PROJECTIVE MODELS OF ALGEBRAIC SURFACES IN SCROLLS

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Preface

During the finishing stages of my thesis I have been thinking a lot on how to write a preface to it. Browsing through other theses I have noted that a typical preface consists of an almost blank page with a few lines where the candidate give thanks to various people. This is of course lame and boring, so I have tried to come up with some better ideas for how to fill this almost compulsory page. I have been reading Alasdair Gray's "Lanark" recently. The biographical notes on the author mentions that he has been editor for "An Anthology of Prefaces". I thought that this would be a perfect book to steal ideas for a preface from. But on closer consideration it occurred to me that this book is probably not available in an abundant number of copies in Bergen and that it would be a waste of time searching for a copy. As a consequence I have no brilliant ideas of what to fill this page with, so I will just stick to the usual list of thank yous.

I would like to thank my thesis advisor, Trygve Johnsen, for his attempt to make a mathematician out of me.

I would like to thank my parents for their support (I hope this page is somewhat intelligible as they will probably understand little of the following).

I would like to thank my fellow students for things of both academical and non-academical nature.

And last but not least a thanks to those who deserve to be thanked but who I have inconsiderately forgotten to mention above.

Happy reading.

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Introduction

Given a polarized surface (S, L)[1] with $L^2 > 0$, we have a morphism

$$\phi_L : S \longrightarrow \mathbb{P}(H^0(S, L)) \cong \mathbb{P}^{h^0(L)-1}.$$

The image of ϕ_L is a projective model of the surface. In this thesis we will study (S, L) and its projective model. There are a lot of questions one can pose: when is ϕ_L birational? what is contracted by ϕ_L? is the projective model contained in any "nice" varieties? etc. We will now look at how these questions are related to my thesis.

We will first look at when ϕ_L is birational. Our discussion will motivate the Clifford index for K3 surfaces and the $\phi(L)$ function for Enriques surfaces.

For K3 surfaces Saint–Donat [SD74] showed that ϕ_L is birational if and only if (the general section of) L is not hyperelliptic. Hence we would like to know when L is hyperelliptic. A curve C is hyperelliptic if and only if it has Clifford index zero (see definition 1.3.2 and [Har77, theorem IV.5.4]). Thus we would like to be able to extend the concept of Clifford index to K3 surfaces. Green and Lazarsfeld [GL87] showed that this can be done (see theorem 1.3.6). For a reformulation of Saint–Donat's result using the Clifford index see theorem 1.3.17. In section 1.3 we will study the Clifford index of a K3 surface. If the Clifford index of L is $\lfloor (g-1)/2 \rfloor$ (where g is the genus of L), we say that L is Clifford general.[2] If L is not Clifford general, then the Clifford index will in a natural way give a (not necessarily unique) decomposition $L \sim D + F$ where D is a divisor computing the Clifford index of L. Most of part I will consist of studying this decomposition. The divisor D can be chosen to have certain "nice" properties and we will call it a Clifford divisor.

The non-Clifford general K3 surfaces are easier to study than the Clifford general K3 surfaces because of the decomposition we get when L is not Clifford general. The notion of BN (Brill–Noether) generality makes it easier to also study the Clifford general case. BN generality has been studied by Mukai [Muk95] and he has been able to get good results for BN general K3 surfaces of low genera. We will see that BN generality implies Clifford generality. Theorem 1.4.10 will however show that there are lots of Clifford

[1] See "Conventions and Notations" for the definition of this and some of the other concepts we use in the introduction.

[2] The name is somewhat misleading; there are more polarized K3 surfaces that are non–Clifford general than Clifford general. In fact for a given genus g the general polarized K3 surface is Clifford general if and only if $g - 1$ is square-free. (See propositions 1.3.9 and 1.3.18.)

general non-BN general K3 surfaces, so the situation is far from reduced to studying the BN general K3 surfaces.[3]

Cossec [Cos83] determined when ϕ_L is birational for an Enriques surface S. In [Cos85] he introduced the $\phi(L)$ function which simplifies his results. In particular ϕ_L will be birational if $\phi(L) \geq 3$ (proposition 4.2.2). We will study the $\phi(L)$ function in section 4.2. We will especially look at possible pairs $(C^2, \phi(C))$ with C an irreducible curve and give existence results when $S$is an unnodal Enriques surface.

We will now look at the curves contracted by ϕ_L. These were studied extensively by Artin [Art62].[4] Let $\mathcal{E}$ be the set of irreducible curves Γ such that $L.\Gamma = 0$. This is the set of curves contracted by ϕ_L. If S is a K3 surface, then the Hodge index theorem gives that every Γ is a smooth rational curve satisfying $\Gamma^2 = -2$. Take a component of the configuration with vertices in $\mathcal{E}$. Using the Hodge index theorem again we see that the intersection matrix of the elements of the component is negative–definite. Thus the component is isomorphic to one of the graphs (A_n), (D_n), (E_6), (E_7), and (E_8) in the classification of semi–simple Lie algebras. (See [SD74, (4.2)] and [Băd01, theorem 3.32] for more details.) In section 1.5 we will classify the components of a subset of $\mathcal{E}$ which is obtained in a natural way from Clifford divisors, more precisely we will classify the components of Δ' in well-behaved pairs (A', Δ') (such pairs will be defined in definition 1.5.3). In table 2.1 we will give the components of $\mathcal{E}$ for the most general non–BN polarized K3 surfaces of genus 12.

We now move on to the third question: is the projective model contained in any "nice" varieties? In this thesis the "nice" variety (except from the obvious $\mathbb{P}^{h^0(L)-1}$) will be a rational normal scroll. An introduction to these are given in section 1.2. A pencil

$$\{D_\lambda\}_{\lambda \in \mathbb{P}^1} \subseteq |D|$$

on S with $h^0(L - D) \geq 2$ gives in a natural way a rational normal scroll containing $\phi_L(S)$. We will use this procedure to get scrolls containing the projective models of K3 surfaces, Enriques surfaces, and Del Pezzo surfaces.

For K3 surfaces these scrolls will be associated to Clifford divisors. In section 1.3 we will see how we can get a pencil contained in $|D|$, where D is a Clifford divisor. This will give a scroll containing $\phi_L(S)$. We will get scrolls in this way as long as the Clifford index is non–zero and L is not Clifford general. A large part of chapter 2 will consist of describing these scrolls when L has genus 12.

On Enriques surfaces the $\phi(L)$ function immediately gives elliptic pencils $|P|$. We can use these pencils to get scrolls when $h^0(L - P) \geq 2$ (see p. 144). These scrolls will be studied in section 4.3.

Also on some polarized Del Pezzo surfaces we find pencils that give scrolls. The pencils will be given by case (a) of proposition 3.2.6. The scrolls will be studied in section 3.3.

[3]Given a fixed genus g our results will imply that i) the moduli space of BN general polarized K3 surfaces of genus g has dimension 19, ii) the moduli space of Clifford general non-BN general polarized K3 surfaces of genus g has dimension 18, and iii) the moduli space of non-Clifford general polarized K3 surfaces of genus g has dimension 18 if $g - 1$is square-free and 19 otherwise.

[4]See also [Băd01, chapter 3] for a more leisurely introduction.

A different way to study the map ϕ_L is by using higher order embeddings. This gives rise to concepts such as k-very ampleness. Geometrically the projective model has no $(k+1)$-secant $(k-1)$-plane $\mathbb{P}^{k-1} \subset \mathbb{P}^{h^0(L)-1}$ if L is k-very ample. We will give an introduction to k-very ampleness and related concepts in section 3.2. We will study the concepts more closely on Del Pezzo surfaces.

This introduction is intended to be a motivation for the rest of the thesis. For a more detailed discussion of the contents and results of this thesis see the introduction to each chapter.

Conventions and Notations

We work over the ground field $\mathbb{C}$. A *surface* is always a reduced and irreducible smooth projective algebraic surface. A *curve* is always reduced and irreducible.[5] A curve on a surface will then necessarily be a prime divisor. A *polarized surface* (S, L) is a surface with a base point free line bundle L.[6] We write $g(L)$ for the arithmetic genus of L. The *genus* of a polarized surface (S, L) is the genus $g(L)$.

Line bundles and divisors are used with almost no distinction.

We will usually write $H^i(L)$ for $H^i(S, L)$, where L is a line bundle on the surface S. $h^i(L)$ is the dimension of $H^i(L)$. Given two divisors A and B we write $A \geq B$ when $|A - B| \neq \emptyset$, i.e. $h^0(A - B) > 0$. Similarly we write $A > B$ when $h^0(A - B) > 0$ and $A \nsim B$.

A *configuration* is a graph where the vertices corresponds to divisors and where the number of edges between two distinct vertices is the intersection number of the corresponding divisors. For all the sets of divisors that we will give the configurations of the intersection numbers will be non-negative so this is well-defined. Note that a configuration says nothing about self-intersections. We say that the configuration of the divisor $m_1A_1 + \cdots + m_rA_r$ is the configuration with vertices $\{A_i\}$.

Given a divisor $m_1A_1 + \cdots + m_rA_r + n_1B_1 + \cdots + n_sB_s$, where $m_i, n_i \in \mathbb{Z}$, $h^0(A_i) > 1$, and $h^0(B_j) = 1$, we can write $B_j \sim k_{1,j}C_{1,j} + \cdots + k_{t,j}C_{t,j}$ uniquely as a sum of prime divisors. Then the *configuration-graph* of $m_1A_1 + \cdots + m_rA_r + n_1B_1 + \cdots + n_sB_s$ is the configuration of $m_1A_1 + \cdots + m_rA_r + \sum_1^s(k_{i,j}C_{1,j} + \cdots + k_{t,j}C_{t,j})$.

An example will clarify these concepts. Take a divisor $A + B + C$ where $A.B = 1$, $A.C = 1$, $B.C = 1$, $C^2 = -2$, $h^0(A) > 1$, $h^0(B) = h^0(C) = 1$, and C is a prime divisor. Assume that $B = C + D$ is a prime decomposition. Then the configuration of $A + B + C$ is

$$\begin{tikzcd} A \arrow[rr, dash] \arrow[dr, dash] & & C \arrow[dl, dash] \\ & B & \end{tikzcd}$$

while the configuration-graph of $A + B + C$ is

$$A \text{ --- } C \equiv D$$

[5]Sometimes we will write irreducible curve instead of just curve to emphasize that it is irreducible.

[6]Note that this differs somewhat from the "usual" definition of a polarized surface. A polarized surface (S, L) is usually defined to be a surface with an ample line bundle L.

Part I

K3 Surfaces

Chapter 1

Some Results on K3 Surfaces

This chapter is about K3 surfaces. We will develop further the theory of Clifford divisors in [JK01]. In the next chapter we will use our results to solve a specific problem: classifying projective models of polarized K3 surfaces of genus 12 in scrolls.

The first three sections include mostly well-known material. Section 1.1 gives an overview of the standard material on K3 surfaces which we will be using later on. The section also includes definitions and results that holds for all surfaces. Several of these results will be used in part II also. Almost all the material in this section is standard and taken from other sources. Lemma 1.1.14 and proposition 1.1.22 are the only results here that I have not seen explicitely stated in the literature.

Section 1.2 is an introduction to rational normal scrolls.

In section 1.3 we introduce the Clifford index of a polarized K3 surface (S, L). This leads naturally to the concept of Clifford divisors. The linear systems of these divisors contains pencils that will give scrolls containing the image of S in $\mathbb{P}^g$ by the natural morphism given by the complete linear system $|L|$. It is these scrolls we will classify for $g = 12$ in chapter 2. We will see that one can choose a Clifford divisor to have certain nice properties. Most of the material in this section is taken from [JK01] though it is a bit rearranged. The most notable new material in this section are propositions 1.3.9 and 1.3.18 with its surrounding material.

In section 1.4 we look at the relationship between Clifford generality and BN (Brill-Noether) generality. This section is in some ways an extension of the ideas in [JK01, section 10]. We will show that BN generality implies Clifford generality. We will also give conditions for a K3 surface to be Clifford general but non-BN general. We will use these conditions to find possible intersection numbers in Clifford general non-BN general K3 surfaces for $g \leq 13$. The main result of this section is theorem 1.4.10, where we show that for $g = 8$ and $g \geq 10$ there exists K3 surfaces that are Clifford general but non-BN general. A good reason for studying the relationship between Clifford and BN generality is the results of Mukai [Muk95]. He finds the projective model of BN general K3 surfaces (S, L) with L ample for $g = 2, \ldots, 10$, and 12. We end the section with looking shortly at how BN generality of K3 surfaces relates to BN theory of curves.

Section 1.5 studies the base point divisor Δ of $L - D$ where L is base point free and

D is a Clifford divisor for L. We introduce the concept of a well-behaved pair (A', Δ') of divisors. The divisor Δ' will have many of the same properties as Δ but will be easier to work with. We show existence of well-behaved pairs for all L and D (proposition 1.5.8) and classify the components of Δ' (theorem 1.5.10).

In section 1.6 we study the relationship between the Clifford divisors of L, $L - D$, and $L + D$. This also gives information on the scroll types associated to L, $L - D$, and $L + D$. The primary motivation of this section was to be able to use some of the results found in [JK01] for $g = 8$ and $g = 9$ to get results for $g = 12$ when $c = 1$ and $c = 2$. Considering the use of this section in the next chapter one sees that this goal has to some extent been fulfilled. But the results of this section are also interesting in themselves and are for large genus much more interesting than in the $g = 12$ case in which we will be using them.[1]

1.1 Preliminaries

We start with the definition of a K3 surface.

Definition 1.1.1. A *K3 surface S* is a smooth regular surface with a trivial canonical bundle, i.e. $h^1(S, \mathcal{O}_S) = 0$ and $K_S = 0$.

The most usual examples of K3 surfaces are Kummer surfaces and the complete intersections (4), $(2, 3)$, and $(2, 2, 2)$. See [BPvdV84, sections V.2, V.16, and V.22] for these and other examples.

K3 surfaces has a natural placing in the classification of surfaces. The *Kodaira dimension* $\kappa(S)$ of a surface is the transcendence degree over $\mathbb{C}$ of the ring

$$R = \bigoplus_{n \geq 0} H^0(S, nK_S).$$

This is the definition given in [Har77, p.421]. See [Băd01, definition 5.6] for an alternative definition. One sees that a K3 surface has Kodaira dimension 0. We know have the following important classification theorem.

Theorem 1.1.2. *(Enriques, Kodaira) [Har77, theorem 6.3], [GH94, p.590] A surface S with $\kappa(S) = 0$ is either*

- *a K3 surface,*
- *an Enriques surface,*
- *an abelian surface,*
- *or a hyperelliptic surface.*

[1] See the last paragraph of section 2.11 for more details.

See the references for definitions of the other types of surfaces (for Enriques surfaces see definition 4.1.1).

Setting $K = 0$ in the Riemann-Roch formula and using Serre duality we get (K3 surfaces have $p_g = 1$)

$$h^0(D) + h^0(-D) = \frac{1}{2}D^2 + 2 + h^1(D) \tag{1.1}$$

for a divisor on a K3 surface. If $D^2 \geq -2$ then the right hand side is larger than or equal to 1. Hence either $|D|$ or $|-D|$ contains an effective member.

For a general surface S we have several equivalence relations between divisors: *linear equivalence* ($D \sim D'$), *algebraical equivalence*, and *numerical equivalence* ($D \equiv D'$). Modulo the equivalence class containing 0 we get the groups $\operatorname{Cl} S$, $\operatorname{NS} S$, and $\operatorname{Num} S$ respectively (see [Har77, V.1] or [Băd01, chapter 4]). Note that, with our definition of a surface, we have $\operatorname{Cl} S \cong \operatorname{Pic} S$.

In general we have that linearly equivalent divisors are algebraically equivalent, and that algebraically equivalent divisors are numerically equivalent (see [Har77, exe.V.1.7]). For K3 surfaces the converse holds.

Proposition 1.1.3. *Let S be a K3 surface. Then two divisors are linearly equivalent if and only if they are algebraically equivalent if and only if they are numerically equivalent. In particular* $\operatorname{Pic} S = \operatorname{NS} S = \operatorname{Num} S$. *Hence* $\operatorname{Pic} S$ *is free $\mathbb{Z}$-module of finite rank.*

Proof. See [SD74, (2.3)] or [Băd01, theorem 10.3]. We have to show that a divisor D numerically equivalent to 0 is linearly equivalent to 0. Assume that $D \nsim 0$. If $D \equiv 0$, then we have in particular that $D^2 = 0$ so $|D|$ or $|-D|$ have to contain an effective divisor, which must be non zero since $D \nsim 0$. Note that if $D \equiv 0$, then $-D \equiv 0$. Hence we have an effective non zero divisor which is numerically equivalent to zero, which is impossible since S is projective.

The last statement follows from the Néron-Severi theorem (which says that $\operatorname{NS} S$ is a finitely generated abelian group) and the fact that $\operatorname{Num} S$ has no torsion. □

For a K3 surface the adjunction formula is particularly simple.

Proposition 1.1.4. *(Adjunction formula) Let C be an effective divisor of arithmetic genus g on a K3 surface, then*

$$D^2 = 2(g-1)$$

Proof. This is just [Har77, exe.V.1.2] with $K = 0$. □

We see that an irreducible curve Γ has negative self-intersection if and only if $\Gamma^2 = -2$ and $g(\Gamma) = 0$. Such a curve is a *smooth rational curve*. By lemma 1.1.16 we also have $h^1(\Gamma) = 0$. Riemann-Roch then gives $h^0(\Gamma) = 1$.

The Hodge Index Theorem is a useful result on surfaces that we will be needing. A divisor D is *big* if $D^2 > 0$.

Proposition 1.1.5. *(Hodge Index Theorem) [Băd01, corollary 2.3] Let H be a big divisor and let D be any divisor on a surface S. with $D.H = 0$. Then $D^2 \leq 0$, with equality if and only if $D \equiv 0$.*

Corollary 1.1.6. *Let H be a big divisor on a surface S, and let D be a divisor. Then*

$$D^2H^2 \leq (D.H)^2,$$

with equality if and only if $(D.H)H \equiv H^2D$.

Proof. Let $E = (D.H)H - H^2D$. Then $E.H = 0$ so by the Hodge index theorem $E^2 = H^2(D^2H^2 - (D.H)^2) \leq 0$, with equality if and only if $E \equiv 0$. □

Remark 1.1.7. By proposition 1.1.3 we can for a K3 surface substitute $\equiv$ with $\sim$ in the last two results. We will later on usually refer to corollary 1.1.6 as the Hodge index theorem also.

A point $P \in S$ is a *base point* of the linear system δ if P is in the union of the prime divisors of D for all $D \in \delta$. We say the a divisor D is *base point free* if $|D|$ has no base points. Note that D is base point free if and only if $\mathcal{O}_S(D)$ is generated by its global sections. A divisor λ is called a *fixed component* of δ if $D - \lambda \geq 0$ for all $D \in \delta$. The union of the fixed components of δ is the *fixed part* of δ. Note that if Δ is the fixed part of a complete linear system then $h^0(\Delta) = 1$.

A divisor D is *numerically effective* (*nef*)[2] if D is effective and $D.E \geq 0$ for all effective divisors E. (Equivalently: D is nef if $D.C \geq 0$ for all curves C on S.) To know that a divisor is nef is useful in many situations. We will now give some conditions for a divisor to be nef.

Proposition 1.1.8. *Let D be an effective divisor on a surface S. Then D is nef if and only if $D.E \geq 0$ for every fixed irreducible component of $|D|$.*

Proof. See [Knu98, proposition 2.17] The intersection number is non-negative between effective divisors if all intersections are transversal [Har77, V, 1.4]. Hence a negative intersection number arises from a common fixed component. □

Corollary 1.1.9. *Let D be an effective divisor on a K3 surface. If D is not nef, then it contains an irreducible curve Γ with $\Gamma^2 = -2$ and $\Gamma.D < 0$.*

Proof. See [Knu98, proposition 2.18] The only way an irreducible component Γ of D gives rise to a negative intersection number $\Gamma.D$ is if it has negative self-intersection. □

Note also that an effective base point free divisor is without fixed components, so it is nef by this corlllary.

Proposition 1.1.10. *[SD74, corollary 3.2] Let $|D|$ be a complete linear system on a K3 surface. Then $|D|$ has no base points outside its fixed components.*

[2] or *numerically eventually free* or something else. There does not seem to be consensus in what the abbreviation nef stands for. Since one normally only uses the abbreviation this should cause no problems.

Proposition 1.1.11. *[SD74, proposition 2.6] Let D be an effective base point free divisor on a K3 surface. Then either*

1. *$D^2 > 0$. Then the generic member of $|D|$ is an irreducible curve of genus $\frac{1}{2}D^2 + 1$. Furthermore $h^1(D) = 0$.*
2. *$D^2 = 0$. Then $D \sim kE$, where $k \geq 1$ is an integer and E is an irreducible curve of genus 1. Furthermore $h^1(D) = k - 1$ and every member of $|D|$ can be written as a sum $E_1 + \cdots + E_k$, where $E_i \in |E|$ for all i.*

We will need to know when a divisor is base point free.

Proposition 1.1.12. *[SD74, subsection 2.7] Let D be a nef divisor on a K3 surface S. Then D is* not *base point free if and only if there exist smooth irreducible curves E and Γ and an integer $k \geq 2$ such that*

$$D \sim kE + \Gamma, \quad E^2 = 0, \quad \Gamma^2 = -2, \quad E.\Gamma = 1.$$

In this case, every member of $|D|$ is of the form $E_1 + \cdots + E_k + \Gamma$, where $E_i \in |E|$ for all i.

Corollary 1.1.13. *If a nef divisor D on a K3 surface is not base point free, then D is big and there exists a curve E such that $E^2 = 0$ and $E.D = 1$.*

If $D^2 \geq 5$ then this corollary is just a special case of Reider's theorem (see [Laz97, theorem 2.1]).

We will now state some results about fixed divisors on K3 surfaces. A graph is a *forest* if it contains no cycles or multiple edges. The next lemma will be very useful to us later on.

Lemma 1.1.14. *Let D be an nonzero effective divisor with $h^0(D) = 1$. Then there exists smooth rational curves $\Gamma_1, \ldots, \Gamma_N$ such that $D = n_1\Gamma_1 + \cdots + n_N\Gamma_N$, where n_i is a positive integer for every i.*

Furthermore the configuration-graph of D is a forest.

Proof. D can be written as a sum of irreducible curves. If D could not be written as in the lemma, then there exists an irreducible curve $0 < C \leq D$ such that $C^2 \geq 0$. But then $h^0(D) \geq h^0(C) \geq 2$, a contradiction.

Suppose the configuration-graph contains a multiple edge. Then $\Gamma_i.\Gamma_j > 1$ for some pair (i, j) $(i \neq j)$. This gives

$$h^0(D) \geq h^0(\Gamma_i + \Gamma_j) \geq (\Gamma_i + \Gamma_j)^2 + 2 \geq 2,$$

a contradiction.

Suppose the configuration-graph contains a cycle. Reordering the vertices if necessary we may assume that it looks as follows:

$$\begin{tikzcd}
\Gamma_1 \arrow[r, dash] \arrow[d, dash] & \Gamma_N \arrow[r, dash] & \cdots \\
\Gamma_2 \arrow[r, dash] & \Gamma_3 \arrow[ur, dash] &
\end{tikzcd}$$

Using Riemann-Roch this gives

$$h^0(D) \geq h^0(\Gamma_1 + \cdots + \Gamma_N) \geq (\Gamma_1 + \cdots + \Gamma_N)^2 + 2 \geq 2,$$

a contradiction. □

Corollary 1.1.15. *Let D be an effective divisor on a K3 surface. The fixed part of $|D|$ can be written (uniquely) as a sum of smooth rational curves.*

Proof. We have already noted that the fixed part Δ of $|D|$ satisfies $h^0(\Delta) = 1$. Hence the proposition follows from the lemma. □

We are in some cases able to say when a smooth rational curve is a fixed component of a complete linear system. We will need a lemma.

Lemma 1.1.16. *[SD74, lemma 2.2] Let D be an effective divisor on a K3 surface, then*

$$h^1(D) = h^0(D, \mathcal{O}_D) - 1$$

Proof. From the exact sequence

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0$$

we get the long exact sequence

$$\begin{aligned} 0 \to H^0(S, \mathcal{O}_S(-D)) \to H^0(S, \mathcal{O}_S) \to H^0(D, \mathcal{O}_D) \to \\ H^1(S, \mathcal{O}_S(-D)) \to H^1(S, \mathcal{O}_S) = 0. \end{aligned}$$

We et the stated result using the additivity of long exact sequences, since $h^0(-D) = 0$ (D effective), $h^0(S, \mathcal{O}_S) = 1$, $h^1(-D) = h^1(D)$ (Serre duality), and $h^1(S, \mathcal{O}_S) = 0$ (definition of K3 surface). □

Proposition 1.1.17. *[SD74, remark 2.7.3] Let D be a big and effective base point free divisor and Γ be a smooth rational curve on a K3 surface. Then Γ is fixed in $|D + \Gamma|$ if and only if $\Gamma.D = 0$ or $\Gamma.D = 1$.*

Proof. By proposition 1.1.11 we have two cases to consider.

i) D an irreducible big curve. Then $h^1(D) = 0$. If $\Gamma.D = 0$, then lemma 1.1.16 gives $h^1(D+\Gamma) = 1$, so $h^0(D) = h^0(D+\Gamma)$ by Riemann-Roch. If $\Gamma.D = 1$, then lemma 1.1.16 gives $h^1(D + \Gamma) = 0$, so $h^0(D) = h^0(D + \Gamma)$ Riemann-Roch. If $\Gamma.D > 1$, then

$$h^0(D) = \frac{1}{2}D^2 + 2 < \frac{1}{2}(D + \Gamma)^2 + 2 \leq h^0(D + \Gamma).$$

ii) $D \sim kE$ where E is an elliptic curve. Then $h^1(D) = k - 1$. If $\Gamma.D = 0$, then lemma 1.1.16 gives $h^1(D + \Gamma) = k$, so $h^0(D) = h^0(D + \Gamma)$ by Riemann-Roch. $\Gamma.D \geq 1$ is ad verbatim as above. □

The concept of numerical connectedness will be of some importance to us.

Definition 1.1.18. Let D be an effective divisor on a surface. We say that D is *numerically m-connected* if for every decomposition $D \sim D_1 + D_2$ of D into a sum of two effective non-zero divisors, we have

$$D_1.D_2 \geq m.$$

Proposition 1.1.19. *(Ramanujam's lemma) [Rei97, lemma 3.11] If D is a big and nef divisor, then D is numerically 1-connected.*

If D is numerically 1-connected, then $h^0(\mathcal{O}_D) = 1$.

In particular if S is a K3 surface and D a numerically 1-connected divisor on S, then $h^1(D) = 0$.

Proof. The first two statements are proven in [Rei97, lemma 3.11]. The last statement follows from lemma 1.1.16 and the second statement. □

Many of the divisors we will be working with will be numerically 2-connected as the next result shows.

Proposition 1.1.20. *[SD74, lemma 3.7] Let C be an irreducible curve on a K3 surface such that $C^2 > 0$. Then C is numerically 2-connected.*

We will need some lattice theory.

Definition 1.1.21. A *lattice* is a free $\mathbb{Z}$-module of finite rank with a $\mathbb{Z}$-valued symmetric bilinear form $b(x, y)$. A lattice is *even* if the associated quadratic form $b(x, x)$ takes on only even values. The *discriminant* of a lattice is the determinant of the matrix of its bilinear form. A lattice is *non-degenerate* if the discriminant is non-zero. If L is a non-degenerate lattice, the *signature* of L is a pair $(s_{(+)}, s_{(-)})$, where $s_{(+)}$, resp. $s_{(-)}$, is the number of positive, resp. negative, eigenvalues of the quadratic form on $L \otimes_{\mathbb{Z}} \mathbb{R}$. A lattice is *unimodular* if the discriminant is ± 1.

Note that both the discriminant and signature is independent on the choice of basis for L.

Every surface S has a lattice associated to it. Just take $\operatorname{Num} S$ as the module, and let the intersection pairing give the symmetric bilinear form. If $d_i, i \in I$, with D_i an element of the numerical equivalence class d_i, is a basis for $\operatorname{Num} S$ then $(D_i.D_j)_{i,j \in I}$ is the matrix of the bilinear form. We call this lattice the *Picard lattice*. The Hodge index theorem says that this lattice has signature $(1, \rho(S) - 1)$, where $\rho(S) := \operatorname{rank} \operatorname{NS} S$ is the *Picard number* (see [Bǎd01, proof of corollary 2.4]).

In the next chapter we will consider many lattices which we among other things have to find the signature of. We will in all of the cases just state the signature without including the computation. If one wants to compute the signature one can of course do this by computing all the eigenvalues (i.e. let Maple compute all eigenvalues) and then count their signs. A smarter way is to use the Descartes rule of signs. See [CLO98, proposition 5.4] for details.

We need a way to decide if certain combinations of elements are possible in a lattice.

Proposition 1.1.22. *Let L be the lattice $\mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_n$. If $b_1, \ldots, b_n$ are elements in L, then the determinant of the matrix given by the bilinear form on $b_1, \ldots, b_n$ is divisible by the determinant of the matrix given by the bilinear form on $a_1, \ldots, a_n$.*

Proof. The determinant of the matrix given by the bilinear form on $a_1, \ldots, a_n$ looks as follows

$$\begin{vmatrix} b(a_1,a_1) & \ldots & b(a_1,a_n) \\ \multicolumn{3}{c}{\dotfill} \\ b(a_n,a_1) & \ldots & b(a_n,a_n) \end{vmatrix}$$

To prove the proposition is enough to show that it is true when we replace a_1 with ca_1 ($c \in \mathbb{Z}$) and with $a_1 + a_2$, since the determinant is unchanged (up to sign) by permuting rows and columns.

If we replace a_1 with ca_1 ($c \in \mathbb{Z}$), then we get

$$\begin{aligned}
\begin{vmatrix} b(ca_1,ca_1) & \ldots & b(ca_1,a_n) \\ \multicolumn{3}{c}{\dotfill} \\ b(ca_n,a_1) & \ldots & b(a_n,a_n) \end{vmatrix} &= \begin{vmatrix} c^2b(a_1,a_1) & \ldots & cb(a_1,a_n) \\ \multicolumn{3}{c}{\dotfill} \\ cb(a_n,a_1) & \ldots & b(a_n,a_n) \end{vmatrix} \\
&= c\begin{vmatrix} cb(a_1,a_1) & \ldots & b(a_1,a_n) \\ \multicolumn{3}{c}{\dotfill} \\ cb(a_n,a_1) & \ldots & b(a_n,a_n) \end{vmatrix} \\
&= c^2\begin{vmatrix} b(a_1,a_1) & \ldots & b(a_1,a_n) \\ \multicolumn{3}{c}{\dotfill} \\ b(a_n,a_1) & \ldots & b(a_n,a_n) \end{vmatrix}
\end{aligned}$$

If we replace a_1 with $a_1 + a_2$, then we get

$$\begin{vmatrix} b(a_1+a_2,a_1+a_2) & b(a_1+a_2,a_2) & \ldots & b(a_1+a_2,a_n) \\ b(a_2,a_1+a_2) & b(a_2,a_2) & \multicolumn{2}{c}{\dotfill} \\ \multicolumn{4}{c}{\dotfill} \\ b(a_n,a_1+a_2) & b(a_n,a_2) & \ldots & b(a_n,a_n) \end{vmatrix}$$

$$= \begin{vmatrix} b(a_1,a_1)+2b(a_1,a_2)+b(a_2,a_2) & b(a_1,a_2)+b(a_2,a_2) & \ldots & b(a_1,a_n)+b(a_2,a_n) \\ b(a_2,a_1)+b(a_2,a_2) & b(a_2,a_2) & \multicolumn{2}{c}{\dotfill} \\ \multicolumn{4}{c}{\dotfill} \\ b(a_n,a_1)+b(a_n,a_2) & b(a_n,a_2) & \ldots & b(a_n,a_n) \end{vmatrix}$$

$$= \begin{vmatrix} b(a_1,a_1)+b(a_1,a_2) & b(a_1,a_2)+b(a_2,a_2) & \ldots & b(a_1,a_n)+b(a_2,a_n) \\ b(a_2,a_1) & b(a_2,a_2) & \multicolumn{2}{c}{\dotfill} \\ \multicolumn{4}{c}{\dotfill} \\ b(a_n,a_1) & b(a_n,a_2) & \ldots & b(a_n,a_n) \end{vmatrix}$$

$$= \begin{vmatrix} b(a_1,a_1) & b(a_1,a_2) & \ldots & b(a_1,a_n) \\ b(a_2,a_1) & b(a_2,a_2) & \multicolumn{2}{c}{\dotfill} \\ \multicolumn{4}{c}{\dotfill} \\ b(a_n,a_1) & b(a_n,a_2) & \ldots & b(a_n,a_n) \end{vmatrix}$$

(where we have first subtracted the second column from the first and then subtracted the second row from the first). ☐

For a Kähler surface $H^2(S,\mathbb{Z})$ is a lattice, with signature $(2h^{2,0}+1, h^{1,1}-1)$. A K3 surface is a Kähler surface with $h^{2,0}=1$ and $h^{1,1}=20$ (see [BPvdV84, chapter VIII]), so $H^2(S,\mathbb{Z})$ is a lattice with signature $(3,19)$. The Hodge decomposition gives

$$H^2(S,\mathbb{C}) \cong H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S).$$

The Hodge index theorem says that the signature of the form on $H^{1,1}(S,\mathbb{R})$ is $(1, h^{1,1}-1) = (1,19)$.

$\mathrm{NS}\,S$ has a natural embedding in $H^2(S,\mathbb{Z})$, and $\mathrm{NS}\,S$ can be identified with $H^2(S,\mathbb{Z}) \cap H^{1,1}(S)$. If S is a K3 surface then $H^2(S,\mathbb{Z})$ has no torsion and is an even lattice, so by [Mor84, 1.3] $H^2(S,\mathbb{Z})$ is isometric the *K3 lattice* $\Lambda = U^3 \oplus E_8(-1)^2$, where U is the lattice whose bilinear form has matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and E_8 is the lattice whose bilinear form has matrix

$$\begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & -1 & & & \\ & & -1 & 2 & 0 & & & \\ & & -1 & 0 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}.$$

Note that Λ is unimodular.

An embedding $M \hookrightarrow L$ of lattices is *primitive* if L/M free. We have the following result.

Proposition 1.1.23. *[Mor84, corollary 1.9] Let Λ be the K3 lattice. Suppose $L \hookrightarrow \Lambda$ is a primitive sublattice of signature $(1, \rho-1)$. Then there exists a K3 surface S and an isometry NS $S \cong L$.*

For $\rho \leq 11$ we have the following stronger result.

Proposition 1.1.24. *[Mor84, corollary 2.9] Let $\rho \leq 11$ and L be a lattice. Then there exists a K3 surface with* $\mathrm{Pic}\, S \cong L$ *if and only if L is an even lattice of signature $(1, \rho-1)$.*

Remark 1.1.25. [Mor84, corollary 2.9] includes only the case $\rho \leq 10$, but we will not need the uniqueness of the primitive embeddings. Hence we can include $\rho = 11$ by [Mor84, remark 2.11].

Proof. The if part is [Mor84, corollary 2.9]. The only if part follows from Hodge index theorem and the adjunction formula. ☐

Let $\Delta := \{\Gamma \in \operatorname{Pic} S | \Gamma^2 = -2\}$ and consider the *Picard-Lefschetz reflection*

$$\begin{array}{rccc} \phi_\Gamma : & \operatorname{Pic} S & \longrightarrow & \operatorname{Pic} S \\ & D & \longmapsto & D + (D.\Gamma)\Gamma \end{array}$$

We see easily that $\phi_\Gamma \circ \phi_\Gamma = \mathrm{id}_{\operatorname{Pic} S}$ so ϕ_Γ is a reflection. Furthermore ϕ_Γ leaves the intersection between divisors invariant. Note that a reflection maps a basis for $\operatorname{Pic} S$ into another basis for $\operatorname{Pic} S$. Let

$$\mathcal{C}_S = \{D \in \operatorname{Pic} S | D \text{ effective and } D^2 > 0\}$$

be the *positive cone* of S and

$$\mathcal{C}_S^+ = \{D \in \mathcal{C}_S | \Gamma.D > 0 \text{ for all } \Gamma \in \Delta\}$$

be the *Kähler cone*. Its closure

$$\overline{\mathcal{C}_S^+} = \{D \in \mathcal{C}_S | \Gamma.D \geq 0 \text{ for all } \Gamma \in \Delta\}$$

is the *big-and-nef cone*. It consists of every big and nef divisor by [BPvdV84, corollary 3.8].

[BPvdV84, proposition VIII.3.9] says that the set $\{\phi_\Gamma\}_{\Gamma\in\Delta}$ leave $\mathcal{C}_S$ invariant and any orbit in $\mathcal{C}_S$ of the group generated by $\{\phi_\Gamma\}_{\Gamma\in\Delta}$ meets $\overline{\mathcal{C}_S^+}$ in exactly one point.

We will now show that given a Picard lattice of a K3 surface we, using this result, can assume that a chosen big divisor in this lattice is nef. Given a big divisor $D \in \operatorname{Pic} S$, we know that either $|D|$ or $|-D|$ contains an effective member. After using, if necessary, the reflection

$$\begin{array}{rccc} \phi_- : & \operatorname{Pic} S & \longrightarrow & \operatorname{Pic} S \\ & D & \longmapsto & -D \end{array}$$

we may assume that $D \in \mathcal{C}_S$. Using the Picard-Lefschetz reflections we may then assume that D is nef.

To end this section we will make some remarks concerning the moduli of K3 surfaces. There is a 20-dimensional family of analytic isomorphism classes of K3 surfaces. Moreover there is a countable union of 19-dimensional families of algebraic K3 surfaces. We have seen that a K3 surface has Picard number between 1 and 20. For a given K3 surface with a specified Picard lattice and Picard number ρ there exists a $(20-\rho)$-dimensional family of isomorphism classes of algebraic K3 surfaces with the same Picard lattice.

1.2 Rational normal scrolls

We will now include some results on scrolls that we will need later on. We start with the definition.

Definition 1.2.1. Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(e_d)$ be a locally free sheaf of rank d on $\mathbb{P}^1$ and let

$$\pi : \mathbb{P}(\mathcal{E}) = \operatorname{Proj} \operatorname{Sym} \mathcal{E} \longrightarrow \mathbb{P}^1$$

denote the corresponding $\mathbb{P}^{d-1}$-bundle. Let $e_1 \geq \cdots \geq e_d \geq 0$ and

$$f := e_1 + \cdots + e_d \geq 2.$$

A *rational normal scroll* $\mathcal{T} := S(e_1, \ldots, e_d)$ of type $(e_1, \ldots, e_d)$ is the image of the map

$$j : \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = \mathbb{P}^n$$

where $n = f + d - 1$.

Remark 1.2.2. Note that there is quite a lot of ambiguity between different authors on the definition of a rational normal scroll. The definition given here is equivalent to the one given in [Sch86] and [JK01]. [EH87] requires that $f \geq 1$, [PS84] requires $e_d > 0$, while [Bra97] only requires $f \geq 0$.

Note also that we will often be sloppy and write *scroll* instead of rational normal scroll. In this thesis scroll will always mean rational normal scroll.

There are several different ways to describe a rational normal scroll. We will include another often used description: Let $e_1, \ldots, e_d$ be integers as above and $n = f + d - 1$. Denote by

$$\overline{a_1, \ldots, a_d}$$

the linear span of the d points $a_1, \ldots, a_d$ in $\mathbb{P}^n$. Choose complementary linear subspaces Λ_i of $\mathbb{P}^n$, each of dimension e_i, rational normal curves $C_i \subset \Lambda_i$, and isomorphisms $\varphi_i : \mathbb{P}^1 \to C_i$. Then

$$\bigcup_{\lambda \in \mathbb{P}^1} \overline{\varphi_1(\lambda), \ldots, \varphi_d(\lambda)} \cong S(e_1, \ldots, e_d).$$

Proposition 1.2.3. *[PS84, lemma 1], [Sch86, section 1], [EH87, section 1], [ACGH85, pp.95-98]*

1. *$\mathcal{T} := S(e_1, \ldots, e_d)$ is a non-degenerate (that is not contained in a hyperplane) irreducible projectively normal variety of degree f and dimension d.*

2. *$S(e_1, \ldots, e_d)$ is nonsingular if and only if $e_d > 0$.*

3. *$j : \mathbb{P}(\mathcal{E}) \to \mathbb{P}H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ is an embedding if and only if $e_d > 0$. (Note that j is always birational.)*

4. *[Rei97, exercise 2.6] Let $\mathcal{E} = (O)_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus (O)_{\mathbb{P}^1}(e_d)$ and $\mathcal{E}' = (O)_{\mathbb{P}^1}(e'_1) \oplus \cdots \oplus (O)_{\mathbb{P}^1}(e'_d)$ be two locally free sheaves of rank d on $\mathbb{P}^1$. Then $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}')$ if and only if there exists an integer c such that $e_i = e'_i + c$ for all i.*

We will now give some examples (taken from [Rei97] and [EH87]).

1. $S(1, 1) \cong \mathbb{P}^1 \times \mathbb{P}^1$

2. $S(a)$ is a rational normal curve of degree a

3. $S(a,0)$ is the cone over a rational normal curve of degree a. (More generally: $S(e_1,\ldots,e_d,0,\ldots,0)$ is a cone over $S(e_1,\ldots,e_d)$).

If X is a non-degenerate variety, then $\deg X \geq 1+\operatorname{codim} X$ ([EH87, proposition 0]). We say that a variety has *minimal degree* if X is non-degenerate and $\deg X = 1+\operatorname{codim} X$. Then one has the following classification result for varieties of minimal degree.

Theorem 1.2.4. *(Del Pezzo, Bertini) [EH87, theorem 1] If $X \subset \mathbb{P}^n$ is a variety of minimal degree, then X is a cone over a smooth such variety. If X is smooth and* $\operatorname{codim} X > 1$, *then $X \subset \mathbb{P}^n$ is either a rational normal scroll or the Veronese surface in $\mathbb{P}^5$.*

It is noted in [Sch86, p. 110] that we may replace $\mathcal{T}$ by $\mathbb{P}(\mathcal{E})$ for most cohomological considerations even when $\mathcal{T}$ is singular. As a consequence of this it is useful to know more about the cohomology on $\mathbb{P}(\mathcal{E})$.

We will now describe the Picard group of $\mathbb{P}(\mathcal{E})$. (This is done in [Har77, exercise II.7.9].) It is generated by the hyperplane class $\mathcal{H} = j^*\mathcal{O}_{\mathbb{P}^n}(1)$ and the fibre[3] $\mathcal{F} = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ of $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ such that

$$\operatorname{Pic}\mathbb{P}(\mathcal{E}) = \mathbb{Z}\mathcal{H} \oplus \mathbb{Z}\mathcal{F}.$$

We have the following important formula for the cohomology on $\mathbb{P}(\mathcal{E})$.

$$h^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(a\mathcal{H}+b\mathcal{F})) = h^0(\mathbb{P}^1, \operatorname{Sym}^a(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(b)). \tag{1.2}$$

(See [Sch86, 1.3] or [EH87, p.7] for proof.) We will use this formula later on when we look at resolutions of projective models of Del Pezzo and Enriques surfaces.

One also has a description of the scroll $S(e_1,\ldots,e_d)$ as a determinantal variety using the homogenous coordinates of $\mathbb{P}^n$: Let $X_{0,0},\ldots,X_{0,e_1},X_{1,0},\ldots,X_{d,a_d}$ be homogenous coordinates of $\mathbb{P}^n$. Then the ideal of $S(e_1,\ldots,e_d)$ is generated by the the 2×2 minors of the following matrix

$$\begin{bmatrix} X_{0,0} & \ldots & X_{0,e_1-1} & X_{1,0} & \ldots & X_{d,a_d-1} \\ X_{0,1} & \ldots & X_{0,e_1} & X_{1,1} & \ldots & X_{d,a_d} \end{bmatrix}$$

(See [ACGH85, p.96] or [Rei97, theorem 2.5] for proof.)

We will now give a summary of the results in [Sch86, section 2]. These will be of importance to us later on.

We start with a smooth variety V (in the cases we consider later on V will always be a surface S) and a line bundle L on V. Consider the natural map

$$\phi_L : V \longrightarrow \mathbb{P}H^0(V,L) = \mathbb{P}^r.$$

We are interested in rational normal scrolls $\mathcal{T} \subset \mathbb{P}^r$ containing $\phi_L(V)$.

[3] We can also view this fibre as a ruling of the cone.

Let $\mathcal{T} \subset \mathbb{P}^r$ be a scroll of degree f containing $\phi_L(V)$. The ruling $\mathcal{F}$ on $\mathcal{T}$ cuts out a pencil of divisors

$$\{D_\lambda\}_{\lambda \in \mathbb{P}^1} \subseteq |D|$$

on V with $h^0(V, L - D) = f \geq 2$.

Conversely from any pencil of divisors $\{D_\lambda\}_{\lambda \in \mathbb{P}^1}$ on V with $h^0(V, L-D) = f \geq 2$ we can construct a scroll of degree f. Each $\phi_L(D_\lambda)$ will span a $(h^0(L) - h^0(L-D) - 1)$-dimensional linear subspace of $\mathbb{P}^r$. The variety swept out by these linear spaces will be a rational normal scroll.

Furthermore we can compute the scroll type rather easily as follows: Decompose the pencil $\{D_\lambda\}$ into its moving part $\{M_\lambda\}$ and fixed part F. Then we have $\mathcal{T} = S(e_1, \dots, e_d)$ with

$$e_i = \#\{j | d_j \geq i\} - 1, \tag{1.3}$$

where

$$\begin{aligned} d = d_0 &= h^0(L) - h^0(L-D) \\ d_1 &= h^0(L-D) - h^0(L-F-2M) \\ \vdots &\ \ \vdots \ \ \vdots \\ d_i &= h^0(L-F-iM) - h^0(L-F-(i+1)M) \end{aligned} \tag{1.4}$$

Note that obviously $d_i = 0$ for all $i \geq n$ for some n. Furthermore the d_i form a non-increasing sequence (see [JK01, remark 2.4]). This is a property we will use several times later on.

1.3 The Clifford index and Clifford divisors

Definition 1.3.1. Let C be a smooth irreducible curve of genus $g \geq 2$. A g^r_d is a linear system of dimension r and degree d. C is *k-gonal* if C possesses a g^1_k but no g^1_{k-1}. (If C is k-gonal we say that k is its *gonality*.) We write gon C for the gonality of C.

Definition 1.3.2. Let C be a smooth irreducible curve of genus $g \geq 2$. If A is a line bundle on C, then the *Clifford index of A* is

$$\operatorname{Cliff} A := \deg(A) - 2(h^0(A) - 1).$$

If $g \geq 4$ we define the *Clifford index of C* as

$$\begin{aligned} \operatorname{Cliff} C &:= \min\{\operatorname{Cliff} A | h^0(A) \geq 2, \deg(A) \leq g-1\} \\ &= \min\{\operatorname{Cliff} A | h^0(A) \geq 2, h^1(A) \geq 2\} \end{aligned}$$

A line bundle A on C *contributes to the Clifford index of C* if a satisfies $h^0(A) \geq 2$ and $\deg A \leq g-1$. A line bundle A on C *computes the Clifford index of C* if in addition $\operatorname{Cliff} C = \operatorname{Cliff} A$.

The next result gives bounds for $\operatorname{Cliff} C$.

Theorem 1.3.3. *Let C be a smooth irreducible curve of genus $g \geq 4$. Then*

$$0 \leq \operatorname{Cliff} C \leq \left\lfloor \frac{g-1}{2} \right\rfloor .$$

Proof. $\operatorname{Cliff} C \geq 0$ is just Clifford's theorem (see [Har77, theorem IV.5.4]).

For the other inequality we use [ACGH85, theorem V.1.1]. This says that if

$$g - 2(g - d + 1) \geq 0,$$

then there exists a g^1_d. In particular there exists a $g^1_{\lfloor (g+3)/2 \rfloor}$. Hence

$$\operatorname{gon} C \leq \left\lfloor \frac{g+3}{2} \right\rfloor . \tag{1.5}$$

Then

$$\operatorname{Cliff} C \leq \left\lfloor \frac{g-1}{2} \right\rfloor$$

follows from the lemma below. □

Lemma 1.3.4. *[CM91, theorem 2.3] The gonality of a smooth irreducible curve C of genus $g \geq 4$ satisfies*

$$\operatorname{Cliff} C + 2 \leq \operatorname{gon} C \leq \operatorname{Cliff} C + 3$$

Proof. We will only prove $\operatorname{Cliff} C + 2 \leq \operatorname{gon} C$, which is the part used in the above proof.

The existence of a g^1_k gives a line bundle A with $h^0(A) = 2$ and $\deg(A) = k \leq \lfloor (g+3)/2 \rfloor$ (using equation (1.5)). Since $\lfloor (g+3)/2 \rfloor \leq g - 1$ (for $g \geq 4$ with equality if and only if $g = 4$) A contributes to the Clifford index of C. Thus

$$\operatorname{Cliff} C \leq \operatorname{Cliff} A = k - 2 = \operatorname{gon} C - 2.$$

□

The curves satisfying $\operatorname{gon} C = \operatorname{Cliff} C + 3$ are conjectured to be very rare and are called *exceptional*.

The Clifford index measures how general C is from the point of view of moduli. We have:

i) $\operatorname{Cliff} C = 0$ if and only if C is hyperelliptic (i.e. C has gonality 2).

ii) $\operatorname{Cliff} C = 1$ if and only if C is trigonal (i.e. C has gonality 3).

iii) $\operatorname{Cliff} C = \lfloor \frac{g-1}{2} \rfloor$ if C is a general curve of genus g.

We will now define the Clifford index of a K3 surface.

Definition 1.3.5. Let L be a base point free big line bundle on a K3 surface, and let $C \in |L|$ be a smooth irreducible curve.Then the *Clifford index of* L is

$$\operatorname{Cliff} L := \operatorname{Cliff} C.$$

If (S, L) is a polarized K3 surface the *Clifford index of* S is

$$\operatorname{Cliff}_L S := \operatorname{Cliff} L.$$

We say that S is *Clifford general* if $\operatorname{Cliff}_L S = \lfloor \frac{g-1}{2} \rfloor$.

Furthermore we set

$$c := \operatorname{Cliff}_L S.$$

The following result of Green and Lazarsfeld shows that the definition is well-defined.

Theorem 1.3.6. *[GL87] Let L be a base point free and big line bundle on a K3 surface. Then* $\operatorname{Cliff} C$ *is constant for all smooth irreducible $C \in |L|$ and if* $\operatorname{Cliff} C < \left\lfloor \frac{g-1}{2} \right\rfloor$, *then there exists a line bundle M on S such that $M_C := M \otimes \mathcal{O}_C$ computes the Clifford index of C for all smooth irreducible $C \in |L|$.*

The following existence theorem shows that $K3$ surfaces with all possible Clifford indices exist.

Theorem 1.3.7. *[JK01, theorem 4.1] Let g and c be integers such that $g \geq 4$ and $0 \leq c \leq \left\lfloor \frac{g-1}{2} \right\rfloor$. Then there exists a polarized K3 surface of genus g and Clifford index c.*

The proof actually shows that there exists at least an 18-dimensional family of polarized K3 surfaces of genus g and Clifford index c in all of the possible cases.

If

$$\mathcal{A}(L) := \{D \in \operatorname{Pic} S | h^0(D), h^0(L-D) \geq 2\} \tag{1.6}$$

is non-empty we set

$$\mu(L) := \min\{D.(L-D) - 2 | D \in \mathcal{A}(L)\}. \tag{1.7}$$

If $\mathcal{A}(L) = \emptyset$ we set $\mu(L) = \infty$. We also write

$$\mathcal{A}^0(L) := \{D \in \mathcal{A}(L) | D.(L-D) - 2 = \mu(L)\}. \tag{1.8}$$

Amazingly we may express $\operatorname{Cliff} L$ by $\mu(L)$:

Theorem 1.3.8. *[Knu01a, lemma 8.3] Let L be a base point free and big line bundle on a K3 surface. Then*

$$\operatorname{Cliff} L = \min\{\mu(L), \left\lfloor \frac{g-1}{2} \right\rfloor\}.$$

One would presume that the general polarized K3 surface is Clifford general. This is not true as the next proposition shows. However if we restrict ourselves to primitive polarized K3 surfaces it is true as proposition 1.3.18 below will show.

Proposition 1.3.9. *Let (S, L) be a polarized K3 surface. Then (S, nL) is not Clifford general for $n > 1$.*

Proof. Set $L' := nL$, $D' := L$, and $d := L^2 > 0$. Note that $2g(L') = n^2d + 2$. Then if $n > 1$ it is obvious that $D' \in \mathcal{A}(L')$. If nL was Clifford general we would have

$$D'.(L - D') = (n-1)d \geq \mu(L') - 2 \geq \left\lfloor \frac{g+3}{2} \right\rfloor = \left\lfloor \frac{n^2d}{4} \right\rfloor + 2 \geq \frac{n^2d}{4} + \frac{5}{4}.$$

This gives

$$4(n-1)d \geq n^2d + 5$$

which is impossible for positive d and real n. Since $f(n) = n^2d - 4nd + (4d + 5)$ has negative discriminant $-20d$ and $f(0) > 0$. $\square$

Looking at this proposition one might conjecture $\operatorname{Cliff} L \geq \operatorname{Cliff} nL$ for $n \geq 1$. This is false: Take for example an ample base point free divisor with $\operatorname{Cliff} L = 0$. Then $\operatorname{Cliff} 2L > 0$ by [SD74, theorem 8.3] and theorem 1.3.17.

We will now look at the case when $\operatorname{Cliff}_L S = \mu(L)$. Then there exist a divisor $D \in \mathcal{A}^0(L)$. We also have $F := L - D \in \mathcal{A}^0(L)$. By interchanging F and D if necessary we may assume that $D.L \leq F.L$ (or equivalently $D^2 \leq F^2$). Hence we have

(C1) $c = D.L - D^2 - 2 = D.F - 2$ and $D \in \mathcal{A}(L)$.

(C2) $D.L \leq F.L$ (or equivalently $D^2 \leq F^2$).

By [JK01, proposition 2.5] we also have

(C3) $h^1(D) = h^1(F) = 0$.

(C4) The (possibly empty) base divisor Δ of F satisfies $L.\Delta = 0$.[4]

If $\mathcal{A}(L) \neq \emptyset$ we can, by [JK01, proposition 2.6], always find $D \in \mathcal{A}^0(L)$ such that

(C5) $|D|$ is base point free and its general member is a smooth irreducible curve.

Definition 1.3.10. A divisor (class) satisfying (C1)-(C4) is a *Clifford divisor* for L. A divisor (class) satisfying (C1)-(C5) is a *free Clifford divisor* for L.

Note that it is enough for a divisor to satisfy (C1)-(C2) to be a Clifford divisor.

Over the next pages we will summarize the most important properties of Clifford divisors.

[4] From now on (that is for the rest of part I) we will always write Δ for the base divisor of F.

Let D be any Clifford divisor. Since $h^0(D) \geq 2$ and $h^1(D) = 0$, Riemann-Roch gives $D^2 \geq 0$. The condition $D.L \leq F.L = (L-D).L$ can be written as $2D.L \leq L^2$. The Hodge index theorem then gives $2D^2(D.L) \leq D^2L^2 \leq (D.L)^2$. Whence $2D^2 \leq D.L = D^2 + D.F$, i.e. $D^2 \leq D.F = c+2$. Thus any Clifford divisor satisfies

$$0 \leq D^2 \leq c+2. \tag{1.9}$$

By the Hodge index theorem we also have

$$D^2L^2 \leq (L.D)^2 = (D^2+c+2)^2 \tag{1.10}$$

We want to say as much as possible about Δ. For this purpose we define

$$\mathcal{R}_{L,D} := \{\Gamma | \Gamma \text{ is a smooth rational curve, } \Gamma.L = 0 \text{ and } \Gamma.D > 0\}. \tag{1.11}$$

Then we have the following proposition.

Proposition 1.3.11. *[JK01, proposition 5.3] Let D be a free Clifford divisor for L and Γ a curve in $\mathcal{R}_{L,D}$. Then $D.\Gamma = -D.A = 1$ and Γ is contained in the base locus Δ of F. In particular $\Delta.D = \#\mathcal{R}_{L,D}$, where the elements are counted with the multiplicity they have in Δ. Furthermore we have that the curves in $\mathcal{R}_{L,D}$ are disjoint.*

We will need the following special cases (where all the Γ's are smooth rational curves):

(E0) $L \sim 2D + \Gamma$, $D^2 = c+1$, $L^2 = 4c+6$, and $\Gamma.D = 1$.

(E1) $L \sim 2D + \Gamma_1 + \Gamma_2$, $D^2 = c$, $L^2 = 4c+4$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma_1 \\ \Gamma_2 & \end{tikzcd}$$

(E2) $L \sim 2D + 2\Gamma_0 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$, $D^2 = c$, $L^2 = 4c+4$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_N \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+1} \\ & & & \Gamma_{N+2} & \end{tikzcd}$$

(E3) $L \sim 2D + 2\Gamma_0 + \Gamma_1$, $c = D^2 = 0$, $L^2 = 6$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \Gamma_1 \end{tikzcd}$$

(E4) $L \sim 4D + 2\Gamma$, $c = D^2 = 0$, $L^2 = 8$, and $\Gamma.D = 1$.

(Q) $L \sim 2D$, $D^2 = c+2$, and $L^2 = 4c+8$.

We then get the following result.

Proposition 1.3.12. *[JK01, proposition 5.6] Let D be a free Clifford divisor. Assume that we are not in the case with $L^2 \leq 4c+6$ and $\Delta = 0$, one of the cases (E0)-(E2), or the case*

$$L^2 = 4c+4, D.\Delta = 1, \text{ and } \Delta^2 = -2.$$

Then

$$h^1(L-2D) \leq \frac{1}{2}c + 1 - D^2.$$

Write $R := L - 2D$. Using Riemann-Roch on R we get that $h^0(R) = 0$ if and only if $L^2 = 4c+4$ and $h^1(R) = 0$ (see [JK01, pp.16-17]). Hence we will mostly be in a situation where $R > 0$. Write F_0 for the moving component of $|F|$. When $R > 0$, we can write $F_0 \sim D + A$ for some effective divisor A. Then we have

$$L \sim 2D + A + \Delta. \tag{1.12}$$

We have the following useful lemma.

Lemma 1.3.13. *[JK01, lemma 6.4] Assume that $R := L - 2D > 0$ and that we are not in one of the cases (E3) or (E4), then*

$$\Delta^2 = -2D.\Delta$$

and

$$\Delta.A = 0.$$

In section 1.5 we will classify the components of a sub-divisor Δ' of Δ. To be able to use this classification we have to know $D.\Delta$. Keeping this and proposition 1.3.12 in mind we see that the following property is useful:[5]

(C6) $h^1(L-2D) = \Delta.D$ or D is of one of the types (E0)-(E4) with $h^1(L-2D) = \Delta.D-1$.

Given a spanned and big divisor L we have a natural morphism

$$\phi_L : S \longrightarrow \mathbb{P}^{h^0(L)-1} = \mathbb{P}^g \tag{1.13}$$

given by the linear system $|L|$. Taking a free Clifford divisor D for L we can choose a subpencil $\{D_\lambda\}_{\lambda\in\mathbb{P}^1} \subset |D|$ as follows: Pick any smooth members D_1 and D_2 in $|D|$ intersecting in D^2 distinct points, such that none of these belong to

$$\bigcup_{\substack{\{\Gamma|\Gamma \text{ is a smooth rational} \\ \text{curve with } \Gamma.L\leq c+2\}}} \Gamma.$$

Then we define

$$\{D_\lambda\}_{\lambda\in\mathbb{P}^1} := \text{ the pencil generated by } D_1 \text{ and } D_2.$$

[5]Note that (C6) here is not what is called (C6) in [JK]. I will later on denote (C6) in [JK] by (C8).

By the results on page 21 this pencil will define in a natural way a scroll containing $\phi_L(S) = S'$, which we will denote by $\mathcal{T} = \mathcal{T}(c, D, \{D_\lambda\}$. We will say that this scroll is *associated* to the Clifford divisor D. We will also say that the scroll is associated to the divisor L and to the polarized K3 surface (S, L). The pencil giving the scroll will not necessarily be base point free, but has D^2 base points.

We will see below (theorem 1.3.17) that if $c > 0$, then ϕ_L is birational. For now we will assume that $c > 0$. By our assumptions on $\{D_\lambda\}$ the D^2 base points of the pencil will be mapped to $n = D^2$ distinct points $x_1, \ldots, x_n$ by ϕ_L. Furthermore let $y_1, \ldots, y_r$ be the images of the contractions of the curves $\Gamma_i \in \mathcal{R}_{L,D}$ and m_i be the multiplicity of Γ_i in Δ. Set $m = \sum m_i$. Then we define

$$Z_\lambda := x_1 + \cdots + x_n + m_1 y_1 + \cdots n_r y_r.$$

By $< Z_\lambda >$ we mean the linear span of the zero-dimensional scheme Z_λ on $\phi_L(S)$. See [JK01, pp.18–19] for more details about these definitions.

With this notation we have the following property which we want to be satisfied.[6]

(C7) $V := \mathrm{Sing}\mathcal{T} =< Z_\lambda >\simeq \mathbb{P}^{n+m-1}$ or D is of one of the types (Q), (E0), (E1), or (E2) with

$$\begin{aligned} V &\simeq \mathbb{P}^{n-2} \text{ if } D \text{ is of type (Q),} \\ V &\simeq \mathbb{P}^{n-1} \text{ if } D \text{ is of type (E0),} \\ V &\simeq \mathbb{P}^{n} \text{ if } D \text{ is of type (E1), and} \\ V &\simeq \mathbb{P}^{n} \text{ if } D \text{ is of type (E2).} \end{aligned}$$

Furthermore V does not intersect (set-theoretically) with S' outside the points in the support of Z_λ and for any irreducible D_λ we have

$$V \cap D_\lambda = Z_\lambda.$$

Here we have used the convention $\mathbb{P}^{-1} = \emptyset$. Note also that property (C7) only gives meaning when $c > 0$. [JK01, proposition 5.11] gives the corresponding properties for $c = 0$.

Definition 1.3.14. A divisor (class) satisfying (C1)-(C7) is a *perfect Clifford divisor* for L.

We have the following existence result.

Theorem 1.3.15. *[JK01, theorem 5.7] For $c > 0$ there exists a perfect Clifford divisor D.*

In fact if $L - 2D > 0$, then any free Clifford divisor satisfying the following two properties is perfect:

[6] Note that (C7) here is not what is called (C7) in [JK]. I will later on denote (C7) in [JK] by (C9).

(C8) If D' is any other free Clifford divisor such that $D' > D$, then $\Delta \neq 0$ and D' is of type $(E1)$ or $(E2)$.

(C9) If D is of type $(E1)$ or $(E2)$, and D' is any other free Clifford divisor satisfying $(C8)$, then $D' \sim D$.

Note that (C8) is called (C6) and (C9) is called (C7) in [JK].

We also have the following result.

Lemma 1.3.16. *[JK01, lemma 6.10] Assume $c > 0$ and let D be a free Clifford divisor, not of type $(E1)$ or $(E2)$. If $h^1(A) = 0$, then D is perfect.*

Given a polarized K3 surface (S, L) it is interesting to ask whether there exists Clifford divisors that are not perfect and to find the number of perfect Clifford divisors (up to linear equivalence). Proposition 1.6.6 will show that in most cases all Clifford divisors are perfect and unique up to linear equivalence. On p. 69 we will give an example of a non-perfect Clifford divisor. See p. 55 for several examples of non-linear perfect Clifford divisors.

We have already mentioned the following result. It shows how different the cases $c = 0$ and $c > 0$ are.

Theorem 1.3.17. *[SD74] Let L be a spanned and big line bundle on a K3 surface S, and denote by ϕ_L the morphism given by $|L|$ and let c be its Clifford index.*

1. *If $c = 0$, then ϕ_L is $2:1$ onto a surface of degree $\frac{1}{2}L^2$.*
2. *If $c > 0$, then ϕ_L is birational onto a surface of degree L^2 (it is in fact an isomorphism outside of finitely many contracted smooth rational curves), and $S' = \phi_L(S)$ is normal and has only rational double points as singularities. In particular $K_{S'} \simeq \mathcal{O}_{S'}$, and $p_a(S') = 1$.*

We will now look at K3 surfaces with Picard number one. As noted in section 1.1, these K3 surfaces are the most general ones. We have

$$\operatorname{Pic} S = \mathbb{Z}D,$$

where $D^2 \in 2\mathbb{Z}^+$. There exists a 19-dimensional family of K3 surfaces with this Picard lattice. Riemann-Roch gives that either D or $-D$ is effective. By a reflection of the Picard lattice, if necessary, we may assume that D is effective. The following proposition is almost immediate.

Proposition 1.3.18. *Let (S, L) be a polarized K3 surface with $\operatorname{Pic} S = \mathbb{Z}D$ and $L = nD$ $(n >\leq 1)$.*

If $n = 1$, then L is Clifford general. Especially if L^2 is square-free, then L is Clifford general.

If $n > 1$, then L is not Clifford general.

Proof. The $n = 1$ case follows from $\mu(L) = \infty$. The statement with L^2 square-free follows since n must equal one if L^2 is square-free. The $n > 1$ case follows from proposition 1.3.9. $\square$

We will now look at the scroll types that are associated to (S, L) when S has Picard number one and L is not Clifford general (i.e. $n > 1$). First we compute $h^0(nD)$ for all n. We know that S contains an ample divisor $A \sim nD$. Since $A.D > 0$ by the Nakai-Moishezon criterion we have $n > 0$. Using the Nakai-Moishezon criterion again we see that mD is effective if and only if $mD.A = nmd \geq 0$, i.e. if $m \geq 0$. Using the Nakai-Moishezon criterion yet another time we get that D is ample and furthermore that nD is ample if and only if $n > 0$. Since $h^1(B) = 0$ if B is ample we have the following equation

$$h^0(nD) = \begin{cases} \frac{1}{2}n^2d + 2, & n > 0 \\ 1, & n = 0 \\ 0, & n < 0 \end{cases} \tag{1.14}$$

We also have to find the perfect Clifford divisors associated to S. Write $L = nD$. Then

$$\mathcal{A}(L) = \{mD | 1 \leq m \leq n\}.$$

Thus

$$\mu(L) = \min\{mD.(L - mD) | 1 \leq m \leq n\} = d \min\{m(n - m) | 1 \leq m \leq n\} = d(n - 1)$$

where the minimum is computed by D and $(n-1)D$. Since (C2) is to be satisfied we see that D is a perfect Clifford divisor and that D is the only one (up to linear equivalence).

We can now compute the scroll type using the equations on p. 21. We get

$$\begin{aligned} d_0 &= \frac{n-1}{2}d \\ \vdots \quad &\vdots \quad \vdots \\ d_i &= \frac{n-i-1}{2}d \\ \vdots \quad &\vdots \quad \vdots \\ d_{n-2} &= \frac{3}{2}d \\ d_{n-1} &= 1 + d/2 \\ d_n &= 1 \\ d_{n+1} &= 0 \end{aligned}$$

and the following scroll type

$$(n, \underbrace{n-1, \ldots, n-1}_{\frac{1}{2}d}, \underbrace{n-2, \ldots, n-2}_{d-1}, \underbrace{n-3, \ldots, n-3}_{d}, \ldots, \underbrace{i, \ldots, i}_{d}, \underbrace{0, \ldots, 0}_{d})$$

We end this section with some short remarks on ampleness. As already noted our definition of a polarized surface (S, L) is with L base point free, while the usual definition is with L ample. We will call a surface S with an ample line bundle L for a *a-polarized surface*.

We first consider the case where (S, L) is both polarized and a-polarized. Since L is ample the Nakai-Moishezon criterion gives $\mathcal{R}_{L,D} = \emptyset$. Especially we get that for any perfect Clifford divisor D we have $h^1(L-2D) = h^1(R) = 0$.[7]

We then consider the case where (S, L) is a-polarized but not polarized. We saw above that the case where (S, L) is both polarized and a-polarized fits nicely into the framework we have done in this section. When (S, L) is a-polarized but not polarized this is no longer true. Remember that we defined the Clifford index of L to be the Clifford index of an irreducible curve in $|L|$. But by [SD74, proposition 8.1] $|L|$ does not contain any irreducible curves. Thus it makes no sense to talk about the Clifford index in this case.

1.4 K3 surfaces which are Clifford general and non-BN general

Given a divisor D on S we set $\mathcal{O}_C(D) := \mathcal{O}_S(D) \otimes \mathcal{O}_C$ and denote the corresponding divisor on C by D_C.

With $F = L - D$ Riemann-Roch for curves gives

$$\deg D_C = h^0(D_C) - h^0(F_C) - 1 + g.$$

Hence

$$\begin{aligned} \operatorname{Cliff} \mathcal{O}_C(D) &= \deg \mathcal{O}_C(D) - 2(h^0(D_C) - 1) \\ &= g + 1 - (h^0(D_C) + h^0(F_C)). \end{aligned} \tag{1.15}$$

We start with the definition of BN generality.

Definition 1.4.1. A polarized K3 surface (S, L) of genus g is said to be *Brill-Noether (BN) general* if the inequality

$$h^0(D)h^0(F) < h^0(L) = g + 1$$

holds for every non-trivial effective decomposition $D + F \sim L$.

[7] The converse does not hold. There exists K3 surfaces with $h^1(R) = 0$ where L is not ample. Consider for example the K3 surface given by the lattice $\mathbb{Z}L \oplus \mathbb{Z}D \oplus \mathbb{Z}\Gamma$ with intersection matrix

$$\begin{bmatrix} L^2 & L.D & L.\Gamma \\ L.D & D^2 & D.\Gamma \\ L.\Gamma & D.\Gamma & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 12 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

One can show that L has Clifford index 1 and perfect Clifford divisor D. It is easy to show that $\mathcal{R}_{L,D} = \emptyset$, so $h^1(R) = 0$. Riemann-Roch gives that either Γ or $-\Gamma$ is effective. Thus L is not ample by the Nakai-Moishezon criterion since $L.\Gamma = 0$.

We will see below how this definition relates to BN theory of curves.

We see that if $\mu(L) = \infty$, then L is BN general. Generalizing this we have the following result.[8]

Proposition 1.4.2. *(Knutsen) Let (S, L) be a polarized K3 surface. If S is BN general, then it is Clifford general.*

Proof. Let D be an effective divisor, with $h^0(D), h^0(F) \geq 2$. Set $d := h^0(D)$, $f := h^0(F)$, and $a := d + f$. Since S is BN general $fd < g + 1$. We want to find an upper bound for a. The expression $df = d(a - d)$ has minimum for fixed a, with $d, a - d \geq 2$, when $d = 2$. Hence a is maximal when $d = 2$ and f is the largest integer such that $2f < g + 1$, i.e. for $f = \lfloor g/2 \rfloor$. This gives

$$h^0(D) + h^0(F) \leq \left\lfloor \frac{g+4}{2} \right\rfloor.$$

If (S, L) is not Clifford general we can find a divisor D that computes the Clifford index for all smooth $C \in |L|$ (proposition 1.3.6). We can choose D such that $h^1(D) = h^1(F) = 0$ (by for example theorem 1.3.8 and [JK, proposition 2.5]). Then we get $h^0(D) = h^0(D_C)$ and $h^0(F) = h^0(F_C)$. By arguing as in the proof of lemma 1.4.5 below. In particular this gives

$$h^0(D_C) + h^0(F_C) \leq \left\lfloor \frac{g+4}{2} \right\rfloor,$$

for all smooth $C \in |L|$. Equation (1.15) then gives

$$\text{Cliff } \mathcal{O}_C(D) \geq \left\lfloor \frac{g-1}{2} \right\rfloor,$$

a contradiction. □

Our next results will concern the cases where the converse does not hold. We will need a couple of lemmas.

Lemma 1.4.3. *Given a polarized K3 surface (S, L), a smooth $C \in H^0(S, L)$, and an effective divisor D on S such that $h^0(D), h^0(L - D) \geq 2$, then*

$$h^0(D_C), h^1(D_C) \geq 2$$

Proof. From the short exact sequence

$$0 \to \mathcal{O}_S(-L) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

we get the exact sequence

$$0 \to \mathcal{O}_S(D - L) \to \mathcal{O}_S(D) \to \mathcal{O}_C(D) \to 0,$$

by tensorizing with D. Hence (with $F := L - D$)

$$0 \to H^0(-F) \to H^0(D) \to H^0(D_C)$$

[8] The result is previously unpublished. I believe it is due to Knutsen. The proof is my own.

is exact. From which we see that $h^0(D_C) \geq h^0(D) \geq 2$, using $H^0(-F) = 0$. A symmetric argument gives $h^0(F_C) \geq h^0(F) \geq 2$.

From Serre duality we get $H^1(D_C) = H^1(L_C - F_C) = H^0(F_C)'$, in particular $h^1(D_C) = h^0(F_C) \geq h^0(F) \geq 2$. □

Lemma 1.4.4. *Given a polarized K3 surface (S, L), a smooth $C \in H^0(S, L)$, and an effective divisor D on S such that $h^0(D), h^0(L - D) \geq 2$, then*

$$\operatorname{Cliff} \mathcal{O}_C(D) \leq g + 1 - (h^0(D) + h^0(L - D)).$$

Proof. Follows immediately from the proof of lemma 1.4.3 and equation (1.15). □

With the additional condition $h^1(D) = 0$ we get a stronger result.

Lemma 1.4.5. *Given a polarized K3 surface (S, L), a smooth $C \in H^0(S, L)$, and an effective divisor D on S such that $h^0(D), h^0(L - D) \geq 2$, and $h^1(D) = 0$ then*

$$\operatorname{Cliff} \mathcal{O}_C(D) = g + 1 - (h^0(D) + h^0(L - D) + h^1(L - D)).$$

Proof. From

$$0 \to \mathcal{O}_S(-F) \to \mathcal{O}_S(D) \to \mathcal{O}_C(D) \to 0$$

we get

$$\begin{array}{ccccccccc} 0 \to & H^0(-F) & \to & H^0(D) & \to & H^0(D_C) & \to & H^1(-F) & \to & H^1(D). \\ & \| & & & & & & \| & & \| \\ & 0 & & & & & & H^1(F)' & & 0 \end{array}$$

Hence $h^0(D_C) = h^0(D) + h^1(F)$. Looking at F instead of D we have the exact sequence

$$0 \to H^0(-D) \to H^0(F) \to H^0(F_C) \to H^1(-D).$$

Using $h^0(-D) = 0$ and $h^1(-D) = h^1(D) = 0$ (be Serre duality) we get $h^0(F) = h^0(F_C)$. The result then follows from equation (1.15). □

Proposition 1.4.6. *Let (S, L) be a polarized K3 surface. Then (S, L) is Clifford general and non–BN general only if there exists an effective divisor D with $h^0(D), h^0(F) \geq 2$ (where $F := L - D$), such that*

$$h^0(D)h^0(F) \geq g + 1 \tag{1.16}$$

and

$$h^0(D) + h^0(F) \leq \left\lfloor \frac{g + 4}{2} \right\rfloor. \tag{1.17}$$

With equality in equation (1.17) only if $h^1(D) = h^1(F) = 0$.

Proof. The first inequality follows directly from the definition of BN generality. The second inequality follows from lemma 1.4.4 and the definition of Clifford generality. We see from equation (1.15) that we have equality in equation (1.17) only if $h^0(D) = h^0(D_C)$ and $h^0(F) = h^0(F_C)$. For this to be the case we must have $h^1(-F) = h^1(-D) = 0$ (see proof of lemma 1.4.5). Hence $h^1(F) = h^1(D) = 0$. □

Let us examine what this proposition gives for small genus g (we assume in the following that $h^0(D) \leq h^0(F)$).

g=2,3,...,7, and 9 In these cases the inequalities have no integral solutions.

g=8 $h^0(D) = h^0(F) = 3$. Since we have equality in equation (1.17) we get $h^1(D) = h^1(F) = 0$. This gives $D^2 = 2$ and $D.L = 7$.

g=10 $h^0(D) = 3, h^0(F) = 4$. Since we have equality in equation (1.17) we get $h^1(D) = h^1(F) = 0$. This gives $D^2 = 2$ and $D.L = 8$.

g=11 $h^0(D) = 3, h^0(F) = 4$. Since we have equality in equation (1.17) we get $h^1(D) = h^1(F) = 0$. This gives $D^2 = 2$ and $D.L = 9$.

g=12 We now have two solutions to the inequalities, both with equality in equation (1.17).

i) $h^0(D) = h^0(F) = 4$. This gives $D^2 = 4$ and $D.L = 11$.

ii) $h^0(D) = 3, h^0(F) = 5$. This gives $D^2 = 2$ and $D.L = 9$.

g=13 We again have two solutions to the inequalities, both with equality in equation (1.17).

i) $h^0(D) = h^0(F) = 4$. This gives $D^2 = 4$ and $D.L = 12$.

ii) $h^0(D) = 3, h^0(F) = 5$. This gives $D^2 = 2$ and $D.L = 10$.

g=14 We now have four different solutions to the inequalities. Two of them without equality in equation (1.17).

i) $h^0(D) = 3, h^0(F) = 6$. Equality in equation (1.17) gives $D^2 = 2$ and $D.L = 10$.

ii) $h^0(D) = 4, h^0(F) = 5$. Equality in equation (1.17) gives $D^2 = 4$ and $D.L = 12$.

iii) $h^0(D) = h^0(F) = 4$.

iv) $h^0(D) = 3, h^0(F) = 5$.

For $g = 8, 10, 11, 12$, and 13 we have a partial converse to the above computations.[9]

Corollary 1.4.7. *Let (S, L) be a Clifford general polarized K3 surface. If $g = 8$ resp. 10 resp. 11, then (S, L) is not BN general if and only if there exists an effective divisor D satisfying $D^2 = 2$ and $D.L = 7$ resp. 8 resp. 9.*

If $g = 12$ resp. 13 then (S, L) is not BN general if and only if there exists an effective divisor D satisfying either $D^2 = 2$ and $D.L = 9$ resp. 10 or $D^2 = 4$ and $D.L = 11$ resp. 12.

Proof. The computations above give the only if part. We will now show the if part. Using Riemann-Roch and the fact that L is nef it is immediate that $h^0(L-D) \geq 2$ in all of the above cases, since $(L - D)^2 \geq 0$ and $(L - D).L > 0$. Likewise we get $h^0(D) \geq 2$.

[9][JK, proposition 10.1] is a special case of the following proposition.

Using

$$\begin{aligned} h^0(D) &= \frac{1}{2}D^2 + 2 + h^1(D) \\ &\geq \frac{1}{2}D^2 + 2 \end{aligned}$$

and

$$\begin{aligned} h^0(F) &= \frac{1}{2}F^2 + 2 + h^1(F) \\ &= \frac{1}{2}(L^2 - D^2 - 2D.F) + 2 + h^1(F) \\ &\geq \frac{1}{2}(2(g-1) + D^2 - 2D.L) + 2 \end{aligned}$$

we get

$$h^0(D)h^0(F) \geq g + 1$$

in all of the cases. □

Remark 1.4.8. The proof of the proposition only holds for those genus g for which the inequalities in proposition 1.4.6 only have solutions with equality in equation (1.17). This only holds for the genera treated in the proposition, so it cannot be extended to higher genus. We will see later that there exists K3 surfaces satisfying every case of the proposition, so none of the cases are superfluous.

The proposition also shows that when $g = 8, 10, 11, 12$, or 13 every non-BN general Clifford general K3 surface (S, L) satisfies

$$\operatorname{Cliff}(L) = \mu(L) = \left\lfloor \frac{g-1}{2} \right\rfloor.$$

This is not necessarily the case if $g > 13$. Consider for example the polarized K3 surface (S, L) of genus 14 with lattice $\mathbb{Z}D \oplus \mathbb{Z}F$ $(L \sim D + F)$ and intersection matrix

$$\begin{bmatrix} D^2 & D.F \\ D.F & F^2 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 9 & 4 \end{bmatrix}$$

Then $\operatorname{Cliff}(L) = 6$, but $\mu(L) = 7$.[10] Because of this it becomes much harder to determine possible scroll types for $g > 13$ than for $g \leq 13$.

We have seen that a polarized K3 surface (S, L) of genus $g = 2, 3, \ldots, 7$ or 9 is Clifford general if and only if it is BN general. The next theorem will show that this equivalence holds for no other g. We start with a lemma which is just a special case of proposition 1.1.22.[11]

[10] Proof: The existence of such a polarized K3 surface follows from proposition 1.1.24. We have $\operatorname{Cliff}(L) \leq 6$. Equations 1.9 and 1.10 together with lemma 1.4.9 gives $\mu(L) = 7$. Thus $\operatorname{Cliff}(L) = 6$ by theorem 1.3.8.

[11] We include it because we will use the proof of the lemma in the proof of the theorem.

Lemma 1.4.9. *Let S be a K3 surface with $Pic\, S = \mathbb{Z}L \oplus \mathbb{Z}D$. If $aL + bD = M$ and $cL + bD = N$ are elements in $Pic\, S$, then the determinant of the intersection matrix of L and D divides the determinant of the intersection matrix of M and N.*

Proof. This follows immediately from computations of the determinants. We have

$$\begin{vmatrix} L^2 & L.D \\ D.L & D^2 \end{vmatrix} = L^2D^2 - (L.D)^2$$

and

$$\begin{vmatrix} M^2 & M.N \\ N.M & N^2 \end{vmatrix} = (a^2d^2 + b^2c^2 - 2abcd)(L^2D^2 - (L.D)^2).$$

$\square$

Theorem 1.4.10. *For $g = 8$ and $g \geq 10$ there exists polarized K3 surfaces that are Clifford general and non–BN general.*

Proof. We will for $g = 8$ and $g \geq 10$ construct K3 surfaces that are Clifford general and non–BN general. The lattice $\mathbb{Z}L \oplus \mathbb{Z}D$ with intersection matrix

$$A = \begin{bmatrix} L^2 & L.D \\ D.L & D^2 \end{bmatrix} = \begin{bmatrix} 2(g-1) & \left\lfloor \frac{g+7}{2} \right\rfloor \\ \left\lfloor \frac{g+7}{2} \right\rfloor & 2 \end{bmatrix}$$

has signature (1,1), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}L \oplus \mathbb{Z}D$.

We will show that (S, L) is Clifford general and non–BN general (after a suitable change of basis) for $g = 8$ and $g \geq 10$.

Using Picard-Lefschetz reflections we may assume that L is nef. L is then also base point free. For if L was not base point free, there would exist a curve C with $C^2 = 0$ and $C.L = 1$ (proposition 1.1.12), a contradiction by lemma 1.4.9. Hence (S, L) is a polarized K3-surface.

We will now show that D is nef by arguing along the lines of the first part of the proof of [Knu01b, proposition 4.4]. Assume D is not nef. Then there exists a smooth rational curve Γ such that $D.\Gamma < 0$ and $\Gamma^2 = -2$ (corollary 1.1.9). Since L is nef $\Gamma.L \geq 0$. If $\Gamma.L > 0$ then we set $e := -D.\Gamma \geq 1$ and $D' := D - e\Gamma$. Then $D'^2 = D^2 = 2$ so by Riemann-Roch either $|D'|$ or $|-D'|$ contains an effective member. Since $h^0(e\Gamma) = 1$ it must be $|D'|$.For if $e\Gamma - D \geq 0$, then $e\Gamma \geq D$. Whence $1 = h^0(e\Gamma) \geq h^0(D) \geq 3$, a contradiction. Hence

$$0 \leq D'.L = D.L - e(\Gamma.L) < D.L.$$

Looking at the proof of lemma 1.4.9 we see that with $L \sim M$ and $D' \sim cL + dD = N$ we have

$$(D'.L)^2 - D^2L^2 \geq 0,$$

with equality only if $d = 0$. If $d = 0$ we get, with $\Gamma \sim xL + yD$, that $cL \sim D' = D - e\Gamma \sim D - e(xL + yD)$, i.e. $e = y = 1$. But then $-1 = \Gamma.D = xD.L + 2$, which is impossible with $x \in \mathbb{Z}$. Hence

$$0 < (D'.L)^2 - D'^2 L^2 = (D'.L)^2 - D^2 L^2 < (D.L)^2 - D^2 L^2,$$

which contradicts lemma 1.4.9.

It remains to look at $\Gamma.L = 0$. We use lemma 1.4.9 again. This gives

$$4(g-1) - \left\lfloor \frac{g+7}{2} \right\rfloor^2 = \det A \left| \begin{vmatrix} L^2 & L.\Gamma \\ \Gamma.L & \Gamma^2 \end{vmatrix} = \begin{vmatrix} 2(g-1) & 0 \\ 0 & -2 \end{vmatrix} = -4(g-1).\right.$$

So

$$\left(4(g-1) - \left\lfloor \frac{g+7}{2} \right\rfloor^2\right) a = 4(g-1)$$

for some integer a. We get

$$4(g-1)(a-1) = \left\lfloor \frac{g+7}{2} \right\rfloor^2 a \leq \left(\frac{g+8}{2}\right)^2 a.$$

One easily sees $a > 1$. Thus

$$\frac{16(g-1)}{(g+8)^2} \geq \frac{a}{a-1} > 1,$$

a contradiction. Hence D is nef.

We will now show that (S, L) is Clifford general. Assume (S, L) is not Clifford general with Clifford index $c < \lfloor \frac{g-1}{2} \rfloor$. Then there exists a free Clifford divisor $M \sim aL + bD$ on S.[12] In particular M and $L - M$ are effective. Since $h^0(M) \geq 2$ and $h^1(M) = 0$ we have $M^2 \geq 0$ by Riemann-Roch. Hence $0 \leq M.L - M^2 - 2 = c \leq M.L - 2$, in particular $M.L \geq 2$. Since L is nef we also have $(L - M).L \geq 0$. Combining these inequalities one gets

$$1 - \frac{bD.L}{L^2} \geq a \geq \frac{2}{L^2} - \frac{bD.L}{L^2}.$$

Hence

$$a = \left\lfloor 1 - \frac{bD.L}{L^2} \right\rfloor. \tag{1.18}$$

Likewise since D is nef we get $M.D \geq 0$ and $(L - M).D \geq 0$. Which leads to the numerical conditions

$$1 - \frac{bD^2}{D.L} \geq a \geq -\frac{bD^2}{D.L}.$$

This gives two cases to consider: (A) $a = \left\lfloor 1 - \frac{bD^2}{D.L} \right\rfloor$ and (B) $a = \left\lceil -\frac{bD^2}{D.L} \right\rceil$.

[12] $M \nsim D$ since $D.(L - D) - 2 = \lfloor \frac{g-1}{2} \rfloor$.

(A) If $a = \left\lfloor 1 - \frac{bD^2}{D.L} \right\rfloor$ we must have (using equation (1.18))

$$-\frac{bD.L}{L^2} - 1 < -\frac{bD^2}{D.L} < -\frac{bD.L}{L^2} + 1.$$

That is

$$\left|b((L.D)^2 - L^2D^2)\right| < (D.L)L^2. \tag{1.19}$$

If $a = 0$, then $0 < b \leq \frac{D.L}{D^2}$ and $M \sim bD$. But (C2) gives $M.L \leq (L - M).L$, so $2bD.L = 2M.L \leq L^2$. Hence

$$b \leq \frac{L^2}{2D.L} = \frac{g-1}{\left\lfloor \frac{g+7}{2} \right\rfloor} < 2.$$

This gives $M = D$, a contradiction.

If $a = 1$, then $b \leq 0$ and $M \sim L + bD$. From $0 \leq M.L - M^2 - 2$ and $0 \leq M^2$ we get

$$b \geq \frac{-2(g-2)}{\left\lfloor \frac{g+7}{2} \right\rfloor} > -4.$$

So $b = -1$, -2, or -3. In all these cases we get a contradiction by looking at $M.L - M^2 - 2 \geq \left\lfloor \frac{g-1}{2} \right\rfloor$. This will get a contradiction for all but a finite number of genera. The remaining genera are treated individually. For example if $b = -3$ we get

$$M.L - M^2 - 2 = 3\left\lfloor \frac{g-1}{2} \right\rfloor - 8 \geq \left\lfloor \frac{g-1}{2} \right\rfloor,$$

for $g \geq 9$. If $g = 8$, then $M.L = -7$ and M is non-effective since L is nef. Thus we get a contradiction for $g = 8$ too.

If $|b| \geq 6$, then

$$\begin{aligned}
\left|b((L.D)^2 - L^2D^2)\right| &\geq 6\left(\left(\frac{g+6}{2}\right)^2 - 4(g-1)\right) \\
&> g^2 + 6g - 7 \\
&\geq 2\left\lfloor \frac{g+7}{2} \right\rfloor (g-1) \\
&= (D.L)L^2,
\end{aligned}$$

which contradicts equation (1.19).

If $g \geq 16$ and $|b| < 6$, then $a = 0$ or $a = 1$ which we have seen is impossible. It remains to look at $g < 17$ and $|b| < 6$.

$g = 8$: $|b| \leq 3$ gives $a = 0$ or $a = 1$. $|b| = 4$ and $|b| = 5$ contradict $4 \leq 2M.L \leq L^2$.

$g = 10$: $|b| \leq 3$ and $b = 4$ give $a = 0$ or $a = 1$. $b = -4$ contradict $M.L - M^2 - 2 < \left\lfloor \frac{g-1}{2} \right\rfloor$. $|b| = 5$ contradicts $4 \leq 2M.L \leq L^2$.

$g = 11, 13, 15$: We either get $a = 0$, $a = 1$ or a contradiction to equation (1.19).

$g = 14$: We only get cases with $a = 0$ or $a = 1$.

(B) If $a = \left\lceil -\frac{bD^2}{D.L} \right\rceil \neq \left\lfloor 1 - \frac{bD^2}{D.L} \right\rfloor$, then $\frac{bD^2}{D.L} \in \mathbb{Z}$, so we can write $a = -\frac{bD^2}{D.L}$. Hence we have

$$1 - \frac{bD.L}{L^2} < 1 - \frac{bD^2}{D.L},$$

which gives $b > 0$, and

$$-\frac{bD^2}{D.L} \leq 1 - \frac{bD.L}{L^2},$$

which gives

$$b \leq \frac{(D.L)L^2}{(D.L)^2 - L^2D^2} < \frac{D.L}{D^2}$$

for $g > 9$. Then $\frac{bD^2}{D.L} \notin \mathbb{Z}$, a contradiction.

For $g = 8$ the smallest value of b with $\frac{bD^2}{D.L} = \frac{2b}{7} \in \mathbb{Z}$ is 7, but then $\frac{(D.L)L^2}{(D.L)^2 - L^2D^2} = \frac{14}{3}$, so $b < 7$ and we get a contradiction in this case too.

It only remains to show that S is non–BN general. By arguing as in the proof of proposition 1.4.7 we see that

$$h^0(D) \geq 3$$

and

$$h^0(L - D) \geq (g + 2) - \left\lfloor \frac{g + 7}{2} \right\rfloor$$

So we need only look at when

$$3\left((g + 2) - \left\lfloor \frac{g + 7}{2} \right\rfloor\right) \geq g + 1.$$

For g odd this happens when $g \geq 11$. For g even this happens when $g \geq 8$. □

Remark 1.4.11. The theorem still holds if we by a polarized K3 surface (S, L) mean a K3 surface with an ample line bundle L. In fact all the L we consider in the proof are ample. This follows from the paragraph starting with "It remains to look at $\Gamma.L = 0$ ..." and the Nakai-Moishezon criterion.

The method of proof here can be used to show Clifford generality for other surfaces with $\operatorname{Pic} S$ of rank 2. Especially we see that the K3 surfaces with intersection matrices

$$\begin{bmatrix} L^2 & L.D \\ D.L & D^2 \end{bmatrix} = \begin{bmatrix} 22 & 11 \\ 11 & 4 \end{bmatrix} \tag{1.20}$$

and

$$\begin{bmatrix} L^2 & L.D \\ D.L & D^2 \end{bmatrix} = \begin{bmatrix} 24 & 12 \\ 12 & 4 \end{bmatrix}$$

are Clifford general. Hence we see that there exist K3 surfaces satisfying every case of proposition 1.4.7.

In the previous section we said quite a lot about the relationship between the Clifford index of curves and of K3 surfaces. The reader may be familiar with BN theory on curves and may be wondering how this relates to our concept of BN generality on a K3 surface. We will end this section with a few comments that will clarify this relationship.

We define the *BN number* $\rho(g, r, d)$ by

$$\rho(g, r, d) := g - (r + 1)(g - d + r).$$

The BN theorem asserts that when $\rho(g, r, d) \geq 0$ every curve of genus g possesses a r^r_d, while when $\rho(g, r, d) < 0$ the general curve of genus g has no g^r_d's. (See [ACGH85, chapter V] for details.)

Given a curve C of genus g we define the *BN index* of a divisor D on C to be

$$\rho(C, D) := g - h^0(D)h^0(K_C - D).$$

If we assume that a g^r_d is a complete linear system $|D|$, then

$$r = r(D) = h^0(D) - 1$$

and by Riemann-Roch on curves

$$d = \deg D = h^0(D) - h^0(K_C - D) - 1 + g,$$

so

$$\rho(C, D) = \rho(g(C), r(D), \deg D).$$

We now define the *BN index* of C as

$$\rho(C) := \min\{\rho(C, D)\}$$

and say that C is *BN general* if $\rho(C) \geq 0$.

We have the following result.

Proposition 1.4.12. *Let (S, L) be a polarized K3 surface and $C \in |L|$ a smooth curve. If C is BN general, then (S, L) is BN general.*

Proof. We prove that if (S, L) is non-BN general, then every smooth curve $C \in |L|$ is non-BN general. If (S, L) is non-BN general, then there exists a divisor D such that

$$h^0(D)h^0(L - D) \geq g + 1.$$

By the proof of lemma 1.4.3 we get $h^0(D_C) \geq h^0(D)$ and $h^0(K_C - D_C) = h^0((L - D)_C) \geq h^0(L - D)$. This gives

$$\begin{aligned} \rho(C) &\leq \rho(C, D_C) \\ &= g - h^0(D_C)h^0(K_C - D_C) \\ &\leq g - h^0(D)h^0(L - D) \\ &\leq -1. \end{aligned}$$

Hence C is non-BN gereral. $\square$

It is an open question whether this proposition can be extended to an analouge of theorem 1.3.6.

1.5 Well-behaved divisors and their configurations

Throughout this section we will assume that (S, L) is a polarized K3 surface with non-general Clifford index and that D is a free Clifford divisor.

Lemma 1.5.1. *Assume we are not in one of the cases (E3) or (E4). Let Δ_0 be $\Delta - R_{L,D}$, where $R_{L,D}$ is the sum of the elements in $\mathcal{R}_{L,D}$ with the multiplicity they have in Δ. Then for every effective divisor $B < \Delta$, resp. $B < \Delta_0$, we have $B.L = 0$, resp. $B.D = 0$.*

Proof. We have $L.\Delta = D.\Delta_0 = 0$ (lemma 1.3.13).[13] Hence if an effective divisor $B < \Delta$, resp. $B < \Delta_0$, satisfies $B.L > 0$, resp. $B.D > 0$, then some other effective divisor $B < \Delta$, resp. $B < \Delta_0$, satisfies $B.L < 0$, resp. $B.D < 0$. This is impossible since L, resp. D, is nef.. □

Lemma 1.5.2. *Assume we are not in one of the cases (E3) or (E4). Let Δ_0 be as in lemma 1.5.1. Then for every smooth rational curve Γ in the support of Δ_0, we have $\Gamma.A = 0$ or 1.*

Proof. Since $D + A$ is the moving component of F and Γ is fixed in F, proposition 1.1.17 gives $(D + A).\Gamma = 0$ or 1. Lemma 1.5.1 gives $D.\Gamma = 0$, hence $A.\Gamma = 0$ or 1.[14] □

Remember that we write

$$L \sim 2D + A + \Delta,$$

where $D + A$ is the moving component of $F := L - D$, and Δ is the base divisor of F. We would like to say as much as possible about Δ. If $\Gamma \in \mathcal{R}_{L,D}$, then $A.\Gamma = 0$ or $A.\Gamma = -1$, by proposition 1.1.17 used on $D + A$ and Γ. It is the case $A.\Gamma = -1$ which causes the most trouble. To get rid of this problem we will introduce new divisors A' and Δ' that behave almost as A and Δ, at the expense of Δ' no longer always being the base divisor of F. We need a precise definition.

Definition 1.5.3. The pair (A', Δ') is *well-behaved* if the following properties are satisfied:

(W1)

$$L \sim 2D + A' + \Delta',$$

(W2)

$$\begin{bmatrix} D^2 & D.A & D.\Delta \\ D.A & A^2 & A.\Delta \\ D.\Delta & A.\Delta & \Delta^2 \end{bmatrix} = \begin{bmatrix} D^2 & D.A' & D.\Delta' \\ D.A' & A'^2 & A'.\Delta' \\ D.\Delta' & A'.\Delta' & \Delta'^2 \end{bmatrix}$$

(W3)

$$R_{L,D} \leq \Delta' \leq \Delta$$

[13] This is true also when $L \not\geq 2D$ since then $\Delta = 0$.

[14] See lemma 1.5.9 for an alternative proof.

(W4)

$$\text{For every effective } \Delta'' \leq \Delta' \text{ we have } A'.\Delta'' = 0.$$

Note that if the pair (A', Δ') is well-behaved then $A \leq A' \leq A + \Delta - R_{L,D}$. Note also that to show (W4) it is enough to show that for every smooth rational curve $\Gamma \leq \Delta'$ we have $A'.\Gamma = 0$. Furthermore (W3) is equivalent to $\Delta' \leq \Delta$ and $\Delta'.D = \Delta.D$. Since $\Delta' \leq \Delta$ several of the properties of Δ will be satisfied by Δ'. In particular Δ' will be fixed in F and $h^0(\Delta') = 1$. Also note that (A, Δ) always satisfy (W1)-(W3).

We will show in proposition 1.5.8 that for every polarized K3 surface (S, L) one can find a well-behaved pair (A', Δ').

Lemma 1.5.4. *Assume we are not in one of the cases (E3) or (E4), and that for every $\Gamma \in \mathcal{R}_{L,D}$ we have $\Gamma.A = 0$. Then for every effective divisor $B \leq \Delta$ we have $B.A = 0$. In particular the pair (A, Δ) is well-behaved.*

Proof. If an effective divisor $B \leq \Delta$ satisfies $A.B \neq 0$, then some smooth rational curve $\Gamma \leq \Delta$ (possibly equal to B), must satisfy $A.\Gamma < 0$. But $(D + A).\Gamma = 0$ or 1, by proposition 1.1.17. Hence $\Gamma \in \mathcal{R}_{L,D}$. But then $A.\Gamma = 0$ by the assumptions, a contradiction. □

By considering the explicit classification we give for $g = 12$ in chapter 2, we find that in most cases the most general family of K3-surfaces associated to a particular scroll type will be such that (A, Δ) is well-behaved. There are exceptions. Take for example the scroll type $(3, 2, 2, 1, 0)$ on page 75.

In particular if D is perfect and$h^1(R) = 0$ (and we are not in one of the cases (E0)-(E4)), then (C6) gives $\Delta.D = 0$. So $(2D + A).\Delta = 0$ which gives $\Delta = 0$, since L is numerically 2-connected. We see that $(A, \Delta) = (A, 0)$ is well-behaved.

The following lemma will be crucial when we show below that for every L and D with $h^1(R) \leq 3$ we can find a well-behaved pair (A', Δ'). Remember our definition of a configuration-graph (see p. 5).

Lemma 1.5.5. *Every component of the configuration-graph of Δ contains a $\Gamma \in \mathcal{R}_{L,D}$ as a vertex. In particular the configuration-graph of $2D + \Delta$ is connected.*

Proof. Suppose there exists a component C of the configuration-graph of Δ without a $\Gamma \in \mathcal{R}_{L,D}$ as a vertex. Let the vertices of this component be $\Gamma'_1, \ldots, \Gamma'_M$, and let n_i be the multiplicity of Γ'_i in Δ. Write $\Delta_1 = n_1\Gamma'_1 + \cdots + n_M\Gamma'_M$. We have $\Delta_1.D = 0$ (by assumption) and $\Delta_1.(\Delta - \Delta_1) = 0$ (since C is a component of the configuration-graph). $\Delta_1.L$=0 gives $\Delta_1.A + \Delta_1^2 = 0$. Using $h^1(F_0) = 0$ we get

$$\frac{1}{2}(D + A)^2 + 2 = h^0(D + A) = h^0(D + A + \Delta_1) \geq \frac{1}{2}(D + A + \Delta_1)^2 + 2,$$

which gives $2\Delta_1.A + \Delta_1^2 \leq 0$. Then $\Delta_1.A + \Delta_1^2 = 0$ gives $\Delta_1^2 \geq 0$. A contradiction since $\Delta_1 \neq 0$ and $h^0(\Delta_1) = h^0(\Delta) = 1$. □

Remark 1.5.6. We have in fact that every component of the configuration-graph of Δ contains exactly one $\Gamma \in \mathcal{R}_{L,D}$ as a vertex. For if $\Gamma_1, \Gamma_2 \in \mathcal{R}_{L,D}$ both were in the same component, then there would be a path from Γ_1 to Γ_2 in the configuration-graph. Let the vertices of the path be $\Gamma_1, \Gamma_2, \Gamma_1', \dots, \Gamma_N'$. Write $\gamma = \Gamma_1 + \Gamma_2 + \Gamma_1' + \dots + \Gamma_N'$. Then $\gamma^2 = -2$, $\gamma.L = 0$, and $\gamma.D = 2$. This contradicts [JK01, lemma 6.3.2(c)].

We are now ready to state the main result of this section: a classification of the Δ' in well-behaved pairs (A', Δ') when $c = 4$ and $D^2 = 0$. Remember that by lemma 1.1.14, any two distinct smooth rational curves, Γ_1 and Γ_2, in the support of Δ satisfies $\Gamma_1.\Gamma_2 = 0$ or 1. This will be used extensively (without mention) in the next proof.

$\mathbf{c = 4, D^2 = 0, L^2 \geq 20}$

$\{4{,}0\}$ $D^2 = 0, D.L = 6$

We can find a well-behaved pair (A', Δ') for every L with $c = 4$ and $D^2 = 0$. Let D be a perfect Clifford divisor.

$h^1(R) \neq 0$ if and only if L is in one of the following cases (where every Γ_i is a smooth rational curve with $\Gamma_i.A' = 0$):

$\{4,0\}^a$ $L \sim 2D + A' + \Gamma, D.A' = 5, 20 \leq L^2 = A'^2 + 22 \leq 72, h^1(R) = 1, \mathcal{R}_{L,D} = \{\Gamma\}$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma \end{tikzcd}$$

$\{4,0\}^b$ $L \sim 2D + A' + \Gamma_1 + \Gamma_2, D.A' = 4, 20 \leq L^2 = A'^2 + 20 \leq 36, h^1(R) = 2, \mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2\}$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma_1 \\ \Gamma_2 & \end{tikzcd}$$

$\{4,0\}^c$ $L \sim 2D + A' + 2\Gamma_0 + \dots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}, D.A' = 4, 20 \leq L^2 = A'^2 + 20 \leq 36, h^1(R) = 2, \mathcal{R}_{L,D} = \{\Gamma_0\}$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_N \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+1} \\ & & & \Gamma_{N+2} & \end{tikzcd}$$

$\{4,0\}^d$ $L \sim 2D + A' + \Gamma_1 + \Gamma_2 + \Gamma_3, D.A' = 3, 20 \leq L^2 = A'^2 + 18 \leq 24, h^1(R) = 3, \mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2, \Gamma_3\}$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma_3 \\ \Gamma_1 & \Gamma_2 \end{tikzcd}$$

$\{4,0\}^e$ $L \sim 2D+A'+\Gamma_{-1}+2\Gamma_0+\cdots+2\Gamma_N+\Gamma_{N+1}+\Gamma_{N+2}, D.A'=3, 20 \leq L^2 = A'^2+18 \leq 24, h^1(R)=3, \mathcal{R}_{L,D}=\{\Gamma_{-1},\Gamma_0\}$, with the following configuration:

$$\begin{tikzcd}
D \arrow[r, dash] \arrow[d, dash] & \Gamma_0 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_N \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+1} \\
\Gamma_{-1} & & & \Gamma_{N+2} &
\end{tikzcd}$$

$\{4,0\}^f$ $L \sim 2D+A'+3\Gamma_0+2\Gamma_1+2\Gamma_2+\Gamma_3+\Gamma_4, D.A'=3, 20 \leq L^2 = A'^2+18 \leq 24, h^1(R)=3, \mathcal{R}_{L,D}=\{\Gamma_0\}$, with the following configuration:

$$\begin{tikzcd}
D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[dr, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_3 \\
& & \Gamma_2 \arrow[r, dash] & \Gamma_4
\end{tikzcd}$$

$\{4,0\}^g$ $L \sim 2D+A'+3\Gamma_0+4\Gamma_1+2\Gamma_2+3\Gamma_3+2\Gamma_4+\Gamma_5, D.A'=3, 20 \leq L^2 = A'^2+18 \leq 24, h^1(R)=3, \mathcal{R}_{L,D}=\{\Gamma_0\}$, with the following configuration:

$$\begin{tikzcd}
D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] \arrow[dr, dash] & \Gamma_2 & & \\
& & & \Gamma_3 \arrow[r, dash] & \Gamma_4 \arrow[r, dash] & \Gamma_5
\end{tikzcd}$$

$\{4,0\}^h$ $L \sim 2D+A'+3\Gamma_0+4\Gamma_1+5\Gamma_2+6\Gamma_3+4\Gamma_4+2\Gamma_5+3\Gamma_6, D.A'=3, 20 \leq L^2 = A'^2+18 \leq 24, h^1(R)=3, \mathcal{R}_{L,D}=\{\Gamma_0\}$, with the following configuration:

$$\begin{tikzcd}
D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_3 \arrow[r, dash] \arrow[dr, dash] & \Gamma_4 \arrow[r, dash] & \Gamma_5 \\
& & & & & \Gamma_6 &
\end{tikzcd}$$

Proof. Every divisor we define throughout this proof is assumed to be effective. Every Γ we introduce is by assumption a smooth rational curve. Assume we are in case $\{4,0\}$. Then $D.L = D^2+c+2 = 6$, and since $c=4$ and L is not Clifford general we must have $g \geq 11$, i.e. $L^2 \geq 20$. Note also that $D.L=6$, $\Delta.L=0$, and $L^2 = L.(2D+A+\Delta)$ give $A.L = L^2-12$. Then especially $h^0(R) \neq 0$. For the rest of the proof we will assume that $h^1(R) \neq 0$.

We will first find the classification in the cases where the conditions of lemma 1.5.4 holds.

For now we will assume $\Gamma.A = 0$ for every $\Gamma \in \mathcal{R}_{L,D}$. In particular the assumptions of lemma 1.5.4 are satisfied.

By proposition 1.3.12 and (C6) we have $1 \leq D.\Delta \leq 3$. Since

$$6 = D.L = D.A + D.\Delta,$$

we have three cases to consider:

(A) $D.\Delta = 1$ and $D.A = 5$,

(B) $D.\Delta = 2$ and $D.A = 4$,

(C) $D.\Delta = 3$ and $D.A = 3$.

In all of the cases we have $\Delta^2 = -2D.\Delta$ and $A.\Delta = 0$, by lemma 1.3.13.

In case (A) there has to exist a smooth rational curve Γ such that

$$\{\Gamma\} = \mathcal{R}_{L,D}.$$

By the assumptions $A.\Gamma = 0$. We write $\Delta = \Gamma + \Delta_1$. Then $\Gamma.L = 0$ gives $\Gamma.\Delta_1 = 0$. We also have $D.\Delta = 1$ and $A.\Delta = 0$, whence $D.\Delta_1 = A.\Delta_1 = 0$. Since

$$L \sim 2D + A + \Gamma + \Delta_1$$

and L is numerically 2-connected we have $\Delta_1 = 0$. This gives $\{4, 0\}^a$.

$L^2 = (2D + A + \Gamma)^2$ gives $A^2 = L^2 - 22$. Using this, with $A.L = L^2 - 12$, we get $L^2 \leq 72$ from the Hodge index theorem.

In case (B) we have either

$$\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2\}$$

or

$$\mathcal{R}_{L,D} = \{\Gamma_0\}.$$

Note that all these Γ_i must satisfy $A.\Gamma_i = 0$ by the assumptions.

If $\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2\}$, then $\Gamma_1.\Gamma_2 = 0$, by proposition 1.3.11. Writing $\Delta = \Gamma_1 + \Gamma_2 + \Delta_1$ we have, as in case (A), $\Delta_1.D = \Delta_1.A = \Delta_1.\Gamma_1 = \Delta_1.\Gamma_2 = 0$. Using that L is numerically 2-connected, as above, we get $\{4, 0\}^b$.

If $\mathcal{R}_{L,D} = \{\Gamma_0\}$, then Γ_0 must have multiplicity two in Δ. We write $\Delta = 2\Gamma_0 + \Delta_1$. Note that since $D.\Delta_1 = A.\Delta_1 = 0$ (by lemma 1.5.4) every divisor in Δ_1 is disjoint from D and A (by lemmas 1.5.1 and 1.5.2). From $0 = \Gamma_0.L = \Gamma_0.(2D + A + 2\Gamma_0 + \Delta_1)$, we get $\Gamma_0.\Delta_1 = 2$. Then $\Delta^2 = (2\Gamma_0 + \Delta_1)^2 = -4$ gives $\Delta_1^2 = -4$. Now there exists either two (and only two) disjoint[15] smooth rational curves Γ_1 and Γ_2 with multiplicity one in the support of Δ_1 such that $\Gamma_0.\Gamma_1 = \Gamma_0.\Gamma_2 = 1$ or one and only one smooth rational curve Γ_1 with multiplicity two in the support of Δ_1 such that $\Gamma_0.\Gamma_1 = 1$. We can now iterate until we get $\{4, 0\}^c$. (See also [JK01, proposition 3.6].)

$L^2 = (2D + A + \Delta)^2$ gives $A^2 = L^2 - 20$ Then $L^2 \leq 36$ follows from the Hodge index theorem, using $A.L = L^2 - 12$.

In case (C) we have either

$$\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2, \Gamma_3\}$$

or

$$\mathcal{R}_{L,D} = \{\Gamma_{-1}, \Gamma_0\}$$

[15] Since $h^0(\Gamma_1 + \Gamma_2) = 1$ we get $\Gamma_1.\Gamma_2 = 0$ or 1 by lemma 1.1.14. Write $\Delta_1 \sim \Gamma_1 + \Gamma_2 + \Delta_2$. Then $0 = \Gamma_1.L = \Gamma_1.\Gamma_2 + \Gamma_1.\Delta_2$. Since Γ_1 has multiplicity 1 in Δ_1 we have $\Gamma_1.\Delta_1 \geq 0$. Hence $\Gamma_1.\Gamma_2 = 0$ (and $\Gamma_1.\Delta_1 = 0$).

or

$$\mathcal{R}_{L,D} = \{\Gamma_0\}$$

Note that these Γ_i must have $A.\Gamma_i = 0$ by the assumptions.

$L^2 = (2D + A + \Delta)^2$ gives $A^2 = L^2 - 18$. Then $L^2 \leq 24$ follows from the Hodge index theorem, with $A.L = L^2 - 12$.

In the first case Γ_1, Γ_2, and Γ_3 are disjoint by proposition 1.3.11. Using that L is numerically 2-connected, as above, we get $\{4,0\}^d$.

In the second case either Γ_{-1} or Γ_0 has multiplicity two in Δ. We assume that Γ_0 has multiplicity two in Δ. Then $\Delta = \Gamma_{-1} + 2\Gamma_0 + \Delta_1$. Since $L.\Gamma_{-1} = 0$, we get $\Delta_1.\Gamma_{-1} = 0$. Similarly $\Delta_1.\Gamma_0 = 2$ and $\Delta_1^2 = -4$. We can now iterate as in the case $\{4,0\}^c$ until we get $\{4,0\}^e$.

In the third case Γ_0 has multiplicity three in Δ. Write $\Delta = 3\Gamma_0 + \Delta_1$. $L.\Gamma_0 = 0$ gives $\Delta_1.\Gamma_0 = 4$. Then $\Delta^2 = -6$ gives $\Delta_1^2 = -12$. There must exist a smooth rational curve $\Gamma' \leq \Delta_1$ such that $\Gamma_0.\Gamma' = 1$. Let Γ' be *any* such curve. Write $\Delta_1 \sim \Gamma' + \Delta''$. Since $0 = L.\Gamma' = 1 + \Delta''.\Gamma'$, we see that Γ' has multiplicity at least two in Δ_1. This gives two cases to consider:

(D) $\Delta_1 \sim 2\Gamma_1 + 2\Gamma_2 + \Delta_2$, where $\Gamma_1.\Gamma_0 = \Gamma_2.\Gamma_0 = 1$ and $\Delta_2.\Gamma_0 = 0$,

(E) $\Delta_1 \sim 4\Gamma_1 + \Delta_2$ where $\Gamma_1.\Gamma_0 = 1$ and $\Delta_2.\Gamma_0 = 0$.

Let us look at (D) first. Then we have either $\Gamma_1.\Gamma_2 = 1$ or $\Gamma_1.\Gamma_2 = 0$. The former is impossible, for in that case $0 = \Gamma_1.L = 1 + \Gamma_1.\Delta_2 > \Gamma_1.\Delta_2$, i.e. $\Gamma_1 < \Delta_2$.[16] The latter gives $\Gamma_1.\Delta_2 = \Gamma_2.\Delta_2 = 1$ and $\Delta_2^2 = -4$. We must then be in one of the following three cases:

(D.α)

$$\begin{tikzcd}
D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[dr, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_3 \arrow[r, dash] & \cdots \\
 & & \Gamma_2 \arrow[r, dash] & \Gamma_4 \arrow[r, dash] & \cdots
\end{tikzcd}$$

(D.β)

$$\begin{tikzcd}
D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[dr, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_3 \arrow[r, dash] \arrow[d, dash] & \cdots \\
 & & \Gamma_2 \arrow[r, dash] & \Gamma_4 \arrow[r, dash] & \cdots
\end{tikzcd}$$

(D.γ)

$$\begin{tikzcd}
D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] & \Gamma_1 \arrow[d, dash] & \\
 & \Gamma_2 \arrow[r, dash] & \Gamma_3 \arrow[r, dash] & \cdots
\end{tikzcd}$$

[16] If $\Gamma_1 < \Delta_2$ then $\Gamma_0.(\Delta_2 - \Gamma_1) = -1$, so $\Gamma_0 < \Delta_2$. This gives that Γ_0 has multiplicity at least four in Δ. Hence $D.\Delta \geq 4$, a contradiction.

Using that L is numerically 2-connected, we get $\Delta_2 = \Gamma_3 + \Gamma_4$ in case (D.α). This is just $\{4,0\}^f$.

The cases (D.β) and (D.γ) are impossible by lemma 1.1.14.

Only case (E) remains. We will show that this gives $\{4,0\}^g$ or $\{4,0\}^h$. We have to consider the more general situation (which we will denote by (E,N)):

$$L \sim 2D + A + 3\Gamma_0 + 4\Gamma_1 + \cdots + (N+3)\Gamma_N + \Delta_{N+1}, \quad N > 0,$$

where all the Γ_i have multiplicity zero in Δ_{N+1} and the (partial) configuration looks as follows

$$D \text{ --- } \Gamma_0 \text{ --- } \Gamma_1 \text{ --- } \cdots \text{ --- } \Gamma_N.$$

Note that $L.\Gamma_N = 0$ gives $\Delta_{N+1}.\Gamma_N = N+4$. We will now show that if we are in case (E,N) the we must be in case (E,$N+1$), $\{4,0\}^g$ or $\{4,0\}^h$. If we are not in case $\{4,0\}^g$ or $\{4,0\}^h$ we will obtain a contradiction by iterating.

Let $\Gamma_{N+1} \leq \Delta_{N+1}$ be a smooth rational curve such that $\Gamma_{N+1}.\Gamma_N \neq 0$. Then $\Gamma_{N+1}.\Gamma_N = 1$. Writing $\Delta_{N+1} = \Gamma_{N+1} + \Delta'_{N+2}$ we have $\Gamma_{N+1}.\Delta'_{N+2} = -(N+1)$, since $L.\Gamma_{N+1} = 0$. Hence Γ_{N+1} has multiplicity at least $\lceil (N+1)/2 \rceil + 1 = \lfloor (N+4)/2 \rfloor$ in Δ_{N+1}. Since $\Delta_{N+1}.\Gamma_N = N+4$, this gives us two cases to consider:

(E.α) $\Delta_{N+1} \sim \left\lfloor \frac{N+4}{2} \right\rfloor \Gamma_{N+1} + \left\lfloor \frac{N+5}{2} \right\rfloor \Gamma_{N+2} + \Delta_{N+2}$,
where $\Gamma_N.\Gamma_{N+1} = \Gamma_N.\Gamma_{N+2} = 1$ and $\Delta_{N+2}.\Gamma_N = 0$,

(E.β) $\Delta_{N+1} \sim (N+4)\Gamma_{N+1} + \Delta_{N+2}$,
where $\Gamma_N.\Gamma_{N+1} = 1$ and $\Delta_{N+2}.\Gamma_N = 0$.

If we are in case (E.α) we have two possibilities:

(E.α1) $\Gamma_{N+1}.\Gamma_{N+2} = 0$

(E.α2) $\Gamma_{N+1}.\Gamma_{N+2} = 1$

In case (E.α1) we have

$$0 = \Gamma_{N+1}.L = (N+3) - 2\left\lfloor \frac{N+4}{2} \right\rfloor + \Gamma_{N+1}.\Delta_{N+2}$$

and

$$0 = \Gamma_{N+2}.L = (N+3) - 2\left\lfloor \frac{N+5}{2} \right\rfloor + \Gamma_{N+2}.\Delta_{N+2}.$$

If N is odd this gives $\Gamma_{N+1}.\Delta_{N+2} = 0$ and $\Gamma_{N+2}.\Delta_{N+2} = 2$. This gives two cases to consider: $\Delta_{N+2} = \Gamma_{N+3} + \Gamma_{N+4} + \Delta_{N+3}$, with $\Gamma_{N+2}.\Gamma_{N+3} = \Gamma_{N+2}.\Gamma_{N+4} = 1$, and $\Delta_{N+2} = 2\Gamma_{N+3} + \Delta_{N+3}$, with $\Gamma_{N+2}.\Gamma_{N+3} = 1$. In the first case $0 = \Gamma_{N+3}.L \geq (N+5)/2 - 2 + \Gamma_{N+3}.\Delta_{N+3} > \Gamma_{N+3}.\Delta_{N+3}$, a contradiction. In the second case we have $0 = \Gamma_{N+3}.L = (N+5)/2 - 4 + \Gamma_{N+3}.\Delta_{N+3}$. For $N \geq 5$ this gives $\Gamma_{N+3}.\Delta_{N+3} < 0$, a contradiction.

For $N = 3$ we get $\Gamma_6.\Delta_6 = 0$. If $\Delta_6 \neq 0$ then $\Delta_6^2 < 0$ (since $h^0(\Delta_6) = 1$), but then $L.\Delta_6 < 0$, a contradiction. If $\Delta_6 = 0$ we get $\{4,0\}^h$.

If $N = 1$, then $\Gamma_4.\Delta_4 = 1$. Let Γ_5 be a smooth rational curve in the support of Δ_4 such that $\Gamma_4.\Gamma_5 = 1$. Then $\Gamma_i.\Gamma_5 = 0$ for $0 \leq i \leq 3$. For if $\Gamma_i.\Gamma_5 \neq 0$, then Γ_i would have larger multiplicity than assumed in Δ. Iterating we would get that Γ_0 has larger multiplicity than three in Δ, a contradiction. Since $L.\Gamma_5 = 0$ we get $\Gamma_5.(\Delta_4 - \Gamma_5) = 0$. We also get $\Gamma_i.(\Delta_4 - \Gamma_5) = 0$ for $0 \leq i \leq 4$. Hence $\Delta_4 = \Gamma_5$, since L is numerically 2-connected. This gives $\{4,0\}^g$.

If N is even, then $\Gamma_{N+1}.\Delta_{N+2} = \Gamma_{N+2}.\Delta_{N+2} = 1$. So there exists $\Gamma_{N+3} < \Delta_{N+2}$ such that $\Gamma_{N+2}.\Gamma_{N+3} = 1$. Write $\Delta_{N+2} = \Gamma_{N+3} + \Delta_{N+3}$. Then we have $0 = \Gamma_{N+3}.L \geq N/2 + \Delta_{N+3}.\Gamma_{N+3} > \Delta_{N+3}.\Gamma_{N+3}$, a contradiction.

In case (E.α2) we have

$$\begin{aligned} 0 &= \Gamma_{N+1}.L \\ &= (N+3) - 2\left\lfloor \frac{N+4}{2} \right\rfloor + \left\lfloor \frac{N+5}{2} \right\rfloor + \Gamma_{N+1}.\Delta_{N+2} \\ &> \Gamma_{N+1}.\Delta_{N+2}, \end{aligned}$$

a contradiction.

It remains to look at case (E.β). But this is equal to case (E,$N+1$). If $N+1 \geq 18$ we get a contradiction, since $\operatorname{Pic} S \leq 20$. If $N+1 < 18$, then we must be in case (E.β), since we have shown that case (E,α) is impossible with $N+1 > 1$. Hence we can iterate until we are in case (E,N'), where $N' \geq 18$.

We now stop assuming $\Gamma.A = 0$ for every $\Gamma \in \mathcal{R}_{L,D}$. Proposition 1.5.8 gives that we can always find a well-behaved pair (A', Δ'). To complete the proof of the theorem interchange every A, resp. Δ, in the above with A', resp. Δ'. □

Remark 1.5.7. We have only used the assumption $c = 4$ to show that $h^1(R) \leq 3$ and to compute $D.A$ and $L.A$. Hence the whole proof holds for any $c < 4$, we only have to restrict ourselves to a lower value of $h^1(R)$ and change the values of $D.A$ and $L.A$. The results we then get for $c = 1, 2$, and 3, are given in [JK01, pp.77–78]. The results as stated there are not entirely correct, since they have overlooked the possibility of $A.\Gamma = -1$ for $\Gamma \in \mathcal{R}_{L,D}$. One gets correct results by substituting A with A' in [JK01, pp.77–78].

If we replace $h^1(R)$ with $D.\Delta$ in the statement of the classification it is enough to assume that D is a free Clifford divisor.

Note the resemblance between the cases $\{4,0\}^d$–$\{4,0\}^f$ and the cases (CG3)-(CG5) in [JK01, pp.74-75]. There should also be a case (CG6) resembling $\{4,0\}^g$ and a case (CG7) resembling $\{4,0\}^h$ These cases have for mysterious reasons been omitted in [JK01]. These cases should be included both on [JK01, p.75] and in the table on [JK01, p.89].

An alternative way to classify Δ' is by using [Bǎd01, theorem 3.22]. See theorem 1.1 for details.

The proof we used to show that $\Gamma.A = 0$ if $\Gamma \in \mathcal{R}_{L,D}$ with multiplicity two or three does not work when the multiplicity is larger than three.

It is possible to find Δ also, but it requires a bit more work. We include the result here without supplying any proof. Note that for our classification purposes in the next chapter it is enough to have found Δ'. In the following we let $\Gamma_0 \in \mathcal{R}_{L,D}$.

If Γ_0 has multiplicity one in Δ and $\Gamma_0.A = -1$, then the component of Δ that contains Γ_0 looks as follows:
$\Gamma_0 + \cdots + \Gamma_N$ with the following configuration

$$\begin{tikzcd} \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_N \end{tikzcd}$$

where $\Gamma_N.A = 1$.

If Γ_0 has multiplicity two in Δ and $\Gamma_0.A = -1$, then the component of Δ that contains Γ_0 looks as follows:
$2\Gamma_0 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2} + \Gamma_{N+3}$ with the following configuration

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & \Gamma_N \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+1} \\ & \Gamma_{N+3} & & \Gamma_{N+2} & \end{tikzcd}$$

where $\Gamma_N.A = 1$, or
$2\Gamma_0 + 3\Gamma_1 + 4\Gamma_2 + 3\Gamma_3 + 2\Gamma_4 + 2\Gamma_5$ with the following configuration

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_2 \arrow[r, dash] \arrow[d, dash] & \Gamma_5 \\ & & & \Gamma_3 \arrow[r, dash] & \Gamma_4 \end{tikzcd}$$

where $\Gamma_4.A = 1$.

If Γ_0 has multiplicity three in Δ and $\Gamma_0.A = -1$, then the component of Δ that contains Γ_0 looks as follows:
$3\Gamma_0 + \cdots + 3\Gamma_N + 2\Gamma_{N+1} + \Gamma_{N+2} + 2\Gamma_{N+3} + \Gamma_{N+4}$ with the following configuration

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[dr, dash] & \cdots \arrow[r, dash] & \Gamma_N \arrow[r, dash] & \Gamma_{N+1} \arrow[r, dash] & \Gamma_{N+2} \\ & & \Gamma_{N+3} \arrow[r, dash] & \Gamma_{N+4} & & \end{tikzcd}$$

where $\Gamma_N.A = 1$, or
$3\Gamma_0 + 5\Gamma_1 + 3\Gamma_2 + 4\Gamma_3 + 3\Gamma_4 + 2\Gamma_5 + \Gamma_6$ with the following configuration

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] \arrow[d, dash] & \Gamma_2 & \\ & & \Gamma_3 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_6 \end{tikzcd}$$

where $\Gamma_2.A = 1$.

As a consequence of the above we see that if D is of type $\{4,0\}^c$ with $N \neq 0$ and $N \neq 2$, then $(A, \Delta) = (A', \Delta')$, i.e. (A, Δ) is well-behaved. Likewise if D is of type $\{4,0\}^h$, then $(A, \Delta) = (A', \Delta')$.

We will now give a proof by Knutsen that shows that for all values of $D.\Delta$ we can find a well-behaved pair (A', Δ').

Proposition 1.5.8. *Given a polarized K3 surface (S, L) with non-general Clifford index and a free Clifford divisor D not of type (E3)-(E4), one can always find a well-behaved pair (A', Δ').*

Proof. We will give an algorithmic proof. We need the following lemma.

Lemma 1.5.9. *Suppose we have a pair of divisors (A_i, Δ_i) such that (W1)–(W3) holds.*[17] *If there exists a smooth rational curve $\Gamma \leq \Delta_i$ such that $\Gamma.A_i > 0$, then $\Gamma.A_i = 1$ and $\Gamma.D = 0$.*

Proof of lemma. Remember that F_0 is the moving component of F. Then $F_0 = F - \Delta \sim D + A$. We write $F_i := D + A_i$. Then $F_0 \leq F_i \leq F$. Hence we have $h^0(F_i) = h^0(F_0)$. Since (A_i, Δ_i) satisfies (W2) we have $F_0^2 = F_i^2$. Riemann-Roch then gives $h^1(F_i) = h^1(F_0) = 0$.[18]

Using Riemann-Roch yet another time gives

$$h^0(F_i + \Gamma) - h^0(F_i) = F_i.\Gamma - 1 + h^1(F_i + \Gamma) = 0.$$

Hence $F_i.\Gamma \leq 1$.

Since $D.\Gamma \geq 0$ we get $\Gamma.A_i \leq 1$. So if $\Gamma.A_i > 0$, then $\Gamma.A_i = 1$ and $\Gamma.D = 0$. □

Write $\Delta_0 := \Delta$ and $A_0 := A$.

Given (A_i, Δ_i) that satisfy (W1)-(W3) assume that there exists a smooth rational curve $\Gamma \leq \Delta_i$ such that $\Gamma.A_i > 0$. Write $A_{i+1} := A_i + \Gamma$ and $\Delta_{i+1} := \Delta_i - \Gamma$. Then (A_{i+1}, Δ_{i+1}) satisfies (W1)-(W3) by the lemma.

We repeat this as long as there exists a smooth rational curve $\Gamma \leq \Delta_i$ such that $\Gamma.A_i > 0$. It is obvious by the definition of divisors that the procedure will stop after finitely many steps. When there no longer exists such a curve (W4) holds also. Hence we have a well-behaved pair of divisors. □

We will end this section with a list of the possible components of Δ'.

Theorem 1.5.10. *Let D be a free Clifford divisor associated to the base point free divisor L and (A', Δ') a well-behaved pair. Then every component of Δ' looks like one of the divisors in table 1.1.*

Proof. We will show this using [Băd01, theorem 3.32]. It says that every component of Δ' has as configuration-graph (A_n), (D_n), (E_6), (E_7), or (E_8). We will find possible components with configuration-graph (A_n), the other cases are left to the reader. With multiplicity we will mean multiplicity in Δ'.

So assume that we have a component $\mathcal{C}$ with configuration-graph (A_n). We may assume $n > 1$ since $n = 1$ is trivial. Let Γ_0 be the vertex of the configuration-graph that is contained in $\mathcal{R}_{L,D}$.[19] Let n_0 be the multiplicity of Γ_0. Let Γ'_1 be an end vertex of (A_n)

[17] Remember that (W1)–(W3) always holds for (A, Δ).
[18] Here we use that we are not in one of the cases (E3) or (E4).
[19] This vertex exists by lemma 1.5.5 and is unique by remark 1.5.6.

that is not contained in $\mathcal{R}_{L,D}$. We may assume that the partial configuration looks as follows ($k \geq 1$)

$$\Gamma_0 \text{ —— } \Gamma'_k \text{ —— } \cdots \text{ —— } \Gamma'_1$$

Let $n'_i > 0$ be the multiplicity of Γ'_i ($1 \leq i \leq k$). Since $L.\Gamma'_1 = 0$ we have $n'_2 = 2n'_1$. Since $L.\Gamma'_2 = 0$ we have $n'_1 + n'_3 - 2n'_2 = 0$, i.e. $n'_3 = 3n'_1$. Continuing like this we get $n'_k = kn'_1$. Then $L.\Gamma'_k = 0$ gives $n_0 = (k+1)n'_1$. Especially

$$n'_k = \frac{k}{k+1} n_0. \tag{1.21}$$

If Γ_0 is an end vertex of (A_n), then

$$0 = L.\Gamma_0 = 2D.\Gamma_0 + n_0\Gamma_0^2 + n'_k\Gamma'_k.$$

This gives

$$\left(2 - \frac{k}{k+1}\right) n_0 = 2.$$

Since n_0 is a positive integer and $2 - k/(k+1) > 1$ we get $n_0 = 1$ and $k = 0$. Hence $n = 1$.

If Γ_0 is not an end vertex of (A_n), then the configuration looks as follows ($k, l > 0$)

$$\begin{array}{ccccccccccccc}
D & \text{——} & \Gamma_0 & \text{——} & \Gamma_1 & \text{——} & \Gamma_2 & \text{——} & \cdots & \text{——} & \Gamma_k \\
 & & & \diagdown & & & & & & & \\
 & & & & \Gamma'_1 & \text{——} & \Gamma'_2 & \text{——} & \cdots & \text{——} & \Gamma'_l
\end{array}$$

Let n_i be the multiplicity of Γ_i and n'_i the multiplicity of Γ'_i. Equation (1.21) gives

$$n_1 = \frac{k}{k+1} n_0$$

and

$$n'_1 = \frac{l}{l+1} n_0.$$

In particular $(l+1) | n_0$.

$L.\Gamma_0 = 2 + n_1 + n'_1 - 2n_0 = 0$ gives

$$n_0 = 2\frac{(k+1)(l+1)}{l+k+2}. \tag{1.22}$$

Then

$$\frac{n_0}{l+1} = 2\frac{k+1}{l+k+2} \in \mathbb{Z}.$$

The number on the right hand side is an integer between 0 and 2 since $0 < k+1 < l+k+2$. That is

$$2\frac{k+1}{l+k+2} = 1,$$

i.e. $k = l$. Equation (1.22) gives $n_0 = k + 1$. This gives the first line of table 1.1 (after a change of indices).

One finds the rest of table 1.1 in a similar fashion: First one fixes a vertex Γ_0 in the configuration-graph and assumes $\Gamma_0 \in \mathcal{R}_{L,D}$. Then one calculate the multiplicities of the vertices using equation (1.21). (We discard the cases where the multiplicities are non–integral. An example of such a case is when we choose Γ_0 to be one of the vertices in (E_6) with degree 2.) $\square$

Table 1.1: Components of Δ'

<table>
<tr><td>component</td><td>configuration</td></tr>
<tr><td>$(N+1)\Gamma_0+N\Gamma_1+N\Gamma_2+(N-1)\Gamma_3+(N-1)\Gamma_4+\cdots+\Gamma_{2N-1}+\Gamma_{2N}$ $(0 \leq N \leq 9)$</td><td>$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[dr, dash] & \Gamma_1 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_{2N-1} \\ & & \Gamma_2 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_{2N} \end{tikzcd}$$</td></tr>
<tr><td>$k\Gamma_0+\cdots+k\Gamma_N+\frac{k}{2}\Gamma_{N+1}+\frac{k}{2}\Gamma_{N+2}+(k-2)\frac{k}{k-1}\Gamma'_1+(k-4)\frac{k}{k-1}\Gamma'_2+\cdots+2\Gamma'_M$ (k even, $M=(k-2)/2$, $M,N \geq 0$, $M+N \leq 16$)</td><td>$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & \Gamma_N \arrow[r, dash] \arrow[dr, dash] & \Gamma_{N+1} \\ & \Gamma'_1 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma'_M & \Gamma_{N+2} \end{tikzcd}$$</td></tr>
<tr><td>$k\Gamma_0+(2k-2)\Gamma_1+(2k-3)\Gamma_2+\cdots+\Gamma_N+(k-1)\Gamma_{N+1}$ $(N=2k-2 \leq 17)$</td><td>$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+1} & \\ & & \Gamma_2 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_N \end{tikzcd}$$</td></tr>
<tr><td>$3\Gamma_0+4\Gamma_1+5\Gamma_2+6\Gamma_3+4\Gamma_4+2\Gamma_5+3\Gamma_6$</td><td>$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_3 \arrow[r, dash] \arrow[d, dash] & \Gamma_6 \\ & & & \Gamma_4 \arrow[r, dash] & \Gamma_5 \end{tikzcd}$$</td></tr>
<tr><td>$4\Gamma_0+6\Gamma_1+4\Gamma_2+4\Gamma_3+2\Gamma_4+2\Gamma_5$</td><td>$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] \arrow[dr, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_4 \\ & & & \Gamma_3 \arrow[r, dash] & \Gamma_5 \end{tikzcd}$$</td></tr>
<tr><td>$4\Gamma_0+6\Gamma_1+8\Gamma_2+10\Gamma_3+12\Gamma_4+6\Gamma_5+8\Gamma_6+4\Gamma_7$</td><td>$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_4 \arrow[r, dash] \arrow[d, dash] & \Gamma_5 \\ & & & \Gamma_6 \arrow[r, dash] & \Gamma_7 \end{tikzcd}$$</td></tr>
<tr><td>$7\Gamma_0+12\Gamma_1+8\Gamma_2+4\Gamma_3+9\Gamma_4+6\Gamma_5+3\Gamma_6$</td><td>$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] \arrow[d, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_3 \\ & & \Gamma_4 \arrow[r, dash] & \Gamma_5 \arrow[r, dash] & \Gamma_6 \end{tikzcd}$$</td></tr>
<tr><td>$8\Gamma_0+4\Gamma_1+10\Gamma_2+12\Gamma_3+6\Gamma_4+8\Gamma_5+4\Gamma_6$</td><td>$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_3 \arrow[r, dash] \arrow[d, dash] & \Gamma_4 \\ & \Gamma_1 & & \Gamma_5 \arrow[r, dash] & \Gamma_6 \end{tikzcd}$$</td></tr>
</table>

$12\Gamma_0 + 6\Gamma_1 + 16\Gamma_2 + 8\Gamma_3 + 12\Gamma_4 + 8\Gamma_5 + 4\Gamma_6$	$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] & \Gamma_2 \arrow[r, dash] \arrow[d, dash] & \Gamma_3 & \\ & \Gamma_1 & \Gamma_4 \arrow[r, dash] & \Gamma_5 \arrow[r, dash] & \Gamma_6 \end{tikzcd}$
$12\Gamma_0 + 6\Gamma_1 + 16\Gamma_2 + 20\Gamma_3 + 24\Gamma_4 + 12\Gamma_5 + 16\Gamma_6 + 8\Gamma_7$	$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] & \Gamma_2 \arrow[d, dash] & \Gamma_5 \arrow[d, dash] & \Gamma_7 \arrow[d, dash] \\ & \Gamma_1 & \Gamma_3 \arrow[r, dash] & \Gamma_4 \arrow[r, dash] & \Gamma_6 \end{tikzcd}$
$15\Gamma_0 + 10\Gamma_1 + 5\Gamma_2 + 18\Gamma_3 + 12\Gamma_4 + 6\Gamma_5 + 9\Gamma_6$	$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] & \Gamma_3 \arrow[r, dash] \arrow[dr, dash] & \Gamma_4 \arrow[r, dash] & \Gamma_5 \\ & \Gamma_1 \arrow[r, dash] & \Gamma_2 & \Gamma_6 & \end{tikzcd}$
$16\Gamma_0 + 30\Gamma_1 + 20\Gamma_2 + 10\Gamma_3 + 24\Gamma_4 + 18\Gamma_5 + 12\Gamma_6 + 6\Gamma_7$	$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] \arrow[dl, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_3 \\ & \Gamma_4 \arrow[r, dash] & \Gamma_5 \arrow[r, dash] & \Gamma_6 \arrow[r, dash] & \Gamma_7 \end{tikzcd}$
$24\Gamma_0 + 12\Gamma_1 + 16\Gamma_2 + 16\Gamma_3 + 8\Gamma_4 + 8\Gamma_5$	$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_4 \\ & \Gamma_1 & \Gamma_3 \arrow[r, dash] & \Gamma_5 \end{tikzcd}$
$24\Gamma_0 + 12\Gamma_1 + 16\Gamma_2 + 8\Gamma_3 + 18\Gamma_4 + 12\Gamma_5 + 6\Gamma_6$	$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_3 & \\ & \Gamma_1 & \Gamma_4 \arrow[r, dash] & \Gamma_5 \arrow[r, dash] & \Gamma_6 \end{tikzcd}$
$24\Gamma_0 + 16\Gamma_1 + 8\Gamma_2 + 30\Gamma_3 + 36\Gamma_4 + 18\Gamma_5 + 24\Gamma_6 + 12\Gamma_7$	$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] & \Gamma_3 \arrow[r, dash] & \Gamma_4 \arrow[r, dash] \arrow[d, dash] & \Gamma_5 \\ & \Gamma_1 \arrow[r, dash] & \Gamma_2 & \Gamma_6 \arrow[r, dash] & \Gamma_7 \end{tikzcd}$
$28\Gamma_0 + 14\Gamma_1 + 40\Gamma_2 + 20\Gamma_3 + 32\Gamma_4 + 24\Gamma_5 + 16\Gamma_6 + 8\Gamma_7$	$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] & \Gamma_2 \arrow[r, dash] \arrow[d, dash] & \Gamma_4 \arrow[d, dash] & \Gamma_7 \arrow[d, dash] \\ & \Gamma_1 & \Gamma_3 & \Gamma_5 \arrow[r, dash] & \Gamma_6 \end{tikzcd}$
$40\Gamma_0 + 30\Gamma_1 + 20\Gamma_2 + 10\Gamma_3 + 48\Gamma_4 + 24\Gamma_5 + 32\Gamma_6 + 16\Gamma_7$	$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[dr, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_3 \\ & \Gamma_5 \arrow[r, dash] & \Gamma_4 \arrow[r, dash] & \Gamma_6 \arrow[r, dash] & \Gamma_7 \end{tikzcd}$
$60\Gamma_0 + 30\Gamma_1 + 20\Gamma_2 + 10\Gamma_3 + 48\Gamma_4 + 36\Gamma_5 + 24\Gamma_6 + 12\Gamma_7$	$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] \arrow[dl, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_3 & \\ \Gamma_1 & \Gamma_4 \arrow[r, dash] & \Gamma_5 \arrow[r, dash] & \Gamma_6 \arrow[r, dash] & \Gamma_7 \end{tikzcd}$

1.6 On the relationship of the Clifford divisors of L, $L - D$, and $L + D$

Remember that given a polarized K3 surface (S, L) with perfect Clifford divisor D the associated scroll type is $(e_1, \dots, e_d)$, with

$$e_i = \#\{j | d_j \geq i\} - 1,$$

where

$$\begin{array}{rcl} d = d_0 & = & h^0(L) - h^0(L - D) \\ d_1 & = & h^0(L - D) - h^0(L - 2D) \\ \vdots & \vdots & \vdots \\ d_i & = & h^0(L - iD) - h^0(L - (i+1)D) \\ \vdots & \vdots & \vdots \\ d_n & = & h^0(L - nD) \\ d_{n+1} & = & 0 \end{array}$$

We will throughout this section let n denote the largest integer such that $h^0(L - nD) > 0$, i.e. $n := e_1$. Note that if $h^1(L - 2D) = h^1(R) = 0$ and $D^2 = 0$, then $d_1 = d_0 = d$. Furthermore $L - D$ is base point free in this case. Hence we can use the machinery on p. 21 to get a scroll containing $\phi_{L-D}(S)$. This scroll has scroll type $(e_1 - 1, \dots, e_d - 1)$.

We will now give some results on the relationship between the perfect Clifford divisors of L and $L + D$. We will need a numerical lemma.[20]

Lemma 1.6.1. *Let (S, L) be a polarized K3 surface with genus g and Clifford index c. If $4(g-1) > (c+4)^2$, then any Clifford divisor D must satisfy $D^2 = 0$.*

Proof. By equation (1.9) we have $0 \leq D^2 \leq c + 2$. Any Clifford divisor must also satisfy equation (1.10):

$$D^2 L^2 \leq (D.L)^2 = (D^2 + c + 2)^2.$$

We have

$$\frac{d}{dx} \frac{(x+a)^2}{x} = \frac{(x+a)(x-a)}{x^2}. \tag{1.23}$$

Hence for even positive integers less than or equal to $c + 2$ the expression $(x + c + 2)^2/x$ has a maximum for $x = 2$.

Thus $D^2 > 0$ gives

$$2(g-1) = L^2 \leq \frac{(D^2 + c + 2)^2}{D^2} \leq \frac{(c+4)^2}{2}.$$

□

[20] I think that that the numerical bound in the lemma is the best possible result. For $c = 1$ and 2 this follows from the classification in [JK, section 11].

Proposition 1.6.2. *Let (S, L) be a polarized K3 surface with genus g, non-general Clifford index $c > 0$, perfect Clifford divisor D, associated scroll type $(e_1, \dots, e_{c+2})$, and $4(g+c+1) > (c+4)^2$. Then D is also a perfect Clifford divisor for $L+D$ with associated scroll type $(e_1+1, \dots, e_{c+2}+1)$.*

Remark 1.6.3. Note that the case $c = 1$ is a stronger version of [JK01, proposition 9.7].

Proof. Write $L' := L + D$. Then L' is base point free and has a well-defined Clifford index c'. Since $D.(L'-D) = D.L = D.(L-D) = c+2$ and $B \in \mathcal{A}(L')$ we see that $c' \leq c$.

Lemma 1.6.1 gives that L' has no Clifford divisor B with $B^2 = 2$, since L' has genus $g' := g + c + 2$.

Assume $c' < c$ and let B be a Clifford divisor of L'. Then we have $B^2 = 0$ and

$$B.(L' - B) = B.(L + D - B) = B.(L + D) = c' + 2 \leq c + 1$$

Since D is nef we have $D.B \geq 0$. Hence $B.L \leq c + 1$. If we can show that $B \in \mathcal{A}(L)$, then this is a contradiction. But $(L - B)^2 = L^2 - 2L.B \geq L^2 - 2c - 2 \geq 0$ (using $c < \lfloor \frac{g-1}{2} \rfloor$), so either $h^0(L - B) \geq 2$ or $h^0(B - L) \geq 2$ by Riemann-Roch. We also have $L.(L - B) \geq L^2 - c - 1 \geq 0$. Hence $h^0(L - B) > 0$, since L is nef. Thus $B \in \mathcal{A}(L)$, and we obtain a contradiction. Hence $c = c'$.

In particular D is a Clifford divisor for $L + D$. By lemma 1.3.16 it is also a perfect Clifford divisor.[21] □

We will now prove a partial converse.

Proposition 1.6.4. *Let (S, L) be a polarized K3 surface with $g \geq c^2 + 4c + 6$ $(c > 0)$. Then*

$$\mathrm{Cliff}(L) = \mathrm{Cliff}(L - D).$$

Let (S, L) be a polarized K3 surface with $g \geq c^2+6c+10$ $(c > 0)$. Then a divisor D is a perfect Clifford divisor for L with $h^1(L - 2D) = 0$ if and only if it is a perfect Clifford divisor for $L - D$. In particular L is associated to the scroll type $(e_1 + 1, \dots, e_{c+2} + 1)$ if and only if $L - D$ is associated to the scroll type $(e_1, \dots, e_{c+2})$.

Remark 1.6.5. I believe that the second part of the proposition still holds for $g \geq c^2+4c+6$ but have been unable to prove this.

We will now make some comments concerning the $c = 2$ case of the proposition.[22]

For $g \leq 17$ the proposition does not hold in general. Especially if $g < 11$, then $L-D$ must have $g < 7$. In this case $L - D$ has Clifford index less than two, so the proposition certainly does not hold in this case.

If $h^1(R) \neq 0$, then the proposition certainly does not hold. For in this case $F = L-D$ is not even base point free, so it makes no sense to talk about its Clifford index.

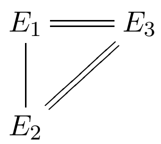
For $11 \leq g \leq 17$ the proposition holds in most cases, but not in all. We will now give some examples of polarized K3 surfaces where the proposition does not hold.

[21] Here $A = F$. Also we can not be in case (E1) or (E2) for then $c = D^2 = 0$.

[22] One can make similar remarks for $c \neq 2$.

Consider $L \sim 2E_1 + 2E_2 + E_3$ ($g = 17$), where E_1, E_2, and E_3 are elliptic curves, with the following configuration:

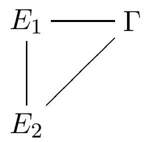
(1.24)



Then both E_1 and E_2 are perfect Clifford divisor with associated scroll type $(2, 2, 2, 2)$. Consider $L - E_1$. We see that $L - E_1$ has Clifford index 1 and perfect Clifford divisor E_2. Hence the proposition does not hold in this case.[23]

Consider $L \sim 3E_1 + 3E_2 + \Gamma$ ($g = 15$), where E_1 and E_2 are elliptic curves, Γ is a smooth rational curve, and we have the following configuration:

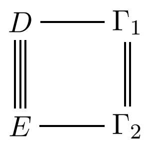
(1.25)



Then both E_1 and E_2 are perfect Clifford divisor with associated scroll type $(3, 3, 3, 3)$. Consider $L - E_1$. We see that $L - E_1$ has Clifford index 1 and perfect Clifford divisor E_2. Hence the proposition does not hold in this case either.

Since we most of the time work with $g = 12$ we will now give an example to show that the proposition does not hold for $g = 12$. Consider $L \sim 3D + \Gamma_1 + \Gamma_2 + E$, where E is an elliptic curve, Γ_1 and Γ_2 are smooth rational curves, and we have the following configuration:

(1.26)



Then both D and $D' := \Gamma_1 + \Gamma_2$ are perfect Clifford divisors. We see that $L - D$ has Clifford index 1, with perfect Clifford divisor D'. Hence the proposition does not hold in this case either.[24]

Proof. Since $g \geq c^2 + 4c + 5$ any perfect Clifford divisor must satisfy $D^2 = 0$, by lemma 1.6.1. The if part is just proposition 1.6.2. We will now prove the only if part.

Assume that D is a perfect Clifford divisor for L. Since $h^1(L - 2D) = 0$ we have $\Delta = 0$. Hence $L - D$ is base-point free, and it makes sense to talk about its Clifford index. We have $D.(L - 2D) = c + 2$. Hence $L - D$ has Clifford index $c' \leq c$. We use here that $D \in \mathcal{A}(L - D)$, i.e. $h^0(D), h^0(L - 2D) \geq 2$. That $h^0(D) \geq 2$ is obvious since D is a perfect Clifford divisor for L. Riemann-Roch gives $h^0(L - 2D) \geq 2$ or $h^0(2D - L) \geq 2$,

[23] We can show that this case and the next two cases all exist by arguing with the intersection matrices as we will do many times in the next chapter.

[24] Configuration (2.58) gives another example with $g = 12$.

since $(L-2D)^2 = L^2 - 4(c+2) \geq 0$ (the last inequality follows from $g \geq c^2+4c+5$). Then $h^0(L-2D) \geq 2$ since L is nef and $L.(L-2D) = L^2 - 4(c+2) > 0$. We will now show that $c' = c$.

Assume $c' < c$. Then there exists a divisor $B \in \mathcal{A}(L-D)$ such that $c'+2 = B.(L-D-B) < c+2$ and $B^2 = 0$.[25] Since L has Clifford index c we must have $B.(L-B) = B.L \geq c+2$. Whence $B.D > 0$. We have $(B+D)^2 = 2B.D$ and $(B+D).L = B.L + D.L = c' + c + 4 + B.D$. Since $B \in \mathcal{A}(L-D)$ we see that $(B+D) + (L-D-B) \sim L$ gives a non-zero effective decomposition of L such that $B+D \in \mathcal{A}(L)$. Hence $(B+D).(L-D-B) \geq c+3$.[26] This gives $c+3 \leq (B+D).(L-D-B) = c'+c+4-B.D$, i.e. $B.D \leq c'+1 \leq c$.

The Hodge index theorem gives

$$(B+D)^2(L-B-D)^2 \leq ((L-B-D).(B+D))^2,$$

so

$$2B.D(L^2-2c-2c'-8) \leq (c'+c+4-B.D)^2 \leq (2c+3-B.D)^2.$$

Thus

$$\begin{aligned} 4(g-1) = 2L^2 &\leq 4c+4c'+16+\frac{(2c+3-B.D)^2}{B.D} \\ &\leq 8c+12+\frac{(2c+3-B.D)^2}{B.D} \\ &< 8c+13+(2c+2)^2 \\ &= 4c^2+16c+17, \end{aligned}$$

where we have used $c' < c$. The last inequality follows from equation (1.23) with $0 < B.D \leq c$.

Hence $c' = c$, so D is a Clifford divisor for $L-D$. If $g(L) \geq c^2+5c+10$, then $g(L-D) \geq c^2+4c+8$ so D is a perfect Clifford divisor by proposition 1.6.6.

Note that since $d = D^2/2+c+2$ (see equation (2.4)) we get $d = c+2$. $\square$

The method of proof we have used here can also be used to show other results.

Proposition 1.6.6. *Let (S,L) be a polarized K3 surface with genus g and non-general Clifford index c. Assume $g \geq c^2+4c+8$. Let D be a perfect Clifford divisor. Then $D^2 = 0$. Furthermore any other Clifford divisor D' is linearly equivalent to D. In particular any Clifford divisor D' is perfect.*

Proof. That $D^2 = 0$ is immediate from lemma 1.6.1. It is also immediate that any other Clifford divisor D' satisfies $D'^2 = 0$.

---

[25] $B^2 = 0$ by lemma 1.6.1.

[26] We cannot have $(B+D).(L-D-B) = c+2$. For then we would have to have $(B+D)^2 = 0$, after arguing as in the proof of lemma 1.6.1. This contradicts $(B+D)^2 = 2B.D > 0$. Hence $(B+D).(L-D-B) > c+2$.

Let D' be a Clifford divisor. Assume $D.D' \neq 0$. We have $D+D' \in \mathcal{A}(L)$,[27] We have seen that any Clifford divisor D' must satisfy $D'^2 = 0$. Hence

$$c+3 \leq (D+D').(L-D-D') = 2(c+2) - 2D.D',$$

so $D.D' \leq (c+1)/2$.

We will now use the Hodge index theorem as in the previous proof to get a bound on L^2. First of all the Hodge index theorem gives

$$(D+D')^2(L-D-D')^2 \leq ((L-D-D').(D+D'))^2,$$

so

$$2D.D'(L^2 - 4(c+2) - 2D.D') \leq 4((c+2) - D.D')^2.$$

Thus

$$\begin{aligned} g = \frac{L^2}{2} + 1 &\leq 5 + 2c + D.D' + \frac{((c+2) - D.D')^2}{D.D'} \\ &\leq 6 + 2c + (c+1)^2 \\ &= c^2 + 4c + 7, \end{aligned}$$

where we have used that the maximum of

$$x + \frac{((c+2)-x)^2}{x},$$

for integers x between 1 and $(c+1)/2$, is when $x = 1$.

If $D.D' = 0$, then Riemann-Roch used on $D - D'$ gives that either $h^0(D-D') \neq 0$ or $h^0(D'-D) \neq 0$. We also have

$$(D-D')^2.L^2 = ((D-D').L)^2 = 0.$$

The equality conditions of the Hodge index theorem then give $L^2(D-D') \sim ((D-D').L)L = 0$. Hence $D \sim D'$. □

[27] It is obvious that $h^0(D+D') \geq 2$. Since $D.(L-D-D') = D.L - D.D' \leq D.L = c+2$ and likewise $D'.(L-D-D') \leq c+2$ we have

$$(L-D-D')^2 = L.(L-D-D') - D.(L-D-D') - D'.(L-D-D') \geq L^2 - 4(c+2) \geq 0.$$

Riemann-Roch then gives $h^0(L-D-D') \geq 2$, so $D+D' \in \mathcal{A}(L)$.

Chapter 2

Projective Models of Polarized K3 Surfaces of Genus 12

We will in this chapter classify projective models of K3 surfaces of genus 12. This chapter is quite long, as a consequence of its computational character.

If $g = 12$, then $g-1$ is square-free. Thus the general polarized K3 surface of genus 12 is BN general (proposition 1.3.18). Mukai [Muk95] has previously described the projective model of BN general polarized K3 surfaces with L ample. We will now quickly describe this projective model. We follow [JK01, pp.72–73].

Let V be a vector space. Write $\mathrm{Grass}(r, V)$ for the Grassmann variety of r-dimensional subspaces of V. Let V be a 7-dimensional vector space and $N \subseteq \wedge^2 V^\vee$ a 3-dimensional vector space of skew-symmetric bilinear forms, with basis $\{m_1, m_2, m_3\}$. We denote by $\mathrm{Grass}(3, V, m_i)$ the subset of $\mathrm{Grass}(3, V)$ consisting of 3-dimensional subspaces U of V such that the restriction of m_i to $U \times U$ is zero. Write $\Sigma^3_{12} = \mathrm{Grass}(3, V, N) := \cap \, \mathrm{Grass}(3, V, m_i)$. It has dimension 3 and degree 12. Then the projective image of a BN general polarized K3 surface of genus 12 with L ample is

$$(1) \cap \Sigma^3_{12} \subseteq \mathbb{P}^{12}.$$

For the rest of this chapter we will be considering the non-BN general case. The non-Clifford general case takes up very much space. We will rely heavily on the configurations we found in section 1.5. The Clifford general and non–BN general case is rather short compared to the non–Clifford general case. We have already done much of the necessary work in section 1.4.

To end this chapter we include a short section 2.11 where we will see how the methods used in this chapter can be extended to other genera.

We summarize the results of this chapter in table 2.1.[1] We use the same conventions as for the tables in [JK01, section 11], i.e. the singularity type listed in the column "sing" indicates that for "almost all" K3 surfaces in question its projective model has singularities exactly as indicated.[2]

[1] For $c = 1$ and $c = 2$ we can find some of this information in the tables in [JK01, p.58 and p.65].

[2] By "almost all" we mean that the moduli of the exceptional set of K3 surfaces is strictly less than the

Table 2.1: Scroll types for $g = 12$

c	D^2	scroll type	#mod	sing
1	0	$(4,3,3)$	18	sm. (2.42)
1	0	$(4,4,2)$	17	sm. (2.45), (2.46)
1	0	$(5,3,2)$	16	A_1 (2.48)
1	0	$(6,3,1)$	17	sm. (2.49)
2	0	$(3,2,2,2)$	18	sm. (2.50)
2	0	$(3,3,2,1)$	17	sm. (2.52), (2.53), (2.54)
2	0	$(4,2,2,1)$	15	A_1 (2.55)
2	0	$(4,3,1,1)$	16	sm. (2.57)
2	0	$(4,3,2,0)$	17	sm. (2.59)
2	0	$(5,3,1,0)$	16	A_1 (2.60)
3	0	$(2,2,2,1,1)$	18	sm. (2.6)
3	0	$(3,2,1,1,1)$	16	A_1 (2.7)
3	0	$(2,2,2,2,0)$	17	A_1 (2.10)
3	0	$(3,2,2,1,0)$	16	A_2 (2.14), A_1 (2.15), (2.16)
3	0	$(3,3,1,1,0)$	15	A_1 (2.17)
3	0	$(4,2,1,1,0)$	14	$A_2 + A_3$ (2.11)
3	0	$(3,3,2,0,0)$	16	$2A_1$ (2.18)
3	0	$(4,2,2,0,0)$	16	$2A_1$ (2.22
3	2	$(3,2,1,1,0,0)$	18	sm. (2.25)
4	0	$(2,1,1,1,1,1)$	18	sm. (2.26)
4	0	$(2,2,1,1,1,0)$	17	A_1 (2.31)
4	0	$(3,1,1,1,1,0)$	16	A_2 (2.32)
4	0	$(2,2,2,1,0,0)$	16	$2A_1$ (2.33)
4	0	$(3,2,1,1,0,0)$	14	$2A_1$ (2.34)
4	0	$(3,2,2,0,0,0)$	15	$3A_1$ (2.35)
4	2	$(1,1,1,1,1,1,0)$	18	sm. (2.36)
4	4	$(2,1,1,1,0,0,0,0)$	18	sm. (2.37)
5	2	$(1,1,1,1,1,0,0,0)$	18	sm.
5	2	$(2,1,1,1,0,0,0,0)$	16	A_1 (2.62), $2A_1$ (2.63)
5	2	$(2,2,1,0,0,0,0,0)$	17	A_1 (2.64), (2.65)
5	4	$(1,1,1,1,0,0,0,0,0)$	18	sm.
5	4	$(2,1,1,0,0,0,0,0,0,0)$	16	$3A_1$ (2.67)

We will now include some preliminary material for our classification of projective models of non-Clifford general polarized K3 surfaces of genus 12. From now on until section 2.8 we will assume that L is a base point free divisor with arithmetic genus 12 ($L^2 = 22$) and non-general Clifford index c. Then $c \leq 4$. Let D be a perfect Clifford divisor. We have

$$D.L = D^2 + c + 2 \tag{2.1}$$

$$0 \leq D^2 < c + 2 \tag{2.2}$$

$$22D^2 \leq (D^2 + c + 2)^2 \tag{2.3}$$

The first equation is just the definition of a Clifford divisor. The next two are just equations (1.9) and (1.10).

The inequalities give the following possible pairs of $\{c, D^2\}$:

$$\begin{array}{cccc} \{1,0\} & \{2,0\} & \{3,0\} & \{3,2\} \\ \{4,0\} & \{4,2\} & \{4,4\} & \end{array}$$

We say that L (and (S, L)) is of *type* $\{c, D^2\}$ if L has Clifford index c and a perfect Clifford divisor with self-intersection D^2. We have seen that D gives a scroll in a natural way. Remember that we say that this scroll is *associated* to the perfect Clifford divisor D. Let d be the dimension of this scroll, and f its degree. Then

$$d = d_0 := h^0(L) - h^0(L - D) = \frac{1}{2}D^2 + c + 2 \tag{2.4}$$

$$d + f \;=\; g + 1 = 13 \tag{2.5}$$

We stumble upon quite a few Diophantine equations in this section, and the following lemma will be useful for solving some of these.

Lemma 2.0.1. *[NZM91, theorem 3.10] Let $f(x, y) = ax^2+bxy+cy^2$ be a binary quadratic form, with integral coefficients and discriminant $d := b^2 - 4ac$. If $d \neq 0$ and d is not a perfect square then the only integral solutions of $f(x, y) = 0$ is $x = y = 0$.*

Proof. Completing the square gives

$$0 = 4af(x, y) = (2ax + by)^2 - dy^2,$$

from which the statement readily follows. □

Let B_1 be a nef divisor and B_2 a divisor such that $B_1.B_2 > 0$ and $B_2^2 \geq -2$. Then by Riemann-Roch either B_2 or $-B_2$ is linearly equivalent to an effective divisor. Since

number of moduli listed in the column "# mod". For the possible singularities see [Băd01, theorem 3.32]. The number(s) after the singularity type denotes the configurations which give rise to this singularity. The computations for this column is non-trivial. We will not include the details except for the first two scroll types we consider.

$B_1.B_2 > 0$ it must be B_2. If B_2 represents an equivalence class in $\operatorname{Pic} S$, then we may assume B_2 to be effective. It is this argument I refer to when I in the following write "B_1 is nef and Riemann-Roch used on B_2 let us assume $B_2 \geq 0$."

In many places we will want to say something about a non-zero divisor B such that $h^0(B) = 1$. In this case lemma 1.1.14 says sufficiently much about B for our purposes. In particular it says that two distinct smooth rational curves, Γ_1 and Γ_2, in the support of B satisfies $\Gamma_1.\Gamma_2 = 0$ or 1. This is a property we will use extensively without referring to lemma 1.1.14 in each instance.

I will take the opportunity to emphasize footnotes 5 and 8 below.

In the following we will find those scroll types associated to a perfect Clifford divisor D. We will give a description of possible configurations in $\operatorname{Pic} S$.[3] In most cases we will only give the most general configurations, though in some cases we will give all possible configurations. If the last non-zero d_i has value larger than one, the situation is too complex for us to give the complete picture. If the last non-zero d_i is equal to one, it is relatively easy to give the complete picture. Still the number of configurations is so large for some of the scroll types that it is not forthwhile to give the complete picture.

Note also that many of the proofs will be quite long and tedious. When these proofs do not present any new and interesting ideas we will feel free to skip the details.[4]

After giving possible configurations we will show existence of the most general one. This too will give long and tedious proofs at times. For the reader who is not interested in checking all the details I recommend reading the existence proofs in subsections 2.1.1, 2.1.2, and 2.1.5. When there is more than one configuration that gives the largest moduli of K3 surfaces (for a given scroll type) we will only do the existence proof for one of the configurations. In all of the cases one can show existence for the other configurations in a similar way, but we will not include the details.

2.1 $c = 3, D^2 = 0$

We have $D.L = 5, d = 5$, and $f = h^0(L - D) = 8$. Since $L.(L - 5D) = -3$ and L is nef we see that $h^0(L - 5D) = 0$. Proposition 1.3.12 gives $h^1(R) \leq 2$. By Riemann-Roch $h^0(L - 2D) = h^0(R) = 3 + h^1(R)$. Table 2.2 gives the possible scroll types.

2.1.1 $(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})$

In this case $h^1(R) = 0$, so $\Delta = 0$. In general A is an irreducible curve of genus 2. Then $L \sim 2D + A$ is a decomposition of L into irreducible curves with the following configuration:

$$D \qquad A \tag{2.6}$$

We will show that there exists an 18-dimensional family of polarized K3 surfaces (S, L) with a perfect Clifford divisor D associated to this scroll type.

[3] We will only number those configurations which are actually possible for a given scroll type.

[4] This is why this chapter is "only" sixty pages!

Table 2.2: Possible scroll types associated to L of type $\{3, 0\}$.

d_0	d_1	d_2	d_3	d_4	scroll type
5	5	3	0	0	$(2,2,2,1,1)$
5	5	2	1	0	$(3,2,1,1,1)$
5	5	1	1	1	$(4,1,1,1,1)$
5	4	4	0	0	$(2,2,2,2,0)$
5	4	3	1	0	$(3,2,2,1,0)$
5	4	2	2	0	$(3,3,1,1,0)$
5	4	2	1	1	$(4,2,1,1,0)$
5	3	3	2	0	$(3,3,2,0,0)$
5	3	3	1	1	$(4,2,2,0,0)$
5	3	2	2	1	$(4,3,1,0,0)$

The lattice $\mathbb{Z}L \oplus \mathbb{Z}D$ with intersection matrix

$$\begin{bmatrix} L^2 & L.D \\ D.L & D^2 \end{bmatrix} = \begin{bmatrix} 22 & 5 \\ 5 & 0 \end{bmatrix}$$

has signature (1,1), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}L \oplus \mathbb{Z}D$.

Using Picard-Lefschetz reflections we may assume that L is nef. We will now show that L is base point free and of type $\{3, 0\}$. It is enough to show that there exists no divisor B such that[5]

$$\begin{aligned} B^2 &= 0 \qquad B.L = 1, 2, 3, \text{ or } 4 \\ B^2 &= 2 \qquad B.L = 7 \end{aligned}$$

This is immediate from lemma 1.4.9.[6]

Assume that B is a perfect Clifford divisor. Let $B \sim xL + yD$. Then $B^2 = 0$ gives $x = 0$. $B.L = 5$ then gives $y = 1$. Hence $B \sim D$, and we may assume that D is a perfect Clifford divisor. Using lemma 2.0.1 we see that there exists no divisor Γ such that $\Gamma^2 = -2$ and $\Gamma.L = 0$. Hence $\mathcal{R}_{L,D} = \emptyset$. We will show below that the associated scroll type is $(3, 2, 1, 1, 1)$ only if $\operatorname{rank} \operatorname{Pic} S > 2$ and that the scroll type $(4, 1, 1, 1, 1)$ is not associated with any perfect Clifford divisor whatsoever. Therefore D must be associated to the scroll type $(2, 2, 2, 1, 1)$, since $h^1(R) = 0$.

Since there exists no divisor Γ such that $\Gamma^2 = -2$ and $\Gamma.L = 0$ we also get that the mapping given by L is smooth.

[5] The non-existence of B such that $B^2 = 0$ and $B.L = 1$ gives that L is base point free (proposition 1.1.12). The non-existence of the rest of the B's give that L is of type $\{3, 0\}$ using equations (2.1) and (2.2). For the rest of this chapter we will write up equations as below without writing the previous two sentences repeatedly (with $\{3, 0\}$ and B substituted suitably).

[6] Alternatively use lemma 2.0.1.

2.1.2 $(\mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$

In this case $h^1(R) = 0$, so $\Delta = 0$. Let Γ' be a smooth rational curve, such that $\Gamma' < L - 3D$. Lemma 1.1.16 and the definition of a free Clifford divisor give $h^0(D, \mathcal{O}_D) = 1$. Then $h^0(D + \Gamma', \mathcal{O}_{D+\Gamma'}) = 1$ or 2. Lemma 1.1.16 and Riemann-Roch then give

$$2 = h^0(D) \leq h^0(D + \Gamma') = D.\Gamma' + h^0(D + \Gamma', \mathcal{O}_{D+\Gamma'}) \leq h^0(L - 3D) = 3.$$

Hence $D.\Gamma' = 0, 1$, or 2.

Since $h^0(L - 3D) = 1$ we can write

$$L \sim 3D + B,$$

where B is a sum of smooth rational curves, $B^2 = -8$, and $B.D = 5$. We can write

$$B \sim B_0 + B_1,$$

where $B_0.D = 5$, $B_1.D = 0$, and B_0 can be written as a sum of smooth rational curves,

$$B_0 = n_1\Gamma_1 + \cdots + n_N\Gamma_N,$$

where $D.\Gamma_i > 0$ and $n_i > 0$ for all i. Then $\Sigma_{i=1}^N n_i \leq 5$. Especially we have $N \leq 5$. If $B_1 \neq 0$, then $B_1^2 \leq -2$ since $h^0(B_1) = 1$ (use Riemann-Roch).

We must have $\operatorname{rank}\operatorname{Pic} S \geq 3$ and the only possibility with $\operatorname{rank}\operatorname{Pic} S = 3$ is configuration 2.7. Over the next couple of pages we will present the ideas of the proof of this statement. Since the proof is so long (and at times uninteresting) we will not include all the details.

$N = 1$: Then we must have $B_0 = 5\Gamma_1$, so $B_0^2 = -50$. L nef and $L.(3D+5\Gamma_1+B_1) = 22$ gives $L.\Gamma_1 = 1$ and $L.B_1 = 2$. Then $L.\Gamma_1 = 1$ gives $\Gamma_1.B_1 = 8$ and $B^2 = -8$ gives $B_1^2 = -38$. Take any smooth rational curve $\Gamma_2 \leq B_1$ such that $\Gamma_1.\Gamma_2 = 1$. Using that $L.\Gamma_2 \geq 0$ we get that Γ_2 has multiplicity at least three in B_1. This gives three possibilities (here $\Gamma_3 \leq B_2$ is another smooth rational curve such that $\Gamma_1.\Gamma_2 = 1$ and B_2 is an effective divisor such that $\Gamma_1.B_2 = 0$):

$$\begin{aligned} B_1 &\sim 3\Gamma_2 + 5\Gamma_3 + B_2 \\ B_1 &\sim 4\Gamma_2 + 4\Gamma_3 + B_2 \\ B_1 &\sim 8\Gamma_2 + B_2 \end{aligned}$$

In the first two case we get $\operatorname{rank}\operatorname{Pic} S \geq 5$ by checking all possibilities. That is the (six different) possibilities given by $\Gamma_2.\Gamma_3 = 0$ or 1 and $L.(\Gamma_2 + \Gamma_3 + B_2) = 1$. These possibilities determine the intersection matrix of $D, \Gamma_1, \Gamma_2, \Gamma_3$, and B_2. Taking determinants we get something non-zero in all of the cases. Thus $D, \Gamma_1, \Gamma_2, \Gamma_3$, and B_2 are linearly independent and $\operatorname{rank}\operatorname{Pic} S \geq 5$.

Similar reasoning in the last case gives $\operatorname{rank}\operatorname{Pic} S \geq 5$ here too. The reasoning here is a bit more complex than in the above cases. That $\operatorname{rank}\operatorname{Pic} S \geq 4$ is easy to show. This follows by checking that D, Γ_1, Γ_2, and B_2 are linearly independent both for $L.\Gamma_1 = 0$

and $L.\Gamma_1 = 1$. To show that $\operatorname{rank}\operatorname{Pic} S \geq 5$ we introduce a new smooth rational curve $\Gamma_3 \leq B_2$ such that $\Gamma_2.\Gamma_3 = 1$, and then show that $D, \Gamma_1, \Gamma_2, \Gamma_3$, and $(B_2 - \Gamma_3)$ are linearly independent.

$N = 2$: Consider first $\operatorname{rank}\operatorname{Pic} S = 3$. We first look at $B_1 = 0$. In this case $B \sim n_1\Gamma_1 + n_2\Gamma_2$. Arguing modulo 2 we see that either $D.\Gamma_1 = 1$ or $D.\Gamma_2 = 1$. Assume that $D.\Gamma_1 = 1$. Then n_1 is odd. A case by case analysis shows that every possible combination of $n_1, n_2, D.\Gamma_2$, and $\Gamma_1.\Gamma_2$, with $n_1 + n_2 \leq 5, 1 \leq D.\Gamma_2 \leq 2$, and $0 \leq \Gamma_1.\Gamma_2 \leq 1$, gives $B^2 \neq -8$, a contradiction. Similarly one can show that $B_1 \neq 0$ gives a contradiction. It is possible to show that the only possibility with $N = 2$ (for all $\operatorname{rank}\operatorname{Pic} S$) is $L \sim 3D + \Gamma_1 + 2\Gamma_2 + \Gamma_3$, with the following configuration:[7]

(2.7)
$$\begin{tikzcd} D \arrow[r, no head] \arrow[d, equal] & \Gamma_1 \\ \Gamma_2 \arrow[r, no head] & \Gamma_3 \end{tikzcd}$$

$N = 3$: We will now "show" that there exists no configurations with $N = 3$ and $\operatorname{rank}\operatorname{Pic} S = 4$. One can easily show that D, Γ_1, Γ_2, and Γ_3 are linearly independent so $\operatorname{rank}\operatorname{Pic} S \geq 4$. The $\operatorname{rank}\operatorname{Pic} S = 4$ case is rather complex. We will show that $\operatorname{rank}\operatorname{Pic} S = 4$ is impossible with $B_1 = 0$. The case $B_1 \neq 0$ is left to the reader. Assume that $\operatorname{rank}\operatorname{Pic} S = 4$ is possible with $B_1 = 0$. Then we have $B \sim n_1\Gamma_1 + n_2\Gamma_2 + n_3\Gamma_3$. Assuming $n_1 \leq n_2 \leq n_3$ we have four possible values of (n_1, n_2, n_3), namely $(1,1,1), (1,1,2), (1,1,3)$,and $(1,2,2)$.

$(1,1,1)$ gives $B^2 = (\Gamma_1 + \Gamma_2 + \Gamma_3)^2 = \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 + \cdots \geq -6 > -8$, a contradiction.

$(1,1,2)$ and $B^2 = -8$ reduces to two cases:

$$\begin{tikzcd} D \arrow[r, equal] \arrow[d, no head] \arrow[dr, no head] & \Gamma_1 \arrow[d, no head] \\ \Gamma_2 & \Gamma_3 \end{tikzcd}$$

and

$$\begin{tikzcd} D \arrow[r, equal] \arrow[d, no head] \arrow[dr, no head] & \Gamma_1 \\ \Gamma_2 \arrow[r, no head] & \Gamma_3 \end{tikzcd}$$

Both configurations give $L.\Gamma_3 = 0$, and therefore $\mathcal{R}_{L,D} \neq \emptyset$. Hence they cannot give scroll type $(3,2,1,1,1)$.

$(1,1,3)$ and $B^2 = -8$ is only possible with the following configuration

$$\begin{tikzcd} D \arrow[r, no head] \arrow[d, no head] \arrow[dr, no head] & \Gamma_3 \arrow[d, no head] \arrow[dl, no head] \\ \Gamma_1 \arrow[r, no head] & \Gamma_2 \end{tikzcd}$$

[7] We will see below that there exists none for $N = 3$, hence this is the only possibility with $\operatorname{rank}\operatorname{Pic} S = 4$ and associated scroll type $(3,2,1,1,1)$.

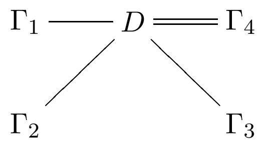
In this case $L.\Gamma_3 = -1$, which contradicts L nef.

$(1, 2, 2)$ and $B^2 = -8$ is impossible.

We have shown that there exists no configurations with $B_1 = 0$ and $1 \leq N \leq 3$. Arguing as above we get the following possible configurations for $N = 4$ and 5 (with $B_1 = 0$):

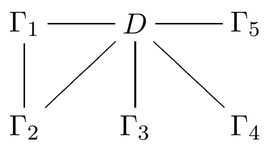
$N = 4$: $L \sim 3D + \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ and configuration

(2.8)



$N = 5$: $L \sim 3D + \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$ and configuration

(2.9)



We will now show that configuration (2.7) gives a 16-dimensional family of polarized K3 surfaces (S, L), with this scroll type.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma_3$ with intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma_1 & D.\Gamma_2 & D.\Gamma_3 \\ D.\Gamma_1 & \Gamma_1^2 & \Gamma_1.\Gamma_2 & \Gamma_1.\Gamma_3 \\ D.\Gamma_2 & \Gamma_1.\Gamma_2 & \Gamma_2^2 & \Gamma_2.\Gamma_3 \\ D.\Gamma_3 & \Gamma_1.\Gamma_3 & \Gamma_2.\Gamma_3 & \Gamma_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & -2 & 0 & 0 \\ 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

has signature (1,3), so by proposition 1.1.24 there exists a K3 surface with Pic $S = \mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma_3$.

Using Picard-Lefschetz reflections we may assume that $L \sim 3D + \Gamma_1 + 2\Gamma_2 + \Gamma_3$ is nef. L nef and Riemann-Roch used on D, Γ_1, and Γ_2 let us assume D, Γ_1, and Γ_2 effective.

We will now show that $B \in \mathcal{A}^0(L)$, with B nef, implies $B \sim D$. This will in particular show that L is base point free and of type $\{3, 0\}$.

Let $B \in \mathcal{A}^0(L)$, with $B \sim xD + y\Gamma_1 + z\Gamma_2 + w\Gamma_3$ and B nef. Then we have $B.L - B^2 \leq (B + H).L - (B + H)^2$ for $H = D$, $H = \Gamma_1$, and $H = \Gamma_2$.[8] This gives $0 \leq 2H.B \leq H.L + 2$, where the first inequality follows from B nef. For $H = \Gamma_3$ we similarly get

[8]For the inequality to be true it is enough that $B+H \in \mathcal{A}(L)$, i.e. $h^0(B+H), h^0(L-B-H) \geq 2$. That $h^0(B+H) \geq 2$ is immediate, since $h^0(B+H) \geq h^0(B) \geq 2$ when H is effective. That $h^0(L-B-H) \geq 2$ is a bit harder to show in general. When $c \leq 3$ and $B \in \mathcal{A}^0(L)$ we have $B^2 - 2B.L \geq -12$ (with inequality if and only if $B^2 = 2$ and $B.L = 7$). With $L^2 = 22$ we then have $(L-B-H)^2 \geq 10 + H^2 - 2L.H + 2H.B \geq 10 + H^2 - 2L.H$. One can easily check that in the cases, $H = D$, $H = \Gamma_1$, and $H = \Gamma_2$, one gets $H^2 - 2L.H \geq -10$. Whence $h^0(L-B-H) \geq 2 + (L-B-H)^2/2 \geq 2$ also. (Later on we will refer to this page when we do similar arguments. In these cases too one can show as we have done above that $h^0(L-B-H) \geq 2$.)

$2|H.B| \leq H.L + 2$.[9] Hence

$$\begin{aligned} 0 \leq \Gamma_1.B &= x - 2y \leq 1 \\ 0 \leq \Gamma_2.B &= 2x - 2z + w \leq 2 \\ -1 \leq \Gamma_3.B &= z - 2w \leq 1 \\ 0 \leq D.B &= y + 2z \leq 3 \end{aligned}$$

A case by case analysis shows that the only integral solution is $B \sim D$. Hence we may assume that D is a perfect Clifford divisor.

From Riemann-Roch we see that either Γ_3 or $-\Gamma_3$ is effective. We will show that we may assume Γ_3 to be effective. If Γ_3 is effective, there is nothing to show. If $-\Gamma_3$ is effective, then we change the basis of $\operatorname{Pic} S$ as follows:

$$\begin{array}{rcll} D & \mapsto & D & := D' \\ \Gamma_1 & \mapsto & \Gamma_1 & := \Gamma_1' \\ \Gamma_2 & \mapsto & \Gamma_2 + \Gamma_3 & := \Gamma_2' \\ \Gamma_3 & \mapsto & -\Gamma_3 & := \Gamma_3' \end{array}$$

We see that $L \sim 3D' + \Gamma_1' + 2\Gamma_2' + \Gamma_3'$ and that the new intersection numbers are equal to the old ones. Furthermore since L is nef we may assume that Γ_1' and Γ_2' is effective. We have assumed that $-\Gamma_3$ is effective, so Γ_3' is effective..

It remains to determine the scroll type given by D. We want to find $\mathcal{R}_{L,D}$. Let $B \sim xD + y\Gamma_1 + z\Gamma_2 + w\Gamma_3 \in \mathcal{R}_{L,D}$. Then we must have $B.D = 1$, $B.L = 0$, and $B^2 = -2$. This is equivalent to $y + 2z = 1$, $5x + y + 3z = 0$, and $-1 = -(y^2 + z^2 + w^2) + xy + 2xz + zw$. These equations are seen to have no simultaneous integral solutions. Hence $\mathcal{R}_{L,D} = \emptyset$. The scroll type must then be either $(3, 2, 1, 1, 1)$ or $(4, 1, 1, 1, 1)$ since $h^0(L - 3D) = h^0(\Gamma_1 + 2\Gamma_2 + \Gamma_3) > 0$. We will show below that the scroll type $(4, 1, 1, 1, 1)$ is not associated to any perfect Clifford divisor, hence the scroll type must be $(3, 2, 1, 1, 1)$.

We will now find the curves contracted by ϕ_L. Let Γ be a divisor such that $\Gamma.L = 0$ and $\Gamma^2 = -2$. Then $5x + y + 3z = 0$ and $-(y^2 + z^2 + w^2) + xy + 2xz + zw = -1$. Inserting $x = -(y + 3z)/5$ into the second equation gives $6y^2 + 11z^2 + 5w^2 = -5xy + 5zw + 5$. Adding $-3y^2 - 6z^2 - 3w^2$ to the left hand side and $5xy - 5zw$ to the right hand side gives

$$3y^2 + 5z^2 + 2w^2 \leq 5,$$

since $-5zy \leq 3y^2 + 3z^2$ and $5zw \leq 3z^2 + 3w^2$. This gives $\Gamma = \pm\Gamma_1$, $\pm\Gamma_2$, or $\pm\Gamma_3$. Only $\Gamma = \pm\Gamma_3$ satisfies $\Gamma.L = 0$. We want Γ to be effective and may by the above base change assume that $\Gamma = \Gamma_3$ is effective. Then Γ has to contain a smooth rational curve γ such that $\gamma.L = 0$, since $\Gamma.L = 0$, $\Gamma^2 = -2$, and Γ is effective.[10] But since Γ_3 is the only effective divisor with $\Gamma.L = 0$ and $\Gamma^2 = -2$ we get $\gamma = \Gamma_3$. The singularity type becomes (A_1).

[9] Since $\Gamma_3^2 = -2$ gives Γ_3 or $-\Gamma_3$ effective.

[10] Since L is nef and $\Gamma.L = 0$ every prime divisor γ contained in Γ must satisfy $\gamma.L = 0$. Since L is big, the Hodge index theorem gives that every prime divisor γ contained in Γ has negative self-intersection. In particular γ must be a smooth rational curve.

2.1.3 $(\mathbf{4}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$

We will show that there exists no perfect Clifford divisor D associated to this scroll type.

Assume that there exists a perfect Clifford divisor D associated to the scroll type $(e_1, \ldots, e_5) = (4, 1, 1, 1, 1)$. $L + D$ gives a mapping ϕ_{L+D} from S to $\mathbf{P}^{g+d} = \mathbf{P}^{16}$ (see [JK01, p.35][11]) Set $S'' = \phi_{L+D}(S)$. Then $|D|$ defines a smooth rational scroll $\mathcal{T}_0$ in $\mathbf{P}^{g+d}$ containing S''.[12] [JK01, proposition 8.4] says that $\mathcal{T}_0$ is smooth of type $(e'_1, \ldots, e'_5) = (e_1 + 1, \ldots, e_5 + 1)$. [JK01, prop.8.14] gives a resolution of $\mathcal{T}_0$. From this resolution we get integers $b^1_1, \ldots, b^5_1$, which (by [JK01, cor.8.19]) satisfies $\sum_{k=1}^{5} b^k_1 = 2g - 2 = 22$. We may order the b^k_1 in a non-increasing way.

From the proof of [JK01, corollary 8.27] we see that

$$b^1_1 \leq 2e'^1_2 = 2(e_2 + 1) = 4.$$

This gives

$$22 = \sum_{k=1}^{5} b^k_1 \leq 5b^1_1 = 20,$$

a contradiction.

2.1.4 $(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{0})$

We must be in case $\{3, 0\}^a$. Hence $\Delta' = \Gamma$, $\mathcal{R}_{L,D} = \{\Gamma\}$, $A'^2 = 4$, $D.A' = 4$, and $L \sim 2D + A' + \Gamma$. In general A' is a curve C of genus 3, and we have the following configuration

(2.10)
$$\begin{array}{ccc} D & \equiv\!\equiv\!\equiv & C \\ | & & \\ \Gamma & & \end{array}$$

We will now show that there exists a 17-dimensional family of polarized K3 surfaces (S, L) with this scroll type.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}A' \oplus \mathbb{Z}\Gamma$ with intersection matrix

$$\begin{bmatrix} D^2 & D.A' & D.\Gamma \\ D.A' & A'^2 & A'.\Gamma \\ D.\Gamma & A'.\Gamma & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 1 \\ 4 & 4 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

has signature (1,2), so by proposition 1.1.24 there exists a K3 surface with Pic $S = \mathbb{Z}D \oplus \mathbb{Z}A' \oplus \mathbb{Z}\Gamma$.

Using Picard-Lefschetz reflections we may assume that $L \sim 2D + A' + \Gamma$ is nef. We will now show that L is base point free and of type $\{3, 0\}$. It is enough to show that there exists no divisor $B \sim xD + yA' + z\Gamma$ such that

$$\begin{aligned} B^2 &= 0 \qquad B.L = 1, 2, 3, \text{ or } 4 \\ B^2 &= 2 \qquad B.L = 7 \end{aligned}$$

[11] Since $D^2 = 0$ we have $\tilde{S} = S$ and $H = L + D$ in the notations of [JK01, p.35].

[12] See [JK01, pp.37–38]. Since $D^2 = 0$ we have $\tilde{D} = D$.

We have $B.L = 5x + 12y$ and $B^2 = 4y^2 - 2z^2 + 8xy + 2xz$. If $B^2 = 0$, then this gives $6z^2 = B.L - (5 - 2B.L)x + 6xz - 10x^2$. If $|z| \leq |x|$, this gives $0 \leq B.L - (5 - 2B.L)x - 4x^2$. For $1 \leq B.L \leq 4$ this gives $x = -1, 0$ or 1, all of which give $y \notin \mathbb{Z}$. If $|x| < |z|$ we get $0 \leq B.L - (5 - 2B.L)x - 10x^2$. For $1 \leq B.L \leq 4$ this gives $x = -1$ or 0, which give $y \notin \mathbb{Z}$. $B^2 = 2$ and $B.L = 7$ is shown in a similar way to have no solutions.

Hence (S, L) is of type $\{3, 0\}$. Let $B \sim xD + yA' + z\Gamma$ be a perfect Clifford divisor. Then $B^2 = 0$ and $B.L = 5$, i.e. $6z^2 = 5 - 5x + 6xz - 10x^2$. This gives $B \sim D$ or $B \sim D + \Gamma$. Changing the basis of $\operatorname{Pic} S$, if necessary, as on p. 67 we may assume $B \sim D$.

If $B \sim D$, then we may assume that D is a perfect Clifford divisor. D nef and Riemann-Roch used on Γ let us assume $\Gamma \geq 0$. But then $L.\Gamma = 0$ and $D.\Gamma = 1$. So $\mathcal{R}_{L,B} \neq \emptyset$.[13]

Hence every perfect Clifford divisor D' must satisfy $\mathcal{R}_{L,D'} \neq \emptyset$. We will see below that every perfect Clifford divisor with $\mathcal{R}_{L,D'} \neq \emptyset$ associated to a scroll type not equal to $(2, 2, 2, 2, 0)$ must have $\operatorname{rank} \operatorname{Pic} S > 3$. Hence the scroll type associated to the perfect Clifford divisor D' on (S, L) must be $(2, 2, 2, 2, 0)$.

We will now stop to answer a question posed on p. 28: Do there exist non-perfect Clifford divisors? We will answer this question in the affirmative by exhibiting an example. We have seen that

$$\mathcal{A}^0(L) = \{D, D + \Gamma, L - D, L - D - \Gamma\}.$$

We have also seen that we may assume that D is a perfect Clifford divisor. Then $D + \Gamma$ satisfies (C1) and (C2), so $D + \Gamma$ is a Clifford divisor. Thus $h^1(D + \Gamma) = 0$ and we get

$$2 = h^0(D) = h^0(D + \Gamma).$$

Thus Γ is fixed in $|D + \Gamma|$, so $D + \Gamma$ is not base point free. Especially (C5) is not satisfied. Hence $D + \Gamma$ is a non-free, and whence a non-perfect, Clifford divisor.

2.1.5 $(\mathbf{4, 2, 1, 1, 0})$

L must be of type $\{3, 0\}^a$. Hence $\Delta' = \Gamma$, where $\mathcal{R}_{L,D} = \{\Gamma\}$. We can write $A' \sim 2D + B_0$, where $h^0(B_0) = 1$.

$\Gamma.L = 0$ gives $\Gamma.B_0 = -2$. Hence we can write $B_0 \sim \Gamma + B_1$, where $D.B_1 = 3, \Gamma.B_1 = 0$, $B_1^2 = -10$, and $L.B_1 = 2$.

Let $\gamma \leq B_1$ be a smooth rational curve such that $D.\gamma \neq 0$. Then $D.\gamma = 1$ by proposition 1.1.17, since $2 = h^0(L - 3D) = h^0(D + 2\Gamma + B_1) = h^0(D)$.

If $\operatorname{rank} \operatorname{Pic} S < 6$ then this scroll type is not associated to any polarized K3 surface.[14] If $\operatorname{rank} \operatorname{Pic} S = 6$ then this scroll type is possible with $L \sim 4D + 3\Gamma + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4$

[13] The argument in footnote 10 gives that Γ is a sum of smooth rational curves γ satifying $\gamma.L = 0$. Since D is nef there must exist a γ such that $\gamma.D = 1$.

[14] The only possible configuration (i.e. a configuration with $h^0(L - 4D) > 0$ and $\mathcal{R}_{L,D}$ consisting of only one curve) we find with $\operatorname{rank}(\operatorname{Pic}(S)) < 6$ is configuration (2.24) with $N = 0$. Then $\Gamma_0.(L - D) = -1$ so Γ_0 is fixed in $F := L - D$. $\Gamma_1.(F - \Gamma_0) = -1$ so Γ_1 is fixed in $F - \Gamma_0$. $\Gamma_2.(F - \Gamma_0 - \Gamma_1) = -1$ so Γ_2 is fixed in $F - \Gamma_0 - \Gamma_1$. $\Gamma_0.(F - \Gamma_0 - \Gamma_1 - \Gamma_2) = -1$ so Γ_0 is fixed in $(F - \Gamma_0 - \Gamma_1 - \Gamma_2)$. Especially $2\Gamma_0$ is fixed in F, hence $h^1(R) \geq 2$. Since $c = 3$ we get $h^1(R) = 2$. Then the associated scroll type cannot be $(4, 2, 1, 1, 0)$.

and the following configuration:

(2.11)
$$\begin{tikzcd} D \arrow[r, dash] \arrow[dr, dash] & \Gamma \arrow[r, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_3 \\ & \Gamma_1 \arrow[r, dash] & \Gamma_4 & \end{tikzcd}$$

This is our first example of a decomposition where (A, Δ) is not well-behaved. For $\Gamma_2.(L - 2D - \Gamma) = -1$, so Γ_2 is fixed in $L - 2D - \Gamma$. Hence $\Delta' < \Gamma + \Gamma_2 \leq \Delta'$.

We will now show that this gives a 14-dimensional family of polarized K3 surfaces (S, L) associated to the scroll type $(4, 2, 1, 1, 0)$.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma \oplus \mathbb{Z}\Gamma_1 \oplus \oplus\mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma_3 \oplus \mathbb{Z}\Gamma_4$ with intersection matrix

$$M = \begin{bmatrix} D^2 & D.\Gamma & D.\Gamma_1 & D.\Gamma_2 & D.\Gamma_3 & D.\Gamma_4 \\ D.\Gamma & \Gamma^2 & \Gamma.\Gamma_1 & \Gamma.\Gamma_2 & \Gamma.\Gamma_3 & \Gamma.\Gamma_4 \\ D.\Gamma_1 & \Gamma.\Gamma_1 & \Gamma_1^2 & \Gamma_1.\Gamma_2 & \Gamma_1.\Gamma_3 & \Gamma_1.\Gamma_4 \\ D.\Gamma_2 & \Gamma.\Gamma_2 & \Gamma_1.\Gamma_2 & \Gamma_2^2 & \Gamma_2.\Gamma_3 & \Gamma_2.\Gamma_4 \\ D.\Gamma_3 & \Gamma.\Gamma_3 & \Gamma_1.\Gamma_3 & \Gamma_2.\Gamma_3 & \Gamma_3^2 & \Gamma_3.\Gamma_4 \\ D.\Gamma_4 & \Gamma.\Gamma_4 & \Gamma_1.\Gamma_4 & \Gamma_2.\Gamma_4 & \Gamma_3.\Gamma_4 & \Gamma_4^2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix}$$

has signature (1,5), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}\Gamma \oplus \mathbb{Z}\Gamma_1 \oplus \oplus\mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma_3 \oplus \mathbb{Z}\Gamma_4$.

Using Picard-Lefschetz reflections we may assume that $L \sim 4D + 3\Gamma + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4$ is nef. L nef and Riemann-Roch used on D and Γ_1 let us assume that D and Γ_1 are effective.

Assume that $B \in \mathcal{A}^0(L)$, with B nef. Let $B \sim xD + y\Gamma + r\Gamma_1 + s\Gamma_2 + t\Gamma_3 + u\Gamma_4$. Then B must satisfy one of the following

$$\begin{aligned} B^2 &= 0 \qquad B.L = 1, 2, 3, \text{ or } 4 \\ B^2 &= 2 \qquad B.L = 7 \end{aligned}$$

Arguing as on page 66 we get:

$$\begin{aligned} 0 \leq B.D &= y + r \leq 3 \\ -1 \leq B.\Gamma &= x - 2y + s \leq 1 \\ 0 \leq B.\Gamma_1 &= x - 2r + u \leq 2 \\ -1 \leq B.\Gamma_2 &= y - 2s + t \leq 1 \\ -1 \leq B.\Gamma_3 &= s - 2t \leq 1 \\ -1 \leq B.\Gamma_4 &= r - 2u \leq 1 \end{aligned}$$

Solving for x we get

$$x = \frac{12}{17}B.D + \frac{9}{17}B.\Gamma + \frac{8}{17}B.\Gamma_1 + \frac{9}{17}B.\Gamma_2 + \frac{3}{17}B.\Gamma_3 + \frac{4}{17}B.\Gamma_4$$

The inequalities then give $-1 \leq x \leq 3$. Similarly we get $0 \leq y \leq 2$, $0 \leq r \leq 2$, $-1 \leq s \leq 2$, $-1 \leq t \leq 1$, and $0 \leq u \leq 1$. Then $1 \leq B.L = 5x + 2r \leq 7$ gives $x = 1$.

Checking all possible values of y, r, s, t, and v we get $B \sim D$, $B \sim D + \Gamma$, $B \sim D + \Gamma + \Gamma_2$, and $B \sim D + \Gamma + \Gamma_2 + \Gamma_3$ as possible solutions. Especially L is of type $\{3, 0\}$.

We will now show that, after a change of basis if necessary, we may assume $B \sim D$. If $B \sim D$, then there is nothing to show.

If $B \sim D + \Gamma$, then we change the basis of $\operatorname{Pic} S$ as follows:

$$\begin{array}{rcccl} D & \mapsto & D + \Gamma & := & D' \\ \Gamma & \mapsto & -\Gamma & := & \Gamma' \\ \Gamma_1 & \mapsto & \Gamma_1 & := & \Gamma'_1 \\ \Gamma_2 & \mapsto & \Gamma + \Gamma_2 & := & \Gamma'_2 \\ \Gamma_3 & \mapsto & \Gamma_3 & := & \Gamma'_3 \\ \Gamma_4 & \mapsto & \Gamma_4 & := & \Gamma'_4 \end{array}$$

We see that $L \sim 4D' + 3\Gamma' + 2\Gamma'_1 + 2\Gamma'_2 + \Gamma'_3 + \Gamma'_4$ and that the new intersection numbers are equal to the old ones. Hence we may assume that $B \sim D$ in this case.

If $B \sim D + \Gamma + \Gamma_2$, then we change the basis of $\operatorname{Pic} S$ as follows:

$$\begin{array}{rcccl} D & \mapsto & D + \Gamma + \Gamma_2 & := & D' \\ \Gamma & \mapsto & -\Gamma_2 & := & \Gamma' \\ \Gamma_1 & \mapsto & \Gamma_1 & := & \Gamma'_1 \\ \Gamma_2 & \mapsto & -\Gamma & := & \Gamma'_2 \\ \Gamma_3 & \mapsto & \Gamma + \Gamma_2 + \Gamma_3 & := & \Gamma'_3 \\ \Gamma_4 & \mapsto & \Gamma_4 & := & \Gamma'_4 \end{array}$$

We see that $L \sim 4D' + 3\Gamma' + 2\Gamma'_1 + 2\Gamma'_2 + \Gamma'_3 + \Gamma'_4$ and that the new intersection numbers are equal to the old ones. Hence we may assume that $B \sim D$ in this case also.

If $B \sim D + \Gamma + \Gamma_2 + \Gamma_3$, then we change the basis of $\operatorname{Pic} S$ as follows:

$$\begin{array}{rcccl} D & \mapsto & D + \Gamma + \Gamma_2 + \Gamma_3 & := & D' \\ \Gamma & \mapsto & -\Gamma_3 & := & \Gamma' \\ \Gamma_1 & \mapsto & \Gamma_1 & := & \Gamma'_1 \\ \Gamma_2 & \mapsto & -\Gamma_2 & := & \Gamma'_2 \\ \Gamma_3 & \mapsto & -\Gamma & := & \Gamma'_3 \\ \Gamma_4 & \mapsto & \Gamma_4 & := & \Gamma'_4 \end{array}$$

We see that $L \sim 4D' + 3\Gamma' + 2\Gamma'_1 + 2\Gamma'_2 + \Gamma'_3 + \Gamma'_4$ and that the new intersection numbers are equal to the old ones. Hence we may assume that $B \sim D$ in this case also.

We will now find $\mathcal{R}_{L,D}$. We will show below that we may assume $h^0(L - 4D) > 0$. Hence $\mathcal{R}_{L,D} \neq \emptyset$, since the scroll type $(4, 1, 1, 1, 1)$ is not associated to any polarized K3 surface of type $\{3,0\}$. Let $B \in \mathcal{R}_{L,D}$. Then $B.D = 1$, $B.L = 0$, and $B^2 = -2$ gives $B \sim \Gamma$, $B \sim \Gamma + \Gamma_2$, or $B \sim \Gamma + \Gamma_2 + \Gamma_3$. Since divisors in $\mathcal{R}_{L,D}$ are disjoint only one of the above can be in $\mathcal{R}_{L,D}$. Consequently $\mathcal{R}_{L,D}$ consists of only one divisor. We will show that we may assume this divisor to be Γ. If $B \sim \Gamma$, then there is nothing to show.

If $B \sim \Gamma + \Gamma_2$, then we change the basis of $\operatorname{Pic} S$ as follows:

$$\begin{array}{rcccl} D & \mapsto & D & := & D' \\ \Gamma & \mapsto & \Gamma + \Gamma_2 & := & \Gamma' \\ \Gamma_1 & \mapsto & \Gamma_1 & := & \Gamma_1' \\ \Gamma_2 & \mapsto & -\Gamma_2 & := & \Gamma_2' \\ \Gamma_3 & \mapsto & \Gamma_2 + \Gamma_3 & := & \Gamma_3' \\ \Gamma_4 & \mapsto & \Gamma_4 & := & \Gamma_4' \end{array}$$

We see that we have $3\Gamma + 2\Gamma_2 + \Gamma_3 = 3\Gamma' + 2\Gamma_2' + \Gamma_3'$ and that the new intersection matrix is equal to the old one. Hence we may assume that $B \sim \Gamma$ in this case too.

If $B \sim \Gamma + \Gamma_2 + \Gamma_3$, then we change the basis of $\operatorname{Pic} S$ as follows:

$$\begin{array}{rcccl} D & \mapsto & D & := & D' \\ \Gamma & \mapsto & \Gamma + \Gamma_2 + \Gamma_3 & := & \Gamma' \\ \Gamma_1 & \mapsto & \Gamma_1 & := & \Gamma_1' \\ \Gamma_2 & \mapsto & -\Gamma_2 - \Gamma_3 & := & \Gamma_2' \\ \Gamma_3 & \mapsto & \Gamma_2 & := & \Gamma_3' \\ \Gamma_4 & \mapsto & \Gamma_4 & := & \Gamma_4' \end{array}$$

We see that we have $3\Gamma + 2\Gamma_2 + \Gamma_3 = 3\Gamma' + 2\Gamma_2' + \Gamma_3'$ and that the new intersection matrix is equal to the old one. Hence we may assume that $B \sim \Gamma$ in this case too. Thus we may assume $\mathcal{R}_{L,D} = \{\Gamma\}$.

We will now show that $h^1(R) = 1$. It is enough to show that we are not in case $\{3,0\}^c$, since $\mathcal{R}_{L,D}$ consists of only one curve.

If $h^1(R) = 2$ we must be in a case equivalent to configuration (2.24) with $N = 0$ or $N = 1$.[15]

If we are in a case equivalent to configuration (2.24) with $N = 1$, then there must

[15] Since $h^0(L - 4D) \geq 0$ (this will be shown below) the scroll type must be $(4,2,2,0,0)$ or $(4,3,2,0,0)$ if $h^1(R) = 2$. It will follow from our treatment of these scroll types below that the configuration must then be as in configuration (2.24). Since $\operatorname{rank} \operatorname{Pic} S = 6$ we have $N = 0$ or $N = 1$.

exist Γ', Γ'_1, Γ'_2 and Γ'_3 in Pic S such that

$$M' = \begin{bmatrix} D^2 & D.\Gamma & D.\Gamma'_1 & D.\Gamma'_2 & D.\Gamma'_3 & D.\Gamma' \\ D.\Gamma & \Gamma^2 & \Gamma.\Gamma'_1 & \Gamma.\Gamma'_2 & \Gamma.\Gamma'_3 & \Gamma.\Gamma' \\ D.\Gamma_1 & \Gamma.\Gamma'_1 & \Gamma'^2_1 & \Gamma'_1.\Gamma'_2 & \Gamma'_1.\Gamma'_3 & \Gamma'_1.\Gamma' \\ D.\Gamma'_2 & \Gamma.\Gamma'_2 & \Gamma'_1.\Gamma'_2 & \Gamma'^2_2 & \Gamma'_2.\Gamma'_3 & \Gamma'_2.\Gamma' \\ D.\Gamma'_3 & \Gamma.\Gamma'_3 & \Gamma'_1.\Gamma'_3 & \Gamma'_2.\Gamma'_3 & \Gamma'^2_3 & \Gamma'_3.\Gamma' \\ D.\Gamma' & \Gamma.\Gamma' & \Gamma'_1.\Gamma' & \Gamma'_2.\Gamma' & \Gamma'_3.\Gamma' & \Gamma'^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

But we have $\det(M) = -17$ and $\det(M') = -12$, which contradicts proposition 1.1.22. Hence we cannot have $N = 1$.

If $N = 0$ there must exist two disjoint curves Γ'_1 and Γ'_2 such that $L.\Gamma'_i = D.\Gamma'_i = 0$, $\Gamma.\Gamma'_i = 1$, and $\Gamma'^2_i = -2$ (for $i = 1$ and 2). Write $\Gamma'_i \sim xD + y\Gamma + r\Gamma_1 + s\Gamma_2 + t\Gamma_3 + u\Gamma_4$. Then $L.\Gamma'_i = 0$ gives $5x+r = 0$, $D.\Gamma'_1 = 0$ gives $y+r = 0$, and $\Gamma.\Gamma'_i = 1$ gives $x-2y+s = 1$. Solving for x and substituting into $B^2 = -2$ gives

$$86x^2 + (13 - 9t + u)x + (t^2 + u^2 - t) = 0.$$

Viewing this as a quadratic equation in x it has integral solutions only if the discriminant

$$169 + 110t - 254t^2 + 26u - 335u^2 - 9(t + u)^2$$

is non-negative. Using $a^2 \geq a$ (for integral a) this gives $0 \leq 199-144t^2-309u^2-9(t+u)^2$, or $144t^2 + 309u^2 + 9(t + u)^2 \leq 199$. Hence $u = 0$ and $t = 0$ or 1. This gives $\Gamma'_i \sim \Gamma_2$ or $\Gamma'_i \sim \Gamma_2 + \Gamma_3$. Since $\Gamma_2.(\Gamma_2 + \Gamma_3) = -1$ there cannot exist a Γ'_2 disjoint to Γ'_1. Hence the case $N = 0$ is impossible too. Therefore $h^1(R) = 1$.

We will now show that $L-4D$ is effective. D nef and Riemann-Roch gives Γ effective. We have already seen that Γ_1 is effective. Therefore it is enough to show that we may assume that Γ_2, Γ_3, and Γ_4 is effective.

Since $\Gamma_2^2 = -2$, Riemann-Roch gives that either Γ_2 or $-\Gamma_2$ is effective. If $-\Gamma_2$ is effective, then $\Gamma.(-\Gamma_2) = -1$ gives that Γ is a fixed divisor in Γ_2. Hence we can write $-\Gamma_2 \sim \Gamma + F$, where F is effective. But then $\Gamma_2.D = 0$ gives $F.D = -1$, a contradiction since D is nef. Hence Γ_2 is effective.

Since $\Gamma_3^2 = -2$, Riemann-Roch gives that either Γ_3 or $-\Gamma_3$ is effective. If Γ_3 is effective, we are done. If $-\Gamma_3$ is effective we change the basis of Pic S as follows:

$$\begin{array}{rcl} D \mapsto & D & := D' \\ \Gamma \mapsto & \Gamma & := \Gamma' \\ \Gamma_1 \mapsto & \Gamma_1 & := \Gamma'_1 \\ \Gamma_2 \mapsto & \Gamma_2 + \Gamma_3 & := \Gamma'_2 \\ \Gamma_3 \mapsto & -\Gamma_3 & := \Gamma'_3 \\ \Gamma_4 \mapsto & \Gamma_4 & := \Gamma'_4 \end{array}$$

We see that $L \sim 4D' + 3\Gamma' + 2\Gamma'_1 + 2\Gamma'_2 + \Gamma'_3 + \Gamma'_4$ and that the new intersection numbers are equal to the old ones. Hence we may assume that Γ_3 is effective.[16]

Since $\Gamma_4^2 = -2$, Riemann-Roch gives that either Γ_4 or $-\Gamma_4$ is effective. If Γ_4 is effective, there is nothing to show. If $-\Gamma_4$ is effective, we change the basis of Pic S as follows:

$$\begin{array}{rcccl} D & \mapsto & D & := & D' \\ \Gamma & \mapsto & \Gamma & := & \Gamma' \\ \Gamma_1 & \mapsto & \Gamma_1 + \Gamma_4 & := & \Gamma'_1 \\ \Gamma_2 & \mapsto & \Gamma_2 & := & \Gamma'_2 \\ \Gamma_3 & \mapsto & \Gamma_3 & := & \Gamma'_3 \\ \Gamma_4 & \mapsto & -\Gamma_4 & := & \Gamma'_4 \end{array}$$

We see that $L \sim 4D' + 3\Gamma' + 2\Gamma'_1 + 2\Gamma'_2 + \Gamma'_3 + \Gamma'_4$ and that the new intersection numbers are equal to the old ones. Hence we may assume that Γ_4 is effective.[17]

Hence we have $h^0(L - 4D) > 0$ and $h^1(R) = 1$. The associated scroll type must be $(4, 2, 1, 1, 0)$.

2.1.6 $(\mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{0})$

This scroll type is possible with many different configurations. Due to the expense of paper and the patience of the reader we will concentrate on the most general ones; which are those that give 16-dimensional families of K3 surfaces.

We must be in case $\{3, 0\}^a$. Hence $\Delta' = \Gamma$, where $\mathcal{R}_{L,D} = \{\Gamma\}$. We can write $A' \sim D + B_0$, where $h^0(B_0) = 1$.

$\Gamma.L = 0$ gives $\Gamma.B_0 = -1$. Hence we can write $B_0 \sim \Gamma + B_1$ with B_1 effective, $D.B_1 = 3, \Gamma.B_1 = 1$, $B_1^2 = -4$, and $L.B_1 = 7$.

There must exist a smooth rational curve $\Gamma_1 \leq B_1$ such that $\Gamma_1.\Gamma = 1$. Write $B_1 \sim \Gamma_1 + B_2$, with B_2 effective. Then $\Gamma.B_2 = 0$.

Note that $D.\Gamma_1 = 3$ does not occur. For then we have $-2 = 2\Gamma_1.B_2 + B_2^2$, $B_2.D = 0$, and $L.B_2 = \Gamma_1.B_2 + B_2^2 \geq 0$, which gives $B_2^2 \geq 0$. This contradicts $h^0(B_0) = 1$. In particular $D.B_2 \neq 0$.

We will now show that $\operatorname{rank} \operatorname{Pic} S \geq 4$. Since D, Γ, and Γ_1 are linearly independent $\operatorname{rank} \operatorname{Pic} S < 4$ only if we can write $B_2 \sim n\Gamma + n_1\Gamma_1 + n_2 D$ $(n, n_1, n_2 \in \mathbb{Q})$. We have $0 \leq \Gamma_1.L \leq 7$ and $0 \leq \Gamma_1.D \leq 2$. Fixing $a = \Gamma_1.L$ and $b = \Gamma_1.D$ the three equations $\Gamma.B_2 = 0$, $\Gamma_1.L = a$, and $L.B_2 = 7 - a$ give three linearly independent equations that we can solve for n, n_1 and n_2. For all possibilities of a and b we get $B^2 \notin 2\mathbb{Z}$, a contradiction. For example if $a = b = 0$ we get $n = \frac{3}{5}, n_1 = \frac{7}{5}$, and $n_2 = \frac{7}{5}$. This gives $B^2 = \frac{28}{25}$. Hence $\operatorname{rank} \operatorname{Pic} S \geq 4$.

We may assume that there exists a smooth rational curve Γ_2 (not equal to Γ or Γ_1) such that $D.\Gamma_2 > 0$ and $\Gamma_2 < B_2$.

[16] Note that we still have Γ_2 effective after the change of basis, for our proof of Γ_2 being effective still holds after the base change.

[17] Note that we may still assume that Γ'_1 is effective, since $L.\Gamma'_1 = 1$.

If $\Gamma_1.B_2 > 0$ we only get configurations which have $\operatorname{rank}\operatorname{Pic} S > 4$, such as:

(2.12)
$$\begin{tikzcd}
\Gamma_2 \arrow[r, dash] & D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma \arrow[d, dash] \\
& \Gamma_3 \arrow[r, dash] & \Gamma_1
\end{tikzcd}$$

with $L \sim 3D + \Gamma_1 + \Gamma_2 + \Gamma_3 + 2\Gamma$.

Hence we are only interested in the case where $\Gamma_1.B_2 = 0$. In this case too we have many possible configurations. Most of these with $\operatorname{rank}\operatorname{Pic} S > 4$, such as

(2.13)
$$\begin{tikzcd}
\Gamma \arrow[r, dash] \arrow[d, dash] & D \arrow[r, dash] \arrow[dl, dash] \arrow[dr, dash] & \Gamma_3 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_{2N+1} \arrow[d, dash] \\
\Gamma_1 & & \Gamma_2 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_{2N}
\end{tikzcd}$$

with $L \sim 3D + \Gamma_1 + \cdots + \Gamma_{N+1} + 2\Gamma$.

If there exists a smooth rational curve $\Gamma_3 \leq B_2$ distinct from Γ, Γ_1, and Γ_2 one can show that $D, \Gamma, \Gamma_1, \Gamma_2$, and Γ_3 are linearly independent so $\operatorname{rank}\operatorname{Pic} S \geq 5$. When $\operatorname{rank}\operatorname{Pic} S = 4$ we may thus assume $B_2 \sim n\Gamma + n_1\Gamma_1 + n_2\Gamma_2$ for non-negative integers n, n_1, and n_2. We got four cases to consider

i) $\Gamma_1.\Gamma_2 = \Gamma.\Gamma_2 = 0$. Since $\Gamma_1.B_2 = \Gamma.B_2 = 0$ we get $n = n_1 = 0$ and $n_2 = 1$.

ii) $\Gamma_1.\Gamma_2 = 0$ and $\Gamma.\Gamma_2 = 1$. From $\Gamma_1.B_2 = \Gamma.B_2 = 0$ we get $n = 2n_1$ and $n_1 + n_2 = n$. Then $B_2^2 = -2$ gives $n_1 = 1/8$. A contradiction.

iii) $\Gamma_1.\Gamma_2 = 1$ and $\Gamma.\Gamma_2 = 0$. This is symmetrical to case ii).

iv) $\Gamma_1.\Gamma_2 = \Gamma.\Gamma_2 = 1$. Since $\Gamma_1.B_2 = \Gamma.B_2 = 0$ we get $n = n_1 = n_2$. But then $B_2^2 = -2$ gives $-2 = (n\Gamma + n_1\Gamma_1 + n_2\Gamma_2)^2 = 0$. A contradiction.

Hence we have $B_2 \sim \Gamma_2$, with $\Gamma_1.\Gamma_2 = \Gamma.\Gamma_2 = 0$. Since $D.\Gamma_1$ equals $0, 1$, or 2 we get the following possible configurations (all with $L \sim 3D + \Gamma_1 + \Gamma_2 + 2\Gamma$):

(2.14)
$$\begin{tikzcd}
D \arrow[r, dash] \arrow[d, dash] \arrow[d, dash, shift left=1] \arrow[d, dash, shift right=1] & \Gamma \arrow[d, dash] \\
\Gamma_2 & \Gamma_1
\end{tikzcd}$$

(2.15)
$$\begin{tikzcd}
D \arrow[r, dash] \arrow[d, equal] \arrow[dr, dash] & \Gamma \arrow[d, dash] \\
\Gamma_2 & \Gamma_1
\end{tikzcd}$$

(2.16)
$$\begin{tikzcd}
D \arrow[r, dash] \arrow[d, dash] \arrow[dr, equal] & \Gamma \arrow[d, dash] \\
\Gamma_2 & \Gamma_1
\end{tikzcd}$$

Note that configuration (2.14) gives an example of a perfect Clifford divisor where (A, Δ) is not well-behaved. For if (A, Δ) was well-behaved, then we would have $\Delta' =$

$\Delta = \Gamma$. But we see that $\Gamma_1.(L - D - \Gamma) = -1$, so Γ_1 is fixed in $L - D - \Gamma$. Hence $\Gamma + \Gamma_1 \leq \Delta$, and we cannot have $\Delta' = \Delta$.[18]

We will now show that there exists a 16-dimensional family of polarized K3 surfaces (S,L) with configuration (2.14) and associated scroll type $(3, 2, 2, 1, 0)$.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$ with intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma & D.\Gamma_1 & D.\Gamma_2 \\ D.\Gamma & \Gamma^2 & \Gamma.\Gamma_1 & \Gamma.\Gamma_2 \\ D.\Gamma_1 & \Gamma.\Gamma_1 & \Gamma_1^2 & \Gamma_1.\Gamma_2 \\ D.\Gamma_2 & \Gamma.\Gamma_1 & \Gamma_1.\Gamma_2 & \Gamma_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 3 & 0 & 0 & -2 \end{bmatrix}$$

has signature (1,3), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}\Gamma \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$.

Using Picard-Lefschetz reflections we may assume that $L \sim 3D + \Gamma_1 + \Gamma_2 + 2\Gamma$ is nef. L nef and Riemann-Roch used on D and Γ_2 let us assume D and Γ_2 effective. Likewise we see that Γ or $-\Gamma$, resp. Γ_1 or $-\Gamma_1$, is effective.

We will now show that L is base point free and of type $\{3, 0\}$. Assume $B \sim xD + y\Gamma + z\Gamma_1 + w\Gamma_2 \in \mathcal{A}^0(L)$.

Arguing as on page 66 we get:

$$\begin{aligned} -1 \leq \Gamma.B &= x - 2y + z \leq 1 \\ -1 \leq \Gamma_1.B &= y - 2z \leq 1 \\ 0 \leq \Gamma_2.B &= 3x - 2w \leq 4 \\ 0 \leq D.B &= y + 3w \leq 3 \end{aligned}$$

A case by case analysis shows that the only integral solutions are $B \sim D$, $B \sim D + \Gamma$, and $B \sim D + \Gamma + \Gamma_1$. Hence $c = 3$. After a change of basis as on the pages 71 and 71 we may assume that $B \sim D$. Hence we may assume that D is a perfect Clifford divisor. D is nef and Riemann-Roch used on Γ let us assume Γ effective. After another change of basis if necessary we may assume that Γ_1 is effective (see page 67).

It remains to determine the scroll type given by D. First we find $\mathcal{R}_{L,D}$. Let $B \in \mathcal{R}_{L,D}$. Then $B.D = 1$, $B.L = 0$, and $B^2 = -2$ give $B \sim \Gamma$. Hence $\mathcal{R}_{L,D} = \{\Gamma\}$. The scroll type must be $(3, 2, 2, 1, 0)$, $(3, 3, 1, 1, 0)$, or $(4, 2, 1, 1, 0)$, since $h^0(L - 3D) = h^0(\Gamma_1 + \Gamma_2 + 2\Gamma) > 0$. We have shown that $(4, 2, 1, 1, 0)$ is only possible with $\operatorname{rank} \operatorname{Pic} S > 4$. We will show the same for the scroll type $(3, 3, 1, 1, 0)$ below. Hence the associated scroll type is $(3, 2, 2, 1, 0)$.

2.1.7 $(3, 3, 1, 1, 0)$

We must be in case $\{3, 0\}^a$. Thus we can write $L \sim 3D + B_0 + \Gamma$, where $A' \sim D + B_0$. $L.\Gamma = 0$ gives $\Gamma.B_0 = -1$. Hence we can write $B_0 \sim \Gamma + B_1$, where $B_1.\Gamma = 1$, $B_1.D = 3$, $B_1.L = 7$, and $B_1^2 = -4$.

[18] In fact we have $A \sim D + \Gamma + \Gamma_2 = A' - \Gamma_1$ and $\Delta = \Gamma + \Gamma_1 = \Delta' + \Gamma_1$.

We have $h^0(L-3D) = h^0(2\Gamma+B_1) = 2$. Γ is fixed in $2\Gamma+B_1$, since $\Gamma.(2\Gamma+B_1) = -3$. Furthermore Γ is fixed in $\Gamma + B_1$, since $\Gamma.(\Gamma + B_1) = -1$. Consequently 2Γ is fixed in $2\Gamma + B_1$. Therefore $h^0(B_1) = 2$.

Assume that we can write $B_1 \sim n\Gamma + n_1 F + n_2 G + n_3 H$, where F, G, and H are prime divisors on S such that F^2, G^2, and H^2 equals -2 or 0. Using $B_1.\Gamma = 1$, $B_1.D = 3$, $B_1.L = 7$, $B_1^2 = -4$, and $h^0(B_1) = 2$ we find no solutions with at least one of the n_i equal to zero.[19] Hence there exists no solutions with $\operatorname{rank} \operatorname{Pic} S < 5$.[20] When all the n_i are non-zero there exists a solution. Let $F^2 = G^2 = H^2 = -2$, then we have the solution $B_1 \sim F + 2G + H$, with the following configuration:

(2.17)
$$\begin{tikzcd} & D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma \arrow[d, dash] \\ F \arrow[r, dash, shift left] \arrow[r, dash, shift right] & G & H \end{tikzcd}$$

We will now show that this gives rise to a 15-dimensional family of K3-surfaces associated to the scroll type $(3, 3, 1, 1, 0)$. The lattice $\mathbb{Z}D \oplus \mathbb{Z}F \oplus \mathbb{Z}G \oplus \mathbb{Z}H \oplus \mathbb{Z}\Gamma$ with intersection matrix

$$\begin{bmatrix} D^2 & D.F & D.G & D.H & D.\Gamma \\ D.F & F^2 & F.G & F.H & F.\Gamma \\ D.G & F.G & G^2 & G.H & G.\Gamma \\ D.H & F.H & G.H & H^2 & H.\Gamma \\ D.\Gamma & F.\Gamma & G.\Gamma & H.\Gamma & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & -2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 2 & 0 \\ 0 & 0 & 2 & -2 & 0 \\ 1 & 1 & 0 & 0 & -2 \end{bmatrix}$$

has signature (1,4), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}F \oplus \mathbb{Z}G \oplus \mathbb{Z}H \oplus \mathbb{Z}\Gamma$.

Using Picard-Lefschetz reflections we may assume that $L \sim 3D + 2\Gamma + F + 2G + H$ is nef. L nef and Riemann-Roch used on D, F, G, and H let us assume that D, F, G, and H is effective.

We will now show that $B \in \mathcal{A}^0(L)$, with B nef, implies $B \sim D$ or $B \sim D + \Gamma$. This will in particular show that L is base point free and of type $\{3, 0\}$. Since $D.L - D^2 - 2 = 3$, we know that $c \leq 3$. Hence $B.L \leq 7$. Assume $B \sim xD + yF + zG + uH + v\Gamma$.

Arguing as on page 66 we get:

$$\begin{aligned} 0 \leq B.D &= y + z + v \leq 3 \\ 0 \leq B.F &= x - 2y + v \leq 2 \\ 0 \leq B.G &= x - 2x + 2u \leq 2 \\ 0 \leq B.H &= 2z - 2u \leq 2 \\ -1 \leq B.\Gamma &= x + y - 2v \leq 1 \end{aligned}$$

[19] This is quite a lot of work. It is not particularly interesting so I will omit the details. Note that there exists several solutions with $h^0(B_1) = 1$. These solutions are associated to other scroll types. In particular we see that the configurations (2.14), (2.15), (2.16), and (2.18) are solutions to $B_1.\Gamma = 1$, $B_1.D = 3$, $B_1.L = 7$, $B_1^2 = -4$, and $h^0(B_1) = 1$.

[20] We may assume that a basis for $\operatorname{Pic} S$ consists of prime divisors.

Adding the inequalities for $B.G$ and $B.H$ we get $0 \leq x \leq 4$. Rearranging

$$B^2 = -2(y^2 + z^2 + u^2 + v^2) + 2x(y + z + v) + 2yv + 4zu \geq 0,$$

we get

$$2(z - u)^2 + y^2 + v^2(y - v)^2 \leq 2x(y + z + v) \leq 8(y + z + v) \leq 24,$$

by the inequality for $B.D$. Hence there are only finitely many possible values of $(y - v)$, $(z - u)$, y and v to check. We can reduce the number of cases further by noting that $u \leq z \leq u + 1$ by the inequality for $B.H$. A case by case analysis (using $1 \leq B.L \leq 7$) gives $B \sim D$ or $B \sim D + \Gamma$.

If $B \sim D + \Gamma$, then after a change of basis of $\operatorname{Pic} S$, as on page 71, we may assume that $B \sim D$.

Hence we may assume that D is a perfect Clifford divisor. It remains to determine the scroll type given by D. We have to find $\mathcal{R}_{L,D}$. Let $B \in \mathcal{R}_{L,D}$. Then $B.D = 1$, $B.L = 0$, and $B^2 = -2$ gives $B \sim \Gamma$. Then $\mathcal{R}_{L,D} = \{\Gamma\}$. Since $h^0(L - 3D) \geq h^0(F + G) \geq 2$, the scroll type must be $(3, 3, 1, 1, 0)$ or $(4, 2, 1, 1, 0)$. We have shown that the scroll type $(4, 2, 1, 1, 0)$ is only possible with $\operatorname{rank} \operatorname{Pic} S > 5$. Hence the scroll type associated to D must be $(3, 3, 1, 1, 0)$.

2.1.8 $(\mathbf{3}, \mathbf{3}, \mathbf{2}, \mathbf{0}, \mathbf{0})$

Let $A' \sim D + B_0$, where B_0 is effective and $h^0(B_0) = 2$. We have $h^1(R) = 2$ so we must be in case $\{3, 0\}^b$ or $\{3, 0\}^c$.

$\{\mathbf{3}, \mathbf{0}\}^{\mathbf{b}}$ Then $L \sim 3D + B_0 + \Gamma_1 + \Gamma_2$. $A'.\Gamma_i = 0$ gives $B_0.\Gamma_i = -1$. Hence we can write $B_0 \sim \Gamma_1 + \Gamma_2 + B_1$, where B_1 is effective, $B_1.\Gamma_i = 1$, and $B_1^2 = 0$. Since $h^0(B_1) \leq 2$ and $B_1 \neq 0$ we get $h^0(B_1) = 2$. In general we have $B_1 \sim E$, where E is an elliptic curve. This gives

$$L \sim 3D + E + 2\Gamma_1 + 2\Gamma_2,$$

with the following configuration:

(2.18)
$$\begin{tikzcd}
D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma_2 \arrow[d, dash] \\
\Gamma_1 \arrow[r, dash] & E
\end{tikzcd}$$

As an example of a less general configuration, where B_1 is not an elliptic curve, we take

$$L \sim 3D + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \cdots + \Gamma_{2N+1},$$

with the following configuration:

(2.19)
$$\begin{tikzcd}
D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma_2 \arrow[d, dash] & \Gamma_5 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_{2N+1} \arrow[d, dash] \\
\Gamma_1 \arrow[r, dash] & \Gamma_3 \arrow[r, dash] \arrow[ur, dash] & \Gamma_4 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_{2N}
\end{tikzcd}$$

$\{\mathbf{3},\mathbf{0}\}^{\mathbf{c}}$ Then $L \sim 3D+B_0+2\Gamma_0+\ldots+2\Gamma_N+\Gamma_{N+1}+\Gamma_{N+2}$. $A'.\Gamma_i = 0$ and $A' \sim D+B_0$ give $B_0.\Gamma_0 = -1$ and $B_0.\Gamma_i = 0$ for $i > 0$. Hence we can write $B_0 \sim \Gamma_0 + B_1$, where B_1 is effective, $D.B_1 = 2$, $B_1\Gamma_0 = 1$, and $B_1.\Gamma_i = -\Gamma_0.\Gamma_i$ for $i > 0$.

$N = 0$. In this case $B_1.\Gamma_1 = B_1.\Gamma_2 = -1$, so we can write $B_1 \sim \Gamma_1 + \Gamma_2 + B_2$, where B_2 is effective, $B_2.D = 2$, $B_2.\Gamma_1 = B_2.\Gamma_2 = 1$, and $B_2.\Gamma_0 = -1$. Hence we can write $B_2 \sim \Gamma_0 + B_3$, where B_3 is effective, $B_3.D = 1$, $B_3.\Gamma_1 = B_3.\Gamma_2 = 0$, $B_3.\Gamma_0' = 1$, and $B_3^2 = 0$. Since $h^0(B_3) \leq 2$ and $B_3 \neq 0$ we get $h^0(B_3) = 2$. In general we have $B \sim E$, where E is an elliptic curve. This gives

$$L \sim 3D + E + 4\Gamma_0 + 2\Gamma_1 + 2\Gamma_2,$$

with the following configuration:

(2.20)
$$\begin{tikzcd}
E \arrow[r, dash] \arrow[dr, dash] & D \arrow[d, dash] & \Gamma_2 \arrow[dl, dash] \\
 & \Gamma_0 \arrow[r, dash] & \Gamma_1
\end{tikzcd}$$

$N > 0$. In this case $B_1.\Gamma_1 = -1$ and $B_1.\Gamma_i = 0$ for $i > 1$. Arguing as for $N = 0$ we get (with E in general an elliptic curve)

$$L \sim 3D + E + 3\Gamma_0 + 3\Gamma_1 + 2\Gamma_2 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2},$$

with the following configuration:

(2.21)
$$\begin{tikzcd}
D \arrow[r, dash] \arrow[dr, dash] & \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] \arrow[dl, dash] & \Gamma_2 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_N \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+1} \\
 & E & & & & \Gamma_{N+2} &
\end{tikzcd}$$

We will now show that there exists a 16-dimensional family of polarized K3 surfaces (S,L) with configuration (2.18) and scroll type $(3, 3, 2, 0, 0)$.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}E \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$ with intersection matrix

$$\begin{bmatrix} D^2 & D.E & D.\Gamma_1 & D.\Gamma_2 \\ D.E & E^2 & E.\Gamma_1 & E.\Gamma_2 \\ D.\Gamma_1 & E.\Gamma_1 & \Gamma_1^2 & \Gamma_1.\Gamma_2 \\ D.\Gamma_2 & E.\Gamma_2 & \Gamma_1.\Gamma_2 & \Gamma_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$$

has signature (1,3), so by proposition 1.1.24 there exists a K3 surface with Pic $S = \mathbb{Z}D \oplus \mathbb{Z}E \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$.

Using Picard-Lefschetz reflections we may assume that $L \sim 3D + E + 2\Gamma_1 + 2\Gamma_2$ is nef. L nef and Riemann-Roch used on D and E let us assume D and E effective. We will now show that L is base point free and of type $\{3, 0\}$. To show all this it is enough to show that there exists no divisor $B \sim xD + yE + z\Gamma_1 + w\Gamma_2$ such that

$$B^2 = 0 \quad B.L = 1, 2, 3, \text{ or } 4$$
$$B^2 = 2 \quad B.L = 7$$

We will now show that there exists no B such that $B^2 = 0$ and $B.L = 1$. The rest of the cases can be handled likewise. $1 = B.L = 5x + 7y$ and $0 = B^2 = -2(z^2 + w^2) + 2(xy + xz + xw + yz + yw)$ give $7(z^2 + w^2) = x - 5x^2 + (z + w)(2x + 1)$. We can without loss of generality assume $|z| \leq |w|$.

$|z| \leq |w| \leq |x|$ gives

$$0 \leq 7(z^2 + w^2) = x - 5x^2 + (z + w)(2x + 1) \leq x - 5x^2 + (2x^2 + 2x) = x + 2|x| - x^2$$

Hence $-1 \leq x \leq 3$. Only $x = 3$ gives $y \in \mathbb{Z}$. But then $7(z^2 + w^2) = 7(z + w) - 42$, which has no integral solutions.

$|z| \leq |x| \leq |w|$ gives $7(z^2+w^2) \leq 2x-3x^2+3w^2$. This gives $0 \leq 4w^2+7z^2 \leq 2x-3x^2$, so $x = 0$. But then $y \notin \mathbb{Z}$.

$|x| \leq |z| \leq |w|$ gives $7(z^2 + w^2) \leq x - 5x^2 + 3z^2 + 3w^2$. This gives $0 \leq 4(w^2 + z^2) \leq x - 5x^2$, so $x = 0$. But then $y \notin \mathbb{Z}$.

We will now show that the scroll type associated to (S, L) is $(3, 3, 2, 0, 0)$. Let $B \sim xD + yE + z\Gamma_1 + w\Gamma_2$ be a perfect Clifford divisor on (S, L). Then we have $B.L = 5$ and $B^2 = 0$. This gives $y = \frac{5}{7}(1 - x)$ and $7(z^2 + w^2) = 5(x + z + w) + 2x(z + w) - 5x^2$. Solving this in the same way that we handled $B^2 = 0$ and $B.L = 1$ above we get: $B \sim D$, $B \sim D + \Gamma_1$, $B \sim D + \Gamma_2$, or $B \sim D + \Gamma_1 + \Gamma_2$. Hence L is of type $\{3, 0\}$.

We will now show that we may assume $B \sim D$. If $B \sim D$ then there is nothing to show. If $B \sim D+\Gamma_1$, then we change the basis of $\operatorname{Pic} S$ as on p. 71. The case $B \sim D+\Gamma_2$ is symmetric.

If $B \sim D + \Gamma_1 + \Gamma_2$, then we change the basis of $\operatorname{Pic} S$ as follows

$$\begin{aligned} D &\mapsto D + \Gamma_1 + \Gamma_2 := D' \\ E &\mapsto E + \Gamma_1 + \Gamma_2 := E' \\ \Gamma_1 &\mapsto \quad -\Gamma_1 \quad := \Gamma_1' \\ \Gamma_2 &\mapsto \quad -\Gamma_2 \quad := \Gamma_2' \end{aligned}$$

We easily see that the lattice $\mathbb{Z}D' \oplus \mathbb{Z}E' \oplus \mathbb{Z}\Gamma_1' \oplus \mathbb{Z}\Gamma_2'$ has the same intersection matrix as $\mathbb{Z}D \oplus \mathbb{Z}E \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$, and that $L \sim 3D' + E' + 2\Gamma_1' + 2\Gamma_2'$. Hence we may assume $B \sim D$ in this case too.

We will now show that the associated scroll type is $(3, 3, 2, 0, 0)$. First of all note that D nef and Riemann-Roch give E, Γ_1, and Γ_2 effective. Hence $h^0(L - 3D) \geq h^0(E) \geq 2$. Since the scroll type $(4, 1, 1, 1, 1)$ is not associated to any polarized K3 surface table 2.2 gives that the associated scroll type must be $(3, 3, 1, 1, 0)$, $(4, 2, 1, 1, 0)$, $(3, 3, 2, 0, 0)$, $(4, 2, 2, 0, 0)$, or $(4, 3, 1, 0, 0)$. The associated scroll type is not $(3, 3, 1, 1, 0)$ or $(4, 2, 1, 1, 0)$ since $\operatorname{rank}\operatorname{Pic} S = 4$. We will show below that the scroll type $(4, 3, 1, 0, 0)$ is not associated to any perfect Clifford divisor. Hence the scroll type is $(4, 2, 2, 0, 0)$ or $(3, 3, 2, 0, 0)$. We will show below that the scroll type $(4, 2, 2, 0, 0)$ is possible with $\operatorname{rank}\operatorname{Pic} S \leq 4$ only with configuration (2.22). Taking determinants, proposition 1.1.22 gives that lattice we are considering does not satisfy configuration (2.22) (compare p. 73). Hence the scroll type must be $(3, 3, 2, 0, 0)$.

2.1.9 $(\mathbf{4}, \mathbf{2}, \mathbf{2}, \mathbf{0}, \mathbf{0})$

Let $A' \sim 2D + B_0$, where B_0 is effective and $h^0(B_0) = 1$. We have $h^1(R) = 2$ so we must be in case $\{3, 0\}^b$ or $\{3, 0\}^c$.

$\{\mathbf{3}, \mathbf{0}\}^{\mathbf{b}}$ Then $L \sim 4D + B_0 + \Gamma_1 + \Gamma_2$. $A'.\Gamma_i = 0$ gives $B_0.\Gamma_i = -2$. Hence we can write $B_0 \sim \Gamma_1 + \Gamma_2 + B_1$, where B_1 is effective with, $B_1.\Gamma_i = 0$, $B_1.D = 1$, $B_1.L = 2$ and $B_1^2 = -2$.

Assume that Γ_1 and Γ_2 have multiplicity zero in B_1. There must exist a smooth rational curve $\Gamma_3 \leq B_1$ such that $\Gamma_3.D = 1$. Write $B_1 \sim \Gamma_3 + B_2$, where B_2 is effective. Then $D.B_2 = \Gamma_1.B_2 = \Gamma_2.B_3 = 0$. We also have $L.\Gamma_3 = 2 + \Gamma_3.B_2$. Since D is nef and $D.B_2 = 0$, we see that Γ_3 has multiplicity zero in B_2. Hence $\Gamma_3.B_2 \geq 0$. Since L is nef and $L.B_1 = 2$ we must have $L.\Gamma_3 \leq 2$. This gives $\Gamma_3.B_2 = 0$. Since L is numerically 2-connected this gives $B_2 = 0$. Hence we have $L \sim 4D + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3$, with the following configuration:

(2.22)
$$\begin{tikzcd}[arrows=-] \Gamma_1 \arrow[r] & D \arrow[r] \arrow[d] & \Gamma_3 \\ & \Gamma_2 & \end{tikzcd}$$

We now assume that either Γ_1 or Γ_2 has non-zero multiplicity in B_1. Without loss of generality we may assume that Γ_1 has multiplicity at least one in B_1. Since D is nef and $D.B_1 = 1$ we get that Γ_1 has multiplicity one in B_1 and that Γ_2 has multiplicity zero in B_1. Write $B_1 \sim \Gamma_1 + B_2$, where B_2 is effective. We have $B_2.D = B_2.\Gamma_2 = 0$ and $L.B_2 = 2$. Since $L.\Gamma_1 = 0$ we get $\Gamma_1.B_2 = 2$. This gives $B_2^2 = -4$.

Assume that there exists two distinct smooth rational curves, Γ_3 and Γ_4, in the support of B_2 such that $\Gamma_1.\Gamma_3 = \Gamma_1.\Gamma_3 = 1$. Write $B_2 \sim \Gamma_3 + \Gamma_4 + B_3$. Then $\Gamma_i.L = 1 + \Gamma_3.\Gamma_4 + \Gamma_i.B_3 \geq 1$ ($i = 3$ or 4). Since L is nef and $B_2.L = 2$ we get $\Gamma_3.\Gamma_4 = \Gamma_3.B_3 = \Gamma_4.B_3 = 0$. We also have $D.B_3 = \Gamma_1.B_3 = \Gamma_2.B_3 = 0$. Hence $B_3 = 0$, since L is numerically 2-connected. This gives $L \sim 4D + 3\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4$, with the following configuration:[21]

(2.23)
$$\begin{tikzcd}[arrows=-] D \arrow[r] \arrow[d] & \Gamma_1 \arrow[r] \arrow[d] & \Gamma_3 \\ \Gamma_2 & \Gamma_4 & \end{tikzcd}$$

Assume that there only exists one smooth rational curve Γ_3 in the support of B_2 such that $\Gamma_1.\Gamma_3 = 1$. Then Γ_3 has multiplicity two in B_2. Write $B_2 \sim 2\Gamma_3 + B_3$. Then $D.B_3 = \Gamma_1.B_3 = \Gamma_2.B_3 = 0$. Furthermore $0 \leq L.\Gamma_3 = -1 + \Gamma_3.B_3 \leq 2$, $0 \leq L.B_3 = \Gamma_3.B_3 + B_3^2 \leq 2$, and $L.B_2 = 3\Gamma_3.B_3 + B_3^2 - 2 = 4$. We see that $\Gamma_3.B_3 = 0$ contradicts $L.\Gamma_3 \geq 0$. Hence $B_3 \neq 0$. Since $h^0(B_3) = 1$ we then get $B_3^2 \leq -2$. To satisfy the conditions given by $L.\Gamma_3$, $L.B_3$, and $L.B_2$ we must then have $\Gamma_3.B_3 = 2$ and $B_3 = -2$.

[21] We see that this is yet another example of a situation where (A, Δ) is not well-behaved. We have $\Delta = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ but $\Delta' = \Gamma_1 + \Gamma_2$.

But $\Gamma_3.B_3 = 2$ and $B_2^2 = -4$ give $-4 = B_2^2 = (2\Gamma_3 + B_3)^2 = -2$, a contradiction. Hence this case is impossible, and $\mathbf{\{3,0\}^b}$ is only possible with the configurations (2.22) and (2.23).

$\mathbf{\{3,0\}^c}$ We have $L \sim 4D + B_0 + 2\Gamma_0 + \ldots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$. $A'.\Gamma_0 = 0$ gives $B_0.\Gamma_0 = -2$. Hence we can write $B_0 = \Gamma_0 + b_1$, with b_1 effective. Then $\Gamma_1.L = 0$ gives $b_1.\Gamma_1 = -1$. Iterating we find that we can write $B_0 = \Gamma_0 + \cdots + \Gamma_N + b_N$, where b_N is effective and $\Gamma_{N+1}.b_N = \Gamma_{N+2}.b_N = -1$. Hence we can write $B_0 = \Gamma_0 + \cdots + \Gamma_{N+2} + b_{N+1}$, with b_{N+1} effective. Then $L.\Gamma_N = 0$ gives $\Gamma_N.b_{N+1} = -1$. Hence we can write $B_0 = \Gamma_0 + \cdots + \Gamma_{N-1} + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2} + b'_1$. Then $\Gamma_{N-1}.L = 0$ gives $\Gamma_{N-1}.b_1 = -1$. Iterating we find that we can write $B_0 = 2\Gamma_0 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2} + b'_N = \Delta' + b'_N$, where b'_N is effective. We easily see that $\Gamma_i.b'_N = 0$ (for $0 \leq i \leq N+2$), $D.b'_N = 1$, $L.b'_N = 2$ and $b'^2_N = -2$. Set $B_1 := b'_N$.

Let $\Gamma' \leq B_1$ be a smooth rational curve such that $\Gamma'.D = 1$. Write $B_1 = \Gamma' + B_2$, where B_2 is effective. Then $D.B_2 = 0$. We have $L.\Gamma' = 2 + 2\Delta'.\Gamma' + B_2.\Gamma' \geq 0$ and $L.B_2 = B_2^2 + 2\Delta'.B_2 + B_2.\Gamma' \geq 0$.

Suppose that $\Delta'.\Gamma' < 0$. Then by iterating as above we get $\Delta' < B_1$, a contradiction since $D.B_1 = 1$. If $\Delta'.B_2 < 0$ we get that $\Delta' < B_2$, also a contradiction. Since $\Delta'.B_1 = \Delta'.\Gamma' + \Delta'.B_2 = 0$, we thus have $\Delta'.\Gamma' = \Delta'.B_2 = 0$.

Since $B_2^2 \leq 0$, $B_2.\Gamma' \geq 0$, and $L.\Gamma' + L.B_2 = 2$ we get $B_2^2 = B_2.\Gamma' = 0$. Since L is numerically 2-connected this gives $B_2 = 0$.

Hence we have $L \sim 4D + 4\Gamma_0 + \cdots 4\Gamma_N + 2\Gamma_{N+1} + 2\Gamma_{N+2} + \Gamma' = 4D + 2\Delta' + \Gamma'$ with the following configuration:

(2.24)
$$\begin{tikzcd}
D \arrow[r, dash] \arrow[d, dash] & \Gamma_0 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_N \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+1} \\
\Gamma' & & & \Gamma_{N+2} &
\end{tikzcd}$$

We will now show that there exists a 16-dimensional family of polarized K3 surfaces (S, L) with configuration (2.22) and associated scroll type $(4, 2, 2, 0, 0)$.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma_3$ with intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma_1 & D.\Gamma_2 & D.\Gamma_3 \\ D.\Gamma_1 & \Gamma_1^2 & \Gamma_1.\Gamma_2 & \Gamma_1.\Gamma_3 \\ D.\Gamma_2 & \Gamma_1.\Gamma_2 & \Gamma_2^2 & \Gamma_2.\Gamma_3 \\ D.\Gamma_3 & \Gamma_1.\Gamma_3 & \Gamma_2.\Gamma_3 & \Gamma_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}$$

has signature (1,3), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma_3$.

Using Picard-Lefschetz reflections we may assume that $L \sim 4D + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3$ is nef. L nef and Riemann-Roch used on D and Γ_3 let us assume D and Γ_3 are effective.

Assume that $B \in \mathcal{A}^0(L)$, with B nef. Let $B \sim xD + y\Gamma_1 + z\Gamma_2 + w\Gamma_3$. Then by arguing as on page 66 we have that

$$0 \leq \Gamma_3.B = x - 2w \leq 1$$

Hence $w = \lfloor x/2 \rfloor$. From $B.L = 5x + 2w$ we then get $B.L = 6x$ (x even) or $B.L = 6x - 1$ (x odd). Hence $B.L$ is congruent 0 or -1 modulo 6. This gives $B.L = 5$, since $D.L - D^2 - 2 = 3$ and $c \leq 3$. Then we must have $x = 1$, $w = 0$, and $B^2 = 0$. $B^2 = 0$ is equivalent to $y(y-1) = z(z-1)$. Hence we find $B \sim D$, $B \sim D + \Gamma_1$, $B \sim D + \Gamma_2$, or $B \sim D + \Gamma_1 + \Gamma_2$. By changing the basis of $\operatorname{Pic} S$ if necessary, as on pages 71 and 80 we may assume that $B \sim D$.

D nef and Riemann-Roch used on Γ_1 and Γ_2 let us assume Γ_1 and Γ_2 effective. Hence $h^0(L - 4D) > 0$. Thus the associated scroll type is $(4,1,1,1,1)$, $(4,2,1,1,0)$, $(4,2,2,0,0)$, or $(4,3,1,0,0)$. We have already shown that $(4,1,1,1)$ is impossible and will show below that $(4,3,1,0,0)$ is impossible. Hence the associated scroll type is $(4,2,1,1,0)$ or $(4,2,2,0,0)$. Since $\operatorname{rank} \operatorname{Pic} S = 4$ the associated scroll type is $(4,2,2,0,0)$.

2.1.10 $(\mathbf{4}, \mathbf{3}, \mathbf{1}, \mathbf{0}, \mathbf{0})$

Reasoning as for the scroll type $(4,2,2,0,0)$, we see that we must be in one of the cases given by configurations (2.22)-(2.24). We will show that none of these configurations have $(4,3,1,0,0)$ as an associated scroll type. We will do this by showing that none of these configurations satisfy $h^0(L - 3D) = 3$, which we must have if the associated scroll type is $(4,3,1,0,0)$, since then $d_3 = 2$ and $d_4 = 1$.

Assume that we have $L \sim 4D + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3$, with configuration (2.22). Then $L - 3D \sim D + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3$, where $h^0(D) = 2$. Γ_3 is fixed in $L - 3D$, since $\Gamma_3.(L - 3D) = -1$. Hence $h^0(L - 3D) = h^0(L - 3D - \Gamma_3)$. Γ_2 is fixed in $L - 3D - \Gamma_3$, since $\Gamma_2.L - 3D - \Gamma_3 = -1$. Consequently $h^0(L - 3D - \Gamma_3) = h^0(L - 3D - \Gamma_3 - \Gamma_2)$. Continuing this way we get

$$\begin{aligned} h^0(L - 3D) &= h^0(L - 3D - \Gamma_3) \\ &= h^0(L - 3D - \Gamma_3 - \Gamma_2) \\ &= h^0(L - 3D - \Gamma_3 - 2\Gamma_2) \\ &= h^0(L - 3D - \Gamma_3 - 2\Gamma_2 - \Gamma_1) \\ &= h^0(L - 3D - \Gamma_3 - 2\Gamma_2 - 2\Gamma_1) \\ &= h^0(D) \\ &= 2. \end{aligned}$$

Hence the scroll type cannot be $(4,3,1,0,0)$.

Reasoning in the same way for the configurations (2.23) and (2.24) we see that $h^0(L - 3D) = 2$ in these cases too. Hence the scroll type $(4,3,1,0,0)$ is not associated to any perfect Clifford divisor D.

2.2 $c = 3, D^2 = 2$

We have $D.L = 7, d = 6$, and $f = h^0(L - D) = 7$. Since $L.(L - 4D) = -6$ and L is nef, we see that $h^0(L - 4D) = 0$. By proposition 1.3.12 we get $h^1(R) = 0$. Hence $\Delta = 0$ and by Riemann-Roch $h^0(L - 2D) = 3$. Since $d_2 \geq d_3$, this gives $h^0(L - 3D) = 0$ or

1. $L.(3D-L) = -1 < 0$ and L nef gives $h^0(3D-L) = 0$. Then Riemann-Roch gives $h^0(L-3D) = 1$. So the only possible scroll type is the one given in table 2.3.

Table 2.3: Possible scroll types associated to L of type $\{3, 2\}$.

d_0	d_1	d_2	d_3	scroll type
6	4	2	1	$(3, 2, 1, 1, 0, 0)$

2.2.1 $(\mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0})$

In this case $h^0(R) = 0$, so $\Delta = 0$. Write $L \sim 3D + B_0$, where B_0 is a sum of smooth rational curves and satisfies $h^0(B_0) = 1$.

Easy computations give $D.B_0 = 1$, $B_0^2 = -2$, and $L.B_0 = 1$. Hence there exists a smooth rational curve Γ such that $D.\Gamma > 0$. If $D.\Gamma > 1$ then there has to exist a smooth rational curve such that $D.\Gamma' < 0$, a contradiction since D is nef. Hence $D.\Gamma = 1$. Since $\Gamma \notin \mathcal{R}_{L,D}$, we have $L.\Gamma > 0$. By the same argument as for $D.\Gamma$ we get $L.\Gamma = 1$.

Write $B_0 \sim \Gamma + B_1$, with B_1 effective. Then $B_1.D = 0$. Since $L.\Gamma = 1$ we get $B_1.\Gamma = 0$. Then $B_1 = 0$ since L is numerically 2-connected. Hence we have

$$L \sim 3D + \Gamma,$$

with the following configuration

$$D \text{ ——— } \Gamma \tag{2.25}$$

We will now show that there exists an 18-dimensional family of polarized K3 surfaces (S,L) with this configuration and scroll type $(3, 2, 1, 1, 0, 0)$.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma$ with intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma \\ D.L & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

has signature (1,1), so by proposition 1.1.24 there exists an 18-dimensional family of K3 surfaces with $\operatorname{Pic} S = \mathbb{Z}L \oplus \mathbb{Z}D$.

Using Picard-Lefschetz reflections we may assume that $L \sim 3D + \Gamma$, is nef. We will now show that L is base point free and of type $\{3, 2\}$. To show all this it is enough to show that there exists no divisor $B \sim xD + y\Gamma$ such that

$$B^2 = 0 \quad B.L = 1, 2, 3, 4, \text{ or } 5$$

since $D.(L-D) = 5$. $B^2 = 0$ gives $0 = 2x^2 + xy - y^2$, which gives $B = 0$, by lemma 2.0.1.[22] Thus L is of type $\{3, 2\}$, and therefore the associated scroll type must be $(3, 2, 1, 1, 0, 0)$.

[22] Alternatively use lemma 1.4.9.

2.3 $c = 4, D^2 = 0$

We have $D.L = 6, d = 6$, and $f = h^0(L - D) = 7$. Since $L.(L - 4D) = -2$ and L is nef, we see that $h^0(L - 4D) = 0$. Proposition 1.3.12 gives $h^1(R) \leq 3$ and by Riemann-Roch $h^0(L - 2D) = h^0(R) = 1 + h^1(R)$. This gives the possible scroll types listed in table 2.4.

Table 2.4: Possible scroll types associated to L of type $\{4, 0\}$.

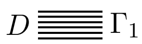
d_0	d_1	d_2	d_3	scroll type
6	6	1	0	$(2,1,1,1,1,1)$
6	5	2	0	$(2,2,1,1,1,0)$
6	5	1	1	$(3,1,1,1,1,0)$
6	4	3	0	$(2,2,2,1,0,0)$
6	4	2	1	$(3,2,1,1,0,0)$
6	3	3	1	$(3,2,2,0,0,0)$
6	3	2	2	$(3,3,1,0,0,0)$

2.3.1 $(\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$

We have $h^1(R) = 0$, so $\Delta = 0$. Furthermore $h^0(A) = 1, D.A = 6, L.A = 10$, and $A^2 = -2$.[23] Let Γ_1 be a smooth rational curve in the support of A, such that $D.\Gamma_1 > 0$. Write $A \sim \Gamma_1 + A_1$.

If $D.\Gamma_1 = 6$, then $A_1 = 0$, since L is numerically 2-connected. Hence we have $L \sim 2D + \Gamma_1$, with the following configuration:

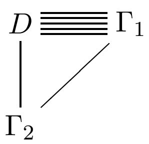
(2.26)



This is the only possible configuration with $\operatorname{rank} \operatorname{Pic} S = 2$. We will now give the possible configurations which give $\operatorname{rank} \operatorname{Pic} S = 3$.

If $D.\Gamma_1 = 5$ and $\operatorname{rank} \operatorname{Pic} S = 3$, then there exists a smooth rational curve Γ_2 such that $D.\Gamma_2 = 1$ and $\Gamma_2 \sim A_1$. Since $A^2 = -2$ we must have $\Gamma_1.\Gamma_2 = 1$. This gives $L \sim 2D + \Gamma_1 + \Gamma_2$, with the following configuration

(2.27)

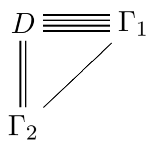


If $D.\Gamma_1 = 4$ and $\operatorname{rank} \operatorname{Pic} S = 3$, then there exists a smooth rational curve Γ_2 such that either $D.\Gamma_2 = 2$ and $\Gamma_2 \sim A_1$ or $D.\Gamma_2 = 1$ and $2\Gamma_2 \sim A_1$. Since $A^2 = -2$ we

[23] Remember that since $h^1(R) = 0$ we can write $A' = A$.

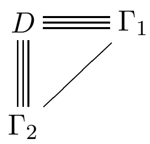
must have $\Gamma_2 \sim A_1$ and $\Gamma_1.\Gamma_2 = 1$. This gives $L \sim 2D + \Gamma_1 + \Gamma_2$, with the following configuration

(2.28)



If $D.\Gamma_1 = 3$ and $\operatorname{rank}\operatorname{Pic} S = 3$, then there exists a smooth rational curve Γ_2 such that $D.\Gamma_2 = 3$, $\Gamma_2 \sim A_1$, and $\Gamma_1.\Gamma_2 = 1$ since $A^2 = -2$. This gives $L \sim 2D + \Gamma_1 + \Gamma_2$, with the following configuration

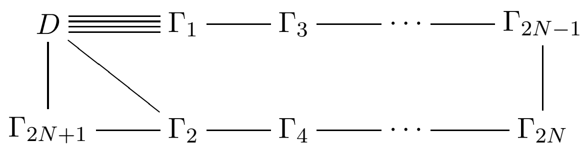
(2.29)



If $D.\Gamma_1 \leq 2$ and $\operatorname{rank}\operatorname{Pic} S = 3$, then there exists a smooth rational curve Γ_2 such that $D.\Gamma_2 \geq 4$, and we are in one of the cases already considered.

There also exists configurations which only are possible with $\operatorname{rank}\operatorname{Pic} S > 3$. As an example take $L \sim 2D + \Gamma_1 + \cdots + \Gamma_{2N+1}$, with the following configuration

(2.30)



We will now show that there exists an 18-dimensional family of polarized K3 surfaces (S, L) of the type given by configuration (2.26), that has a perfect Clifford divisor associated to the scroll type $(2, 1, 1, 1, 1, 1)$.

The lattice $\mathbb{Z}L \oplus \mathbb{Z}D$ with intersection matrix

$$\begin{bmatrix} L^2 & L.D \\ D.L & D^2 \end{bmatrix} = \begin{bmatrix} 22 & 6 \\ 6 & 0 \end{bmatrix}$$

has signature (1,1), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}L \oplus \mathbb{Z}D$.

Using Picard-Lefschetz reflections we may assume that L is nef. We will now show that L is base point free and is of type $\{4, 0\}$. It is enough to show that there exists no divisor B such that

$$\begin{aligned} B^2 &= 0 \quad B.L = 1, 2, 3, 4, \text{ or } 5 \\ B^2 &= 2 \quad B.L = 7 \text{ or } 8 \\ B^2 &= 4 \quad B.L = 10 \end{aligned}$$

This is immediate from lemma 1.4.9.[24]

[24] Alternatively use lemma 2.0.1.

We will see below that if the scroll type associated to a perfect Clifford divisor D' of type $\{4, 0\}$ is not $(2, 1, 1, 1, 1, 1)$, then $\operatorname{rank} \operatorname{Pic} S > 2$. Hence the scroll type must be $(2, 1, 1, 1, 1, 1)$.

2.3.2 $(\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{0})$

We must be in the situation $\{4, 0\}^a$. Hence $\Delta' = \Gamma$, $\mathcal{R}_{L,D} = \{\Gamma\}$, $A'^2 = 0$, $D.A' = 5$, and $L \sim 2D + A' + \Gamma$. In general A' is an elliptic curve E. We then have $L \sim 2D + E + \Gamma$, with the following configuration:

(2.31)
$$\begin{array}{ccc} D & \equiv\!\equiv & E \\ | & & \\ \Gamma & & \end{array}$$

We will now show that there exists a 17-dimensional family of polarized K3 surfaces (S, L), with this scroll type.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}E \oplus \mathbb{Z}\Gamma$ with intersection matrix

$$\begin{bmatrix} D^2 & D.E & D.\Gamma \\ D.E & E^2 & E.\Gamma \\ D.\Gamma & E.\Gamma & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 1 \\ 5 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

has signature (1,2), so by proposition 1.1.24 there exists a K3 surface with Pic $S = \mathbb{Z}D \oplus \mathbb{Z}E \oplus \mathbb{Z}\Gamma$.

Using Picard-Lefschetz reflections we may assume that $L \sim 2D + E + \Gamma$ is nef. We will now show that $B \in \mathcal{A}^0(L)$, with B nef, implies $B \sim D$ or $V \sim D + \Gamma$. This will in particular show that L is base point free and of type $\{4, 0\}$. Let $B \in \mathcal{A}^0(L)$, with $B \sim xD + yE + z\Gamma$. Since D satisfies $D.L - D^2 - 2 = 4$ we must have $c \leq 4$. Hence B must satisfy one of the following

$$\begin{array}{ll} B^2 = 0 & B.L = 1, 2, 3, 4, 5, \text{ or } 6 \\ B^2 = 2 & B.L = 7 \text{ or } 8 \\ B^2 = 4 & B.L = 10 \end{array}$$

We have $B.L = 6x + 10y \equiv 0 \pmod 2$. If $B.L = 2$ and $B^2 = 0$, then we get $y = (1 - 3x)/5$ and $z^2 - 2xz + (3x^2 - x) = 0$. Considering this last equation as a polynomial of degree two in the variable z, we see that it has a real solution if and only if the discriminant $4x^2 - 12x^2 + 4x$ is larger than or equal to zero. This gives $x = 0$, but then $y = 1/5$, a contradiction. The other cases are treated similarly. The only solutions are $B \sim D$ and $B \sim D + \Gamma$. By changing the basis of Pic S if necessary (as on p. 71) we may assume that D is a perfect Clifford divisor.

We will now show that $\mathcal{R}_{L,D} \subseteq \{\Gamma\}$. Let $B \sim xD + yE + z\Gamma$ be a divisor in $\mathcal{R}_{L,D}$. Then $B.L = 0$ gives $x = -\frac{5}{3}y$ and $B.D = 1$ gives $z = 1 - 5y$. Substituting this into $B^2 = -2$ we get $y = 0$ or $y = 2/5$. $y = 1$ gives $B \sim \Gamma$ and $y = 2/5$ is impossible, hence $\mathcal{R}_{L,D} \subseteq \{\Gamma\}$.

We will now show that $h^1(R) = 1$. We first show $h^1(R) \leq 1$. If $h^1(R) > 1$, then $(h^1(R) - 1)\Gamma$ must be fixed in $D + E$, i.e. $h^0(D + E) = h^0(D + E - (h^1(R) - 1)\Gamma)$. We assume $h^1(R) > 1$. Then

$$h^1(D + E - (h^1(R) - 1)\Gamma) \neq 0$$

by Riemann-Roch. But we also have $A \sim D + E - (h^1(R) - 1)\Gamma$ and may assume $h^1(A) = 0$,[25] a contradiction. Hence $h^1(R) \leq 1$. L nef and Riemann-Roch give E effective. Hence $h^0(L - 2D) \geq h^0(E) \geq 2$. Table 2.4 then gives $h^1(R) = 1$.

We will show below that the scroll type $(3, 1, 1, 1, 1, 0)$ is only associated to K3 surfaces with $\operatorname{rank}\operatorname{Pic} S > 3$. Hence the scroll type associated to D must be $(2, 2, 1, 1, 1, 0)$.

2.3.3 $(\mathbf{3, 1, 1, 1, 1, 0})$

We must be in case $\{4, 0\}^a$. Hence $\Delta' = \Gamma$, $\mathcal{R}_{L,D} = \{\Gamma\}$, $A'^2 = 0$, and $D.A' = 5$. Hence we have $L \sim 2D + A' + \Gamma$. Since $h^0(L - 3D) = 1$, we can write $A' \sim D + B_0$, where $h^0(B_0) = 1$, $D.B_0 = 5$, $L.B_1 = 4$, and $B_0^2 = -10$. $\Gamma.A' = 0$ gives $\Gamma.B_0 = -1$, so we can write $B_0 \sim \Gamma + B_1$, where $h^0(B_1) = 1$, $D.B_1 = L.B_1 = 4$, $\Gamma.B_1 = 1$, and $B_1^2 = -10$. There must exist a smooth rational curve Γ_1 in the support of B_1 such that $\Gamma.\Gamma_1 = 1$.

We will now give all[26] possible configurations such that $\operatorname{rank}\operatorname{Pic} S \leq 4$. Since D, Γ_1, and Γ are linearly independent we get $\operatorname{rank}\operatorname{Pic} S \geq 3$. We can show that B_1 must contain another smooth rational curve Γ_2. Then one can show that D, Γ, Γ_1, and Γ_2 are linearly independent. If $\operatorname{rank}\operatorname{Pic} S = 4$ we can write

$$B_1 \sim n_1\Gamma_1 + n_2\Gamma_2 + n_3\Gamma + n_4 D,$$

where $n_i \in \mathbb{Z}$. Arguing as on p. 74 we get only one possibility with $\operatorname{rank}\operatorname{Pic} S = 4$: $L \sim 3D + \Gamma_1 + 2\Gamma_2 + 2\Gamma$, with the following configuration:[27]

(2.32)
$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash, shift left] \arrow[d, dash, shift right] & \Gamma \arrow[d, dash] \\ \Gamma_2 & \Gamma_1 \end{tikzcd}$$

We will now show that there exists a 16-dimensional family of polarized K3 surfaces (S,L) with configuration (2.32) and associated scroll type $(3, 1, 1, 1, 1, 0)$.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma$ with intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma_1 & D.\Gamma_2 & D.\Gamma \\ D.\Gamma_1 & \Gamma_1^2 & \Gamma_1.\Gamma_2 & \Gamma_1.\Gamma \\ D.\Gamma_2 & \Gamma_1.\Gamma_2 & \Gamma_2^2 & \Gamma_2.\Gamma \\ D.\Gamma & \Gamma_1.\Gamma & \Gamma_2.\Gamma & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & -2 & 0 & 1 \\ 2 & 1 & -2 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$$

[25] By [JK01, section 6] there exists a perfect Clifford divisor with $h^1(A) = 0$. We have seen that the only possible perfect Clifford divisors are D and $D + \Gamma$. We may change the basis of $\operatorname{Pic} S$, if necessary, such that D is the perfect Clifford divisor with $h^1(A) = 0$.

[26] Actually there exists only one.

[27] This is yet another example of a situation where (A, Δ) is not well-behaved. We have $\Delta = \Gamma + \Gamma_1$ but $\Delta' = \Gamma$.

has signature (1,3), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma$.

Using Picard-Lefschetz reflections we may assume that $L \sim 3D + \Gamma_1 + 2\Gamma_2 + 2\Gamma$ is nef. L nef and Riemann-Roch used on D and Γ_2 let us assume D and Γ_2 effective.

We will now assume that $B \sim xD + y\Gamma_1 + z\Gamma_2 + w\Gamma$ is a perfect Clifford divisor. Since D satisfies $D.L - D^2 - 2 = 4$ we must have $c \leq 4$. Hence B must satisfy one of the following

$$\begin{array}{ll} B^2 = 0 & B.L = 1, 2, 3, 4, 5, \text{ or } 6 \\ B^2 = 2 & B.L = 7 \text{ or } 8 \\ B^2 = 4 & B.L = 10 \end{array}$$

Arguing as on page 66 we get:[28]

$$\begin{aligned} -1 \leq \Gamma.B &= x + y - 2w \leq 1 \\ -1 \leq \Gamma_1.B &= -2y + w \leq 1 \\ 0 \leq \Gamma_2.B &= 2x + y - 2z \leq 2 \end{aligned}$$

We also have $1 \leq B.L = 6x + 2z \leq 10$. Since x and z are integers this gives $2 \leq B.L = 6x + 2z \leq 10$. Solving for x we have

$$x = \frac{1}{25}\Gamma.B + \frac{3}{25}\Gamma_2.B + \frac{3}{25}B.L + \frac{2}{25}\Gamma_1.B.$$

The inequalities then give

$$\frac{3}{25} \leq x \leq \frac{39}{25},$$

so $x = 1$. Similarly we get $0 \leq y \leq 1$, $0 \leq z \leq 1$, and $-1 \leq w \leq 1$. Checking all the possibilities gives $B \sim D$, $B \sim D + \Gamma$, or $B \sim D + \Gamma + \Gamma_1$. Changing the basis, if necessary, we may assume that $B \sim D$.

D nef and Riemann-Roch used on Γ give Γ effective. After another change of basis, if necessary, we may assume that Γ_1 is effective also. Thus $h^0(L-3D) = h^0(\Gamma_1+2\Gamma_2+2\Gamma) \geq 1$. We will show below that the other possible scroll types with $h^0(L-3D) \geq 1$ have $\operatorname{rank} \operatorname{Pic} S > 4$. Hence the associated scroll type is $(3, 1, 1, 1, 1, 0)$.

2.3.4 $(\mathbf{2, 2, 2, 1, 0, 0})$

L must be of type $\{4, 0\}^b$ or $\{4, 0\}^c$. We will only consider type $\{4, 0\}^b$, since this is the most general case. We have

$$L \sim 2D + A' + \Gamma_1 + \Gamma_2,$$

[28] Here we have to consider $c = 4$, which we did not have to do on page 66. Thus we no longer automatically have $h^0(L - B - D) \geq 2$. (For example if $B^2 = 4, B.L = 10$, and $B.D \leq 2$ then $(L - B - D)^2 < 0$.) For $H = \Gamma, \Gamma_1$, and Γ_2 our argument still holds.

where A' satisfies $h^0(A') = 3$ and $h^1(A') = 0$. In general A' will be an irreducible curve of genus 2. We have the following configuration:

$$\begin{tikzcd}[arrows=dash] D \arrow[r, equal] \arrow[d] \arrow[dr] & A' \\ \Gamma_1 & \Gamma_2 \end{tikzcd} \tag{2.33}$$

We will show that there exists a 16-dimensional family of polarized K3-surfaces (S, L), with this scroll type.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}A' \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$ with intersection matrix

$$\begin{bmatrix} D^2 & D.A' & D.\Gamma_1 & D.\Gamma_2 \\ D.A' & A'^2 & A'.\Gamma_1 & A'.\Gamma_2 \\ A'.\Gamma_1 & A'.\Gamma_1 & \Gamma_1^2 & \Gamma_1.\Gamma_2 \\ D.\Gamma_2 & A'.\Gamma_2 & \Gamma_1.\Gamma_2 & \Gamma_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 1 & 1 \\ 4 & 2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}$$

has signature (1,3), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}A' \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$.

Using Picard-Lefschetz reflections we may assume that $L \sim 2D + A' + \Gamma_1 + \Gamma_2$ is nef. L is nef and Riemann-Roch used on D and A' let us assume D and A' effective.

We will now assume that $B \sim xD + y\Gamma_1 + z\Gamma_2 + wA'$ is a perfect Clifford divisor. Since D satisfies $D.L - D^2 - 2 = 4$ we must have $c \leq 4$. Hence B must satisfy one of the following

$$\begin{aligned} B^2 &= 0 \qquad B.L = 1, 2, 3, 4, 5, \text{ or } 6 \\ B^2 &= 2 \qquad B.L = 7 \text{ or } 8 \\ B^2 &= 4 \qquad B.L = 10 \end{aligned}$$

Arguing as we have done several times already we find that we may assume $B \sim D$ to be a perfect Clifford divisor (possibly after a change of basis). D nef and Riemann-Roch gives Γ_1 and Γ_2 effective. Hence $h^0(L - 2D) \geq h^0(A') \geq 3$. Table 2.4 give $h^1(R) \geq 2$. We will see below that the other possible scroll types with $h^1(R) \geq 2$ only arise when $\operatorname{rank} \operatorname{Pic} S \geq 5$. Thus the associated scroll type is $(2, 2, 2, 1, 0, 0)$.

2.3.5 $(\mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0})$

We must be in case $\{4, 0\}^b$ or $\{4, 0\}^c$. Once again we will only consider case the $\{4, 0\}^b$, since this is the most general case. We have $h^0(L - 3D) = 1$, so we can write $A' \sim D + B_0$, where $h^0(B_0) = 1$, $D.B_0 = 4$, and $B_0^2 = -6$. From $L.\Gamma_1 = 0$ we get $\Gamma_i.B_0 = -1$ ($i = 1$ or 2). Hence we can write $B_0 \sim \Gamma_1 + \Gamma_2 + B_1$, where $D.B_1 = 2$, $\Gamma_1.B_1 = \Gamma_2.B_1 = 1$, and $B_1^2 = -6$. Continuing this line of reasoning, as we have done in multiple cases above, we find that the most general case has $L \sim 3D + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4$, with the following configuration

(2.34)
$$\begin{tikzcd}
\Gamma_4 \arrow[r, dash] & D \arrow[r, dash] \arrow[dl, dash] \arrow[d, dash] & \Gamma_1 \arrow[d, dash] \\
\Gamma_5 & \Gamma_2 \arrow[r, dash] & \Gamma_3
\end{tikzcd}$$

We will now show that this gives a 14-dimensional family of polarized K3-surfaces (S, L), associated to the scroll type $(3, 2, 1, 1, 0, 0)$.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \oplus\mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma_3 \oplus \mathbb{Z}\Gamma_4 \oplus \mathbb{Z}\Gamma_5$ with intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma_1 & D.\Gamma_2 & D.\Gamma_3 & D.\Gamma_4 & D.\Gamma_5 \\ D.\Gamma_1 & \Gamma_1^2 & \Gamma_1.\Gamma_2 & \Gamma_1.\Gamma_3 & \Gamma_1.\Gamma_4 & \Gamma_1.\Gamma_5 \\ D.\Gamma_2 & \Gamma_1.\Gamma_2 & \Gamma_2^2 & \Gamma_2.\Gamma_3 & \Gamma_2.\Gamma_4 & \Gamma_2.\Gamma_5 \\ D.\Gamma_3 & \Gamma_1.\Gamma_3 & \Gamma_2.\Gamma_3 & \Gamma_3^2 & \Gamma_3.\Gamma_4 & \Gamma_3.\Gamma_5 \\ D.\Gamma_4 & \Gamma_1.\Gamma_4 & \Gamma_2.\Gamma_4 & \Gamma_3.\Gamma_4 & \Gamma_4^2 & \Gamma_4.\Gamma_5 \\ D.\Gamma_5 & \Gamma_1.\Gamma_5 & \Gamma_2.\Gamma_5 & \Gamma_3.\Gamma_5 & \Gamma_4.\Gamma_5 & \Gamma_5^2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

has signature (1,5), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \oplus\mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma_3 \oplus \mathbb{Z}\Gamma_4 \oplus \mathbb{Z}\Gamma_5$.

Using Picard-Lefschetz reflections we may assume that $L \sim 3D + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4$ is nef. L nef and Riemann-Roch used on D, Γ_3, Γ_4, and Γ_5 let us assume that D, Γ_3, Γ_4, and Γ_5 are effective.

We will now assume that $B \sim xD + y\Gamma_1 + z\Gamma_2 + u\Gamma_3 + v\Gamma_4 + w\Gamma_5$ is a perfect Clifford divisor. Since D satisfies $D.L - D^2 - 2 = 4$ we must have $c \leq 4$. Hence B must satisfy one of the following

$$\begin{aligned} B^2 &= 0 \qquad B.L = 1, 2, 3, 4, 5, \text{ or } 6 \\ B^2 &= 2 \qquad B.L = 7 \text{ or } 8 \\ B^2 &= 4 \qquad B.L = 10 \end{aligned}$$

Arguing as on page 66 we get:[29]

$$\begin{aligned} 2 \leq B.L &= 6x + 2u + v + w \leq 10 \\ -1 \leq \Gamma_1.B &= x + u - 2y \leq 1 \\ -1 \leq \Gamma_2.B &= x + u - 2z \leq 1 \\ 0 \leq \Gamma_3.B &= y + z - 2u \leq 2 \\ 0 \leq \Gamma_4.B &= x - 2v \leq 1 \\ 0 \leq \Gamma_5.B &= x - 2w \leq 1 \end{aligned}$$

[29] Remember footnote 28.

The two last inequalities give $v = w = \lfloor x/2 \rfloor$. Reasoning as on page 89 we get $B \sim D$, $B \sim D + \Gamma_1$, $B \sim D + \Gamma_2$, or $B \sim D + \Gamma_1 + \Gamma_2$. Hence we must be in case $\{4, 0\}$. If $B \nsim D$ then after a change of basis of $\operatorname{Pic} S$, as on page 71, we may assume that $B \sim D$.

D nef and Riemann-Roch used on Γ_1 and Γ_2 let us assume that Γ_1 and Γ_2 are effective. We can now show that $\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2\}$. The scroll type must then be $(3, 2, 1, 1, 0, 0)$, since $h^0(L - 3D) = h^0(2\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4) > 0$.

2.3.6 $(\mathbf{3}, \mathbf{3}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0})$

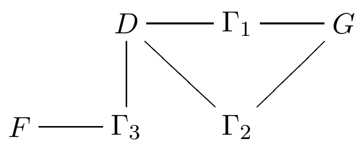
We will show that this scroll type is not associated to any polarized K3 surface. This will be done by showing that if L has a perfect Clifford divisor D associated to this scroll type, then L will violate the Hodge index theorem.

We have $h^1(R) = 3$ and $h^0(L - 3D) = 2$. Write $A' \sim D + B_0$, with B_0 effective. Then $B_0^2 = -2$. By arguing as for the scroll type $(4, 3, 1, 0, 0)$ we see that $2 = h^0(L - 3D) = h^0(B_0)$ in each of the cases $\{4, 0\}^d$, $\{4, 0\}^e$, $\{4, 0\}^f$, $\{4, 0\}^g$, and $\{4, 0\}^h$.

$\{\mathbf{4}, \mathbf{0}\}^{\mathbf{d}}$ We have $\Delta' = \Gamma_1 + \Gamma_2 + \Gamma_3$. $L.\Gamma_i = 0$ gives $B_0.\Gamma_i = -1$. Hence we can write $B_0 \sim \Gamma_1 + \Gamma_2 + \Gamma_3 + B_1$, where we show as on p. 83 that $h^0(B_1) = 2$. Furthermore $B_1^2 = -2$. Hence $h^1(B_1) \neq 0$. By Ramanujam's lemma we may write $B_1 \sim F + G$ (with F and G effective) where $F.G = 0$. Since $F^2 \leq 0$,[30] $G^2 \leq 0$ and $B_1^2 = F^2 + G^2 = -2$ we may assume that $F^2 = -2$ and $G^2 = 0$.

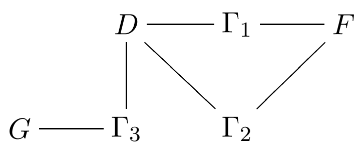
Since L is numerically 2-connected, $B_1.D = 0$, and $B_1.\Gamma_i = 1$ $(i = 1, 2, 3)$ we have two possibilities (up to symmetry):

(A) $L \sim 3D + F + G + 2\Gamma_1 + 2\Gamma_2 + 2\Gamma_3$, with the following configuration:



Write $B \sim 2D - 3G$. Then $B.L = B^2 = 0$ and $B \neq 0$, which contradicts the Hodge index theorem.

(B) $L \sim 3D + F + G + 2\Gamma_1 + 2\Gamma_2 + 2\Gamma_3$, with the following configuration:



Write $B \sim D - 3G$. Then $B.L = B^2 = 0$ and $B \neq 0$, which contradicts the Hodge index theorem.

$\{\mathbf{4}, \mathbf{0}\}^{\mathbf{e}}$ We have $\Delta' = \Gamma_{-1} + 2\Gamma_0 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$. $L.\Gamma_i = 0$ gives $B_0.\Gamma_i = -1$ for $i = -1$ and $i = 0$. Hence we can write $B_0 \sim \Gamma_{-1} + \Gamma_0 + B_1'$. $L.\Gamma_1 = 0$ gives $B_1'.\Gamma_1 = -1$. Hence $\Gamma_1 \leq B_1'$. Iterating we get $\Gamma_1 + \cdots + \Gamma_{N+2} \leq B_1'$. Hence we can write $B_0 \sim \Gamma_{-1} + \cdots + \Gamma_{N+2} + B_1$, with B_1 effective. $L.\Gamma_N = 0$ gives $B_1.\Gamma_N = -1$.

[30] Since $h^0(F) \leq 2$.

Hence $\Gamma_N \leq B_1$. Iterating we get $\Gamma_1 + \cdots + \Gamma_N \leq B_1$. Hence we can write $B_0 \sim \Gamma_{-1} + 2\Gamma_1 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2} + B_2 = \Delta' + B_2$, with B_2 effective. Then $L.\Gamma_i = 0$ gives $B_2.\Gamma_{-1} = B_2.\Gamma_0 = 1$ and $B_2.\Gamma_i = 0$ $(0 < i \leq N+2)$. We also find that $h^0(B_2) = 2$ and $B_2^2 = -2$. Hence $h^1(B_1) \neq 0$. As above we can write, by Ramanujam's lemma, $B_2 \sim F + G$, with $F.G = 0$, $F^2 = -2$, and $G^2 = 0$. As above this gives two possibilities.

(C) $L \sim 3D + F + G + 2\Delta'$, with the following configuration:

$$\begin{tikzcd}
\Gamma_{-1} \arrow[r, dash] \arrow[d, dash] & D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & \Gamma_N \arrow[r, dash] \arrow[dr, dash] & \Gamma_{N+1} \\
G & & F & & & \Gamma_{N+2}
\end{tikzcd}$$

Write $B \sim 2D - 3G$. Then $B.L = B^2 = 0$ and $B \neq 0$, which contradicts the Hodge index theorem.

(D) $L \sim 3D + F + G + 2\Delta'$, with the following configuration:

$$\begin{tikzcd}
\Gamma_{-1} \arrow[r, dash] \arrow[d, dash] & D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & \Gamma_N \arrow[r, dash] \arrow[dr, dash] & \Gamma_{N+1} \\
F & & G & & & \Gamma_{N+2}
\end{tikzcd}$$

Write $B \sim D - 3G$. Then $B.L = B^2 = 0$ and $B \neq 0$, which contradicts the Hodge index theorem.

$\{\mathbf{4,0}\}^{\mathbf{f}}$ We have $\Delta' = 3\Gamma_0 + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4$. $L.\Gamma_0 = 0$ gives $B_0.\Gamma_0 = -1$. Hence we have $\Gamma_0 \leq B_0$ and $B_0 \sim \Gamma_0 + B_0'$, with B_0' effective. Then $\Gamma_1.B_0' = \Gamma_2.B_0' = -1$. Hence we may write $B_0 \sim \Gamma_0 + \Gamma_1 + \Gamma_2 + B_0''$. Then $\Gamma_3.B_0'' = \Gamma_4.B_0'' = -1$. Hence we may write $B_0 \sim \Gamma_0 + \cdots + \Gamma_4 + B_1$.

The intersection numbers between the Γ_i and B_1 are equal to the intersection numbers between the Γ_i and B_0. Hence we may iterate and write $B_1 \sim \Gamma_0 + \cdots + \Gamma_4 + B_2$, with B_2 effective. Iterating this procedure n times gives $B_0 \sim n\Gamma_0 + \cdots + n\Gamma_4 + B_n$, with B_n effective, for all n. A contradiction.

$\{\mathbf{4,0}\}^{\mathbf{g}}$ and $\{\mathbf{4,0}\}^{\mathbf{h}}$ These cases are treated in the same way as $\{\mathbf{4,0}\}^{\mathbf{f}}$.

2.3.7 $(\mathbf{3,2,2,0,0,0})$

We have $h^1(R) = 3$ and $h^0(L - 3D) = 1$. Write $A' \sim D + B_0$, with B_0 effective. We must be in one of the cases $\{4,0\}^d$, $\{4,0\}^e$, $\{4,0\}^f$, $\{4,0\}^g$, or $\{4,0\}^h$. We will only consider the case $\{4,0\}^d$ since this gives the most general family of K3 surfaces associated to this scroll type.

$\{\mathbf{4,0}\}^{\mathbf{d}}$ We have $\Delta' = \Gamma_1 + \Gamma_2 + \Gamma_3$. $L.\Gamma_i = 0$ gives $B_0.\Gamma_i = -1$. Hence we can write $B_0 \sim \Gamma_1 + \Gamma_2 + \Gamma_3 + B_1$, where B_1 is effective. Then $B_1^2 = -2$, $B_1.D = 0$, $B_1.L = 4$, and $B_1.\Gamma_i = 1$ $(i = 1,2,3)$. Note that $B_1.D = 0$ gives that Γ_i $(i = 1,2,3)$ has multiplicity 0 in B_1.

There must exist a smooth rational curve Γ_4 in the support of B_1 such that $\Gamma_1.\Gamma_4 = 1$. Write $B_1 \sim \Gamma_4 + B_2$. Then Γ_4 has multiplicity 0 in B_2, for if it had non-zero multiplicity then Γ_1 would also have non-zero multiplicity, a contradiction. Hence $B_2.\Gamma_4 \geq 0$.

If $\Gamma_2.\Gamma_4 = \Gamma_3.\Gamma_4 = 1$, then $L.\Gamma_4 = 4 + B_2.\Gamma_4 \leq 4$ gives $B_2.\Gamma_4 = 0$. Then $B_2 = 0$, since L is numerically 2-connected. This situation gives the most general family of K3 surfaces.

We have $L \sim 3D + 2\Gamma_1 + 2\Gamma_2 + 2\Gamma_3 + \Gamma_4$, with the following configuration

(2.35)
$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] \arrow[rd, dash] & \Gamma_1 \arrow[rdd, dash] & \\ \Gamma_2 \arrow[rrd, dash] & \Gamma_3 \arrow[rd, dash] & \\ & & \Gamma_4 \end{tikzcd}$$

We will now show that this gives rise to a 15-dimensional family of K3-surfaces associated to the scroll type $(3, 2, 2, 0, 0, 0)$. The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma_3 \oplus \mathbb{Z}\Gamma_4$ with intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma_1 & D.\Gamma_2 & D.\Gamma_3 & D.\Gamma_4 \\ D.\Gamma_1 & \Gamma_1^2 & \Gamma_1.\Gamma_2 & \Gamma_1.\Gamma_3 & \Gamma_1.\Gamma_4 \\ D.\Gamma_2 & \Gamma_1.\Gamma_2 & \Gamma_2^2 & \Gamma_2.\Gamma_3 & \Gamma_3.\Gamma_4 \\ D.\Gamma_3 & \Gamma_1.\Gamma_3 & \Gamma_2.\Gamma_3 & \Gamma_3^2 & \Gamma_3.\Gamma_4 \\ D.\Gamma_4 & \Gamma_1.\Gamma_4 & \Gamma_2.\Gamma_4 & \Gamma_3.\Gamma_4 & \Gamma_4^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & -2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 1 & 1 & -2 \end{bmatrix}$$

has signature (1,4), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma_3 \oplus \mathbb{Z}\Gamma_4$.

Using Picard-Lefschetz reflections we may assume that $L \sim 3D+2\Gamma_1+2\Gamma_2+2\Gamma_3+\Gamma_4$ is nef. L nef and Riemann-Roch used on D and Γ_4 let us assume that D and Γ_4 is effective.

Assume that $B \in \mathcal{A}^0(L)$, with B nef. Write $B \sim xD + y\Gamma_1 + z\Gamma_2 + v\Gamma_3 + w\Gamma_4$. Then B must satisfy one of the following

$$\begin{array}{ll} B^2 = 0 & B.L = 1, 2, 3, 4, 5, \text{ or } 6 \\ B^2 = 2 & B.L = 7 \text{ or } 8 \\ B^2 = 4 & B.L = 10 \end{array}$$

Arguing as we have done many times already we get $B \sim D$, $B \sim D+\Gamma_1$, $B \sim D+\Gamma_2$, $B \sim D+\Gamma_3$, $B \sim D+\Gamma_1+\Gamma_2$, $B \sim D+\Gamma_1+\Gamma_3$, $B \sim D+\Gamma_2+\Gamma_3$, or $B \sim D+\Gamma_1+\Gamma_2+\Gamma_3$.

If $B \nsim D + \Gamma_1 + \Gamma_2 + \Gamma_3$, then after a change of basis of $\operatorname{Pic} S$ (if necessary), as on pages 71 and 80, we may assume that $B \sim D$.

If $B \sim D + \Gamma_1 + \Gamma_2 + \Gamma_3$ then we change the basis of $\operatorname{Pic} S$ as follows:

$$\begin{array}{rcl} D \mapsto & D + \Gamma_1 + \Gamma_2 + \Gamma_3 & := D' \\ \Gamma_1 \mapsto & -\Gamma_1 & := \Gamma_1' \\ \Gamma_2 \mapsto & -\Gamma_2 & := \Gamma_2' \\ \Gamma_3 \mapsto & -\Gamma_3 & := \Gamma_3' \\ \Gamma_4 \mapsto & \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 & := \Gamma_4' \end{array}$$

We easily see that the lattice $\mathbb{Z}D' \oplus \mathbb{Z}\Gamma'_1 \oplus \mathbb{Z}\Gamma'_2 \oplus \mathbb{Z}\Gamma'_3 \oplus \mathbb{Z}\Gamma'_4$ has the same intersection matrix as $\mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma'_3 \oplus \mathbb{Z}\Gamma_4$ and that $L \sim 3D' + 2\Gamma'_1 + 2\Gamma'_2 + 2\Gamma'_3 + \Gamma'_4$ Hence we may assume in this case too that $B \sim D$.

We see that $h^1(R) = 3$.[31] Since we have shown that no polarized K3 surface is associated to the scroll type $(3, 3, 1, 0, 0, 0)$ the associated scroll type is $(3, 2, 2, 0, 0, 0)$.

2.4 $c = 4, D^2 = 2$

We have $D.L = 8, d = 7$, and $f = h^0(L - D) = 6$. Since $L.(L - 3D) = -2$ and L is nef, we see that $h^0(L - 3D) = 0$. By proposition 1.3.12 we see that $h^1(R) \leq 1$ Riemann-Roch gives $h^0(L - 2D) = h^0(R) = h^1(R)$. This gives the two possible scroll types of table 2.5.

Table 2.5: Possible scroll types associated to L of type $\{4, 2\}$.

d_0	d_1	d_2	scroll type
7	6	0	$(1, 1, 1, 1, 1, 1, 0)$
7	5	1	$(2, 1, 1, 1, 1, 0, 0)$

2.4.1 $(\mathbf{1, 1, 1, 1, 1, 1, 0})$

In this case $h^0(R) = 0$, so $\Delta = 0$ and F is base point free. Since $F^2 = 8$ we can by proposition 1.1.11 assume that F is an irreducible curve of genus 5. We have $L \sim D + F$ with the following configuration:

(2.36) $$D \equiv F$$

We will now show that there exists an 18-dimensional family of polarized K3 surfaces (S, L) with a perfect Clifford divisor D associated to this scroll type.

The lattice $\mathbb{Z}L \oplus \mathbb{Z}D$ with intersection matrix

$$\begin{bmatrix} L^2 & L.D \\ D.L & D^2 \end{bmatrix} = \begin{bmatrix} 22 & 8 \\ 8 & 2 \end{bmatrix}$$

has signature (1,1), so by proposition 1.1.24 there exists a K3 surface with Pic $S = \mathbb{Z}L \oplus \mathbb{Z}D$.

Using Picard-Lefschetz reflections we may assume that L is nef. We will now show that L is base point free and is of type $\{4, 2\}$. To show all this it is enough to show that

[31] D nef and Riemann-Roch give Γ_1, Γ_2, and Γ_3 effective. thus $h^0(L - 3D) > 0$. Table 2.4 gives $h^0(L - 3D) = 1$ (since we have shown that no polarized K3 surface is associated to the scroll type $(3, 3, 1, 0, 0, 0)$). Especially $h^0(\Gamma_1) = 1$. Since $\Gamma_1.D = 1$ there must exist a smooth rational curve $\Gamma \leq \Gamma_1$ such that $\Gamma.D_1 = 1$. Since L is nef and $L.\Gamma_1 = 0$ we have $L.\Gamma = 0$ and $\Gamma \in \mathcal{R}_{L,D}$. Similar reasoning holds for Γ_2 and Γ_3. Thus $h^1(R) \geq 3$. Since $c = 4$ we get $h^1(R) = 3$.

there exists no divisor $D \sim xL + yD$ such that

$$\begin{array}{ll} B^2 = 0 & B.L = 1,2,3,4,5, \text{ or } 6 \\ B^2 = 2 & B.L = 7 \\ B^2 = 4 & B.L = 10 \end{array}$$

We have $B.L = 22x + 8y \equiv 0 \pmod 2$, so $B^2 = 0$ or $c = 4$. If $0 = B^2 = 22x^2 + 16xy + 2y^2$, then lemma 2.0.1 gives $x = y = 0$, i.e. $B = 0$, a contradiction. We also easily see that $B^2 = 4$ and $B.L = 10$ have no simultaneous integral solution. Hence L is of type $\{4, 2\}$.

We will show below the scroll type $(2,1,1,1,1,0,0)$ does not occur. Hence the scroll type must be $(1,1,1,1,1,1,0)$.

2.4.2 $(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0})$

In this case $h^0(R) = 1$, so $\mathcal{R}_{L,D} = \Gamma$. If we write $\Delta \sim \Gamma + \Delta_1$, and set $A' \sim A + \Delta_1$ and $\Delta = \Gamma$, then we have

$$L \sim 2D + A' + \Gamma,$$

Easy computations give $A'^2 = 0$ and $D.A' = 3$. $1 \leq h^0(A') \leq h^0(A'+\Gamma) = h^0(L-2D) = 1$ gives $h^0(A') = 1$. Then $A'^2 = 0$ gives $A' = 0$. This contradicts $D.A' = 3$, so this scroll type is not associated to any perfect Clifford divisor of type $\{4, 2\}$.

2.5 $c = 4, D^2 = 4$

We have $D.L = 10, d = 8$, and $f = h^0(L - D) = 5$. Since $L.(L - 3D) = -8$ and L is nef, we see that $h^0(L - 3D) = 0$. By proposition 1.3.12 we get $\Delta = 0$. Riemann-Roch gives $h^0(L - 2D) = 1$. Thus we only get the possible scroll type in table 2.6.

Table 2.6: Possible scroll types associated to L of type $\{4, 4\}$.

d_0	d_1	d_2	scroll type
8	4	1	$(2,1,1,1,0,0,0,0)$

2.5.1 $(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$

We can write $L \sim 2D + A$, where $D.A = 2$, $A^2 = -2$, $L.A = 2$, and $h^0(A) = 1$, since $\Delta = 0$. Hence we can write A as a non-zero sum of smooth rational curves (lemma 1.1.14).

We have two cases to consider:

(a) There exists a smooth rational curve, Γ, with multiplicity 1 in A, such that $D.\Gamma = 2$ and $\Gamma \leq A$.

(b) There exist two smooth rational curves, Γ_1 and Γ_2, with multiplicity 1 in A, such that $D.\Gamma_i = 1$ and $\Gamma_i \leq A$.

In case (A) we can write $A \sim \Gamma + A_1$, with A_1 effective. Then $D.A_1 = 0$. Since $L.A = 2$ and L is nef we get $L.\Gamma = 2 + A_1.\Gamma \leq 2$, i.e. $A_1.\Gamma \leq 0$. But $A_1.\Gamma \geq 0$ since Γ has multiplicity zero in A. Hence $A_1.\Gamma = 0$. Thus $A_1 = 0$, since L is numerically 2-connected. The configuration is:

(2.37)
$$\begin{tikzcd} D \arrow[r, equal] & \Gamma \end{tikzcd}$$

In case (B) we can write $A \sim \Gamma_1 + \Gamma_2 + A_1$, with A_1 effective. Then $D.A_1 = 0$. Since $\Gamma_1, \Gamma_2 \notin \mathcal{R}_{L,D}$, we have $L.\Gamma_1 > 0$. $L.A = 2$ then gives $L.\Gamma_1 = L.\Gamma_2 = 1$ and $L.A_1 = 0$. This gives $\Gamma_1.(\Gamma_2 + A_1) = \Gamma_2.(\Gamma_1 + A_1) = 1$ so we have two cases to consider: $\Gamma_1.\Gamma_2 = 1$ and $\Gamma_1.\Gamma_2 = 0$.

In the first case we have $A_1.\Gamma_i = 0$. We then get $A_1 = 0$, since L is numerically 2-connected. This case has the following configuration:

(2.38)
$$\begin{tikzcd} D \arrow[r, no head] \arrow[d, no head] & \Gamma_1 \arrow[dl, no head] \\ \Gamma_2 & \end{tikzcd}$$

In the second case we have $A_1.\Gamma_i = 1$ and $A_1^2 = -2$. Let $\Gamma_3 \leq A_1$ be a smooth rational curve such that $\Gamma_1.\Gamma_2 = 1$. Write $A_1 \sim \Gamma_3 + A_2$. Then $\Gamma_1.A_2 = 0$ and $\Gamma_2.A_2 = 1 - \Gamma_2.\Gamma_3$.

If $\Gamma_2.\Gamma_3 = 1$, then $\Gamma_2.A_2 = 0$ and $\Gamma_3.A_2 = 0$ (since $\Gamma_3.L = 0$). So $A_2 = 0$, since L is numerically 2-connected. Hence $L \sim 2D + \Gamma_1 + \Gamma_2 + \Gamma_3$, with the following configuration

(2.39)
$$\begin{tikzcd} D \arrow[r, no head] \arrow[d, no head] & \Gamma_1 \arrow[d, no head] \\ \Gamma_2 \arrow[r, no head] & \Gamma_3 \end{tikzcd}$$

If $\Gamma_2.\Gamma_3 = 0$, then there exists a smooth rational curve $\Gamma_4 \neq \Gamma_3$ such that $\Gamma_2.\Gamma_4 = 1$ and $\Gamma_4 \leq A_2$. Write $A_1 \sim \Gamma_3 + \Gamma_4 + A_3$. Then $A_1.L = 0$ gives $\Gamma_3.L = \Gamma_4.L = 0$, which gives $\Gamma_3.(\Gamma_4 + A_3) = \Gamma_4.(\Gamma_3 + A_3) = 1$. This is the same situation as above, so we can iterate. There are then two possible types of configurations:

(I) $L \sim 2D + \Gamma_1 + \Gamma_2 + \cdots \Gamma_{2N+2}$, where $0 \leq N \leq 9$, and the following configuration

(2.40)
$$\begin{tikzcd} D \arrow[r, no head] \arrow[dr, no head] & \Gamma_1 \arrow[r, no head] & \cdots \arrow[r, no head] & \Gamma_{2N+1} \arrow[d, no head] \\ & \Gamma_2 \arrow[r, no head] & \cdots \arrow[r, no head] & \Gamma_{2N+2} \end{tikzcd}$$

Note that $N = 0$ gives configuration (2.38).

(II) $L \sim 2D + \Gamma_1 + \Gamma_2 + \cdots \Gamma_{2N+3}$, where $0 \leq N \leq 8$, and the following configuration

$$\begin{tikzcd} D \arrow[r, dash] \arrow[dr, dash] & \Gamma_1 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_{2N+1} \arrow[r, dash] & \Gamma_{2N+3} \\ & \Gamma_2 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_{2N+2} \arrow[ur, dash] & \end{tikzcd} \tag{2.41}$$

Note that $N = 0$ gives configuration (2.39).

We will show that there exists an 18-dimensional family of polarized K3 surfaces (S, L) with a perfect Clifford divisor D associated to this scroll type and configuration as in (2.37).

The lattice $\mathbb{Z}L \oplus \mathbb{Z}D$ with intersection matrix

$$\begin{bmatrix} L^2 & L.D \\ D.L & D^2 \end{bmatrix} = \begin{bmatrix} 22 & 10 \\ 10 & 4 \end{bmatrix}$$

has signature (1,1), so by proposition 1.1.24 there exists a K3 surface with Pic $S = \mathbb{Z}L \oplus \mathbb{Z}D$.

Using Picard-Lefschetz reflections we may assume that L is nef. We will now show that L is base point free and of type $\{4, 4\}$. To show this it is enough to show that there exists no divisor $B \sim xL + yD$ such that

$$\begin{array}{ll} B^2 = 0 & B.L = 1, 2, 3, 4, 5, \text{ or } 6 \\ B^2 = 2 & B.L = 7 \text{ or } 8 \end{array}$$

$B.L$ is even, since $B.L = 22x + 10y$. If $B^2 = 0$ then $0 = 22x^2 + 20xy + 4y^2$, but this gives $B = 0$, by lemma 2.0.1. $B^2 = 2$ and $B.L = 8$ have no simultaneous integral solution. Hence L is of type $\{4, 4\}$ and the associated scroll type must be $(2, 1, 1, 1, 0, 0, 0, 0)$.

2.6 $c = 1, D^2 = 0$

We have $D.L = 3, d = 3$, and $f = h^0(L - D) = 10$. Since $L.(L - 8D) = -2$ and L is nef, we see that $h^0(L - 8D) = 0$. The Hodge index theorem gives $h^1(R) = 0$ (see [JK01, p. 77]) By Riemann-Roch $h^0(L - 2D) = h^0(R) = 4$. This gives the possible scroll types of table 2.7.

We will not go into quite as much detail as we have done for $c = 3$ and $c = 4$. Much of the information we give can be found in [JK01, pp.58–59] and [Ste00, pp.8–10]. Furthermore one can translate results from $g = 9$ to $g = 12$, so [JK01, pp.90–93] is also a good reference.

2.6.1 $(4, 3, 3)$

In this case $h^0(L - 4D) = 1$. We can write $L \sim 4D + B$, where $h^0(B) = 1$, $D.B = 3$, $B.L = 10$, and $B^2 = -2$.

Table 2.7: Possible scroll types associated to L of type $\{3, 0\}$.

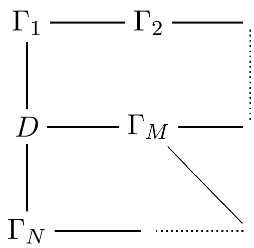
d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	scroll type
3	3	3	3	1	0	0	0	$(4, 3, 3)$
3	3	3	2	2	0	0	0	$(4, 4, 2)$
3	3	3	2	1	1	0	0	$(5, 3, 2)$
3	3	3	1	1	1	1	0	$(6, 2, 2)$
3	3	2	2	2	1	0	0	$(5, 4, 1)$
3	3	2	2	1	1	1	0	$(6, 3, 1)$

In the most general case we have $B \sim \Gamma$, where Γ is a smooth rational curve. This gives $L \sim 4D + \Gamma$, with the following configuration:

(2.42) $$D \equiv \Gamma$$

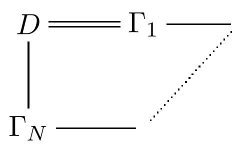
There also exists several less general situations. Such as $L \sim 4D + \Gamma_1 + \Gamma_2 + \cdots + \Gamma_N$, with the following configuration:

(2.43)



or $L \sim 4D + \Gamma_1 + \Gamma_2 + \cdots + \Gamma_N$, with the following configuration:

(2.44)



We will now show that configuration (2.42) gives a 18-dimensional family of polarized K3 surfaces (S, L), with this scroll type.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma$ with intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma \\ D.\Gamma & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & -2 \end{bmatrix}$$

has signature (1,1), so by proposition 1.1.24 there exists a K3 surface with Pic $S = \mathbb{Z}D \oplus \mathbb{Z}\Gamma$.

Using Picard-Lefschetz reflections we may assume that $L \sim 4D + \Gamma$ is nef. Assume that $B \sim xD + y\Gamma$ is in $\mathcal{A}^0(L)$. Then B must satisfy

$$B^2 = 0 \qquad B.L = 1, 2, \text{ or } 3$$

It is immediate from lemma 1.4.9[32] that $B \sim D$. Hence L is of type $\{1,0\}$.

We will show below that the associated scroll type is not $(4,3,3)$ only if $\operatorname{rank}\operatorname{Pic} S > 2$. Hence the scroll type associated to (S,L) must be $(4,3,3)$.

2.6.2 $(\mathbf{4},\mathbf{4},\mathbf{2})$

We can write $L \sim 4D + B$, where $h^0(B) = 2$. Furthermore $B^2 = -2$, $D.B = 3$, and $B.L = 10$.

We can write $B \sim F + G$, where F and G are effective and non-zero with $F.G = 0$, since $h^1(B) \neq 0$ (Ramanujam's lemma). Since $h^0(B) = 2$ we must have $F^2 \leq 0$ and $G^2 \leq 0$. We may assume that $F^2 = 0$ and $G^2 = -2$, since $B^2 = -2$. Since L is numerically 2-connected this gives two possibilities:

(2.45)
$$\begin{tikzcd} D \arrow[r, equal, no head] \arrow[d, no head] & F \\ G & \end{tikzcd}$$

and

(2.46)
$$\begin{tikzcd} D \arrow[r, no head] \arrow[d, equal, no head] & F \\ G & \end{tikzcd}$$

In general we have $G \sim \Gamma$, where Γ is a smooth rational curve, and $F \sim E$, where E is an elliptic curve.

We will now show that configuration (2.45) gives a 17-dimensional family of polarized K3 surfaces (S,L), with this scroll type.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}E \oplus \mathbb{Z}\Gamma$ with intersection matrix

$$\begin{bmatrix} D^2 & D.E & D.\Gamma \\ D.E & E^2 & E.\Gamma \\ D.\Gamma & E.\Gamma & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

has signature (1,2), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}E \oplus \mathbb{Z}\Gamma$.

Using Picard-Lefschetz reflections we may assume that $L \sim 4D+E+\Gamma$ is nef. Assume that $B \sim xD + yE + z\Gamma$ is in $\mathcal{A}^0(L)$. Then B must satisfy

$$B^2 = 0 \qquad B.L = 1, 2, \text{ or } 3$$

Substituting $B.L = 3x + 4y + 6z$ into $B^2 = 0$ gives $15z^2 + (14y - 2B.L)z + (4y^2 - B.Ly) = 0$. This is a quadratic equation in the variable z with discriminant $225 - 223y^3 + 88B.Ly^2 - 8(B.L)^2y$. Since we are only interested in integral solutions this discriminant

[32] Alternatively use lemma 2.0.1.

must be non-negative. This gives $y = 0$ or $y = 1$. $y = 0$ gives $z(15z + 2B.L) = 0$, i.e. $z = 0$. Then we have $B.L = 3x$. Since $1 \leq B.L \leq 3$ this gives $x = 1$ and $B \sim D$. $y = 1$ gives no solutions. Consequently D is a perfect Clifford divisor.

D nef and Riemann-Roch used on E and Γ let us assume E and Γ effective. Then $h^0(L - 4D) \geq h^0(E) \geq 2$, so the scroll type cannot be $(4, 3, 3)$. We will show below that the associated scroll type can be $(5, 3, 2)$ only if $\operatorname{rank} \operatorname{Pic} S > 3$. We will also show below that the scroll types $(6, 2, 2)$ and $(5, 4, 1)$ are not associated to any perfect Clifford divisor. Hence the associated scroll type to D must be either $(4, 4, 2)$ or $(6, 3, 1)$.

If the associated scroll type is $(6, 3, 1)$, then we must be in a situation equivalent to configuration (2.49). There must then exist a divisor $\Gamma_2 \sim xD + yE + z\Gamma$ such that $\Gamma_2^2 = -2$, $\Gamma_2.L = 1$, and $\Gamma_2.D = 0$. Solving for z this gives $3z^2 = 3$, i.e. $z \pm 1$. Then $\Gamma_2.D = 0 = y + 2z$ gives $y = \mp 2$. So $\Gamma_2.L = 1 = 3x + 4y + 6z = 3x \mp 2$. Thus $x \notin \mathbb{Z}$, a contradiction. The associated scroll type to D must be $(4, 4, 2)$.

2.6.3 $(\mathbf{5}, \mathbf{3}, \mathbf{2})$

We can write $L \sim 5D + B$, where $h^0(B) = 1$. Furthermore $B^2 = -8$, $D.B = 3$, and $B.L = 7$. We see that B satisfies exactly the same conditions as $L - 4D$ in [JK01, (i) p.92].[33] We have the following possibilities:

$L \sim 5D + 2\Gamma_1 + \Gamma_2 + \Gamma_3$, with the following configuration:

(2.47)
$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_2 \\ \Gamma_3 & & \end{tikzcd}$$

or $L \sim 5D + 3\Gamma_1 + \cdots + 3\Gamma_N + 2\Gamma_{N+1} + \Gamma_{N+2} + \Gamma_{N+3}$ $(1 \leq N \leq 16)$, with the following configuration:

(2.48)
$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_1 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_N \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+1} \arrow[r, dash] & \Gamma_{N+2} \\ & & & \Gamma_{N+3} & & \end{tikzcd}$$

Note that we may view configuration (2.47) as configuration (2.48) with $N = 0$.

We will now show that configuration (2.47) gives a 16-dimensional family of polarized K3 surfaces (S, L), with this scroll type.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma_3$ with intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma_1 & D.\Gamma_2 & D.\Gamma_3 \\ D.\Gamma_1 & \Gamma_1^2 & \Gamma_1.\Gamma_2 & \Gamma_1.\Gamma_3 \\ D.\Gamma_2 & \Gamma_1.\Gamma_2 & \Gamma_2^2 & \Gamma_2.\Gamma_3 \\ D.\Gamma_3 & \Gamma_1.\Gamma_3 & \Gamma_2.\Gamma_3 & \Gamma_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}$$

[33] This is natural since the associated scroll type there is $(4, 2, 1) = (5 - 1, 3 - 1, 2 - 1)$.

has signature (1,3), so by proposition 1.1.24 there exists a K3 surface with Pic $S = \mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma_3$.

Using Picard-Lefschetz reflections we may assume that $L \sim 3D+2\Gamma_1+\Gamma_2+\Gamma_3$ is nef. L nef and Riemann-Roch used on D, Γ_1, and Γ_3 let us assume D, Γ_1, and Γ_3 effective.

We will now show that $B \in \mathcal{A}^0(L)$, with B nef, implies $B \sim D$. This will in particular show that L is base point free and of type $\{1,0\}$. Let $B \sim xD + y\Gamma_1 + z\Gamma_2 + w\Gamma_3$. Then B must satisfy

$$B^2 = 0 \qquad B.L = 1, 2, \text{ or } 3$$

Furthermore by arguing as on page 66 we get:

$$\begin{aligned} 0 \leq \Gamma_1.B &= x - 2y + z \leq 2 \\ -1 \leq \Gamma_2.B &= y - 2z \leq 1 \\ 0 \leq \Gamma_3.B &= x - 2w \leq 2 \\ 0 \leq D.B &= y + w \leq 2 \end{aligned}$$

Arguing as on page 71 we get $0 \leq x \leq 4$, $0 \leq y \leq 2$, $0 \leq z \leq 1$, and $0 \leq w \leq 1$. Checking all values, using $1 \leq B.L \leq 3$ and $B^2 = 0$, gives $B \sim D$. Hence we may assume that D is a perfect Clifford divisor.

We will now show that we may assume Γ_2 effective. Since $\Gamma_2^2 = -2$ either Γ_2 or $-\Gamma_2$ is effective. If Γ_2 is effective, then we are done. If $-\Gamma_2$ is effective, then we change the basis of Pic S as follows:

$$\begin{array}{rcl} D \mapsto & D & := D' \\ \Gamma_1 \mapsto & \Gamma_1 + \Gamma_2 & := \Gamma_1' \\ \Gamma_2 \mapsto & -\Gamma_2 & := \Gamma_2' \\ \Gamma_3 \mapsto & \Gamma_3 & := \Gamma_3' \end{array}$$

We see that $L \sim 5D' + 2\Gamma_1' + \Gamma_2' + \Gamma_3'$ and that the new intersection numbers are equal to the old ones. Hence we may assume that Γ_2 is effective.

This gives $h^0(L - 5D) = h^0(2\Gamma_1 + \Gamma_2 + \Gamma_3) \geq 1$. We will show below that the scroll types $(6,2,2)$ and $(5,4,1)$ are not associated to any perfect Clifford divisor. Hence the scroll type is either $(5,3,2)$ or $(6,3,1)$.

If the associated scroll type is $(6,3,1)$, then we must be in a situation equivalent to configuration (2.49). Then there must exist a divisor $\Gamma_1' \sim y\Gamma_1 + z\Gamma_2 + w\Gamma_3$ such that $\Gamma_1'^2 = -2$, $\Gamma_1'.L = 1$, and $\Gamma_1'.D = 1$. Solving for w in $\Gamma_1'.L = 1 = 3x + 2y + 3w$ and $\Gamma_1'.D = 1 = y + z$, and substituting into $\Gamma_1'^2 = -2$ give

$$z^2 + (w-1)z + (\frac{1}{3} - \frac{5}{3}w + 2w^2) = 0$$

This is a quadratic equation in the variable z with discriminant $-7w^2 + 14w/3 - 1/3$. Since we are only interested in integral solutions this discriminant must be non-negative. But $-7w^2 + 14w/3 - 1/3$ is negative for all integers, so there exists no integral solutions. Thus the scroll type cannot be $(6,3,1)$. Hence the associated scroll type to D must be $(5,3,2)$.

2.6.4 $(\mathbf{5}, \mathbf{4}, \mathbf{1})$

This scroll type is not associated to any perfect Clifford divisor.

Reasoning as for the scroll type $(5, 3, 2)$, we see that we must be in one of the cases given by configurations (2.47)-(2.48).

Let $B \sim L - 5D$. By arguing as for the scroll type $(4, 3, 1, 0, 0)$ we get $h^0(D+B) = 2$, a contradiction, since $d_3 = 2$, $d_4 = 1$, and $d_5 = 0$ gives $h^0(L - 4D) = 3$.

See [JK01, p.59] for an alternative proof.

2.6.5 $(\mathbf{6}, \mathbf{3}, \mathbf{1})$

We can write $L \sim 6D + B$, where $h^0(B) = 1$, $D.B = 3$, $B^2 = -14$, and $B.L = 4$.

Note that $h^0(L-5D) = h^0(D+B) = 2 = h^0(D)$. Hence we see that for every smooth rational curve, Γ' in B such that $D.\Gamma' \neq 0$ we must have $D.\Gamma' = 1$, by proposition 1.1.17.

We have three cases to consider:

(A) There exists three distinct smooth rational curves, Γ_1, Γ_2, and Γ_3, with multiplicity one in B such that $\Gamma_i.D = 1$.

In this case we have $\Gamma_i.L \geq 2$. Hence $B.L \geq 6$, a contradiction.

(B) There exists two distinct smooth rational curves, Γ_1 and Γ_2, with multiplicity one and two respectively in B such that $\Gamma_i.D = 1$.

Then $\Gamma_1.L \geq 2$ and $\Gamma_2.L \geq 3$. Hence $B.L \geq 5$, a contradiction.

(C) There exists a smooth rational curve, Γ_1, with multiplicity three in B such that $\Gamma_1.D = 1$.

Write $B = 3\Gamma_1 + B_1$.

If $L.\Gamma_1 = 0$, then we get $B_1^2 = -12$ and $B_1.D = B_1.\Gamma_1 = 0$. A contradiction since L is numerically 2-connected.

If $L.\Gamma_1 = 1$, then there exists a smooth rational curve Γ_2 with multiplicity one in B_1 such that $\Gamma_1.\Gamma_2 = 1$. Write $B \sim 3\Gamma_1 + \Gamma_2 + B_2$. Then we have $L.\Gamma_2 \geq 1$ and $3\Gamma_1.L + \Gamma_2.L = 3 + \Gamma_2.L \leq B.L \leq 4$. Hence $L.\Gamma_2 = 1$ and $\Gamma_2.B_2 = 0$. We also have $D.B_2 = \Gamma_1.B_2 = 0$. Hence $B_2 = 0$, since L is numerically 2-connected.

If $L.\Gamma_1 \geq 2$, then $L.B \geq 3L.\Gamma_1 \geq 6$. A contradiction.

Thus the only possibility is $L \sim 6D + 3\Gamma_1 + \Gamma_2$, with the following configuration:

$$D \text{ —— } \Gamma_1 \text{ —— } \Gamma_2 \tag{2.49}$$

We will now show that this configuration gives a 17-dimensional family of polarized K3 surfaces (S, L) associated to the scroll type $(6, 3, 1)$.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$ with intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma_1 & D.\Gamma_2 \\ D.\Gamma_1 & \Gamma_1^2 & \Gamma_1.\Gamma_2 \\ D.\Gamma_2 & \Gamma_1.\Gamma_2 & \Gamma_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

has signature (1,2), so by proposition 1.1.24 there exists a K3 surface with Pic $S = \mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$.

Using Picard-Lefschetz reflections we may assume that $L \sim 6D + 3\Gamma_1 + \Gamma_2$ is nef. L nef and Riemann-Roch used on D, Γ_1, and Γ_2 let us assume D, Γ_1, and Γ_2 effective.

Assume that $B \sim xD + y\Gamma_1 + z\Gamma_2$ is in $\mathcal{A}^0(L)$. Then B must satisfy

$$B^2 = 0 \qquad B.L = 1, 2, \text{ or } 3$$

Furthermore by arguing as on page 66 we get:

$$\begin{aligned} 0 \le \Gamma_1.B &= x - 2y + z \le 1 \\ 0 \le \Gamma_2.B &= y - 2z \le 1 \\ 0 \le D.B &= y \le 2 \end{aligned}$$

The second inequality gives $z = \lfloor y/2 \rfloor$. Using $0 \le y \le 2$ and checking all three possible cases gives $B \sim D$.

Since $h^0(L - 6D) = h^0(3\Gamma_1 + \Gamma_2) \ge 1$ the scroll type must be $(6, 3, 1)$.

2.6.6 $(\mathbf{6}, \mathbf{2}, \mathbf{2})$

This scroll type is not associated to any perfect Clifford divisor.

By arguing as above for the scroll type $(6, 3, 1)$, we get that $L \sim 6D + 3\Gamma_1 + \Gamma_2$, with the following configuration:

$$D \,\text{——}\, \Gamma_1 \,\text{——}\, \Gamma_2$$

To show that D is not associated to the scroll type $(6, 2, 2)$ it is enough to show that $h^0(L - 3D) = h^0(3D + 2\Gamma_1 + \Gamma_2) \ge 5$. But this follows from Riemann-Roch.

See [JK01, p.59] for an alternative proof.

2.7 $c = 2, D^2 = 0$

We have $D.L = 4, d = 4$, and $f = h^0(L - D) = 9$. Since $L.(L - 6D) = -2$ and L is nef, we get $h^0(L - 6D) = 0$. Proposition 1.3.12 gives $h^1(R) \le 1$. By Riemann-Roch $h^0(L - 2D) = h^0(R) = 5 + h^1(R)$. This gives the possible scroll types of table 2.8.

Much of the information we give can be found in [JK01, pp.63–67] and [Ste00, pp.8–10]. Furthermore we can translate results for $g = 8$ to $g = 12$, so [JK01, p.88-90] is also a good reference.

2.7.1 $(\mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{2})$

We have $\Delta = 0$ and can write $L \sim 3D + B$, where $h^0(B) = 1$, $D.B = 4$, $L.B = 10$, and $B^2 = -2$.

There exists several possible configurations. The most general one is when B equals Γ, a smooth rational curve. Then we have $L \sim 3D + \Gamma$, with the following configuration:

$$D \equiv\!\equiv \Gamma \tag{2.50}$$

Table 2.8: Possible scroll types associated to L of type $\{2, 0\}$.

d_0	d_1	d_2	d_3	d_4	d_5	scroll type
4	4	4	1	0	0	$(3, 2, 2, 2)$
4	4	3	2	0	0	$(3, 3, 2, 1)$
4	4	3	1	1	0	$(4, 2, 2, 1)$
4	4	2	2	1	0	$(4, 3, 1, 1)$
4	4	2	1	1	1	$(5, 2, 1, 1)$
4	3	3	3	0	0	$(3, 3, 3, 0)$
4	3	3	2	1	0	$(4, 3, 2, 0)$
4	3	3	1	1	1	$(5, 2, 2, 0)$
4	3	2	2	2	0	$(4, 4, 1, 0)$
4	3	2	2	1	1	$(5, 3, 1, 0)$

As an example of a less general situation we take $B \sim \Gamma_1 + \Gamma_2$, with the following configuration:

(2.51)
$$\begin{tikzcd} D \arrow[r, equal] \arrow[d, equal] & \Gamma_1 \arrow[dl, dash] \\ \Gamma_2 & \end{tikzcd}$$

We will now show that configuration (2.50) gives a 18-dimensional family of polarized K3 surfaces (S, L), with this scroll type.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma$ with intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma \\ D.\Gamma & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & -2 \end{bmatrix}$$

has signature (1,1), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}\Gamma$.

Using Picard-Lefschetz reflections we may assume that $L \sim 3D + \Gamma$ is nef. Assume that $B \sim xD + y\Gamma$ is in $\mathcal{A}^0(L)$. Then B must satisfy

$$B^2 = 0 \qquad B.L = 1, 2, 3, \text{ or } 4$$

It is immediate by lemma 1.4.9[34] that $B \sim D$. Hence L is of type $\{2, 0\}$.

We will show below that the associated scroll type is not $(3, 2, 2, 2)$ only if $\operatorname{rank} \operatorname{Pic} S > 2$. Hence the scroll type associated to (S, L) is $(3, 2, 2, 2)$.

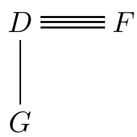
2.7.2 $(\mathbf{3}, \mathbf{3}, \mathbf{2}, \mathbf{1})$

We have $\Delta = 0$ and can write $L \sim 3D + B$, where $h^0(B) = 2$, $D.B = 4$, $L.B = 10$, and $B^2 = -2$.

[34] Alternatively use lemma 2.0.1.

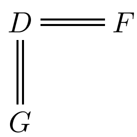
Since $h^1(B) \neq 0$ we can write $B \sim F + G$, where F and G are effective and non-zero with $F.G = 0$ (Ramanujams's lemma). Since $h^0(B) = 2$ we must have $F^2 \leq 0$ and $G^2 \leq 0$. We may then assume that $F^2 = 0$ and $G^2 = -2$, since $B^2 = -2$. Since L is numerically 2-connected this gives the following possibilities:

(2.52)



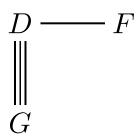
and

(2.53)



and

(2.54)



In general we have $G \sim \Gamma$, where Γ is a smooth rational curve, and $F \sim E$, where E is an elliptic curve.

We will now show that configuration (2.52) gives a 17-dimensional family of polarized K3 surfaces (S, L), with this scroll type.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}E \oplus \mathbb{Z}\Gamma$ with intersection matrix

$$\begin{bmatrix} D^2 & D.E & D.\Gamma \\ D.E & E^2 & E.\Gamma \\ D.\Gamma & E.\Gamma & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

has signature (1,2), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}E \oplus \mathbb{Z}\Gamma$.

Using Picard-Lefschetz reflections we may assume that $L \sim 3D+E+\Gamma$ is nef. Assume that $B \sim xD + yE + z\Gamma$ is in $\mathcal{A}^0(L)$. Then B must satisfy

$$B^2 = 0 \qquad B.L = 1, 2, 3, \text{ or } 4$$

Furthermore by arguing as on page 66 we get:

$$\begin{aligned} 0 \leq D.B &= 3y + z \leq 3 \\ 0 \leq \Gamma.B &= x - 2z \leq 1 \\ 0 \leq E.B &= 3x \leq 5 \end{aligned}$$

The last inequality gives $0 \leq x \leq 1$. The second inequality then gives $z = \lfloor x/2 \rfloor = 0$. Then the first inequality gives $0 \leq y \leq 1$. Since $1 \leq B.L = 4x + 9y + z \leq 3$, we get $B \sim D$. Hence we may assume that D is a perfect Clifford divisor.

L nef and Riemann-Roch used on Γ and E let us assume Γ and E effective. Hence $h^0(L - 3D) \geq h^0(E) \geq 2$.

Since $\operatorname{rank} \operatorname{Pic} S = 3$ the scroll type must be $(3, 3, 2, 1)$ or $(4, 3, 2, 0)$.[35] Calculating determinants we see that $(4, 3, 2, 0)$ is impossible by proposition 1.1.22. Hence the associated scroll type is $(3, 3, 2, 1)$.

2.7.3 $(\mathbf{4}, \mathbf{2}, \mathbf{2}, \mathbf{1})$

We can write $L \sim 4D + B$, where $h^0(B) = 1$. Furthermore $B^2 = -10$, $D.B = 4$, $h^0(D + B) = 2$, and $B.L = 6$. We see that B satisfies exactly the same conditions as $L - 2D$ in [JK01, (i) p.89-90].[36] We have the following possibilities:

$L \sim 4D + 2\Gamma + \Gamma' + \Gamma_1 + \Gamma_2$, with the following configuration:

$$\begin{tikzcd}[arrows=dash] D \arrow[r] \arrow[d] \arrow[dr] & \Gamma \arrow[r] & \Gamma' \\ \Gamma_1 & \Gamma_2 & \end{tikzcd} \tag{2.55}$$

or $L \sim 4D + 2\Gamma + \Gamma' + 2\Gamma_0 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$ $(N \geq 0)$, with the following configuration:

$$\begin{tikzcd}[arrows=dash] D \arrow[r] \arrow[d] & \Gamma \arrow[r] & \Gamma' & \\ \Gamma_0 \arrow[r] & \cdots \arrow[r] & \Gamma_N \arrow[r] \arrow[d] & \Gamma_{N+1} \\ & & \Gamma_{N+2} & \end{tikzcd} \tag{2.56}$$

We see that all of these configurations give $h^0(D + B) = 2$. Hence if there exists K3 surfaces with these configurations they must be associated to the scroll type $(4, 2, 2, 1)$.

We will now show that configuration (2.55) gives a 15-dimensional family of polarized K3 surfaces (S, L), with this scroll type.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma \oplus \mathbb{Z}\Gamma' \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$ with intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma & D.\Gamma' & D.\Gamma_1 & D.\Gamma_2 \\ D.\Gamma & \Gamma^2 & \Gamma.\Gamma' & \Gamma.\Gamma_1 & \Gamma.\Gamma_2 \\ D.\Gamma' & \Gamma.\Gamma' & \Gamma'^2 & \Gamma'.\Gamma_1 & \Gamma'.\Gamma_2 \\ D.\Gamma_1 & \Gamma.\Gamma_1 & \Gamma'.\Gamma_1 & \Gamma_1^2 & \Gamma_1.\Gamma_2 \\ D.\Gamma_2 & \Gamma.\Gamma_2 & \Gamma'.\Gamma_2 & \Gamma_1.\Gamma_2 & \Gamma_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & -2 \end{bmatrix}$$

[35] This will follow from our work below.
[36] This is natural since the associated scroll type there is $(3, 1, 1, 0) = (4 - 1, 2 - 1, 2 - 1, 1 - 1)$.

has signature (1,4), so by proposition 1.1.24 there exists a 15-dimensional family of K3 surfaces with Pic $S = \mathbb{Z}D \oplus \mathbb{Z}\Gamma \oplus \mathbb{Z}\Gamma' \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$.

Using Picard-Lefschetz reflections we may assume that $L' \sim 3D + 2\Gamma + \Gamma' + \Gamma_1 + \Gamma_2$ is nef.[37]

By arguing as we have done many times we get $c = 2$ and that L' has associated scroll type $(3,1,1,0)$. We may after a change of basis, if necessary, assume that D is a perfect Clifford divisor.

Then proposition 1.6.2 gives that $L \sim L' + D$ has associated scroll type $(4,2,2,1)$.[38]

2.7.4 $(\mathbf{4,3,1,1})$

We write $L \sim 4D + B$, where $D.B = 4$, $B.L = 6$, $B^2 = -10$, and $h^0(B) = 1$.

The most general possible configuration is $L \sim 4D+2\Gamma_1+2\Gamma_2+\Gamma_3$, with the following configuration:

(2.57)
$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma_1 \arrow[d, dash] \\ \Gamma_2 \arrow[r, dash] & \Gamma_3 \end{tikzcd}$$

As an example of a less general configuration we take $L \sim 4D+2\Gamma_1+2\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5$, with the following configuration:

(2.58)
$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_5 \\ \Gamma_2 \arrow[r, dash] \arrow[dr, dash] & \Gamma_3 & \\ & \Gamma_4 & \end{tikzcd}$$

Note that this is a configuration for which the result of proposition 1.6.4 does not hold. For if configuration (2.58) satisfies proposition 1.6.4, then we would be able to find the polarized surface $(S, L-D)$ in the classification for $g = 8$ in [JK, p.89] with scroll type $(4-1, 3-1, 1-1, 1-1) = (3,2,0,0)$. But there we see that $(3,2,0,0)$ is of type $\{2,0\}^b$ or $\{2,0\}^c$. We have $(L-D).\Gamma_1 = 0$ and $(L-D).\Gamma_2 = 1$, so $\mathcal{R}_{L-D,D} \subseteq \{\Gamma_1\}$. We must then be in case $\{2,0\}^c$. This is impossible since in the case $\{2,0\}^c$ there must either exists either two disjoint smooth rational curves Γ_1' and Γ_2' with multiplicity one in Δ such that $\Gamma_1.\Gamma_1' = \Gamma_1.\Gamma_2' = 1$ or there exists a smooth rational curve Γ_1' with multiplicity two in Δ such that $\Gamma_1.\Gamma_1' = 1$.

[37] We are looking at $L' := L - D$ not L! See next footnote for our reason for dong this.

[38] If we had looked on L instead on $L - D$ it would be much more difficult to determine the scroll type. We would get $h^0(L-4D) > 0$ and $\mathcal{R}_{L,D}$, so the scroll type is $(4,3,1,1)$ or $(4,2,2,1)$. To exclude the scroll type $(4,3,1,1)$ we would have to know all possible configurations for $(4,3,1,1)$ and show that (S, L) does not satisfy any of these. This is considerably more time-consuming than looking at $L - D$.

We will now show that configuration (2.57) gives a 16-dimensional family of polarized K3 surfaces (S, L), with this scroll type.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma_3$ with intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma_1 & D.\Gamma_2 & D.\Gamma_3 \\ D.\Gamma_1 & \Gamma_1^2 & \Gamma_1.\Gamma_2 & \Gamma_1.\Gamma_3 \\ D.\Gamma_2 & \Gamma_1.\Gamma_2 & \Gamma_2^2 & \Gamma_2.\Gamma_3 \\ D.\Gamma_3 & \Gamma_1.\Gamma_3 & \Gamma_2.\Gamma_3 & \Gamma_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

has signature (1,3), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2 \oplus \mathbb{Z}\Gamma_3$.

Using Picard-Lefschetz reflections we may assume that $L \sim 4D + 2\Gamma_1 + \Gamma_2 + \Gamma_3$ is nef. L nef and Riemann-Roch used on Γ_1, Γ_2, and Γ_3 let us assume that Γ_1, Γ_2, and Γ_3 is effective.

Assume that $B \sim xD + y\Gamma_1 + z\Gamma_2 + u\Gamma_3$ is in $\mathcal{A}^0(L)$. Then B must satisfy

$$B^2 = 0 \qquad B.L = 1, 2, 3, \text{ or } 4$$

Furthermore by arguing as on page 66 we get:

$$\begin{aligned} 0 \le D.B &= y + z \le 3 \\ 0 \le \Gamma_1.B &= x - 2y + u \le 1 \\ 0 \le \Gamma_2.B &= x - 2z + u \le 1 \\ 0 \le \Gamma_3.B &= y + z - 2u \le 2 \end{aligned}$$

Checking all possible values we get $B \sim D$. Furthermore one gets $\mathcal{R}_{L,D} = \emptyset$. Since $h^0(L - 4D) = h^0(2\Gamma_1 + \Gamma_2 + \Gamma_3) > 0$, the associated scroll type must then be $(4, 2, 2, 1)$ or $(4, 3, 1, 1)$. The scroll type $(4, 2, 2, 1)$ does not occur when $\operatorname{rank} \operatorname{Pic} S = 4$. Hence the associated scroll type is $(4, 3, 1, 1)$.

2.7.5 $(5, 2, 1, 1)$

There exists no perfect Clifford divisors associated to this scroll type.

Assume that there exists a perfect Clifford divisor D associated to this scroll type. Then we can write $L \sim 5D + B$, where $D.B = 4$, $B.L = 2$, $B^2 = -18$, and $h^0(B) = 1$. As usual we may write B as a sum of smooth rational curves. Using proposition 1.1.17 we also have that $0 \le D.\Gamma \le 1$ for all smooth rational curves in the support of B, since $h^0(B) = 1$.

There exists (up to multiplicity) four smooth rational curves Γ_i ($i = 1, \ldots, 4$) in the support of B such that $D.\Gamma_i = 1$. Then $L.\Gamma_i \ge 1$ since $\mathcal{R}_{L,D} = \emptyset$. Hence $2 = L.B \ge L.\Gamma_1 + L.\Gamma_2 + L.\Gamma_3 + L.\Gamma_4 \ge 4$, a contradiction.

The methods of [JK01, section 9] give an alternative proof.

2.7.6 $(\mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{0})$

We will now show that this scroll type is not associated to any perfect Clifford divisor.

Assume otherwise. We can write $L \sim 3D+B+\Gamma$, where $\mathcal{R}_{L,D} = \{\Gamma\}$ and $h^0(B+\Gamma) = 3$. Then $\Gamma.B = -1$, so we can write $B \sim B_1 + \Gamma$, where $B_1.\Gamma = 1$, $D.B_1 = 2$, $L.B_1 = 10$, and $B_1^2 = 2$.

Then $(B_1 - D)^2 = -2$ and $L.(B_1 - D) = 6$. Thus L nef and Riemann-Roch used on $B_1 - D$ gives $h^0(B_1 - D) > 0$. So $h^0(L - 4D) > 0$, a contradiction.

Alternative proofs can be found using the methods of [JK01, section 9], [Ste00], or [Bra97].

2.7.7 $(\mathbf{4}, \mathbf{3}, \mathbf{2}, \mathbf{0})$

We can write $L \sim 4D + \Gamma + B$, where $\mathcal{R}_{L,D} = \{\Gamma\}$ and $h^0(B + \Gamma) = 1$. Then $\Gamma.B = -2$, so we can write $B \sim B_1 + \Gamma$, where $B_1.\Gamma = 0$, $D.B_1 = 2$, $L.B_1 = 6$, and $B_1^2 = -2$.

In general B_1 is linearly equivalent to a smooth rational curve Γ_1, and we have the following configuration

$$\begin{tikzcd} D \arrow[r, equal] \arrow[d, dash] & \Gamma_1 \\ \Gamma & \end{tikzcd} \tag{2.59}$$

We will now show that configuration (2.59) gives a 17-dimensional family of polarized K3 surfaces (S, L), with this scroll type.

We write $A \sim D + \Gamma_1$ to ease notation. The lattice $\mathbb{Z}D \oplus \mathbb{Z}A \oplus \mathbb{Z}\Gamma$ with intersection matrix

$$\begin{bmatrix} D^2 & D.A & D.\Gamma \\ D.A & A^2 & A.\Gamma \\ D.\Gamma & A.\Gamma & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

has signature (1,2), so by proposition 1.1.24 there exists a K3 surface with Pic $S = \mathbb{Z}D \oplus \mathbb{Z}A \oplus \mathbb{Z}\Gamma$.

Using Picard-Lefschetz reflections we may assume that $L \sim 3D + \Gamma_1 + 2\Gamma$ is nef.

Assume that $B' \in \mathcal{A}^0(L)$, with B' nef. Write $B' \sim xD + yA + z\Gamma$. Then B' must satisfy one of the following

$$B'^2 = 0 \qquad B'.L = 1, 2, 3, \text{ or } 4$$

Since $B'.L = 4x + 10y$ we have $B'.L = 2$ or 4.

Furthermore by arguing as on page 66 we get:

$$\begin{aligned} 0 \le D.B' &= 2y + z \le 3 \\ 0 \le B_1.B' &= 2x + 2y + z \le 6 \end{aligned}$$

These inequalities give $-3 \le x \le 3$. For $B'.L$ to be 2 or 4 we must have $x = 1$ with $y = 0$ or $x = 3$ with $y = -1$. In the first case $B'^2 = 0$ gives $z = 0$ or $z = 1$. In the second case $B'^2 = 0$ gives $z^2 + 2z + 5 = 0$, which has no integral solutions. Hence $B' \sim D$ or $B' \sim D + \Gamma$. If $B' \sim D + \Gamma$, then we may change the basis of $\operatorname{Pic} S$ as we have done many times already. Thus we may assume that $B' \sim D$.

Since $L.\Gamma = 0$ and $D.\Gamma = 1$ we see that $\mathcal{R}_{L,D} \neq \emptyset$.[39] Hence $h^1(R) = 1$, since $h^1(R) \leq 1$.

Since $\operatorname{rank} \operatorname{Pic} S = 3$ the scroll type must be $(4, 3, 2, 0)$, by our treatment of the other possible scroll types below.

2.7.8 $(5, 2, 2, 0)$ and $(4, 4, 1, 0)$

There exists no perfect Clifford divisors associated to these scroll types.

For the scroll type $(5, 2, 2, 0)$ one can show this by the method we used to show that the scroll type $(4, 3, 1, 0, 0)$ is not associated to any perfect Clifford divisor. To do this we would have had to find all possible configurations, and then show that these configurations actually are associated to the scroll type $(5, 3, 1, 0)$ by looking at $h^0(L - 3D)$. This is a lot of work which we will not include here.

For the scroll type $(4, 4, 1, 0)$ one can show this by using Ramanujam's lemma as we have done for several other scroll types. This is also quite a lot of work, so we will not include the details.

Instead we will refer to the tables in [Bra97, A.2] and [Ste00, p.10], which say that these scroll types are not associated to any K3 surface.

2.7.9 $(5, 3, 1, 0)$

We can write $L \sim 5D + \Gamma + B$, where $\mathcal{R}_{L,D} = \{\Gamma\}$ and $h^0(B + \Gamma) = 1$. Then $\Gamma.B = -3$, so we can write $B \sim B_1 + 2\Gamma$.

Suppose there exists smooth rational curve Γ_1 in the support of B_1 such that $D.\Gamma_1 \neq 0$ and $\Gamma_1 \neq \Gamma$. Then since $L.B_1 = 1$ we see that Γ_1 must have multiplicity one in B_1. Also $D.\Gamma_1 = 1$, since $2 = h^0(D) \leq h^0(D + \Gamma_1) \leq h^0(L - 4D) = 2$. Furthermore $\Gamma_1.B_1 \geq -2$ and $\Gamma_1.\Gamma \geq 0$. Hence $L.\Gamma_1 = 5D.\Gamma_1 + B_1.\Gamma_1 + \Gamma.\Gamma_1 \geq 3$, a contradiction since $L.B_1 = 1$.

Hence Γ has multiplicity one in B_1 and we can write $B_1 \sim B_2 + \Gamma$, where $B_2.\Gamma = 3$, $D.B_2 = 0$, $L.B_2 = 1$, and $B_2^2 = -10$.

There exists several possible configurations. The most general one is $L \sim 5D + 4\Gamma + 2\Gamma_1 + \Gamma_2$,[40] with the following configuration:

(2.60)
$$\begin{tikzcd}[arrows=dash]
D \arrow[r] & \Gamma \arrow[r] \arrow[d] & \Gamma_1 \\
& \Gamma_2 &
\end{tikzcd}$$

Another possible configuration is $L \sim 5D + 4\Gamma + 3\Gamma_1 + \Gamma_2 + \Gamma_3$, with the following

[39] See footnote 31.

[40] Where $B_2 \sim 2\Gamma_1 + \Gamma_2$.

configuration:

$$
\begin{array}{ccccccc}
D & \text{——} & \Gamma & \text{——} & \Gamma_1 & \text{——} & \Gamma_2 \\
 & & & & | & & \\
 & & & & \Gamma_3 & &
\end{array}
\tag{2.61}
$$

We will now show that configuration (2.60) gives a 16-dimensional family of polarized K3 surfaces (S, L), with this scroll type associated to it.

The lattice $\mathbb{Z}D \oplus \mathbb{Z}\Gamma \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$ with intersection matrix

$$
\begin{bmatrix} D^2 & D.\Gamma & D.\Gamma_1 & D.\Gamma_2 \\ D.\Gamma & \Gamma^2 & \Gamma.\Gamma_1 & \Gamma.\Gamma_2 \\ D.\Gamma_1 & \Gamma.\Gamma_1 & \Gamma_1^2 & \Gamma_1.\Gamma_2 \\ D.\Gamma_2 & \Gamma.\Gamma_2 & \Gamma_1.\Gamma_2 & \Gamma_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}
$$

has signature (1,3), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}\Gamma \oplus \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$.

Using Picard-Lefschetz reflections we may assume that $L \sim 5D + 4\Gamma + 2\Gamma_1 + \Gamma_2$ is nef.

Assume that $B' \in \mathcal{A}^0(L)$, with B' nef. Arguing as we have done many times we get $B' \sim D$, $B' \sim D + \Gamma$, $B' \sim D + \Gamma + \Gamma_1$, or $B' \sim D + \Gamma + \Gamma_2$. Then we may change the basis of $\operatorname{Pic} S$ to assume that $B' \sim D$. Changing the basis of $\operatorname{Pic} S$ yet another time (if necessary) lets us assume that Γ, Γ_1, and Γ_2 is effective.[41] We also find $\mathcal{R}_{L,D} \neq \emptyset$. Hence $h^1(R) = 1$, since $c = 2$. Since $h^0(L - 5D) = h^0(4\Gamma + 2\Gamma_1 + \Gamma_2) > 0$, the associated scroll type must then be $(5, 3, 1, 0)$.

2.8 Clifford general non-BN general polarized K3 surfaces of genus 12

We have seen (proposition 1.4.7) that every Clifford general non-BN general K3 surfaces of genus 12 satisfies

$$\operatorname{Cliff}(L) = \mu(L) = 5.$$

But there is a huge difference from the non-Clifford general cases in that we are no longer is guaranteed the existence of perfect Clifford divisors. In particular we are no longer guaranteed that (C6) holds. Hence it becomes much harder to determine the base divisor Δ of $F = L - D$.

Using lemma 1.4.6 and proposition 1.4.7 we see that there exists a divisor D such that

$$\mu(L) = 5 = D.(L - D) + 2$$

and

$$h^1(D) = h^1(L - D) = 0.$$

Using [JK01, lemma 2.6] we get the existence of a free Clifford divisor,[42] i.e. a divisor

[41] See p.73 for details.

[42] Compare [JK01, corollary 10.4].

that satisfies (C1)-(C5).

Proposition 1.4.7 gives us two cases to consider:

$$D^2 = 2 \text{ and } D.L = 9$$

and

$$D^2 = 4 \text{ and } D.L = 11.$$

2.9 $D^2 = 2$ and $D.L = 9$

We have $h^0(L) = 13$, $h^0(D) = 3$, and $h^0(L - D) = 5$. Since $L.(L - 3D) = -5 > 0$ and L is nef, we see that $h^0(L - 3D) = 0$. Hence we have

$$\begin{aligned} d_0 &= 8, \\ d_1 &= 5 - h^0(L - 2D), \\ d_2 &= h^0(L - 2D). \end{aligned}$$

Since $d_1 \geq d_2$ we get $0 \leq h^0(L - 2D) \leq 2$. We have the possible scroll types of table 2.9.

Table 2.9: Possible scroll types associated to non-BN general L with $D^2 = 2$ and $D.L = 9$.

d_0	d_1	d_2	scroll type
8	5	0	$(1, 1, 1, 1, 1, 0, 0, 0)$
8	4	1	$(2, 1, 1, 1, 0, 0, 0, 0)$
8	3	2	$(2, 2, 1, 0, 0, 0, 0, 0)$

2.9.1 $(\mathbf{1, 1, 1, 1, 1, 0, 0, 0})$

We will first find $\mathcal{R}_{L,D}$. To do this we will argue along the lines of [JK01, lemma 10.5]. First note that we have $h^1(R) = 1$ and $h^0(R) = 0$.

Choose a smooth curve $D_0 \in |D|$ and set $F_{D_0} = F \otimes \mathcal{O}_{D_0}$. Then

$$\deg F_{D_0} = F.D = c + 2 = 7 = D^2 + 3 = 2g(D_0) + 1.$$

Then [Har77, corollary 3.2(b)] gives that F_{D_0} is very ample.

Tensoring the exact sequence

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_{D_0} \to 0$$

with F we get the exact sequence

$$0 \to R \to F \to F_{D_0} \to 0.$$

Tensoring this sequence with $-\Delta$ we get the exact sequence

$$0 \to R - \Delta \to F - \Delta \to (F - \Delta)_{D_0} \to 0.$$

Thus (using that Δ is fixed in F and $h^0(R) = h^0(R - \Delta) = h^1(F) = 0$) we get the exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & H^0(F) \arrow[r] \arrow[d, equal] & H^0(F_{D_0}) \arrow[r] & H^1(R) \arrow[r] & 0 \\ 0 \arrow[r] & H^0(F - \Delta) \arrow[r] & H^0((F - \Delta)_{D_0}) & & \end{tikzcd}$$

Whence $h^0((F - \Delta)_{D_0}) \geq h^0(F_{D_0}) - 1$, so $\Delta.D = 0$ or 1 by [Har77, proposition 3.1(b)].

From the definition of $\mathcal{R}_{L,D}$ we find that $\mathcal{R}_{L,D}$ is either empty or consists of only one curve. Both of these cases arise.

The lattice of proposition 1.4.10 gives an 18-dimensional family of K3 surfaces with $c = 5$ and $D^2 = 2$. Lemma 1.4.9 gives $\mathcal{R}_{L,D} = \emptyset$ for this lattice. We will see below that the scroll types $(2,1,1,1,0,0,0,0)$ and $(2,2,1,0,0,0,0,0)$ always have $\operatorname{rank} \operatorname{Pic} S > 3$. Hence the scroll type must be $(1,1,1,1,1,0,0,0)$.[43]

There exists a 17-dimensional family of K3 surfaces with $\mathcal{R}_{L,D} = \{\Gamma\}$ and associated scroll type $(1,1,1,1,1,0,0,0)$. Consider the lattice $\mathbb{Z}D \oplus \mathbb{Z}F \oplus \mathbb{Z}\Gamma$ with intersection matrix

$$\begin{bmatrix} D^2 & D.F & D.\Gamma \\ D.F & F^2 & F.\Gamma \\ D.\Gamma & F.\Gamma & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 1 \\ 9 & 2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$$

It has signature (1,2), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}F \oplus \mathbb{Z}\Gamma$.

Using the methods of the previous sections we find that we may assume that D is a free Clifford divisor. Furthermore one gets $\mathcal{R}_{L,D} = \{\Gamma\}$.

We will show below that the scroll types $(2,1,1,1,0,0,0,0)$ and $(2,2,1,0,0,0,0,0,0)$ always have $\operatorname{rank} \operatorname{Pic} S > 3$. Hence the associated scroll type is $(1,1,1,1,1,0,0,0)$.

2.9.2 $(\mathbf{2,1,1,1,0,0,0,0})$

We have $h^0(L - 2D) = 1$. Hence we can write $R := L - 2D$ as a sum of smooth rational curves. We also have $R.D = 5$ and $R^2 = -6$. There exists two different configurations[44] that both give 16-dimensional families of K3 surfaces (all the Γ_i are smooth rational curves):

$L \sim 2D + \Gamma_1 + \Gamma_2 + \Gamma_3$, with the following configuration:

(2.62)

$$\begin{tikzcd} D \arrow[r, equal] \arrow[d, no head] \arrow[dr, equal] & \Gamma_3 \\ \Gamma_1 & \Gamma_2 \end{tikzcd}$$

[43] Note that this give an example of a K3 surface with a free Clifford divisor that is not a perfect Clifford divisor. We have $h^1(R) = 1 \neq \Delta.D = 0$.

[44] One can show that both of these configurations in fact exists with the prescribed properties.

and $\mathcal{R}_{L,D} = \{\Gamma_1\}$.

$L \sim 2D + \Gamma_1 + \Gamma_2 + \Gamma_3$, with the following configuration:

(2.63)
$$\begin{tikzcd} D \arrow[r, dash, triple] \arrow[d, dash] \arrow[dr, dash] & \Gamma_3 \\ \Gamma_1 & \Gamma_2 \end{tikzcd}$$

and $\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2\}$.

2.9.3 $(\mathbf{2, 2, 1, 0, 0, 0, 0, 0})$

We have $h^0(L - 2D) = 2$. We also have $R.D = 5$ and $R^2 = -6$. In this case there exists two configurations[45] that give 16-dimensional families of K3 surfaces (all the Γ_i are smooth rational curves and E is an elliptic curve):

$L \sim 2D + E + 2\Gamma_1 + \Gamma_2$, with the following configuration:

(2.64)
$$\begin{tikzcd} \Gamma_2 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] & D \arrow[r, dash, triple] & E \end{tikzcd}$$

and $L \sim 2D + E + 2\Gamma_1 + \Gamma_2$, with the following configuration:

(2.65)
$$\begin{tikzcd} \Gamma_2 \arrow[r, dash] & \Gamma_1 \arrow[r, equal] & D \arrow[r, dash] & E \end{tikzcd}$$

The following configuration also satisfy $h^0(L - 2D) = 2$, $R.D = 5$, and $R^2 = -6$:

$L \sim 2D + E + \Gamma_1 + 2\Gamma_2$, with the following configuration:

$$\begin{tikzcd} \Gamma_2 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] & D \arrow[r, dash, quadruple] & E \end{tikzcd}$$

$L \sim 2D + E + \Gamma_1 + 2\Gamma_2$, with the following configuration:

$$\begin{tikzcd} \Gamma_2 \arrow[r, dash] & \Gamma_1 \arrow[r, dash, triple] & D \arrow[r, equal] & E \end{tikzcd}$$

$L \sim 2D + E + \Gamma_1 + 2\Gamma_2$, with the following configuration:

$$\begin{tikzcd} \Gamma_2 \arrow[r, dash] & \Gamma_1 \arrow[r, equal] & D \arrow[r, dash, triple] & E \end{tikzcd}$$

$L \sim 2D + E + \Gamma_1 + 2\Gamma_2$, with the following configuration:

$$\begin{tikzcd} \Gamma_2 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] & D \arrow[r, dash, quadruple] & E \end{tikzcd}$$

None of these four configurations are associated to the scroll type $(2, 1, 1, 1, 0, 0, 0, 0)$. One can show this as follows: The configurations with $D.E = 4$ give

$$5 = h^0(L - D) = h^0(D + R) \geq h^0(D + E) = 6,$$

a contradiction.

[45] All with $\mathcal{R}_{L,D} = \emptyset$.

The configuration with $D.\Gamma_1 = 3$ gives

$$5 = h^0(L - D) = h^0(D + R) \geq h^0(D + E + \Gamma_1) \geq 6,$$

a contradiction.

The configuration with $D.E, D.\Gamma_1 > 1$ gives $h^0(D + E) = 5$, with $D + E$ base point free, and $\Gamma_1.(D + E) = 2$. Proposition 1.1.17 gives

$$5 = h^0(L - D) = h^0(D + R) \geq h^0(D + E + \Gamma_1) > 5,$$

a contradiction.

There also exists configurations with $\mathcal{R}_{L,D} \neq \emptyset$. For example $L \sim 2D + E + \Gamma_1 + \Gamma_2 + \Gamma_3$, with the following configuration:

$$\begin{tikzcd} \Gamma_1 \arrow[r, dash] & D \arrow[r, dash] \arrow[d, dash] \arrow[dr, equal] & E \\ & \Gamma_2 & \Gamma_3 \end{tikzcd} \tag{2.66}$$

and $\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2\}$.

2.10 $D^2 = 4$ and $D.L = 11$

We have $h^0(L) = 13$, $h^0(D) = 4$, and $h^0(L - D) = 4$. Since $L.(L - 3D) = -11 > 0$ and L is nef, we see that $h^0(L - 3D) = 0$. Hence we have

$$\begin{aligned} d_0 &= 9 \\ d_1 &= 4 - h^0(L - 2D) \\ d_2 &= h^0(L - 2D) \end{aligned}$$

Since $d_1 \geq d_2$ we get $0 \leq h^0(L - 2D) \leq 2$. Hence we have the possible scroll types of table 2.10.

Table 2.10: Possible scroll types associated to non-BN general L with $D^2 = 4$ and $D.L = 11$.

d_0	d_1	d_2	scroll type
9	4	0	$(1,1,1,1,0,0,0,0,0)$
9	3	1	$(2,1,1,0,0,0,0,0,0)$
9	2	2	$(2,2,0,0,0,0,0,0,0)$

2.10.1 $(\mathbf{1, 1, 1, 1, 0, 0, 0, 0, 0})$

Arguing as in section 2.9.1 we get $\mathcal{R}_{L,D} = \emptyset$ or $\mathcal{R}_{L,D} = \{\Gamma\}$.

The lattice of equation (1.20) gives an 18-dimensional family of K3 surfaces with $c = 5$ and $D^2 = 4$. Lemma 1.4.9 gives $\mathcal{R}_{L,D} = \emptyset$. We will see below that the other possible configurations with $c = 5$ and $D^2 = 4$ all have $\operatorname{rank} \operatorname{Pic} S > 3$. Hence the scroll type must be $(1, 1, 1, 1, 0, 0, 0, 0, 0)$.

There exists a 17-dimensional family of K3 surfaces with $\mathcal{R}_{L,D} = \{\Gamma\}$ and associated scroll type $(1, 1, 1, 1, 0, 0, 0, 0, 0)$.

Consider the lattice $\mathbb{Z}D \oplus \mathbb{Z}F \oplus \mathbb{Z}\Gamma$ with intersection matrix

$$\begin{bmatrix} D^2 & D.F & D.\Gamma \\ D.E & F^2 & F.\Gamma \\ D.\Gamma & F.\Gamma & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 4 & 11 & 1 \\ 11 & 4 & -1 \\ 1 & -1 & -2 \end{bmatrix}$$

It has signature (1,2), so by proposition 1.1.24 there exists a K3 surface with $\operatorname{Pic} S = \mathbb{Z}D \oplus \mathbb{Z}F \oplus \mathbb{Z}\Gamma$.

Using the methods of the previous sections we find that we may assume that D is a free Clifford divisor. Furthermore one gets $\mathcal{R}_{L,D} = \{\Gamma\}$.

We will show below that for the other possible scroll types we always have $\operatorname{rank} \operatorname{Pic} S > 3$. Hence the associated scroll type is $(1, 1, 1, 1, 1, 0, 0, 0)$.

2.10.2 $(\mathbf{2, 1, 1, 0, 0, 0, 0, 0, 0})$

We have $h^0(L - 2D) = 1$. Hence we can write $R := L - 2D$ as a sum of smooth rational curves. We also have $R.D = 3$, $R.L = 0$, and $R^2 = -6$. This is just the conditions we used to classify Δ in section 1.5. Hence we can use the results we got there.

We get the following configurations:[46]

$L \sim 2D + \Gamma_1 + \Gamma_2 + \Gamma_3$ with the following configuration

(2.67)
$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma_3 \\ \Gamma_1 & \Gamma_2 \end{tikzcd}$$

and $\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2, \Gamma_3\}$.

$L \sim 2D + \Gamma_{-1} + 2\Gamma_0 + \cdots 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$ with the following configuration

(2.68)
$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma_0 \arrow[r, dash] & \cdots \arrow[r, dash] & \Gamma_N \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+1} \\ \Gamma_{-1} & & & \Gamma_{N+2} & \end{tikzcd}$$

and $\mathcal{R}_{L,D} = \{\Gamma_{-1}, \Gamma_0\}$.

[46] Note that in all of the following cases we can show that R is fixed in $F = L - D$. Hence $R = \Delta$. We also have $h^1(R) = 2 \neq D.\Delta = 3$.

$L \sim 2D + 3\Gamma_0 + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4$ with the following configuration

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[dr, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_3 \\ & & \Gamma_2 \arrow[r, dash] & \Gamma_4 \end{tikzcd} \tag{2.69}$$

and $\mathcal{R}_{L,D} = \{\Gamma_0\}$.

$L \sim 2D + 3\Gamma_0 + 4\Gamma_1 + 2\Gamma_2 + 3\Gamma_3 + 2\Gamma_4 + \Gamma_5$ with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] \arrow[dr, dash] & \Gamma_2 & & \\ & & & \Gamma_3 \arrow[r, dash] & \Gamma_4 \arrow[r, dash] & \Gamma_5 \end{tikzcd} \tag{2.70}$$

and $\mathcal{R}_{L,D} = \{\Gamma_0\}$.

$L \sim 2D + 3\Gamma_0 + 4\Gamma_1 + 5\Gamma_2 + 6\Gamma_3 + 4\Gamma_4 + 2\Gamma_5 + 3\Gamma_6$ with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_3 \arrow[r, dash] \arrow[dr, dash] & \Gamma_4 \arrow[r, dash] & \Gamma_5 \\ & & & & & \Gamma_6 & \end{tikzcd} \tag{2.71}$$

and $\mathcal{R}_{L,D} = \{\Gamma_0\}$.

2.10.3 $(\mathbf{2,2,0,0,0,0,0,0,0})$

We will show that this case does not exist. We have $h^0(R) = 2$ and $h^1(R) = 3$. We also have $R.D = 3$, $R.L = 0$, and $R^2 = -6$.

By Ramanujam's lemma we can write $R \sim G + H$, where G and H are non-zero with $G.H = 0$.[47] Since $L.R = 0$ we get $L.G = L.H = 0$. Using lemma 1.1.16 and $G.H = 0$ we get

$$\begin{aligned} h^1(G+H) &= h^0(G+H, \mathcal{O}_{G+H}) - 1 \\ &= h^0(G, \mathcal{O}_G) + h^0(H, \mathcal{O}_H) - 1 \\ &= h^1(F) + h^1(G) + 1. \end{aligned}$$

Then Riemann-Roch gives

$$\begin{aligned} 2 = h^1(G+H) &= \frac{1}{2}G^2 + \frac{1}{2}H^2 + 2 + h^1(G+H) \\ &= \frac{1}{2}G^2 + \frac{1}{2}H^2 + 3 + h^1(F) + h^1(G) \\ &= h^0(F) + h^0(G) - 1. \end{aligned}$$

[47] Since $h^0(R) = 2$ we must have $G^2 \leq 0$ and $H^2 \leq 0$.

Hence either $h^0(G) = 2$ or $h^0(H) = 2$.

Assume $G^2 = 0$. Since $G.L = 0$ we get $G = 0$ by the Hodge index theorem, a contradiction.

Assume $G^2 = -2$. Then we have $H^2 = -4$. Furthermore $L.G = L.H = 0$ gives $D.G = 1$ and $D.H = 2$. If $h^0(G) = 2$, then $h^1(G) \neq 0$ and we can write $G \sim G_1 + G_2$, where G_1 and G_2 are non-zero with $G_1.G_2 = 0$. Since $D.G = 1$ we either have $D.G_1 = 0$ or $D.G_2 = 0$. If $D.G_1 = 0$, then since $L.G_1 = 0$ we also get $H.G_1 = 0$. Consequently $G_1 = 0$, since L is numerically 2-connected, a contradiction. We similarly get a contradiction if $D.G_2 = 0$. Hence we cannot have $h^0(G) = 2$. Therefore $h^0(H) = 2$. Then $h^1(H) \neq 0$ and we can write $H \sim H_1 + H_2$, where H_1 and H_2 are non-zero with $H_1.H_2 = 0$. We cannot have $H_i^2 = 0$ for the same reason as for $G^2 = 0$. Therefore $H_1^2 = H_2^2 = -2$, since $H^2 = -4$. Since H_1 and H_2 are non-zero and L is numerically 2-connected we get $H_1.D = H_2.D = 1$. Arguing as above we must have $h^0(H_1) = 2$ or $h^0(H_2) = 2$. Iterating the argument for $h^0(G) = 2$ we get a contradiction.

If $G^2 = -4$ or $G^2 = -6$, then we get $H^2 = -2$ or $H^2 = 0$ respectively. Hence we are in a case already considered.

If $G^2 \leq -8$, then $H^2 > 0$, a contradiction.

So the scroll type $(2, 2, 0, 0, 0, 0, 0, 0, 0)$ is not associated to any free Clifford divisor.

2.11 $g \neq 12$

In this chapter we have only considered $g = 12$. We will now give an overview over what has to be done for other genera.

The BN general case is the hardest to classify, because we can not use $\mu(L)$ to automatically reduce the problem to a lattice-theoretical one. On the other hand Mukai [Muk95] has been able to say very much in this case for low genera (with L ample). I have only been able to obtain a rather poorly translation of his article, so it is difficult for me to say whether his techniques may be extended to higher genera. Since he has not done the $g = 11$ case it seems probable that his techniques does not easily extend to other genera. The base point free but non-ample situation also has to be considered. It looks like no work has been done in this area.

For the Clifford general and non-BN general case almost everything that is known at present is included in section 1.4. The big problem that remains to show here is whether one always can assume $h^1(F) = h^1(D) = 0$. Proposition 1.4.6 gives conditions for $h^1(F)$ and $h^1(D)$ to vanish but one would like stronger conditions if possible. For $g \geq 14$ the conditions of proposition 1.4.6 are not strong enough to guarantee $h^1(F) = h^1(D) = 0$. Even for the case $g = 14$ iii) treated on p. 33 it is difficult to decide whether $h^1(F) = h^1(D) = 0$ always holds. The example with $\mathrm{Cliff}(L) = 6$ and $\mu(L) = 7$ in remark 1.4.8 is of this type with $h^1(D) = h^1(F) = 0$. I have been able to show that if $h^1(D) \neq 0$ or $h^1(F) \neq 0$, then $\operatorname{rank}\operatorname{Pic} S > 4$ by arguing with Ramanujans's lemma and the Hodge index theorem. If one wants to use this procedure to show that $h^1(D) = h^1(F) = 0$ always holds, then one has to check that $\operatorname{rank}\operatorname{Pic} S = n$ is impossible for each n with $1 \leq n \leq 20$. This is computationally very long-winded, and it does not seem feasible to do this in every case where the conditions of proposition 1.4.6 are not satisfied. So if one wants to study fully the Clifford general and non-BN general case for $g \geq 14$ it seems that one has to try a different approach.

The non-Clifford general case is easier. For $g \neq 12$ this can be treated just as we have done in this chapter for $g = 12$. Chapter 1 includes all the necessary theory. For higher g there are more possibilities for c, this will make the classification more time consuming. But for a fixed c the classification will actually become easier as we increase g. There are several reasons for this. Firstly when g is high enough we may assume $D^2 = 0$ (lemma 1.6.1). Secondly when g is high enough we always get $h^1(R) = 0$ by the Hodge index theorem. See for example our classification of Δ' for $c = 4$ where we showed that $h^1(R) = 0$ for $L^2 \geq 74$ and $c = 4$. Thirdly the results of section 1.6 implies that for large g we may just make small modifications in the classification for $g - c - 2$ to get the classification for g. For example when $c = 4$ it is enough to know the classification from $g = 74$ to 86 to know the classification for all $g \geq 74$.[48]

[48]Proof: When $g \geq 74$ we have $h^1(R) = 0$. Propositions 1.6.2 and 1.6.4 then give a one-to-one correspondence between perfect Clifford divisors associated to polarized K3 surfaces of genus $g(L) = g$ and $g(L + D) = g + 6$ with $c = 4$.

Part II

Other Surfaces

Chapter 3

Del Pezzo Surfaces

We will now consider Del Pezzo surfaces. Section 3.1 gives an introduction to Del Pezzo surfaces and contains results we need later on.

In section 3.2 we introduce some concepts used to study higher order embeddings: k-jet ampleness, k-very ampleness, k-spannedness, birational k-very ampleness, and birational k-spannedness. We give a brief introduction to these concepts before discussing k-very ampleness and birational k-very ampleness in more detail on Del Pezzo surfaces. Di Rocco [Roc96] has completely characterized k-very ampleness on Del Pezzo surfaces. Knutsen [Knu02] has done the same for birational k-very ampleness. We will translate Knutsen's result to numerical conditions in the Picard group. This will enable us to compare the results of Di Rocco and Knutsen.

In many cases the results of Knutsen give in a natural way scrolls containing projective models of Del Pezzo surfaces. In section 3.3 we will study these scrolls. In the first half of the section we compute scroll types, while we in the second half look at the resolutions that arise from the inclusion of the projective model in the scroll.

3.1 Preliminaries

Definition 3.1.1. A *Del Pezzo surface* S is a surface with an ample anticanonical bundle $-K_S$. The *degree* of S is $\deg S = K_S^2$.

Remark 3.1.2. This definition differs slightly from the one in [Har77, remark V.4.7.1]. In [Har77] a Del Pezzo surface is a surface with a very ample anticanonical bundle $-K_S$.

We write $B_{P_1,\ldots,P_n}(S)$ for the blow up of S at the points $P_1, \ldots, P_n$. We say that the points $P_1, \ldots, P_n$ are in *general position* if no 3 of the P_i are collinear and no 7 of them lie on a conic.

We have a complete description of the isomorphism classes of Del Pezzo surfaces.

Theorem 3.1.3. *(see [Dem80]) Let S be a Del Pezzo surface. Then*

$$1 \leq \deg S \leq 9.$$

Furthermore S is of one of the following types:

(a) $\deg S = 9$ *if and only if* $S \cong \mathbb{P}^2$

(b) if $\deg S = 8$*, then either* $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ *or* $S \cong B_P(\mathbb{P}^2)$*.*

(c) if $1 \leq \deg S \leq 7$*, then* $S \cong B_{P_1,\ldots,P_{9-\deg S}}(\mathbb{P}^2)$ *where* $P_1, \ldots, P_{9-\deg S}$ *are points in general position on* $\mathbb{P}^2$*.*

We will from now on denote a Del Pezzo surface of degree $\deg S$ by S_n, where $n = 9 - \deg S$, when $S \ncong \mathbb{P}^2$ and $S \ncong \mathbb{P}^1 \times \mathbb{P}^1$.

We will now describe the Picard lattice on a Del Pezzo surface.

If $S \cong \mathbb{P}^2$, then

$$\operatorname{Pic} S = \mathbb{Z}l,$$

where l is a line bundle such that $l^2 = 1$. Furthermore $K_S = -3l$.

If $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, then

$$\operatorname{Pic} S = \mathbb{Z}l_1 \otimes \mathbb{Z}l_2,$$

where l_1 and l_2 are line bundles such that $l_i^2 = 1$ and $l_1.l_2 = 0$.

For S_n the Picard lattice is a bit more involved.[1] Let

$$\pi : S_n \to \mathbb{P}^2$$

be the blowing up of $\mathbb{P}^2$. We denote by l the class of $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and by e_i the class of $\pi^{-1}(P_i)$. Then

$$l^2 = 1, \qquad e_i.e_j = -\delta_{ij}, \qquad e_i.l = 0,$$

and

$$\operatorname{Pic} S_n = \mathbb{Z}l \otimes \mathbb{Z}e_1 \otimes \cdots \otimes \mathbb{Z}e_n.$$

Furthermore $-K_{S_n} = 3l - \sum_1^n e_1$. Note that $-K_{S_n}$ is base point free unless $n = 8$. Also $-K_S$ is very ample if and only if $\deg S \geq 3$. Thus we see that the definition in [Har77] corresponds to our definition when $\deg S \geq 3$.

We will usually write line bundles on the form $al - \sum_1^n b_i e_i$. Note that the the self-intersection of $al - \sum_1^n b_i e_i$ is $a^2 - \sum_1^n b_i^2$.

Riemann-Roch on Del Pezzos is as follows

$$h^0(D) - h^1(D) + h^0(K - D) = \frac{1}{2} D.(D - K) + 1.$$

Since K is ample (and hence especially effective) there arises situations where both $h^0(D)$ and $h^0(K - D)$ are positive.

We will need some vanishing results.

Proposition 3.1.4. *[Par91, (0.4.5) and (0.4.6)] Let* S *be a Del Pezzo surface.*

If C *is a connected reduced curve, then*

$$h^1(-C) = h^1(C + K) = 0.$$

[1] See [Har77, section V.3] for proofs of the following.

If L is base point free, then

$$h^1(L) = h^2(L) = 0.$$

If L is nef with $g(L) \geq 1$, then

$$h^1(L) = h^1(L+K) = h^2(L) = h^2(L+K) = 0.$$

The adjunction formula is

$$g(L) = \frac{1}{2}L.(L+K) + 1.$$

Using Riemann-Roch and the previous proposition we see that this gives $h^0(L+K) = g(L)$ when L is nef and $g(L) \geq 1$.

Almost all nef divisors on Del Pezzos are base point free. In fact L is nef if and only if L is base point free or $L \sim -K_{S_8}$ [Roc96, corollary 4.7].

A (-1)–curve is a curve Γ such that $\Gamma^2 = -1$. Such a curve is necessarily a smooth rational curve and satisfies $\Gamma.K = -1$[2] (use the adjunction formula). We write

$$I_n := \{\Gamma \in \operatorname{Pic} S_n | \Gamma^2 = -1 \text{ and } L.K_{S_n} = -1\}.$$

All of the members of his set are irreducible effective divisors. The cardinality of I_n is finite (see table [Roc96, p.4]).

Proposition 3.1.5. *Let L be a divisor on a Del Pezzo surface, such that $L \nsim -K_{S_8}$, $L \nsim -2K_{S_8}$, and $L \nsim -K_{S_7}$. Then L is ample if and only if it is very ample.*

Proof. If $S \cong \mathbb{P}^2$ or $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, then the result follows from [Har77, examples II.7.6.1 and II.7.6.2].

Assume that $S \cong S_r$, $L \nsim -K_{S_8}$, $L \nsim -2K_{S_8}$, and $L \nsim -K_{S_7}$. Then [Roc96, corollary 4.5] gives that if $L.\xi > 0$ for any $\xi \in I_r$ and $L.(l - e_1) > 0$, then L is very ample (since 1-very ample is equivalent to very ample).[3] By definition very ample implies ample. The Nakai-Moishezon criterion gives that if L is ample, then $L.\xi > 0$ for any $\xi \in I_r$ and $L.(l - e_1) > 0$. Thus we get the result. □

3.2 k-very ampleness and birational k-very ampleness

After introducing the concepts of k-very ampleness and birational k-very ampleness we study these concepts on Del Pezzo surfaces. The first half of this section is influenced by lectures held by T. Szemberg on "Higher order embeddings" in Bedlewo, March 2002. We will only consider the algebraic properties of the concepts we introduce. For geometric properties see the references.

[2] This property in fact characterizes (-1)-curves almost completely: if D is an effective irreducible divisor on S_r such that $D.K_{S_r} = -1$, then D is a (-1)-curve or $D = -K_{S_8}$ [Dem80, lemma 9].

[3] See next section.

Definition 3.2.1. Let L be a line bundle on a smooth projective variety X and $k \geq 1$ an integer.

L is *k-jet ample* if the evaluation map

$$H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_X/(\mathfrak{m}_{y_1}^{k_1} \otimes \cdots \otimes \mathfrak{m}_{y_r}^{k_r}))$$

is surjective for any choice of distinct points $y_1, \ldots, y_r$ in X and positive integers $k_1, \ldots, k_r$ such that $\sum k_i = k + 1$.

L is *k-very ample* (resp. *k-spanned*) if for every zero-dimensional subscheme (resp. curvilinear zero-dimensional subscheme) $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \subseteq (X, \mathcal{O}_X)$ of length $k+1$ the natural map

$$H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_{\mathcal{Z}})$$

is surjective.

These concepts where introduced by Beltrametti and Sommese to study higher order embeddings, but have lately been much studied for their own sake. We list some examples and first properties.

1. A line bundle L is 0-jet ample if and only if it is 0-very ample if and only if it is 0-spanned if and only if it is base point free.

2. A line bundle L is 1-jet ample if and only if it is 1-very ample if and only if it is 1-spanned if and only if it is very ample.

3. A line bundle L is k-jet ample only if it is k-very ample only if it is k-spanned.

4. There exists line bundles that are k-very ample but not k-jet ample. Take for example an abelian surface with Picard number 1 and let L be a primitive line bundle of type $(1, 7)$. Then L is 2-very ample and 1-jet ample but not 2-jet ample (see [BS97]). There exists examples of k-spanned line bundles that are not k-very ample, but none are known on surfaces.

5. Let L be k-jet ample and M be l-jet ample. Then $L + M$ is $k + l$-jet ample. In particular let L be an ample line bundle and fix a positive integer k, then pL is k-jet ample for all sufficiently large p. (Corresponding statements holds for k-spannedness.[4])

6. Let C be a curve of genus g and L a line bundle of degree d. Then [Ha, corollary IV.3.2] says that if $\deg L \geq 2g + k$, then L is k-jet ample ($k = 0$ or 1). This result also holds for all $k > 1$.

Proposition 3.2.2. *[BS88a, 0.5.1] If L is a k-very ample line bundle on a surface S, then $L.C \geq k$ for every curve C on S, with equality only if $C \cong \mathbb{P}^1$.*

[4] But not for k-very ampleness. See [BS93].

Theorem 3.2.3. *[BS88b, theorem 2.1] Let M be a nef line bundle on a smooth surface S, such that $M^2 \geq 4k+5$, for $k \geq 0$. Then either $K = M + K_S$ is k-very ample or there exists an effective divisor D such that $M - 2D$ is $\mathbb{Q}$-effective, D contains some zero-dimensional subscheme where the k-very ampleness fails and*

$$M.D - k - 1 \leq D^2 \leq \frac{M.D}{2} < k+1.$$

Di Rocco has completely characterized k-very ampleness on Del Pezzo surfaces [Roc96, corollary 4.6].

Definition 3.2.4. (Knutsen) Let L be a line bundle on a smooth projective variety X and $k \geq 1$ an integer. L is *birationally k-very ample* (resp. *birationally k-spanned*) if there exists a non-empty Zariski-open subset of S where L is k-very ample (resp. k-spanned).

Knutsen gives in [Knu02] conditions for the adjoint bundle on a Del Pezzo surface to be birationally k-very ample.

Theorem 3.2.5. *[Knu02, theorem 1.1] Let L be a nef line bundle of sectional genus $g(L) \geq 2$ on a Del Pezzo surface S and $k \geq 1$ an integer. The following is equivalent:*

1. *$L + K_S$ is birationally k-very ample,*
2. *$L + K_S$ is birationally k-spanned,*
3. *$L.D \geq k+2$ (resp. $L.D \geq k+3$) for all smooth rational curves D with $D^2 = 0$ (resp. $D^2 = 1$) and $-K_S.L \geq k + 2 + m(L) + K_S^2$ whenever $K_S + m(L) \leq \min\{k+1, 4\}$, where $m(L)$ is the cardinality of*

$$\mathcal{R}(L) := \{\Gamma | \Gamma = (-1)\text{-}curve \text{ and } \Gamma.L = 0\}$$

We will be working with the following conditions:

$$(\dagger) \qquad \begin{gathered} h^0(D) \geq 2,\ h^0(L-D) \geq 2, \\ D.(L-D) \leq k+1,\ L + K_S \geq D, \\ \text{and if } L^2 \geq 4k+3, \text{ then } L \geq 2D \end{gathered}$$

With (†) we have the following proposition, which is a main ingredient in the proof of theorem 3.2.5.

Proposition 3.2.6. *[Knu02, proposition 3.7] Assume that L is a big and nef line bundle of sectional genus $g(L) \geq 2$ on a Del Pezzo surface S such that $L \not\sim -2K_{S_8}$. Assume that there are divisors satisfying the condition (†) for $k = k_0$, but none for $k < k_0$. Then $k_0 \geq 1$, and any divisor D satisfying (†) for $k = k_0$ has the following properties (with $M := L - D$):*

1. *$D.M = k_0 + 1$,*

2. $h^1(D) = h^1(M) = h^1(D + K_S) = h^1(M + K_S) = 0$,

3. M *and* D *are nef with* $g(M) \geq 1$,

4. the general members of $|D|$ *and* $|M|$ *are smooth curves.*

Furthermore D *is of one of the following types (with* $g(D) = 0$ *in (a)-(b) and* $g(D) = 1$ *in (c)-(g)):*

(a) $D^2 = 0$, $D.K_S = -2$,

(b) $D^2 = 1$, $D.K_S = -3$,

(c) $D \sim -K_{S_8}$,

(d) $D^2 = -K_S.D = 2$, $K_S^2 \leq 2$ *with* $D \sim -K_{S_7}$ *if* $K_S^2 = 2$,

(e) $D^2 = -K_S.D = 3$, $K_S^2 \leq 3$ *with* $D \sim -K_{S_6}$ *if* $K_S^2 = 3$,

(f) $D^2 = -K_S.D = 4$, $g(L) = 5, k_0 = 3$, $K_S^2 \leq 4$ *with* $D \sim -K_{S_5}$ *if* $K_S^2 = 4$.

Remark 3.2.7. If $S \cong \mathbb{P}^2$ or $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, then it is easily seen that we must be in case (b) of the proposition. Then we have $D \sim l$ if $S \cong \mathbb{P}^2$ and $D \sim l_i$ if $S \cong \mathbb{P}^1 \times \mathbb{P}^1$.

When $S \cong S_r$ tables 3.1 and 3.2 give the possible equivalence classes of D in the cases (a) and (b).[5]

Possibilities for D in the cases (c)-(f) are discussed in [Knu02, remark 3.8]. Here we just note that in most cases when we have a divisor L with an associated divisor D as in the proposition of type (c)-(f), then in most cases there also exists a divisor D' of type (a) or (b) associated to L. For example if $S \cong S_5$, then we must be in case (a),(b), or (f). The only big and nef divisor L with $g(L) \geq 2$ which is not of type (a) or (b) is $L \sim -2K_{S_5}$.[6]

We will now look at the numerical conditions the results of [Knu02] give for $\deg S \geq 7$. We first consider $S \cong \mathbb{P}^2$ and $S \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 3.2.8. *Let* $S \cong \mathbb{P}^2$ *and* $L \sim al$ *be a divisor on* S*. Then* L *is* k*-very ample if and only if* L *is birationally* k*-very ample if and only if* $a \geq k$.

Let $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ *and* $L \sim al_1 + bl_2$ *be a divisor on* S*. Then* L *is* k*-very ample if and only if* L *is birationally* k*-very ample if and only if* $a \geq k$ *and* $b \geq k$.

[5] Table 3.1 is equal to the table on top of p.5 in [Roc96]. It is not immediate that they should be equal since the table in [Roc96] lists divisors such that $D^2 = 0$ and $D.K_S = -2$, while table 3.1 lists nef divisors such that $D^2 = 0$ and $D.K_S = -2$. Table 3.2 lists nef divisors such that $D^2 = 1$ and $D.K_S = -3$. In this case there exists five non-nef divisors that satisfy $D^2 = 1$ and $D.K_S = -3$, such as $L \sim 7l - 3e_1 - 3e_2 - 3e_3 - 3e_4 - 3e_5 - e_6 - e_7 - e_8$. These have been excluded from the table.

[6] This is not immediate, but the proof is not particularly difficult either. It consists more or less of finding all L such that $-K_{S_5}.(L + K_{S_5}) = 4$ and $D.(L - D) > 4$ for all possible D in tables 3.1 and 3.2.

Table 3.1: Possible divisors D in case (a)

$a, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8$
$1, 1, 0, 0, 0, 0, 0, 0, 0$
$2, 1, 1, 1, 1, 0, 0, 0, 0$
$3, 2, 1, 1, 1, 1, 1, 0, 0$
$4, 3, 1, 1, 1, 1, 1, 1, 1$
$4, 2, 2, 2, 1, 1, 1, 1, 0$
$5, 2, 2, 2, 2, 2, 2, 1, 0$
$5, 3, 2, 2, 2, 1, 1, 1, 1$
$6, 3, 3, 2, 2, 2, 2, 1, 1$
$7, 3, 3, 3, 3, 2, 2, 2, 1$
$7, 4, 3, 2, 2, 2, 2, 2, 2$
$8, 3, 3, 3, 3, 3, 3, 3, 1$
$8, 4, 3, 3, 3, 3, 2, 2, 2$
$9, 4, 4, 3, 3, 3, 3, 3, 2$
$10, 4, 4, 4, 4, 3, 3, 3, 3$
$11, 4, 4, 4, 4, 4, 4, 4, 3$

Table 3.2: Possible divisors D in case (b)

$a, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8$	$a, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8$
$1, 0, 0, 0, 0, 0, 0, 0, 0$	$9, 4, 4, 4, 4, 2, 2, 2, 2$
$2, 1, 1, 1, 0, 0, 0, 0, 0$	$9, 5, 4, 3, 3, 3, 2, 2, 2$
$3, 2, 1, 1, 1, 1, 0, 0, 0$	$10, 5, 5, 3, 3, 3, 3, 3, 2$
$4, 3, 1, 1, 1, 1, 1, 1, 0$	$10, 5, 4, 4, 4, 3, 3, 2, 2$
$4, 2, 2, 2, 1, 1, 1, 0, 0$	$10, 5, 5, 3, 3, 3, 3, 3, 2$
$5, 2, 2, 2, 2, 2, 2, 0, 0$	$10, 6, 3, 3, 3, 3, 3, 3, 3$
$5, 3, 2, 2, 2, 1, 1, 1, 0$	$11, 5, 5, 4, 4, 4, 3, 3, 2$
$5, 3, 3, 1, 1, 1, 1, 1, 1$	$11, 6, 4, 4, 4, 3, 3, 3, 3$
$6, 3, 3, 2, 2, 2, 2, 1, 0$	$12, 5, 5, 5, 4, 4, 4, 4, 2$
$6, 3, 3, 3, 2, 1, 1, 1, 1$	$12, 5, 5, 5, 5, 4, 3, 3, 3$
$6, 4, 2, 2, 2, 2, 1, 1, 1$	$12, 6, 5, 4, 4, 4, 4, 3, 3$
$7, 3, 3, 3, 3, 2, 2, 2, 0$	$13, 6, 5, 5, 5, 4, 4, 4, 3$
$7, 4, 3, 3, 2, 2, 2, 1, 1$	$13, 6, 6, 4, 4, 4, 4, 4, 4$
$8, 3, 3, 3, 3, 3, 3, 3, 0$	$14, 6, 5, 5, 5, 5, 5, 5, 3$
$8, 4, 3, 3, 3, 3, 3, 1, 1$	$14, 6, 6, 5, 5, 5, 4, 4, 4$
$8, 4, 4, 3, 3, 2, 2, 2, 1$	$15, 6, 6, 6, 5, 5, 5, 5, 4$
$8, 5, 3, 2, 2, 2, 2, 2, 2$	$16, 6, 6, 6, 6, 6, 5, 5, 5$
$9, 4, 4, 4, 3, 3, 3, 2, 1$	$17, 6, 6, 6, 6, 6, 6, 6, 6$

Proof. Assume first that $S \cong \mathbb{P}^2$. Then it is shown in [BS93, (2.1.1)] that L is k-very ample if and only if $a \geq k$. By remark 3.2.7 we see that $L \sim al$ is not birationally k-very ample if and only if $(L-K).D = (L-K).l \leq k+2$.[7] Since $K_{\mathbb{P}^2} \sim -3l$, this condition is equivalent to $L.l = a \leq k-1$.

The proof for $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ is similar. □

We will now consider $S \cong S_1$.

Lemma 3.2.9. *Let $S \cong S_1$ and $L \sim al - be_1$ be a divisor on S. Then L is big and nef with $g(L) \geq 2$ if and only if $a > b+1 \geq 1$, $L \nsim 2l$, and $L \nsim 3l$.*

Proof. L is nef if and only if $a \geq b \geq 0$, by [Roc96, theorem 3.4]. L is big if and only if $a^2 > b^2$. Hence L is big and nef if and only if $a > b \geq 0$.

Assume that L is a big and nef divisor. If $a = b+1$ we get

$$g(L) = \frac{1}{2}L.(L+K)+1 = \frac{1}{2}((2b+1)+(-2b-3))+1 = 0.$$

If $a = b+2$, then

$$g(L) = \frac{1}{2}L.(L+K)+1 = \frac{1}{2}((4b+4)+(-2b-6))+1 = 2b$$

is less than 2 if and only if $b = 0$.

If $a = b+3$, then

$$g(L) = \frac{1}{2}L.(L+K)+1 = \frac{1}{2}((6b+9)+(-2b-9))+1 = 4b+1$$

is less than 2 if and only if $b = 0$.

If $a > b+3$, then $g(L) \geq 2$. □

Proposition 3.2.10. *Let $S \cong S_1$ and $L \sim al - be_1$ be a big and nef divisor with $g(L) \geq 2$.*

1. *If $b = 0$, then $L+K$ is birationally k-very ample if and only if $k \leq a-3$.*

2. *If $b \neq 0$, then $L+K$ is birationally k-very ample if and only if $k \leq a-b-2$.*

Proof. Let L be a big and nef divisor with $g(L) \geq 2$ that is birationally (k_0-1)-very ample but not k_0-very ample. Then we must be in case (a) or (b) of proposition 3.2.6. If we are in case (b), then $D^2 = 1$ and $D.K = -3$ gives $D \sim l$. If we are in case (a), then $D^2 = 0$ and $D.K = -2$ gives $D \sim l - e_1$.

Assume that we are in case (b), then we have

$$D.(L-D) = l.((a-1)l - be_1) = a-1 = k_0+1.$$

[7] If $a < 0$, this is trivial. If $a = 0$, then L is easily seen to be birationally 0-very ample but not birationally 1-very ample. If $a > 0$, then $L-K$ is nef and big with $g(L-K) \geq 2$ so the conditions of proposition 3.2.6 holds for $L-K$ (see lemma 3.2.11).

Hence $a = k_0 + 2$. If $b \neq 0$, then $D' = l - e_1 < L$ and

$$k_0 + 1 \leq D'.(L - D') = (l - e_1).((a-1)l - (b-1)e_1) = a - b \leq k_0 + 1,$$

thus we are also in case (a).

Assume that we are in case (a). Then $b \neq 0$ since $D \sim l - e_1 < L$ and $M = L - D$ is nef. We have

$$D.(L - D) = (l - e_1).((a-1)l - (b-1)e_1) = a - b = k_0 + 1.$$

Hence $L \sim (k_0 + 1 + b)l - be_1$.

This gives the if part of the proposition. That it also gives the only if part follows from theorem 3.2.5. $\square$

Lemma 3.2.11. *Let L be a nef divisor on a Del Pezzo surface. Then $L - K$ is big and nef with $g(L - K) \geq 2$.*

Proof. Since both L and $-K$ is nef, it is immediate that $L - K$ is nef. Since L is nef and $-K$ is ample we also gave that $L^2 \geq 0$, $-L.K > 0$, and $-K^2 > 0$. Hence $(L - K)^2 > 0$ so $L - K$ is big. Using $L^2 \geq 0$ and $-L.K > 0$ we get

$$g(L - K) = \frac{1}{2}(L - K).L + 1 > 1.$$

$\square$

Corollary 3.2.12. *Let $S \cong S_1$ and $L \sim al - be_1$ be a nef (equivalently a base point free) divisor. Then L is birationally k-very ample if and only if $k \leq a - b$.*

Proof. By the lemma $L' = L - K$ is nef and big with $g(L) \geq 2$ for all nef L. The result then follows from proposition 3.2.10. $\square$

Corollary 3.2.13. *Let $S \cong S_1$ and $L \sim al - be_1$ be a nef (equivalently a base point free) divisor. Then L is birationally k-very ample but not k-very ample if and only if $b < k \leq a - b$.*

Proof. Follows from [Roc96, theorem 4.6] and the preceding corollary. $\square$

This corollary gives the existence of line bundles that are $(k-1)$-very ample and birationally k-very ample but not k-very ample, namely $L \sim (k-1)l - ke_1$.

For $S \cong S_2$, as for $S \cong S_1$, we only have (up to symmetry) the possibilities $D \sim l$ and $D \sim l - e_1$ to consider. The proof of proposition 3.2.10 holds almost ad verbatim for $S \sim S_2$. We get the following results.

Proposition 3.2.14. *Let $S \cong S_2$ and $L \sim al - b_1e_1 - b_2e_2$ be a big and nef divisor with $g(L) \geq 2$ and $b_1 \geq b_2$.*

1. *If $b_1 = 0$, then $L + K$ is birationally k-very ample if and only if $k \leq a - 3$.*

2. If $b_1 \neq 0$, then $L+K$ is birationally k-very ample if and only if $k \leq a - b_1 - 2$.

Corollary 3.2.15. *Let $S \cong S_2$ and $L \sim al - b_1e_1 - b_2e_2$ be a nef (or equivalently a base point free) divisor with $b_1 \geq b_2$. Then L is birationally k-very ample if and only if $k \leq a - b_1$.*

Corollary 3.2.16. *Let $S \cong S_2$ and $L \sim al - b_1e_1 - b_2e_2$ be a nef (or equivalently a base point free) divisor with $b_1 \geq b_2$. Then L is birationally k-very ample but not k-very ample if and only if $b_1 + k \leq a < b_1 + b_2 + k$ or $b_2 < k \leq a - b_1$.*

Proof. Follows from [Roc96, theorem 4.6] and the preceding corollary. □

Remark 3.2.17. One can make similar statements as in the preceding results for Del Pezzo surfaces with $\deg S < 7$ using [Roc96, theorem 4.6] and proposition 3.2.6. There will arise no new technical difficulties, but the proofs will be much longer with many more cases to consider. For example for $\deg S = 1$ there is over a hundred different possibilities (up to symmetry) for D which have to be considered.

3.3 Scroll types and resolutions

We will now look at the scroll types that arise when $S \cong S_1$ and we are in case a) of proposition 3.2.6. We have already seen in the proof of proposition 3.2.10 that this case arises if and only if $L \sim (k_0 + 1 + n)l - ne_1$ with $n > 0$ and $D \sim l - e_1$. Then $L + K \sim (k_0 + n - 2)l - (n-1)e_1$. By [Roc96, corollary 4.6] we see that $L + K$ is base point free (since $k_0 \geq 1$ and $n > 0$). Hence we get a morphism

$$\phi_{L+K} : S \longrightarrow \mathbb{P}^{h^0(L+K)-1} = \mathbb{P}^{g(L)-1} \tag{3.1}$$

Since $h^0(D) = 2$ we see that $|D|$ is a pencil. We have seen (p. 21) that this gives a scroll $\mathcal{T} = \mathcal{T}(D)$ containing $\phi_{L+K}(S)$. We would like to determine its type. For this we need the following lemma.

Lemma 3.3.1. *Let $S \cong S_r$ and $L \sim al + \sum b_ie_i$ a divisor on S with $a, b_1, \dots, b_r \geq 0$. Then $\sum b_ie_i$ is fixed in L and $h^0(L) = \frac{1}{2}a(a+3) + 1$.*

Proof. Assume first that $\sum b_ie_i$ is fixed in L, then

$$h^0(L) = h^0(al) = \frac{1}{2}a(a+3) + 1,$$

where we have used Riemann-Roch and the fact that al is nef so $h^1(al) = h^2(al) = 0$.

We will now show that $\sum b_ie_i$ is fixed in L. By induction on the number of non-zero b_i it is enough to show that b_je_j is fixed in L. We have

$$L.e_j = -b_j.$$

Since e_j is an irreducible curve this gives that e_j is fixed in L by [Har77, proposition V.1.4]. Now

$$(L - e_j).e_j = -(b_j - 1),$$

so e_j is fixed in $L - e_j$ (if $b_j \geq 2$). Continuing this we get that e_j is fixed in $L - be_j$ for $b < b_j$. Hence $b_j e_j$ is fixed in L. $\square$

To find the scroll type it is enough to calculate

$$d_i = h^0(L + K - iD) - h^0(L + K - (i+1)D).$$

We have

$$L + K - iD \sim (k_0 + n - 2 - i)l - (n - 1 - i)e_1.$$

[Roc96, theorem 3.4] gives that this divisor if nef if and only if $i \leq n - 1$. This gives $h^1(L + K - iD) = h^2(L + K - iD) = 0$ for $i \leq n - 1$, so we can use Riemann-Roch to calculate $h^0(L + K - iD)$.

When $n - 1 < i \leq k_0 + n - 2$ we can calculate $h^0(L + K - iD)$ using lemma 3.3.1.

For $i > k_0 + n - 2$ we get $l.(L + K - iD) < 0$, so $h^0(L + K - iD) = 0$ since l is nef.

Combining the above we get

$$\begin{array}{rcl} d = d_0 & = & k_0 \\ d_1 & = & k_0 \\ \vdots & \vdots & \vdots \\ d_{n-1} & = & k_0 \\ d_n & = & k_0 - 1 \\ d_{n+1} & = & k_0 - 2 \\ \vdots & \vdots & \vdots \\ d_{k_0+n-2} & = & 1 \\ d_{k_0+n-1} & = & 0 \end{array}$$

Which gives the scroll type

$$(k_0 + n - 2, k_0 + n - 3, \ldots, n, n - 1). \tag{3.2}$$

Note that when $S \cong S_1$ and we are in case (b) of proposition 3.2.6 we get no similar results, for then $L + K$ is not even base point free and we do not even get a morphism ϕ_{L+K}.

We will now generalize some of the arguments made above while finding the scroll type. First note that the only fact about al we used in the proof of lemma 3.3.1 to show that $\sum b_i e_i$ was fixed was that $h^0(al) > 0$. Hence the same proof gives:

Proposition 3.3.2. *Let $S \cong S_r$ and $L \sim al - \sum b_i e_i$ be a divisor on S with $h^0(L) > 0$. Furthermore write $b'_i = \max\{0, b_i\}$ and $b^\dagger_i = \min\{0, b_i\}$. Then $\sum b^\dagger_i e_i$ is fixed in L and $h^0(L - \sum b'_i e_i) = h^0(L) > 0$.*

Secondly we showed that $al - be_i$ is not effective when $a < 0$. By similar arguments we can give a complete description of when $h^0(L) > 0$ for a divisor on S_1.

Proposition 3.3.3. *Let $al - be_1$ be a divisor on S_1. Then $h^0(L) > 0$ if and only if $a \geq 0$ and $a \geq b$.*

Proof. Assume that L is effective. Then since l and $l - e_1$ are nef divisors on S_1, we get $L.l \geq 0$ and $L.(l - e_1) \geq 0$. This gives $a \geq 0$ and $a - b \geq 0$.

For the converse assume that $a \geq 0$ and $a \geq b$. If $b < 0$, then $L \sim (al) + (-be_1)$ is linearly equivalent to the sum of two effective divisors and hence $h^0(L) > 0$. If $b \geq 0$, then L is nef by [Roc96, theorem 3.4] and hence $h^0(L) > 0$. $\square$

We will now look at scroll types that arise when $S \cong S_2$. We can handle this situation almost as we did $S \cong S_1$, there are only minor computational differences. This time also we are in case (a) or (b) of proposition 3.2.6. Case (b) happens if and only if $L \sim (k_0+2)l$, in which case $L+K$ is not even base point free and we do not get a morphism ϕ_{L+K}. Case (a) happens only if $D \sim l - e_1$ or $D \sim l - e_2$. We can assume without loss of generality that $D \sim l - e_1$. Then we get $L \sim (k_0 + 1 + n_1) - n_1 e_1 - n_2 e_2$ where $n_1 \geq n_2 \geq 0$ and $n_1 \geq 1$. For $L + K$ to be base point free we also have to assume $k_0 \geq n_2$ by [DR, theorem 3.4]. Then as above we get a morphism ϕ_{L+K} and a scroll $\mathcal{T} = \mathcal{T}(D)$ containing $\phi_{L+K}(S)$.

We have

$$L + K - iD \sim (k_0 + n - 2 - i)l - (n - 1 - i)e_1 - (n_2 - 1)e_2.$$

Using [Roc96, corollary 4.6] again we see that $L + K - iD$ is nef if and only if $i \leq n - 1$. As above this gives $h^0(L + K - iD)$ when $i \leq n - 1$.

For $i > n - 1$ we can calculate $h^0(L + K - iD)$ using proposition 3.3.2. We get

$$h^0(L + K - iD) = h^0((k_0 + n_1 - 2 - i)l - (n_2 - 1)e_2).$$

$A := (k_0 + n_1 - 2 - i)l - (n_2 - 1)e_2$ is nef for $i \leq k_0 + n_1 - n_2 - 1$ and we can calculate $h^0(A)$ using Riemann-Roch and $h^1(A) = h^2(A) = 0$. For $i > k_0 + n_1 - n_2 - 1$ we get $h^0(A) = 0$, since then $A.(l - e_2) > 0$ and $l - e_2$ is nef.

Combining the above we get

$$\begin{array}{rcl}
d = d_0 & = & k_0 \\
d_1 & = & k_0 \\
\vdots & \vdots & \vdots \\
d_{n_1-1} & = & k_0 \\
d_{n_1} & = & k_0 - 1 \\
d_{n_1+1} & = & k_0 - 2 \\
\vdots & \vdots & \vdots \\
d_{k_0+n_1-n_2-2} & = & n_2 + 1 \\
d_{k_0+n_1-n_2-1} & = & n_2 \\
d_{k_0+n_1-n_2} & = & 0
\end{array}$$

Which gives the scroll type

$$(\underbrace{k_0+n_1-n_2-1,\ldots,k_0+n_1-n_2-1}_{n_2},k_0+n_1-n_2-2,\ldots,n_1,n_1-1). \tag{3.3}$$

One can make similar statements for $S \cong S_r$ with $r \geq 3$, but in these cases the computations will be more difficult and there are more cases to consider as we will get different scroll types for each choice of D in table 3.1.

We will now take a closer look at the resolutions that arise from the inclusion $\phi_{L+K}(S) \subset \mathcal{T}$. We will use the techniques of [Sch86] and the results will be similar to those in [JK01, section 7].

We will assume that we are in case a) of proposition 3.2.6. To make things easier we will also assume that $L+K$ is very ample. Then the morphism

$$\phi_{L+K} : S \longrightarrow \mathbb{P}^{h^0(L+K)-1} = \mathbb{P}^{g(L)-1} \tag{3.4}$$

is an embedding. As above we get a canonical scroll $\mathcal{T} = \mathcal{T}(D)$ containing $S' := \phi_{L+K}(S)$. We will construct a resolution of the structure sheaf $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathcal{T}}$-module.

If $S \cong S_1$, S_2, or S_3, then $D \sim l-e_i$ and $|D|$ consists of the pullback, by the morphism

$$\pi : S \longrightarrow \mathbb{P}^2,$$

of the lines in $\mathbb{P}^2$ that passes through P_i. Hence every divisor in $|D|$ is a smooth rational curve.

For general S every divisor in $|D|$ is not a smooth rational curve. Take for example $S \cong S_4$ and $D \sim 2l-e_1-e_2-e_3-e_4$. Then $|D|$ consists of the pullback, by the morphism

$$\pi : S \longrightarrow \mathbb{P}^2,$$

of the curves of degree 2 in $\mathbb{P}^2$ that passes through P_1, P_2, P_3, and P_4. Generically (actually in all but three cases) these curves will be irreducible conics in $\mathbb{P}^2$ (their pullback will then be smooth rational curves). But we also have the three exceptional cases shown in figure 3.1, thus not all members of $|D|$ are smooth rational curves. See also [Roc96, proposition 3.1].

[Roc96, second table p. 5] shows that $|D|$ will always consist of only smooth rational curves except for a finite number of exceptions.

Since ϕ_{L+K} is an embedding we get that when $D_0 \in |D|$ is a smooth rational curve the image $\phi_{L+K}(D_0)$ is also a smooth rational curve. The degree of $\phi_{L+K}(D_0)$ is k_0-1 and

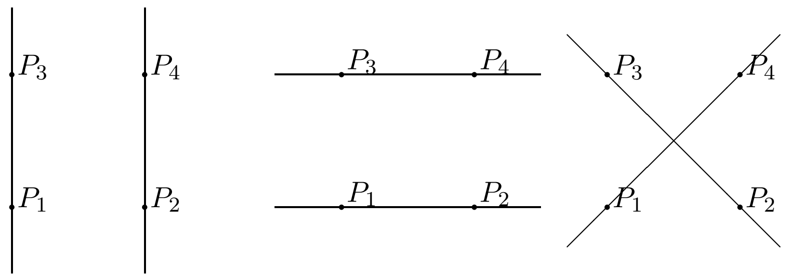
$$\overline{D_0} \cong \mathbb{P}^{h^0(L)-h^0(L-D)-1} = \mathbb{P}^{k_0-1}.$$

If we now argue as in the proof of [JK01, lemma 7.1] (using [Sch86, lemma 5.2] instead of [Sch86, lemma 4.2]) we get the following resolution[8]

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^{k_0-1}}(-(k_0-1))^{\beta_{k_0-2}} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^{k_0-1}}(-2)^{\beta_1} \\ \longrightarrow \mathcal{O}_{\mathbb{P}^{k_0-1}} \longrightarrow \mathcal{O}_{\phi_{L+K}(D_0)} \longrightarrow 0, \end{aligned}$$

[8]Note that the resolution gives $k_0 \geq 3$. We can also show this using the results of [Roc96], though the proof is much longer in this case. We will show how the proof goes for $S \cong S_1$ as an example. When $S \cong S_1$, we have $L+K \sim (k_0+n-2)l-(n-1)e_1$. Since we have assumed that $L+K$ is very ample, [Roc96, theorem 4.6] gives $k_0 \geq 3$.

Figure 3.1: Reducible curves of degree 2 in $\mathbb{P}^2$ passing through P_1, P_2, P_3, and P_4.



where

$$\beta_i = i\binom{k_0 - 1}{i+1}. \tag{3.5}$$

Then [Sch86, theorem 3.2] (compare [Sch86, corollary 4.4.i]) gives the following resolution of $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathcal{T}}$-module

$$\begin{aligned} 0 \longrightarrow \bigoplus_{j=1}^{\beta_{k_0-2}} \mathcal{O}_{\mathcal{T}}(-(k_0-1)\mathcal{H} + b^j_{k_0-2}\mathcal{F}) \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{\beta_i} \mathcal{O}_{\mathcal{T}}(-(i+1)\mathcal{H} + b^j_i\mathcal{F}) \\ \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{\beta_1} \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + b^j_1\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0. \end{aligned} \tag{3.6}$$

The problem now is to determine the b^j_i. As noted in [Sch86, remark 3.3][9] we may find linear equations that b^j_i satisfy by using the additivity of the Hilbert polynomials. We will find these equations as in the proof of [JK01, proposition 7.2(d)], i.e. we will use equation (1.2). For this we need the following lemma.

Lemma 3.3.4. *Set* $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(e_d)$*, with* $e_1, \ldots, e_d \geq 0$ *and* $f = e_1 + \cdots + e_d$*. Then with* $a \geq 0$ *and* $b \geq 0$ *we get*

$$h^0(\mathbb{P}^1, \mathrm{Sym}^a(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(b)) = \binom{a+d-1}{a}\left(\frac{af}{d} + b\right)$$

Proof. We first calculate $h^0(\mathbb{P}^1, \mathrm{Sym}^a(\mathcal{E}))$. We have

$$\mathrm{Sym}^a(\mathcal{E}) = \sum_{\substack{a_1+\cdots+a_d=a \\ a_1,\ldots,a_d \geq 0}} \mathcal{O}_{\mathbb{P}^1}(a_1e_1 + \cdots + a_de_d)$$

[9] Where there is a typo.

The average value of $a_1e_1 + \cdots + a_de_d$ is

$$a\frac{\sum e_i}{d} = a\frac{f}{d}.$$

Using that the number of summands is

$$\binom{a+d-1}{a}$$

we get

$$h^0(\mathrm{Sym}^a(\mathcal{E})) = \sum_{\substack{a_1+\cdots+a_d=a \\ a_1,\ldots,a_d\geq 0}} h^0(\mathcal{O}\mathbb{P}^1(a_1e_1+\cdots+a_de_d)) = \binom{a+d-1}{a}\left(\frac{af}{d}\right).$$

Twisting with b we get the wanted result. □

In our case $\mathcal{T} \cong \mathbb{P}(\mathcal{E})$ with $d = \deg \mathcal{T} = h^0(L+K) - h^0(L+K-D) = k_0$ and $f = h^0(L+K-D) = h^0(L+K) - k_0 = g(L) - k_0$.

Write

$$F_i = \bigoplus_{j=1}^{\beta_i} \mathcal{O}_{\mathcal{T}}(-(i+1)\mathcal{H} + b_i^j\mathcal{F}).$$

Then the additivity of the Hilbert polynomial gives

$$\chi(\mathcal{O}_{\mathcal{T}}(n\mathcal{H})) - \chi(S'_{\mathcal{T}}(n\mathcal{H})) = \sum_{i=1}^{k_0-2}(-1)^{i+1}\chi(F_i(n\mathcal{H})).$$

For large n we have $\chi(\mathcal{O}_{\mathcal{T}}(n\mathcal{H})) = h^0(\mathcal{O}_{\mathcal{T}}(n\mathcal{H}))$ and $\chi(F_i(n\mathcal{H})) = h^0(F_i(n\mathcal{H}))$, by [Har77, III.5.3] using that $\mathcal{H}$ is very ample on $\mathcal{T}$. We can calculate h^0 using (this is equation (1.2))

$$h^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(a\mathcal{H} + b\mathcal{F})) = h^0(\mathbb{P}^1, \mathrm{Sym}^a(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(b)).$$

For large n we also have

$$\begin{aligned}\chi(S'_{\mathcal{T}}(n\mathcal{H})) &= h^0(S'_{\mathcal{T}}(n\mathcal{H})) \\ &= h^0(n(L+K)) \\ &= n^2(g(L)-1) + \binom{n}{2}K.(L+K) + 1.\end{aligned}$$

Combining the above and using the lemma we get

$$\begin{aligned}&\binom{n+k_0-1}{n}\left(n\frac{g-k_0}{k_0}+1\right) - n^2(g-1) - \binom{n}{2}K.(L+K) - 1 \\ &= \sum_{i=1}^{k_0-2}(-1)^{i+1}\binom{n+k_0-i-2}{k_0-1}\left(\left(\frac{(n-i-1)(g-k_0)}{k_0}+1\right)\beta_i + \sum_{j=1}^{\beta_i} b_i^j\right)\end{aligned} \tag{3.7}$$

This is a polynomial equation in n that holds for all large n. Consequently it holds for all n. Inserting different values of n we get enough linear equations to determine $\sum_{j=1}^{\beta_i} b_i^j$ for all i.

Example We will examine $k_0 = 3$, which is the simplest case. It is also the only case where the equation above gives enough information to determine all the b_i^j.

When $k_0 = 3$ we have $\beta_1 = 1$ and the only unknown is b_1^1. Inserting $n = 2$ in the above equation gives

$$b_1^1 = -K.(L + K) - 4.$$

For $S \cong S_1$ and $L \sim (n+4)L - ne_1$ $(n > 1)$ this gives the following resolution

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + 2n\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

Chapter 4

Enriques Surfaces

In this chapter we consider Enriques surfaces. The first section consists of preliminary material. The next two sections contain new material.

In section 4.2 we study the function $\phi(C)$ introduced by Cossec. The function is defined purely lattice-theoretically, but it encodes much geometric information. We include several results that show the importance of this function. The main part of this section deals with existence of pairs $(C^2, \phi(C))$ with C an irreducible curve. To get existence results we restrict ourselves to unnodal Enriques surfaces. We get complete results for small values of $\phi(C)$ (table 4.1) and give a conjectural picture for large $\phi(C)$. It does not seem computationally feasible to give complete results for large $\phi(C)$.

The function $\phi(C)$ gives elliptic pencils that give scroll containing the image $\phi_C(S)$ for S an Enriques surface. In section 4.3 we study these scrolls. We first try to compute scroll types. Proposition 4.3.4 is an extension of [Cos83, lemma 5.2.8]. The proof shows how difficult it is to compute the scroll type for nodal Enriques surfaces. For unnodal Enriques surfaces the situation is easier. We will in fact show that for unnodal Enriques surfaces we are able to compute the scroll type almost completely just by knowing C^2 and $\phi(C)$. For small C^2 our results are listed in table 4.4. We end the section with some resolutions that arise from the inclusion of the projective model in the scroll.

4.1 Preliminaries

In this section we define Enriques surfaces and state results that we need later on in this chapter. Most of the material in this section will be taken from [Cos83, Cos85, CD89]. We start with the definition of an Enriques surface.

Definition 4.1.1. A *(classical) Enriques surface* is a smooth projective surface S such that $h^1(S, \mathcal{O}_S) = 0$, $2K_S = 0$, and $K_S \neq 0$ where K_S is the canonical divisor class.

Remark 4.1.2. An alternative definition of Enriques surface (which follows naturally from the proof of the classification theorem 1.1.2) is a surface with $\kappa(S) = 0$ and $b_2 = 10$. Since we work over the complex numbers our definition is equivalent to this one. But if one works over an algebraically closed field with characteristic 2, then these definitions are not equivalent. See [CD89, chapter 1] for details.

Note that $K = K_S$ is numerically equivalent to zero.

For examples of Enriques surfaces see [Cos83, 2.5], [Bǎd01, 10.16], [BPvdV84, V.23], and [CD89, 1.6].

Looking at the definitions and theorem 1.1.2 it is not surprising that there is a close relationship between Enriques surfaces and K3 surfaces. In fact K3 surfaces are double covers of Enriques surfaces, as the next result says.

Proposition 4.1.3. *[Bǎd01, propositions 10.14 and 10.15], [CD89, 1.3]. Let S be an Enriques surface. Then there exists an étale covering of degree 2, $\pi : S' \to S$, with S' a K3 surface and the structural group of π isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

Conversely let S'' be a surface that admits an étale and connected covering of degree 2, $\pi : S' \to S''$, with S' a K3 surface. Then S'' is an Enriques surface.

We also include the following important theorem before moving on to results we will be using later on.

Theorem 4.1.4. *[Bǎd01, theorem 10.17], [BPvdV84, VIII.17] Every Enriques surface admits an elliptic fibration.*

Riemann-Roch on Enriques surfaces looks as follows (with L a divisor on S)

$$h^0(L) + h^0(K - L) = \frac{1}{2}L^2 + 1 + h^1(L).$$

When L is effective and non-zero, this gives

$$\begin{aligned} h^0(L + K) &= \frac{1}{2}L^2 + 1 + h^1(L + K) \\ h^0(L) &= \frac{1}{2}L^2 + 1 + h^1(L) \end{aligned}$$

There is a special case in which $h^1(D)$ always vanishes.

Proposition 4.1.5. *[Cos83, proposition 1.3.1] Let C be an irreducible curve on an Enriques surface such that $C^2 > 0$. Then $h^1(D) = 0$ and*

$$\dim |C| = h^0(C) - 1 = \frac{1}{2}C^2$$

We include another vanishing result that we will need later on.

Proposition 4.1.6. *[Cos85, theorem 2.6] Let D be a big and nef divisor on an Enriques surface. Then*

$$h^1(D) = h^1(-D) = 0.$$

Remember that given a divisor D on a K3 surface with $D^2 \geq -2$, then either $|D| \neq \emptyset$ or $|-D| \neq \emptyset$. For a Enriques surface one must have $D^2 \geq 0$. That is given a divisor D on an Enriques surface with $D^2 \geq 0$, then either $|D| \neq \emptyset$ or $|-D| \neq \emptyset$ (see [BPvdV84, proposition VIII.16.1(ii)] for proof).

The Bertini theorem gives the following for Enriques surfaces (compare theorem 1.1.11)

Theorem 4.1.7. *[Cos83, theorem 1.5.1] Let L be an effective divisor without fixed component on an Enriques surface S. Then either*

1. *$L^2 > 0$ and the generic member of $|L|$ is an irreducible curve, or*
2. *$L^2 = 0$ and there exists an elliptic pencil $|P|$ and an integer $k \geq 1$ such that $L \sim kP$, $h^1(L) = k$, and every member of L is of the form $P_1 + \cdots + P_k$ where $P_i \in |P|$.*

Proposition 4.1.8. *[Cos83, proposition 1.5.2] Let D be an effective divisor on an Enriques surface S such that $D^2 > 0$. Then the moving part of $|D|$ is irreducible.*

The adjunction formula on an Enriques surface reduces to

$$g(D) = \frac{1}{2}D^2 + 1.$$

In particular $C^2 \geq -2$ for any irreducible curve C and $C^2 = -2$ if and only if C is a smooth rational curve. To be consistent with Cossec we will call a smooth rational curve a *nodal curve* in this chapter.

Let C be a nodal curve. Then $h^0(C) = 1$ since C has negative self-intersection and $h^0(C + K) = 0$ by Riemann-Roch (see [Cos83, proposition 1.6.1(ii)]).

An Enriques surface S is of *special type* if it contains an elliptic pencil $|P|$ and a nodal curve θ such that $P.\theta = 2$. By [Cos85, theorem 4.1] an Enriques surface is of special type if and only if it contains a nodal curve. We say that an Enriques surface S is *nodal* if it contains a nodal curve. An Enriques surface S is *unnodal* if it does not contain a nodal curve. A generic Enriques surface is unnodal.[1] We have the following useful result.

Proposition 4.1.9. *[Cos85, proposition 2.4] Let D be a big and nef divisor on an Enriques surface S. Then $|D|$ has no fixed components unless $|D| = |P + \theta|$ for an elliptic pencil $|P|$ and a nodal curve θ such that $P.\theta = 2$.*

In particular if S is unnodal, then every big and nef divisor is without fixed components.

Definition 4.1.10. [Cos83, definition 1.6.2.1] Let D be an effective non-zero divisor on an Enriques surface S. Then D is of *canonical type* if its support is connected, $D^2 = 0$, and D is nef.

A divisor of canonical type is said to be a *primitive divisor of canonical type* if there is no divisor D' such that $D = mD'$ for an integer $m \geq 2$.

One easily sees that a primitive divisor of canonical type is primitive lattice theoretically also.[2] Important properties of canonical and primitive divisors are summarized in [Cos83, 1.6.2.2-1.6.2.5]. For Enriques surfaces we have the following result.

[1] [BP83] shows this using the global Torelli theorem for K3 surfaces and proposition 4.1.3 above. It also follows from [Cos85, theorem 4.1].

[2] D gives an element $[D]$ of the Néron-Severi group which we have seen can be viewed as a lattice. In a lattice M a *primitive element* is an element such that $M/m\mathbb{Z}$ is a free abelian group. See also p.17.

Proposition 4.1.11. *[Cos83, propositions 1.6.4. and 1.6.8] Let D be a non-zero effective divisor with $D^2 = 0$ on an Enriques surface S. Then*

1. *there exists a divisor E of canonical type such that $D - E \geq 0$, and*
2. *if D is a primitive divisor of canonical type, then $|2D|$ is an elliptic pencil.*

The following proposition gives a converse to the last part of this proposition.

Proposition 4.1.12. *(Šafarevic) [Cos83, proposition 1.6.3] Let $|P|$ be an elliptic pencil on an Enriques surface S. Then there exists two primitive divisors of canonical type, E and E', satisfying the following properties:*

1. *$|P| = |2E| = |2E'|$, $E' \in |E + K|$.*
2. *$2E$ and $2E'$ are the only multiple fibers of $|P|$.*
3. *$h^0(E) = h^0(E') = h^0(E + E') = 1$.*

Note especially that the intersection number of an elliptic pencil with any divisor is even. Because of this proposition we will also call primitive divisors of canonical type for *halfpencils*.

We will now examine the Enriques lattice. On the surfaces we have studied earlier, K3 surfaces and Del Pezzo surfaces, numerical equivalence is equal to linear equivalence. On an Enriques surface this is no longer true. This is immediate since K is numerically equivalent to 0, but not linearly equivalent to 0. One can show that K is the only non-zero divisor (up to linear equivalence) that is numerically equivalent to 0. This is the second part of the following result.

Proposition 4.1.13. *[Cos83, theorem 2.3 and proposition 2.1] $\operatorname{Pic} S$ is generated by the class of nodal curves and irreducible curves of arithmetic genus zero.*

The torsion $\operatorname{Pic}^\tau S$ of $\operatorname{Pic} S$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and generated by the class of the canonical divisor.

Since K is algebraically equivalent to zero this result gives

$$\operatorname{NS} S = \operatorname{Num} S = \operatorname{Pic} S / \operatorname{Pic}^\tau S.$$

Theorem 4.1.14. *[Cos83, theorem 2.2], [Cos85, (1.1)–(1.3)], [CD89, proposition 2.5.7][3] All Enriques surfaces have isomorphic Picard lattices $\mathbb{E}$. In fact[4]*

$$\operatorname{NS} S \cong U \oplus E^8(-1) =: \mathbb{E}.$$

We also have the following description of $\mathbb{E}$:

$$\mathbb{E} = \bigoplus_{i=0}^{9} \omega_i$$

[3]There is a typo in [CD89, proposition 2.5.7]: the first 6 in the last row of the matrix should be a 4.
[4]See p. 17 for the definitions of U and E^8.

with the following intersection matrix

$$\begin{bmatrix} 10 & 7 & 14 & 21 & 18 & 15 & 12 & 9 & 6 & 3 \\ 7 & 4 & 9 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 14 & 9 & 18 & 28 & 24 & 20 & 16 & 12 & 8 & 4 \\ 21 & 14 & 28 & 42 & 36 & 30 & 24 & 18 & 12 & 6 \\ 18 & 12 & 24 & 36 & 30 & 25 & 20 & 15 & 10 & 5 \\ 15 & 10 & 20 & 30 & 25 & 20 & 15 & 12 & 8 & 4 \\ 12 & 8 & 16 & 24 & 20 & 16 & 12 & 9 & 6 & 3 \\ 9 & 6 & 12 & 18 & 15 & 12 & 9 & 6 & 4 & 2 \\ 6 & 4 & 8 & 12 & 10 & 8 & 6 & 4 & 2 & 1 \\ 3 & 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

We will call $\mathbb{E}$ the *Enriques lattice*. The second description of the Enriques lattice will be very useful to us later on.

The dual basis $\{r_i\}$ of $\{\omega_i\}$ consists of elements such that $r_i^2 = -2$. These r_i give Picard-Lefschetz reflections of the Enriques lattice. The *Weyl group* $W(E)$ of E is the group generated by these reflections. The *fundamental chamber* $\mathcal{C}$ of E is the subset of $L := E \otimes_{\mathbb{Z}} \mathbb{R}$ given by

$$\mathcal{C} := \{x \in L | x.r_i > 0 \text{ for all } i\}.$$

The closure $\bar{\mathcal{C}}$ of $\mathcal{C}$ is the convex polyhedral cone spanned by the vectors $\{\omega_i\}$, i.e.

$$\bar{\mathcal{C}} = \{\sum_i a_i\omega_i | a_i \geq 0\}.$$

Let

$$H := \{x \in L | x^2 > 0\}.$$

Then H is the disjoint union of two components H_1 and $H_2 = -H_1$, where H_1 is the component which contains the fundamental chamber. By the reflection $x \longmapsto -x$, if necessary, we may assume that H_1 is the positive cone. Then $\bar{\mathcal{C}}$ is contained in the big and nef cone.

[Bou68] shows that $\mathcal{C}$ is the fundamental domain for the action of $W(E)$ on E_1. Thus when considering nef divisors it will for many purposes be enough to consider $D \in \bar{\mathcal{C}}$.[5]

4.2 The function $\phi(C)$

We will now introduce a function $\phi(C)$ that will help us to classify projective models of polarized Enriques surfaces.

Definition 4.2.1. Let $\mathbb{E}_0' = \{E \in \mathbb{E} |\, E^2 = 0, E \neq 0\}$, and let C be a big divisor. Then

$$\phi(C) := \min_{E \in \mathbb{E}_0'} |E.C|.$$

[5]See [Cos85, section 1] and [CD89, sections 2.1-2.5] for a more complete treatment of $W(E)$ and $\mathcal{C}$.

Note that when C is an irreducible curve one actually has

$$\phi(C) = \min_{E \in \hat{\mathbb{E}}'_0} (E.C),$$

where $\hat{\mathbb{E}}'_0 = \{E \in \mathbb{E}_0 | E \text{ is an halfpencil}\}$.[6] Take a halfpencil E calculating $\phi(C)$. There is associated a unique pencil $2E = P$. Assume C base point free and $h^0(C - P) \geq 2$. Then this pencil gives a scroll containing the image of S given by the morphism $\phi(C)$ (see p. 21). Thus we get a *canonical scroll* associated to (almost) every polarized Enriques surface (S, C).

The next two propositions shows that classifying polarized Enriques surfaces according to the ϕ-function gives a lot more geometric information than one would suppose. As usual we write ϕ_C for the natural map

$$\phi_C : S \longrightarrow \mathbb{P}^{h^0(C)-1}$$

given by $|C|$.

Proposition 4.2.2. *(Cossec) Let C be an irreducible curve on an Enriques surface S. Then*

1. *$|C|$ has a base point if and only if $\phi(C) = 1$. In fact if $\phi(C) = 1$, then $|C|$ has exactly two base points of multiplicity 1.*

2. *ϕ_C is a birational map into a surface with at most rational double points as singularities if and only if $\phi(C) \geq 3$.*

Proof. This is just [CD89, theorem 4.4.1] and [CD89, theorem 4.6.1.]. The results are also contained in [Cos83] and [Cos85]. □

For $\phi(C) = 2$ the situation is a bit more complex. See [CD89, theorem 4.6.3 and proposition 4.7.1].

The next result shows the relationship between $\phi(C)$ and birational k-very ampleness. We will later on consider $\phi(C)$ and k-very ampleness.

Proposition 4.2.3. *(Knutsen[7]) Let L be a base point free divisor on an Enriques surface. Then L is birationally $(\phi(C) - 2)$-very ample but not birationally $(\phi(C) - 1)$-very ample.*

We will now consider the existence of pairs $(C^2, \phi(C))$ with C an irreducible curve. The next proposition gives some bounds.

---

[6] Proof: Let D be an element in $\mathbb{E}'_0$ such that $C.D = \phi(C)$ (using Riemann-Roch and the fact that C is nef we may assume D effective and $|C.D| = C.D$). We may write (by [CD89, theorem 3.2.1]) $D \sim E + \sum m_i R_i$ $(m_i \geq 0)$, where E is canonical and R_i are nodal curves. Then $E.C \leq D.C$, so we may assume $E = D$. It is easily seen that we may also assume E primitive. Then E is an halfpencil by proposition 4.1.11.

[7] Private correspondence 7. May 2002.

Proposition 4.2.4. *[CD89, corollary 2.7.1][8] For any C (with $C^2 > 0$) we have*

$$0 < (\phi(C))^2 \leq C^2.$$

Moreover there exists C such that we have $(\phi(C))^2 = C^2$.

We will look at which of the pairs $(C^2, \phi(C))$ satisfying the above inequalities actually exists. We will restrict ourselves to unnodal Enriques surfaces to make things simpler.[9] What makes the unnodal case simpler is proposition 4.1.9. This says that a big and nef divisor on an unnodal Enriques surface has no fixed components. Using that every big and effective divisor on an unnodal Enriques surface is nef together with theorem 4.1.7 we see that *every* big and effective divisor on an unnodal Enriques surface is linearly equivalent to an irreducible curve. Hence to show that a pair $(C^2, \phi(C))$ exists with C an irreducible curve it is enough to find a pair $(C^2, \phi(C))$ with C effective.

Since both C^2 and $\phi(C)$ are conserved by reflections we may furthermore assume that $C \in \bar{\mathcal{C}}$. As we have already noted we may assume that every divisor in $\bar{\mathcal{C}}$ is effective. It is actually very easy to calculate $\phi(C)$ for $C \in \bar{\mathcal{C}}$. By [CD89, lemma 2.7.1] we have

$$\phi(C) = C.\omega_9.$$

Proposition 4.2.5. *Let S be an unnodal Enriques surface, and let $\phi \geq 2$ be an even integer. Then there exists an irreducible curve C with $\phi(C) = \phi$ on S such that*

$$C^2 = \phi^2.$$

Proof. Pick an irreducible curve C such that $C \sim \frac{\phi}{2}\omega_1$. Then

$$\phi(C) = C.\omega_9 = \phi$$

and

$$C^2 = \phi^2.$$

□

For odd integers ϕ it is obvious that we never obtain equality in $(\phi(C))^2 \leq C^2$ since $\mathbb{E}$ is an even lattice. One would hope for the existence of pairs $(C^2, \phi(C)) = (\phi^2 + 1, \phi)$. This is not always the case. I have only been able to find pairs $(\phi^2 + 1, \phi)$ for $\phi = 1$ and 3. Numerical computations for $\phi \leq 25$ suggests the following conjecture.

[8] There is a slight misprint in the proof of [CD, corollary 2.7.1]. It is supposed to be

$$\max((\omega_i.\omega_9)/\omega_i^2) = ((\omega_2.\omega_9)/\omega_2^2) = 1.$$

[9] Since the general Enriques surface is unnodal this is no great restriction. Furthermore note that in the unnodal case all the restrictions we make below still holds: the "only" problem being that we do not necessarily have existence. I.e. if a pair $(C^2, \phi(C))$ exists on a nodal Enriques surface, then the same pair always exists on an unnodal Enriques surface but not necessarily conversely.

Table 4.1: Properties of $\phi(C)$ for $\phi(C) \leq 12$.

ϕ	$b(\phi)$	$c(\phi)$	$d(\phi)$
1	2	2	$\emptyset$
2	4	4	$\emptyset$
3	10	10	$\emptyset$
4	16	16	$\emptyset$
5	28	28	$\emptyset$
6	36	48	$\{38, 46\}$
7	54	54	$\emptyset$
8	64	70	$\{66, 68\}$
9	88	88	$\emptyset$
10	100	112	$\{102, 104, 106, 110\}$
11	130	54	$\{132, 134\}$
12	144	160	$\{146, 148, 150, 152, 156, 158\}$

Conjecture 4.2.6. *Let S be an unnodal Enriques surface, and let $\phi > 1$ be an odd integer. Let C be an irreducible curve on S such that $\phi(C) = \phi$. Then*

$$C^2 \geq \phi^2 + \phi - 2.$$

Furthermore for every ϕ there exists an irreducible curve C on S such that $\phi(C) = \phi$ and $C^2 = \phi^2 + \phi - 2$.

The last part of the conjecture is easily seen to be true. Just pick an irreducible curve C such that

$$C \sim \omega_0 + \frac{\phi - 3}{2}\omega_1.$$

The hard part is to show that this divisor actually computes the minimal possible value.

Another interesting problem to consider is whether there exists a number $c = c(\phi)$ such that for fixed $n \geq c$ (n even) there always exists an irreducible curve C with $C^2 = n$ and $\phi(C) = \phi$. Table 4.1[10] gives the value of $c(\phi)$ for $\phi \leq 12$. According to Maple there always exists such a c for $\phi \leq 100$. This gives the following conjecture.

Conjecture 4.2.7. *Let S be an unnodal Enriques surface. Then there exists a number $c = c(\phi)$ such that for fixed $n \geq c$ (n even) there always exists an irreducible curve C with $C^2 = n$ and $\phi(C) = \phi$.*

A proof of the conjecture will also probably at the same time give a upper bound to $c(\phi)$. A procedure for showing that $c(\phi)$ exists for fixed ϕ is given below. I have not been able to prove the conjecture for large ϕ except for rather rare classes of numbers. I have for example shown the conjecture for odd prime ϕ with 2 a quadratic non-residue

[10] Here $b(\phi) :=$ minimal value of C^2 such that there exists C with $\phi(C) = \phi$. And $d(\phi)$ is the set of integers C^2 between $b(\phi)$ and $c(\phi)$ such that there does not exist C with $\phi(C) = \phi$.

by considering $C \sim a_1\omega_1 + a_8\omega_8 + a_9\omega_9$ and $C \sim a_7\omega_7 + a_8\omega_8 + a_9\omega_9$. I also conjecture that $b(\phi) < c(\phi)$ when $\phi \geq 10$.

We will now say some words about how table 4.1 was computed. As noted above we may assume $C \in \bar{\mathcal{C}}$. Then we can write

$$C \sim a_0\omega_0 + a_1\omega_1 + a_2\omega_2 + a_3\omega_3 + a_4\omega_4 + a_5\omega_5 + a_6\omega_6 + a_7\omega_7 + a_8\omega_8 + a_9\omega_9,$$

with $a_i \geq 0$.

We see that

$$\phi(C) = C.\omega_9 = 3a_0 + 2a_1 + 4a_2 + 6a_3 + 5a_4 + 4a_5 + 3a_6 + 2a_7 + a_8$$

is independent of a_9. Thus

$$\phi(C + a\omega_9) = \phi(C).$$

We also have

$$(C + a\omega_9)^2 = C^2 + 2a\phi \equiv C^2 (\mod 2\phi).$$

Hence if we for every even congruence class modulo 2ϕ have found a C with C^2 in the congruence class and $\phi(C) = \phi$, then $c(\phi)$ exists. It is obvious that if we are interested in minimal values of C^2 we may assume $a_9 = 0$.

Example We will sketch the computations for $\phi = 6$. There exists 26 different equivalence classes $C \in \bar{\mathcal{C}}$ with $a_9 = 0$ such that $\phi(C) = 6$. Computing C^2 for all of these we see that $\omega_0 + \omega_6$, $3\omega_1$, and $2\omega_1 + \omega_7$ all give the minimal value $C^2 = 36$ (this is of course consistent with proposition 4.2.5).

Table 4.2: Finding $c(\phi)$ for $\phi = 6$.

C	C^2	$C^2 \mod 12$
$3\omega_1$	36	0
$\omega_0 + \omega_1 + \omega_8$	50	2
$2\omega_0$	40	4
ω_3	42	6
$\omega_1 + \omega_4$	44	8
$\omega_2 + 2\omega_8$	58	10

We will now show that $c(\phi)$ exists and compute it. Looking at the 26 divisors we have modulo $2\phi = 12$ one sees that every even congruence class is filled for some divisor C, so $c(6)$ exists. To compute it we have to find the divisors C with minimal self-intersection in each congruence class. These are given in table 4.2. We see that $c(6) = 48$.

We will make some comments on the ϕ function and k-very ampleness. For the rest of this section we will not necessarily assume that S is unnodal.

Proposition 4.2.8. *[Sze01, theorem 2.4] and [Knu01a, theorem 1.2]. Let L be an ample divisor on an Enriques surface. If L is k-very ample, then $\phi(L) \geq k+2$.*

If S is unnodal, then the converse is also true.

Comparing with proposition 4.2.3 we see that for unnodal Enriques surfaces k-very ampleness is equivalent to birationally k-very ampleness.

Proposition 4.2.9. *For $k = 0$ and $k = 1$ the converse statement of proposition 4.2.8 is true even if S is nodal.*

Proof. For $k = 0$ this is just [CD89, theorem 4.4.1].

For $k = 1$ this is just [CD89, theorem 4.6.1] by using the Nakai-Moishezon criterion.

Alternatively use [Knu01a, theorem 1.2]. Since L is assumed ample the Nakai-Moishezon criterion holds. Hence every effective divisor D with $D^2 = -2$ must satisfy $D.L > 0$. Thus the $D.L \leq k - 1$ case of [Knu01a, theorem 1.2] is not possible for $k < 2$. □

For $k \geq 2$ I am unsure whether the converse holds. Because of the nodal curve in [Knu01a, theorem 1.2] I do not think so, but I have been unable to come up with a counterexample.

Szemberg also has the following result.

Proposition 4.2.10. *[Sze01, theorem 2.4] Let L be an ample divisor on a smooth Enriques surface. Then for $n \geq k + 2$ the divisor nL is k-very ample.*

The result here is the best possible, which the divisor L of the next proposition shows.

Proposition 4.2.11. *Let L be an ample line bundle on S with $\phi(L) = 1$ (or equivalently an ample divisor with base points). Then $L \sim dE_1 + E_2$ for two halfpencils E_1 and E_2 with $E_1.E_2 = 1$.*

Such an ample divisor always exists if S is unnodal.

Proof. L is now given by [CD89, prop. 3.6.1], since $\phi(L) = 1$. We can not be in case (ii): for then $L.R = 0$, which contradicts L ample.

We now prove existence. Existence of E_1 and E_2 follows from [Cos85, theorem 3]. Since we have assumed S unnodal we see that $dE_1 + E_2$ is ample.[11] □

4.3 Scroll types and resolutions

Let C be an irreducible curve and P a pencil such that its halfpencil E computes $\phi(C)$. We want to study the canonical scroll $\mathcal{T}$ containing $\phi_C(S)$ and given by $|P|$.

To find possible scroll types we are interested in calculating $h^0(C - nP)$ for $n \geq 1$. We first look at $h^0(C - P)$. One easily sees that $C - P$ is effective (see lemma 4.3.1). Thus to calculate $h^0(C - P)$ it is enough to calculate $h^1(C - P)$. Our first proposition will extend [Cos83, lemma 5.2.8] to all values of $\phi(C)$. For this we need some lemmas.

[11] Since any effective divisor L with $L^2 > 0$ on an unnodal Enriques surface is ample (use the Hodge index theorem to show that the Nakai-Moishezon criterion is satisfied).

Lemma 4.3.1. *Let C be an irreducible curve on S such that $C^2 \geq 2$ and let $|P|$ be an elliptic pencil, with halfpencil E.*

Then the multiplicity m of E in C is bounded as follows

$$\left\lfloor \frac{C^2}{2E.C} \right\rfloor \leq m \leq \left\lfloor \frac{C^2}{E.C} \right\rfloor$$

Proof. We have

$$\left(C - \frac{C^2}{2E.C} E \right)^2 = 0.$$

Hence either $C - \left\lfloor \frac{C^2}{2E.C} \right\rfloor E \geq 0$ or $\left\lfloor \frac{C^2}{2E.C} \right\rfloor E - C \geq 0$. Since $C.(C - \frac{C^2}{2E.C} E) = \frac{C^2}{2}$ and C is nef, we get $C - \left\lfloor \frac{C^2}{2E.C} \right\rfloor E \geq 0$. This gives the lower bound.

For the upper bound note that

$$C.\left(C - \frac{C^2}{E.C} E \right) = 0.$$

The result then follows since C is nef. □

Note that this lemma says that for $n \leq \frac{C^2}{4\phi(C)}$ we have $h^0(C - nP) > 0$ and for $n \geq \frac{C^2}{2\phi(C)}$ we have $h^0(C - nP) = 0$. Also note that if S is unnodal the proof actually gives $m = \left\lfloor \frac{C^2}{2E.C} \right\rfloor = \left\lfloor \frac{C^2}{2\phi(C)} \right\rfloor$.

Lemma 4.3.2. *Let C be an irreducible curve on S such that $C^2 \geq 2$, let $|P|$ be an elliptic pencil with halfpencil E, let m be the multiplicity of E in C, and let θ be a nodal curve such that $E.\theta \geq 1$. Then $\theta.(C - mE) \geq -P.C$.*

Proof. We note that the multiplicity k of θ in $C - mE$ is less than $E.C$ since E is nef and thus $E.(C - mE - k\theta) = E.C - E.k\theta \geq 0$. Hence

$$\theta.(C - kE) \geq \theta.k\theta \geq -2E.C = -P.C,$$

where the first inequality follows from the fact that θ intersects every prime divisor unequal to θ non-negatively. □

For $P.C = 2$ or 4 we have $\theta.(C - mE) \geq -P.C/2$.[12] I believe, but have not been able to prove, that this also holds for larger values of $\phi(C)$. If it does then one can automatically get better bounds in lemma 4.3.3 and proposition 4.3.4.

The following lemma is similar (both in proof and statement) to [Cos83, lemma 4.7].

Lemma 4.3.3. *Let C be an irreducible curve on S such that $C^2 \geq 2$ and let $|P|$ be an elliptic pencil. Then for any effective divisor Δ such that $P.\Delta > 0$ we have*

$$C.\Delta \geq \left\lfloor \frac{C^2}{P.C} \right\rfloor - P.C.$$

[12] This follows from studying all the cases of [Cos83, theorem 4.2 and theorem 5.3.6].

For $P.C = 2$ or 4 we have the improved bound

$$C.\Delta \geq \left\lfloor \frac{C^2}{P.C} \right\rfloor - \frac{P.C}{2}.$$

Proof. Let E be a halfpencil of P, let m be the multiplicity of E in C, and Θ be a component of Δ such that $E.\Theta \geq 1$. If Θ contains a nodal curve θ such that $E.\theta \geq 1$, then we assume $\Theta = \theta$. If not, then we may assume that Θ contains no nodal curves. In the first case we get (using lemma 4.3.1 and lemma 4.3.2)

$$C.\Delta \geq C.\Theta \geq m - P.C \geq \left\lfloor \frac{C^2}{P.C} \right\rfloor - P.C.$$

In the latter case we get

$$C.\Delta \geq C.\Theta \geq m \geq \left\lfloor \frac{C^2}{P.C} \right\rfloor.$$

The improved bound for $P.C = 2$ and 4 is arrived at by the same argument using the remark right after lemma 4.3.2. □

Note that if S is unnodal, then the proof actually shows

$$C.\Delta \geq \left\lfloor \frac{C^2}{P.C} \right\rfloor.$$

When $P.C = 2$, we see that

$$\frac{C^2}{P.C} = \left\lfloor \frac{C^2}{P.C} \right\rfloor,$$

since C^2 is even. This will improve the bound in the next proposition.

We are now ready to prove the following

Proposition 4.3.4. *Let $|C|$ be an irreducible curve on S such that $\phi(C) \geq 2$, and $C^2 \geq 2\phi(C)(2\phi(C) - 1)$. If $|P|$ is an elliptic pencil such that $C.P = 2\phi(C)$, then $h^1(C - P) = 0$.*

For $\phi(C) = 2$ it is enough to assume $C^2 \geq 10$. For $\phi(C) = 3$ it is enough to assume $C^2 \geq 22$.

Proof. We argue as in the first half of the proof of [Cos83, lemma 5.2.8]. Let $|C - P|$ be the decomposition of $|C - P|$ into its moving part $|M|$ and fixed part F. Assume $h^1(C - P) \neq 0$. Then

$$h^0(M) = h^0(C - P) \geq \frac{(C - P)^2}{2} + 2 = \frac{C^2}{2} + 2 - P.C.$$

If $(C-P)^2 = C^2 - 4\phi(C) > 0$, then $h^0(M) \geq 3$ and $|M|$ is irreducible (by [Cos83, proposition 1.5.2]), especially $M^2 > 0$ and $h^1(M) = 0$. We have assumed $C^2 \geq 2\phi(C)(2\phi(C)-1)$ and $\phi(C) \geq 2$, so $(C - P)^2 = C^2 - 4\phi(C) > 0$ is obviously satisfied. Thus we get

$$M^2 = 2(h^0(M) - 1) \geq C^2 + 2 - 2P.C.$$

Using this we get

$$\begin{aligned} h^0(C) - 1 = \frac{C^2}{2} &\geq h^0(M+P) - 1 \\ &\geq \frac{(M+P)^2}{2} \\ &\geq \frac{C^2}{2} + 1 - P.C + P.M, \end{aligned}$$

which gives $P.M < P.C$ and thus $P.F > 0$. Hence there exists an irreducible component Θ of F such that $P.\Theta \geq 2$ and $P.M \leq P.C - 2$ (remember that every elliptic pencil intersects every divisor evenly). Then

$$\begin{aligned} h^0(C) - 1 = \frac{C^2}{2} &\geq h^0(M+P+\Theta) - 1 \\ &\geq \frac{(M+P+\Theta)^2}{2} \\ &\geq \frac{C^2}{2} + 1 - P.C + M.\Theta + P.M + P.\Theta + \frac{\Theta^2}{2}. \end{aligned}$$

This gives

$$\begin{aligned} P.C - 1 &\geq M.\Theta + P.M + P.\Theta + \frac{\Theta^2}{2} \\ &\geq \frac{M^2}{P.M} + P.\Theta + \frac{\Theta^2}{2} \\ &\geq \frac{M^2}{P.M} + \frac{\Theta^2}{2} + 2, \end{aligned}$$

where the first inequality follows from lemma 4.3.3. This gives

$$\begin{aligned} P.C - 2 &\geq \left\lfloor \frac{M^2}{P.M} \right\rfloor \\ &\geq \frac{M^2}{P.M} - \frac{P.M - 2}{P.M} \\ &\geq \frac{M^2}{P.C - 2} - \frac{P.C - 4}{P.C}, \end{aligned}$$

where we have used that $P.M$ is even and $P.M \leq P.C - 2$, and

$$C^2 + 2 - 2P.C \leq M^2 \leq \left(P.C - 1 - \frac{2}{P.C - 2}\right)(P.C - 2).$$

Thus if $C^2 > 2\phi(C)(2\phi(C) - 1) - 2$ we can not have $h^1(C - P) \neq 0$.

The last two statements follows from the $P.C = 2$ and 4 cases of lemma 4.3.3 (used on M). $\square$

If S is unnodal, then things are much easier than the proof of this proposition suggests and we get much better results. We have in fact that every effective and big divisor D on an unnodal Enriques surface S satisfies $h^1(D) = 0$.We have already noted that every effective divisor on an unnodal Enriques surface is nef. The result then follows from proposition 4.1.6.

With this in mind we can say quite a lot about possible scroll types for unnodal Enriques surfaces S. As usual we let C be an irreducible curve and $|P|$ be an elliptic pencil such that $P.C = 2\phi(C)$. We must restrict ourselves to C with C base point free and $h^0(C-P) \geq 2$, by [Sch86, 2.2] (see also p. 21). Then ϕ_C is always a birational morphism except for $C^2 = 8$ and C superelliptic. First of all note that for

$$n < \frac{C^2}{4\phi(C)}$$

$C - nP$ is effective and $(C-nP)^2 > 0$ (lemma 4.3.1). Riemann-Roch and proposition 4.1.6 then gives

$$h^0(C-nP) = \frac{1}{2}(C-nP)^2 + 1.$$

Since S is unnodal we also have

$$h^0(C-nP) = 0$$

for

$$n > \frac{C^2}{4\phi(C)}.$$

This gives the following proposition

Proposition 4.3.5. *Let S be an unnodal Enriques surface. Let C be an irreducible base point free curve, with ϕ_C birational, and $|P|$ be an elliptic pencil such that $P.C = 2\phi(C)$ and $h^0(C-P) \geq 2$. If $\frac{C^2}{4\phi(C)} \notin \mathbb{Z}$, then the scroll type associated to P is*

$$(\underbrace{\overbrace{\left\lfloor\frac{C^2}{4\phi(C)}\right\rfloor, \dots, \left\lfloor\frac{C^2}{4\phi(C)}\right\rfloor}^{\frac{C^2 - 4\phi(C)\left\lfloor\frac{C^2}{4\phi(C)}\right\rfloor}{2}+1}, \left\lfloor\frac{C^2}{4\phi(C)}\right\rfloor - 1, \dots, \left\lfloor\frac{C^2}{4\phi(C)}\right\rfloor - 1}_{2\phi(C)}).$$

Proof. We have already done most of the work. We just have to calculate the d_i ($i < \frac{C^2}{4\phi(C)}$) from the above expressions for $h^0(C-nP)$. Since $P \leq C$ we have $\left\lfloor\frac{C^2}{4\phi(C)}\right\rfloor \neq 0$[13] and we get

$$d_0 = d_1 = \cdots = d_{\left\lfloor\frac{C^2}{4\phi(C)}\right\rfloor - 1} = 2\phi(C)$$

[13]For if $\left\lfloor\frac{C^2}{4\phi(C)}\right\rfloor = 0$, then $C^2 < 4\phi(C)$. Proposition 4.2.4 gives $C^2 = 6$ with $\phi(C) = 2$ or $C^2 = 10$ with $\phi(C) = 3$. In both of these cases $h^0(C-P) = 0$ since S is unnodal.

and

$$d_{\left\lfloor \frac{C^2}{4\phi(C)} \right\rfloor} = \frac{C^2 - 4\phi(C)\left\lfloor \frac{C^2}{4\phi(C)} \right\rfloor}{2} + 1.$$

□

An interesting observation is that the scroll type is uniquely defined even if there exists several distinct pencils with halfpencils that compute $\phi(C)$.

When $\frac{C^2}{4\phi(C)} \in \mathbb{Z}$ it is a lot harder to find the scroll type. We still have

$$d_0 = d_1 = \cdots = d_{\left\lfloor \frac{C^2}{4\phi(C)} \right\rfloor - 2} = 2\phi(C),$$

but now we have

$$\left(C - \left\lfloor \frac{C^2}{4\phi(C)} \right\rfloor P\right)^2 = \left(C - \frac{C^2}{4\phi(C)} P\right)^2 = 0.$$

So we do not necessarily have $h^1(C - \frac{C^2}{4\phi(C)}P) = 0$.[14]

The last two non-zero d_i are as follows

$$d_{\frac{C^2}{4\phi(C)} - 1} = 2\phi(C) - h^1\left(C - \frac{C^2}{4\phi(C)}P\right)$$

and

$$d_{\frac{C^2}{4\phi(C)}} = h^1\left(C - \frac{C^2}{4\phi(C)}P\right) + 1.$$

So the problem of finding the scroll type is reduced to computing $h^1(C - \frac{C^2}{4\phi(C)}P)$. Write $C - \frac{C^2}{4\phi(C)}P = G$.

We will now show that $G \sim mF$ for some primitive divisor F of canonical type. Since we are assuming S unnodal every effective divisor is nef. Thus an effective divisor D with $D^2 = 0$ is of canonical type if and only if its support is connected. We can write G as a sum of distinct divisors G_i ($1 \leq i \leq k$) of canonical type. By [Cos83, 1.6.2.2 and 1.6.2.3] we may assume that $G_i = m_i F_i$ where F_i is primitive. Assume $i > 1$. Then $F_1.F_2 = 0$ and by Riemann-Roch either $F_1 - F_2$ or $F_2 - F_1$ is effective. By [Cos83, 1.6.2.4] $F_1 = F_2$, a contradiction. Hence $i = 1$.

Now

$$mF.C = \left(C - \frac{C^2}{4\phi(C)}P\right).C = \frac{C^2}{2}$$

gives

$$m = \frac{C^2}{2F.C}. \tag{4.1}$$

[14] In fact we have $h^1(C - \frac{C^2}{4\phi(C)}P) = 0$ if and only if $C - \frac{C^2}{4\phi(C)}P$ equals some primitive divisor F.

Since $m \geq 1$ we have $F.C \leq \frac{C^2}{2}$. By the definition of $\phi(C)$ we have $F.C \geq \phi(C)$. Thus we have only finitely many possibilities for m.

If m is even [Cos83, proposition 1.6.4 and theorem 1.5.1] gives

$$h^1\left(C - \frac{C^2}{4\phi(C)}P\right) = h^1(mF) = \frac{m}{2} = \frac{C^2}{4F.C}.$$

If m is odd, then $h^1(mF) < h^1((m+1)F)$.[15] Thus

$$h^1\left(C - \frac{C^2}{4\phi(C)}P\right) = h^1(mF) = \left\lfloor \frac{m}{2} \right\rfloor = \left\lfloor \frac{C^2}{4F.C} \right\rfloor.$$

Since m determines $h^1(C - \frac{C^2}{4\phi(C)}P)$ uniquely we get only finitely many different possible scroll types.[16]

We get a further restriction on the possible scroll types by noting that

$$\phi = E.C = E.\left(C - \frac{C^2}{4\phi(C)}P\right) = mE.F, \tag{4.2}$$

i.e. $m|\phi(C)$.

Until now we have only considered which scroll types are possible. For the unnodal case we can use the results of the previous section to get existence results as well. It is again the case $\frac{C^2}{4\phi(C)} \in \mathbb{Z}$ which is the hardest. We start with an example.

Example We will find the scroll types that exists when $C^2 = 60$ and S is an unnodal Enriques surface. This is the hardest and most interesting case with C^2 comparatively small. We will see that *all* the scroll types that are numerically possible by our computations above actually exists.

First of all note that $2 \leq \phi(C) \leq 7$ by proposition 4.2.4.[17] For $\phi(C) = 2, 4, 6$, and 7 the possible scroll types are given uniquely by proposition 4.3.5. That there a actually exists C with these scroll types and $C^2 = 60$ follows from table 4.1.

We now look at $\phi(C) = 3$. Then equation 4.2 gives $m = 1$ or $m = 3$. Assume $C \in \bar{\mathcal{C}}$. We also assume $P \sim 2\omega_9$.[18]

An easy calculation gives

$$C \sim 3\omega_8 + 7\omega_9$$

or

$$C \sim \omega_7 + 8\omega_9.$$

In the first case $mF \sim 3(\omega_8 - \omega_9)$. This divisor cannot be primitive so we must have $m = 3$. In the second case $mF \sim \omega_7 - \omega_9$. Then $m = 3$ contradicts $\omega_7.mF = 4$. Hence $m = 1$. Thus we see that both $m = 1$ and $m = 3$ are possible.

[15]For if $h^1(mF) = h^1((m+1)F)$ (m odd), we could reduce to $h^1(F) = h^1(2F)$, a contradiction.

[16]Of course this also follows from the fact that only two of the d_i are unknown and that the d_i form a non-increasing sequence.

[17]Remember that we are only interested in C being base point free so we exclude $\phi(C) = 1$.

[18]Since we get existence later on this assumption is valid. If we later on did not get existence then this assumption would not necessarily be valid.

Table 4.3: Scroll types when $C^2 = 60$.

$\phi(C)$	scroll type
2	$(7, 7, 7, 6)$
3	$(5, 4, 4, 4, 4, 4)$
3	$(5, 5, 4, 4, 4, 3)$
4	$(3, 3, 3, 3, 3, 3, 3, 2)$
5	$(3, 2, 2, 2, 2, 2, 2, 2, 2, 2)$
5	$(3, 3, 3, 2, 2, 2, 2, 2, 1, 1)$
6	$(2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1)$
7	$(2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$

It remains to consider $\phi(C) = 5$. We do this almost exactly as we did $\phi(C) = 3$. Equation 4.2 gives $m = 1$ or $m = 5$. We assume again $C \in \vec{\mathcal{C}}$ and $P \sim 2\omega_9$. Then there are three possibilities for C:

$$C \sim 5\omega_8 + \omega_9,$$
$$C \sim \omega_4 + 3\omega_9,$$

and

$$C \sim \omega_1 + \omega_7 + \omega_8 + 2\omega_9.$$

In the first case $mF \sim 5(\omega_8 - \omega_9)$. This gives $m = 5$. The second possibility gives $mF \sim \omega_4 - 3\omega_9$. Then $\omega_0.mF = 3$ gives $m = 1$. Thus both $m = 1$ and $m = 5$ are possible.

The scroll types we get are given in table 4.3.

Looking just at this example one may conjecture that *all* scroll types that are numerically possible by equations 4.1 and 4.2 exist. This is not true because one must also make sure that $F.C \geq \phi(C)$. Take for example $C^2 = 12$ and $\phi(C) = 3$. Then $\frac{C^2}{4\phi(C)} = 1$. Equation 4.2 gives $m = 1$ or 3. If $m = 3$ then equation 4.1 gives $F.C = 2$, which contradicts $\phi(C) = 3$. I am unsure whether all scroll types that are numerically possible by equations 4.1 and 4.2 with $F.C \geq \phi(C)$ exists, if one excludes the pathological cases where $(C - P)^2 = 0$.[19]

By arguing as in the example we get table 4.4. This table shows all the scroll types that exists for $C^2 \leq 30$. The pairs $(C^2, \phi(C))$ that are not in the table does not occur since we have assume C base point free and $h^0(C - P) \geq 2$. Since S is assumed unnodal all the C in the table are ample. The map ϕ_C has degree 1 for all C in the table except for the C in the first line. The C in the first line is superelliptic and gives a morphism

[19] By proposition 4.2.4 these are $C^2 = 8$ with $\phi(C) = 2$, $C^2 = 12$ with $\phi(C) = 3$, and $C^2 = 16$ with $\phi(C) = 4$. Take for example $C^2 = 8$ with $\phi(C) = 2$. In this case we have $(C - P)^2 = 0$. $m = 1$ gives $h^1(C - P) = 0$, which gives $h^0(C - P) = 1$. But we are only interested in $h^0(C - P) \geq 2$ so we have excluded this case. Note also that $m = 2$ gives C superelliptic by [CD89, theorem 4.7.1].

of degree 2. Note that for *every* isomorphism class of unnodal surfaces every line in this table exists.

We will end this chapter with a little bit about the resolutions we get from the inclusions $\phi_C(S) := S' \subset \mathcal{T}$.[20] We no longer assume that S is unnodal, but to make the resolutions nicer we will assume $h^1(C-P)=0$ and $\phi(C)\geq 3$. Then for any $P_0 \in |P|$ we have

$$\overline{P_0} \cong \mathbb{P}^{h^0(C)-h^0(C-P)-1} = \mathbb{P}^{2\phi(C)-1}.$$

Lang [Lan79] showed that the generic member of an elliptic pencil on S is a smooth elliptic curve. The proof of [JK01, lemma 7.1] works ad libatim in the Enriques surface case.[21] Thus for all $P_0 \in |P|$ we get a resolution of the type

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^{2\phi(C)-1}}(-2\phi(C)) \longrightarrow \mathcal{O}_{\mathbb{P}^{2\phi(C)-1}}(-2\phi(C)+2)^{\beta_{2\phi(C)-3}} \longrightarrow \cdots \\ \longrightarrow \mathcal{O}_{\mathbb{P}^{2\phi(C)-1}}(-2)^{\beta_1} \longrightarrow \mathcal{O}_{\mathbb{P}^{2\phi(C)-1}} \longrightarrow \mathcal{O}_{\phi_C(P_0)} \longrightarrow 0, \end{aligned}$$

where

$$\beta_i = i\binom{2\phi(C)-1}{i+1} - \binom{2\phi(C)-2}{i-1}. \tag{4.3}$$

As in the Del Pezzo case [Sch86, theorem 3.2] (see also [Sch86, corollary 4.4.i]) now gives a resolution of $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathcal{T}}$-module

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-2\phi(C)\mathcal{H} + (C^2/2 - 2\phi(C)-1)\mathcal{F}) \\ \longrightarrow \textstyle\bigoplus_{j=1}^{\beta_{2\phi(C)-3}} \mathcal{O}_{\mathcal{T}}(-(2\phi(C)-2)\mathcal{H} + b^j_{2\phi(C)-3}\mathcal{F}) \\ \longrightarrow \cdots \longrightarrow \textstyle\bigoplus_{j=1}^{\beta_1} \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + b^j_1\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0. \end{aligned} \tag{4.4}$$

By arguing as in chapter 3 we get the following analogue to equation (3.7)

$$\begin{aligned} \tbinom{n+2\phi(C)-1}{n}\left(n\tfrac{C^2/2-2\phi(C)+1}{2\phi(C)}+1\right) - n^2(n^2C^2/2+1) - 1 = \\ \textstyle\sum_{i=1}^{2\phi(C)-3}(-1)^{i+1}\tbinom{n-i-2\phi(C)-2}{2\phi(C)-1}\left(\left(\tfrac{(n-i-1)(C^2/2-2\phi(C)+1)}{2\phi(C)}+1\right)\beta_i + \sum_{j=1}^{\beta_i} b^j_i\right) + \\ (-1)^{2\phi(C)-1}\tbinom{n-1}{2\phi(C)-1}\left(\tfrac{(n-2\phi(C))(C^2/2-2\phi(C)+1)}{2\phi(C)} + C^2/2 - 2\phi(C)\right) \end{aligned} \tag{4.5}$$

[20] This will be very similar to what we did with Del Pezzos in the previous chapter so we will not include as much details.

[21] Just remember to disregard the first paragraph of the proof.

Table 4.4: Scroll types for $C^2 \leq 20$ with C base point free and $h^0(C-P) \geq 2$.

C^2	$\phi(C)$	scroll type
8	2	$(1,1)$
10	2	$(1,1,0,0)$
12	2	$(1,1,1,0)$
14	2	$(1,1,1,1)$
14	3	$(1,1,0,0,0,0)$
16	2	$(2,1,1,1)$
16	2	$(2,2,1,0)$
16	3	$(1,1,1,0,0,0)$
16	4	$(1,1,0,0,0,0,0)$
18	2	$(2,2,1,1)$
18	3	$(1,1,1,1,0,0)$
18	4	$(1,1,0,0,0,0,0,0)$
20	2	$(2,2,2,1)$
20	3	$(1,1,1,1,1,0)$
20	4	$(1,1,1,0,0,0,0,0)$
22	2	$(2,2,2,2)$
22	3	$(1,1,1,1,1,1)$
22	4	$(1,1,1,1,0,0,0,0)$
24	2	$(3,2,2,2)$
24	2	$(3,3,2,1)$
24	3	$(2,1,1,1,1,1)$
24	3	$(2,2,1,1,1,0)$
24	4	$(1,1,1,1,1,0,0,0)$
26	2	$(3,3,2,2)$
26	3	$(2,2,1,1,1,1)$
26	4	$(1,1,1,1,1,1,0,0)$
28	2	$(3,3,3,2)$
28	3	$(2,2,2,1,1,1)$
28	4	$(1,1,1,1,1,1,1,0)$
28	5	$(1,1,1,1,1,0,0,0,0,0)$
30	2	$(3,3,3,3)$
30	3	$(2,2,2,2,1,1)$
30	4	$(1,1,1,1,1,1,1,1)$
30	5	$(1,1,1,1,1,1,0,0,0,0)$

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