes to games, as the difference between what constitutes a formal property and a non-formal property can be vague. Nevertheless, these are usable criteria for talking in terms of "formal properties" of language games, while giving room for non-formal, contingent properties which fluctuate from instance to instance without causing an 'essential change'.

Admittedly, extending the terminology of 'form' to games involves an element of stipulation, going beyond Wittgenstein's own writings. However, this move achieves three things. Firstly, it helps emphasize the point of his remark in *PI* §108, where he stated that there is no formal *unity* of all languages, only a family of structures. The adjective 'formal' is here used without an implication of unity: a formal property serves as a distinguishing mark, and formal properties can be more or less essential. Secondly, it highlights the difference, for the later Wittgenstein, between the 'formal' and the logically necessary and sufficient. The use of sound may be an essential formal property of music, as a language game, but John Cage nevertheless produced a silent composition. Thirdly, it brings into prominence the idea that there need not be any fixed

[130] The example "volleyball becoming volleyball*" alludes to the aforementioned scenario of volleyball going from not having goals to 'it' (i.e., the new game by the same name) having goals (which is to say: generally being played with goals). For a historical example, chess underwent an essential change (i.e., chess* becoming chess) in Medieval times, when the queen piece, with its modern moves, replaced the weaker vizier.

and clear gap between linguistic and mathematical practices, on the one hand, and extra-linguistic/extra-mathematical facts, on the other. Linguistic and mathematical practices are embedded in both physical environments and evolving cultures.

Hence, this notion of 'form' fits with Wittgenstein's appeal to recurring characteristics (e.g. *PI* §35), internal/external properties and relations (*PI POP*, xi-247, *RFM*, VII-6, cf. *TLP* 4.122) and his understanding of 'family resemblances' (*PI* §§67-68), especially given the fact that a formal property of a language game is a mark of its difference from, and kinship to, other language games, depending on whether they share that formal property. This is so because we draw on a formal property to identify and teach a game, thereby either aligning it with other games with the same feature or distinguishing it from those that lack it.

The notion of 'formal properties' relates to 'families of games' (*PI* §67) as follows: If a set of formal properties can be jointly brought up to identify two or more games as such, then those games are part of the same family, but the reverse does not hold: a family of games need not be identifiable by any set of formal properties. Two or more games stand in a 'formal relation' to each other if they can be jointly identified through a set of formal properties.[131]

Following Kuusela (2019, pp. 176-177), calculi constitute a subset of language games that are characterized by rules. However, this is not to suggest that mathematical activity is no more than rule-governed manipulation of signs, any more than chess is no more than pushing pieces on a board (*PI* §33). In doing mathematics, as when playing chess, we take for granted various features of the context, not just syntactic rules. Nevertheless, mathematics *is*, for Wittgenstein, comprised of calculi which involve following and extending rules. The upcoming sections will say more about the kind of rule-following that characterizes mathematical calculi.

[131] As will be discussed in the upcoming section, a pattern of formal properties constitutes a forms of life. Note that the rules of football might subtly change, "football" being used in a new way, but it *might* still be the same game despite having new formal properties. Inversely, football might retain exactly the same set of rules but go from being a pastime and a sport to, say, a solemn ritual performed only on holidays. In this case, the change could be essential enough for this new activity to be considered entirely distinct from football in the old sense. Analogous cases can be found throughout history. For example, the role of art and the process of art-production have changed to such an extent that it is questionable whether a person from the 17th century would have considered a given painting or installation in a contemporary gallery to be 'art' (see Sedivy, 2014).

4.1.2 Forms of life and language games

The above redeployment of 'form' is fashioned to fit with the concept 'form of life'. This redeployment is motivated by a need to clarify the relationship between a form of life and a language game. Specifically, it rules out the idea that this relationship is like that of a set to its elements or a container to its contents. If that had been the case, it would make sense for the same language game(s) to be played, without alteration, in *any* form of life. But, as we have seen, Wittgenstein (*PI* §19, §23) took language games to be interdependent with forms of life.

On the present interpretation, when Wittgenstein spoke of 'form of life' as in *PI* §19, §23, §241 and *PI POP,* xi-345, he meant something like a pattern of formal properties of language games. That is to say, a form of life is *how* language games are played, in general, in some population or in a given setting, and this, in turn, delineates *which* language games are played (again, generally speaking) among that population or in that setting. When Wittgenstein said that what has to be accepted [*Hinzunehmende*] is forms of life (*PI* §345), he was *not* contrasting forms of life with arbitrary language games or taking the former to be a causal precondition of the latter. The concept 'formal properties' clarifies the idea that forms of life are interwoven with language without being reducible to a mere *set* of language games.

A language game can come to an end without this necessarily changing the form of life to which it belongs, provided other language games of the same family endure, or the language game returns after some time. A language game also need not be tied to a single form of life, provided its formal properties are exhibited elsewhere. Clearly, human beings can engage in cross-cultural communication (cf. *PI* §206). We causally interact with and perceive one another, our bodies are all vulnerable to sickness and injury, we obey and disobey orders and rules, etc. Formal properties, and language games, are shared across forms of life. Again, that is *not* to say that any given language game is untouched by the form of life of which it is part.

A given form of life can be understood in terms of a single language game ("*the* language game", as Wittgenstein put it in *PI* §41 and §293). This could perhaps be compared to Mauss' (1950, p. 3) notion of a 'total social phenomenon', a key institution or cultural feature that ramifies out to *all* aspects of a society. The language game will in that case have enough characterizable detail, have enough formal properties, to be stamped with the unique signature of the entire form of life (cf. *PI POP,* i-1). Burotti (2018, p. 58) writes that '*the* language game' is the whole of language and the activities into which it is woven (*PI* §7), whereas individual language games are simplified objects of comparison.

Similarly to what has been suggested so far, Norman Malcolm (1963, p. 93; 1986, pp. 237-239) reads forms of life and language games as interlinked, stressing the practical nature of language-use. Gier (1980, p. 247) argues explicitly against Malcolm's interpretation, stating: "It is the language game and its related intentions, emotions, etc. that are embedded in the human situations, customs, and institutions of forms of life." For Gier, a form of life is the bedrock of language, and not *vice versa*. As motivation for this one-directional construal, he adduces a remark of Wittgenstein's from 1937: "I want to say: it is characteristic of our language that the foundation [*Grund*] on which it grows consists in steady ways of living [*fester Lebensformen*], regular ways of acting" (Ms-119,74v). Gier (1980, p. 247) also cites Wittgenstein's motto expressed in *OC* §402: "*Im Anfang war die Tat*" ("in the beginning was the deed", from Goethe's *Faust*, in contrast to "in the beginning was the word").[132]

These remarks are taken to show that forms of life are more fundamental than language games. However, by reading these remarks as indicating that language is dependent on forms of life but not *vice versa*, 'language' becomes in principle distinct from activity, and relatively artificial or formal. Wittgenstein generally melded the concept of 'language' with a form of activity (*PI* §492), the *use* of language, which suggests that this is a false dichotomy. Indeed, seeing language as practice is, at least in part, the *point* of the analogy with games (*PI* §23).

Moreover, Wittgenstein continued the remark from 1937 (Ms-119,74v) by writing that "[t]he simple form (and that is the prototype [*Urform*]) of the cause-effect game is determining the cause, not doubting". Here, he used the word "game" to describe the very foundation [*Grund*] on which (more complex) causally explanatory language grows. The word 'game' ['*Spiel*'] commonly suggests a trivial and artificial

[132] See Hacker (2010, p. 19) on the import of this sentence. Related remarks in *On Certainty* include §204, where Wittgenstein wrote that "it is our acting, which lies at the bottom of the language-game".

activity, but this plays little role in Wittgenstein's later use of the analogy with games. Indeed, the opposite is often the case, as in *PI* §§654-656, where he gave the advice: "Regard the language game as the primary thing."

Moyal-Sharrock (2015, p. 37) writes that "Wittgenstein is clear […] that language is *not* a form of life but *part* of a form of life", quoting *PI* §23. However, once again, this remark should not be read as suggesting that language is merely a *component* of a form of life, as if it were in principle detachable from it. Rather, the point is that a language is understood *together with* the form(s) of life in which it is used. The practical concerns and endeavors of human beings inform the meaning of their words, but the inverse is also true. Ribes-Iñesta (2006, p. 118) nicely articulates this aspect of Wittgenstein's philosophy:

> *[H]uman activity is impregnated with language because it takes place in an environment that is built up through language and as language, and because it always occurs along the routes fixed up by language, regardless of whether or not it is morphologically linguistic.*

In 1939, Rush Rhees translated an earlier draft of the *Investigations*, rendering *PI* §19 as follows: "And to imagine a language means to imagine a way of living" (Ts-226). Wittgenstein accepted this translation of "*Lebensform*" into "*way* of living". This might be taken to invalidate the kind of association between forms of life and formal properties of games that was outlined in the previous section. After all, if *form* played no important role in Wittgenstein's concept of '*Lebensform*', then that concept should not be conceived in terms of formal properties. Boncompagni (2022, p. 17) addresses Rhees' translation and appears to draw a conclusion along those lines; this translation shows that Wittgenstein "did not intend to point to formal characteristics of our life, such as a model or a structure inherent in it", but also that Wittgenstein "was interested in the ways in which humans conduct their activities".

In the first of these statements, the word "structure" appears to be intended in a logical sense, given its inclusion alongside "model". However, if the word is taken in a less technical sense, as including social or biographical structure, then there is a way of reconciling the upshots. After all, structures of human life are ways of living, ways in which humans conduct their activities. Wittgenstein's acceptance of the translation of "*Form*" into "way" in Rhees' translation would then be explained by the fact that the latter is the English word for a modality of behavior; a way of acting is a possible pattern of action. In other words, by letting the translation into "way" stand, Wittgenstein can be taken to have *equated* an interest in the formal character of our lives with an interest in the ways in which we conduct our lives.

If so, as a precise repurposing of Tractarian terminology (cf. *TLP* 2.033), the notion of a 'form of life' is *formal*, pertaining to a *possible* life-structure. It does not merely pertain to actual societies or lives. That does not make 'form of life' an abstract, Platonist notion, referring to a thing existing over and above the lives which have that form, which may be the conception that Boncompagni (2022, p. 17) rightfully seeks to avoid. However, the notion *does* pertain to possibilities rather than strictly to facts.[133] A form of life can be more or less prevalent, and can emerge, die out, return, etc., even in a given community, as that community changes its form. This corresponds with changes in the language games that people engage in.

The verdict of this exegetical assessment is that the way people engage in practices *determines* (i.e., both *shapes*, for them, and *settles*, for the sake of philosophy) what form of life these people manifest. Crucially, this is not merely a matter of *which* language games are played, as if each practice existed in isolation, but of how people's lives are organized, how their practices are structured and flow into one another. Wittgenstein (*PI* §92) indicated that he was still concerned with the structure of language, but he now saw the structure as "*surveyable* through a process of ordering" linguistic practices which participants are more or less familiar with, rather than as something to be unearthed through logical analysis.

4.2 Mathematical application and 'deference'

The above terminological considerations outlined the notion of 'formal properties of games' as features that characterize language games and together constitute forms of life. This gives a foundation for approaching Wittgenstein's later view of the relationship between mathematics and forms of life. The previous chapter connected mathematics to language games, but the role of these language games within forms of life has been left undetermined. Building on the preceding discussion, that line of

[133] Floyd (2020a, pp. 118-119) makes a similar point by contrasting *Lebensform* with *Lebenswelt* ('life-world').

inquiry now takes the following form: which pattern(s) of formal properties, if any, characterizes mathematical forms of life? The following passage of Wittgenstein's (*RFM*, VII-6) is exemplary of his later views, and useful in this connection:

> *There is no doubt at all that* in certain language-games *mathematical propositions play the part of rules of description, as opposed to descriptive propositions. / But that is not to say that this contrast does not shade off in all directions. And* that *in turn is not to say that the contrast is not of the greatest importance. / We feel that mathematics stands on a pedestal – this pedestal it has because of a particular role that its propositions play in our language game. / What is proved by a mathematical proof is set up as an internal relation and withdrawn from doubt.*

One thing to note about this passage is that Wittgenstein distinguished between 'certain language-games' and 'our language game', that is, language in general. Further, he claimed that mathematical formulae are rules of description within *certain* linguistic activities. He here appears to refer to the fact that we apply equations meditatively, for instance drawing the inference "I have 4 cups of milk and 7 cups of water [and 4 + 7 = 11], so I have 11 cups of liquid". This is similar to how we appeal to grammatical rules in cases like "my car is blue and yours is yellow [and blue is darker than yellow], so mine is darker than yours". The bracketed content plays a comparable role in both of these examples. That is, if we were to justify either of them, we would cite the content in the brackets, though this is commonly left tacit.

However, when seen on the level of an entire language, Wittgenstein writes that mathematics 'stands on a pedestal', even apparently standing above other rules such as those involved in the grammar of color-description. What he meant by this is, at this point, still obscure, other than that it has to do with the role that mathematical propositions play in language in general. However, his allusion to proof setting up an 'internal relation', at the end of *RFM*, VII-6, serves as a clue as to what he had in mind. In light of the above stipulations, internal relations can be regarded as formal properties shared by two or more language games. The 'withdrawing from doubt' Wittgenstein mentioned in the final sentence can thus be taken to mean a ruling out of *practical alternatives* that would otherwise leave room for hesitation, the condition that something is treated as a fixed part of a form of activity. Thus, another way of saying that something is 'withdrawn from doubt' is to say that a language game is characterized by a given formal property. If so, Wittgenstein in *RFM*, VII-6 is suggesting that mathematical propositions (or their proofs) set up formal properties of language games.

This will be explored further in the sections to follow. First, it is worth reflecting on the idea that the 'local' role of a mathematical equation in a specific situation differs from the 'global' role of mathematics, and that it is at least not uncharitable to read Wittgenstein as drawing such a distinction. A basic illustration might help to bring this out. Consider a group that has gathered 32 berries in the forest and now wants to share the berries evenly among themselves.[134] There are various ways in which they could perform this task, some of which look more like a mathematical procedure (e.g., a series of arithmetical operations) than others.

On the *more* mathematical end: One person recites numerals while pointing at berries one by one, before doing the same for members of the group. Then, she divides the number of berries by the number of people to get n. Counting berries anew, she stops at n and gives those berries to a berry-less individual; this action is repeated until all the berries are distributed.

On the *less* mathematical end: The person gathers the berries together in a pile, then everyone eats from it to his/her own satisfaction while potentially yielding or having an apologetic attitude in the event of being accused of eating too many berries.

After either of these two kinds of scenarios, and any other kind of procedure where the group divides the 32 berries between themselves, two different questions can be asked:

A. Did anyone attempt to apply the equation '$32/x = y$'?
B. Does the equation '$32/x = y$' model how the berries were actually distributed?

Although it is important to note that both questions can be taken in either way, it is probably most natural to read A as an empirical question, pertaining to the ways in which the individuals in the group in fact thought or behaved, while B can be read as a mathematical problem of its own. To elaborate on this distinction, A can be answered

[134] This can be compared to the case of 'Mother' attempting to distribute 23 strawberries evenly to her 3 children and being unable to do so, discussed by Lange (2013), Braine (1972), and Bangu (2021) as an illustration of the explanatory role of mathematics. The *angle* of the example here is different, aiming to draw out what constitutes mathematical practice, in accordance with Wittgenstein's practical emphasis.

affirmatively on the mere condition that someone used some expression for 32 and attempted to divide that number by some other number, even if the wrong answer was obtained as a result. Inversely, if a collection of 32 berries were *not* divided into x piles of y berries such that $32/x = y$ (whether or not x = the number of people), then that would by itself be grounds for answering B in the negative.

Here, the distinction between the local role and the global status of mathematics should be evident. Even if no one actually appealed to the concrete equation, or even used a numeral at all, we say that the equation in question actually *does* apply, or does *not* apply, in this case depending on how the berries were distributed. Hence, to paraphrase Wittgenstein (*RFM*, VII-6), mathematics is given an honorific role; it seems to "stand on a pedestal" above the language game, not necessarily being explicitly applied within it but still, somehow, being relevant to how it is played. This will be unpacked further in the next section.

4.2.1 Experiment and calculation

For the later Wittgenstein, the pertinent distinction in the above scenario is whether the distribution of berries was an *experimental* process, with the outcome of the distribution being incidental, or a *calculation*, with the distribution of berries being determined mathematically. Still, these descriptions remain essentially ambiguous. In order to draw the distinction between experimental process and calculation, both the concrete details and the context of the episode must be considered. In the case of the group simply gathering the berries in a pile and picking from this pile at a relatively even rate, the answer to question A is ostensibly negative, but the answer to question B depends on our evaluation of the outcome of the process.

In *RFM* I Appx. II-75, Wittgenstein wrote that mathematics functions strictly as a *measure* on the level of individual intent and action. A formula, technique, or number series serves as a means by which we judge results in particular cases. Even when mathematics is evaluated, it is treated as an ideal of general action; it is the human being *qua* counter, calculator, or geometrician who is evaluated, in light of whatever form of mathematics is taken to be relevant. In order to answer B, whether the equation '$32/x = y$' models the scenario of berry-distribution, we would effectively ourselves calculate

$32/x = y$ and compare whatever result we got to whatever outcome came of the group's activity. In doing so we would be judging that activity in a mathematical way, effectively "taking it" to be mathematical.[135]

Inversely, question A, the question of whether anyone actually attempted to apply the equation '$32/x = y$', is not merely a matter of psychology, but of whether the procedure of distributing berries allows for mathematical evaluation. What would allowing for mathematical evaluation mean? For Wittgenstein this would be a characteristic *deference* shown within the particular language game to mathematical practice, the pertinent family of mathematical language games. The word "deference" is here used to elaborate on Wittgenstein's remark that "mathematics stands on a pedestal" (*RFM*, VII-6), which is taken to signify a formal relation.

The idea of 'deference' introduced here should not be understood in a strictly psychological way. In the scenario given, the individuals dividing berries among themselves would not necessarily abide by the authority of any given mathematician, even if they *did* seek to distribute berries evenly with, as it were, mathematical precision. Wittgenstein (*PI* §§232-235) hypothesized individuals calculating who were not led by instruction and intersubjective correction, instead simply doing arithmetic by following their own inner voice. Notably, he appears to allow for the idea that we might still consider this 'mathematics':

> *Wouldn't it be possible for us, however, to calculate as we actually do (all agreeing, and so on), and still at every step to have a feeling of being guided by the rules as by a spell, astonished perhaps at the fact that we agreed? (Perhaps giving thanks to the Deity for this agreement.)* (*PI* §234)

His point was not that rule-following is inessential to mathematics, but rather to emphasize the depth of "the physiognomy of what we call 'following a rule' in everyday life" (*PI* §235). Since following rules is not merely a psychological matter, for Wittgenstein, the relevant formal relation should be characterized *modally*. 'Deference',

[135] Cf. Barton (1996, p. 1037): "It is hard to describe mathematical aspects of a culture using the knowledge categories of that culture without imposing the 'realities' of the 'capital-M' Mathematics which has been developed, and is largely accepted, as an international, academic discipline." Wittgenstein pinpoints a difference between mathematics and other bodies of knowledge by calling the former an *activity* (*PI* §349). On this view, it is no wonder that acknowledging a form of knowledge *as* 'mathematics' may require the *use* of mathematics.

then, means yielding to others in guiding a procedure *should it* produce practically significant deviations. When calculating, we do not insist that our own result holds *if* it conflicts with that of others. In Wittgenstein's (*PI* §234) scenario of miraculous agreement, an absence of significant deviation is simply assumed.

Summarizing, a practically deferential relationship holds, not between individuals, but between a language game in which mathematics is applied and the family of language games in which that form of mathematics is used. We are disinclined to *see* the language game as belonging to that family unless this deference is present. That is, unless people defer to established practice, they are not engaged in calculation, but rather in what Wittgenstein called 'experiment' (*RFM*, I-75-105, *PI* §291).[136] Unless the hypothetical group would be willing to *change* their practice by reference to an established formula, proof, or mathematical definition, we may at best surmise that they *unwittingly* did mathematics. If they *would* be willing to appeal to a formula, proof, or mathematical definition to change their behavior, even if they happened to miscalculate, we would still say that they were attempting to do mathematics.

This way of understanding mathematics contrasts with both Platonist and Aristotelian conceptions, according to which the application of a formula either aims to describe imperfect physical imitations of abstract entities or involves arriving at such entities by abstracting from concrete particulars. These approaches assume that mathematical propositions represent an abstract reality, in some way being comparable to empirical propositions.[137] For Wittgenstein, by contrast, mathematics is not representational. Hence, although norms of precision come into play in practice, there is no theoretical limit to our *imprecision* in the use of mathematical concepts.[138] In comparison to the aforementioned conceptions, this way of understanding applications makes sense of the fluidity and pervasiveness of mathematical vocabulary, such as "divide", in everyday life. A group merely going through the process of sharing berries

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[136] Although Wittgenstein called calculation and experiment "poles between which human activities move" (*RFM*, VII-30), he also recognized that experiments presuppose particular circumstances (*LFM*, X, p. 93).

[137] Representationalist approaches need not take an explicitly Platonist or Aristotelian form. For instance, Lange (2013, 2017) accounts for 'distinctively mathematical scientific explanations' as forms of explanation that make indispensable appeal to 'mathematical facts', facts that are modally stronger than ordinary laws of nature.

[138] Railton (2000) surveys the role of tools used to adapt and establish local "*a priori*" standards in practice. Tools and frameworks are used to guide, make conform, and structure. Standards of precision are established and maintained for practical purposes, and mathematics is applied on the basis of, and as part of, this process.

evenly among themselves may *in fact* be performing the operation of division; what is required is an openness to correction by reference to established mathematical practice.

Orders and the institutional nature of rule-following

The above suggests that language games involving a mathematical concept are *organized around* the relevant form of mathematics, activities of pure mathematics such as techniques of proof, calculation, geometric construction, etc., that involve the mathematical concept(s) in question. For the group attempting to distribute berries, the pertinent mathematical concept was 'division' (i.e., 'sharing evenly'). Although mathematical concepts are considered abstract, describing language games involving mathematical vocabulary brings out how mathematical techniques are locally adapted to fulfil different functions in different settings. Language games featuring mathematical concepts are set up in deference to established mathematical practices, but precisely what this entails is still shaped by the context.

Wittgenstein described the embedding of mathematics in the structure of human life on multiple levels. In *PI* §212, he wrote: "When someone of whom I am afraid orders me to continue a series, I act quickly, with perfect assurance, and the lack of reasons does not trouble me." The mention of being afraid of someone [*fürchte*] is here apparently meant to suggest subservience. His point, then, is that even entertaining the idea that the next step in following the rule is open to multiple interpretations would go against the authority of the person who began reciting the series. For example, say that someone wrote "1, 2, 4, 8" and told you to continue in the same way. In other kinds of cases, dwelling on how best to follow an order might potentially demonstrate subservience. Not so in this case. On the contrary, the very idea of taking the next step on the basis of an independently derived interpretation would be deemed subversive. The next term in the series is "withdrawn from doubt" (*RFM*, VII-6).

By contrast, say that the authority-figure began writing "1, 2, 9, 6" and told you to continue in the same way. In this case, it would not be clear what function or rule (if any) the sequence is meant to satisfy. Making an independent decision would be *required* in order to comply with this person's order. Provided no rule was intended (that is, provided the authority simply wrote down an arbitrary list of numerals), then 'go on in the same way' would mean writing down additional numerals at will. At that point, the details of the process would be in your hands. So, while in the previous case adding the term "16" to the sequence "1, 2, 4, 8" would be to *continue* the initial act of writing down a series, complying with the command to extend "1, 2, 9, 6" (i.e., an arbitrary sequence of digits) would necessarily be discontinuous.

It might be argued that this distinction derives from the fact that, in the former case, the order can be made more explicit: continuously write down the result of multiplying the previous term by 2. What this imperative entails can be cashed out for every term in the series: 16 follows 8, 32 follows 16, and so on. Meanwhile, in the latter case, no explicit imperative has been aired at all, except perhaps: continue writing down arbitrary numerals. The latter cannot be elaborated any further. So, it might be argued, it is the *former* order that gives *reasons* for acting as one does, while continuing to append random numerals to a list is simply an arbitrary, ill-defined task. By this logic, Wittgenstein (*PI* §212) apparently had it backwards: it is precisely when continuing a series that one *does* have reasons for acting as one does.

However, there are orders which are analogous to "continue the series 1, 2, 4, 8" which clearly rule out any reasoning about one's actions. For example, say that an authority figure wrote down "A, B, C, D" and told you to continue in the same way. If you have learned the alphabet, you simply write "E, F, G, H" and so on until told to stop or you reach the end of the alphabet. Again, in contrast, if the authority instead wrote down "A, G, M, C" and told you to go on, you *would* seemingly have to reason about what following this order might require. Teaching someone the alphabet, or reciting it after having learned it, is not about reasoning; there is no convincing reason for *why* one term (e.g., "E") follows another (e.g. "D"). Along these lines, Wright (2007, p. 496) distinguishes between rule-following which *does* involve justification, such as castling in chess (which is done by taking into account both the rules and the specifics of the chess position at hand), and rule-following in what he calls "basic cases", such as when applying the concept 'red': the latter is "uninformed by anterior reason-giving judgement". Wittgenstein's (*PI* §212) order-example shows that this distinction is manifested in social relationships and can be explicated through the extension of someone else's authority.

Wittgenstein here attempted to highlight that a number sequence is akin to the alphabet in the sense that, even if we have to calculate to arrive at the next term in the sequence, ultimately there are no further justifications: "Explanations come to an end somewhere" (*PI* §1, cf. *PI* §109, §§143-144). The attempt to explain any given term is superseded by anthropological description, even in the case of following orders to do mathematics, orders that are imbricated in a network of rational norms and evaluations. If ordered to continue a series, the reason for writing *one* term as opposed to another (e.g., the fact that "1, 2, 4, 8" is followed by "16", not "15") is ultimately a question of what motivates the order itself, as well as why the order was followed.[139] In and of itself, nothing about 16 being the result of 2 × 8 justifies writing the term "16", continuing just *that* series in this way. That answer has to be coupled with "because so-and-so ordered me to continue '1, 2, 4, 8' in the same way".

The context of the action informs any answer as to *why* any given step is taken, in the course of complying with an order. That *why*-question is not merely about motivations but interrogates the kind of action that is involved in taking this step. Both the example of continuing a series and reciting the alphabet involve a redirection of agency to the source of the series and/or the authority who ordered the recitation/continuation, away from whoever was merely ordered to take the subsequent steps. This kind of dispersion of agency is common in rule-governed practices, even when there is no identifiable individual authority. In football, for instance, following the rules depends on immediate coordination with other players.

Wittgenstein's mention of 'blindly' following a rule (*PI* §219), discussed in the previous chapter (section 3.1.1), is relevant here. Zeroing in on a specific behavior, any *justification* for taking that behavior to be governed by a rule falls out of view, and one might as well say: "This is simply what I do" (*PI* §217). With this remark, Wittgenstein was not making the claim that rule-following is *reducible* to unthinking reaction. He drew on the metaphor of blindness as a way of formulating an alternative to the

[139] It might be objected that the fact that the numeral "16" should follow after uttering "1, 2, 4, 8" is patently obvious, that the formula $a_n = a_{n-1} \times 2$ is directly manifested in this sequence. However, that depends on the context. If "1, 2, 4, 8" is uttered while counting objects, the next term might very well be 15. The point is that the fact that "1, 2, 4, 8" is to be understood to exemplify the function of multiplication by 2 would be expressed, for instance, precisely in it being followed by an open-ended order to continue in the same way.

'mythological' picture of a rule as already being completed in advance (*PI* §221; see also Boncompagni, 2022, p. 46, cf. Williams, 2010, p. 190). 'Blind' rule-following, mathematical or otherwise, contrasts with Wright's (2007, pp. 490-491) "*modus ponens* model of rule-following". Above-ground-level action (cf. *RFM*, VI-31) corresponds to what could be simulated by a universal Turing machine. However, the distinction is not absolute. A musician can make 'blind' moves when composing a piece of music, but the steps involved could be justified by rules which are retrospectively apparent.

Instead of explaining rule-following as a matter of individual unthinking reaction, Wittgenstein was essentially making the point that rule-following has to be seen in context. Taken out of context, any reason for acting in a given way accords with an indefinite number of rules. When considered purely causally and in isolation, an individual might write "A, B, C, D" etc., not because that is the alphabet, but because that is the sequence written on the blackboard by this person's elementary school teacher decades ago. For this person, *in isolation*, "write the ABC" might only incidentally entail what we call 'writing the ABC', as the action would comply with not just the rules for alphabetical ordering but for 'blackboard ordering'; if he/she had memorized another sequence of letters as a child, the discrepancy between the individual's disposition and the rule itself would have been apparent.

Still, we *do* say that an individual follows a rule when writing the alphabet, and often do so regardless of the causes, reasons, or motivations behind this behavior. It follows that attributing rule-following is not necessarily to attribute causes, reasons, or motivations for behavior. Whether some individual's behavior justifies the attribution of rule-following instead depends on whether it makes sense to follow the rule in the situation at hand. As Wittgenstein's remark in *PI* §212 and the alphabet illustration show, an example of it making sense to follow a rule would be a relevant authority figure ordering the person to continue in a given way; such an action sets up the "institution of the rule" in these circumstances (*PI* §199, §337, §380).

According to the later Wittgenstein, then, the concept of 'rule' is used in particular practical circumstances in which something turns on distinguishing between correct and incorrect behavior. Taken out of such circumstances, the concept goes idle, in this being comparable to the concept of 'ordering someone' (cf. *RFM* I-1, *RFM*, I, Appx. I-8). Consider the example of Robinson Crusoe.[140] Whether it makes sense for us to attribute rule-following *or rule-breaking* to this isolated character depends, not just on what *he* hypothetically does, but on what *we* are doing. That is not merely to agree with Kripke (1982, p. 110) in stating that we are "applying our criteria for rule-following to him". Schroeder (2021, pp. 90-91) points out that Kripke here offers a vacuous description, since, no matter what, whenever we (whoever 'we' might be) take something to be an *F*, we apply *our* criteria for 'what it is to be an *F*'.[141]

Rather, the point is that there are no *general* criteria for rule-following to begin with, which is a good thing, since such criteria would themselves be rules which in turn apparently require criteria, inviting a vicious infinite regress (cf. *PI* §239). Rather, to attribute rule-following is an unregulated part of our practices. What it entails differs "from case to case of application" (*PI* §201). That makes it unclear what it means to say that we can describe Crusoe as following a rule. To say *that* is so far not to exclude anything, since there could be a language game describing him as following a rule no matter *what* he does. Again, the concept of 'rule' begins to idle when talking in such generalities (cf. *PI* §132). The next section will go further into this, extending the comparison of mathematical vocabulary with deictic language and showing that mathematical practices involve rules of a particular kind.

4.2.2 Iterative rules and techniques

If attributions of rule-following *in general* must be part of a rich context, it might be thought that what Wittgenstein said in *PI* §212 about continuing a series unhesitatingly as an extension of an authority's agency does not have anything specifically to do with mathematics, or number series, simply serving as an example. If rule-following in any case amounts to an institution, it might appear that rule-following *as such* obviates the

[140] As Schulte (2011, p. 435-436) points out, Wittgenstein discussed the figure of Robinson Crusoe, an isolated castaway character, repeatedly in his manuscripts. He distinguished between Crusoe exhibiting (what we would recognize as) regular and irregular behavior; Ms-165,103: "Well, we would only call such a phenomenon language if the behavior of this person were at all similar to that of humans & if we understood in particular the language of their gestures and facial expressions of sadness, resentment, joy etc." Cf. *PI* §206.

[141] However, there is a distinction between describing someone's physical movements and attributing a specific intention or understanding to someone. While the former can be understood merely extensionally, the latter is intension-dependent. In effect, Wittgenstein (*PI* §650, *PI PoP* i-1) drew this distinction when noting that we would not generally ascribe to a dog the belief that something will happen *tomorrow*. The issues here are relevant to techniques and their context-specific significance, as will be elaborated in the next section.

agency of the rule-follower, or at least any *justification* for acting, at a ground level. Along these lines, Baker and Hacker (2009, p. 5) describe mathematical necessity as based on the unquestioning acceptance of a network of conventionally adopted rules, likening mathematical compulsion to an unwillingness to break laws or codes of conduct in other areas of society. This could make it seem as if any differences between different kinds of rules are of secondary importance when characterizing mathematics.

Strikingly, though, what Wittgenstein focused on in *PI* §212 and surrounding remarks was cases in which a 'segment' of a 'sequence' or 'pattern' is to be extended. The rules involved in these remarks are connected to expressions such as "etc." and ellipses, whether these are used simply as 'dots of laziness' marking an abbreviation of a finite enumeration (see Marion, 1998, p. 96; *LFM* XVIII, pp. 170-171) or as a sign for open-endedness like in the case of arithmetic progressions. Wittgenstein was here exploring the phrases "continue like this" and "go on in the same way". That implies that the notion of a rule discussed in these remarks is, at least first and foremost, *iterative*, involving the execution of a given repeatable technique.[142] As Wittgenstein indicated in *PI* §208, part of learning a technique is to see its results as uniform in a particular way, that is, as manifesting an *iterative* rule (cf. *RFM*, IV-28).

When it comes to their role in practices, iterative rules differ in important ways from other kinds of rules. Consider traffic laws. It is possible to drive incorrectly, not following the traffic laws, but nevertheless successfully drive to one's destination. Similarly, it is possible to fail to follow a recipe but nevertheless cook a decent meal. The notions of *making a mistake* and *not doing something* are distinct when it comes to these rules. Regulations, rules-of-thumb, and heuristics pertaining to an activity, like driving or cooking, can be followed or not followed within those activities. The rules of mathematics are different. There is no sharp distinction between not following mathematical rules and simply *not* doing mathematics (*RFM*, IV-26). Generally, performing calculating techniques simply *is* to follow mathematical rules.

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[142] The fact that the pertinent form of rule (e.g., multiply by 2, when given an order to expand the series 1, 2, 4, 8, …) concerns the application of a technique also explains why Wittgenstein often switched between talk of rules and orders, such as in *PI* §212. To learn a technique enables recognition of the regularity of its results. We see how to comply with an order to continue from where the sample (e.g., "1, 2, 4, 8") leaves off. See Floyd & Mühlhölzer (2020, pp. 196-197) on Wittgenstein's understanding of technique.

That previous sentence *could* be taken to imply that doing mathematics is in some sense a uniquely rule-bound endeavor, but it could *alternatively* be taken to imply that the rules of mathematics are in some sense equivalent to techniques. Which is the correct perspective? Recall that the decimal numerals are a system of abbreviated techniques, and that mathematics itself is a mixture of "techniques of proof" (*RFM*, III-46). Wittgenstein clearly gravitated towards a technique-based perspective in his later period. Hence, the pertinent point is not just that mathematics contains *constitutive* rules, rules the following of which constitute the entities (symbols, concepts) at work in mathematical reasoning and so jointly determine the meaning of "mathematics". Rather, mathematics involves rules which emerge *through* the invention and application of techniques. This can be seen as a deepening and pluralizing of Wittgenstein's earlier commitment to the equivalence of internal relations and operations (*TLP* 5.232).[143]

This leads back to the topic of calculation and mathematical propositions, considering that "the connection between a mathematical proposition and its application is roughly that between a rule of expression and the expression itself in use" (*LFM*, IV, p. 47). As mentioned in Chapter 3, there are different ways to spell this out. In *PI* §226, Wittgenstein wrote that

> *Suppose someone continues the sequence 1, 3, 5, 7, . . . in expanding the series 2x − 1. And now he asks himself, "But am I always doing the same thing, or something different every time?" If, from one day to the next, someone promises: "Tomorrow I'll come to see you" – is he saying the same thing every day, or every day something different?*

Here, Wittgenstein himself drew on a comparison with indexical language. The question he posed is rhetorical, because the person says the same thing in one sense but something different in another. It is the same in the sense that the person follows the same rule. If, on the 10th day in a row, the person makes a promise to return the next day, this person would have made that very same promise even if it had been day 11. Generalizing, the person makes the very *same* promise every day; to 'take the next step' does *not* require fresh commitments depending on whatever step is reached, that is, on which day it is, or which number comes next in the series.

[143] This highlights a limit of the analogy between mathematical practice and chess due to a distinction in the rules involved (cf. von Wright, 1965, pp. 6-7): in a given chess game, the rules are premade and never extended.

At the same time, though, each step is also specific. The extension of "the next step" depends on context, like "tomorrow". Say that the person, on day 10, promises to return the next day. If so, he promises to return on day 11. If it had been day 11, he would actually have promised to return on day 12, rather than day 11. In this sense, the meaning of "take the next step" is determined by what precisely is, for the person following the rule, the previous and the subsequent step. Wittgenstein's simile illustrates that iterative rule-following is inherently tied to the use of indexical phraseologies such as "next", "previous", and "the same".

The point of the remark is to clear away the idea that a technique is a *mechanism* behind a rule. So, Wittgenstein continued, "[w]ould it make sense to say: 'If he did something *different* every time, we wouldn't say he was following a rule?' That makes *no* sense" (*PI* §227). This denies the suggestion that the relation between a technique and a rule is merely *causal*. Instead, the two notions are *grammatically* interlinked, in the same way in which the intensional and extensional aspects of indexical language are woven together:[144] the concept of a technique is not divorced from the sameness of its results (*PI* §225). Just as there can be no promising to visit every day, apart from promising on any given day to come the day following that, there can be no expanding the series $2x - 1$ apart from continuing the sequence 1, 3, 5, 7, and so on.

Saying that someone 'does the same thing every time' is not merely a description of a behavioral regularity in a physical sense. It can be considered a statement that the individual is performing a technique and, in so doing, iteratively following a rule. As the previous section suggests, this is a contextual matter. To say this might be to say something about what the person is attentive to and responsible for; for instance, the person may be willing to disregard the *individuality* of what comes out in the course of the rule. If a person each day, after deliberation, only incidentally lands on the promise to return the following day, then this person would generally not be said to follow a rule.[145] So, in this example, following a rule is a kind of long-term engagement in a

[144] On Kaplan's (1977/1989) theory, the context-specific content of an expression belongs to its intension.
[145] That is not to say the distinction must be clear-cut. It may not be obvious, to anyone, whether to regard a given behavior as rule-bound (cf. *PI* §82). As anthropologist S. F. Moore (1978, p. 1) notes: "[T]he making of rules and social and symbolic order is a human industry matched only by the manipulation, circumvention, remaking, replacing and unmaking of rules and symbols in which people seem almost equally engaged."

language game: it does not necessarily *transcend* the daily visits, but in its practical and social ramifications it is not *reducible* merely to daily visits, either.

When Wittgenstein stated that "the connection between a mathematical proposition and its application is roughly that between a rule of expression and the expression itself in use" (*LFM*, IV, p. 47) this indicates that the mathematical proposition and its applications should be understood together, similarly to the intensional and extensional aspects of a deictic expression. And, likewise, mathematical propositions are not applied as if it were through the operation of a mechanism, through a causal regularity. Rather, its application is a form of engagement in a language game. However, what this means remains to be elaborated with further precision.

Arithmetic in isolation

With these considerations in mind, it might be asked whether the rules of mathematics are essentially *social* and whether, for instance, an isolated figure like Robinson Crusoe could do arithmetic.[146] The preceding section shows that the question is overly general. To attempt to pin it down, it is clear that the author Daniel Defoe could characterize Crusoe's behavior by doing arithmetic himself, writing that "Crusoe plucked 4 bananas and 3 coconuts, therefore gathering 7 pieces of fruit". It is ambiguous whether this calculation, as described, is meant to be attributed to Crusoe. Recalling Wittgenstein's (*PI* §226) allusion to days, it is noteworthy that this would be similar to writing that Crusoe planned to do something 'on Monday'; if Defoe had written this, he might just have meant that Crusoe intended to do something on a given day, or he might have meant that Crusoe intended to do something on *Monday*, as such, in which case he would have been attributing the full practice of planning via weekdays to him.

Knorpp (2003, 2015) argues that the idea of rules as a mere matter of intersubjective correction, suggested by Kripke's (1982) reading of Wittgenstein, entails an unintended symmetry: on such a view, it is just as (im-)possible for a community-member to follow a rule as it is for an isolated individual to follow a rule. If rules derive from correction, they emerge from self-correction just as much as they do from

[146] As will become clear, these are two importantly different questions; on this, see Canfield (1996).

intersubjective correction.[147] However, it is unclear what 'an isolated self-correcting rule-follower' might mean, other than to say that some individual behaves in a way that we recognize as similar to following a rule in our form of life. Knorpp (2015, p. 41) describes individuals following rules in isolation, but we recognize such situations as rule-governed due to the way they are set up and described in English.

Moreover, the ways in which we *can* describe completely isolated individuals as rule-followers are limited. It might be stipulated that Crusoe remembered the weekday on which he was first stranded, and has maintained a record of the weeks as the time went by. However, even if he managed to keep track of 7-day intervals, the link between this record and our weekdays would become more and more tenuous. If Crusoe intended to hold an event on a Monday, as such, what he *meant* by this would depend strictly on his memory or on a calendar retained from before isolation (cf. *PI* §267). That is not how the situation is for us. A plan for an event to be held on a Monday does not normally rest on a single record or calendar. We can announce an event to be held on Monday, meaning the day which people generally recognize as such, while being wrong about how many days are left until that event. So, there is at least some difference between Robinson Crusoe and us, in this respect, and this difference would only increase over time, as Crusoe's link to established social practices would dissipate.

Perhaps more interestingly, a 'super-Crusoe', someone hypothetically born and living alone on an island for his entire life (cf. Stern, 2011, p. 232; Pears, 2006, pp. 60-62), could not be said to plan to do something on a Monday as such, not knowing the weekdays. That is so even if he does something every (what we would call) Monday. By stipulation, he would not be taking part in the practices which give weekdays their role in our lives, even if – as *we* could verify with our calendars – he did something every 7 days, lining up with our Mondays.

It could be argued that this analogy with weekdays brings in culture-specific *realia* that is irrelevant to the question of abstract calculation. Etiologically, however,

[147] Arguably, then, Kripke's 'sceptic' is really a rule *nihilist*, reducing rules to arbitrary reactions, as Knorpp, (2015) suggests. Lynch (1992, p. 242) highlights how the skeptic eschews talk of rules in favor of extrinsic notions: "The skeptical solution invokes psychological dispositions and/or extrinsic social factors to explain how an agent can unproblematically extend the rule to cover new cases. The antiskepticist reading treats the rule as an expression in, of, and as the orderly activity in which it occurs. The rule formulates an orderly activity insofar as order is already produced within the activity, and the rule's use elaborates that order."

the weekdays are no more ‘cultural’ than mathematical symbols. The symbols “+” and “–”, in their current usage, originated as part of 16th century merchant practices (Cajori, 1993, p. 230). In any case, it has been suggested that we can imagine a super-Crusoe endowed with mathematical-grammatical ‘understanding’ through an act of God or a chance confluence of atoms (see Forster, 2004, p. 94).[148] Say that such an individual repeatedly combines coconuts and, every time he does so, writes a number of strokes in the sand corresponding to the sum of coconuts. This can be seen as equivalent to the operation of addition. Wittgenstein himself questioned how to respond to someone raising the possibility of such an isolated calculator (cf. Pears, 2006, p. 62):

> *But what about this consensus – doesn’t it mean that* one *human being by himself could not calculate? Well,* one *human being could at any rate not calculate just once in his life.* (*RFM*, III-67)

Although the last sentence might seem somewhat hesitant, Wittgenstein here made a strong point. To elaborate, note that the criterion for addition in the case of the super-Crusoe described above is given by *us*, as hypothetical hidden observers of this isolated individual, as we sum up the coconuts and compare the result to the number of strokes left in the sand. That is not how the situation is for people who are not in total isolation. When people do addition, it is not the case that the criterion for them doing so is the calculation of someone they are not in contact with. This is significant, since following a rule is not only to exhibit regularity in behavior but a way of engaging in a practice.[149] In particular, to regard someone as performing a technique, such as addition, is to regard that behavior as *exemplary* of the performance of that technique.

Once again, the deixis involved in iterative rule-following helps illuminate this point, and the connection can be brought out via a parallel: Could one isolated person intend to *turn leftwards*? Here it might seem as if the answer is obviously yes, but note that an isolated person could not intend to turn leftwards just *once* in her life; she would intend to turn in *whatever* particular direction she turns to face (such as towards the Sun, or westwards, or both, if they happen to be co-extensional). The meaning of deictic

[148] Cf. Robinson (2003, pp. 162-166) for a critique of the idea of appealing to God or chance as ‘causing’ understanding, and (ibid. pp. 167-169) on *solecisms* as mistakes that are characteristic of rule-following.
[149] See Sidnell (2003) for an anthropological account of Wittgenstein’s remarks on rule-following along these lines, coupled with cases exemplifying the role of rules *within* various practices.

concepts, such as 'left', presupposes not just the possibility of repetition but *actual* repetition in practice, not in spite of but *because* that repetition involves applications across different contexts. Without repetition, the behavior is not exemplary of the cross-contextually applicable technique we call "turning leftwards".

The same holds for mathematics. Unless a behavior is, in fact, repeated in a language game, its repeatability is not in play in that language game. In that case it is not a mathematical technique, but an idiosyncratic behavior. On this basis, insofar as 'calculate' means something that *in general* involves other people, which it does in, for instance, commercial contexts, we are not justified in attributing calculation to an isolated individual. When you calculate that 10 + 5 = 15, you do not merely perform a sequence of physical movements, you also provide a model for what someone is doing when paying for a $15 item with a $10 and $5 bill (cf. *PI* §268). But super-Crusoe physically could not provide a model for such interactions. That is not to say that an entirely isolated person would be *unable* to calculate, but that, as Wittgenstein put it, "the phenomena now gravitate towards another paradigm" (*PI* §385).

This acknowledges that rule-following may be possible for isolated individuals, without implying that practices involving rules are viable in contexts of isolation. Calculations can be done by isolated individuals even if mathematics is essentially social, as Canfield (1996) makes clear. Nevertheless, mathematics is rooted in social practices, including commerce, science, and various kinds of everyday settings in which people defer to mathematical techniques. A person faced with total isolation may go on to calculate, but this kind of behavior will gradually lose its point, severed from any form of life in which mathematics has its home.

4.3 Mathematics and formal relations

The above delimits the topic of Wittgenstein's so-called 'rule-following considerations', at least insofar as they are relevant to the topic of formal properties of mathematics, to the notion of iterating a given technique. The forms of rules pertinent to mathematical practice exemplify a technique in the iterative sense, informing the meaning of deictic phraseologies such as "go on in the same way". This technique does not have to be explicitly numerical – Wittgenstein wrote, in *Zettel* §706, that "[n]umbers are not fundamental to mathematics" – but there has to be a possibility of manifesting it through a repeatable task, a kind of activity which we can call "following the rule". For example, one might follow the rule $a_n = 2a_{n-1}$ by expanding the sequence of numerals "1, 2, 4, 8", or by predicting the resource requirements of a consistently growing population, or in order to outline a family tree of someone's ancestors.

Though there is a distinction to be made between rules in the sense of number series and repeating ornamental patterns, on the one hand, and in the sense of socially enforced guidelines or rules of conduct, such as traffic laws or cooking recipes, on the other, this is a grammatical rather than ontological distinction, for Wittgenstein. A rule is always contingent on human activity. The difference, to reiterate, is that when we follow an iterative rule in expanding the sequence 1, 3, 5, 7, … we are 'doing the same thing' repeatedly, e.g., calculating $2x - 1$ (cf. *PI* §224). Varying kinds of techniques, or none at all, might be involved when someone drives, cooks, or tends to a garden (cf. *RFM*, I-167), even when they do so in accord with a rule, while the rules of mathematical practice and techniques are grammatically 'interwoven' (*PI* §§224-225) with the repeated application of techniques.

In accordance with the discussion on 'deference' in section 4.2, the techniques called for, constituting 'doing the same thing' when applying mathematics, are determined by broader mathematical practices. To give an illustration of this, say that a person describes the steering wheel of a vehicle as "circular". Then, in a context in which details are important, someone measures the distance between diametrically opposite points along the outer edge of the wheel, and these distances are found to vary *significantly*. The technique of measuring the distance from one point along the edge to the center would not be available, there being no 'center' to speak of. We can tell from this description that the adjective "circular" should be withdrawn.

Here, there is a conflict between the use of a mathematical concept (in this case, the concept 'circle') and this person's idiosyncratic use. However, this conflict would not be due to a mismatch between the properties of a circle, as if it were an independently conceived abstract object, and the description of the wheel. After all, in this case there would simply be no 'circle' that could potentially conflict with the description. Rather, the person deviated grammatically. When the person described the wheel as "circular", the use of the term deviated from its general use since, as it turned out, relevant geometric techniques were not applicable.

Terminology can often be applied in idiosyncratic ways without involving any *clear-cut* deviation – e.g., to say "the price of butter rises" is not to leave the physical meaning of "rise" behind, but to extend its use (Diamond, 1991, p. 227) – but the case of mathematics is different. The reason, to return to the example, is that "circle" and "circular", when used for describing physical objects or structures, are linked to specific geometric techniques. These techniques are unavailable when the terms are used idiosyncratically. This is not to say that there is an absolute standard of what counts as 'deviant' or 'normal'; within a given practice there are standards for what is meant by the application of mathematics.[150] Still, using the word "circular" here deviated from the *rule* for the repeatability of geometric techniques.

This deviation would have further ramifications, depending on the context in which the wheel was meant to be used. We could, for instance, not draw inferences pertaining to the circumference of the wheel. Note that the rule is not a definition, or a verbal expression in general, though attempts could be made to formulate it as such, but a general way of applying techniques. Practices of using mathematics are grounds for the 'same' concepts to be applied whenever we describe the 'same' geometric shape in two different contexts, for Wittgenstein. This grounding is not causal, but institutional or cultural. That is, in our (attempt to) perform the pertinent techniques we defer to these practices inasmuch as we can expect to be taken seriously from a mathematical perspective. This deference, moreover, is a formal property of any language game of mathematical application. If the person in the example decided *not* to withdraw or modify the term "circular", we would conclude that the term was being put to a nonmathematical use or that the person exhibited a failure to understand geometry.

4.3.1 The non-epistemic character of mathematical applicability

The preceding section might make it seem as if pure mathematics in some sense has logical priority over applied mathematics, on Wittgenstein's view. The description of mathematics as "standing on a pedestal" (*RFM*, VII-6) over our language games could

[150] On the ways rules are "responsible to reality", see Conant (1997, p.220), Railton (2000), and Sidnell (2003).

be taken to lend weight to such an interpretation. However, recall that Wittgenstein was, in *RFM*, VII-6, talking about how we '*feel*' about mathematics, as if it stands on a pedestal. On the present reading, the nature of this feeling can be unpacked and demystified in terms of deference shown to mathematical practice being a formal property of language games in which mathematical vocabulary is used and accompanying techniques are attempted. That mathematics stands on a pedestal is something we might be "tempted to say" as we engage in these language games; as such, it is "raw material" for philosophical investigation, rather than an insight or theory (*PI* §254).

At several points, Wittgenstein suggested that the feeling of the untouchable nature of pure mathematics, though its sources may be benign, can give rise to philosophical distortions. An example of this is discussed in *RFM*, III-87 (and, relatedly, in *RFM*, I-121). This feeling is distortive when combined with a tendency to model our commitment to forms of mathematical and logical reasoning on our empirical knowledge about the situations or objects within them (Schroeder, 2013, pp. 164-166). Mathematical knowledge then gains a peculiar solidity, as if it reflects the absolutely inflexible nature of its subject matter. The idea of the untouchable nature of mathematics may in some situations play useful roles in people's lives, insofar as it manifests the kind of deference described above,[151] but confusions arise whenever this deference is interpreted as a reflection of the *certainty* or *necessity* of mathematical propositions.

According to Wittgenstein, when we begin to think of the feeling of mathematics standing on a pedestal as a reflection of the status of mathematical knowledge, we are inevitably led to misconstrue the relation between pure mathematics and its applications. Roughly, we are led to think that, since pure mathematics is unassailable, its applicability must be *a priori* and its applications must involve propositions that are necessarily true. Wittgenstein diagnosed this as the mistaking of an anthropological phenomenon – the fact that we take specific ways of acting as constituting mathematical

[151] The feeling that pure mathematics is 'absolutely true' can be seen as a psychological manifestation of the deference to learned procedures in language games of mathematical application (cf. Bloor, 1983, p. 93). As suggested in section 4.2 with the example of dividing berries evenly in more or less mathematical ways, that kind of deference should be conceived as a modal notion and as a criterion for understanding activity as mathematical in the first place. It therefore need not be expressed as an actual attitude or feeling.

activity – for an epistemic attitude. In making this mistake, we conflate different language games, with differing formal properties, similarly to the example of believing there are goals in volleyball (cf. *PI PoP*, xiii-369). The deference required by mathematics is a matter of potentially yielding to others in practice, not to be conflated with the kind of epistemic trust people might have in, say, testimonies made by expert witnesses.

For Wittgenstein, it is not the unassailability and generality of their *content* that makes propositions of mathematics applicable across a wide range of contexts. In *RFM*, IV-15-17 he considered the example of applying the commutative law, $a + b = b + a$ and $ab = ba$, when dealing with collections. Such formulae are not *descriptions* that apply *a priori* to all physical structures or collections, nor do they form necessary truths about physical structures or collections when supplemented with terms for units or quantities. Even axioms such as the commutative law have to be learned. Learning them is not simply a matter of self-evidence, but a matter of putting the self-evidence into practice (*RFM*, IV-3; Friederich, 2011). We apply axioms and definitions in stage-setting to structure new language games (*RFM*, IV-5).

To illustrate, say that someone has stored 2 stacks of 3 boxes, each containing 5 apples, but intends to organize the boxes into 3 stacks. After counting the boxes and stacks thereof, the person might explicitly recall the commutative law, calculating that $2 \times 3 \times 5 = 3 \times 2 \times 5$. However, in reality, that formula would likely not be expressed here, and, even if it were, that by itself would not properly speaking constitute its *application*. Instead, applying the formula might involve a physical procedure of reorganizing these collections of apples, in this case storing them in 3 stacks of 2 boxes. The axiom would teach how the counting nouns ("stack" and "box") should be applied in both of these configurations. This being so, the person concerned with the number of stacks, disregarding whether they consist of 2 or 3 boxes, would *in practice* be applying the formula $abc = bac$ by intentionally reorganizing the boxes.

Wittgenstein (*RFM*, IV-17) aired the possibility of people not even having a conception of propositions of pure mathematics. As Dawson (2014, p. 4144) explains, he here allowed that people move "from empirical statement to empirical statement without ever formulating the rules by which they make the transitions as propositions".

Even mathematical progress can be conceived in this non-propositional setting. Wittgenstein drew a 5-by-4 grid of dots, claiming that such a grid might convince people of the commutative law by itself, as a replacement of the formula $ab = ba$. Mathematical relationships can be demonstrated by arranging objects and symbols in well-structured rows and columns, given that we count them *as* elements distributed along an x-axis and y-axis. Wittgenstein thus expanded his early focus on mediative, inferential applications of mathematics (*TLP* 6.2-6.211) by describing various ways in which mathematics connects, not just propositions, but language games (*RFM*, VII-62-67). By applying a simple mathematical formula, a 2-dimensional table can be used to translate between contrasting language games, as in the example of counting boxes of apples and counting stacks thereof. In Wittgenstein's view, then, to apply a mathematical proposition is not to entertain it. We apply it whenever certain formal relations hold between things we do or sentences we utter.[152]

Accordingly, applying mathematics need not express beliefs; it can equally serve to express uncertainty or disbelief. Consider the sentence "I do not know whether I have eaten half of the 10 apples, so I cannot say whether 5 apples remain or not". A person uttering this has tacitly calculated that $10/2 = 5$ in order to articulate a specific form of uncertainty. It might be objected that the calculation nevertheless produces certainty: $10/2 = 5$ is used to infer that *if* you knew that you had eaten half of the 10 apples, and that no more were removed, you *would* be justified in believing that 5 apples remain. However, that fails to explain why, or how, $10/2 = 5$ is relevant to the original sentence expressing uncertainty. That sentence does not have the form "I know that $10/2 = 5$, so I do not know whether $10 - x = 5$". Rather, it can be paraphrased as "I do not know which value to assign to x in $10 - x = 5$, so I do not know whether *to apply* $10/2 = 5$ here". So, the equation $10/2 = 5$ functions as what Wittgenstein called a "rule of description" (*RFM*, VII-6), not a certificate for transferring an epistemic stance.[153]

[152] In line with this, mathematical applications need not conform to patterns of logical inference, often involving dialogue and action (cf. *RFM*, I-17). Consider the following exchange: A: "Did you buy at least 2 liters of milk?" B: "I bought 3 liters." A: "Good." Here, A voices neither premise nor conclusion, but the approbation tacitly applies the inequality $3 \geq 2$; A effectively attributes deference to that calculation to B.

[153] Cf. Kusch (2016) and Wright (2004) for an argument that Wittgenstein held mathematics to express both grammatical and epistemic rules. The present reading aligns more with McGinn (2022, pp. 104-106) and Moyal-Sharrock (2005); not, that is, in denying the idea of mathematical propositions, for Wittgenstein, but in emphasizing their distinctively practical function; mathematical propositions are rules for ways of proceeding.

This illustrates the difference between believing an empirical proposition and applying mathematics. The equation 10/2 = 5 reminds us that we cannot infer that half of the 10 apples remain after the operation described by "10 – *x*" unless we are willing to replace "*x*" with "5". Mathematics affords us with ways of reformulating sentences which contain mathematical concepts, translating from one form of expression to another, such as from "half of the 10 apples" to "5 apples". These methods maintain their applicability irrespective of an individual's level of certainty or uncertainty with respect to the sentences themselves. As this goes to show, to apply mathematics is not to justify a belief on the basis of the certainty of a pure mathematical proposition. For the later Wittgenstein, on the contrary, it is *pure* mathematics that depends on its own applications in practices *outside* mathematics. The previous sections have contextualized this and highlighted its relevance for Wittgenstein's later philosophy. However, precisely what it means, in what sense even *pure* mathematics 'depends on' its own applicability, and why he held this, will be explored in the upcoming section.

4.3.2 The necessity of practical application

In *RFM*, V-2, written 1942, Wittgenstein made the following claims:

> *I want to say: it is essential to mathematics that its signs are also employed in* mufti. */ It is the use outside mathematics, and so the* meaning *of the signs, that makes the sign-game into mathematics.*

This remark might at first blush appear paradoxical, tracing the difference between a 'sign-game' and mathematics to the use of signs *outside* mathematics. In context, the remark is part of an investigation into the extent to which mathematics can be called a "game". Wittgenstein was responding to Russell's logical conception of arithmetic[154] as well as Frege's critical comments on formalism in the second volume of the *Grundgesetze* (cf. Floyd, 2002, p. 310).[155] According to Frege, formalism makes a

[154] Wittgenstein's later view of Russell's logicism is encapsulated by the following remark from *RFM*, III-4: "The reduction of arithmetic to symbolic logic is supposed to shew the point of application of arithmetic, as it were the attachment by means of which it is plugged in to its application. As if someone were shewn, first a trumpet without the mouthpiece – and then the mouthpiece, which shews how a trumpet is used, brought into contact with the human body. But the attachment which Russell gives us is on the one hand too narrow, on the other hand too wide; too general and too special. The calculation takes care of its own application." Wittgenstein argued that, once you describe arithmetic, you already also describe its domains of application, and *vice versa*.
[155] In *Grundlagen*, Frege (1953, pp. 107-112) criticized the notion that mathematics advances through purely formal postulations, and hence that it can be understood as a system of rules for the manipulation of symbols, on the grounds that such postulations introduce entirely new symbols. In the case of rational, negative, complex,

mystery of mathematical applicability:

> *"Why can one make applications of arithmetical equations? Solely because they express thoughts. How could an equation which expressed nothing, which was nothing but a group of figures that could be transformed according to certain rules into other groups of figures, be applied? Now, it is applicability alone which elevates arithmetic above a game to the rank of a science. Applicability thus necessarily belongs to it. Is it appropriate, then, to exclude from arithmetic what it needs to be a science?"* (Frege, 2013, vol. 2, p. 100)

While Wittgenstein sympathized with the analogy between mathematics and a game, drawing on it himself, he also criticized the formalist understanding of the concept of a 'game'. Through this critique he effectively responded to, and reoriented, Frege's comments on formalism.[156] The first move Wittgenstein made in *RFM* part V was to distinguish between a *game* and a *machine*. A game is an anthropological phenomenon that generally features rules but is not completely bounded by them. It exists insofar as, and in the specific ways that, it is sustained through human practice. A machine, by contrast, exists as a designed physical structure. What they have in common is that they can both generally be explained by reference to the respective functions or roles that their elements are supposed to serve, that is, the role of players or pieces in the case of games and the function of components in the case of machines.

Wittgenstein (*RFM*, V-1-4) noted that the normativity which is typically obvious in the case of games, e.g. that pieces must be moved (and not moved) in specific ways to count as part of chess, is present in a less obvious way in the case of machines. The components of a machine are *meant* to function in specific ways. If it fails to function in these ways, we say the machine is "malfunctioning" and in need of repair. The function of machines can be described in the absence of any particulars: coffee machines make coffee, cars drive, etc. Through such forms of speech we idealize machines, speaking of what they *would* do if powered up and ideally functional. We even talk about machines as if they were logically inseparable from their predefined

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and irrational numbers, he held that formalism fails to address the relation between formally introduced terms (e.g. symbols for operations with these numbers), and their relation to extant symbols, creating the false impression of godlike summoning of objects and operations through sheer definition (ibid., p. 110).

[156] However, as Whiting (2017, p. 424) stresses, Wittgenstein "does not mention circumstances of significant activity into which language-games are woven merely to address the 'mere-game' objection". In later writings, Wittgenstein *consistently* highlights the embeddedness of language-games in larger contexts.

movements.[157] The word "machine" is then used as a picture of a *hypothetical* mechanism identified with an intended function, not a contingent physical structure.

Wittgenstein warned against conflating the game-analogy with the metaphor of an idealized machine. To say that mathematics is a game is not to say that it functions autonomously according to a set of rules. Regarding mathematics as an idealized machine is to see mathematics as a model or a picture of its own 'operation' (*RFM*, IV-48). The rules of mathematics are thereby seen, not as part of mathematical practice in the form of justifications people actually appeal to when calculating, but as laws determining how mathematics *must* be used and *must* develop, even in the absence of any human agent. We are at this point misled by the metaphor: we come to see mathematics as an 'ethereal mechanism' (*RFM*, I-119-123) fixed by rules from the outside, rather than as a human activity developing through its own standards (Shanker, 1998, p. 31). For example, we come think that there must be an answer to whether a given pattern ever occurs in the decimal expansion of π, even if we possess no technique for determining whether it *does* occur (*RFM*, V-9), as if it were a machine that has been designed to operate in some way, even if it is beyond the capacities of human beings to see how.

The chess analogy revisited

At this point it might seem as if Wittgenstein's view is, if anything, farther from Frege than Thomae and Heine (cf. Frege, 2013, vol. 2, p. 98). If mathematics is a human activity developing according to its own standards, and those standards are contingent, arbitrariness seems to loom. That is, this loosely-speaking constructivist perspective seems to imply that mathematics could, as a matter of historical contingency, have taken on *any form* whatsoever without room for complaint. If the standards for what counts as 'mathematical' are internal to the mathematical language game, there is nothing that tethers the game to any particular form, let alone to the form of mathematics at the present. Like chess, mathematics could change over time and attain altogether different formal properties. This could already be seen as a *reductio ad absurdum*; such a

[157] As Wittgenstein noted in *RFM*, I-102, talking of the 'internal properties' of an object or structure, as he did in the *Tractatus*, can be compared to, or explained as, talking of an idealized machine in this sense.

constructivist view seems untenable given that mathematical progress is meant to be anything but arbitrary. The ideal of sound and valid argumentation is to leave the goalposts of reasoning unmoved, so such a critical response is not unfounded.

However, Wittgenstein did *not* argue that standards for what counts as 'mathematical' are strictly internal to mathematics as a 'sign-game'. That is precisely what he denied by stating that the signs used in mathematics must also be employed "in *mufti*" (*RFM*, V-2), that is, in logically untidy contexts outside of any strictly rule-bound formalism. This is where the analogy of mathematics and the game of chess breaks down. In contrast to signs for numbers, shapes, operations, etc., which are used both within and outside of calculi, chess-pieces are not employed for practical purposes, indeed *at all*, outside of the game. In line with Gustafsson (2020, pp. 217-218) on the limits of the chess analogy, if we insist on comparing a piece of mathematical terminology (e.g. "add") to a chess-piece *outside* of a calculus, then it would effectively constitute indefinitely many *different* 'pieces' depending on its specific uses.

This is not to say that there *is* no analogy to be made between chess and mathematics. Taking our cue from Wittgenstein (*LFM*, XV, pp. 143-144, *RFM*, I, Appx. III-2, VI-32), we can imagine chess not being a closed system. For example, we could imagine that chess was played between generals, and that the pieces remaining after a given game were used to decide, between two sides potentially going into war, which sections of their respective armies would be allowed to be used in battle. In that case, the wooden pieces would have gained a "plainclothes" use, outside of the game, to refer to sections of an army. However, the link would not be merely referential. *Through* setting up chess and playing on the board, links would be forged between the pieces in the game and military personnel in a real-world conflict, this link being analogous to the link between calculations and empirical propositions.[158]

On more than one occasion, Wittgenstein aired precisely such a hypothetical scenario, imagining the point of chess to be expanded in such a way that the playing of

[158] It could equally be imagined the other way around: the result of real-life conflicts could impact the starting conditions of any game of chess afterwards conducted between representatives of the opposing parties.

that game would ramify out to other activities. Thus, in a conversation recorded by Waismann in 1931:[159]

> *"Think of the game of chess. Today we call it a game. Suppose, however, a war were waged in such a way that the troops fought one another on a field in the form of a chess-board and that whoever was mated had lost the war. Then the officers would be bending over a chessboard just as they now do over an ordnance map. Then chess would not be a game any longer; it would be a serious business."* (WVC, p. 170)

Note that the chess board is said to stand to the battlefield like a map stands to a territory. Given that this hypothetical 'war-chess' is analogous to mathematics, which Wittgenstein took it to be since he introduced it to illustrate the difference between calculi and (mere) games, there must be a relation between formulae written on paper and applications of mathematics in practice. Focusing on the 'map'-part of the quote might lead one to assume that Wittgenstein regarded this relation as one of physical mapping. However, already in 1931, he evidently saw the relation in practical terms, as can be seen from the follow-up remark in which he identifies looking up entries in a diary or timetable (and acting on the information gained) with a calculus.

Dawson (2014, p. 4141) comments on Wittgenstein's war-chess analogy and affords it a significant degree of relevance, understanding this as Wittgenstein's argument for *why* applicability is a necessary feature of core regions of mathematics:

> *Wittgenstein suggests that we might imagine wars being fought using chess. The suggestion is that if that were to happen then chess would no longer be just a game. Mathematics is not just a sign game (as Sudoku is) because we do not treat it as just a game. It does not matter what the intentions were behind chess as regards whether it is a game or not, it matters how we actually use it in our lives. To put the idea another way (not Wittgenstein's own way), we might say that a religious text is marked out as a religious text by the way that it is consulted and used rather than by its content.*

Dawson's reading is sensible, but the radically anthropological import of the analogy deserves a greater emphasis. In Wittgenstein's hypothesized scenario, chess is not just 'consulted' but is actually *enacted* on the battlefield. That is, the actions that we would

[159] See also *LFM*, XV, pp. 143-144. Cf. *LFM*, III, p. 34 on another possible application of chess.

otherwise regard as mere chess moves would, in Wittgenstein's imagined scenario, correspond to movements of troops on the field. So, in hypothesizing this situation, Wittgenstein was not just highlighting how we '*treat*' the game of chess, our having a different attitude towards it than we do to mathematics. Rather, he was highlighting the *integration* of mathematics in the practical structure of a culture or civilization. It is precisely that structural integration which is absent from chess in real life. Dawson (2014, p. 4141) goes some way towards acknowledging this by saying that it is not a matter of which intentions were behind chess, but a matter of how it is used. However, 'use' is overly anemic to capture the role given to (what would otherwise be) chess in Wittgenstein's imagined scenario; the entire war would be molded into the shape of chess, after all.

Granted, it could be replied that this strange scenario would make war, or, at any rate, the aspect of it which involves the positioning of an army, into an extension of the game of chess. So, not only 'chess', but the concept 'war' itself would be changed in this scenario. This is somewhat true, but it seems to be open whether to regard the matter this way, or whether to say that war and chess are still separate. If we *do* choose to regard the two concepts as fused together, this hypothetical variety of 'chess' would still be autonomous, in a sense, because it would *encompass* its own applications beyond the board, that is, its role in war. The concept of 'victory' in chess may in that case be regarded as a victory on the battlefield, and *vice versa*. A real life analogy here is whether we should call, for example, the quantitative methods taught to engineering or economy students 'mathematics', or merely applications thereof. The answer is again that both perspectives are available; it can be said that the students are taught mathematics by being taught its applications and *vice versa*.

Formal relations between calculi and other language games

From the hypothetical analogy of people regarding chess not merely as a pastime activity but treated the practice as meaningful with respect to actual war, Wittgenstein extrapolated that chess would be invested with the highest degree of seriousness, even among people not otherwise interested in the game: "It might not be a game at all. It might be done merely in the Foreign Office; there would be no chess clubs" (*LFM*, XV,

p. 144). In such a scenario, the formal properties of chess (e.g. how the different pieces are moved) would to some extent be shared with language games involving military strategy. How pieces move in the one context would reflect the movement of troops in the other context; in other words, there would be formal *relations* between the game of chess and certain language games involved in war.

The reason behind Wittgenstein's remarks (*RFM*, V-2, V-41, *LFM*, pp. 140-170) on the requirement that mathematical terminology has a broader use in order for its mathematical use to constitute 'mathematics' can be clarified by drawing on this hypothetical 'chess'. If the signs involved in (what is for us) mathematical activity had no other function, what we call "mathematics" *would* be a mere 'sign-game', a pastime akin to chess, crosswords, or sudoku.[160] It is crucial for the analogy to hold that the playing of chess is systematically related to real-life conditions, so that the wooden pieces are correlated to real-life phenomena *through* the activity of playing chess. That is in contrast to the pieces having merely conventional secondary uses without any relation to what happens *in the course of* games of chess. An example of that would be if the wooden pieces happened to serve as mascots for branches of an army, or if we merely 'thought of' the pieces as relating to war. In that case, chess would remain a pastime (or professional) game, albeit with external associations. Mathematical activity has to be *systematically* related to decision-making in order for mathematics not to be a mere game.

Wittgenstein's (*RFM*, V-2, V-41) later position that the signs in mathematics must also have an extra-mathematical use is often discussed as if it were a criterion *he* laid down. Rodych (2018, §3.5; 1997) reads Wittgenstein here as returning to a weakened form of formalism, taking mathematics to be a syntactic formal system (a system of signs and their rule-governed manipulation), but now stipulating that the formal system must have an extrasystemic application in order to count as "mathematics". The decision to restrict the label "mathematics" in this way is

[160] Indeed, when playing sudoku we are applying discrete mathematics or logic; see Floyd (2012, p. 236). Nevertheless, sudoku is strictly a pastime activity insofar as it is disconnected from other applications of the combinatorics involved, other ways of using the same transformations. Wittgenstein's continuation of *RFM*, V-2 makes this clear: "Just as it is not logical inference either, for me to make a change from one formation to another (say from one arrangement of chairs [or, in sudoku, 2-dimensionally related numerals] to another) if these arrangements have not a linguistic function apart from this transformation."

interpreted as an attempt to rule out pseudo-mathematical formalisms, on the one hand, and as part of an emerging interest in, or emphasis on, the role of mathematics in human activities, on the other.[161] Still, on this view, "mathematics" is defined through what amounts to a discretionary stipulation, if not a dogmatic call for interpretability in concrete terms.

Hacking (2011, p. 156) contrasts Wittgenstein's early 1930s view that the application of arithmetic takes care of itself with the 'silent revolution' occurring in 1937, an idea introduced by Steiner (2009). Hacking suggests that Wittgenstein in 1937 finally dropped the idea of auto-applicability and began to regard empirical applications as a necessary condition for mathematics. However, this discontinuous reading depends on a particular interpretation of the idea that 'applications take care of themselves'. This appears to be interpreted by Hacking to mean, not that the applicability of a given form of calculation is guaranteed, but that the applicability of a calculus is a purely extrinsic matter, irrelevant to mathematics as such.[162] This reading does not sit well with the fact that Wittgenstein *continued* to maintain the idea of auto-applicability after 1937, as in *RFM*, III-4: "The application of the calculation must take care of itself. And that is what is correct about 'formalism'."[163] He elaborated on this in *RFM*, IV-7: "One application of a mathematical proposition must always be the calculating itself. That determines the relation of the activity of calculating to the sense of mathematical propositions."

Since the later Wittgenstein additionally saw extra-mathematical applicability as essential to mathematics, the idea of auto-applicability should be interpreted as being consistent with this. The expanded chess analogy already allows for such an interpretation: the moves that constitute mere manipulations of signs *within* mathematical calculi also have an immediate role *outside* the calculi. On this reading, the idea that Wittgenstein simply stipulated as a criterion that mathematical signs must

[161] Rodych (1997, p. 218-220) argues that the later Wittgenstein sought to maintain his intermediate criticisms of transfinite set theory. Limiting the use of "mathematics" to formal systems with extrasystemic applications rules out idiosyncratic systems for manipulating signs, systems without practical application and/or contact with forms of mathematics that have a practical application. However, understood as a restriction, this requirement is difficult to pin down. The need for '*practical* application' is both too restrictive and mutable, and mere 'application' is vague enough to border on emptiness. Wittgenstein offers several examples, such as hanging equations up as ornamentation (*RFM*, VII-40), which involve 'applications' but arguably not mathematics.
[162] Nakano (2020) highlights the relevance of the auto-applicability of arithmetic and argues convincingly that this theme does not imply that a calculus is 'cut off' from its applications, even in the middle period.
[163] Ms-122,12v[2]et13r[1], written November 1939.

have additional uses outside of mathematics ignores the reason *why* he saw such uses as necessary. That reason is at the core of Wittgenstein's understanding of the relation between mathematics and forms of life. The formal properties of mathematical activity, the characteristics which, as Wittgenstein put it in *RFM*, V-2, are "*essential*" to mathematics, are the formal relations between activities of pure mathematics and the various empirical or otherwise non-mathematical settings in which mathematical signs are used.

On this reading, there are automatic links between 'playing the (sign-)game' of mathematics and engaging in (otherwise) nonmathematical activities, given that the the very same moves are involved in them both. These links are not based on decision. As mentioned, Wittgenstein (*RFM*, III-4) wrote, "[t]he *application* of the calculation must take care of itself" and in the following paragraph he added "[t]he calculation takes care of its own application". In other words, any given calculation manages to 'apply itself', as it were, because it is part of a calculus, a 'sign-game' where the ways in which its moves relate to other language games have already been established. Consider the relation between geometry and cartography, for instance, where in concrete cases this relation literally becomes a matter of projection.

What distinguishes calculations from arbitrary ways of manipulating signs, then, is that calculations do not have to wait for an application. Any given calculus serves a role in society *through* the grammatical role(s) that calculus already serves in a form of life. Mathematics remains a family of activities in its own right (*RFM*, V-5, V-33); however, just as chess in the above scenario played a direct role in war, mathematical activities play a direct role within various other practices and domains, without need for intermediating acts of correlation. A way of calculating, in mathematics, is immediately also a way of calculating with respect to its applications, say, in designing a map or determining the dimensions of a building.

If this understanding of the role of the game-analogy when it comes to mathematics is along the right lines, it becomes crucial not to read Wittgenstein's philosophy of mathematics in an overly formalist and conventionalist way, as if he saw signs used in mathematical activity as part of a calculus and then, in addition, as having a separate career in the course of empirical description and prediction. That reading severs the links Wittgenstein found to hold between pure mathematics and practices of ordinary language, links consisting in formal relations between language games rather than mere associations amenable to stipulation. In sum, in his later philosophy, Wittgenstein did not strictly speaking see mathematics as a formal system at all. He saw it as an anthropological phenomenon that *does* include the manipulation of signs according to strict rules, but in ways that ramify out to other practices in a form of life.

The applicability of mathematics across forms of life

In the final section of this chapter, Wittgenstein's way of understanding mathematics through formally related language games will be tied more explicitly to forms of life. If the applicability of mathematics consists in a pattern of formal relations between activities of pure mathematics (including calculating sums, constructing geometric shapes, devising proofs in higher mathematics, etc.) and surrounding language games in which mathematics is actually applied (including building houses, predicting the movement of astronomical objects, buying groceries, etc.), when it comes to comparing the mathematics of two cultures or civilizations, there is no need for an exact match so long as the general patterns are the same. This implies that two societies can have different practices while using the same mathematics.[164]

To illustrate, say one form of life, A, has language games which another form of life, B, does not. For example, A engages in astronomy, being able to predict the movement of the planets of the solar system, while B does not. So, in A, mathematics is applied in a kind of context which is not present in B. Nevertheless, as long as the *way* that mathematics is applied for the purpose of astronomical prediction in A has an analogue in B, this difference between them does not amount to a difference in their respective mathematics. B might analogously apply the same or similar formulae in order to calculate properties of, say, musical frequencies and harmony. It is the *role* of mathematics, a given pattern of formal relations between language games, that must be shared between A and B, not the formulae or empirical applications considered on their own. Just as A and B can have different contexts of empirical application (e.g.

[164] The converse, however, does not hold. If two societies have the same practices, they have the same mathematics.

astronomy and music), they can also have different notations or conventions for calculating. What is analogous or identical, when forms of life A and B agree in their mathematics, is the role of their mathematical activity, the pattern of formal relations between calculi and other language games.[165]

For the later Wittgenstein, the above has to be considered in grammatical terms, that is to say, in terms of what we call 'having the same mathematics'. The exegetical claim here is that, for Wittgenstein, we describe two cultures as having broadly the same mathematics if the role of what we call "mathematical activity" between them is analogous, meaning that there is a similar pattern of relationships between their 'sign-games' and their *other* activities. People in two forms of life might defer to their respective sign-games when using relevant signs, and, if they have the same mathematics, the patterns of deference are analogous.

A consequence of this account is that the development of mathematics in a given society is not a purely intellectual or theoretical matter, depending on just pure mathematics, but also involves the broader use of mathematical terminology. To an extent, the development of mathematics might be contingent on the development of science or technology, or on changes in the economy, just as such developments may come after innovations in mathematics. New *possible* relations of application can open up with intellectual breakthroughs, inventions, or changes. It is not as if change comes automatically, however. New mathematical trails must actually be blazed, whether it be by theoreticians or by practitioners who arrive at novel forms of application; mathematical methods have to actually be *invented* (*RFM*, I, Appx. II-2).

Still, with this, the arbitrariness and relativism dissuading many from entertaining any form of constructivism in the philosophy of mathematics is, to an extent, avoided. Mathematics is *not* deeply arbitrary, since it is fixed from all sides by formal relationships between mathematical calculi and other activities. If we were to radically change how we used mathematics, we would immediately sever these links,

[165] It might be objected that even if A and B use different notations, the mathematics they actually apply must be the same in order for us to tell that they have the same mathematics (all else being equal). However, this seems to beg the question. After all, the claim is not that A and B need not apply the same mathematics in order to have the same mathematics, but that whether A and B apply the same mathematics (in any case) depends on the respective relationships between calculi and the applications of those calculi.

which, for Wittgenstein (*RFM*, V-2), would constitute an essential change. Mathematics *is* arbitrary, however, to the extent that the formal relations constituting mathematics, exemplified by the way an equation is applied in various contexts, depend on a form of life. These formal relations (like any formal property of a game) are a consequence of human history, and are not determined by natural laws or metaphysical principles; mathematics is not, ultimately, a matter of discovery (*RFM*, I-168).

Similarly, mathematics is *not* deeply relative, since formal properties are culturally nonspecific. They can be, and often *are*, shared between cultures. It is possible for intercultural communication in the domain of mathematical inquiry, and with respect to its results, since mathematics as such is not strongly sensitive to the practices in which it is applied; it is dependent only on the *role* of practices in forms of life. We can see this, for example, from our ability to use the decimal system of Arabic numerals to characterize, in some level of detail, ancient forms of mathematics and the role it might have had in people's practices. Mathematics *is* relative, however, to the extent that it is not a matter of knowledge of abstract objects or propositions that would be true independently of mathematical practice (cf. *RFM*, I-25). Mathematics, like language, is wholly and inextricably part of the history of humanity.

5 The two faces of mathematics

Two pictures of mathematics were brought up in the previous chapter which Wittgenstein critiqued in his philosophical career, especially so in *RFM* and *LFM*: the idea that mathematics is an idealized machine and that it is a repository of certain knowledge. The present chapter begins by discussing these two pictures, before they are used to contextualize Wittgenstein's transition from the middle to the later period as part of his evolving concern with the genealogy and morphology of concepts. Finally, Wittgenstein's later writings on proof will be brought to the fore, along with his criticisms of nominalization and generalization of mathematics.

According to Wittgenstein, the picture of an *idealized machine* presents mathematical symbolisms as mechanisms that exist and function irrespective of human activity (*LFM*, XX, p. 196). On this picture, rules of pure mathematics are taken as part of a machine which, due to contingent human limitations, we cannot concretely operate or realize. As an example of this general picture, it has been held that the geometric concept of a 'circle' can never actually be instantiated due to the ineliminable microscopic discrepancies that are at play in the physical world. So, it is inferred, our applications of geometry are at best a pale imitation of *actual*, pure geometry (*PI* §342). On the model of mathematics as an idealized machine, the concept of 'circle' has an *abstract* meaning which any application merely approximates, similarly to how a physical machine at best approximates the purely abstract blueprints of its design.

Irrespective of the insistence that a concept such as 'circle' has no literal descriptive applications in the physical world, the concept of a 'circle' *is* used in pure mathematics. We say, for instance, that the circumference of a circle satisfies the function 2πr. According to the picture of mathematics as an idealized machine, however, the values of that function could not possibly be physically determined by mere humans. On that picture, it must be possible to conceive of a 'rule' that could not be followed in a literal sense, the rule for determining the circumference of a circle. Finite beings such as humans may have mathematical knowledge, but they inherently fall short in their mathematical practice.[166] Strictly speaking, only an omnipotent being would be able to

[166] Along these lines, see for instance Webb (1980, p. 236), who conceives of humans as "abstract machines". Inspired by Turing, Webb abstracts away from human's internal states, but importantly leaves us with the

properly operate with mathematical concepts.

Wittgenstein's rejoinder to that model, in *RFM*, *LFM*, and the *Investigations*, was to emphasize that mathematics *is* an activity conducted by physical human beings. He rejected the notion that mathematical concepts are abstract in the sense of being unrealizable (*PI* §426). In actual practice, the division between pure and applied mathematics, between the abstract and concrete, is not absolute, but relative to the context.[167] We can calculate and apply the result empirically, or get an empirical result and use it for calculation. Techniques of approximation and concretization can be part of pure mathematics, just as abstraction is part of applied mathematics. For instance, a teacher telling her pupils to "draw another shape with the same number of edges as this one" is giving an abstract instruction: draw a polygon with *n* edges. Though abstract, such a task is taken literally when the pupils go on using a ruler.[168]

Wittgenstein argued that mathematics is more akin to a (family of) game(s) than to an idealized machine, and pure mathematics is not, as it were, a 'meta-game'; it is part of the same family (*RFM*, VII-33). Activities we call "pure mathematics" relate directly to what we call "applied mathematics" through formal relations. In particular, mathematical activities *defer* to pure mathematical techniques. The relation of deference does not stand *apart* from activities of pure mathematics, or its applications, but is an internal aspect of these activities. For example, like the pupils learning the concept of an *n*-gon just mentioned, when students first learn linear algebra, this is taught in terms of connections with previously learned forms of mathematics, such as geometry. They are taught ways in which linear algebra relate to these other activities, and, in that sense, they are immediately taught potential applications.[169]

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capacity to perform only a limited (i.e. 'finite') number of tasks. On Wittgenstein's attitude to technology, machinery, and what we would now call the 'language of thought'-hypothesis, see Raleigh (2018).

[167] Pérez-Escobar & Sarikaya (2022) conceive of it as a sociological division between groups. As they go on to acknowledge, for Wittgenstein, the distinction should be drawn relatively to the intended use of mathematics.

[168] If, in an unusual circumstance, some pupil insisted that the true number of edges of the shape that the teacher had drawn is a matter of nanometer-scale discrepancies among subatomic particles, and cannot be discerned let alone replicated by humans, *that* student would be showing a misunderstanding of the terms "edge", "shape", "number", and/or "discern". Such a response would be like answering "what time is it?" with "that cannot be precisely measured by humans". Though such responses might seem merely pedantic, in Wittgenstein's view they are not; they express philosophical (grammatical) misunderstandings: see *PI* §342.

[169] The concept of 'application' thus can be, and is, extended from empirical prediction and description into pure mathematics (cf. Schwayder, 1969, p. 81). In mathematics it is common to talk of the 'application' of some technique or result in relation to other subfields. Considering that calculi of pure mathematics are considered in

Another upshot of Wittgenstein's perspective on mathematics as a *family* of games is that pure mathematics is multifaceted and cannot be reduced to a single form of activity. The formal relations between calculi and other activities, the ways in which pure mathematical procedures relate to, and are used in, more or less non-mathematical contexts, are variegated. Still, we defer to *practices* of pure mathematics, considered in terms of well-established and widely taught techniques such as elementary arithmetic and Euclidean geometry, these being areas in which we do not come into conceptual conflict (cf. *RFM*, VI-21). Any significant deviation, when an individual applies some technique in an aberrant way, is simply seen as nonmathematical, or as a mistake, or perhaps – if done in the right way and in the right circumstances – as an attempt at some form of mathematical innovation.

This latter phrasing might seem overly conventionalist. Talk of "regarding" someone as innovative or innumerate might seem as if concedes too much to a reading such as Kripke's (1982), according to which there is *nothing more* to following a rule than what we take to be rule-following. The point is not that innumeracy is *nothing more* than disagreement over how to apply signs from case to case. As was explored in Chapter 3, capacity for using mathematical vocabulary (broadly, including numerals) is far from an isolated, frivolous matter:

> *"Counting and calculating are not – e.g. – simply a pastime. Counting (and that means: counting like* this*) is a technique that is employed daily in the most various operations of our lives. And that is why we learn to count as we do: with endless practice, with merciless exactitude."* (*RFM*, I-4)

As Wittgenstein emphasized, our particular ways of using mathematics play an essential role in various activities in our everyday lives, some of which are physically beneficial or even necessary. Deviation in the use of signs would not be an isolated preference, but would make a clear practical and social difference. Developmental dyscalculia, lack of calculating ability, can lead to social dysfunction (Kaufmann et al., 2020). Empirically speaking, the demands of social and practical life pressure people to conform in their use of mathematical terminology, to accept the rules of mathematics. However, this is a

terms of *practice*, the distinction between 'theoretical' and 'practical' applications is a matter of degrees. This should not make us lose sight of Wittgenstein's critiques of *merely* theoretical application, however.

conformity in practice, not opinions (*PI* §241), and the acceptance is not a matter of individual decision (McGinn, 2022, p. 105).

So much for the picture of mathematics as an idealized machine, or a logical machinery (*LFM*, XX, p. 194), and Wittgenstein's emphasis on agreement in mathematical practice. The *second* picture brought up in the previous chapter concerned the *epistemic* status of mathematics. According to that picture, the usefulness of mathematics reflects the certainty of our belief in, or knowledge of, propositions of pure mathematics. On that view, a mathematical proof establishes the certainty of a proposition and for that reason guarantees its applicability. For instance, on an epistemic model, the proof that there is no largest prime number serves to make us *certain* that, try as we might, we will fail to find a largest prime number.

In contrast to this epistemic picture of applicability, Wittgenstein (*RFM*, III-22-25) stressed that the usefulness of mathematics comes, not from the certainty of its results, but from the process of calculation itself. We rely on theorems not because they express certainties, but *because* they are the fruits of mathematical labor. Hence, a proof convinces us, not merely of its result, but of a way of proceeding *to* that result: "Proof, one might say, does not merely shew *that* it is like this, but: *how* it is like this. It shows *how* 13 + 14 yield 27" (*RFM*, III-22).

This latter remark could be seen as giving mathematician an exaggerated centrality, as if we wait have to wait for mathematicians to tell us how to do elementary arithmetic. Unless we interpret "proof" in a broad way so as to include calculations performed by laypeople, several of Wittgenstein's examples of proofs, such as in *RFM*, III-22, are both unrealistic and out of step with his comments on pure mathematics, like his discussion of "Russell's calculus" (*RFM*, III-3-8, III-12-20). We have no need for abstruse proofs to know that 13 + 14 yields 27, but we *do* perform calculations to make use of such elementary propositions.

So, "proof" should be understood broadly, in this context. Wittgenstein's point in *RFM*, III-22 is that, when we talk of a proof, or of an operation having a given result, we mean that a certain pattern of behavior is accepted. Wittgenstein's writings point towards the fact that proof does not merely serve to give credence to a proposition. Proving and calculating are *activities*, and as such are best understood in terms of verbs and adverbs: mathematics affords us with techniques, *ways* of doing things. Wittgenstein thereby drained the appeal out of seeing mathematics as a repository of absolutely certain results, *a priori* true propositions. Whereas the idea of mathematics as an idealized machine overemphasizes the *rule-bound* aspect of mathematics, the idea of a repository of certainties overemphasizes its *propositional* aspect.

There is, for the later Wittgenstein, a broadly mistaken assumption underlying both of these conceptions: the idea that mathematics is fundamentally a *theoretical* subject to be studied from the outside, rather than a family of practices that people engage in, in their everyday and professional lives. There is no metamathematical perspective, if that is taken to mean standing apart from mathematics and deriving rules *for*, or propositions *about*, calculi (*TBT*, p. 376; cf. Mühlhölzer, 2012; Berto, 2009, p. 194). By prioritizing practice, Wittgenstein sought to circumnavigate a dichotomy forced on us by regarding mathematics as a fundamentally theoretical subject matter, through which mathematical progress comes to seem either arbitrary or predetermined: as if mathematics is either the result of designing whatever systems of rules or instructions we want, as self-contained formalisms, or all cultures are bound to the same path, gradually unveiling a domain of universal mathematical insights.

As Frege (2013, vol. 2, p. 100) argued, the former view makes a mystery of the useful applicability of mathematics. It also fails to capture the integration of mathematical concepts in everyday practice. On the latter view, because mathematical propositions are seen as pertaining to abstract objects, they have the property of being true *no matter what*, and we can know that they are, or would be, true in any possible world (Lewis, 2001, pp. 108-110). This raises questions about how we know that mathematical propositions *are* true (see Benacerraf, 1973, pp. 671-672). Given that their subject matter is taken to be abstract, we could even ask what difference it would make if they were false, and so what is *meant* by calling them 'true'.

As indicated in Chapter 3 (p. 102), in the case of *p* being a mathematical formula, Wittgenstein held that we use formulations such as "*p* is true" or "I know that *p*" as regulative statements or formulations of the rules of a calculus.[170] Such sentences express grammatical rules which could be compared to "it is true that Tuesday follows Monday" or "I know that 3 left turns equal 1 right turn". When phrasing matters in this way, talking of our knowledge of *p* or the truth-value of *p*, we are contrasting *p* from pseudo-equations which are relevantly similar. The sentence "*p* is true" is *false* only if *p* is itself a pseudo-equation, not being part of the relevant calculus. So, the 'truth' or 'falsity' of a (pseudo-)formula is equivalent to its correctness or incorrectness, and these notions are invoked in the context of instruction, that is, when teaching someone a calculus or reminding them of how it works.

5.1 The way and the goal

In his so-called "transitional period", Wittgenstein wrote on the nature of mathematics in ways that foreshadow much of what he would go on to say later, in *RFM* and *LFM*. The earlier writings can be understood as Wittgenstein working his way towards an accurate understanding of the practices of mathematics. He had yet to settle on what an anthropologically grounded perspective on mathematics entailed, and how it should be expressed. Accordingly, some of his remarks from the early 1930s can be taken as radically constructivist and revisionist with respect to mathematical practice in ways that he would later avoid. The extent to which he retained some of the more radical-seeming views espoused in this period is, however, a matter of dispute (see e.g., Rodych, 1997; Floyd, 1995, 2001, 2020; Lampert, 2008; Dawson, 2015, especially pp. 28-32 and pp. 88-96; Marion, 1995, 1998; and Bangu, 2020).

Regardless of the precise extent of the continuity of Wittgenstein's views on mathematical practice, his earlier remarks occasionally help clarify his mature thought both through differences and similarities. In particular, the theme of the (apparent) two-sidedness of mathematics, the rule-aspect and the proposition-aspect, emerges in Wittgenstein's writings in the middle period.[171] *Philosophical Remarks* (written 1929-1930) is especially relevant here:

[170] See *RFM*, III-39. Wittgenstein suggested that, in philosophical contexts, we avoid using "*p* is true/false" when it comes to mathematics and logic (*LFM*, XIX, pp. 188-190). As will be discussed below, 'understanding a mathematical proposition' is a *vague* concept (*RFM*, V-46); it involves understanding the *point* of a calculus.
[171] It can also be taken to derive, in a perhaps less obvious way, from Wittgenstein's early dichotomy of functions and operations (on the latter, see Lampert, 2008).

> *"In my opinion, no way can be found in mathematics which isn't also a goal.* […] *Wouldn't this imply that we can't learn anything new about an object in mathematics, since, if we do, it is a new object?* […] *There can't be two independent proofs of one mathematical proposition."* (*PR* §155)

Wittgenstein (*PR* §156) added an analogy of a knot; trying to untie loops of thread that one believes to be tied into a knot, before realizing there is no knot after all. This is meant to be analogous to trying to solve what is thought to be a mathematical problem before realizing that it is not a problem at all. When you see that a mathematical problem is insoluble, you find that there was actually nothing to solve. For example, when we proved that there is no largest prime number, we not only showed that there can be no such thing as 'finding the largest prime number', but that it is unclear what such a discovery would be. The proof did not merely rule out the existence of such an 'object', it ruled out our possessing the *concept* 'largest prime'.

However, Wittgenstein (*PR* §156, §158) went on to push this point further:

> *But now I want to say that the analogy with a knot breaks down, since I can have a knot and get to know it better and better, but in the case of mathematics I want to say it isn't possible for me to learn more and more about something which is already given me in my signs, it's always a matter of learning and designating something new. / I don't see how the signs, which we ourselves have made for expressing a certain thing, are supposed to create problems for us.*
>
> *Where a connection is now known to exist which was previously unknown, there wasn't a gap before, something incomplete which has now been filled in! – (At the time, we weren't in a position to say 'I know this much about the matter, from here on it's unknown to me.') / That is why I have said there are no gaps in mathematics. This contradicts the usual view.*

These two remarks not only contradict 'the usual view', they can be taken to conflict with the history of mathematics as such. By saying that there are no gaps in mathematics, Wittgenstein appears to imply that there are no unsolved problems, nothing that mathematicians have not yet settled or discovered. However, mathematicians *are* ostensibly working on unsolved problems. Taken in a strict or naïve sense, then, the claim that there are no gaps in mathematics can be read as conflicting with the later recommendation of looking at mathematics from an anthropological point of view and acknowledging actual practice (*RFM,* III-87).

However, these remarks should be taken to express something less counterintuitive, particularly as indicated by the following sentence of *PR* §156: "I don't see how the signs, which we ourselves have made for expressing a certain thing, are supposed to create problems for us". Wittgenstein was here not outright denying that there *are*, in some sense, unsolved mathematical problems; he was denying a particular understanding of their origin. Problems in mathematics do not emerge from the signs themselves, because, if they did, we could simply fill in our knowledge of the signs, or adopt a different notation, and thereby make the mathematical problem vanish.[172] Even *if* there are open mathematical problems, solving them is not merely a matter of attending to our use of signs. *PR* §158 thus rejects the idea that mathematical problems signal a lack of knowledge of our own mathematical symbolism.

This observation makes sense in light of Wittgenstein's tendency, in the middle period, to distinguish strictly between two classes of problems: (1) problems *within* a calculus, the solution to which is a matter of calculation; and (2) problems requiring a change of the calculus itself.[173] The former may include problems that are beyond the capabilities of any individual, but their solutions are nevertheless determined by the rules of a calculus, discoverable via a systematic method.[174] By contrast, the second class of problems, which would include open problems such as the Goldbach Conjecture (whether every even integer greater than 2 is the sum of 2 primes), do not *have* solutions; a new calculus must be invented to solve it.

Solving an open problem requires a modification and innovation of mathematical practice. However, since Wittgenstein in the early 1930s took calculi to be *determined* by a set of rules, syntactically defining the mathematical sense of the signs employed, he would regard words such as "modification" or "expansion" as misnomers. Instead, a change in mathematics requires the complete *substitution* of a calculus, conceived as a

[172] That would perhaps bring mathematical problems closer to philosophical problems, which Wittgenstein *did* take to revolve around artificial demands that we, often unwittingly or tacitly, place on our words. Unlike philosophical problems, mathematical problems do not rest on *confusions*. They are not *dissolved* by giving grammatical reminders that successfully show "the fly the way out of the fly-bottle" (*PI* §309).
[173] See e.g. Ms-305,1[2] (1930): "Fundamentally, one can only pose a question if one already knows the answer, if the bridge between question and answer has already been built." [*Im Grunde lässt sich eine Frage nur stellen, wenn man die Antwort schon weiß, wenn die Brücke zwischen Frage und Antwort schon geschlagen ist*.]
[174] After 1932, Wittgenstein moved away from the stress on systematicity; see Säätelä (2012, p. 10); *TLP* 6.2.

closed syntactic system, for another. This being so, when it comes to unsolved problems in pure mathematics, the so-called ‘problem’ is better understood as a form of *stimulus* (WVC, p. 144). Led by certain analogies, mathematicians seek to put a new calculus in place of an old one, in a way that is arbitrary from the point of view of the rules of that, or indeed any, calculus (Rodych, 2008).

During this period, Wittgenstein also emphasized the gap between propositions and systems of rules. While a proposition represents an external subject matter, a system of rules *defines* a logical form (*PR* §143). Hence, propositions can be more or less similar, representing subtly different things, but distinctions between systems of rules are absolute and categorical. To illustrate, he contrasted the calculus of cardinal arithmetic, which he saw as ‘complete’ in the sense of being conceptually determinate, to the variety in apples, adding that there are no subtle distinctions between logical forms as there are between the tastes of apples (*BBB*, p. 19).

In section 5.3.1 (p. 198) it will be argued that the later Wittgenstein partly retained the critical edge of this perspective. However, he came to realize that he had operated with an overly rigid understanding of the nature and origin of mathematical problems (cf. Schroeder, 2021, p. 174). After all, the region of mathematics in which a conjecture is first stated does, at least *in some sense*, constitute the ‘same’ region as the one in which the problem is later solved.[175] If a solution involved the wholesale *reinvention* of a calculus, we would likely deny that it was an adequate solution. By thinking of mathematical calculi as determined by rules, and proofs as adopting new rules,[176] Wittgenstein had in effect earlier denied this.

Already in the middle period, Wittgenstein was clearly concerned with accounting for the nature of mathematical practice and its differences from empirical science. However, he was limited in this by *anthropological* misapprehensions.[177] His

[175] See Gerrard (1991, p. 132). A proof can cross into several different domains, but the point is that a solution is at least in contact with the ‘same’ region of mathematics in which the problem was stated.
[176] Note that the issue is not that Wittgenstein conflated “what’s logically undetermined and what’s arbitrary”, as Hersh (1997, p. 205) alleges. The issue is that he (in the early 1930s) saw calculi as determined by rules.
[177] The distinctiveness of Wittgenstein’s views in this period is here exaggerated for the sake of brevity and clarity; it is complicated by several remarks, such as the following from his lectures in the early to mid-1930s: “In what sense can one say that a question in mathematics makes no sense? […] Ask yourself, What uses does one make of the question? It does stand for a certain activity by the mathematician, of trying, of messing about. If the question did not stand for something, one would expect *any sort* of activity” (Wittgenstein, 1979a, p. 222;

philosophy of mathematics was influenced by the idea that calculi are to be regarded as 'closed', their signs being the product of internal rules governing syntactic operations. Wittgenstein thereby underestimated the interconnections between mathematical calculi and other practices, as well as the possibility for one calculus to be 'related' to another calculus in a conceptually significant way.

Historically speaking, mathematical subjects often begin with intuitive techniques and heuristics and are only later systematized and axiomatized (Hersh, 1997, p. 8). Even sharply defined and regulated systems are in principle *porous*, especially when they involve signs that are also used elsewhere, as is typically the case. A sign might travel from one context (say, some part of number theory, or geometry) to gain a related use in another context (say, a family of programming languages, or cartography), and the use of the sign in the first context might be updated in response to its use in the second.[178] In effect, it can be difficult to say where the one context ends and the other begins. The dividing lines shift with time, and whether sharp distinctions are drawn depends on the application of relevant pieces of mathematics.

Calculi and their interconnections

The later Wittgenstein gained a richer perspective. For him, the way and the goal – or calculation/proof/technique and object/proposition, respectively – were interrelated, but not identical, as he had declared them to be in *PR* §155. As part of this shift, he was able to recognize that there *can* be multiple, distinct proofs of one and the same mathematical proposition. More will be said about his later view of mathematical proofs in sections 5.2 and 5.3. This change of mind came about as a realization that mathematical systems, that is, calculi or subfields of mathematics, are not logically *determined* by their rules – or that, in *that* strictly logical sense of 'system', mathematics, like language, does not consist of *systems* at all. This is what mathematics not being an idealized machine means; though it is rule-governed, it is not *determined by* a set of rules irrespective of how they are followed. Hence, the later Wittgenstein

Säätelä, 2012). Indeed, an incipient concern with the *practice* of mathematics (as opposed to either syntax or 'ontology'/'semantics') can be discerned already in the *Tractatus*, most explicitly *TLP* 6.211.

[178] Wilson (2006) provides several examples of traveling concepts in this vein. Similarly, Lakatos' (1976) discussion of the Euler characteristic shows how different aspects of a mathematical concepts like 'polyhedron' come to stand out in different contexts, escaping exhaustive definition.

maintained the term "calculus", but he now by this meant a *practice*, a kind of language game, which is clear from his frequent intersubstitution of those terms.

Thus, in the Investigations, Wittgenstein wrote that "new types of language, new language-games, as we may say, come into existence, and others become obsolete and get forgotten", adding that "[w]e can get a *rough picture* of this from the changes in mathematics" (*PI* §23). The shift from thinking of calculi in terms of systems of rules to thinking of them in terms of practices relates to his understanding of how rule-following is shaped by context. He realized that whether two calculi relate to each other, in any given way, is potentially vague. The vagueness of "relation" here mirrors the vagueness of "interpretation" that Wittgenstein discussed with respect to rule-following in the *Investigations*, cf. *PI* §38 and §§198-202.

In these remarks, "interpretation" can be taken to have psychological implications, meaning that a person interprets a rule by *intending* the rule in a certain way. It was *part* of Wittgenstein's aim to undermine the assumption that following a rule is necessarily a psychologically interpretive process. However, the word "interpret" carries a far more general meaning: it means 'to enact', 'execute', 'apply', or 'follow in a particular way'. By restricting the word "interpretation" to the substitution of alternative expressions of a rule (*PI* §201), then, Wittgenstein made the point that we can follow a rule *full stop*, without specification of *how* the rule is followed.[179] That is relevant because it provides philosophical ammunition against the idea that formal relations, particularly relations between calculi, must themselves be rules.

To see this, assume that mathematical calculi is determined by rules and that we necessarily go by rules whenever we convert from one calculus to another. Then, given that any two calculi *do* relate to one another, a regress is generated. Assume that calculus A relates to calculus B in some way. If so, a 3rd calculus is implied, C, constituted by A and B as well as the rules that govern their interaction. If, again, C is related in some way to a 4th calculus, D, this immediately implies the existence of a 5th calculus, E, containing the rules governing *their* interaction, and so on. The *calculi of calculi* expand outwards, and eventually all of mathematics is considered *one* overarching calculus. At this point, however, the question remains how, specifically, mathematics is to be

[179] Cf. Fogelin (2009), who sees what he calls "*defactoism*" as a central pillar of Wittgenstein's later philosophy.

enacted, implemented, or applied.

Famously, Wittgenstein (*PI* §201) denied that a viable answer here would be to posit further rules for interpreting the rules, since, again, this leads to a vicious regress: one rule is taken to be required to interpret another, which requires another rule, etc., *ad infinitum*. The point is simply that we do not need a rule for how to enact a rule in the first place, and this ends up undermining the rationale for equating mathematical practice with what is determined by rules. On the contrary, a rule presupposes a practice, a way it is normally followed (*PI* §199). Likewise, the way one system of rules relates to another system of rules is a matter of practice. The role of calculi within forms of life is a matter of how their relevant pieces of mathematics have developed historically, as well as how they are actually used in the present.

5.2 Genealogy and morphology

As broached already in Chapter 3, Wittgenstein viewed mathematical proof in terms of the formation of mathematical concepts. This section will explore and attempt to give an anthropological account of this idea. First, it has to be noted that some of his writings on this topic in *RFM*, perhaps especially part III, are vague. Notes from his lectures collected in *LFM* are often more discursive, adding clarifications. Additionally, the anthropological basis of mathematics presented in the previous two chapters, which centers on relations between language games, still guides Wittgenstein's remarks on concept formation in mathematics. Some of the vagueness of his writing in *RFM* is then accounted for as arising from the intended connections between these remarks and material from *PI*. The aim is for these latent connections to become clearer over the course of the upcoming sections.

The distinction between the way and the goal that Wittgenstein discussed in *PR* §155, but also later, as indicated in the previous section, can be rephrased in a different register as a concern with genealogy and morphology, respectively. Roughly, "genealogy" means a kind of origin or a process of development, while "morphology" means a shape or structure. Things with the same genealogy have originated in the same way, and things with the same morphology have the same structure. For a pertinent example, two family members, though they share (at least to some extent) a genealogy, tend to differ in physical morphology.

A combination of genealogical and morphological concerns is implicit in Wittgenstein's (*PI* §67) elaboration on family resemblance, with 'game' and 'number' as examples, through the use of an ostensibly genealogical metaphor. He illustrated the development of the concept of 'number' as a process of twisting fiber upon fiber, thereby creating a thread without a single fiber running through the entire length. In this metaphor, the fibers are various uses of the expression "number" which, in turn, might be categorized into more specific sub-concepts, such as 'cardinal number', 'rational number', 'real number', etc. The concept of 'number' is the thread formed by the *intertwining* of these different uses of the term "number".

In *PI* §67, Wittgenstein also distinguished *direct* and *indirect* affinities, seemingly talking about uses of terms that are conceptually equivalent or interchangeable, in the direct case, and uses of terms that are transitively related through one or several conceptual links, in the indirect case. To draw out the genealogical implications of family resemblance, Wittgenstein's remarks can be compared to a biological principle attested to Ernst Haeckel: the life of an individual organism, both in a biographical and physical sense, mirrors the evolution of the species to which it belongs. That principle has a structural similarity to Wittgenstein's metaphor that the meaning of a concept reflects the intertwining out of which it emerged.[180]

The metaphorical allusions to intertwining and spinning together demand a more concrete explanation that shows how they give rise to affinities and family resemblances. One hint in this direction can be found in Wittgenstein's insistence that there is a *direct* affinity between "things that have hitherto been called 'number'" (*PI* §67). This can be read as an echo of his earlier emphasis on the indivisible nature of atomic facts. In the *Tractatus*, atomic facts are said to be composed of objects which "fit into one another like the links of a chain" (*TLP* 2.03), being part of each other and

[180] See Boncompagni (2022, p. 4; p. 8) for more on the relation to Haeckel as well as Wittgenstein's (and Waismann's) 'Goethean' approach to 'form' as a contextual and comparative concept. Boncompagni quotes Waismann (1965, p. 81) on the methodology he associates with Wittgenstein: "We are collating one form of language with its environment, or transforming it in imagination so as to gain a view of the whole space in which the structure of our language has its being."

thus not requiring any further intermediating links in order to form an atomic fact.[181]

Wittgenstein's imagery of intertwining in *PI* §67 presents a similar notion, but in the context of a historical process. In both cases, he denied the idea that the connection between elements – things that are called "number", in *PI* §67, and objects constituting an atomic fact, in *TLP* 2.03 – is an external relation holding as a matter of contingent fact. Rather, his point is that each element is in part determined by interrelated elements. Just as it was, for the early Wittgenstein, *because* one object 'fits' together with a determinate set of other objects (and thereby serving as a constituent in a determinate range of atomic facts) that it *is* the object that it is, it is *because* a given use of the term "number" is spun together with a specific range of other uses of that term that it *is* an instance of the concept of 'number'.

This is not to ignore the differences between the two metaphors. The later Wittgenstein (*PI* §67) added that "the strength of the thread resides not in the fact that some one fiber runs through its whole length, but in the overlapping of many fibers", meaning that concepts are not reducible to a single condition or criterion. The metaphor of spinning or intertwining accommodates *degrees of similarity* in meaning, which was notably absent from the *Tractatus* and, as discussed in the previous section, explicitly denied in Wittgenstein's writings on calculi from his middle period. Since two uses of a term might be more or less tightly intertwined, the concepts they express (if any) might be said to resemble one another more or less.

Wittgenstein also went on to deny that a concept should necessarily be regarded as the logical sum of its sub-concepts, although we *might* also restrict a concept in that way (*PI* §68). In contrast to the metaphor of objectual fit in the *Tractatus*, there need not be a single answer as to whether or not one use of a phrase expresses the same concept as another use; this might vary from case to case depending on conditions that go beyond the syntax of the respective sentences. Comparing utterances to moves in 'language games' was meant to illustrate this point; a given form of expression can be used to make different moves in different settings.

In any case, the metaphor of conceptual intertwining or spinning serves an

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[181] This doctrine was Wittgenstein's early response to F. H. Bradley's regress argument against relations, allowing the process of logical analysis to terminate at the level of simple names of objects; see Copi (1958).

important role in Wittgenstein's later philosophy (cf. Floyd, 2016, p. 70), and this centrality is another similarity it has to the notion of objectual fit in the *Tractatus*. Wittgenstein was in *PI* §67 expressing himself in general terms, with 'number' and 'game' serving as examples. *All* concepts can be regarded as more or less strictly defined, more or less loosely extendible, which is not to deny that strictness of terminology is a standard of scientific and mathematical practice.[182] Furthermore, in the case of 'game', since Wittgenstein was already (in *PI* §7; *RFM*, V-15; VII-33) comparing the use of language and mathematics *in general* to the playing of games, the example comes with a layer of *metonymy* which serves to make a point that stretches much farther than the extent of the example itself. For, if the concept of 'game' is used and extended through intertwining, and language and mathematics *themselves* are (at least largely) understandable in terms of games or families of games, then we (at least largely) use and extend the concepts of 'language' and 'mathematics' through intertwining. In light of this, the next section seeks to spell out this metaphor of conceptual intertwining more concretely, while relating it to the equally important notion of an intermediate link (*PI* §122). This, in turn, will help illuminate Wittgenstein's later writings on mathematical proof.

Intermediate links and affinities

To describe the origin and the structure of a physical entity (an object or a process) is one thing, but describing the origin and structure of a *concept* is something else entirely. In this context, Wittgenstein consistently stressed the need to avoid conflating the two. For example, we should avoid thinking of *number* as if it were a type of object, such as a mineral dug from the ground, and to think of mathematics as if it investigated the structure of said objects, as if it were the 'mineralogy of numbers' (*RFM*, IV-11). A tendency for hypostatization and reification in the philosophy of mathematics is, however, understandable given that any talk of the development or the properties of a

[182] Rather than "strict" on the one hand and "loose" on the other, we might also say "literal" or "concrete", on the one hand, and "comparative" or "figurative", on the other. At issue is the extent to which we are willing to regard two different utterances as having a close enough *kinship* to express the same concept, which is not quite the same as the degree of precision we take to be required for using that concept. A figure of speech can be precise, turning on fine details, but common; it can have kinship to many different concepts despite (or because of) its high level of precision. Conversely, a precise or specific concept (such as 'needle in a haystack') can lend itself to broadly applicable metaphors, sweeping comparisons, and a wide range of analogies.

concept involves objectual formulations, akin to that which seems to force us to read "concept" as part of a name for an object (e.g. "the concept 'horse' is easily attained": Frege, 1951; cf. Jolley, 2007: Proops, 2013). Objectual formulations are part of mathematical language, as in "there are 4 prime numbers between 10 and 20". Nevertheless, Wittgenstein warned against reading such formulations in an overly representationalist way and so conflating the genealogy and morphology of mathematical concepts with the study or construction of independently existing 'abstract objects' (*PI* §339) or 'ideal rules' (*PI* §88, *PoP* xi §133). This critical aspect of his writings will be explored further in section 5.3 (p. 194).

This section will focus on the notion of an intermediate link [*Zwischengliedern*], which Wittgenstein emphasizes explicitly in *PI* §122; it will be related to mathematical proof in the subsequent section. The notion of intermediate links is introduced as follows:

> *A main source of our failure to understand is that we don't have an* overview *of the use of our words. – Our grammar is deficient in surveyability. A surveyable representation produces precisely that kind of understanding which consists in 'seeing connections'. Hence the importance of finding and inventing* intermediate links. */ The concept of a surveyable representation is of fundamental significance for us. It characterizes the way we represent things, how we look at matters. (Is this a 'Weltanschauung'?)* (*PI* §122; cf. §161)

As this makes clear, the notion of an intermediate link, like other concepts the later Wittgenstein introduced, is principally methodological, devised to do philosophical work. Roughly, intermediate links can be understood as concepts which resemble, in some relevant respect, another (cluster of) concept(s). Philosophically, the value of intermediate links comes when dealing with an entrenched picture of a concept. Such links show the possibility of different ways of understanding. Wittgenstein sought to diffuse the dualism between the pictures of mathematics as an ethereal machine and a repository of certainties, broached in the introductory section of this chapter, through giving 'reminders' of mathematical practice (cf. *PI* §125).

A good example of a concept for the purposes of intermediate linking is that of a *multiset*. A 'multiset' is a data structure akin to a set, an unordered collection of items, with the difference being that multiple instances of a type are recognized as such

(Blizard, 1989).[183] For example, [*a*, *b*, *c*, *c*] is a multiset containing two instances of *c*. The arithmetical operation of addition (as opposed to union or Minkowski addition) is undefined for ordinary sets, but it can be defined for multisets: for instance, adding [*a*, *b*] to [*a*, *b*, *c*, *c*] yields [*a*, *a*, *b*, *b*, *c*, *c*]. In the case of multisets, we might describe the operation of addition as *simultaneously* numerical and objectual; it adds a number (of things), but also things (of a certain number), to one another. The concept 'multiset' thereby gives us an intermediate link between the operation of addition and the use of "add" in physical contexts, such as "add the contents of that box to this one". It does this without requiring a reference to abstract objects or appealing to cardinal arithmetic operating on cardinal numbers in contrast to ordinary natural numbers.

It could be argued that multisets play a far more prominent role in people's lives than they are given credit for. Any type of physical container potentially holding multiple instances of a type, such as a box or a drawer, functions more like a multiset than a set. When we count the contents in a box, we do not regard them as 'objects' as such, but as things of *various* kinds, and thereby effectively regard them as multiset members. The same goes for the apples in the drawer Wittgenstein described in *PI* §1 and in *RFM*, I-100, even though he was not phrasing this in terms of either multisets or sets. What is worth noting is that, when presented with physical demonstrations or hypothetical scenarios, pupils are shown multiset addition. However, they are immediately also taught a significant aspect of the *numerical* concept of 'addition', especially the summing of natural numbers. Hence, for a significant range of uses of "add", it does not matter whether we mean multiset addition, or the addition of quantities, or the addition of natural numbers, because the action comes to the same thing.

This, it seems, illustrates what Wittgenstein meant by "affinity" in *PI* §67. Direct affinity holds when two uses of terms constitute essentially the same move(s) in the same range of language games. The intersubstitutability of expressions (leaving the sense of *any* relevant sentence unaffected) can thus be considered a symptom of direct

[183] "Multiset" is an overly technical-seeming term for what is effectively a more general concept than 'set' (sets can be seen as multisets with only a single instance of any type). Knuth (1996, p. 473, 483, 694) lists historical alternative naming candidates: "bag", "list", "bunch", "heap", etc., noting that these all have more or less misleading associations. He credits "multiset" to N. G. de Bruijn (Knuth, 1996, p. 694).

affinity. For example, two different signs used for the same natural number, such as "two" and "2", have direct affinity in this sense. Far more prevalent, however, is the phenomenon of *indirect* affinity, which by extension can be taken to mean the condition that a given use of an expression constitutes the same move in (at least) two relevant language games, but not in general.

Multisets can be used to demonstrate such affinities, replacing sheer set-membership with counts while highlighting the context-dependent nature of counting (cf. Blizard, 1989, p. 37). For example, we might use the word "add" to describe someone pouring items from one box into another. If we restrict our focus to the quantity of apples ending up in the latter box, then the distinction between the multiset and numerical concepts of addition makes no difference. Here, [*a*, *a*] may be added to [*a*, *a*] to get [*a*, *a*, *a*, *a*]. However, the context can alter the operation performed. Suppose the boxes contain 3 distinct cultivars of apples: [*a*, *b*] and [*a*, *c*]. With the cultivars in focus, the result of addition is alternatively [*a*, *a*, *b, c*] or [*a*, *b*, *c*].

Despite its interest, 'multiset' is just one concept exemplifying intermediate links. It can, in turn, be compared to ordered sets and strings: the notion of 'adding' another character to a string of text is not quite the same as, but similar to, the notion of 'adding' an item to a container, which in turn is similar to 'adding' one number to another. Overall, these concepts resemble one another in limited ways, in particular contexts, given that they function interchangeably within a limited range of language games. Finding intermediate links requires paying attention to the language games and practices that *surround* a given expression. This kind of insight led to the methodology and "*Weltanschauung*" Wittgenstein mentioned in *PI* §122, expressed also in *RFM*, II-6: "The motto here is always: Take a *wider* look round."[184]

When two concepts are interchangeable in a particular range of cases but are otherwise distinct, there is an *aspectual* difference involved (*PI* PoP xi §113). Focusing on a way in which two concepts are alike, on the one hand, and a way in which they are dissimilar, on the other, is to pay attention to two different aspects. So, for example, one

[184] With this contextualist methodology and its concomitant understanding of proof, Wittgenstein was reversing his earlier attack, alongside Frege, on 'piecemeal definitions': "If a primitive idea has been introduced, it must have been introduced in all the combinations in which it ever occurs. It cannot, therefore, be introduced first for *one* combination and later reintroduced for another" (*TLP* 5.451). Cf. White (2017, pp. 297-298).

aspect of multisets is that two multisets can be added together, which makes multisets akin to numbers, but the same aspect also makes them dissimilar from sets. Another aspect of multisets is that they have members, which makes them dissimilar from numbers but more akin to sets.

On this reading, Wittgenstein saw aspects as comparative and as based on linguistic practice. To see an aspect is to see a way in which one concept relates to another in a/across language game(s). Hence, the pertinent point about aspects is not that perception is 'inherently' aspect-laden, making us describe what we see now one way, now another. Rather, different language games are connected, and overlap, in various ways in practice. A single description, action, sign, or item can be used either as part of one (family of) language games or another. Individually shifting from the one aspect to the other alters the salience of different conceptual connections (cf. Baz, 2000; Agam-Segal, 2023; Pompa, 1967). This suggests that the treatment of aspects in Wittgenstein's later writings on mathematics is a corollary of his remarks on concepts, which is consistent with how the same topic came up in earlier writings (*TLP* 5.5423).

In particular, then, Wittgenstein's discussion of the role of aspects in mathematical practice is not grounds for attributing to him any kind of (Platonist) realism about mathematical proofs or mathematical objects (which is not to say that he was an *empiricist*, either; see *RFM*, VI-23, and Floyd & Mühlholzer, 2020, p. 159). There need not be anything to understand, perceive, or discern, about mathematical 'objects' independently of what we broadly speaking do with mathematical techniques, that is, independently of how we calculate, do geometry, read proofs, etc. As will be explored below, this notion of aspects, with the distinction between direct and indirect affinity, in combination with the notion of formal properties of language games discussed in the previous chapter, provides a way of answering critiques of Wittgenstein's later remarks on mathematical proof as entailing an untenable form of radical constructivism.

Proofs and intermediate links

The later Wittgenstein came to see mathematical proofs as part of a practical *context*, that is, a whole host of formal relations between practices which revolve around the proof (and involve concepts that occur in the proof), without which a would-be 'proof' would not be a proof at all. It is because he recognized that a proof has a particular role in mathematical practice that Wittgenstein came to think that we can have multiple proofs of the same theorem. Wright (2001, p. 378) writes that "the rule-following considerations may do a great deal to explain Wittgenstein's antipathy to a Platonist conception of the subject matter of mathematics, but they do little, unsupplemented, to explain his distinctive views about proof". Regardless of the interpretation of the rule-following remarks, it seems right that Wittgenstein's remarks on proof outstrip a direct application of considerations of rules, recognizing the role of context.

Wittgenstein saw proofs as built up with intermediate links, that is, (sub-)concepts with indirect affinity to each other. Although the previous sections described these notions from the *Investigations* as methodological tools for philosophical clarification, Wittgenstein also described activities of calculation and proof in *RFM* precisely as involving 'seeing connections' (*RFM*, VII-18, *PI* §122), calling for 'surveyability' (*RFM*, I-154, III-43, *PI* §122). Connections are seen through the invention and recollection of intermediate links, (sub-)concepts serving as bridges between one (cluster of) concept(s) and another. Mathematical practice, even on an elementary level, is formally related to activities of proof. It is because we recognize how terms or techniques are used in a context which goes *beyond* the immediate step of a proof that we can understand how terms are *extended* in that step in a way that is nevertheless *consistent* (in the historical sense of that word), potentially leading to a result of mathematical significance.

The fact that Wittgenstein's views on proof changed from his middle period into his later period is illustrated clearly by the following passage from *RFM*, written in 1941:

> *Now how about this–ought I to say that the same sense can only have* one *proof? Or that when a proof is found the sense alters? / Of course some people would oppose this and say: "Then the proof of a proposition cannot ever be found, for, if it has been found, it is no longer the proof of* this *proposition". But to say this is so far to say nothing at all. – It all depends* what *settles the sense of a proposition, what we choose to say settles its sense. The use of the signs must settle it; but what do we count as the use? – / That these proofs prove the*

> *same proposition means, e.g.: both demonstrate it as a suitable instrument for the same purpose. / And the purpose is an allusion to something outside mathematics.* (*RFM*, VII-10)

We here find him acknowledging that two different proofs might prove the same proposition, in contrast to what he held in 1930, because it is not necessarily the case that we view a proof *on its own* as enough to determine the sense of whatever proposition it proves. The sense of the proposition might depend on its purposes, which are independent of the proof – indeed, Wittgenstein remarked that 'purpose' here goes beyond mathematics in general.

This being said, Wittgenstein retained much of the motivation *behind* the denial of the possibility of there being multiple proofs of the same proposition. He still held that a proof changes the conceptual structure of a calculus, and that an unproven conjecture requires a proof in order to fix its mathematical meaning.[185] Put in terms of the game-analogy, he still held that a proof is akin to a process of modifying the rules of a game. The modification did not exist *prior* to the process of the rules being modified, and the process of modification was not *merely* a matter of working out the consequences of, or filling in gaps in, the pre-existing game. Mathematical proof is inherently creative; it is ampliative, not merely explicative.

At this point it should be noted that the idea of mathematical proof being inherently creative and ampliative, even though what is expanded by proof is practice, not knowledge in a propositional sense, does not entail that proof must be *construct* a specifiable mathematical 'object'. For example, the proof that for any positive integer *n*, there is always at least one prime number between *n* and 2*n*, does not construct such a prime. Proof, for Wittgenstein, involves the invention or extension of *technique*. The emphasis is accordingly on *how* an 'object' is (or, with indirect proofs, is not) constructed *or* proven to exist, with the objectual phraseology being understood metaphorically. Accordingly, Wittgenstein wrote:

> *In order to see how something can be called an 'existence-proof', though it does not permit a construction of what exists, think of the*

[185] *RFM*, VII-10. This is somewhat simplified and will be unpacked below. The point is that Wittgenstein came to hold that a mathematical hypothesis, a conjecture, lacks meaning *in the mathematical context in which we would like to situate it*. We do not (yet) understand its *role* in the relevant calculus/calculi of mathematics, because we have not yet *given* it that role in a way that satisfies the function of those calculi in our lives.

different meanings of the word "where". (For example the topological and the metrical.) For it is not merely that the existence-proof can leave the place of the 'existent' undetermined: there need not be any question of such a place. (*RFM*, V-26)

A proof can yield a reproducible pattern of 'calculation' in a broad sense without specifically leading to something we might call a "construction". Wittgenstein (*RFM*, V-46) *did* go on to suggest, however, that constructive proofs will be regarded as more useful, and nonconstructive proofs potentially as less interesting, given a wider recognition of the fact that the function of a proof is essentially to contribute to our stock of mathematical techniques.

In any case, it is not that Wittgenstein adhered to a principle of constructability as a criterion of mathematical sense and *therefore* saw nonconstructive proofs as illegitimate (see Floyd & Mühlhölzer, 2020, p. 207; cf. Wang, 1958, pp. 471-472). Rather, the relevant point is that his later writings on proof de-emphasizes the role of logical methods (e.g. *RFM*, III-47) as part of a denial of any attempt to reduce proof merely to a means for deriving a given domain of propositions (cf. *RFM* III-46). Deduction is often *used* in mathematics, as it is in all kinds of reasoning, but it is not *distinctive* of mathematics. On the contrary, mathematics is distinguished by a *plurality* of different but interrelated methods (Diamond, 1991, p. 34).

Accordingly, Wittgenstein's point with calling mathematics a 'colorful mixture' (or a 'motley'; *RFM*, III-48) was not simply to emphasize the diversity of proof techniques and of mathematical practices more broadly, in and of itself. Rather, that remark comes as a corollary; the diversity of proof techniques must be acknowledged because proof is essentially *practical*. To prove is already to do mathematics, that is, already to engage in calculation. This explains, to an extent, why Wittgenstein frequently talked of "proofs" in cases involving elementary calculations, such as 200 + 200 = 400 (*RFM*, III-23). We make use of proofs *in* mathematical practices, and that is why proofs in general contain, and distinctively exhibit, particular techniques. There is a difference between proving and merely calculating, but the difference depends on the roles of these activities, not on the morphology of the behaviors themselves.

Shifting his focus from the internal structure of formal systems – from seeing mathematical propositions as deriving from disjoint sets of rules – to a more contextual or systems-theoretical understanding of mathematical practice, we find Wittgenstein in *RFM* and *LFM* thinking about the way systems of rules emerge, interact, and change in society. In line with this, a new conception of proof emerged, one in which a proof essentially relates to the way calculations are used and understood. Though he still rejected the idea that proofs are mere means of discovering the truth-value of a pre-existing proposition, he recognized that two different proofs can serve the same anthropological role and thus have the same 'point', and it is this *point* that fixes the proposition that is proved. He illustrated this idea *via negativa* by asking us to imagine that the circumstances which give proofs their 'point' disappeared:

> *"When two proofs prove the same proposition it is possible to imagine circumstances in which the whole surrounding connecting these proofs fell away, so that they stood naked and alone, and there were no cause to say that they had a common point, proved the same proposition. / One has only to imagine the proofs without the organism of applications which envelopes and connects the two of them: as it were stark naked. (Like two bones separated from the surrounding manifold context of the organism; in which alone we are accustomed to think of them.)"* (RFM, *V-10*)

Wittgenstein here resorts to metaphor, describing the circumstances which give proofs their point as an 'organism of applications'. A proof serves a function, so a comparison could be made with tools and instruments (cf. *PI* §569). We have several types of tools which do the same jobs in disparate ways, such as hammers and pneumatic nail guns. These tools may be functionally equivalent, but their manner of physical operation is different. The surrounding circumstances, the materials involved in application, and the practical purposes for which applications are customarily made is what gives such distinct tools and instruments the same point. Two distinct mathematical proofs may have one and the same point in a similar way.

Proofs in advanced mathematics are rarely *directly* invoked in (e.g. scientific) practice. However, they are still referenced in some context or other, even if strictly by mathematicians, for purposes which, in the end, *do* relate to practical decisions. So, in *RFM*, V-10 (cf. PI §578) and elsewhere, Wittgenstein described these patterns of use, the customary invocations of a proof in various practices, as belonging to the sense of the proof. Notably, *RFM*, III-36:

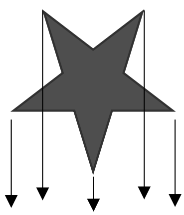
> *If I were to see the standard metre in Paris, but were not acquainted with the institution of measuring and its connexion with the standard metre – could I say, that I was acquainted with the concept of the standard metre? / Is a proof not also part of an institution in this way?*

Wittgenstein here posed two rhetorical questions, the first of which is apparently meant to be answered 'no' and the second 'yes'.[186] The phrase "acquainted with" does arguably also have a looser use – someone uneducated in mathematics could, perhaps, be acquainted with a proof simply by e.g. having heard of it, or having seen it written down, and thereby knowing of its existence – but Wittgenstein's claim that there is an important aspect of a proof that is not immediately given with what is written down nevertheless still stands. This aspect is the *institutional* aspect, that is, the role the proof plays in our practices. The upcoming section explores Wittgenstein's relating of the concept 'proof' to that of 'picture', and how this relates to the notions of indirect affinity and intermediate links from the *Investigations*.

5.2.1 Proofs as pictures and pictures as proofs

Investigating the difference between proofs and empirical experiments, Wittgenstein drew our attention to pictorial or diagrammatic representations of elementary geometric relationships. In so doing, he extended the word "proof" to cases in which it is questionable whether we would normally see anything requiring proof. For instance, he drew a picture similar to the one below, vertically tracing each outer point of a pentagram to its own line (*RFM*, I-25):

(A)



This is to be taken as a visual proof that the star has 5 outer points. Someone first encountering this picture might be inclined to posed several questions. (1) Why does

[186] Wittgenstein here affirms Müller's (2023) point, discussed in section 3.2.1 (p. 92), that a measurement standard is a material object *together with* the (legal/practical) institution surrounding it. See Fogelin (2009, p. 106). Note also the tentative tone, which is evidence of the *developmental* nature of Wittgenstein's thinking on proof and conceptual change/persistence, this line of thinking developing into remarks of *On Certainty*.

Wittgenstein treat this as an example of a 'proof', when the numerical correspondence between the points and the lines seems like an immediately perceivable fact? That the above shape has 5 points seems more likely to be *taken for granted* than to be investigated or demonstrated mathematically. In particular, it is unclear how the lines are supposed to help at all, since, when simply counted visually, the number of lines seems equally open to question as the number of points of the star. (2) What makes a picture into a (part of a) proof, that is, what is it about a picture (such as the above) that potentially serves to demonstrate a result? Do we have sensory perception of mathematical truths, and/or do such pictures serve as visual or physical *evidence* of mathematical (at least geometric) truths? (3) What is the role of the pictorial in a proof? Was Wittgenstein saying that *all* proofs are reducible to visual proofs?

These questions are answered by Wittgenstein's detailed follow-up remarks. First, he noted that he made the example 'memorable' on purpose; that is, its elementary nature is intentional. The example of a pentagram is contrasted with an example of a polygon with randomly many points (call it an "arbitrary polygon"), for which the procedure of vertically correlating points with lines would fail to have the same effect (*RFM*, I-27). With an arbitrary polygon, we might still go through the effort of tracing each point, correlating all the points with vertical lines. However, this would constitute a kind of *experiment*, where the result would give us empirical, timebound knowledge. At the time of producing the arbitrary polygon the number of points was unknown to us, but after correlating the points to lines we would know that, at the present moment, the points match the lines in number. It may be *unlikely* that the arbitrary polygon changes from the time it was produced to the time its points were correlated, but the activity of correlating the points to lines would not rule this out. Since we are not going to reproduce the arbitrary polygon, counting its outer points offers us only *ad hoc* knowledge.

With the example of the outer points of the pentagram in *RFM*, I-25, by contrast, Wittgenstein suggests that we do not gain *temporal* knowledge at all. Rather, by demonstrating equinumerosity between the points and the lines, we gain a *conceptual insight* about the pentagram. Here, Wittgenstein continued, saying that we have gained "conceptual insight" means that something new can now be *done* using the figure (*RFM*, I-47).

To illustrate this, Wittgenstein (*RFM*, I-25-26) described a ceremony or séance of some kind: given a group of individuals in a star formation, and a set of 5 wands, someone can use figure A to infer that there is a wand available for each individual in the formation. Wittgenstein added the upshot that "I could regard [figure A] as a schematic *picture* of my giving the five men a wand each". Then, Wittgenstein (*RFM*, I-27) added, this schematic picture can be *conceived* as a mathematical proof pertaining to the pentagram shape, on the one hand, and the lines, on the other, proving that there are as many lines (set up in this way) as there are points on such a star. The proof specifies what is involved in an activity set up in a specific way.

Now the second question can be addressed: what makes a picture into a (part of a) proof? Wittgenstein (*RFM*, I-28) wrote that "[a] proof – I might say – is a *single* pattern, at one end of which are written certain sentences and at the other end a sentence (which we call the 'proved proposition')". In other words, a picture, pattern, or design (words which Wittgenstein used interchangeably in this context) constitutes the *essential* part of a proof. The sentences before and after the picture might be necessary for motivating or communicating the proof, but they are not strictly speaking *part* of the proof. In the same way, a description of a design is not itself part of that design, and the sentence "in the proof on the blackboard, the proposition *p* follows from *q* and *r*" is not part of the proof that *p* follows from *q* and *r* (*RFM*, I-28).

In contrast to these contingencies, a picture is the exigent (part of) a proof, the part which serves to guide a procedure. However, that is not to say that the picture can be understood in isolation. As Wittgenstein prominently argued (e.g. *PI PoP xi* §167, §173, 175), a picture can be taken in various ways depending on the circumstances and the *use* to which it is put. He provided an example of people using white and black patches for the purpose of differentiating 'light' and 'dark' objects (*RFM*, I-28-29). In the case of the figure A, we are told that a ritual or séance involves the formation of a pentagram by the participants, as well as the use of wands. The concept of a 'pentagram' is precisely what is instantiated by drawing A, and physical possession of wands is schematized by tracing lines in the figure. So, the very concepts that describe the production of the figure are also involved in the ritual institution. *That*, it seems, is why the figure serves as a proof specifically within the context of that ritual.

Without that institutional context, a context involving analogous concepts to the ones used when forming the figure, the figure would not serve as a proof at all. Wittgenstein's critical remarks on Cantor's 'diagonal number' can be seen as exemplifying this emphasis on the context-bound nature of proof: "Is the question not really: What can this number be used for? True, that sounds queer. – But what it means is: what are its mathematical surroundings?" (*RFM*, II-3, cf. *RFM*, II-1-22). What is relevant here is that Wittgenstein talked of mathematical surroundings, and went on to call a proof "puffed-up" (*RFM*, II-21; Han, 2010) precisely when it lacks the "means" to do what it is presented as doing, as a consequence of not being seen in its proper surroundings. In other words, a puffed-up proof relies on techniques other than those which belong to the regions of mathematics it has become associated with.

Simplicity and reproducibility

Schroeder (2021, p. 142) observes that Wittgenstein in *RFM*, I-25 was aiming to 'disperse the philosophical fog' by constructing a simple scenario in which one commands a clear view of the use of words (cf. *PI* §5). As an additional feature, however, it is clearly important for Wittgenstein that the figure is connected to a setting in which it might actually be used. The proof presupposes that the elements of the picture (the 'hand' and 'pentacle' patterns) recur. Wittgenstein's example is one in which these patterns are not merely encountered as regular natural phenomena, but in which they are already conceptualized and intentionally reproduced.

Thus, when drawing a contrast between the pentagram (and the wands that have a ceremonial use), on the one hand, and an arbitrary polygon, on the other, he stressed the non-reproducibility of the latter, the fact that the latter plays no reliable role in a practice. Wittgenstein made the same point in other settings, suggesting that the relevance of reproducibility for a proof is its serving a purpose, e.g. for calculation, for example in *RFM*, III-11. The proof in *RFM*, I-25 does not merely *describe* a relation among patterns, but shows a way of doing something in a new way, giving a procedure for drawing a star in such a way as to guarantee *that* it is a pentagram. As explored in the next sections, this *practical* focus has implications for how we should understand "the element of stipulation" (Schroeder, 2021, p. 144), Wittgenstein's insistence that a proof involves a change in our concepts.

First, this leads to the 3rd question raised above: are *all* proofs visual, in some sense? The answer to this is no, because the later Wittgenstein extended the concept 'picture' (and its synonyms) metaphorically, so as to include idioms and grammatical *motifs*, the components of which are verbal in nature. In *LFM*, I, pp. 20-21, he described a picture as something we use as a stand-in for (the meaning of) an expression. A picture is "a piece of the application, a representative piece" of a concept, something we can use as a criterion for *understanding* the concept. So, for example, both a specific definition of "multiplication" and a mental image of someone doing arithmetic on paper might serve as *pictures* of the concept of 'multiplication'.

Hence, not just visual imagery but any potential device or fact that is taken to be more or less representative for a given (use of) an expression would, for Wittgenstein, constitute a picture. However, the adequacy of our pictures can depreciate, depending on how the picture compares to the evolving way(s) in which we *in fact* use our expressions:

> *"We learn our ordinary everyday language; certain words are taught us by showing us things, etc. – and in connexion with them we conjure up certain pictures. We can then change the use of words gradually; and the more we change it, the less appropriate the picture becomes, until finally it becomes quite ridiculous."* (*LFM*, I, p. 18)

As touched on in section 5.1 (p. 170), philosophy, in Wittgenstein's preferred sense, employs intermediate links in an attempt to release us from arresting 'pictures' ('*Bild*'); see *PI* §1, §23, §59, etc. There appears to be no reason to *sharply* distinguish 'pictures' as the components of proofs from 'pictures' as the subject matter of philosophy.[187] Although pictures are useful in mathematics, even here this is not inherently so, since they can vary in clarity and adequacy. In *LFM*, XXV, pp. 239-240, Wittgenstein described Thomas Hardy as talking of "the reality corresponding to a mathematical

[187] As evidence that pictures can play a similar role in mathematical proofs as they do in philosophical argumentation, for Wittgenstein, note that he first used the 'fly-bottle'-metaphor in 1937 to describe the effect of proof: "Can't we say: the figure which shews you the solution removes a blindness, or again changes your geometry? It as it were shews you a new dimension of space. (As if a fly were shewn the way out of the fly-bottle.)" (*RFM*, I-44; *MS* 118). A few days later, he used the very same metaphor to describe his aim in philosophy (*PI* §309). See also Diamond (1991, p. 254).

proposition”, rejecting this as a misleading *picture* which belongs to physics, not mathematics. An example of a mathematical picture that has changed in appropriateness over time can be seen in *RFM*, IV-2. Wittgenstein here presents an illustration meant to convince someone of the parallel axiom (cf. Steiner, 2000, p. 338). The advent of non-Euclidean geometry changed the extent of the relevance of that kind of picture.

Perhaps the only constant which characterizes all of the later Wittgenstein’s uses of “picture” (and its synonyms) is that it alludes to a *recurring* arrangement. A picture is something *re*cognizable because it is *re*producible; it is a pattern we can reliably form. We can organize our activities around pictures and use them to orient ourselves with *because* they are so easily reproduced (hence the conflation of ‘picture’ with ‘paradigm’ in *RFM*, e.g. I-6, I-28-29, I-32, and I-41). As mentioned, although a picture may have a visually perceivable form, this is not necessary; a set of interrelated terms, or a prevalent movement of thought, might constitute a picture even if it were only ever uttered orally (hence a ‘picture’ is also a ‘pattern’).

One may worry, here, that the identification of proofs with recurring arrangements downplays the creative aspect of proofs, suggesting that proofs simply reproduce pictures and bring nothing new to the table. On the contrary, Wittgenstein was attuned to the creative nature of proof. He compared open mathematical problems to riddles (*LFM*, VIII, p. 84, cf. *RFM*, V-6, *TBT*, p. 431). A riddle is solved by gaining a new understanding of the sense of the question itself, which often calls for creatively combining apparently unrelated forms of language (see Säätelä, 2012, p. 14; Diamond, 1991, pp. 270-285). Similarly, in mathematical research, it is not unusual for practitioners to be surprised to find that the solution to a problem requires techniques, or veers into regions, that would initially seem to be unrelated to it. Surprise at *how* mathematical methods connect or fit together, however, should be distinguished from astonishment *that* something is the case, which instead signals a lack of clarity (Floyd, 2012).

The later Wittgenstein retained the stress on the fact that at least an important class of open problems is not solvable by merely ‘working out’ something already given and decidable, a process which could be completed by a pre-conceived algorithm. But equally, he also did not see a proof as reducible to an act of stipulating (a system of) jointly compatible rules. By saying that a proof reproduces pictures, and that a picture is an institutionally anchored common reference-point, this is not to deny that mathematicians can act creatively in how they *use* pictures, that is, in *what* kinds of mathematics they apply and *how* they apply them.

There are affinities between these aspects of Wittgenstein's writings on proof and areas of recent research on mathematical practice. For example, Larvor (2012, p. 721) argues that informal proof involves inferential actions that do not strictly act on propositions, but also involve modifying "diagrams, notational expressions, physical models, mental models and computer models". Without suggesting that there is an overall agreement, this can be taken to align with Wittgenstein's understanding of proof as an activity of operating with pictures. Similarly, Tanswell (forthcoming) analyzes the prevalent use of imperative language in proofs to argue that proofs can be understood as recipes, seeing "proofs themselves as a kind of artefact of the way we structure the discipline of mathematics. It is the proving activities which are primary, with proofs themselves functioning to record, store and transfer mathematical knowledge, techniques, concepts, structures, reasoning patterns and ideas" (ibid. p. 10). Tanswell draws on Ryle's (1946) concept of 'knowing how', but the analysis coheres with Wittgenstein's view of pictures as paradigms of ways of calculating in a broad sense.

Pictures and mathematical experimentation

The preceding remarks concern Wittgenstein's understanding of the *logic* of mathematical proof, showing that he used 'picture' not just to emphasize a visual element (though this is not to deny the importance of visualization in mathematics) but to illuminate the institutional nature of the concept of proving in mathematics.[188] By talking of 'pictures', Wittgenstein was not merely attempting to describe what might be termed the 'rhetorical style' typically employed in, and to some extent characteristic of, proof – a subject of psychological and linguistic interest. In particular, Wittgenstein did not claim that a proof is a *mere* arrangement of accepted signs or conventions, or that a proof merely rehashes recognizable ways of doing things.

[188] Cf. Shanker (1987, p. 115), contrasting Wittgenstein's writings and that of Lakatos (1976) with respect to a related distinction.

Rather, Wittgenstein's remarks here are grammatical. The idea is that something is called a "proof" in part due to its connection with some institution or a set of activities, since a proof serves to *demonstrate* that some form of action is already part of that institution. The proof 'impresses' a picture on us (forming a paradigm; *RFM*, I-41) insofar as we subsequently engage in our activities in certain ways, being willing to *employ* the proof in relevant settings.

With this in mind, a comparison could be made to the readings advanced by Ernest (1998) and Bloor (1983). These authors take Wittgenstein to champion a philosophical theory according to which, among other things, mathematical results are conceived as social constructs (cf. Bloor, 1997; Schatzki, 1996). According to these authors, Wittgenstein was not first and foremost emphasizing the practical nature of proof in order to counter hypostatizing and representationalist philosophical views of mathematics. Rather, Wittgenstein was arguing that mathematical knowledge is socially constructed, and that proofs are the vehicle or process of this construction. Accordingly, for these authors, pictures (as used in a proof) tend to serve the role of rhetorical devices leading people to accept a given *result*, a proposition or object.

Ernest (1998, p. 80) writes that "Wittgenstein characterizes the acceptance of a new mathematical theorem as a decision. […] Consequently, in effect, we have decided to accept the theorem as a new piece of mathematical knowledge" and so "Wittgenstein's view is that a mathematical proof serves to justify an item of mathematical knowledge by its persuasiveness, not by its inherent logical necessity" (*ibid*., p. 83).[189] It should be noted that Ernest highlights the necessity of justification underlying the persuasiveness of a proof, so the point is not that he attributes to Wittgenstein an unduly subjectivist or relativist theory. Rather, the point is that Ernest describes proof as leading to a result in the form of an item of knowledge: a formula, sentence, or abstract object. This, however, understates a crucial theme in the later Wittgenstein's writings on mathematics: the point of proofs, rather than to discover or

[189] Ernest (1998, p. 83) quotes the following remarks in support of his interpretation: "I go through the proof and say: 'Yes, this is how it has to be; I must fix the use of my language in *this* way.' / I want to say that the *must* corresponds to a track which I lay down in language." (Wittgenstein, RFM, III-30)". Wittgenstein here talked about a change in how language is used, the allusion to a 'track' signifying a form of action as demonstrated in the proof. Ernest follows with the quote: "*In* a demonstration we get *agreement* with someone" (Wittgenstein, *RFM*, I-66). It is explicitly stated here that we get agreement *in* a demonstration, that is, in a way of acting.

produce ‘destinations’, is to produce *paths* of action (*RFM*, I-34, III-59, III-69, IV-8, V-42, VII-74).

Although, again, the differences with the present reading should not be exaggerated, similar concerns apply to Bloor’s comments on Wittgenstein on proof in the following:[190]

> *“The results reached about the essential relationships between say, pyramids and cubes, are, as Wittgenstein's* finitist *approach requires, created rather than discovered: a sentence asserting an internal relation between two objects, such as a mathematical sentence, is not describing objects but constructing concepts (*LFM*, p.73)”* (Bloor, 1983, p. 98; cf. p. 94).

Here, proof is described as leading to a result *about* essential relationships. A sentence, a verbal formulation of the content of a proof, is taken to construct a concept. In support of this reading, Bloor references *LFM*, lecture VII, p. 73, where Wittgenstein said the following:

> *We may say: We accept this figure as a proof that the hand and the pentagram have the same number. This means that we accept a new way of finding out that two things have the same number. We don’t coordinate things one with the other now; we just look at this figure.*

However, Wittgenstein’s claim on this occasion, in effect elaborating *RFM*, I-25, was that a proof compels us to accept, not a (quasi-propositional) result about an essential relationship, but an action or form of behavior, together with a way of regarding this action or behavior.

Here it should be noted that Wittgenstein rejected the idea that a proof ‘compels’ us in the sense that there is a *logical compulsion* to accept a valid proof independently of whether or not people *do* happen to accept the proof (*RFM*, I-113-142). That idea expresses the myth of mathematics as an ethereal machine, operating according to inexorable laws which are thought to apply regardless of whether any humans ever exhibit them in their calculations. A proof does not compel us in that abstract, timeless sense. However, Wittgenstein also noted that a proof *does* compel us in the sense that

[190] That said, in several cases Bloor lands on the side of a more thoroughgoing practical interpretation, such as in his comments on ‘mathematical essences’: “It is as if the work that society puts into sustaining a technique returns to its users in the phenomenological form of an essence.” (Bloor, 1983, p. 93).

accepting the proof means that we *act in accord* with it: "And how does it come about that the proof *compels* [*zwingt*] me? Well, in the fact that once I have got it, I go ahead in such-and-such a way, and refuse any other path" (*RFM*, I-34).

So, a picture as involved in a proof is not (quasi-)propositional and more or less convincing or persuasive with respect to a result, but 'schematic' and more or less *practically* compelling (*RFM*, I-26). As Shwayder (1969, p. 110) observes, then, Wittgenstein's view of proof could be seen as (pre-)classical: he equated proof with perspicuous demonstration, within a given form of activity, rather than with strictly logical derivation.[191] A picture in mathematics demonstrates (i.e. showcases) a pattern of action and, when successfully used in or as a proof, leads us to accept this model of action in practice. Considered as a proposition (with that word now not being understood strictly representationally; cf. *RFM*, V-46), the 'result' of the proof could be considered a label or descriptor of the action(s) which the proof leads us to take into our repertoire of accepted/acceptable mathematical techniques:

> *The proof taught me e.g., a technique of approximation. But still it proved* something*, convinced me of something.* That *is expressed by the proposition: It says what I shall now do on the strength of the proof."* (*RFM*, VII-74)

This reading also coheres with Wittgenstein's claim that the verbal expression of a result is liable to *mislead* us about the nature of the proof (*RFM*, II-7, III-26). The verbal expression, after all, is not of primary import; the way in which it was *reached* is. So, almost paradoxically, there is no theory of mathematical proofs or theorems as mere socially constructed knowledge, or as arbitrarily adopted stipulations, in Wittgenstein's writings, because his emphasis on social practice is *deeper* than what such a theory would sustain. There is no *understanding* a mathematical result independently of it being incorporated into our practices.

According to Nordmann (2010), in addition to viewing proofs as pictures, the later Wittgenstein did acknowledge that we can regard proofs as *experiments*. These two ways of seeing proof, Nordmann suggests, are complementary, though they ostensibly

[191] "Demonstrate" is thus perhaps more apposite than "prove", and closer to the Greek verb "*deíknūmi*" meaning 'I show' (Graves-Gregory, 2014, p. 31). As for the scope of the remarks on proof, recall his description of the dichotomy of experiment and calculation as "poles between which human activities move" (*RFM*, VII-30).

vary in relevance, with the experiment-conception being more relevant to indirect proofs, such as proofs of the infinity of the prime numbers, which assume what is to be rejected (e.g. that there are finitely many primes) for the sake of contradiction.

On the one hand, on Nordmann's (2010, pp. 191-192) reading, "a proof can and ought to be regarded as a picture that meets the requirement of being surveyable […], as exemplified by a calculation on a sheet of paper". With this pictorial way of understanding proof, only the proof itself can *show* what is proven, which is why proofs must be surveyable. On the other hand, Nordmann (ibid.) adds, "a proof can be regarded as an experiment, necessarily so if one wants to understand the productive and creative aspects of proof". Given the latter perspective, which we paradigmatically employ to understand *reductio ad absurdum* reasoning, the function of a proof is to change the domain of the imaginable; "[t]he proof shows us what was proved in that it implicates us in a certain experience at the end of which we see things differently" (ibid.), giving us new mathematical commitments and evaluations.

While this dichotomy serves well as a way of framing a philosophical discussion of proofs, the later Wittgenstein's anthropological understanding of proof indicate that 'picture' and 'experiment' are mutually related aspects of mathematical activity. Pictures are recurring, recognizable patterns within a given context. A proof pictures an arrangement *as a result of* certain actions, something that can be seen against a backdrop of mathematical techniques and conventions with which at least some mathematicians are already familiar. Accordingly, for a proof to be surveyable means that it successfully brings out *the way the proof itself was formed* relative to a given state of mathematical practice. It is to that end that a proof must be memorable and instantly apprehensible (*RFM*, III-55; cf. Mühlhölzer, 2005). Wittgenstein emphasized surveyability *because* he emphasizes the reproducibility of the proof as something other than the reproduction of e.g. a physical figure on a piece of paper (*RFM*, III-1).

As Nordmann (2010, p. 194) notes, in identifying proofs with memorable pictures (*RFM*, III-9), Wittgenstein is frequently read as *contrasting* proofs (atemporal, formal) from experiments (temporal, empirical).[192] However, there is a clear relation

[192] See Frascolla (1994, p. 132-134). Wittgenstein *did* prominently draw a contrast between calculation and empirical or causal experiment, as in *RFM*, VII-67: "How about the following: You aren't calculating if, when

between the two; for Wittgenstein, a proof *pictures* an experiment (*RFM*, I-36, III-1). What a mathematician does when searching for a proof involves several kinds of circumstantial factors that are irrelevant to any proof as such; say, the proof may be written in ink of a certain color. Mathematical proficiency requires an ability to disregard such factors in order to appreciate the *mathematical* picture that is left by the mathematician's activity, at which point this activity is not seen as an idiosyncratic experiment but as a *model* of, or for, potential mathematical behavior(s).

Mathematical activity can involve experimentation, but this nevertheless has to be distinguished from empirical experiments as used in the natural sciences because, as Nordmann (2010, p. 197) puts it, "the mathematician's experiment immediately yields a surveyable picture of itself". The mathematical 'experiment' is immediately frozen into a pattern or (potential) model which we do, or do not, adopt as part of mathematics. However, for Wittgenstein, the point is not strictly speaking that the mathematician's experimentation *yields* a picture; rather, the mathematician's calculations *are* the picture that the proof conveys. The process of solving a mathematical problem (open or not) can involve experimentation, but that is not to say that the *noun* 'experiment' serves to describe this process after the fact (cf. Wheeler, 2022, p. 9).

In upshot, to prove is something *done*, and it is only by extension that the artefacts on paper resulting from this activity may be called "the proof". Proving is, in this respect, akin to a trend-setting act of improvisation, a performance to be imitated, not akin to an act of creation, such as writing a book or making a cabinet.[193] This implies that Wittgenstein should be taken loosely when talking of hardening a proposition into a rule (*RFM*, VI-22-23; note that he prefaces this with "it is as if we" and "so to speak"). This phrasing might appear to suggest that the result of a certain 'calculating

you get now this, now that result, and cannot find a mistake, you accept this and say: this simply shews that certain circumstances which are still unknown have an influence on the result. / This might be expressed: if calculation reveals a causal connexion to you, then you are not calculating."

[193] Corroborating this reading, the later Wittgenstein compared proof to musical composition: "One might ask: what arrangement of themes together has a *point*, and what has *no* point? Or again: *Why* has this arrangement a point and *this* one none? That may not be easy to say! Often we may say: 'This one corresponds to a gesture, this one doesn't.')" (*RFM*, I-171). He also writes, in an unusually emphatical tone: "The *exact* correspondence of a correct (convincing) transition in music and in mathematics." (*RFM*, III-63). Fully exploring Wittgenstein's views on music would go too far afield, but it suffices to note that the parallel he sees is between the activity of proof and that of musical composition. Both the mathematician and musician *manifests* arrangements and is led by considerations *other* than causal limitations or affordances. As he writes elsewhere, "[*t*]*here are no* causal connexions in a calculation, only the connexions of the pattern" (*RFM*, VII-18).

experiment' is already a quasi-mathematical object before undergoing a process of sublimation and being turned into an 'object of mathematics'.[194] Wittgenstein's perspective, if the present reading is correct on this point, is that mathematicians experiment *behaviorally*; the behavior is then accepted as a rule, *as a paradigm* (*RFM*, III-28), which is to say, is endorsed and imitated by the mathematically proficient.

5.2.2 Forming a new concept

The above throws some light on Wittgenstein's view that a proof does not merely reveal new information about our concepts, but *changes* our concepts or, what might to some extent be considered the same, creates *new* concepts. Schroeder (2021) distinguishes between the later Wittgenstein's newfound 'grammar-conception' of mathematics, according to which a proof endorses a new grammatical rule (i.e. introduces a new concept (*RFM*, III-30)), and the 'calculus-conception' deriving from his middle period, according to which the meaning of a mathematical proposition is given by its proof. Schroeder highlights a tension between these two conceptions in how they deal with the nature of proof: On the one hand, "If we take seriously the idea of proof (as showing [a mathematical proposition] to be *true*), the endorsement as a grammatical rule appears redundant", however, "If, on the other hand, such an endorsement is crucial (as the grammar view seems to suggest), should it not be possible even without a proof"? (Schroeder, 2021, p. 58; cf. Kreisel, 1958, p. 40).

To give an illustration, it might be argued that a proof showing *how to* calculate 2 + 2 = 4 is *already* all the justification we might ever want for us to accept the equation 2 + 2 = 4.[195] So, on the one hand, it seems that a proof that 2 + 2 = 4 would render superfluous any *ratification* of "2 + 2 = 4" as a rule. On the other hand, if we ratify "2 + 2 = 4" as a rule, it is difficult to see why we should need a proof showing that we can calculate 2 + 2 = 4. Schroeder (2021, p. 174) argues that Wittgenstein aimed to resolve this tension by conceiving of proofs as demonstrating potential applications of a rule (i.e. the mathematical proposition), showing its potential usefulness and compatibility

[194] Schroeder's (2021, p. 139) assessment is relevant: "Where the metaphor of hardening empirical propositions goes wrong is in locating that process of attuning only at the level of propositions, when in fact it occurs already at the level of choosing suitable mathematical concepts."

[195] This could be expressed as: a proof of 2 + 2 = 4 already shows it to be *true* that 2 + 2 = 4, and that is already enough to justify its acceptance. However, that way of putting it is, in my view, farther from Wittgenstein.

in practice, and thereby motivating us to adopt it.

In order to appreciate this tension, it seems natural to read the later Wittgenstein as saying that a rule adopted on the basis of a proof is a quasi-proposition or an abstract decree of the kind "two plus two is four" or "if you have two objects and get two more, you now have four objects" (I use verbal language to underline their propositional character). On *that* reading, the question is *how* a mathematical proof supports its result, understood as a kind of decree, without this collapsing into the traditional notion of showing *that* the result is true.

Frascolla (1994, p. 141) presents Wittgenstein's solution to this tension as follows:

> *A fundamental characteristic of our form of life is that a linguistic rule regarding the meaning of a description like "the result of the correct multiplication of 12 by 12" is adopted only as a part of another accepted rule that sets up the meaning of the predicate "figure yielded by the correct multiplication of 12 by 12". In so far as adopting the latter definition means making a certain sign construction the paradigm of the correct multiplication of 12 by 12, which contains the result of the operation, the passage through the proof reveals itself to be an anthropological condition for the ratification of the rule-theorem.*

The role of pictures in proof is brought out by this passage. The proof that 12 × 12 = 144 constitutes a construction of the sign "144", and this construction-picture is accepted as the paradigm for correct multiplication of 12 by 12. However, on Frascolla's reading, what the proof achieves seems to go beyond manifesting a paradigm. The proof also lends support to a definition of "the result of the correct multiplication of 12 by 12" and "figure yielded by the correct multiplication of 12 by 12", in terms of "144" and "'144'", respectively. People are persuaded to ratify these definitions, and so the "rule-theorem" 12 × 12 = 144, on the basis of the proof. In response, it might be questioned why we need to go through with ratifying these rules as it were *in addition* to accepting the paradigm for correctly multiplying 12 by 12.

When we can (and when it matters whether we can) choose to 'endorse' or 'ratify' a rule, we are thinking of a 'rule' in the *de jure* sense, that is, as a decree, convention, or definition that a community may or may not adopt in practice. However, if we apply the reading of the role of pictures in proofs advanced above, Wittgenstein suggests that a proof fundamentally compels us to accept, not a kind of decree, but a pattern of behavior. For example, at some point, a proof might have guided us to confidently engage in patterns of *inferring* of the form "there were 2 objects and then 2 more" to "there are 4 objects". Note that this is a *de facto* practical tendency, as distinct from a 'pattern of *inference*' considered abstractly.

This interpretation is supported by several passages in *RFM* and *LFM*, but the following remarks bring out the point. First, *RFM*, III-41: "Every proof is as it were an avowal of a particular employment of signs." Second, *RFM*, IV-35: "Once more: we do not look at the mathematical proposition as a proposition dealing with signs, and hence it is not that."[196] In combination, what Wittgenstein was saying is that a proof does not result in a proposition which predicates anything *about* (how we should employ) signs, or in a statement *about* an internal relation between concepts. Rather, the proof *itself* is accepted as a paradigm for how to use signs, which is why it *itself* is likened to an avowal of an employment of signs.

If this is along the right lines, it also throws the so-called 'calculus-conception' into a different light. On this interpretation, a calculus is not fundamentally a system algorithmically generating mathematical propositions as outputs. Rather, Wittgenstein's concern with calculi was a concern with the role of human actions as part of systems. This tendency can already be seen in his middle period conversations with the Vienna Circle, such as in his 1931 classification of the behavior of looking things up in a diary and acting on the information thereby obtained with steps in a calculus (*WVC*, p. 171). With the predominance of the analogy of language-games by the mid-1930s, he tied human action, including mathematical reasoning, more explicitly to overlapping *social* systems, and finally to interwoven forms of life.

However, we need not assume that a proof merely leads us to endorse a rule, in the sense of a stated decree, in order to be faced with something like the problem raised by Kreisel (1958) and the tension addressed by Schroeder (2021, p. 58) and Frascolla

[196] See also *RFM*, III-24: "'This is the model for the addition of 200 and 200' – not: 'this is the model of the fact that 200 and 200 added together yield 400'. The process of adding *did* indeed yield 400, but now we take this result as the criterion for the correct addition – or simply: for the addition – of these numbers."

(1994, pp. 140-142). Thinking of proof in terms of being guided by a picture or a series of pictures, as outlined in the previous sections, the question becomes why, or in what sense, a proof involves anything *new*. If the picture(s) conveyed by a proof demonstrate(s) an action as already belonging to a practice, why is a proof required? That is, what is it that a proof leads us to *accept*?

To give a simple illustration that highlights the issue, in chess, only pawns and knights can move on the first turn. This, it can be assumed, is not an officially recognized rule of FIDE chess, but happens to be the case given the way the game is played (that is, its formal properties: all the rules for how the pieces are moved and everything involved in how the game is arranged and played). Now, say that someone *proved* this, either visually or by citing the starting positions and the allowed and disallowed moves of the various pieces. The question is, when it is proven that only the pawn and the knight can move from the initial setup, what is it that remains to be accepted or decided on the basis of the proof? The proof simply reflects the way the game is, as a matter of fact, already played, so whence the conceptual change?

Thus, Schroeder (2021, p. 58) highlights a challenge for Wittgenstein's philosophy of proof, namely to understand how to account for the *justificatory* role of proof without conceiving of justification as based on *discovery*. Wittgenstein understood proof as involving a decision on a new language game, or (what may come to the same) a modification of the formal properties of a language game (*RFM*, IV-23). And yet, if endorsing a proof involves the recognition of a picture as already being part of a language game, it would seem more accurate to say that a proof highlights the features and relations that our practices already embody.

This was a tension that Wittgenstein wrestled with in his middle period, but, in my view, one he eventually resolved. His solution can be seen, for example, in *RFM*, IV-36, written in 1942. This remark is also helpful in pulling together key concepts that have been considered in this and previous chapters, namely *proof*, *pictures*, *rules*, and *techniques*:

> *A proposition may describe a picture and this picture be variously anchored in our way of looking at things, and so in our way of living and acting. […] The effect of proof is, I believe, that we plunge into the new rule. / Hitherto we have calculated according to such and such a*

> *rule; now someone shews us the proof that it can also be done in another way, and we switch to the other technique – not because we tell ourselves that it will work this way too, but because we feel the new technique as identical with the old one* […]

The *overall* point made in this passage is contained in the very first sentence, in which Wittgenstein flags the anthropological nature of a picture. A picture may be "variously anchored in our way of living and acting", that is, a picture may come up in multiple different kinds of settings, and may serve multiple different roles in people's lives. Wittgenstein then turned to the concept of 'technique' and 'the way something is done' immediately after mentioning rules, highlighting that he conceived of rules not as abstract decrees but as practices. That we "plunge into" a new rule means that we begin acting in a different way.

Wittgenstein then compared insight into mathematical relations to seeing an identity between different ways of doing things. As this illustrates, Wittgenstein saw concepts, aspects, and rules as a result of practices and techniques. We form a new concept via a proof, we endorse a new rule, and *in so doing* we let the picture it contains guide us to do things in a new way. This being so, there actually *is* an element of discovery in proof, since it contains a picture which convinces us of there being a *new* way of doing something that we were already doing.

To return to the chess example, on this account of proof, a picture showcasing that only the pawn and knight can be moved on the first turn would properly speaking constitute a *proof* only if the picture in question is then 'anchored' in our way of living and acting, serving a function in how chess is subsequently set up, helping us differentiate chess from other games, or having some other function. The proof changes the concept of 'chess' by changing the formal properties and relations of the game, that is, what it means to engage in that activity.[197]

So, even when the proof is *obvious*, a proof is not just a foregone conclusion; it forges a formal relation between language games, allowing for new conceptualizations

[197] There are variants of chess that have differing starting setups, with pieces starting in different squares than in standard chess (e.g. 'Fischer random chess'). Depending on the proof, any game variant that allows other pieces to move on the first turn might be separated out and no longer taken to be in the same 'family' of games as chess. This could be compared to the star example from *RFM*, I-25. The proof, in that case, defines what counts as the star-formation for the given practice (i.e. that its outer points correlate with the 5 lines/wands).

of a terrain of calculating activity. Here, Wittgenstein's discussion of exponentiation, its comparison to multiplication, and its role in the history of mathematics is also illuminating:

> *[T]he* same *proof as shews that $a \times a \times a \times a... = b$, surely also shews that $a^n = b$; it is only that we have to make the transition according to the definition of 'a^n'. – But this transition is exactly what is new. But if it is only a transition to the old proof, how can it be important? 'It is only a different notation.' Where does it stop being – just a different notation? / Isn't it where only the one notation and not the other can be used in such-and-such a way?* (RFM, III-47)[198]

Proofs alter the order in our actions by setting up new connections between practices. In this case, it does this by showcasing a way of going from one action to another action. This makes a substantial difference insofar as it has a practical effect. In order for something to be a proof it must do more than just leave us with, say, a new terminology describing whatever we already were doing. Two different ways of doing the same thing, such as calculating that $a \times a \times a...$ (n occurrences of a) = b and calculating that $a^n = b$, can subsequently be considered part of two distinct language games. That is important, and the new notation requires proof, to the extent that some of our practices are substantially altered by showcasing the connection.

For example, whenever we operate with a number of factors, we might find an expression of the form "$a^n = b$" to be conspicuous and manageable in cases when "$a \times a \times a...$ (n occurrences of a) = b" is not. Historically, exponentiation derives from geometric reasoning, initially tied to measuring land, explaining the words "squared" and, later, "cubed". In 1637, Descartes introduced the contemporary notation, treating exponentiation not as pertaining to area or volume but, now in conformity with other arithmetical operations, as pertaining to lengths of line segments. Wittgenstein (*RFM*, III-47) suggested that, as we began to focus on the powers of numbers, we came to see a new aspect, e.g. seeing 64 *as* 4^3. Aspects of a notation accord with ways we *do* calculate as well as directions in which we might take our calculi.

It might be argued that there is an ambiguity in Wittgenstein's later work over

[198] This remark leads into the statement that "[i]t is not the introduction of numerical signs as abbreviations that is important, but the *method* of counting", discussed in Chapter 3 (p. 69).

whether a proof demonstrates a connection or forges a connection that was not already there (cf. Shwayder, 1969, p. 113). His view of exponentiation is again relevant. Although he alluded to a demonstration showing that the definition of "$a^n = b$" allows for a transition from $a \times a \times a...$ (n as) $= b$ to $a^n = b$, Wittgenstein stated that "I [i.e. someone introducing the notation of exponentiation] am surely setting up a new connexion! – A connexion – between what objects? Between the technique of counting factors and the technique of multiplying." So, the answer to the ambiguity is that the mathematician is an inventor, not a discoverer (*RFM*, I-168).

Proof as practical invention

A proof, then, invents connections between techniques that were not there before. However, we should keep in mind that invention is a genuine achievement and, as a concept, presupposes constraints. Setting up connections between forms of action, or formal properties, need not be easily done. Changes are admitted only in those cases, and to the extent that, we "feel the new technique as identical to the old one", which depends on the way the techniques are "anchored in our way of living and acting" (*RFM*, IV-36). Wittgenstein also noted that we differentiate between features of a calculus that we consider more essential (being part of the *point* of the calculus, its purpose) and features we consider more accidental and expendable (*RFM,* I, Appx. I-18). So, the mathematician is in a position of *altering* the structure of mathematical practice, in a broad sense, but this ramifies to the applicability of calculi in both pure mathematical, empirical, and colloquial contexts. For example, the notation of exponents enabled modeling of exponential rates of change, and this has become part of the *point* of this notation.

In *RFM* part III, Wittgenstein suggested that the need for perspicuity and exact reproducibility of a proof (i.e. the exact copyability of its picture(s): the steps in the proof) has to do with a need to rule out irrelevant factors, ruling out causal contingencies that are not generally pertinent to how our language games are played. That is, reproducibility of proof rules out *non-formal properties* in our rule-bound activities. To return to chess, the exact physical positioning of a piece within a given square is not relevant to the game, so any would-be proof presupposing such an exact placement would effectively not be a proof about *chess*. The point is that, for Wittgenstein, a proof *as such* must be reproducible because its role is not merely to give a one-time answer to a question '*about*' the formal properties of a practice, but to serve an atemporal role *in* practice; "the proof serves as a measure" (*RFM*, III-21).

This deserves a more elaborate illustration. Say that a scientist studies chess games while recording information about the exact physical placement of the pieces within their squares. The pieces, in this hypothetical scenario, are used according to the normal rules of chess. The scientist finds a statistical link between the exact physical placement of pawns and, for instance, the rate at which those pawns capture other pieces. This finding is communicated with a picture of a chess square divided into 5 regions, one of which is said to be the most effective position. Now, if we were to take this hypothetical report as a *proof about chess*, that would itself mean that what we call "chess" would have to change. That is, by accepting the report as proving something about chess, we sign up to treating the placement of a piece on one of the 5 regions *inside* a field as a formal property of chess, the complexity of chess now matching the picture.[199] If we fail to treat these 5 subdivisions of the squares as part of chess, the picture could not possibly be reproduced in chess games, and so the proof would not go through. It would have been a kind of empirical study, a temporal experiment.

That is of course how we would likely respond. We would not accommodate such a picture *in* chess. Professional players might make use of the research, but, in teaching chess, we would continue to ignore spatial relations beyond the 64 squares. That is not just due to an arbitrary choice, on our part. Chess simply does not have a technique to *demonstrate* that placing a piece within one region over another, on a given square, might somehow be preferable. So, no such picture, considered as a picture of chess as we currently practice it, could have the force of proof to demonstrate the result. As Wittgenstein (*RFM*, III-55) put it:

> *To repeat a proof means, not to reproduce the conditions under which a particular result was once obtained, but to repeat every step and the result. And although this shews that proof is something that must be*

[199] This is linked to the point Wittgenstein makes in *PI* §47, which is not just that there are innumerably many ways in which we *can* draw a distinction between 'simple' and 'composite', but also that the ways we *do* draw that distinction express what differentiations are *meaningful* for us in retaining a grip on our activities.

> *capable of being reproduced* in toto *automatically, still every such reproduction must contain the force of proof, which compels acceptance of the result.*

Given that a proof does not demand a transformation of the basic point of our activities (e.g. chess), the picture(s) it contains must be reproducible *from within our activities* as currently practiced. There are no logical or metaphysical barriers here, but there *are* relational and practical limits. We could not admit a form of 'proof' that changed our basic concepts in such a way that we could no longer recognize them, because in attempting to reproduce it we would alienate ourselves from what we were attempting to do. If we were to take such a proof seriously, to the extent it *could* be taken seriously, we would in effect withdraw from practices that we previously engaged in. This is why Wittgenstein (*RFM*, I-74) wrote, in response to a hypothetical interlocutor holding the view that mathematical statements express the essence of concepts, "to the *depth* that we see in the essence there corresponds the deep need for the convention". Practically speaking, people rely on the recognizability of basic concepts and the persistence of practices that play a prominent role in their lives. In other words, in a certain sense Wittgenstein was not denying the necessary features of mathematical concepts, for a given community, but he was denying that their necessity is a matter of logic or metaphysics.[200]

The following section will consider the more critical implications of this practical view of mathematics, proof, and concept formation. The dualism broached in the introduction of this chapter, which conceives of mathematics in either fundamentally machinelike or epistemic terms, will be seen to result from an underappreciation of the anthropological character of mathematics, and this underappreciation is in turn connected to the view that mathematics should fundamentally be understood as abstract theoretical activity.

5.3 Nominalization and abstraction

In Wittgenstein's view, the question "does the number three exist?" is analogous to the question "does the color red exist?" (*LFM*, XXVI, pp. 247-248, cf. ibid., lecture XXIV).

[200] Cf. Robinson's (1998, pp. 50-51) discussion of practical necessity inherent to a form of life.

His answer to such questions was not a denial of the existence of the 'objects' concerned, but a highlighting of the grammar involved. Such questions are unusual, not just because they address the existence of *abstracta*, but because their respective nouns are ordinarily used in other ways, as verb phrases or as adjectives. We typically use "three" and "red" when quantifying things and talking of something being/not being red, respectively. Nominalization can be harmless, often serving as a grammatical convenience, but there is potential for unclarity whenever a sentence turns *crucially* on the contrived noun. Just as it is unclear what, for example, "the orbit of Mars" might mean aside from Mars' potential *movement* (i.e. the action of orbiting other objects), it is unclear what "the number three" or "the color red" might mean aside from numbering certain things or something being a certain color. Through these nominalized turns of phrase, the concepts ('orbit'/'three'/'red') go on holiday (cf. *PI* §38).

By analogy with an abstract view of pure mathematics, we could imagine that we had a 'pure color theory', a hypothetical field concerned with colors independently of instantiations of said colors in actual cases. Such a field would contain such sentences as "the color red is more like yellow than blue". Instead of formulating such sentences for the purposes of teaching aesthetic and visual language, or as a heuristic for mixing paints, however, the hypothetical pure color theory would treat such sentences as independent *insights about colors as such*. In other words, the field would essentially revolve around nominalized forms of color-adjectives, like "the color red", and would explicitly *detach* such forms of expression from constructions of the form "*x* is red", even though the latter is the grammatical basis of the former.

The analogy with mathematics is the following: modern pure mathematics, insofar as it is built on imagery of e.g. 'the number line', essentially revolves around nominalized forms of numerical language in a similar way. Just as Wittgenstein exposed the emptiness of the idea of a pure color theory by highlighting our reliance on *samples* in our usage of color language (*PI* §16, §50, §§56-58), he developed a similar critique against the view that calculi are determined by abstract theoretical activity, through discovering facts/objects or enacting ideal rules.[201]

[201] Several passages on the emptiness of the frictionless ideal are relevant, encapsulated by *PI* §100: "'Well, perhaps you'll call it a game, but at any rate it isn't a perfect game.' This means: then it has been contaminated,

Rather than being concerned with methods of calculation or geometric constructions, pure mathematics is often conceived as engaging in a study of abstract objects. The distinction between pure and applied mathematics is presented as a gap between metaphysical realms. The picture of a line composed of uncountably many points, each specified by a real number with an infinitely long decimal expansion, is a prominent mythology of *abstracta* waiting to be discovered independently of human capabilities (*RFM*, V-32). To some extent, this imagery serves a sociological role in the self-conception of modern mathematics.[202]

It could be argued that, with the advent of symbolic algebra, modern mathematics has been transformed into a purely symbolic discipline, unhindered by ontological misconceptions. Nevertheless, Wittgenstein's critiques of the idea of mathematics being a matter of exploration, and of "mathematical alchemy" (*RFM*, IV-11, V-16), do not suggest that he harbored a special affinity for modern mathematics (cf. Stenlund, 2015, p. 63). Underlying Wittgenstein's critical remarks on foundationalism and set theory (e.g. *RFM*, VII-16) is an anthropological perspective that is as far removed from abstract formalism as it is from Platonism.

The accuracy of the claim that modern mathematics is relatively free of ontological assumptions can also be questioned. The ideal of exploring a pre-existing mathematical realm is, of course, not modern in origin. Both this ideal, the mythology of discovering abstract facts/objects, *and* the ramifications of this ideal in motivating mathematical research, can be traced at least as far back as to the ancient Greeks. However, the ideal has (again, arguably) taken on a more important and expansive role over the last centuries. The mathematician Jean Dieudonné, for one, traces the roots of the theoretical approach characteristic of modern mathematics to the ancient Greeks and the philosophical context of Platonism:[203]

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and what I am interested in now is what it was that was contaminated. – But I want to say: we misunderstand the role played by the ideal in our language. That is to say: we too would call it a game, only we are dazzled by the ideal, and therefore fail to see the actual application of the word 'game' clearly."

[202] Thurston (1994) presents a relevant ethnography of professional mathematical activity, including the role of what he calls "polite fictions" with respect to foundations. Thurston describes a mismatch between an official picture of mathematical activity and the relatively discursive, informal nature of much mathematical progress.

[203] See also Lakoff & Núñez (2000, pp. 107-110, pp. 338-341), who, from a perspective somewhat closer to Wittgenstein's, trace what they call "the romance of mathematics" to the ancient Greeks.

> *"While all ancient civilisations, in order to satisfy the needs of daily life, had to develop procedures of arithmetical calculation and spatial measurement, only the Greeks, from the sixth century B.C., thought of analysing the chain of reasoning behind these procedures, and thus created an entirely new mode of thinking"* (Dieudonné, 1992, p. 29).

There are two ideas that characterized ancient Greek mathematics and which have left a major mark on intellectual life ever since, according to Dieudonné (ibid.). Firstly, the idea of proof as axiomatic and encompassed in logical derivation. Secondly, the sharp semantic separation of the names of objects of mathematical interest – numbers and geometric figures, objects which are immaterial and obtained by abstraction – from the homophones that humans happen to use in practical calculations. Both of these ideas, and tendencies, emerged around the time of Plato, according to Dieudonné, and these are aspects of mathematics, and philosophy of mathematics, that appear to have only become increasingly important in the modern era.

If Dieudonné is right, modern mathematics, far from freeing itself from antiquated ontological doctrines, is "*constrained* by the essential nature […] of *classical* objects" (ibid., p. 2), or, put in a less Platonist way, carries with it assumptions that have characterized the subject since ancient times. Of course, there are problems with framing the history of mathematics in any such general way. Nevertheless, Wittgenstein held that the foundational status given to mathematical logic has led both mathematicians and philosophers astray by continuing "to build on the Aristotelian logic" (*RFM*, V-48). This appears to signal that the influence of the ancient Greeks' philosophical understanding of mathematics has not abated.

One counterargument here might be that modern mathematics has obvious practical applications, in science and technological development, and that its rigor and complexity is in part *due to* the detachment of pure mathematics from any requirement of direct applicability. So, though modern mathematics might be more abstract in one sense, it is nevertheless all the more useful. We saw progress in mathematics precisely by letting go of any demand for physical, intuitive interpretability, the argument goes, for instance through Viète's (1591) notational innovations allowing for calculations with magnitudes irrespective of what they represent (see Domski, 2021, §1.2) and Descartes' fusing of geometry with algebra (Boyer, 1956, p. 84; cf. Landini, 2011, p.

30, pp. 54-55). As argued by Stenlund (2015), the development from verbal, geometric mathematics to symbolic mathematics can be taken to align with Wittgenstein's intermediate stress on the distinction between calculation and prose.

Indeed, modern mathematics *could* be considered less representational ancient mathematics due to symbolic algebra using decimal numbers and variables, in contrast to verbal and syncopated algebra (cf. Merzbach & Boyer, 2011, p. 162). However, it should be recalled that symbolic algebra serves to construct *models* of empirical phenomena (Dieudonné, 1998, pp. 19-20). That is, although numbers do not refer to physical entities, they *are* taken as parameters of physical systems, and thus go proxy for elements of empirical description. From a Wittgensteinian perspective on calculating activities, then, there is no need to posit any great historical leap in the way in which mathematics is applied. Rather, there have been myriad practical developments connected with broader history, including a shift from solving static numerical equations to using dynamic functions to describe movement or change.

Again, even though there is a contrast between a modern symbolic understanding of numbers and the ontological understanding of *arithmos* as exhibited in the writings of ancient philosophers such as Plato and Aristotle,[204] that need not entail that there is a greater chasm between modern and ancient mathematics in their applications. Judging by some of our earliest archaeological evidence of mathematical activity – such as Plimpton 322, a Babylonian clay tablet listing Pythagorean triples or numbers satisfying $a^2 + b^2 = c^2$ (see Abdulaziz, 2010) – ancient mathematics already employs symbolism for practical purposes. It was, at least in some instances, used for administration and accounting, and written down following symbolic rules.

In any case, as an objection to reading Wittgenstein as a *critic* of the trajectory of pure mathematics on the basis of his rejection of the model of abstract theoretical activity, the argument from the practical success of modern pure mathematics has several weaknesses. Crucially, if pure mathematics is useful, it is not absolutely pure.

[204] Cf. Stenlund (2015), who draws on Klein (1936/1968). Klein emphasized the philosophical imprint on the mathematics of ancient Greece and argued that, at that time, numerals qualified objects. For instance, Aristotle compared the difference between ten sheep and ten dogs to the difference between a scalene and equilateral triangle (Klein, 1936/1968, p. 47). In modern mathematics, for Klein, pure symbolism has replaced such ontological teachings, and Stenlund takes this development to align with Wittgenstein's anti-Platonism.

To return to the analogy with color, the sentence "red is more like yellow than blue" might be expressed by someone teaching the grammar of color-words in English; the students can then use this rule to draw inferences pertaining to concrete objects. In fact, the students would *only* learn the rule if they come to use it in this way.[205] So, such a sentence would normally not be taken as absolutely pure.

What would constitute a hypothetical pure color theory would be a treatment of colors in some strictly ideal and/or purely formal sense, independently of any possibility of color-description. The reason such a notion would be empty, from Wittgenstein's perspective, is that, conceived of as pertaining only to abstract forms, e.g. *red* or the color red '*as such*', a sentence like "red is more like yellow than blue" does not express a rule, as there is no occasion to follow it. Thus, the main contention Wittgenstein had with Platonism is the cutting off the rules of mathematics from practice, from the uses of mathematical concepts in real life. That means that his approach differs from *any* philosophy or historiography of mathematics narrowly conceiving of mathematics as an abstract deductive system or detached field of study.

This also implies that the primary question for Wittgenstein is not whether a given formalism has *useful* applications, but whether or not it has *any* implementation in the way that people conceive of, and motivate, it at all. His concern with "mathematical prose", most prominent in the middle period, was the tendency for people to be misled by extrinsic forms of expression to a false understanding of their calculus, forms of expressions which do not reflect how the calculus is actually used and/or applied (e.g. *RFM*, VII-41).[206] At bottom, Wittgenstein sought to expose a mythology of mathematical practice based on a tendency to nominalize mathematical verbs, adjectives, and adverbs. He took that mythology to be the only *grounding* for certain tendencies in mathematical practice that are not wrong, but unmotivated.

[205] This reverses the typical course of learning; informal familiarization with color(-words) is the basis for explicit use of such rules (Waismann, 1965, p. 231). If anything that supports the point being made.

[206] Cf. Stenlund (2015, p. 28), who sees the critique of prose as an attack on outdated verbal mathematics: "Equations, as well as addition and multiplication were expressed in verbal language, in mathematical prose (to use Wittgenstein's expression)." However, Wittgenstein did not distinguish prose from calculation on the basis of mere syntax. A sentence such as "adding 2 apples to 3 apples gives 5 apples" can constitute a calculation, no less so than "2 + 3 = 5" (*LFM*, XII, p. 113; *RFM*, VI-9). See also Kienzler & Sunday Grève (2016, pp. 80-81).

5.3.1 The critique of generalization

In the *Blue Book*, Wittgenstein described the craving for generality as a source of philosophical confusion, and he here included a mathematical tendency: “I mean the method of reducing the explanation of natural phenomena to the smallest possible number of primitive natural laws; and, in mathematics, of unifying the treatment of different topics by using a generalization” (*BBB*, p. 18). Wittgenstein saw mathematical generalization as an example of scientific method serving as a misleading source of inspiration for philosophers. Clearly, though, he was in these remarks *not* attacking scientific method as such, but a certain form of abuse or misuse of it. Mathematical methods in particular might be regarded as unimpeachable from the point of view of Wittgenstein’s later writings, in which he called for philosophers not to embroil themselves in the business of mathematicians (*PI* §124, *RFM*, V-52; cf. Dawson, 2015).

This being so, if the *Blue Book* is to be made to cohere with Wittgenstein’s later, more explicitly non-revisionistic writings, it could seem like he has to be read as saying that it is merely the over-generalizing way that philosophers *interpret* mathematics that involves philosophical confusion. However, in the *Blue Book*, Wittgenstein went on to suggest that he was targeting a tendency that is rooted in the *motivation* to generalize, a motivation which is also prevalent in parts of contemporary mathematics (*BBB*, pp. 18-19):

> *If, e.g., someone tries to explain the concept of number and tells us that such and such a definition will not do or is clumsy because it only applies to, say, finite cardinals I should answer that the mere fact that he could have given such a limited definition makes this definition extremely important to us. (Elegance is* not *what we are trying for.) For why should what finite and transfinite numbers have in common be more interesting to us than what distinguishes them? Or rather, I should not have said “why should it be more interesting to us?”—it* isn't*; and this characterizes our way of thinking.*

Misguided interests or motivations, a “contemptuous attitude towards the particular case”, are involved in the tendency of unifying the treatment of different mathematical topics *for its own sake*. Again, it is not that Wittgenstein saw this tendency as erroneous, mathematically speaking, but that he saw it as generally ill-motivated. The *reason* he found this tendency to be ill-motivated can be connected to his critique of mathematics as abstract theoretical activity.

Mathematics is not just a body of knowledge, but a family of activities. It therefore involves what Robinson (2003, p. 35) describes as "functional or structural unities which imply completion". That is, mathematics is comprised of activities which people *already* engage in, a fact which cannot subsequently be overturned by making changes to the activities in question (Shanker, 1987, p. 254). Wittgenstein's later comments on the methodology of language games thus to an extent align with his middle-period stress on the completeness of calculi: "How do we compare games? By describing them […] and *emphasizing* their differences and analogies" (*RFM*, II-49). In considering different definitions of 'number' we are in effect distinguishing between techniques in different calculi, or language-games, not trying to pin down independent categories of objects. Any definition of 'number' we might produce characterizes a game, whereas it is not as if all demarcations of physical properties describe objects. That is why we have to distinguish the aim of defining 'number', where all differences we *can* make are inherently significant, from the aim of defining a category of objects, such as 'apple'.[207]

It might be questioned whether this distinction is in tension with the later Wittgenstein's writings on family resemblances between concepts (e.g. *PI* §68). That is, it might seem as if we should avoid talking of completeness in the case of concepts, like 'number', that have multiple open-ended areas of application, and can very well be expected to develop further. However, in this context, 'completeness' does not imply temporal finality. It is simply the opposite of *incomplete*. There are no gaps that we still have to fill in when we are dealing with a mathematical technique. The upcoming section explores this distinction further.

Infinity and formalization

In *RFM*, II-45, Wittgenstein made the point that techniques do not require an established *endpoint* for them to be distinctive and usable: "To say that a technique is unlimited

[207] Note that this example illustrates abstraction as an effect of a nominalizing tendency. Treating the term of interest, "number", as *first and foremost* a common noun (thinking of it as essentially naming an abstract object, category, etc.), we are motivated to find a definition with a form suited to a common noun.

does not mean that it goes on without ever stopping – that it increases immeasurably; but that it lacks the institution of the end, that it is not finished off." He gave the following comparisons: "As one may say of a sentence that it is not finished off if it has no period. Or of a playing-field that is unlimited, when the rules of the game do not prescribe any boundaries – say by means of a line." The first example here is questionable. A sentence which lacks a punctuation mark might be confusing or even incomprehensible, and therefore *incomplete*, especially if is not obvious that it is meant to end from context. The second example is more helpful.

As this shows, Wittgenstein's conception of infinity is closer to the Aristotelian tradition than that of Cantor (Marion, 1998, pp. 181-182). However, there are differences worth noting. For Wittgenstein, infinity is a property of a rule that is unlimited; it describes a technique without an institution of ending (RFM, II-45). Aristotle in book 3 of the *Physics* appears to have viewed the infinite in terms of potentially unlimited magnitudes and quantities. Granted, he conceptualized the potential infinite in terms of the unlimited applicability of operations (e.g. indefinitely adding or dividing line segments), but he still at least arguably conceived of the infinite in terms of potential results of the application of operations.

By contrast, a rule that has no end, which can be repeated or iterated indefinitely, is not a potential (set of) extension(s). That is, for Wittgenstein, it is important to distinguish the rule from the potential results of that rule. Infinity is strictly speaking a property of rules which do not prescribe a final result, or a property of the corresponding technique. That is not to say that Wittgenstein was out to deny that there are other formulations involving "infinite", aside from the strictly adverbial. This can even be seen in his discussion of infinity in the *Philosophical Remarks*, written in 1930. For example, the following remark is consistent with his later views, given that the emphasis is on the adverbial use of "infinite", which accords with the idea that the infinite should in literal contexts be ascribed to a rule or technique:

> *You could put it like this: it makes sense to say that there can be infinitely many objects in a direction, but no sense to say that there are infinitely many… The 'infinitely many' is so to speak used adverbially and is to be understood accordingly.* (*PR* §142)

Moore (2011, pp. 109-110) quotes from this passage and takes it to indicate that Wittgenstein was closely aligned with Aristotle. Wittgenstein recognized syncategorematic references to infinity, that is, talk of potentially infinite quantities or potentially infinite magnitudes, e.g. sentences such as "a ruler can measure infinite distances". However, again, such language is secondary. This is important because the prioritization of the adverbial use of "infinite" informs Wittgenstein's view of the nonsensicality of non-lawlike infinities (cf. Moore, 2011, p. 111). The reason 'infinite' implies 'rule-governed' is that infinity is a property of a rule. To say that a ruler can measure infinite distances is best understood to mean that we have no established limit to the measuring of units of distance: it is measuring as such that is infinite, not the distances potentially measured. The role of measuring within our form of life is unrestricted.[208]

Several of Wittgenstein's later remarks can also be seen as consistent and continuous with the critique of unreflective generalization in the *Blue Book*. For example, he advanced the following critique of the reliance on mathematical logic in *RFM*, V-48:

> *'Mathematical logic' has completely deformed the thinking of mathematicians and of philosophers, by setting up a superficial interpretation of the forms of our everyday language as an analysis of the structures of facts. Of course in this it has only continued to build on the Aristotelian logic.*

The point Wittgenstein made in the remarks on mathematical logic in *RFM* part V, from 1942-1943, is that possessing a general means for translating formulae does not necessarily aid our understanding. To illustrate this, say that we have the string of letters, "*ABC*", along with the rules $A = 12$, $B = A + 2$, $C = B + 2$. We can now translate "*ABC*" into "121416" and back again. However, this tells us nothing about how to understand "*ABC*". This is so even if we begin operating with "121416" in a calculus, for example if we calculate that 121416 – 2 = 121414 and translate the result back into "*ABB*". The whole process amounts to encoding and decoding a string, and whether or not we *understand* that string is a separate question entirely.

[208] Just as 'infinite' is taken adverbially, 'continuum' should be thought of in terms of the *method* of interpolation, or "the art of reading between the lines of a table" as Han (2010, p. 228) puts it. Cf. *RFM*, II-44.

Wittgenstein (*RFM*, V-46) rejected the response that any question of *understanding* is irrelevant since mathematical propositions are just positions in a game, a mere sign-game in this case. He rebutted by highlighting that 'understanding a mathematical proposition' is a "vague concept". We can now see that he meant that mathematics is not simply a matter of knowing rules, but of knowing *how* and *why* they are followed: their role in practice. Our understanding of a mathematical proposition is bound up with its proof (*RFM*, V-42-46), but its proof is not merely a vehicle of verification. It gives the proposition a role to play.

In order for a process of translation to add to our understanding, then, the translation would have to reflect or enhance the way we use the formula, sentence, or string that we want to translate. In the above example, the translation would have to extend our understanding of "*ABC*" in some way. For example, say that *A*, *B*, and *C* are frequencies played on a musical instrument. Following the above rule, "*ABC*" is translated into "121416", which is entered into a computer program, which then renders notes in staff notation. The numbers 12, 14, and 16 here function as instructions for outputting note symbols. In this case we have a useful rule, a method of translation, which enhances our understanding of "*ABC*": we understand it as a set of frequencies which, via an algorithm we now know, can be represented in musical notation.

In the case of mathematical logic, the aim is to analyze propositions, but everything turns on the motivations for this. According to Wittgenstein, there is a prevalent but unfounded assumption that mathematical propositions should be made to conform to the subject-predicate pattern (*RFM*, V-40, V-47); they are incomplete as they stand, and analysis reveals their logical structure. However, similarly to above, provided there is no specific *use* intended for e.g. "there is no prime number, p, such that $7 < p < 11$", translating it into "there is no x such that $P(x)$ and $G(x,7)$ and $L(x,11)$" yields no further understanding. The result, in the absence of further calculation or application, is just a "translation of vague ordinary prose" (*RFM*, V-46; VII-41).

Wittgenstein's strongest critique of the idea that a system such as Russell's *Principia Mathematica*[209] forms a foundation for arithmetic is that this logical formalism, rather than *justifying* the distinctions that are significant in arithmetic,

[209] Or, more generally, "Russell's prose" (*RFM*, VII-41; Kienzler & Sunday Grève, 2016 pp. 81-82).

generalizes mathematical techniques and thereby *paves over* significant differences. "The logical notation swallows the structure" (*RFM*, V-24). Moreover, the motivation behind analyzing mathematical formulae is to render their truth-value explicit, in the same way an analyzed tautology bears its truth on its face. However, the correctness of a formula is already calculable from *how* it is constructed, and any analysis could *at best* reflect this construction procedure. A recognition of the mode of construction of a mathematical proposition, and a knowledge of the construction as being mathematically *correct*, is therefore presupposed by any logical analysis of it.[210]

Accordingly, the anti-generalizing critique from Wittgenstein's middle period, directed towards an overly theoretical philosophy of mathematics, *does* survive and persist into his later period. The critique can be taken to adhere to his later maxim of leaving mathematics as it is (*PI* §124) so long as that remark is read as bearing on the results of mathematics, not the social phenomenon of mathematics in general. As Wittgenstein noted in the early 1940s about the conception of mathematics as a mineralogy of numbers, he was not just up against an obscure philosophical theory, but rather a widespread and deeply entrenched mythology: "Our whole thinking is penetrated with this idea" (*RFM*, V-11). In upshot, even though Wittgenstein was not concerned with directly *revising* mathematics, his writings, middle *and* late, contain a profound critique of modern mathematics in its relation to philosophy.

Mathematics, technology, and methods of projection

The previous section discussed an appeal to modern technology, taking the success of technological applications as a way of challenging the interpretation of Wittgenstein's critique of a common understanding of pure mathematics as a purely abstract and theoretical pursuit without practical implications. The following passage is pertinent to this argument:

> *The sickness of a time is cured by an alteration in the mode of life of human beings, and it was possible for the sickness of philosophical problems to get cured only through a changed mode of thought and of life, not through a medicine invented by an individual. / Think of the use of the motor-car producing or encouraging certain sicknesses, and*

[210] On the circularity of Russell's attempt to ground arithmetic on logic, see Marion (2011, p. 142).

mankind being plagued by such sickness until, from some cause or other, as the result of some development or other, it abandons the habit of driving (*RFM*, II-23).

In this remark from 1938, Wittgenstein addressed philosophical problems in general. The first sentence is significant in showing that he had a social understanding of the root of philosophical problems.[211] However, it is the second sentence that is more relevant for present purposes.[212] From Wittgenstein's perspective, arguing that the motivations behind mathematical activity are philosophically beyond reproach whenever the mathematics has led to technological results (scientific, financial, and other uses could be added) is potentially to put the cart before the horse. After all, technology is not just a *product* of mathematics; the way a given community uses and develops technology can also influence its interests in mathematics.

The interplay of mathematics and technology in history shows that is not always easy, or desirable, to untangle mathematical progress from technological innovation. Technical and social changes have repeatedly led to new uses and goals for mathematics, changing generally accepted views about what is 'achieved' and what is 'yet to be achieved', in which case standards of mathematical progress are shifted or updated. For a relatively recent example, consider Shannon's (1948) founding of information theory, which led to technical innovations and new mathematical problems, while itself being stimulated by technical developments.

It is important to note that Wittgenstein examined the underlying motivations and interests that drive pure mathematical activity, and not specific techniques or outcomes in isolation. If these interests become ungrounded and untethered, as he saw it, then they also disconnect from any comprehension of how mathematics is employed in developing and utilizing technology. Therefore, it cannot be assumed that the abstract, theoretical model of mathematics is necessary for technological progress. On the contrary, it could be argued that the model gives a distorted view of the relationship between mathematics and technology. Wittgenstein's discussion of conflations of

[211] This is contrary to what is suggested by Gellner (2004, pp. 164-168).

[212] Note that, despite its inclusion in *RFM*, it is questionable whether Wittgenstein in this passage meant to address the topic of *mathematics*, as it is disconnected from his other writings on the subject. Regardless, the passage tells us that he saw philosophical problems as analogous to problems generated by technology.

models of machines with actual machinery targets precisely such a distortion (*RFM*, I-122, cf. IV-21; Shanker, 1998, p. 6).

In general, comparisons between the understanding and use of technology and the understanding and application of calculi feature prominently throughout Wittgenstein's work on mathematics. In both cases we tend to take the customary applications for granted, so much so that thinking of the technology or calculus without them can be difficult:

> *There is a way of looking at electrical machines and installations (dynamos, radio stations, etc., etc.) which sees these objects as arrangements of copper, iron, rubber etc. in space, without any preliminary understanding.* […] *It is quite analogous to looking at a mathematical proposition as an ornament. (Z, §711)*

Wittgenstein's background as an engineer should be kept in mind here. At the time of his education, engineering in the German-speaking world turned towards graphical methods (see Kallenberg, 2012; cf. Nordmann, 2002). Wittgenstein's later appeal to the use of pictures in mathematics can be seen in light of the use of technical drawings as plans and schematics for the workings of machines. Such drawings function as paradigms in the sense that they are models for the construction and operation of devices, comparable to how calculations function as paradigms for using language and calibrating measuring instruments (cf. *LFM*, VII-73).

The relation between a model and its applications can be thought of as internal, in the sense that both *relata* are described at once: schematics and technology are jointly understood *via* a particular method of projection. It is similar with mathematical calculi. Both the calculus and its domains of empirical applications are understood, together, via a given 'method of projection', a given intended use of the calculus. Calculi are developed to fulfil certain roles, so that, by calculating, certain purposes are satisfied, inside and/or outside mathematics. A form of mathematics is normally coupled with a range of uses, to the extent that, as Wittgenstein (*Z* §711) pointed out, an uncanny effect occurs when they are taken apart. Of course, when a new kind of mathematics emerges it need not have any direct *utility* at all, but there are nevertheless some forms of use intended for it, some *purpose* involved, if only with respect to other calculi. Inasmuch as mathematics is empirically applicable overall, given that a new kind of mathematics is *mathematically* useful, is also indirectly empirically applicable.[213]

It might be worried that talking of the 'interpretation' of subfields of mathematics, and of formal relations between calculi and domains of application, conflates mathematics with representational language, which Wittgenstein consistently contrasted. However, there is an everyday sense in which a form of mathematics has an interpretation that does not not necessarily involve a commitment to the idea that the mathematics is representational or has a semantics (cf. Stenlund (2015, p. 50)). Interpretation in this sense should not be assimilated to ostensive explanation, an assimilation that Wittgenstein took to be confused (*PI* §28). In other words, the interpretation of a region of mathematics is not necessarily a matter of individuals mentally assigning semantic (denotative) value to numerals or letters. Rather, in a fairly ordinary sense, to say that a mathematical calculus is interpreted simply means that a community coordinates the *workings* of that calculus with a family of language games organized around a given purpose. The mathematics in question is put to a given use.

For Wittgenstein, the workings of a calculus, like the workings of technology, are the workings of human beings *applying* it (*RFM*, IV-20). Mathematics is as if it were a social machinery, each component of which is dependent on others. This can be illustrated by the history of scientific practice. In the 17th century, natural philosophers began replacing unwieldy verbal descriptions of the curved paths of projectiles and celestial objects with algebraic equations (Katz, 2007, p. 195). This process of replacement did not divest the scientific enterprise of its descriptive and predictive purposes, as if the use of modern algebra untethered it from the world. Rather, calculations were done to describe actual movement (cf. *RFM*, IV-15). Rule-bound symbolic operations inherited the *role* of (rule-bound) verbal geometric reasoning for the purpose of precisely describing and predicting the motions of objects.

Although the Scientific Revolution greatly expanded the role of mathematics in our understanding of nature, it did not gain this new role as an inherent effect of the invention of new symbolism. Rather, this was the historical result of a process of integrating new algebraic methods, coupled with associated forms of speech (involving

[213] In other words, in the peripheries of advanced mathematics, applicability becomes a matter of purpose with respect to mathematics itself (cf. *RFM*, V-15, VII-32). Cf. also Schroeder (2021, p. 118).

terms like "variable", "constant", "tendency", "increase", "curve", "parameter", etc.), *into* observational and experimental traditions. Hadden (1994, pp. 71-94) details how these new mathematical methods can be traced to commercial practices of banking and double-entry bookkeeping which proliferated over the 14th and 15th centuries in Europe, eventually having a major role in the transformation out of the feudal system. Thus, commercial and financial practices belong to the core of the social machinery of mathematics (cf. Graeber, 2011, pp. 237-238; *RFM*, I-53).

For Wittgenstein, the way we coordinate a calculus with other practices depends on treating certain pictures (applications, equations, etc.) as *representative* of moves in that calculus (*LFM*, I, pp. 18-21). He highlighted that a picture requires a method of projection to be applied and understood, and that a picture may vary in adequacy depending on the stability of the role(s) we expect it to perform (cf. *PI* §139, *LFM*, I, p. 18). That implies that we can come to believe that a calculus is coordinated with a given practice or another form of mathematics in a way which it, in fact, is not. This was Wittgenstein's view of the techniques and symbolism of real numbers and transfinite set theory, which he saw as disconnected from the use of other number systems to an extent that is not sufficiently recognized:

> *"There is a* muddle *at present, an unclarity. But this doesn't mean that certain mathematical propositions are* wrong*, but that we think their interest lies in something in which it does not lie. I am not saying transfinite propositions are false, but that the wrong pictures go with them. And when you see this the result may be that you lose your interest."* (*LFM,* VIII, p. 84)

The *RFM*, II-23 metaphor of the car "producing or encouraging certain sicknesses" can, by analogy, be understood to indicate the possibility of a kind of application promoting a misleading understanding of a region of language or a calculus. As Diamond (1991, pp. 286-287) puts it, "there is no support in Wittgenstein for the idea that if a form of words has a place in some activity, that form of words is not expressive of deep confusion". In at least some cases, an improved understanding is attained by *replacing our pictures*, replacing the ways we represent what is done in the calculus. However, it might also go the other way around: we might retain our pictures but make changes in the calculi to fit them. A calculus might even be abandoned, if it is recognized that its interest for us were completely ungrounded: we might find ourselves unable to give it any of the applications which motivated its development.

The roads of mathematics

Wittgenstein's criticisms of ideas surrounding real numbers and transfinite cardinality, albeit important, have to be placed in their proper context. The focus in this thesis has been on his view of mathematics as a human phenomenon. As such, one important aspect of what lies behind Wittgenstein's critical writings on these subjects has been emphasized, namely his rejection of the widespread understanding of mathematics as abstract theoretical activity. To recap, it has been argued that the abstract view of mathematics comes in two general forms, conceiving mathematics either as a field of knowledge of abstract objects waiting to be discovered, or as an ethereal machine, a formal system essentially disconnected from practices of application. Either way, humans are given a passive role, and mathematics a purely abstract one. Wittgenstein naturally tended to address this view in piecemeal ways. Some passages are, however, more explicitly framed in response to an overall abstract view of mathematics:

> *"The mathematical proposition says: The road goes there. Why we should build a certain road isn't because mathematics says that the road goes there – because the road isn't built until mathematics says it goes there. What determines it is partly practical considerations and partly analogies in the present system of mathematics."* (*LFM*, XIV, p. 139)

Quite simply, mathematics is a multifaceted human activity, and it does not exist before we make it, which means that we are not predetermined in how we make it. As this chapter has shown, the topic on which Wittgenstein's distance from the abstract theoretical view of mathematics comes out most clearly is that of proof, where he rejects the dichotomy of passive discovery and arbitrary invention. Wittgenstein understood proofs as practical demonstrations involving pictures. A mathematical proposition expresses the *calculating technique* introduced by a proof. It automatically indexes that technique, placing it in a system of other techniques.

In producing a proof of, say, 23 × 17 = 391, it is not as if we necessarily also have to explicitly show how this proof relates to further results of the same sort, e.g. a proof of 23 × 16 = 368. Rather, the proof of 23 × 17 = 391 might already also suffice to prove that $23 \times 16 = 368$. Once we have an adequate picture of a family of calculations, all involving the same symbolism and general calculating procedure, proofs of further results in the same area become potentially superfluous: "Suppose I put into the archives a general rule and a few examples; and you now give a new example. This might be a new rule – and we need not put this into the archives, but we might do so" (*LFM*, XI, p. 106).[214] When and where we *do* need proofs depends on how well our given pictures guide us in further calculations.

A mathematical proposition can be read as a decree regulating the technique (i.e. "do such-and-such", e.g. "infer 4 from 2 times 2", "do not look for a largest prime", or "do not try to trisect an arbitrary angle"). However, we have to keep in mind that such formulations obscure the proof and are therefore potentially misleading (*RFM*, III-26). The actual process by which a proof is accepted consists, not in the endorsement of a mere decree or stipulation, but in the practical acceptance of an actual *rule*, the pattern of calculation that the proof models.

Finally, the chapter has clarified why Wittgenstein held that the idea that proofs determine the *truth value* of conjectures, in that way informing us of a separate mathematical reality, is decidedly misleading. Recalling his analogy of mathematical reasoning with musical composition (*RFM*, III-63), this would be akin to the idea that composing music amounts to discovering a preexisting, ideal composition. Note, though, that such phraseologies can be tempting even in the case of music, since we can *hear* an 'unresolved' sequence in the progression of a melody, requiring an 'answer' going one way or the other. The answer, however, is not a piece of knowledge but a *resolution*, a pattern decided upon.

Of course, a musical composition might adhere to certain rules of a genre, just as a mathematician employs a recognizable strategy of proof, but the actual composition, the actual proof, is nevertheless not a *consequence* of the rules of its genre. Even if a given proof exhibits a well-known pattern of inference, the proof is still decided upon as a *new* proof that further exhibits this pattern. Our recognition of the

[214] Cf. Wittgenstein's discussion of 'proof of relevance' and 'proof of verifiability' in *TBT*, pp. 378-379. See also *LFM*, XIV, p. 135: "You might ask what the difference is between proving that $3 \times 0 = 0$ and proving that $126 \times 631 =$ -----. Well, one is taught a technique which applies easily to things which are obviously not exceptions; but 0 is clearly an exception in one way or another."

reproducible patterns in a calculus – the pictures we create with it – serves as a basis for further developments in mathematics. Pictures are modified by way of analogy, by the linking up of techniques, and extended into new regions. This very act of extension is the invention of a new technique to be taken further.

What has also been brought out, however, is that, for all its radicalness in the context of analytic philosophy of mathematics, Wittgenstein's anthropological point of view did not saddle him with an unreasonably arbitrary account of mathematics. Mathematics is bound up with its applications, in practice and in language, and is therefore far from arbitrary with respect to human interests, even though our operations with formulae within calculi must be distinguished from our assertions of empirical propositions. The authority of a proof comes from the practices, the language games, in which its techniques and pictures have their home, and from the role those language games have in our lives. However, this anthropological perspective makes it all the more important to achieve and maintain perspicuity about mathematical methods, since we are left without any guarantee of satisfying the motivations or interests at play in our development of formalism. Neither abstract objects nor ideal rules dictate *for us*, ahead of time, the directions in which the roads of mathematics are to be paved.

6 Conclusion

The goal of this thesis has been to investigate Wittgenstein's later writings on mathematics by focusing on the fact that he saw mathematics as a *human* practice. The overriding claim has been that, by calling mathematics an "anthropological phenomenon", Wittgenstein was not just indicating how mathematics should be approached, as an expression of his overall anthropological methodology in the later period. Rather, he was airing the upshot of a certain view of mathematics. I have argued that, for the later Wittgenstein, in order to understand the phenomenon of mathematics, we must account for it in terms of human practice.

This calls for a *strongly* anthropological reading, which sees in Wittgenstein's writings on calculation, proof, number, etc. not simply evidence of a generic anthropological outlook but, rather, specific ways of understanding these concepts through their role in human life. The suitability and cogency of this idea has been explored and elaborated over the course of the thesis by investigating different topics in Wittgenstein's writings which pertain to mathematics.

I began by outlining some general desiderata of a philosophical account of mathematics that are relevant to the reading under consideration, so as to have a benchmark with which to evaluate its strengths and weaknesses. These desiderata relate to the use and understanding of mathematics on a basic level, as befitting Wittgenstein's focus on practice:

1. *Normativity*. Mathematical practice is normally rule-bound; the (in-)correctness of any act of calculation is predetermined according to how it 'should' go.
2. *Significance.* Mathematical propositions are translingual and transhistorically understandable, and any proof is in principle surveyable and reproducible.
3. *Grounding*. If a calculation or its expression in an equation is correct, it is in principle beyond dispute and incontestable, though it is neither empirically confirmable nor disconfirmable.
4. *Coherence*. Mathematical expressions are systematically relatable and

interchangeable across disparate contexts, provided certain conditions are met.[215]

5. *Applicability*. Mathematical propositions can in many cases be applied in order to form empirical descriptions and predictions.
6. *Constancy*. A mathematical problem, if solvable, is always solvable in the same way.

All of these desiderata are accounted for by the anthropological reading of Wittgenstein on mathematics. Of course, it might be argued that they are cherry-picked to fit this account in particular. However, *any* philosophy of mathematics could be accused of weighing some features over others. Here, the point is not to emphasize these features, nor to suggest that they are jointly constitutive of mathematics. Rather, they are simply acknowledged as aspects of mathematical practice, and the point of recalling the features is not to draw on them as evidence in favor of, or against, Wittgenstein's perspective, but to *elaborate* how that perspective helps illuminate mathematics. The choice of features is made with that aim in mind.

Over the course of the thesis, I began by discussing prominent interpretations of the later Wittgenstein's philosophy of mathematics, his view of mathematics as a human phenomenon in particular. In focus, initially, were the topics of anthropology, conventionalism, and naturalism. I argued that Wittgenstein talked of 'human beings' and 'anthropology' not in a technical, either purely social or purely biological sense, but as covering both characteristics of the human physical constitution and more local characteristics related to cultural upbringing.

For Wittgenstein, there is a sense in which one can call the human being a "ceremonial animal" (2018, p. 42), an animal which, as a matter of fact, attributes cultural significance to behaviors regardless of any theoretical causes or effects of those behaviors. This 'ceremonial' or 'ritual' aspect, which he saw as simultaneously cultural and biological, is a distinctive feature of the forms of life of human beings. Rituals can develop into entire systems of significant action and interaction, and Wittgenstein's writings on language games emphasize irreducibility and contingency which implies

[215] Relatability and interchangeability here mean *grammatical* relatability and interchangeability. If S(ϕ) (a sentence S containing a numerical expression ϕ) makes sense, then S(ψ) makes sense with any ψ satisfying certain conditions. If S(ϕ) makes sense, and T(ψ) makes sense as well as satisfies certain conditions, then T(ϕ) makes sense. If '$a = b$' is a mathematical proposition and S('a') makes sense, then S('b') makes sense.

that he saw them as ritualistic in some respects.

This exegetical discussion turned towards language games and rules, and I rejected the idea that Wittgenstein's remarks on rule-following imply that our use of languages and mathematics is contingent on explicitly recognized rules or decrees. Rather, in most cases when Wittgenstein wrote about "rules", he meant open-ended, iterative actions which come to characterize language games, in this being comparable to turns in turn-based games. As was discussed in Chapter 4, the rules of a given language game may in some cases change while the game remains the same, provided its overall point or nature remains the same. This reading aligns with Jaakko Hintikka's (1989, p. 284, cf. 1996, pp. 209-232) account of the evolution of Wittgenstein's philosophy from 1929 onwards, who infers from the rule-following considerations that language games have conceptual primacy over rules.[216]

"Language game" was at once an analogy and a methodological tool, for Wittgenstein. The literal meaning of the expression is an actual or hypothetical linguistic and/or mathematical *practice*. Even though, strictly speaking, we can distinguish and compare language games in indefinitely many ways, because, in the abstract, "everything is analogous to everything else" (*LFM*, XIV, p. 135), it is useful to stipulate language games of specific sorts, depending on the philosophical problems at hand. Typically, for Wittgenstein, we should consider simple practices in which an expression or a rule is clearly manifested. In the case of mathematics, as Wrigley (1977, p. 59) points out, it can be especially useful to limit ourselves to simple calculi in order to avoid conflating calculation with inessential 'prose'.

However, one might still wonder about the choice to focus on games as an extended analogy, instead of talking about practices. Wittgenstein used the phrase "language game" to make several mutually reinforcing points. Two of these points have been most relevant in the foregoing discussions. Firstly, Wittgenstein emphasized the sense in which an expression gets its meaning *from* a practical surrounding (which is more or less vaguely distinct from other surroundings), similarly to how, say, a piece in chess gets its significance *as* a piece from being employed in the context of a particular

[216] However, this does not imply an agreement with further exegetical claims, such as the idea that the concept of 'language game' had a theoretical function. Cf. Hintikka & Hintikka (1986, p. 189).

chess game, or a dollar gets its value from being embedded in an economy. In other words, for Wittgenstein, the notion of language games actuated a general *context principle* (cf. Frege, 1980, xxii), bringing out the fact that expressions are internally related to linguistic and mathematical practices.

The second reason has already been alluded to, as it relates to rule-following. In essence, the philosophical problem of rule-following can be stated as the question of what it is that renders two instances of human behavior the *same* action. Unless we recognize a 'home' for a given action, a background which it stands out against, effecting a recognizable *kind* of change, the action can be understood in indefinitely many ways, as an instance of indefinitely many different rules. Examples of this phenomenon include forming sequences such as "1, 2, 3, 4" or "1, 2, 4, 8". These sequences, in the abstract, are compatible with *any* continuation.

We can understand our recognition of actions *as* the kinds of action they are by understanding them in the context of distinctions that are meaningful to humans. The two aforementioned sequences have specific continuations, as part of distinct series, due to the fact that we distinguish between a finite number of significant uses of numerals. The expressions "1, 2, 3, 4, etc." and "1, 2, 4, 8, etc." are, in other words, perfectly exact. They are used in mathematics and ordinary language, e.g., to inform someone of a way to count, or to indicate a rate of change. A rule-following paradox arises when such expressions are mistakenly taken to be *inexact* due to their open-ended use, as if they were abbreviations of infinite sequences.

The point generalizes. Expressions are not abbreviations of infinitely complicated uses, a "body of meaning" (cf. *PI* §559) to which they are associated. They are more like pieces in a game which are used in definite *ways*, regardless of whether those uses are open-ended – and so can be iterated to generate indefinitely many results – or closed. Thus, for Wittgenstein, games serve as a model for the *locus* of meaning. We recognize games as institutions. They are taught and explicitly organized, involving discrete and identifiable states, conditions, pieces, roles, and/or rules. We see actions *as* the actions they are because they are like moves in games, changing a situation in a humanly significant way. The appeal to language games can therefore be understood as an elaboration of the idea of the human being as a 'ceremonial animal'.

Chapters 4 and 5 combined the notion of 'deference' with the terminology of formal properties/relations. The deference to mathematics constitutes a formal relation between language games applying mathematics and activities of pure mathematics, with the latter being understood as including the teaching of mathematics all the way down to the elementary school level. The concepts involved here need not be – and with respect to everyday applications of mathematical concepts often *are not* – particularly complex. For example, we can easily recognize when someone is drawing, or attempting to construct, a triangle. By contrast, it is more difficult to recognize someone as cooking a fine meal, or as building an apartment complex; 'triangle' is a simpler concept than 'fine meal' and 'apartment complex'.

The strictness involved in mathematical application, rather than being a matter of any inherent complexity or specificity in the logic of mathematics, is an expectation of deference to a degree that depends on the purpose or use of the mathematics in question. For example, an engineer designing a bridge is typically expected to act with great exactitude when applying mathematics. High standards apply to any triangles constructed as part of this process. By contrast, a child can demonstrate understanding of 'triangle' by drawing a rough shape. We demand greater precision to the extent that we apply mathematics for *serious* purposes.

To apply mathematics, therefore, does not mean to redirect our messy empirical questions to pure mathematics. A calculus gives no answers, only ways of proceeding. Careful application of mathematics involves precise employment of techniques. That being so, Wittgenstein suggests, *any* strictly rule-bound activity would in principle be applicable as mathematics, provided it was given a serious role to play. Some calculi would be more cumbersome than others, and might for that reason be less useful, but that depends on its intended use. For example, Wittgenstein (1979, p. 170) suggested that if, for historical reasons, wars were conducted according to the principles of chess, then generals would be bending over chess boards similarly to how they now consult ordnance maps, in which case chess would genuinely be a form of pure mathematics; it would be a functional calculus, not just a game.

Since calculi are understood as rule-bound activities which reproduce pictures, there are no grounds to fear that a calculus might be in conflict with reality. The way a calculus is applied for empirical purposes depends on judgement, but this is so no less for, say, basic arithmetic than it would be when operating with an inconsistent system. For Wittgenstein, if a system of rules leads to contradiction, it could still be useful, even for serious applications, since further techniques could be adopted to prevent arbitrary inferences from contradictions.

Rather than the traditional worry over contradiction, Wittgenstein worried that 'prose' obscures the significance of a calculus. The notion of prose is linked to a *passive* idea of proofs, an idea that the result of a proof determines what was proven, rather than the actual procedure employed in the proof. For Wittgenstein, a passive understanding of proof was core to an overall mythology of mathematics as abstract theoretical activity. With an overemphasis on the 'upshot' of proofs, mathematics is conceived as either a matter of approximating an ideal (the ethereal machine view) or as seeking to discover and represent properties of abstract objects (the epistemological view). The mythology of mathematics as abstract theoretical activity engenders confusion by suggesting that, in doing mathematics, we are pinning down principles or properties of objects that hold independently of what is actually done with the calculi.

This confusion does not give rise to *incorrect* mathematics, as far as Wittgenstein is concerned, but rather *unmotivated* mathematics. Thinking in terms of abstract theoretical activity, we are liable to develop interests in calculi that depend on an understanding of rules or symbols which bears no relation to what role those rules or symbols have in actual practice. This could be compared to developing a board game for the purpose of playing with pieces that have certain properties, when these properties are not reflected in how the pieces are actually used in the game. The resulting calculus is disconnected from the motivations behind it.

With this conclusion in mind, the six basic features of mathematics that were outlined in the introductory chapter (p. 11) can be approached anew, to see how Wittgenstein accounts for them on the anthropological reading that has been defended:

1. *Normativity*. Calculation takes place in the context of a calculus which consists in the reproduction of pictures, symbolic patterns which are demonstrative of given techniques or procedures in that calculus. Individuals are trained to defer to how things are done in calculi, modifying their behavior to reproduce said

pictures. A given calculation appears to be pre-determined because a deviating result simply does not count as the outcome of a calculation.

2. *Significance.* Counting and basic geometric constructions are easily transmitted, constituting rudimentary techniques. Due to the paradigmatic role of pictures, calculi have a self-reproducing structure. Insofar as calculating techniques survive, the pictures earlier instances left behind remain surveyable. This is reflected by our varying degree of understanding of mathematical artefacts, their role in historical calculi being understood to a varying extent. The trans*lingual* transmission of mathematics can be understood by the fact that numerals and basic shapes are demonstrative of counting and elementary geometric constructions. Techniques involving rudimentary techniques can be incorporated into most language games. Numerals are used precisely to convert from one language game to another through the mediative application of equations. In particular, the Hindu-Arabic system of numerals has diffused internationally, and is used to translate equations expressed with local and/or archaic numerals.
3. *Grounding*. To calculate is not to reason about matters of fact, but to arrive at forms of expression enabling specific moves in language games. A formula of pure mathematics serves as a model for transforming empirical propositions, and that is why there are no grounds for disputing its empirical veracity. It can, however, be disputed *as* a part of a calculus, in which case it contends with other potential formulae which would be given the same role. A formula being 'true' is comparable to e.g. it being 'true' that the letter E follows the letter D in the alphabet, or that the chess-bishop moves diagonally. This should not be thought of as a matter of mere compliance with stipulations. 'Truth' in mathematics means for something to be *done* in a certain way in contrast to others. In its applications, mathematics is systematically coordinated with more or less essential practices, so judgements of mathematical truth and falsity are not a frivolous matter.
4. *Coherence*. Different methods of counting are generally comparable because counting is rudimentary and its results are generally congruous across linguistic contexts. Methods of counting are often numerically comparable, allowing for

applications of equations with multiple units (e.g. 10 cm + 3 feet = 101.44 cm). Mathematical proofs form pictures which guide us in calculating further. We take their result for granted, using them when inferring in other contexts. A result marks a significant practical distinction. It signifies *how* it itself was produced, through a distinctive calculating method. Still, analogous methods may be available in two different calculi, so analogous pictures can be produced in two different mathematical contexts. The verbal expression of a mathematical proof is not a proposition, but a way of placing the result in a system of techniques. To describe a proof as "showing that 25 times 25 equals 625" relates its result to other techniques, such as dividing 625 by 25. This systematization guides us when applying mathematics. However, the verbal expression of a proof can also be misleading, potentially hiding and conflating differences in techniques.

5. *Applicability*. A mathematical formula stands to its applications in the same way a deictic sentence stands to its contextual employment. Formulae belong to calculi which are systematically linked with other language games, equations of elementary arithmetic being linked broadly to language games featuring countable nouns. The general applicability of a calculus reflects general tendencies relevant to its domain of applicability, e.g. that objects do not 'randomly' evaporate or duplicate. In a given situation, we can apply the rules of a calculus in order to formulate a description. A description formed using a calculus can be false, but, whenever this is the case, the calculus either already yields an applicable alternative description, or the calculus is inapplicable as a whole. For example, education in engineering does not teach equations in isolation, but teaches the use of entire calculi as such, that is, ways of calculating so as construct forms of description that are applicable in a given domain.
6. *Constancy*. A mathematical problem, if solvable, remains solvable in the same way. If a given problem varies, its solution is not mathematical. That is because calculation is not experimental; its result *cannot* vary, since the result is part of the calculation. To calculate in a given way requires getting a given result. That also applies to proof, which is a form of calculation that extends a calculus in a certain way. To accept a proof is to accept it as a calculation, in Wittgenstein's

> broad sense of that word, which means that the procedure along with the result is incorporated as part of the calculus. It follows that proofs do not solve empirical problems. When a form of mathematics is used in an empirical theory, it provides us with ways of constructing and connecting empirical propositions which are supported or predicted by that theory. Any falsification of the empirical theory would no more discredit the mathematics than it would the measuring instruments used to set up the experiments; it might at worst weaken our interest in the mathematics (and in the measuring instruments, for that matter).

With this, it is clear that the later Wittgenstein's picture of mathematics is one revolving around practice. However, mathematics is not simply a product of nature, or a direct expression of utility. Mathematics is a part of human forms of life, similarly to language, which means that philosophers have to approach it first and foremost as an anthropological phenomenon. This affirms the hypothesis aired in the introductory chapter, namely that Wittgenstein's descriptions of mathematics as an anthropological phenomenon (*RFM*, VII-33) should be taken in a strong sense. This was not merely an example of his overall anthropological outlook in his later period. Mathematics is a human activity. By contrast, although philosophical problems surrounding the concept of 'color' call for an anthropological methodology insofar as it requires paying attention to the use of color words, that does not make color an anthropological phenomenon as such. Animal species can perceive differences in color, and objects can be said to emit color even if no one sees them. Mathematics *is* specifically anthropological, for Wittgenstein. It is embedded in our language games and in the structure of our forms of life. As such, its existence is contingent on the historical emergence and persistence of certain forms of human practice. Philosophically, therefore, mathematics has to be understood *as a human phenomenon* in its own right, not as a merely human attempt to grasp ethereal rules or abstract objects.

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