Valuation of Guaranteed Investment Contracts

Cand. Scient. Thesis in Statistics
Mathematical Finance

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January 20, 2003

This thesis is written with the use of LaTeX2ε, reportstyle,
11 point text, a4wide layout and twoside format.
Numerical results are provided with the use of the statistical
package S-Plus6.

Acknowledgements

Especially, I want to thank my supervisor, Trygve Nilsen, for a instructive and inspiring subject. The freedom I was given forced me to get a perspective and really think through my own way the various problems occuring. In the beginning many dead ends were encountered but when it comes to my own learning I am convinced this was the right way to go. I appreciate him always being available and encouraging when things did not work out as they should.
The other employees at Kroepelin have also been helpful and understanding when I knocked on their door.

Many thanks also to my fellow students at Kroepelin for their contributions making the (sometimes long) lunch-breaks memorable, especially "Bondebridge" will never be the same.

Finally, many thanks to my family for their support during the years, and especially Kari for her moral support and always being positive.

Contents

Introduction

Guaranteed investment products have over the last decade made its entrance into the world financial markets. In general these products have been concentrated around rate of return guarantees on investments in stocks, funds and indices[1]. Guarantees where the underlying rate of return is the short-term interest rate have also been popular and we try to separate guarantees where the underlying rate of guarantee is either the rate of return on "stock-like" investments or the short-term interest rate. Today financial calculus is also used to price contracts on foreign currencies, electric power and contracts in many other markets.

Guaranteed contracts obviously find applications in other parts of our lifes than investments in "traditional" financial assets as stocks, bonds etc. Guarantees in life insurance is not a new concept but the development in financial theory has done it possible to price guarantees more consistently. In traditinal life insurance the guarantees have in practice been set far below the short-term interest rate at contract inception, i.e. the guarantees have been far out-of-money or in other words almost worthless. Companies in several countries encountered then severe problems in the 90's after neglecting the impact of such interest-rate guarantees, as the fall of the high short-term interest rate from the late 80's made the relatively high guarantees valuable or in-the-money. Obviously such interest rate guarantees may have substantial market values.

In order to price these guarantees the financial theory, which developed after the famous papers done by Black and Scholes [7] in 1973 and by Merton [18] the very same year, has shown to be a valuable tool. This theory uses Itô-diffusions[2] and geometric Brownian motion, GBM, to model price processes such as stocks, indices etc. A GBM-model turns out to be very handy when trying to price options on underlying price processes and the price of such options is central in valuing guarantees. But the simplicity of the GBM-model also seem to exhibit some weaknesses with regard to empiral observations of the price processes, especially the log-prices of many stocks seem to be heavytailed and skewed. This is not properties compatible with the GBM-model which is a log-normal process. More advanced models have thus been proposed in order to model heavy tails and skewness, but these do not hold the simplicity of the GBM-model. In this thesis we will therefore concentrate on the "classic" financial theory and the GBM-model is therefore preferred.

[1] Also called Long Guarantees
[2] See Appendix A

As mentioned above option pricing is central in order to value guaranteed contracts. A guaranteed contract gives the investor a guaranteed amount at maturity of the contract, in addition the contract gives the investor a **positive** stochastic amount if e.g. the stock or index has a terminal value above the initial value[3]. We also consider contracts where this additional rate of return is path-dependent or partly path-dependent, in the sense that we consider the whole or parts of the trajectory of the underlying process and not only its terminal value.

The recent years banks and financial institutions have offered products with an upper bound on the stochastic amount, i.e. contracts with a maximal rate of return, and products where the invested amount is not entirely guaranteed. In order to bound the probability space of the rate of return, **barrier options** and **collar options** have been implemented into the contracts. These options have the property of lowering the value of the option-part and thereby the entirely guaranteed contract. The terminal value can also be calculated as a **mean** of e.g. the index value in a period before termination of the contract, but then we encounter the problem of pricing an **Asian option**. These are the rate of return structures which are considered in this thesis, other structures which do not guarantee the initial amount or with coupon bearing bonds have also been popular but are not considered here.

The guaranteed contract consists of the guaranteed amount, which value can be represented as a certain number of zero coupon bonds[4], and the positive stochastic amount, which can be represented as a certain number of options on the underlying process, e.g. options on a index.

$$\text{Guaranteed Contract} = \text{Zero Coupon Bonds} + \text{Options}$$

The financial calculus presented in Chapter 1 and in Appendix A gives us a tool to value the guaranteed contract above consisting of bonds and options.
We will consider options which can not be exercised at any other time than at the contract end, named European style options. This is in the spirit of the contracts considered being investment and savings products with a relatively long horizon and is not meant for the typical investor with a short investment horizon.

So in this thesis we present the financial calculus which has developed over the recent thirty years and some of the work done on interest-rate theory and derivative theory. We then define various guarantees where the underlying rate of return is the rate of return on a stock, mutual fund or index when these processes are modelled as GBM, and guarantees on the short-term interest rate.

A summary of the content is as follows:

Chapter 1 Contains general theory with important financial definitions like contingent claims, numeraire, arbitrage, completeness and combines several results to the important

[3]Priced with a European option
[4]A bond which does not pay a coupon rate

pricing formula in section 1.4. This is the tool we need to valuate options and bonds in the preceding chapters. This formula gives the "fair" price of different claims as an expectation under a equivalent probability measure Q. This martingale-based theory was developed by Harrison & Kreps [11] in 1979 and Harrison & Pliska [12] in 1981.

Chapter 2 Here we define the zero coupon bond and other interest rates used in the financial market. In section 2.3 we use the pricing result from Chapter 1 to price the zero coupon bond. Stochastic models of the short-term interest rate are also presented.

Chapter 3 This is the most extensive chapter and values different options connected to the eventually positive stochastic rate of return generated. Even if they are different they still have something in common, they are all of "European" style since the only time they can be exercised is at the predeterminated maturity time. European, Collar, Asian and Barrier options are presented, with degree of difficulty in (probably) that order.

Chapter 4 Here the guaranteed contracts considered are defined. We show how the contracts consist of a sum of bonds and options, and we present the market prices in percent of the invested amount.

Chapter 5 Finally some numerical results are presented. We are content with a few remarks regarding these since they already have been heavily analyzed by existing literature.

Chapter 1

General Theory

First we give a brief introduction to a financial market modelled by Brownian motion, that is we consider a continous-time market model. Combining the results in this chapter gives us an valuation formula and we get some very appealing analytic results regarding pricing of different derivatives.
Most of the chapter is taken from Øksendal [24], except of section 1.5 which basicly is from Bjørk [6] and Musiela & Rutkowski [20].

1.1 Market and Self-Financing Portfolios

Definition 1.1.1 *A* **market** *is an* $\mathcal{F}_t^{(m)}$*-adapted (n+1)-dimensional Itô process* $S(t) = (S_0(t), S_1(t), \ldots, S_n(t))$; $\quad 0 \le t \le T$ *which we will assume has the form*

$$dS_0(t) = r(t,\omega)S_0(t)dt, \qquad S_0(0) = s \tag{1.1}$$

where r(t) is a given, possibly stochastic, process, usually assumed to be the price process of a locally risk free asset. S_0 *describes a bank deposit or a money account with the stochastic short rate of interest r(t).*

and

$$\begin{aligned} dS_i(t) &= \mu_i(t,\omega)dt + \sum_{j=1}^{m} \sigma_{ij}(t,\omega)dW_j(t) \\ &= \mu_i(t,\omega)dt + \sigma_i(t,\omega)dW(t) \ ; \qquad S_i(0) = x_i, \end{aligned} \tag{1.2}$$

where σ_i *is row number* i *of the* $n \times m$ *matrix* $[\sigma_{ij}]$*;* $\quad 1 \le i \le n \in \mathbf{N}$

m refers to the number of Wiener processes generating the uncertainty. These processes may e.g. be assumed to be independent or correlated.

A special case appears when r is a deterministic constant, then we can interpret $S_0^{-1}(t)$ as the price of a bond at time t.

We think of $S_i(T)$ as the price of some security or asset.

Definition 1.1.2 *The market $S(t)_{t\in[0,T]}$ is called* **normalized** *if $S_0(t)\equiv 1$*

The market can be normalized by defining

$$Z(t)=\frac{S(t)}{S_0(t)}=S_0^{-1}(t)S(t)=(1,Z_1(t),\ldots,Z_n(t)) \tag{1.3}$$

where $Z(t)$ can be viewed as the **discounted price process**
By normalization of the market with the price $S_0(t)$, i.e. the safe investment, we get the prices of the other securities as units of $S_0(t)$. We regard the price of $S_0(t)$ as the *unit* price or the *numeraire*. Solving the differential equation (1.1) we get the solution

$$S_0(t)=\exp\Big(\int_0^t r(s)ds\Big), \qquad S_0(0)=1$$

Thereby we get

$$P(t):=S_0^{-1}(t)=\exp\Big(-\int_0^t r(s)ds\Big)>0 \quad \forall t\in[0,T] \tag{1.4}$$

and the dynamics of the normalizations (by the Itô formula)

$$\begin{aligned} dZ_i(t) &= d(P(t)S_i(t))=dP(t)S_i(t)+P(t)dS_i(t)\\ &= -r(t)P(t)S_i(t)dt+P(t)dS_i(t)\\ &= P(t)[dS_i(t)-r(t)S_i(t)dt] \end{aligned} \tag{1.5}$$

Definition 1.1.3 *Let the (n+1)-dimensional price process $S(t)_{t\in[0,T]}$ be given*

1. *A* **portfolio** *strategy is any $\mathcal{F}_t^{(m)}$-adapted (n+1)-dimensional stochastic process*

$$h(t)=h(t,S(t))=(h_0(t,S(t)),h_1(t,S(t)),\ldots,h_n(t,S(t))); \qquad 0\le t\le T. \tag{1.6}$$

   The portfolio h is said to be **Markovian** *when it is on this form.*

2. *The* **value process** *V^h corresponding to the portfolio h is given by*

$$V^h(t)=\sum_{i=0}^n h_i(t)S_i(t) \tag{1.7}$$

   with the corresponding **discounted value process** *to the portfolio h*

$$V^z(t;h) = \frac{V(t,h)}{S_0(t)}. \tag{1.8}$$

3. *A portfolio h is said to be* **self-financing** *if*

$$dV^h(t) = \sum_{i=0}^{n} h_i(t)dS_i(t) \tag{1.9}$$

i.e. if

$$dV^h(t) = h(t)dS(t) \tag{1.10}$$

That is when the market value of the old portfolio equals the purchase value of the new portfolio.

4. *For a given portfolio h the corresponding* **relative portfolio** *u is given by*

$$u_i(t) = \frac{h_i(t)S_i(t)}{V^h(t)}, \quad i = 0, 1, \ldots, n \tag{1.11}$$

and

$$\sum_{i=0}^{n} u_i(t) \equiv 1$$

Remark 1 Note that if $h(t)$ is a self-financing portfolio for $S(t)$ and

$$V^z(t;h) = h(t)Z(t) = h(t)P(t)S(t) = P(t)V^h(t)$$

is the value process of the *normalized* market, then we have by Itô 's formula, (1.5) and (1.10)

$$\begin{aligned}
dV^z(t,h) &= d(P(t)V^h(t)) = P(t)dV^h(t) + dP(t)V^h(t) \\
&= P(t)h(t)dS(t) - r(t)P(t)V^h(t)dt \\
&= P(t)h(t)dS(t) - r(t)P(t)h(t)S(t)dt \\
&= P(t)h(t)[dS(t) - r(t)S(t)dt] \\
&= h(t)dZ(t) \qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.12)
\end{aligned}$$

The portfolio $h(t)$ is self-financing even for the normalized market.

Remark 2 If the portfolio "weights" $h_1(t), h_2(t), \ldots, h_n(t)$ already are chosen, we can make the portfolio $h(t)$ self-financing by choosing the "right" $h_0(t)$. If we combine (1.7) and (1.10) by solving (1.10) we get

$$\begin{aligned} V^h(t) &= \sum_{i=0}^{n} h_i(t)S_i(t) = h_0(t)S_0(t) + \sum_{i=1}^{n} h_i(t)S_i(t) && (1.13)\\ &= V(0) + \int_0^t \sum_{i=0}^{n} h_i(s)dS_i(s) && (1.14)\\ &= V(0) + \int_0^t h_0(s)dS_0(s) + \sum_{i=1}^{n} \int_0^t h_i(s)dS_i(s) && (1.15) \end{aligned}$$

Put

$$Y_0(t) = h_0(t)S_0(t) \tag{1.16}$$

Then we get from (1.13) and (1.15) that

$$dY_0(t) = d(h_0(t)S_0(t)) = r(t)Y_0(t)dt + dA(t) \tag{1.17}$$

where

$$A(t) = \sum_{i=1}^{n} \Big(\int_0^t h_i(s)dS_i(s) - h_i(t)S_i(t) \Big)$$

and

$$\begin{aligned} d\Big(\int_0^t h_0(s)dS_0(s) \Big) &= h_0(t)dS_0(t) = h_0(t)r(t)S_0(t)dt \\ &= r(t)Y_0(t)dt; \qquad dV(0) = 0 \end{aligned}$$

Using $\exp(-\int_0^t r(s)ds) = P(t)$ as integrating factor we get the solution of (1.17) by

$$P(t)Y_0(t) - P(0)Y_0(0) = \int_0^t P(s)dA(s)ds; \qquad P(0) = 1, \quad Y_0(0) = h_0(0)S_0(0) = h_0(0)$$

which gives

$$h_0(t) = P(t)Y_0(t) = h_0(0) + \int_0^t P(s)dA(s)ds$$

Solving the integral in the last equation by integration by parts we get

$$
\begin{aligned}
h_0(t) &= h_0(0) + P(t)A(t) - P(0)A(0) - \int_0^t A(s)dP(s) \\
&= h_0(0) + \sum_{i=1}^{n} h_i(0)S_i(0) + P(t)A(t) + \int_0^t r(s)A(s)P(s)ds \\
&= V(0) + P(t)A(t) + \int_0^t r(s)A(s)P(s)ds
\end{aligned} \tag{1.18}
$$

Therefore if we already have chosen e.g. the number of stocks, $h_1(t), h_2(t) \ldots, h_n(t)$, in the portfolio; then we can make it self-financing by investing $h_0(t)$ according to (1.18) in the bank.

Remark 3 Note that the changes in the wealth are due to capital gains. For simplicity we assume that the securities do not generate dividends.

1.2 Admissibility, Arbitrage and Martingale Measures

Definition 1.2.1 *A portfolio h(t), which is self-financing, is said to be* **admissible** *if the corresponding value process $V^h(t)$ is a.s. (almost surely) lower bounded.*

This is a natural condition since in real life the investors are "tied" to the creditors and debt accumulation. Without this condition it could be possible to generate any terminal value $V^h(t)$ from $V(0) = 0$ with a risky price process modelled as Brownian motion. See [24], example on p251.

Definition 1.2.2 *An* **arbitrage portfolio** *is a self-financed portfolio with the following properties*

(i) $V(0) = 0$

(ii) $P(V^h(T) > 0) > 0$

(iii) $P(V^h(T) < 0) = 0$

If no arbitrage portfolios exists we say that the market is free of arbitrage.

Later on we will assume that the respective market is arbitrage free; this is very appealing from a economic point of view since the existence of arbitrage portfolios is no market equilibrium. We have also assumed that all investors prefer more money to less and thereby will "exploit" these possibilities of making money out of nothing. This will (hopefully) create a market equilibrium.

Definition 1.2.3 *The probability measure Q is called a* **martingale measure** *if*

(i) $P \sim Q$

(ii) The discounted price process is a Q local martingale <u>or</u>

(iii) The discounted price process is a Q martingale

If we denote it a Q-martingale we say that we have a **strong martingale measure**

Definition 1.2.4 *Consider a given martingale measure Q and a self-financing portfolio h. Then h is called Q-**admissible** if $V^z(t;h)$ is a Q-martingale.*

Since by definition $Z(t)$ is a Q-martingale and since the $V^z(t)$-process is the stochastic integral of h with respect to $Z(t)$, we see that every sufficiently integrable self-financing portfolio is in fact admissible.

Lemma 1.2.1 *Assume there exists a martingale measure Q on $\mathcal{F}_T^{(m)}$ such that $P \sim Q$. Then the market $\{S(t)\}_{t\in[0,T]}$ has no arbitrage in the sense that there exists no Q-admissible portfolio.*

Conversely if the market $S(t)_{t\in[0,T]}$ has no arbitrage, then there exists an equivalent martingale measure Q.

For proof see e.g. [24] p253-254 and references therein.

1.3 Attainability and Completeness

In this section we will assume that that the market is free of arbitrage, i.e. there exists an equivalent martingale measure Q.
We will now give some results which are important in pricing theory and regarding attainability and completeness.

The following important lemma is taken from Yor [23], Proposition 17.1

Lemma 1.3.1 *Suppose a process $u(t,\omega) \in \mathcal{V}^m(0,T)$ satisfies the condition*

$$E\Big[\exp\Big(\frac{1}{2}\int_0^T u^2(s,\omega)ds\Big)\Big] < \infty \tag{1.19}$$

Define then the measure $Q = Q_u$ on $\mathcal{F}_T^{(m)}$ by

$$\frac{dQ(\omega)}{dP(\omega)} = \exp\Big(-\int_0^T u(t,\omega)dW(t) - \frac{1}{2}\int_0^T u^2(t,\omega)dt\Big) \tag{1.20}$$

Then

$$\widetilde{W}(t) := \int_0^t u(s,\omega)ds + W(t) \tag{1.21}$$

is an $\mathcal{F}_t^{(m)}$-martingale and an $\mathcal{F}_t^{(m)}$-Brownian motion w.r.t. Q and any $F \in L^2(\mathcal{F}_T^{(m)}, Q)$ has a unique representation

$$F(\omega) = E^Q[F] + \int_0^T \phi(t,\omega) d\widetilde{W}(t) \tag{1.22}$$

where $\phi(t,\omega)$ is an $\mathcal{F}_t^{(m)}$-adapted process such that

$$E^Q\Big[\int_0^t \phi^2(t,\omega)dt\Big] < \infty. \tag{1.23}$$

Lemma 1.3.2 *Let $Z(t) = \frac{S(t)}{S_0(t)} = P(t)S(t)$ be the normalized price process as in (1.3). Suppose $h(t)$ is an admissible portfolio for the market $\{S(t)\}_{t\in[0,T]}$ with value process*

$$V^h(t) = h(t)S(t) \tag{1.24}$$

Then $h(t)$ is also an admissible portfolio for the normalized market $\{Z(t)\}$ with value process

$$V^z(t;h) = h(t)Z(t) = h(t)S_0^{-1}(t)S(t) = P(t)V^h(t) \tag{1.25}$$

and vice versa.
Assuming a self-financing portfolio h(t) we get

$$V^h(t) = V^h(0) + \int_0^t h(s)dS(s); \qquad 0 \le t \le T \tag{1.26}$$

$$\Updownarrow$$

$$P(t)V^h(t) = V^h(0) + \int_0^t h(s)dZ(s); \qquad 0 \le t \le T \tag{1.27}$$

Proof Since $r(t)$ is bounded which implies that $V^h(t)$ is lower bounded; then is also $V^z(t,h)$ lower bounded. That is $h(t)$ is an admissible portfolio. This is an important observation when proving that the existence of a equivalent measure results in a no-arbitrage market.
Further we have

$$V^z(t,h) = h(t)Z(t) = P(t)V^h(t) \tag{1.28}$$

Assuming that $h(t)$ is self-financing for the market $\{S(t)\}$ then we have by (1.12)

$$dV^z(t,h) = h(t)dZ(t) \tag{1.29}$$

Hence $h(t)$ is also admissible for the normalized market $\{Z(t)\}$ which proves the lemma.

Lemma 1.3.3 *Suppose there exists an m-dimensional process $u(t,\omega) \in \mathcal{V}^m(0,T)$ such that*

$$\sigma(t,\omega)u(t,\omega) = \mu(t,\omega) - r(t,\omega)S(t,\omega) \text{ for a.a. } (t,\omega) \tag{1.30}$$

and

$$E\Big[\exp\Big(\frac{1}{2}\int_0^T u^2(s,\omega)ds\Big)\Big] < \infty. \tag{1.31}$$

Define then the measure $Q = Q_u$ and the process $\widetilde{W}(t)$ as in (1.20) and (1.21) respectively. Then $\widetilde{W}$ is a Brownian motion w.r.t. Q and in terms of $\widetilde{W}$ we have the following representation of the normalized market $Z(t) = P(t)S(t)$:

$$dZ_0(t) = 0 \tag{1.32}$$
$$dZ_i(t) = P(t)\sigma_i(t)d\widetilde{W}(t); \qquad 1 \leq i \leq n. \tag{1.33}$$

The normalized value process $V^z(t,h)$ of an admissible portfolio $h(t)$ is a local Q-martingale given by

$$dV^z(t,h) = P(t)\sum_{i=1}^{n} h_i(t)\sigma_i(t)d\widetilde{W}(t) \tag{1.34}$$

Proof The first statement follows from lemma 1.3.1 and the Girsanov theorem. We prove (1.34) by computing

$$\begin{aligned}
dZ_i(t) &= d(P(t)S_i(t)) = P(t)dS_i(t) + S_i(t)dP(t)\\
&= P(t)[\mu_i(t)dt + \sigma_i(t)dW(t)] - r(t)P(t)S_i(t)dt\\
&= P(t)[\mu_i(t)dt - r(t)S_i(t)dt + \sigma_i(t)dW(t)]\\
&= P(t)[\mu_i(t)dt - r(t)S_i(t)dt + \sigma_i(t)(d\widetilde{W}(t) - u_i(t)dt)]\\
&= P(t)\sigma_i(t)d\widetilde{W}(t)
\end{aligned}$$

The last two equations above follows from the assumption of existence of a process $u(t,\omega)$. If $\int_0^T E_Q[P^2(t)\sigma_i^2(t)]dt < \infty$, then Q is a equivalent martingale measure and $Z_i(t)$ is a martingale w.r.t. Q by Theorem A.1.2.
Equation (1.34) follows now easily from (1.29) and (1.33).

Definition 1.3.1

- *A (European)* **contingent T-claim** *is a lower bounded $\mathcal{F}_T^{(m)}$-measurable stochastic variable $F(\omega) = \mathcal{X}$ of the form*

$$F(\omega) = \mathcal{X} = \Phi(S_T), \tag{1.35}$$

  where the **contract function** *Φ is some given real valued funtion.*

- *A fixed claim is said to be* **attainable**, *in the market $\{S(t)\}_{t\in[0,T]}$, if there exists an admissible, self-financing portfolio h(t) and a real number $y = V(0)$ such that the corresponding value process has the following property*

$$V^h(T) = V(T;h) = \mathcal{X} = y + \int_0^T h(t)dS(t), \qquad P - a.s.$$

  and such that

$$V^z(t,h) = y + \int_0^t P(s)\sigma_i(s)d\widetilde{W}(s); \quad 0 \leq t \leq T \text{ is a (strong) } Q - martingale.$$

  That is the existence of such a portfolio $h(t)$ makes it a replicating or hedging portfolio in the sense that it replicates the value of the claim $F(\omega) = \mathcal{X}$

- *The market $\{S(t)\}_{t\in[0,T]}$ is called* **complete** *if every bounded T-claim is attainable.*

The claim $\mathcal{X}$ is attainable or in other words can be replicated by the portfolio $h(t)$ if there exists a real number $V(0)$ such that if we start with this as our initial fortune then we can find an admissible portfolio $h(t)$ which generates a value $V^h(T)$ at time T which a.s. equals $\mathcal{X}$.
Here we have to require that $V^z(t)$ is a (strong) martingale and not just a local martingale w.r.t. Q.
Dropping the martingale condition in the definition above gives us a replicating portfolio which does not need to be unique.

(For more about strong and local martingales see e.g. [24].)

In order to say anything about completeness of the market the following results are useful

Theorem 1.3.1 *The market $\{S(t)\}$ is complete iff $\sigma(t,\omega)$ has a left inverse $\Lambda(t,\omega)$ for a.a.(t,ω), i.e. if there exists an $\mathcal{F}_t^{(m)}$-adapted matrix valued process $\Lambda(t,\omega) \in \mathbf{R}^{m\times n}$ such that*

$$\Lambda(t,\omega)\sigma(t,\omega) = I_m \qquad a.a.\ (t,\omega). \tag{1.36}$$

Which is equivalent to the property

$$rank\ \sigma(t) = m \quad for\ a.a.\ (t,\omega) \tag{1.37}$$

For proof see [24] p263.

Remark 1 Assume that (1.36) holds. Let $F = \mathcal{X}$ be a bounded T-claim. We now want to show that we can find an admissible portfolio $h(t) = (h_0(t), \ldots, h_n(t))$ and a real number y such that if we put

$$V_y^h(t) = y + \int_0^t h(s)dW(s) \tag{1.38}$$

Then by Lemma 1.3.1 $V_y^z(t,h)$ is a Q-martingale and

$$V_y^h(t) = F(\omega)$$

By (1.34) this is equivalent to

$$P(T)F(\omega) = V_y^z(T,h) = y + \int_0^T P(t)\sum_{i=1}^n h_i(t)\sigma_i(t)d\widetilde{W}(t).$$

And by Lemma 1.3.1 we have the unique representation

$$P(T)F(\omega) = E^Q[P(T)F] + \int_0^T \phi(t,\omega)d\widetilde{W}(t) = E^Q[P(T)F] + \int_0^T \sum_{i=1}^m \phi(t,\omega)d\widetilde{W}_j(t)$$

for some $\phi(t,\omega) = (\phi_1(t,\omega), \phi_2(t,\omega), \ldots, \phi_m(t,\omega)) \in \mathbf{R}^m$. Hence by putting

$$y = E^Q[P(T)F]$$

and choosing $\widehat{h}(t) = (h_1(t), h_2(t), \dots, h_n(t))$ such that

$$P(t)\sum_{i=1}^{n} h_i(t)\sigma_{ij}(t) = \phi_j(t) \; ; \qquad 1 \le j \le m$$

i.e. such that

$$P(t)\widehat{h}(t)\sigma(t) = \phi(t).$$

We then get by (1.36) the solution of this equation

$$\widehat{h}(t,\omega) = S_0(t)\phi(t,\omega)\Lambda(t,\omega).$$

By choosing $h_0(t)$ according to (1.18) the portfolio becomes self-financing.
By definition and by using (1.34) we obviously have

$$P(t)V_y^h(t) = y + \int_0^t h(s)dZ(s) = y + \int_0^t \phi(s)d\widetilde{W}(s)$$

and

$$P(T)V_y^h(T) = P(t)V_y^h(t) + \int_t^T \phi(s)d\widetilde{W}(s) \tag{1.39}$$

Conditioning on the information generated by $\{W(t)\}$ and taking expectation under the measure Q in the last equation we get the important result

$$P(t)V_y^h(t) = E^Q[P(T)V_y^h(T)|\mathcal{F}_t] = E^Q[P(T)F|\mathcal{F}_t] \tag{1.40}$$

Since $V_y^h(t)$ is lower bounded, every claim in the market is attainable in the sense that the value of the claim can be replicated by the portfolio $h(t)$ and the market is by definition complete.

Remark 2 Note that the filtration $\{\widetilde{\mathcal{F}}_t^{(m)}\}$ generated by $\{\widetilde{W}(t)\}$ is <u>contained</u> in $\{\mathcal{F}_t^{(m)}\}$ by (1.21) but not necessarily equal to $\{\mathcal{F}_t^{(m)}\}$.
Conditioning on the information generated by $\{\widetilde{W}(t)\}$ will by definition give the same result.

Corrollary 1.3.1

(i) If n=m, i.e. the number of price processes equals the number of Wiener processes generating the uncertainty, then the market is complete if and only if $\sigma(t,\omega)$ is invertible for a.a.(t,ω).

(ii) If the market is complete, then

$$rank\ \sigma(t,\omega) = m \qquad for\ a.a.\ (t,\omega)$$

In particular, $n \ge m$.
Moreover the process $u(t,\omega)$ satisfying (1.33) is unique.

1.4 Pricing of Contingent Claims

We now turn to the problem of finding av "reasonable" price process $\Pi(t, X)$ of the given contingent claim given in (1.35).
The holder of the contract Φ receives the stochastic amount $\mathcal{X}$ at time T. A European claim is a claim where the value of the claim only depends on the value of $S(T)$ of the securities at the final time T.
It is also possible to consider path-dependent claims, i.e. claims where the payoff also depends on the value of the market $S(t)$ where $t \in [0, T]$. These claims are in general more complicated to handle and we will return to such claims later on.

Let us now consider a T-claim $F(\omega)$. A European option on the claim F guarantees the owner the amount $F(\omega)$ at time $t = T > 0$.
What is a "reasonable" price for this guarantee?

Consider the following argument:
If the buyer of the option pays the price y for the guarantee, then she has an initial fortune of $-y$ in her portfolio. Using this negative value to hedge a value $V_{-y}^h(T, \omega)$ to time T and adding the guaranteed payoff $F(\omega)$ then this has to give a non-negative result if this deal is going to give her an intencive to buy the option:

$$V_{-y}^h(T, \omega) + F(\omega) \geq 0 \quad \text{a.s}$$

The maximal price $p(F)$ she is willing to pay for the option is $p(F) = \sup y$ when there exists an admissible portfolio h such that

$$V_{-y}^h(T, \omega) := -y + \int_0^T h(s) dS(s) \geq -F(\omega) \quad \text{a.s.} \tag{1.41}$$

A similar argument can be given to find the minimum price $q(F)$ the seller is willing to accept for the option:
If the seller receives z for the guarantee then he could regard this as his (positive) initial fortune in his portfolio. Using this positive value to hedge a value $V_z^h(T, \omega)$ to time T and which is not smaller than the guaranteed amount $F(\omega)$ promised the buyer, gives also the seller a non-negative result:

$$V_z^h(T, \omega) \geq F(\omega) \quad \text{a.s.}$$

The minimum price he is willing to accept for the option is then $q(F) = \inf z$ when there exists an admissible portfolio h such that

$$V_z^h(T, \omega) := z + \int_0^T h(s) dS(s) \geq F(\omega) \quad \text{a.s.} \tag{1.42}$$

Theorem 1.4.1 *Using Lemma* 1.3.3 *and assuming that* (1.30) *and* (1.31) *holds and that* $F(\omega)$ *is a bounded* T*-claim, letting* Q *be as in* (1.20) *and assuming that this martingale measure is unique, i.e. that the market* $\{S(t)\}$ *is complete and all claims in the market can be replicated, then the price of the European claim at time* t *is given by*

$$p(F) = q(F) = \frac{1}{P(t)} E^Q[P(T)F|\mathcal{F}_t] := S_0(t) \cdot E^Q[\frac{F}{S_0(T)}|\mathcal{F}_t] \tag{1.43}$$

Proof For simplicity let $t = 0$. Since the market is complete we can find unique y and z, both elements in $\mathbf{R}$ such that

$$\begin{aligned} -y + \int_0^T h(s)dS(s) &= -F(\omega) \quad \text{a.s.} \\ z + \int_0^T h(s)dS(s) &= F(\omega) \quad \text{a.s.} \end{aligned}$$

From Lemma (1.3.2) and Lemma (1.3.3) we obviously get

$$\begin{aligned} -y + \int_0^T \sum_{i=1}^n h_i(s)P(s)\sigma_i d\widetilde{W}(s) &= -P(T)F(\omega) \quad \text{a.s.} \\ z + \int_0^T \sum_{i=1}^n h_i(s)P(s)\sigma_i d\widetilde{W}(s) &= P(T)F(\omega) \quad \text{a.s.} \end{aligned}$$

Taking expectation under the measure Q and observing that $\int_0^t \sum_{i=1}^n h_i(s)P(s)\sigma_i d\widetilde{W}(s)$ is a Q-martingale, we get

$$\begin{aligned} y &= E^Q[P(T)F] \\ z &= E^Q[P(T)F] \end{aligned}$$

Hence

$$p(F) \leq y = E^Q[P(T)F]$$

and

$$q(F) \geq z = E^Q[P(T)F]$$

For a time $0 < t < T$ we get from (1.39) and the completeness assumption that $V_{-y}^h = y$ and $V_z^h = z$

$$\begin{aligned} E^Q[P(T)V_{-y}^h(T)|\mathcal{F}_t] &= P(t) \cdot y \\ E^Q[P(T)V_z^h(T)|\mathcal{F}_t] &= P(t) \cdot z \end{aligned} \tag{1.44}$$

Since $V_{-y}^h(T) = F(\omega) = V_z^h(T)$ this proves the theorem.

1.5 Change of Numeraire

In order to get analytical pricing formulas in situations where we want to compute derivatives of several underlying assets or processes, we can instead of the asset price $B(t)$ use e.g. the T-bond as numeraire or other assets as e.g. price processes.
When using a T-bond as numeraire, the maturity date of the bond coincides with the maturity date of the derivative and we call the equivalent probability measure the **T-forward neutral measure**.
When using a price process as $S(t)$ as numeraire we will denote it the martingale measure for the numeraire process $S(t)$.

Especially when the short interest rate is modelled as a stochastic process, a different numeraire will become handy.

We assume the existence of a stochastic process $\beta(t)$, referred to as the numeraire process, with the following dynamics under the martingale measure Q

$$d\beta(t) = \mu_\beta(t,\omega)dt + \sigma_\beta(t,\omega)d\widetilde{W}(t)$$

Defining the market as in Definition 1.1.1 we now normalize the market by the "new" numeraire $\beta(t)$

$$Z(t) = \frac{S(t)}{\beta(t)} = \left[\frac{S_0(t)}{\beta(t)}, \frac{S_1(t)}{\beta(t)}, \cdots, \frac{S_n(t)}{\beta(t)}\right]$$

Where the dynamics of $S_i(t)$ under the martingale measure Q is defined as

$$dS_i(t) = r(t)S_i(t)dt + \sigma_i(t,\omega)d\widetilde{W}(t)$$

We also define the stochastic process of the short-term rate as an Itô diffusion

$$dr(t) = \alpha(t, r(t))dt + \nu(t, r(t))d\widetilde{W}(t)$$

Consider contingent claims of the form

$$\mathcal{X} = \Phi(S(T), r(T))$$

Denote the non-normalized economy as the S-economy and similary the normalized economy as the Z-economy.
It is now easy to prove[1] that the following assertions are true

(i) A portfolio strategy $h(t, S(t))$ is self-financing in the S-economy iff it is self-financing in the Z-economy.

(ii) The value processes V^S and V^Z are connected trough

$$V^Z(t;h) = \frac{V^S(t;h)}{\beta(t)}$$

(iii) The claim $\mathcal{X}$ is reachable in the S-economy iff the claim

$$\frac{\mathcal{X}}{\beta(T)}$$

is reachable in the Z-economy.

(iv) The S-market is arbitrage free iff the Z-market is arbitrage free

[1] See [6] page 280-281

(v) The S-market is complete iff the Z-market is complete

(vi) The process $\Pi(t, \mathcal{X})$ is an arbitrage free price process in the S-model iff the process

$$\frac{\Pi(t, \mathcal{X})}{\beta(t)}$$

is an arbitrage free price process in the Z-model.

Pricing

From the assertions above we can now price a claim $\mathcal{X}$ either in the S-market or the in Z-market. From (i), (iii) and (vi) we obviously have that

$$\frac{\Pi(t; \mathcal{X})}{S_0(t)} = \Pi^{Q^\beta}\Big(t; \frac{\mathcal{X}}{S_0(T)}\Big)$$

which gives

$$\Pi(t; \mathcal{X}) = \beta(t) \cdot \Pi^{Q^\beta}\Big(t; \frac{\mathcal{X}}{\beta(T)}\Big) \tag{1.45}$$

The well known price $\Pi(t; \mathcal{X})$ in the Q-martingale measure, with the money account as numeraire, can now equivalently be found using another process $\beta(t)$ as numeraire.
Noticing that whatever price process we use as numeraire, the arbitrage free price of a claim in their respective normalized economies have to be equal. In other words

$$\Pi^{Q_2^\beta}\Big(t; \mathcal{X}\Big) = \Pi^{Q_1^\beta}\Big(t; \mathcal{X}\Big).$$

Finally from (1.45) and Theorem 1.4.1 we get

$$\Pi(t; \mathcal{X}) = E^Q[e^{-\int_t^T r(s)ds} \cdot \mathcal{X}|\mathcal{F}_t] = \beta(t) \cdot E^{Q^\beta}\Big[\frac{\mathcal{X}}{\beta(T)}|\mathcal{F}_t\Big] \tag{1.46}$$

Modelling with e.g. stochastic interest rates, this formula becomes important in order to get analytical results.

In general in a no-arbitrage market we can find a (possibly) stochastic process $u(t, \omega)$, which satisfies Novikov's condition trough the Girsanov theorem.
Then we can compute explitcitly the Radon-Nykodym derivative

$$M_T = \frac{dQ(\omega)}{dP(\omega)}, \text{ on } \mathcal{F}_T$$

By the theorem we can now move between two equivalent probability measures P and Q, which gives for a given process Y_t

$$E^Q[Y_t] = E^P[M_t \cdot Y_t]$$

and important for our purpose

$$E^Q[Y_T|\mathcal{F}_t] = E^P\Big[\frac{M_T}{M_t} \cdot Y_T|\mathcal{F}_t\Big] = M_t^{-1} \cdot E^P[M_T \cdot Y_T|\mathcal{F}_t]$$

Letting e.g. $Y(T) = e^{-\int_t^T r(s)ds} \cdot \mathcal{X}$ and define the measure P as Q^β we get the situation above.

1.6 Summary

From an idealistic point of view we assume from now on that the economy and thereby our market is free of arbitrage, or in other words an quilibrium is prevailing in the market. This is an important assumtion in order to have a functioning price generating market.
Further we can summarize this chapter as follows

- We model the market in general as an multi-dimensional Itô-process. The stock or price process number i would usually be modelled as

  $$dS_i(t) = \mu_i(t,\omega)(t)dt + \sigma_i(t,\omega)dW(t).$$

  When we later on consider options/claims on a stock we usually assume a 1-dimensional Itô-diffusion on the form

  $$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

  or what is called **geometric Brownian motion**.

- In order to price claims/options on a underlying stock or index we consider portfolios based on this underlying process and a locally risk-free asset, e.g. a bank deposit. We then use the non arbitrage argument to show that if we can find a self-financing portfolio which at any time replicates the value of the claim, then a "fair" price of the claim should equal the value of this portfolio at the time when the contract is agreed to start.

- If we can replicate every bounded claim at any time with by changing our self-financing portfolio, the market is said to be **complete** and there exists a unique price for the claim/option.

- It can be shown that the normalized valu process, $V^z(t,h)$, of a portfolio h isa martingale and thereby its dynamics can be written as an Itô-integral. Solving this integral and taking expectation gives us the arbitrage-free price of a contingent claim $\mathcal{X}$

$$\Pi(t;\mathcal{X}) = S_0(t) \cdot E^Q_{t,s}[S_0^{-1}(T) \cdot \mathcal{X}]$$

where $S_0(t)$ is the discounting process normalizing the market.

- By a change of numeraire or discounting process we can also price claims when modelling with a stochastic interest rate.

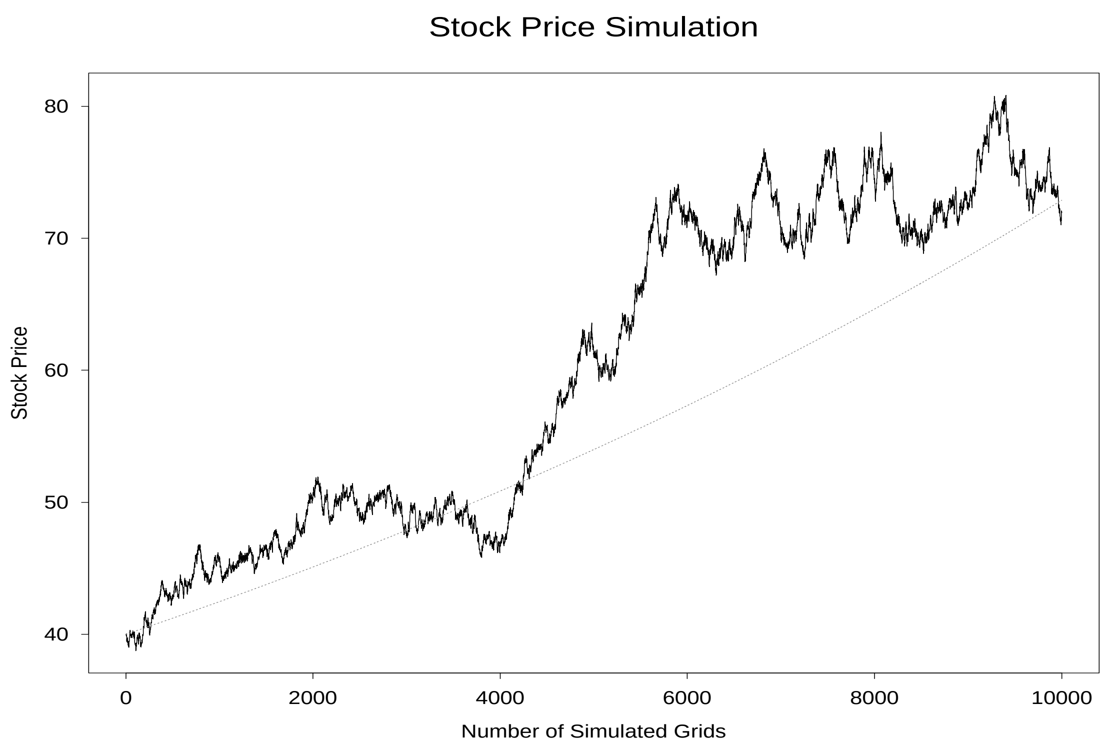

Figure 1.1: Simulated stock price and its expectation over a 10-year period with 4 grids a day when assuming 250 trading days a year.

Important Assumptions

- Short positions and fractional holdings are allowed.
- The selling price equals the buying price, i.e. no bid-ask spread.
- No transactions costs of trading.
- Arbitrage possibilities do not exist since investors prefer more money to less and will take advantage of arbitrage possibilities, creating a market equilibrium.

Chapter 2

Bonds and Interest Rates

We have so far considered the process $S_0(t)$ to be a bank deposit with a locally risk-free rate of return $r(t)$ and the processes $S_i(t)$ to be any Brownian-motion driven assets.
Now we turn to the bond market and try to apply arbitrage theory to bond pricing. We consider a single price process and let $S_i(t) = S(t)$ be a bond.
The theory and results in this chapter is basicly from Bjørk [6], but Musiela & Rutkowski [20] and Miltersen & Persson [19] have also been useful.

2.1 Zero Coupon Bonds

In this thesis the primary object w.r.t. bonds is zero coupon bonds or pure discount bonds. That is a bond which is not coupon bearing.

Definition 2.1.1 *A* **zero coupon bond** *with maturity date T, called a T-bond, is a contract which guarantees the holder 1 monetary unit to be paid out on the date T. The price of the T-bond at time t is denoted by $B(t,T)$.*

In order to guarantee the existence of a sufficiently rich and regular bond market we have to make the following assumptions:

(i) There exists a (frictionless) market for T-bonds for every $T > 0$.

(ii) The relation $B(t,t) = 1$ holds for all t.

(iii) For every fixed t, the bond price $B(t,T)$ is differentiable w.r.t. time of maturity T. $B(t,T)$ is then a function of T which provides the prices, at the fixed time t, for all bonds of all possible maturities. This function is called the term structure t.

(iv) For fixed maturity T, $B(t,T)$ as a function of t will be a scalar stochastic process which gives the prices at different times of the bond. The trajectory will typically be irregular like a Wiener-process.

Since the bond market consists of an infinite number of assets this market is different from the market in the previous chapter.
In general we are interested in the relation which must exist between all these bonds in order

to avoid arbitrage possibilities. Also we are of course interested in computing prices of interest rate derivatives.

2.2 Interest Rates

Given the bond market above we can now define a number of interest rates. In order to do this consider the following construction.
Suppose that we are at time t and let us fix two other points of time, S and T, with $t < S < T$. We now want to write a contract at time t which allows us to make an investment of one monetary unit at time S and which gives us a **deterministic** rate of return, determined at the contract time t, over the intervall $[S, T]$
This can be done by the following procedure.

1. At time t we sell one S-bond. This will give us $B(t, S)$ monetary units which we use to buy exactly $\frac{B(t,S)}{B(t,T)}$ T-bonds. Thus our net investment equals zero.
2. At time S the S-bond matures, so we are obliged to pay out one monetary unit.
3. At time T the T-bonds mature at one monetary unit a piece, so we will receive the amount of $\frac{B(t,S)}{B(t,T)}$ monetary units.
4. Based on a contract made at time t, an investment of one monetary unit at time S has yielded $\frac{B(t,S)}{B(t,T)}$ monetary units at time T.
5. Thus at time t, we have made a contract guaranteeing a riskless rate of interest over the future interval $[S, T]$. This is called a forward rate of interest.

From this argument we can define the following interest rates:

- The mean rate of return per unit of time contracted at t, referred to as the **LIBOR forward rate** for $[S, T]$, is defined as

$$L(t, S, T) = \frac{\frac{B(t,S)}{B(t,T)} - 1}{T - S} = \frac{B(t, S) - B(t, T)}{B(t, T)(T - S)}$$

- The **LIBOR spot rate** for $[S, T]$ contracted at time $t = S$ is defined as

$$L(S, T) = \frac{1 - B(t, T)}{B(t, T)(T - S)}$$

- In order to define a continous rate of return consider the following argument:
  If the short-term interest rate, $r(t)$, is deterministic the price of a T-bond at time t will obviously be given as $B(t, T) = e^{-\int_t^T r(s)ds} = e^{-(T-t)R(t,T)}$ by letting $R(t, T) = \frac{1}{(T-t)} \int_t^T r(s)ds$ be the mean rate of return for $[t, T]$. On the contrary if the short-term interest rate, $r(t)$, is stochastic, then obviously $\int_t^T r(s)ds$ will <u>not</u> be $\mathcal{F}_t$-measurable. But still we are able to define the **continously compounded spot rate**, $R(S, T)$ for the period $[S, T]$ as

$$Y(S,T) = -\frac{\log P(S,T)}{T-S}$$

$Y(S,T)$ is also known as the **yield**, and when relating $Y(S,T)$ to time to maturity T we get a funcion called the **yield curve** or the **term structure of interest rates**. This is an adapted process and has become important in calibrating short rate models.

This can easily be done by taking the logarithm of the expression $B(t,T) = e^{-(T-t)R(t,T)}$.

- In a similar way we can define the **forward rate**. Assume that we follow the procedure above. At step 2 we have to pay out one monetary unit. Regarding this as an investment at time S, we get $\frac{B(t,S)}{B(t,T)}$ monetary units at time T. Then the mean forward rate, $R(t,S,T)$, is the solution to $\frac{B(t,S)}{B(t,T)} = e^{(T-S)R(t,S,T)}$. Taking the logarithm of this expression we can define the **continously compounded forward rate** contracted at time t as

$$Y(t,S,T) = \frac{\log B(t,S) - \log B(t,T))}{T-S}$$

- We also define an instantaneous forward rate which plays an important role in the HJM-methodology.
  Let $f(t,T)$ be the forward interest rate at date $t < T$ for instantaneous riskfree borrowing or lending at date T. $f(t,T)$ is to be interpreted as the interest rate over the infinitesimal intervall $[T, T+dT]$, seen from time t.
  This is “only” a formal mathematical definition and this forward rate is not observable in the market.
  Given this definition for a family $f(t,T)$, $\quad t \leq T \leq T^*$, of forward rates, the bond prices can be defined by setting

$$B(t,T) = \exp\Big(-\int_t^T f(t,s)ds\Big) \quad \forall t \in [0,T] \tag{2.1}$$

  where T^* is our time horizon.
  Assuming that the family of bond prices is differentiable w.r.t. time of maturity T, (2.1) can be solved for $f(t,T)$.
  This gives the **instantaneous forward rate** with maturity T, contracted at time t as

$$f(t,T) = -\frac{\partial \log B(t,T)}{\partial T}$$

- More traditional stochastic interest models are based on the stochastic short-time interest rate or the spot rate $r(t)$. Analogous to the forward rate the spot rate is to be interpreted as the interest rate for risk free borrowing or lending over the infinitesimal time interval $[t, t+dt]$ at time t.
  From the notation above we define the **instantaneous short rate** at time t as

$$r(t) = f(t,t) \tag{2.2}$$

If $r(t)$, is stochastic, then $\int_t^T r(s)ds$ will still not be $\mathcal{F}_t$-measurable and the bond prices can not be given as $B(t,T) = \exp\Big(- \int_t^T r(s)ds\Big)$.

Still we can define the **money account** or **bank deposit** as in (1.1)

$$S_0(t) := B(t) = \exp\Big(- \int_0^t r(s)ds\Big)$$

with the following dynamics

$$dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = s$$

2.3 Bond Pricing

We have already defined the bond price in (2.1) as

$$B(t,T) = \exp\Big(- \int_t^T f(t,s)ds\Big)$$

The forward rate $f(t,s)$ is an adapted process but since it is not observable we have to price the bond through specifying the dynamics of the short rate.

We will now assume throughout that for any fixed maturity $T < T^*$, the price process $B(t,T), \quad t \in [0,T]$, follows a strictly positive and adapted process on a filtered probability space. Suppose also that the adapted process $r(t)$ models the short-term interest rate and that $r(t)$ is defined on the same filtered probablity space as $B(t,T),\ t \in [0,T]$.

In order for the bond market to be arbitrage free, bonds with different maturities have to satisfy certain internal consistency relations. So by this reason we make the following definition

Definition 2.3.1 *A family $B(t,T)$, $t \in [0,T]$ of adapted processes is called an* arbitrage free family of bond prices relative to r *if the following conditions are satisfied*

(i) $B(T,T) = 1$ *for every* $T \in [0,T^*]$ *and*

(ii) There exists an equivalent probability measure such that the relative bond price

$$Z(t,T) = \frac{B(t,T)}{S_0(t)} = \frac{B(t,T)}{B(t)}, \quad \forall t \in [0,T],$$

follows a martingale under this equivalent measure.
This is called the martingale measure*, Q, for the family $B(t,T)$ relative to r.*

Assuming that the bond market is free of arbitrage, then there exists a process $\lambda(t)$ according to Lemma 1.2.1 and Lemma 1.3.1, such that the price of a T-claim with the bond as the underlying price process $B(t,T)$ is given by

$$\Pi(t,T) = E^Q[B(T)^{-1} \cdot \mathcal{X}|\mathcal{F}_t] = E^Q[e^{-\int_t^T r(s)ds} \cdot \mathcal{X}|\mathcal{F}_t], \quad \forall t \in [0,T] \tag{2.3}$$

In particular the price of a bond is given by letting $X = 1$, that is a claim of one bond

$$B(t,T) = E^Q[B(T)^{-1}|\mathcal{F}_t] = E^Q[e^{-\int_t^T r(s)ds}|\mathcal{F}_t], \quad \forall t \in [0,T] \tag{2.4}$$

The last two equations easily follow from Theorem 1.4.1.

In our contionous framework we model the short rate of interest as an Itô process and thereby, according to the Girsanov theorem, it possess the following Q-dynamics

$$dr(s) = \big(\mu(s,\omega) - \sigma(s,\omega)\cdot\lambda(s)\big)ds + \sigma(s,\omega)d\widetilde{W}(s) \qquad r(t) := r \tag{2.5}$$

Since the only exogenous given asset is the risk free one, it is quite clear that the bond market is incomplete. (We are then not able to form hedging portfolios.)
If we take one particular bond as benchmark we can decide a unique $\lambda(t)$ such that the bond market becomes complete and we get a unique, arbitrage free pricing process.

If we assume deterministic interest rates the claim-and bond price will obviously be given as

$$\begin{aligned}\Pi(t,T) &= e^{-r(T-t)}\cdot E^Q[\mathcal{X}|\mathcal{F}_t]\\ B(t,T) &= e^{-r(T-t)}\end{aligned}$$

2.4 Interest Rate Models

The focus in this thesis is on deterministic interest rates. This is because we concentrate on guarantees where the underlying rate of return is the rate of return on a stock or index and not the rate of return of a short-term interest rate. Though we give an simple example where we model the short-term interest rate due to the model of Vasiček [22].

Assuming the existence of instantaneous interest rates may become problematic because it requires smoothness w.r.t. maturity. An alternative is to directly specify the dynamics of all possible bonds. (See [20] Chapter 14).

The Heath-Jarrow-Morton methodology of term structure modelling is based on exogenous specification of the dynamics of instantaneously compounded forward rates $f(t,T)$. i.e the observed forward rated curve $f(t,T); \forall T \geq 0$ will be the initial condition. This will relieve us the task off inverting the yield curve in order to calibrate a interest rate model. The HJM-approach to interest rates is not a specific model, like e.g. the short-rate models, but it is a general framework to be used for analyzing different interest rate models.
In section 2.4.2 we show that the short-term interest rate model of Vasiček is a special case in the HJM-framework.
For more about interest rate models see e.g. [14], [6], [20] and [13].

2.4.1 Forward Rate Models

We take the observed forward rated curve $f(t,T)$; $\forall T \geq 0$ as given and define the forward rate dynamics as follows

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t) \tag{2.6}$$
$$f(0,T) = f^*(0,T) \tag{2.7}$$

where α and σ are assumed to be continously differentable in the T variable and adapted stochastic processes and W is a d-dimensional independent P-Wiener process .
We also assume that all processes are regular enough so that we can differentiate under the integral sign and changing the order of integration.
Later we will restrict ourselves to **Gaussian** models. i.e. models where all volatilities of assets, forward rates and short-term rates are deterministic functions. This will give us an computational advantage w.r.t analytical results since the market price of risk then will become deterministic.
From the definition of the forward rates in (2.1) and (2.2) there must be some relations between the bond prices and the short-term rate when we assume that $f(t,T)$ satisfies the dynamics above

By definition we have

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(u,t)du + \int_0^t \sigma(u,t)dW(u) \tag{2.8}$$

By the stochastic Fubini theorem we have

$$\alpha(u,t) = \alpha(u,u) + (\alpha(u,t) - \alpha(u,u)) = \alpha(u,u) + \int_u^t \alpha_T(u,s)ds$$
$$\sigma(u,t) = \sigma(u,u) + (\sigma(u,t) - \sigma(u,u)) = \sigma(u,u) + \int_u^t \sigma_T(u,s)ds$$

Inserting this in (2.8) we get

$$r(t) = f(0,t)+\int_0^t \alpha(u,u)du+\int_0^t \int_u^t \alpha_T(u,s)dsdu+\int_0^t \sigma(u,u)dW(u)+\int_0^t \int_u^t \sigma_T(u,s)dsdW(u)$$

Changing the order of integration we get

$$r(t) = f(0,t)+\int_0^t \alpha(u,u)du+\int_0^t \int_0^s \alpha_T(u,s)duds+\int_0^t \sigma(u,u)dW(u)+\int_0^t \int_0^s \sigma_T(u,s)dW(u)ds$$

Since

$$f(0,t) = r(0) + \int_0^t f_T(0,u)du$$

the short-term rate is given as

$$r(t) = r(0) + \int_0^t \{f_T(0,s) + \alpha(s,s) + \int_0^s \alpha_T(u,s)du + \int_0^s \sigma_T(u,s)dW(u)\}ds + \int_0^t \sigma(s,s)dW(s)$$

or

$$r(t) = r(0) + \int_0^t \zeta(s)ds + \int_0^t \sigma(s,s)dW(s) \tag{2.9}$$

where

$$\zeta(s) = f_T(0,s) + \alpha(s,s) + \int_0^s \alpha_T(u,s)du + \int_0^s \sigma_T(u,s)dW(u) = \alpha(s,s) + f_T(s,s)$$

and with the dynamics

$$dr(t) = \zeta(t)dt + \sigma(t,t)dW(t)$$

This framework can now be used to evaluate short-term interest rate models.

Likewise we can find the dynamics of the bond prices by assuming that $f(t,T)$ satisfies (2.6) By the definiton of forward rates we have

$$B(t,T) = e^{-\int_t^T f(t,s)ds} = e^Y(t,T) = e^Y$$

where $Y(t,T)$ obviously is given by

$$Y(t,T) = -\int_t^T f(t,s)ds$$

Using the Itô formula on the bond price we get

$$dB(t,T) = d(e^Y) = e^{Y(t,T)}dY(t,T) + \frac{1}{2}e^{Y(t,T)}(dY(t,T))^2 \tag{2.10}$$

Here we have to compute $dY(t,T)$ and $(dY(t,T))^2$

$$dY(t,T) = d\Big(-\int_t^T f(t,s)ds\Big) = f(t,t)dt - \int_t^T df(t,s)ds = f(t,t)dt - \int_t^T \alpha(t,s)dtds - \int_t^T \sigma(t,s)dW(t)ds$$

by changing dt and $dW(t)$ with ds and recognizing $f(t,t)$ as $r(t)$ we obtain

$$\begin{aligned}dY(t,T) &= r(t)dt - \int_t^T \alpha(t,s)dsdt - \int_t^T \sigma(t,s)dsdW(t) = r(t)dt + A(t,T)dt + S(t,T)dW(t)\\ A(t,T) &= -\int_t^T \alpha(t,s)ds, \quad S(t,T) = -\int_t^T \sigma(t,s)ds\end{aligned}$$

Using the Itô calculus we derive $(dY(t,T))^2$ from the equation above when we assume a d-dimensional factor model.(d sources of uncertainty).

$$(dY(t,T))^2 = ||S(t,T)||^2 dt$$

Substituting $dY(t,T)$ and $(dY(t,T))^2$ into (2.10) we obtain

$$\begin{aligned}dB(t,T) &= B(t,T)(r(t)dt + A(t,T)dt + S(t,T)dW(t)) + \frac{1}{2}B(t,T)||S(t,T)||^2 dt\\ &= B(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}||S(t,T)||^2\right\}dt + B(t,T)S(t,T)dW(t) \qquad (2.11)\end{aligned}$$

For a more formal proof see [20] p306.

Absence of Arbitrage

Deriving the two relations above that must hold between forward rates and bond prices and between forward rates and short-term rates in order for consistency, we did **not** assume that the market was free of arbitrage.
Since we have d sources of randomness, one from every Wiener process, and an infinite number of traded assets, one bond for every maturity T, we run a risk for introducing arbitrage possibilities in the market. In order to avoid this we present the HJM drift condition, which gives a conditon on the drift term $\alpha(t,T)$ of the forward rate.

Theorem 2.4.1 HJM drift condition *Assume that the family of forward rates given by* (2.6) *and that the induced bond market is free of arbitrage. Then there exists a d-dimensional column-vector process*

$$\lambda(t) = [\lambda_1(t), \dots, \lambda_d(t)]' \qquad (2.12)$$

with the property that for all $T \geq 0$ and for all $t \leq T$, we have

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)'ds - \sigma(t,T)\lambda(t) \qquad (2.13)$$

Where $'$ denotes tranpose.

Proof From (2.11) we have the bond dynamics under the P-measure, when we take the

forward rate as given.
Assuming no-arbitrage, then there must exist an equivalent probability measure Q and an $\mathcal{F}_t^{(d)}$ -adapted and (t,ω)-measurable process $\lambda(t,\omega)$ for the normalized bond market.
Using the bank account, $B(t)$, as numeraire, the dynamics of the normalized bond market becomes

$$dZ(t,T) = Z(t,T)\Big\{A(t,T) + \frac{1}{2}(S(t,T))^2\Big\}dt + Z(t,T)S(t,T)dW(t)$$

Using Girsanov's theorem this becomes an martingale by letting

$$\begin{aligned} -S(t,T)\cdot\lambda(t) &= A(t,T) + \frac{1}{2}||S(t,T)||^2 \\ &\Updownarrow \\ -\sum_{i=1}^{d} S_i(t,T)\lambda_i(t) &= A(t,T) + \frac{1}{2}||S(t,T)||^2 \end{aligned}$$

Where the right part of the above equation is the **risk premium** for the T-bond.
Taking the T-derivative of the last equation gives us

$$\sigma(t,T)\lambda(t) = -\alpha(t,T) + \sigma(t,T)\int_t^T \sigma(t,s)'ds$$

Rearranging we get (2.13).

Corrollary 2.4.1 HJM drift condition under Q *Under the martingale measure Q, the processes α and β must satisfy the following relation for every t and every $t \leq T$*

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)'ds \tag{2.14}$$

Proof *Defining the forward rate dynamics under Q we have*

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\widetilde{W}(t)$$

The bond price dynamics must then satisfy

$$dB(t,T) = B(t,T)\Big\{r(t) + A(t,T) + \frac{1}{2}||S(t,T)||^2\Big\}dt + B(t,T)S(t,T)d\widetilde{W}(t)$$

But since we are under a martingale measure, every assets local rate of return has to equal the short rate r. This gives us

$$r(t) + A(t,T) + \frac{1}{2}||S(t,T)||^2 = r(t)$$

Taking the T-derivative the result follows.

2.4.2 A Short-Rate Model

We consider here the short-term interest rate model proposed by Vasiček [22] in 1977. This is a HJM-model in the sense that there exists a mathematical transformation that makes the Vasiček model a special case of the framework in (2.6) and (2.7).

The Vasiček Model

This is one of the simplest short-rate models. Assuming time-independent parameters we define the dynamics of the short-rate under the P-measure by

$$dr(t) = a(\bar{r} - r(t))dt + \sigma dW(t)$$

for some constants $a, \bar{r}$ and σ.
This is a **Ornstein-Uhlenbeck process** with a mean reverting structure and since the Vasiček model above only depends on one source of uncertainty it is classified as a **single-factor model**. Assuming a non-arbitrage market there must exist a equivalent martingale measure Q and a process λ such that

$$d\widetilde{W}(t) = \lambda dt + dW(t).$$

Instead of specifying the dynamics of r under the objective probability measure P we can now specify the dynamics under the martingale measure Q

$$\begin{aligned} dr(t) &= a(\bar{r} - \frac{\sigma\lambda}{a} - r(t))dt + \sigma d\widetilde{W}(t) \\ &= a(\hat{r} - r(t))dt + \sigma d\widetilde{W}(t) \end{aligned} \tag{2.15}$$

where $\hat{r} = \bar{r} - \frac{\sigma\lambda}{a}$.
Since solving the differential equation in (2.15) is equivalent to solve

$$d\big(e^{at} r(t)\big) = e^{at} a\hat{r} dt + e^{at} \sigma d\widetilde{W}(t) \tag{2.16}$$

we get the solution of (2.15) to be

$$r(t) = e^{-at} r(0) + \int_0^t e^{-a(t-s)} a\hat{r} dt + \int_0^t e^{-a(t-s)} \sigma d\widetilde{W}(s) \tag{2.17}$$

$$= e^{-at} r(0) + \hat{r}(1 - e^{-at}) + \int_0^t e^{-a(t-s)} \sigma d\widetilde{W}(s). \tag{2.18}$$

This solution is a Gaussian process since $\widetilde{W}(t)$ is Brownian motion and the integral by definition is a sum. Using (A.1) and (A.2) we can find the repective expectation and variance functions under the measure Q

$$\mu_r(t) := E^Q[r(t)] = e^{-at}r(0) + \hat{r}(1-e^{-at}) \quad \text{and} \quad \sigma_r^2(t) := Var^Q[r(t)] = \frac{\sigma^2}{2a}(1-e^{-2at}).$$

and with the following long term values

$$\begin{aligned} E^Q[r(\infty)] &= \hat{r} \qquad & Var^Q[r(\infty)] &= \frac{\sigma^2}{2a} \\ E^P[r(\infty)] &= \bar{r} \qquad & Var^P[r(\infty)] &= \frac{\sigma^2}{2a} \end{aligned}$$

From (2.4) we have the bond price

$$B(t,T) = E^Q[e^{-\int_t^T r(s)ds}]. \tag{2.19}$$

Since $r(s)$ is normally distributed, with expectation $\mu_r(s)$ and variance $\sigma_r^2(s)$, then the integral $\int_t^T r(s)ds$ is also normally distributed. In order to find the expectation and variance we use (2.18) and try to find an expression for the integral above. Assume $0 \le t \le s$, then $r(s)$ is iven by

$$r(s) = e^{-a(s-t)}r(t) + \hat{r}(1-e^{-a(s-t)}) + \sigma\int_t^s e^{-a(s-u)}d\widetilde{W}(u)$$

and the integral is obtained by changing the order o integration

$$\begin{aligned} \int_t^T r(s)ds &= r(t)\int_t^T e^{-a(s-t)}ds + \hat{r}\int_t^T (1-e^{-a(s-t)})ds + \sigma\int_t^T\int_t^s e^{-a(s-u)}d\widetilde{W}(u)ds \\ &= \frac{r(t)}{a}(1-e^{-a(T-t)}) + \hat{r}(T-t) - \frac{\hat{r}}{a}(1-e^{-a(T-t)}) + \sigma\int_t^T\int_u^T e^{-a(s-u)}dsd\widetilde{W}(u) \\ &= \frac{r(t)}{a}(1-e^{-a(T-t)}) + \hat{r}(T-t) - \frac{\hat{r}}{a}(1-e^{-a(T-t)}) + \frac{\sigma}{a}\int_t^T (1-e^{-a(T-u)})d\widetilde{W}(u) \end{aligned}$$

Taking expectation and variance of this last equation under Q and using the Itô isometry and (A.2) we get

$$\mu_I(t,T) := E\Big[\int_t^T r(s)ds\Big] = \hat{r}(T-t) + \frac{(r(t)-\hat{r})(1-e^{-a(T-t)})}{a} \tag{2.20}$$

and

$$\begin{aligned}
\sigma_I^2(t,T) := Var\Big[\int_t^T r(s)ds\Big] &= \frac{\sigma^2}{a^2}Var\Big[\int_t^T (1-e^{-a(T-u)})d\widetilde{W}(u)\Big] \\
&= \frac{\sigma^2}{a^2}\Big\{E\Big[\Big(\int_t^T (1-e^{-a(T-u)})d\widetilde{W}(u)\Big)^2\Big] - \Big(E\Big[\int_t^T (1-e^{-a(T-u)})d\widetilde{W}(u)\Big)^2\Big\} \\
&= \frac{\sigma^2}{a^2}E\Big[\int_t^T (1-e^{-a(T-u)})^2 du\Big] = \frac{\sigma^2}{a^2}\int_t^T (1-2e^{-a(T-u)}+e^{-2a(T-u)})du \\
&= \frac{\sigma^2}{a^2}\Big[T-t-\frac{2}{a}(1-e^{-a(T-t)})+\frac{1}{2a}(1-e^{-2a(T-t)})\Big]. \qquad (2.21)
\end{aligned}$$

We then have

$$-\int_t^T r(s)ds \sim \mathcal{N}\big[-\mu_I(t,T),\sigma_I^2(t,T)\big] \qquad (2.22)$$

such that

$$e^{-\int_t^T r(s)ds} \sim \mathcal{LN}\big[-\mu_I(t,T),\sigma_I^2(t,T)\big] \qquad (2.23)$$

where $\mathcal{LN}$ is to be interpreted as an log-normal distributed variable.
The **bond price** given in (2.19) is then easily computed as

$$B(t,T) = E^Q[e^{-\int_t^T r(s)ds}] = e^{-\mu_I(t,T)+\sigma_I^2(t,T)/2}. \qquad (2.24)$$

In order to find bond prices when modelling with more complex short-rate models, other techniques can be applied. See e.g. the section on **affine term structures** in [6].

The Vasiček Model in HJM-terms

We want to show that the Vasiv̌ek model is a special case in the HJM framework. Assume that the forward rates in (2.6) are specified directly under a martingale measure Q. Solving the differential under this measure for $0 \le t \le s \le T$ we get

$$f(t,s) = f(0,s) + \int_0^t \alpha(u,s)du + \int_0^t \sigma(u,s)d\widetilde{W}(u).$$

From (2.14) we get the HJM drift condition under Q and the solution above then becomes

$$f(t,s) = f(0,s) + \int_0^t \sigma(u,s)\int_u^s \sigma(u,v)'dvdu + \int_0^t \sigma(u,s)d\widetilde{W}(u).$$

Letting $s = t$ we get the short-term interest rate by definition, which any short-rate model must satisfy if the HJM framework is assumed.

$$r(t) = f(t,t) = f(0,t) + \int_0^t \sigma(u,t) \int_u^t \sigma(u,v)' dv du + \int_0^t \sigma(u,t) d\widetilde{W}(u). \tag{2.25}$$

Comparing this with the solution of the Vasiček model in (2.17) and (2.18) we must have that

$$\sigma(u,t) = e^{-a(t-u)}\sigma$$

where σ is the volatility in the short-rate model. Inserted in (2.25) this gives

$$r(t) = f(t,t) = f(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2 + \sigma \int_0^t e^{-a(t-u)} d\widetilde{W}(u)$$

It remains to compare the other terms in the solutions for the Vasiček and the forward rate. From (2.18) and the above equation we must have

$$f(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2 = e^{-at}r(0) + \hat{r}(1-e^{-at})$$

which gives the **HJM-representation**

$$f(0,t) = e^{-at}r(0) + \hat{r}(1-e^{-at}) - \frac{\sigma^2}{2a^2}(1-e^{-at})^2.$$

This is our initial condition in the HJM framework for the forward rate curve, thus automatically gives us a perfect fit between model and observed data. And finally this shows that the Vasiček model is a special case of the HJM framework.

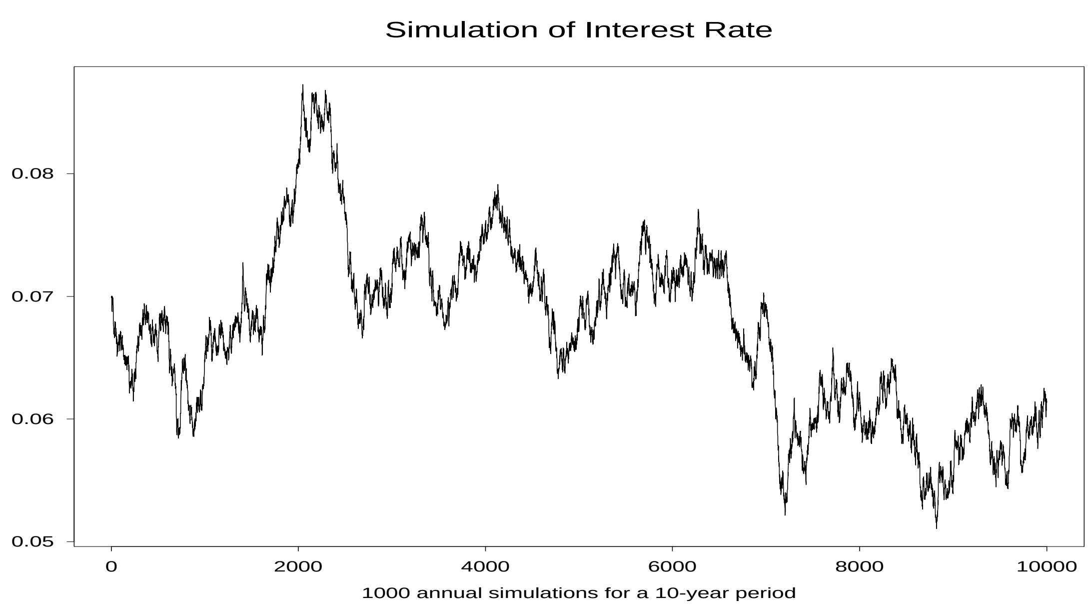

Figure 2.1: Vasiček model with $r_0 = 7\%$.

Chapter 3

Options

For simplicity we will in this thesis consider **Gaussian** models. i.e. models where the volatilities of all assets and the forward rate are deterministic or constant.
This assumption violates to some extent empirical research where e.g. the implied volatility shows to be non-deterministic. In order to model this feature there has been suggested many "improved" models such as time series, stochastic volatility through an SDE and jump-diffusion models both on the underlying asset and the volatility.
The theory and results presented in this chapter from beginning to section 3.4.3 is taken from Bjørk [6]. The last sections is taken from Kunitomo & Ikeda [16] while some proofs regarding hit probabilities are taken from the article of Anderson [2].

We will only consider the price of call options since these are central in guaranteed investment contracts. From the put-call parity the put price can in most cases easily be derived.

We also restrict ourselves to options of "European" type, i.e. where the only exercise time is at the contract end, i.e. at maturity time T.

For simplicity we consider a market consisting of one traded asset $S(t)$, modelled as **geometric** Brownian motion, and a bank account $B(t)$, with the following dynamics under the measure P

$$
\begin{aligned}
dS(t) &= \mu(t)S(t) + \sigma(t)S(t)dW(t); \qquad S(0) := s \\
dB(t) &= r(t)B(t)dt; \qquad B(0) := 1
\end{aligned}
$$

making the dynamics of the asset an Itô-diffusion and thereby Markovian.
Here $W(t)$ is a possibly multi-dimensional Wiener process.
The solutions of the above differential equations are given as

$$
\begin{aligned}
S(t) &= s\exp\Big(\int_0^t (\mu(v) - \frac{1}{2}\sigma^2(v))dv + \int_0^t \sigma(v)dW(v)\Big) \\
B(t) &= \exp\Big(\int_0^t r(v)dv\Big)
\end{aligned}
$$

For constant drift-and diffusion term we have

$$S(t) = s\exp\big((\mu - \frac{1}{2}\sigma)t + \sigma W(t)\big).$$

3.1 European Options

From definition (1.3.1) we know that a European option is a T-claim in the sense that the value of the claim only depends on the time of maturity T.
We now use the pricing formula (1.43) and the change of numeraire theory to obtain closed form solutions of a European claim, considering both deterministic and stochastic interest rates.

A European call option/claim is on the form

$$\mathcal{X} = \Phi(S(T)) = [S(T) - K]^+ = \max[S(T) - K, 0].$$

3.1.1 Deterministic Interest Rates

Assume non-stochastic interest rates and let the bank account $B(t)$ be the numeraire $S_0(t)$. It is easily seen that with deterministic interest rates the bank account equals bond prices.i.e. $B(t)/B(T) = B(t,T)$.
From (1.43) the price of the claim is given as

$$\Pi(t;\mathcal{X}) = B(t)E^Q[\frac{\mathcal{X}}{B(T)}|\mathcal{F}_t] = \exp\Big(-\int_t^T r(v)dv\Big)\cdot E^Q[\mathcal{X}|\mathcal{F}_t] = B(t,T)\cdot E^Q[\mathcal{X}|\mathcal{F}_t] \quad (3.1)$$

where we have to compute $E^Q[\mathcal{X}|\mathcal{F}_t] = E^Q[\max[S(T) - K, 0]|\mathcal{F}_t]$.
The normalized market have the following dynamics under P

$$dZ(t) = d(S(t)/B(t)) = Z(t)(\mu(t) - r(t))dt + Z(t)\sigma(t)dW(t)$$

Assume arbitrage free market and let

$$\lambda(t) = \frac{\mu(t) - r(t)}{\sigma(t)}$$

The dynamics of $Z(t)$ and $S(t)$ under the equivalent measure Q are then

$$\begin{aligned} dZ(u) &= Z(u)\sigma(u)d\widetilde{W}(u) \\ dS(u) &= r(u)S(u)du + S(u)\sigma(u)d\widetilde{W}(u), \qquad S(t) = s. \end{aligned}$$

Writing $S(T) = se^X$, $X(u)$ is a stochastic variable, normally distributed

$$\begin{aligned} X &\sim \mathcal{N}[\mu_x(u), \sigma_x(u)] \\ X(t) &= \int_0^t (r(v) - \frac{1}{2}||\sigma(v)||^2)dv + \int_0^t \sigma(v)d\widetilde{W}(v) \end{aligned}$$

where

$$\mu_x(u) = \int_0^u (r(v) - \frac{1}{2}||\sigma(v)||^2)dv, \qquad \sigma_x^2(u) = \int_0^u ||\sigma(v)||^2 dv.$$

since the expextation of an Itô -integral equals zero, this follows from the Itô isometry. We then have

$$E^Q[\mathcal{X}|\mathcal{F}_t] := E^Q_{t,s}[\mathcal{X}] = E^Q_{t,s}[\Phi(S(T))] \tag{3.2}$$

$$= E^Q_{t,s}[\Phi(se^X)] = \int_{-\infty}^{\infty} \Phi(se^x)f(x)dx \tag{3.3}$$

For a few particular choices of Φ we can solve the above equation analytically, e.g. when $\Phi(x) = \max[x - K, 0]$, a European claim form. Otherwise it must evaluated numerically. Normalizing $X(t,T)$ we get (without confusing the normalized market $Z(t)$ with the variable $Z \sim \mathcal{N}[0,1]$)

$$Z = \frac{X(t,T) - \mu_x(t,T)}{\sigma_x(t,T)} = \frac{\int_t^T \sigma(t,s)d\widetilde{W}(s)}{\int_t^T \sigma^2(t,s)ds} \sim \mathcal{N}[0,1]$$

Under the measure Q we can now write $S(T)$ as

$$S(T) = se^{\mu_x(t,T)+\sigma_x(t,T)Z}$$

The integral in equation (3.3) now becomes

$$\int_{-\infty}^{\infty} \max\left[se^{\mu_x(t,T)+\sigma_x(t,T)z} - K, 0\right]\varphi(z)dz \tag{3.4}$$

where φ is the density of the $\mathcal{N}[0,1]$ distribution, i.e

$$\varphi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}.$$

The integrand in the integral in (3.4) obviously vanishes when

$$se^{\mu_x(t,T)+\sigma_x(t,T)z} < K,$$

i.e. when $z < z_0$, where

$$z_0 = \frac{\ln\left(\frac{K}{s}\right) - \mu_x(t,T)}{\sigma_x(t,T)}. \tag{3.5}$$

We can now write the integral as

$$\begin{aligned}\int_{z_0}^{\infty}\left(se^{\mu_x(t,T)+\sigma_x(t,T)z} - K\right)\varphi(z)dz &= \int_{z_0}^{\infty} se^{\mu_x(t,T)+\sigma_x(t,T)z}\varphi(z)dz - \int_{z_0}^{\infty} K\cdot\varphi(z)dz \\ &= A - B\end{aligned}$$

Using the symmetry of the normal distribution the integral B can be written

$$B = K\cdot Pr(Z \geq z_0) = K\cdot Pr(Z \leq -z_0)$$

Since the cumalative distribution function of the $\mathcal{N}[0,1]$ is involved we denote it

$$B = K\cdot N[-z_0]$$

where $N[x]$ is defined as

$$N[x] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}dz.$$

Completing the square in the component of integral A we get

$$\begin{aligned}A &= se^{\mu_x(t,T)}\cdot\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{\sigma_x(t,T)z-\frac{1}{2}z^2}dz = se^{\mu_x(t,T)}\cdot\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{-\frac{1}{2}(z-\sigma_x(t,T))^2+\frac{1}{2}\sigma_x^2(t,T)}dz \\ &= se^{\mu_x(t,T)}e^{\frac{1}{2}\sigma_x^2(t,T)}\cdot\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{-\frac{1}{2}(z-\sigma_x(t,T))^2}dz = se^{\int_t^T r(s)ds}\cdot Pr(Z' \geq z_0)\end{aligned}$$

We easily identify $e^{\int_t^T r(s)ds}$ as the inverted bond price and Z' as $\mathcal{N}[\sigma_x(t,T),1]$ distributed. Normalizing Z' we get

$$A = s(B(t,T))^{-1}\cdot Pr(Z \geq z_0-\sigma_x(t,T)) = \frac{s}{B(t,T)}\cdot Pr(Z \leq \sigma_x(t,T)-z_0) = \frac{s}{B(t,T)}\cdot N[\sigma_x(t,T)-z_0]$$

Since

$$\mu_x(t,T) = \int_t^T (r(s) - \frac{1}{2}||\sigma(s)||^2)ds = -\ln B(t,T) - \frac{1}{2}\sigma_x^2(t,T)$$

and putting the last results into equation (3.1), we now present the famous **Black-Scholes formula**.

Theorem 3.1.1 Black-Scholes formula *The price* $\Pi(t,\mathcal{X})=\mathbf{C}(t,s;K)$ *of a European call option,* $\mathcal{X} = \max[S(T)-K,0]$, *is given by*

$$\Pi(t,\mathcal{X}) = sN[M_1] - KB(t,T)N[M_2] \tag{3.6}$$

Where

$$M_1 = \frac{-\ln(K/s) - \ln B(t,T)}{\sigma_x(t,T)} + \frac{1}{2}\sigma_x(t,T) \tag{3.7}$$

$$M_2 = \frac{-\ln(K/s) - \ln B(t,T)}{\sigma_x(t,T)} - \frac{1}{2}\sigma_x(t,T) = M_1 - \sigma_x(t,T) \tag{3.8}$$

and

$$\sigma_x^2(t,T) = \int_t^T ||\sigma(s)||^2 ds$$

3.1.2 Stochastic Interest Rates

We now turn to the computation of European options when we are modelling with stochastic interest rates. The price of a claim $\mathcal{X}$ is given as

$$\Pi(t;\mathcal{X}) = E^Q\Big[\frac{B(t)}{B(T)}\mathcal{X}|\mathcal{F}_t\Big] := E^Q_{t,s}\big[e^{-\int_t^T r(s)ds}\mathcal{X}\big]$$

Where $\mathcal{X} = \max[S(T)-K,0]$ The obvious problem here is that we have to compute the expectation of the product of two stochastic variables; $\int_t^T r(s)ds$ and $\mathcal{X}$. This expectation is to be evaluated under the measure Q, where these stochastic variables are **not** independent. The obvious reason is that the drift term of the process $S(T)$ includes $\int_t^T r(s)ds$ under Q. Using the results from section 1.5 we can avoid this problem.

Using the result (1.46) and letting a T-bond be the numeraire process $\beta(t)$ we get the following

$$\begin{aligned}\Pi(t;\mathcal{X}) &= E^Q_{t,s}\big[e^{-\int_t^T r(s)ds}\mathcal{X}\big] = B(t,T)E^{Q^T}_{t,s}\Big[\frac{\mathcal{X}}{B(T,T)}\Big] \\ &= B(t,T)E^{Q^T}_{t,s}[\mathcal{X}] \end{aligned} \tag{3.9}$$

where $B(T,T) := 1$. We now face a similar problem to deterministic interest rates, the only difference is the new martingale measure Q^T. Using a HJM forward rate model the P-dynamics of the forward rate, $f(t,T)$, $S(t)$ and $B(t,T)$ are given by

$$\begin{aligned} df(t,T) &= \alpha_f(t,T)dt + \sigma_f(t,T)dW(t) \\ dS(t) &= \mu(t)S(t)dt + \sigma(t)S(t)dW(t) \\ dB(t,T) &= B(t,T)\big(r(t) + A(t,T) + \frac{1}{2}||\sigma_B(t,T)||^2\big) + B(t,T)\sigma_B(t,T)dW(t) \end{aligned}$$

where $A(t,T)$ and $\sigma_B(t,T) := S(t,T)$ are defined in (2.11).
We still let $W(t)$ be a possibly multi-dimensional Wiener process.
Under the Q-measure we have the following dynamics

$$\begin{aligned} dS(t) &= r(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t) \\ dB(t,T) &= r(t)B(t,T)dt + B(t,T)\sigma_B(t,T)d\widetilde{W}(t) \end{aligned}$$

Now using the T-bond as numeraire we find the dynamics of normalized process $Z(t,T) = S(t)/B(t,T)$ by Itô 's formula and noting that $Z(T,T) = S(T)/B(T,T) = S(T)$

$$\begin{aligned} dZ(t,T) &= d\Big(\frac{S(t)}{B(t,T)}\Big) = \frac{dS(t)}{B(t,T)} - \frac{S(t)dB(t,T)}{(B(t,T)^2} - 2\cdot\frac{1}{2}\frac{dS(t)dB(t,T)}{(B(t,T)^2} + \frac{S(t)\big((dB(T,T)\big)^2}{(B(t,T))^3} \\ &= \frac{S(t)}{B(t,T)}\{(r(t)dt + \sigma(t)d\widetilde{W}(t) - (r(t)dt + \sigma_B(t,T)d\widetilde{W}(t)) - \sigma(t)\cdot\sigma_B(t,T)dt \\ &\quad +||\sigma_B(t,T)||^2 dt\} \\ &= Z(t,T)\big\{||\sigma_B(t,T)||^2 - \sigma(t)\cdot\sigma_B(t,T)\big\}dt + Z(t,T)\cdot\big\{\sigma(t) - \sigma_B(t,T)\big\}d\widetilde{W}(t) \end{aligned}$$

Using Girsanov's theorem we change the probability measure such that this normalized process becomes a martingale.
From the non-arbitrage assumption there must exist such a equivalent martingale measure.
Let

$$\lambda(t) = \frac{||\sigma_B(t,T)||^2 - \sigma(t)\sigma_B(t,T)}{\sigma(t) - \sigma_B(t,T)} = -\sigma_B(t,T)$$

we now get the following Q^T-dynamics

$$\begin{aligned} dS(u) &= S(u)\big\{r(u) + \sigma(t)\sigma_B(u,T)\big\}du + S(u)\sigma(u)d\widetilde{W}^T(u) \\ dZ(u,T) &= Z(u,T)\big\{\sigma(u) - \sigma_B(u,T)\big\}d\widetilde{W}^T(u) \end{aligned}$$

Solving the differential equation for the normalized process we get

$$\begin{aligned} Z(T,T) &= Z(t,T)e^{-\int_t^T \frac{1}{2}(\sigma(s)-\sigma_B(s,T))^2 ds + \int_t^T (\sigma(s)-\sigma_B(s,T))d\widetilde{W}^T(s)} \\ &= \frac{s}{B(t,T)}e^{-\int_t^T \frac{1}{2}(\sigma(s)-\sigma_B(s,T))^2 ds + \int_t^T (\sigma(s)-\sigma_B(s,T))d\widetilde{W}^T(s)} \end{aligned}$$

Letting $Z(T,T) = \frac{s}{B(t,T)}e^X$, where $X(t,T)$ is a stochastic variable, normally distributed

$$X(t,T) = -\frac{1}{2}\int_t^T (\sigma(s) - \sigma_B(s,T))^2 ds + \int_t^T (\sigma(s) - \sigma_B(s,T))d\widetilde{W}^T(s)$$

where

$$\mu_x(t,T) = -\frac{1}{2}\int_t^T (\sigma(s)-\sigma_B(s,T))^2 ds, \qquad \sigma_x^2(t,T) = \int_t^T (\sigma(s)-\sigma_B(s,T))^2 ds$$

and

$$X \sim \mathcal{N}\big[\mu_x(t,T), \sigma_x(t,T)\big]$$

We can now calculate the expectation in the pricing equation (3.9).

$$\begin{aligned}
E_{t,s}^{Q^T}[\mathcal{X}] &= E_{t,s}^{Q^T}[\Phi(S(T))] = E_{t,s}^{Q^T}[\Phi(S(T)/B(T,T))] = E_{t,z}^{Q^T}[\Phi(Z(T,T))] \\
&= E_{t,z}^{Q^T}[\Phi(ze^X)], \qquad z = S(t)/B(t,T) \\
&= E_{t,z}^{Q^T}[\Phi(ze^{\mu_x(t,T)+\sigma_x(t,T)V})] \\
&= \int_{-\infty}^{\infty} \max\big[ze^{\mu_x(t,T)+\sigma_x(t,T)v} - K\big]\varphi(v)dv \qquad (3.10)
\end{aligned}$$

where $V \sim \mathcal{N}[0,1]$.
From the integral in (3.10) we proceed as before

$$\int_{v_0}^{\infty} ze^{\mu_x(t,T)+\sigma_x(t,T)v}\varphi(v)dv - \int_{v_0}^{\infty} K\varphi(v)dv = A - B$$

where

$$v_0 = \frac{\ln\frac{K}{z} - \mu_x(t,T)}{\sigma_x(t,T)}$$

B obviously equals

$$B = K \cdot N[-v_0]$$

Completing the square which we have in A we have

$$\begin{aligned}
A &= ze^{\mu_x(t,T)} \cdot \frac{1}{\sqrt{2\pi}}\int_{v_0}^{\infty} e^{-\frac{1}{2}(v-\sigma_x(t,T))^2 + \frac{1}{2}\sigma_x^2(t,T)} dv \\
&= ze^{\mu_x(t,T)}e^{\frac{1}{2}\sigma_x^2(t,T)} \cdot Pr(V' \geq v_0)
\end{aligned}$$

where $V' \sim \mathcal{N}[\sigma_x(t,T),1]$
Normalizing V' we get

$$A = z \cdot Pr(Z \leq \sigma_x^2(t,T) - v_0) = \frac{s}{B(t,T)}N[\sigma_x(t,T) - v_0]$$

Since $\mu_x(t,T) + \frac{1}{2}\sigma_x^2(t,T) = 0$ then $\mu_x(t,T) = -\frac{1}{2}\sigma_x^2(t,T)$, and we get by putting the formulas for A and B into (3.9) the **Black-Scholes formula** for stochastic interest rates:

Theorem 3.1.2 *The price* $\Pi(t;\mathcal{X}) = \mathbf{C}(t,s;K)$ *of a European call option, with stochastic interest rates, is given by*

$$\Pi(t;\mathcal{X}) = sN[M_1] - KB(t,T)N[M_2] \tag{3.11}$$

where

$$M_1 = \frac{-\ln(K/s) - \ln B(t,T)}{\sigma_x(t,T)} + \frac{1}{2}\sigma_x(t,T) \tag{3.12}$$

$$M_2 = \frac{-\ln(K/s) - \ln B(t,T)}{\sigma_x(t,T)} - \frac{1}{2}\sigma_x(t,T) = M_1 - \sigma_x(t,T) \tag{3.13}$$

and

$$\sigma_x^2(t,T) = \int_t^T (\sigma(s) - \sigma_B(s,T))^2 ds.$$

3.2 Collar Contract

In order to lower the price of an European option we could consider an option which "rules out" unlikely results of the underlying price process.
Let $K_2 > K_1 > 0$ be fixed real numbers. A **collar contract** is a "european type" contract, i.e. not path dependent, on the form

$$\mathcal{X} = \Phi\big(S(t)\big) = \min\big[\max[S(T), K_1], K_2\big].$$

By drawing a figure it is easily seen that this can be rewritten to

$$\begin{aligned} \mathcal{X} &= K_1 + \max[S(T) - K_1, 0] - \max[S(T) - K_2, 0] \\ &:= K_1 + [S(T) - K_1]^+ - [S(T) - K_2]^+. \end{aligned} \tag{3.14}$$

Or, by observing

$$\begin{aligned} \min[F,G] &= G + \min[F-G,0] = G + [F-G]^- \\ \max[F,G] &= G + \max[F-G,0] = G + [F-G]^+ \end{aligned}$$

and

$$\min[F-G,0] = -\max[G-F,0].$$

This gives

$$\begin{aligned}
\mathcal{X} &= \min\big[\max[S(T),K_1],K_2\big] = \max[S(T),K_1] + \min\big[K_2 - \max[S(T),K_1],0\big] \\
&= K_1 + [S(T)-K_1]^+ - \max\big[\max[S(T),K_1]-K_2,0\big] \\
&= K_1 + [S(T)-K_1]^+ - \big[\max[S(T),K_1]-K_2\big]^+
\end{aligned}$$

Since $K_1 < K_2$ by assumption we obviously have that

$$\begin{aligned}
\big[\max[S(T),K_1]-K_2\big]^+ &= [S(T)-K_2]^+ \text{ when } S(T) > K_1 \\
\big[\max[S(T),K_1]-K_2\big]^+ &= [K_1-K_2]^+ = 0 \text{ when } S(T) < K_1
\end{aligned}$$

but when $S(T) < K_1$ we also have $S(T) < K_2$.
Then we have

$$\big[\max[S(T),K_1]-K_2\big]^+ = [S(T)-K_2]^+, \text{ when we assume } K_1 < K_2,$$

which gives (3.14).
In other words, the Collar contract consists of in this case a sum of two European options and a fixed value K_1.
Using the valuation formula in (1.43) we have from the linear property of mathematical expectation that the price of the claim $\mathcal{X}$ is given by

$$\begin{aligned}
\Pi(t;\mathcal{X}) &= \Pi\big(t;K_1 + [S(T)-K_1]^+ - [S(T)-K_2]^+\big) \\
&= K_1 \cdot B(t,T) + \mathbf{C}(t,s;K_1) - \mathbf{C}(t,s;K_2) \qquad (3.15)
\end{aligned}$$

where $B(t,T)$ is the bond pricwe of a zero-coupon bond with maturity date T and $\mathbf{C}(t,s;K_1)$ and $\mathbf{C}(t,s;K_2)$ is the Black-Scholes price at time t for a European option with exercise K_1 and K_2, respectively.

When we are considering guarantees the collar contract intuitively reduces the price of the guarantee since extreme values of the the underlying price process are ruled out.

3.3 Asian Options

An Asian option is a class of options considered both as European and American style. The last type gives the owner of the contract the right to exercise the contract at any time of the contract period and not only at the time of contract termination. We will consider European style contracts, i.e the only time the contract can be exercised is at time of contract termination.
An Asian option is a **average** option in the sense that the payoff depends on a average of the underlying process during the whole or parts of the contract period. This averaging feature makes the Asian option more robust with regard to manipulations, like speculators

wanting to make money, near the date of contract termination. The Asian option is typcally less expensive than the European option. Typically in our continous-time framework the avereging part is a **arithmetic** average of e.g. the stock price $S(u)$, for $t \leq u \leq T$

$$A_S(t,T) = \frac{1}{T-t}\int_t^T S(u)du$$

with payoff or contract function $C^A(t,s;K)$

$$C^A(t,s;K) = \max\left[A_S(t,T) - K, 0\right] = (A_S(t,T) - K)^+.$$

The distribution of $A_S(t,T)$ can now be found to be non-lognormal which makes a closed form solution for the price of a Asian option hard to find. No general solution for the price of the contract $C^A(t,s;K)$ is known, but Geman & Yor [10] give a closed-form expression for the Laplace transform of the price which must be found through numerical techniques. A variety of different techniques have been developed to analyze different arithmetic average Asian options, see Musiela &Rutkowski [20], Dufresne [9] and references therein.

Using the pricing theory from Chapter 1 the Black-Scholes price for the contract $C^A(t,s;K)$ is given by

$$\mathbf{C}^A(t,s;K) = e^{-r(T-t)} \cdot E^Q_{t,s}\left[(A_S(t,T) - K)^+\right] \tag{3.16}$$

when assuming a constant interest rate and

$$\mathbf{C}^A(t,s;K) = B(t,T) \cdot E^{Q^T}_{t,s}\left[(A_S(t,T) - K)^+\right] \tag{3.17}$$

when assuming stochastic interest rates.
The obviuos problem with both pricing functions is the expectation expression. **Monte Carlo** simulation applies the Strong Law of Large Numbers and we have

$$\frac{1}{N}\sum_{i=1}^{N}(A_{S_i}(t,T) - K)^+ \xrightarrow{a.s.} E^Q_{t,s}\left[(A_S(t,T) - K)^+\right] \tag{3.18}$$

when r is assumed to be constant and where $(A_{S_i}(t,T) - K)^+$, for $1 \leq i \leq N$, are independent, identical distributed and with finite expectation. And evaluated under the equivalent probability measure Q. We get an similar expression for stochastic interest rates, but now we evaluate under the measure Q^T.

So the approximate price of the Asian option with constant and stochastic interest rates, respectively, is then given by

$$\mathbf{C}^A(t,s;K) \approx e^{-r(T-t)} \cdot \frac{1}{N}\sum_{i=1}^{N}(A_{S_i}(t,T)-K)^+ \tag{3.19}$$

$$\mathbf{C}^A(t,s;K) \approx B(t,T) \cdot \frac{1}{N}\sum_{i=1}^{N}(A_{S_i}(t,T)-K)^+ \tag{3.20}$$

Remark The strenghts of the Monte Carlo simulation is that it can be used when the payoff is path-dependent and when the contract depends on more than one underlying process. The approximation in (3.20) seems to be unnecesssary complex in the sense that the dynamics of the underlying process S_i is directly specified as a function of the stochastic interest rate under the measure Q^T. This can be avoided when we instead consider the "original" pricing function with stochastic interest rate

$$\mathbf{C}^A(t,s;K) = E^Q_{t,s}\big[e^{-\int_t^T r(v)dv} \cdot (A_S(t,T)-K)^+\big]. \tag{3.21}$$

By the Law of Large Numbers we can approximate the price of the Asian option with stochastic interest rate by

$$\mathbf{C}^A(t,s;K) \approx \frac{1}{N}\sum_{i=1}^{N} e^{-\int_t^T r_i(v)dv} \cdot (A_{S_i}(t,T)-K)^+ \tag{3.22}$$

where $e^{-\int_t^T r_i(v)dv} \cdot (A_{S_i}(t,T)-K)^+$ for $1 \le i \le N$ are independent, identical distributed and with finite expectation.

3.4 Barrier Options

Barrier options belongs to a class of options whose payoff is path-dependent in the sense that it depends on if the underlying price(s) hit a predescribed barrier during the lifetime of the option.
We will here give analytical formulas for single-and double suitable barriers. For simplicity we consider price processes modelled as geometric Brownian motion with constant coefficients.
We also assume deterministic interest rates. Even though these assumptions is rather restrictive the closed form solutions we achieve are advantageous when calculating replicative strategies and comparative statistics like sensitivity analysis.
In order to give some closed form solutions to single barrier options we first present some results from absorbed probability distributions. (See [8]).

3.4.1 Single Barrier

Definition 3.4.1 *The* **hitting time** *of y, $\tau(X,y)$, for the process $X(t)$, denoted by $\tau(y)$, is defined by*

$$\tau(y) = \inf\{t \geq 0 | X(t) = y\}.$$

And the X-process **absorbed** *at y is defined by*

$$X_y(t) = X(t \wedge \tau)$$

where the notation $\alpha \wedge \beta = \min[\alpha, \beta]$. *We also define the* **running maximum** *and* **minimum** *processes,* $M_X(t)$ *and* $m_X(t)$, *by*

$$\begin{aligned} M_X(t) &= \sup_{0 \leq s \leq t} X(s) \\ m_X(t) &= \inf_{0 \leq s \leq t} X(s) \end{aligned}$$

Let now β be a upside barrier in the sense that $X(0) < \beta$ and let $\tau(\beta) = \inf\{t \geq 0 | X(t) = \beta\}$. We then define the absorbed process by

$$X_\beta(t) = \begin{cases} X(t), & t < \tau \\ \beta, & t \geq \tau, \quad X(0) = \alpha \end{cases}$$

We are now interested in finding the density at time t of the absorbed process X_β since this will become handy when evaluating the price of different barrier options. We also give the distributions of the running maximum and minimum processes.

First we give an important result known as the **reflection principle**

Lemma 3.4.1 *If* $W(t)$ *is Brownian motion then the following formula is valid for every* $t > 0,\ y \geq$ *and* $x \leq y$

$$P(W(t) \leq x, M_W(t) \geq y) = P(W(t) \geq 2y - x, M_W(t) \geq y) = P(W(t) \geq 2y - x) \tag{3.23}$$

and if $x \geq y$

$$P(W(t) \geq x, m_W(t) \leq y) = P(W(t) \leq 2y - x, m_W(t) \leq y) = P(W(t) \leq 2y - x) \tag{3.24}$$

In the first statement, the last equality is obvious since the the condition that supremum be above y is superfluous in view of the condition that the path end be above $2x - y \geq y$.
A similar argument "explains" the last statement with infimum.

Modelling the $X(t)$-process with **drift** we have

$$dX(t) = \mu dt + \sigma dW(t), \qquad X(0) = \alpha$$

First we find the distribution of X_β when $\alpha < \beta$, i.e. when we have a barrier from "above"

$$\begin{aligned} F_\beta(x; t, \alpha) &= P(X_\beta(t) \le x | X(0) = \alpha) = P(X(t) \le x, M_X(t) < \beta) \\ &= P(X(t) \le x) - P(X(t) \le x, M_X(t) \ge \beta) \quad \text{by total probability} \\ &= P\Big(W(t) \le \frac{x - \alpha - \mu t}{\sigma}\Big) - B = A - B = N\Big(\frac{x - \alpha - \mu t}{\sigma \sqrt{(t)}}\Big) - B \end{aligned}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz$$

i.e. the cumulative distribution function of a standard normally distributed variable.

We can not immediately use the reflection principle in the B term since the process $X(t)$ is modelled with a drift term. But by using Girsanov's theorem we can "kill" this drift and calculate under a new equivalent probability measure $\widetilde{P}$.

Letting $u(t) = \frac{\mu}{\sigma}$, put

$$Z(t) = e^{-\int_0^t u(s)dW(s) - \frac{1}{2}\int_0^t u^2(s)ds} = e^{-\frac{\mu}{\sigma}W(t) - \frac{1}{2}(\mu/\sigma)^2 t}$$

and by Girsanov

$$d\widetilde{P} = Z(T)dP. \tag{3.25}$$

Then

$$\widetilde{W}(t) = \int_0^t u(s)ds + W(t) = (\mu/\sigma) \cdot t + W(t)$$

is Brownian motion w.r.t. the measure $\widetilde{P}$. Under this new measure the $X(t)$-process is without drift: $dX(t) = \sigma d\widetilde{W}(t),\ X(0) = \alpha$, and

$$X(T) = \alpha + \mu T + \sigma W(T) \Leftrightarrow W(T) = \frac{X(T) - \alpha - \mu T}{\sigma}$$

Under this equivalent measure $Z(T)$ is given as

$$Z(T) = e^{-\frac{\mu}{\sigma}X(T)+\frac{1}{2}(\mu/\sigma)^2T+\frac{\mu\alpha}{\sigma^2}} \tag{3.26}$$

Inserting (3.26) in (3.4.4) we get

$$\frac{d\widetilde{P}}{dP} := Z(T) = e^{-\frac{\mu}{\sigma}X(T)+\frac{1}{2}\frac{\mu^2}{\sigma^2}T+\frac{\mu\alpha}{\sigma^2}} \Longleftrightarrow \frac{dP}{d\widetilde{P}} := \frac{1}{Z(T)} = e^{\frac{\mu}{\sigma}X(T)-\frac{1}{2}\frac{\mu^2}{\sigma^2}T-\frac{\mu\alpha}{\sigma^2}}$$

From this we obviously have

$$\begin{aligned} P(G) &= \int \frac{1}{Z(T)} \cdot \mathbf{1}_{\{G\}}(x)d\widetilde{P}(x) \\ &= \widetilde{E}\Big[\mathbf{1}_{\{G\}} \cdot \frac{1}{Z(T)}\Big] \end{aligned}$$

In our context we have $G = X(t) \leq x \bigcap M_X(t) > \beta$ which gives

$$\begin{aligned} B &= P(X(t) \leq x, M_X(t) \geq \beta) = \widetilde{E}\Big[\frac{1}{Z(T)} \cdot \mathbf{1}_{\{X(t)\leq x \bigcap M_X(t)>\beta\}}\Big] \\ &= \widetilde{E}\big[e^{\frac{\mu}{\sigma}W(T)+\frac{1}{2}\frac{\mu^2}{\sigma^2}T} \cdot \mathbf{1}_{\{X(t)\leq x \bigcap M_X(t)>\beta\}}\big] \\ &= \widetilde{E}\big[e^{\frac{\mu}{\sigma^2}X(T)-\frac{\mu\alpha}{\sigma^2}-\frac{1}{2}\frac{\mu^2}{\sigma^2}T} \cdot \mathbf{1}_{\{X(t)\leq x \bigcap M_X(t)>\beta\}}\big] \\ &= e^{-\frac{\mu\alpha}{\sigma^2}}\widetilde{E}\big[e^{\frac{\mu}{\sigma^2}X(T)-\frac{1}{2}\frac{\mu^2}{\sigma^2}T} \cdot \mathbf{1}_{\{X(t)\leq x \bigcap M_X(t)>\beta\}}\big] \end{aligned}$$

Since by Girsanov the process $(X(t)-\alpha)/\sigma$ follows a standard Brownian motion under the equivalent measure $\widetilde{P}$, we can finally apply the reflection principle

$$\begin{aligned} B &= e^{-\frac{\mu\alpha}{\sigma^2}}\widetilde{E}\big[e^{\frac{\mu}{\sigma^2}(2\beta-X(T))-\frac{1}{2}\frac{\mu^2}{\sigma^2}T} \cdot \mathbf{1}_{\{2\beta-X(t)\leq x \bigcap M_X(t)>\beta\}}\big] \\ &= e^{-\frac{\mu\alpha}{\sigma^2}}\widetilde{E}\big[e^{\frac{\mu}{\sigma^2}(2\beta-X(T))-\frac{1}{2}\frac{\mu^2}{\sigma^2}T} \cdot \mathbf{1}_{\{X(t)\geq 2\beta-x \bigcap M_X(t)>\beta\}}\big] \\ &= e^{-\frac{\mu\alpha}{\sigma^2}}\widetilde{E}\big[e^{\frac{\mu}{\sigma^2}(2\beta-X(T))-\frac{1}{2}\frac{\mu^2}{\sigma^2}T} \cdot \mathbf{1}_{\{X(t)\geq 2\beta-x\}}\big] \\ &= e^{\frac{\mu(2\beta-\alpha)}{\sigma^2}} \cdot \widetilde{E}\big[e^{-\frac{\mu}{\sigma^2}X(T)-\frac{1}{2}\frac{\mu^2}{\sigma^2}T} \cdot \mathbf{1}_{\{X(t)\geq 2\beta-x\}}\big] \end{aligned} \tag{3.27}$$

since $2\beta - x \geq \beta$. Under the $\widetilde{P}$-measure we have that

$$\widetilde{W}(t) = \frac{X(t)-\alpha}{\sigma}; \quad \text{is Brownian motion}$$

solving this for $X(t),\ t = T$ and inserting it in (3.27) we get

$$B = e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot \widetilde{E}[e^{-\frac{\mu}{\sigma^2}\widetilde{W}(T)-\frac{1}{2}\frac{\mu^2}{\sigma^2}T} \cdot \mathbf{1}_{\{X(t)\geq 2\beta-x\}}]$$

In order to calculate the expectation above we can define a new equivalent probability measure $\widehat{P}$ by setting

$$\frac{d\widehat{P}}{d\widetilde{P}} := N(T) = e^{-\frac{\mu}{\sigma}\widetilde{W}(T)-\frac{1}{2}\frac{\mu^2}{\sigma^2}T}$$

Then there exists a process $v(t)$ by Girsanov, which equals

$$v(t) = \frac{0-(-\mu)}{\sigma} = \frac{\mu}{\sigma} = u(t)$$

such that

$$\widehat{W}(t) = \frac{\mu t}{\sigma} + \widetilde{W}(t) = \frac{\mu t}{\sigma} + \frac{X(t)-\alpha}{\sigma} \tag{3.28}$$

is Brownian motion under the measure $\widehat{P}$.
By Girsanov we now get

$$\begin{aligned}
B &= e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot \widetilde{E}\big[e^{-\frac{\mu}{\sigma^2}\widetilde{W}(T)-\frac{1}{2}\frac{\mu^2}{\sigma^2}T} \cdot \mathbf{1}_{\{X(t)\geq 2\beta-x\}}\big] \\
&= e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot \widehat{P}(X(t) \geq 2\beta - x) \\
&= e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot \widehat{P}(\widehat{W}(t) \geq \frac{2\beta - x - \alpha + \mu t}{\sigma}) \quad \text{from (3.28)} \\
&= e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot \widehat{P}(\widehat{W}(t) \leq \frac{x - 2\beta + \alpha - \mu t}{\sigma}) \\
&= e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot N(\frac{x - 2\beta - \mu t + \alpha}{\sigma\sqrt{t}})
\end{aligned} \tag{3.29}$$

Finally summing A and B we get the distribution of $X_\beta(t)$, assuming $\alpha < \beta$

$$F_\beta(x,t,\alpha) = A - B = N\Big(\frac{x-\mu t-\alpha}{\sigma\sqrt{t}}\Big) - e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot N\Big(\frac{x-2\beta-\mu t+\alpha}{\sigma\sqrt{t}}\Big) \tag{3.30}$$

Taking the derivative of this distribution w.r.t. x, we get the density of X_β on the interval $(-\infty, \beta)$. Later we will show that this also is density for X_β on the interval (β, ∞)

$$f_\beta(x,t,\alpha) = \frac{\partial F_\beta(x,t,\alpha)}{\partial x} = \varphi(x, \mu t + \alpha, \sigma\sqrt(t)) - e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot \varphi(x, 2\beta + \mu t - \alpha, \sigma\sqrt(t)) \tag{3.31}$$

We now find the distribution of the **running maximum and minimum processes**, they will become handy finding $F_\beta(x,t,\alpha)$ when $\alpha > \beta$, i.e. when the barrier is from “below”.

By total probability and the reflection principle we have

$$
\begin{aligned}
P(M_X(t) \geq y) &= P(X(t) \leq y, M_X(t) \geq y) + P(X(t) > y, M_X(t) \geq y) \\
&= P(X(t) \leq y, M_X(t) \geq y) + P(X(t) > y) = B' + 1 - N\Big(\frac{x - \alpha - \mu t}{\sigma\sqrt{t}}\Big)
\end{aligned}
$$

Where B' is on the same form as B, only $x = y,\ \beta = y$, which gives

$$
B' = e^{\frac{2\mu(y-\alpha)}{\sigma^2}} \cdot N\Big(\frac{-y - \mu t + \alpha}{\sigma\sqrt{t}}\Big)
$$

We can now give the distribution of the **running maximum**, $M_X(t)$

$$
\begin{aligned}
F_{M_{X(t)}}(y) &= P(M_X(t) \leq y) = 1 - P(M_X(t) \geq y) = P(X(t) \leq y) - B' \\
&= N\Big(\frac{y - \alpha - \mu t}{\sigma\sqrt{t}}\Big) - e^{\frac{2\mu(y-\alpha)}{\sigma^2}} \cdot N\Big(\frac{-y - \mu t + \alpha}{\sigma\sqrt{t}}\Big) \qquad (3.32)
\end{aligned}
$$

Assume now that $\alpha > \beta$. Using total probability twice the distribution of X_β can now be found by

$$
\begin{aligned}
F_\beta(x, t, \alpha) &= P(X_\beta(t) \leq x) = P(X(t) \leq x, m_X(t) > \beta) \\
&= P(X(t) \leq x) - P(X(t) \leq x, m_X(t) \leq \beta) \\
&= P(X(t) \leq x) - P(m_X(t) \leq \beta) + P(X(t) \geq x, m_X(t) \leq \beta) \qquad (3.33)
\end{aligned}
$$

First we have to calculate $P(m_X(t) \leq \beta) = F_{m_{X(t)}}$ and $P(X(t) \geq x, m_X(t) \leq \beta) = C$.
We start with the last one

$$
\begin{aligned}
C &= P(X(t) \geq x, m_X(t) \leq \beta) = \widetilde{E}\Big[\frac{1}{Z(T)} \mathbf{1}_{\{X(t) \geq x \bigcap m_X(t) \leq \beta\}}\Big] \\
&= e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot \widetilde{E}\big[e^{-\frac{\mu}{\sigma^2}\widetilde{W}(T) - \frac{1}{2}\frac{\mu^2}{\sigma^2}T} \cdot \mathbf{1}_{\{X(t) \leq 2\beta - x \bigcap m_X(t) \leq x\}}\big] \\
&= e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot \widehat{P}(X(t) \leq 2\beta - x) = e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot \widehat{P}\Big(\widehat{W}(t) \leq \frac{2\beta - x - \alpha + \mu t}{\sigma}\Big) \\
&= e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot N\Big(\frac{-x + 2\beta + \mu t - \alpha}{\sigma\sqrt{t}}\Big) \qquad (3.34)
\end{aligned}
$$

Before we calculate $P(m_X(t) \leq \beta)$ observe the following:

For a set A we have $\inf(A) = -\sup(-A)$, which in our context gives $m_X(t) = -M_{(-X)}(t)$.
Since

$$
X(t) = \alpha + \mu t + \sigma W(t) \Longleftrightarrow -X(t) = -\alpha - \mu t - \sigma W(t)
$$

we have

$$\begin{aligned} m_X(t) &:= \inf_{0\le s\le t} X(s) = -\sup_{0\le s\le t}(-X(s)) = -\sup_{0\le s\le t}(-\alpha - \mu t - \sigma W(t)) \\ &= -\sup_{0\le s\le t}(\alpha' + \mu' t - \sigma W(t)) \\ &= -\sup_{0\le s\le t}(\alpha' + \mu' t + \sigma W(t)) := -M_{X'}(t) \end{aligned} \tag{3.35}$$

where $\alpha' = -\alpha$ and $\mu' = \mu$.
The last equation follows from the symmetry of Brownian motion since $W(t) \sim -W(t)$.

Using (3.35) and (3.32) we get the distribution of the **running minimum**, $m_X(t)$

$$\begin{aligned} F_{m_{X(t)}}(y) &= P(m_X(t) \le y) = P(-M_{X'}(t) \le y) = P(M_{X'}(t) \ge -y) \\ &= 1 - P(M_{X'}(t) \le -y) \\ &= 1 - N\Big(\frac{(-y) - \alpha' - \mu' t}{\sigma\sqrt{t}}\Big) + e^{\frac{2\mu'((-y)-\alpha')}{\sigma^2}} \cdot N\Big(\frac{-(-y) - \mu' t + \alpha'}{\sigma\sqrt{t}}\Big) \\ &= N\Big(\frac{y + \alpha' + \mu' t}{\sigma\sqrt{t}}\Big) + e^{\frac{2\mu'((-y)-\alpha')}{\sigma^2}} \cdot N\Big(\frac{-(-y) - \mu' t + \alpha'}{\sigma\sqrt{t}}\Big) \\ &= N\Big(\frac{y - \alpha - \mu t}{\sigma\sqrt{t}}\Big) + e^{\frac{2\mu(y-\alpha)}{\sigma^2}} \cdot N\Big(\frac{y - \alpha + \mu t}{\sigma\sqrt{t}}\Big) \end{aligned} \tag{3.36}$$

Or, what is easier

$$\begin{aligned} F_{m_{X(t)}}(y) &= P(m_X(t) \le y) = P(X(t) \le y, m_X(t) \le y) + P(X(t) \ge y, m_X(t) \le y) \\ &= P(X(t) \le y) + C'; \quad x = y,\ \beta = y \\ &= N\Big(\frac{y - \alpha - \mu t}{\sigma\sqrt{t}}\Big) + e^{\frac{2\mu(y-\alpha)}{\sigma^2}} \cdot N\Big(\frac{y - \alpha + \mu t}{\sigma\sqrt{t}}\Big) \end{aligned}$$

Finally we use the above result together with (3.33) and (3.34) to get the distribution of the absorbed process X_β when $\alpha > \beta$

$$\begin{aligned} F_\beta(x,t,\alpha) &= P(X(t) \le x) - P(m_X(t) \le \beta) + P(X(t) \ge x, m_X(t) \le \beta) \\ &= N\Big(\frac{x - \mu t - \alpha}{\sigma\sqrt{t}}\Big) - \Big\{N\Big(\frac{\beta - \alpha - \mu t}{\sigma\sqrt{t}}\Big) + e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot N\Big(\frac{\beta + \mu t - \alpha}{\sigma\sqrt{t}}\Big)\Big\} \\ &+ e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot N\Big(\frac{-x + 2\beta - \alpha + \mu t}{\sigma\sqrt{t}}\Big) \\ &= N\Big(\frac{x - \alpha - \mu t}{\sigma\sqrt{t}}\Big) - e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot N\Big(\frac{x - 2\beta + \alpha - \mu t}{\sigma\sqrt{t}}\Big) \\ &+ e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} - N\Big(\frac{\beta - \alpha - \mu t}{\sigma\sqrt{t}}\Big) - e^{\frac{2\mu(\beta-\alpha)}{\sigma^2}} \cdot N\Big(\frac{\beta - \alpha + \mu t}{\sigma\sqrt{t}}\Big) \end{aligned}$$

Taking the derivative of this last equation w.r.t. x we get the same density as in 3.31, i.e. whether $\alpha < \beta$ or the other way around we get the same density for the absorbed process X_β.

3.4.2 Out and In Contracts

Modelling the price process as standard brownian motion we have the dynamics

$$dS(v) = \alpha S(v)dv + \sigma S(v)dW(v), \quad S(t) = s$$

We now consider a contingent claim of the form $\mathcal{X} = \Phi\big(S(T)\big)$, i.e. of "european type".
In other words the value of the claim is path-dependent but the claim can not be exercised before time T.
Denote the pricing function of the claim $\mathcal{X}$ as before as $\Pi(t, s; \mathcal{X})$, where $S(t) = s$.

We will consider barrier contracts of the form

1. Down-and-Out

$$\mathcal{X}_{\beta O} = \begin{cases} \Phi(S(T)) & \text{if } \inf_{t<v\leq T} S(v) > \beta \\ 0 & \text{else} \end{cases} = \Phi(S(T) \cdot \mathbf{1}_{\{\inf_{t<v\leq T} S(v) > \beta\}}$$

2. Up-and-Out

$$\mathcal{X}^{\beta O} = \Phi(S(T) \cdot \mathbf{1}_{\{\sup_{t<v\leq T} S(v) < \beta\}}$$

3. Down-and-In

$$\mathcal{X}_{\beta I} = \Phi(S(T) \cdot \mathbf{1}_{\{\inf_{t<v\leq T} S(v) \leq \beta\}}$$

4. Up-and-In

$$\mathcal{X}^{\beta I} = \Phi(S(T) \cdot \mathbf{1}_{\{\sup_{t<v\leq T} S(v) \geq \beta\}}$$

Definition 3.4.2 *For a fixed contract function Φ we define*

$$\Phi_\beta(x) = \begin{cases} \Phi(x) & \textit{if } x > \beta \\ 0 & \textit{if } x \leq \beta \end{cases} \tag{3.37}$$

and

$$\Phi^\beta(x) = \begin{cases} \Phi(x) & \textit{if } x \leq \beta \\ 0 & \textit{if } x > \beta \end{cases} \tag{3.38}$$

In-Out Parity

Consider the same barrier β for e.g. a down-and-in contract and a down-and-out contract. Let $\Pi_{\beta I}(t,s,\Phi)$ and $\Pi_{\beta O}(t,s,\Phi)$ be the prices of these two contracts, then from a total probability argument we have

$$\Pi(t,s,\Phi) = \Pi_{\beta I}(t,s,\Phi) + \Pi_{\beta O}(t,s,\Phi) \tag{3.39}$$

This gives in general that

$$\Pi_{\beta I}(t,s,\Phi) = \Pi(t,s,\Phi) - \Pi_{\beta O}(t,s,\Phi) \quad \text{and} \quad \Pi^{\beta I}(t,s,\Phi) = \Pi(t,s,\Phi) - \Pi^{\beta O}(t,s,\Phi)$$

So we "only" need to to find the prices of either the out-price or the in-price.

We now find the price of the **down-and-out** contract without specifying the claim contract $\Phi\big(S(T)\big)$ in advance

$$\begin{aligned}
\Pi_{\beta O}(t,s,\Phi) &= e^{-r(T-t)} \cdot E^Q_{t,s}[\mathcal{X}_{\beta O}], \quad \text{by (1.43)} \\
&= e^{-r(T-t)} \cdot E^Q_{t,s}[\Phi(S(T)) \cdot \mathbf{1}_{\{\inf_{t<v\le T} S(v)>\beta\}}] \\
&= e^{-r(T-t)} \cdot E^Q_{t,s}[\Phi_\beta(S(T)) \cdot \mathbf{1}_{\{\inf_{t<v\le T} S(v)>\beta\}}] \\
&= e^{-r(T-t)} \cdot E^Q_{t,s}[\Phi_\beta(e^{X(T)}) \cdot \mathbf{1}_{\{\inf_{t<v\le T} e^{X(v)}>\beta\}}]
\end{aligned}$$

where under the Q-measure

$$dX(v) = \hat{r}dv + \sigma d\widetilde{W}(v) X(t) := \ln s \quad \text{and} \quad \hat{r} = r - \frac{1}{2}\sigma^2.$$

Using this and the density of the absorbed process X_β, the last equation above then equals

$$\begin{aligned}
&= e^{-r(T-t)} \cdot E^Q_{t,s}[\Phi_\beta(e^{X(T)}) \cdot \mathbf{1}_{\{\inf_{t<v\le T} e^{X(v)}>\beta\}}] \\
&= e^{-r(T-t)} \cdot \int_{-\infty}^{\infty} \Phi_\beta(e^x) f_{\ln\beta}(x,t,\ln s)dx \\
&= e^{-r(T-t)} \cdot \int_{-\infty}^{\infty} \Phi_\beta(e^x) \varphi(x, \hat{r} + \ln s, \sigma\sqrt{t})dx \\
&- e^{-r(T-t)} \cdot e^{\frac{2\hat{r}(ln\beta - \ln s)}{\sigma^2}} \cdot \int_{-\infty}^{\infty} \Phi_\beta(e^x) \varphi(x, \hat{r} + 2\ln\beta + \ln s, \sigma\sqrt{t})dx \\
&= e^{-r(T-t)} \cdot E^Q_{t,s}[\Phi_\beta S(T)] - e^{-r(T-t)} \cdot \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot E^Q_{t,\frac{\beta^2}{s}}[\Phi_\beta S(T)] \\
&= \Pi(t,s,\Phi_\beta) - \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \Pi(t, \beta^2/s, \Phi_\beta).
\end{aligned} \tag{3.40}$$

So the problem of computing the price of an down-and-out contract can be "reduced" to computing the price of an related claim without barrier.

Using the same arguments we can similary find the price of the **up-and-out** contract $\mathcal{X}^{\beta O}$

$$
\begin{aligned}
\Pi^{\beta O}(t,s,\Phi) &= e^{-r(T-t)} \cdot E^{Q}_{t,s}[\mathcal{X}^{\beta O}] \\
&= e^{-r(T-t)} \cdot E^{Q}_{t,s}[\Phi(S(T)) \cdot \mathbf{1}_{\{\sup_{t<v\leq T} S(v)<\beta\}}] \\
&= e^{-r(T-t)} \cdot \int_{-\infty}^{\infty} \Phi^{\beta}(e^{x})\varphi(x,\hat{r}+\ln s, \sigma\sqrt{t})dx \\
&- e^{-r(T-t)} \cdot e^{\frac{2\hat{r}(ln\beta - \ln s)}{\sigma^2}} \cdot \int_{-\infty}^{\infty} \Phi^{\beta}(e^{x})\varphi(x,\hat{r}+2\ln\beta+\ln s, \sigma\sqrt{t})dx \\
&= \Pi(t,s,\Phi^{\beta}) - \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \Pi(t,\beta^2/s,\Phi^{\beta}). \qquad (3.41)
\end{aligned}
$$

We can now by the in-out parity find the respective in-prices, but first note that from (3.45) and (3.46) we obviously get $\Phi = \Phi_{\beta} + \beta^{\beta}$. From the linearity of the pricing formula Π we have

$$
\Pi(t,s,\Phi) = \Pi(t,s,\Phi_{\beta}) + \Pi(t,s,\Phi^{\beta}). \qquad (3.42)
$$

Using the in-out parity and (3.42) we get the price for a **down-and-in** contract

$$
\begin{aligned}
\Pi_{\beta I}(t,s,\Phi) &= \Pi(t,s,\Phi) - \Pi_{\beta O}(t,s,\Phi) \\
&= \Pi(t,s,\Phi) - \left\{\Pi(t,s,\Phi_{\beta}) - \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \Pi(t,\beta^2/s,\Phi_{\beta})\right\} \\
&= \Pi(t,s,\Phi^{\beta}) + \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \Pi(t,\beta^2/s,\Phi_{\beta}) \qquad (3.43)
\end{aligned}
$$

Similary we find the price of a **up-and-in** contract

$$
\begin{aligned}
\Pi^{\beta I}(t,s,\Phi) &= \Pi(t,s,\Phi) - \Pi^{\beta O}(t,s,\Phi) \\
&= \Pi(t,s,\Phi) - \left\{\Pi(t,s,\Phi^{\beta}) - \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \Pi(t,\beta^2/s,\Phi^{\beta})\right\} \\
&= \Pi(t,s,\Phi_{\beta}) + \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \Pi(t,\beta^2/s,\Phi^{\beta}). \qquad (3.44)
\end{aligned}
$$

3.4.3 Pricing of European Single Barrier Contracts

Yet we have considered a claim Φ in general without specifying a specific contract type. We now present the result when we assume the contract to be a European option as long as the barrier is not reached during the contract time. We will consider a European call option with exercise time T and exercise price K. Since we mainly are interested in call options we restrict ourselves to their pricing functions, but from the put-call parity the put prices can easily be derived. See [6] p191.

A **down-and-out contract** with barrier β then becomes

$$\mathcal{X}_{\beta O} = C_{\beta O} = \max\left[S(T) - K, 0\right] \cdot \mathbf{1}_{\{\inf_{t<v\leq T} S(v) > \beta\}}.$$

Using (3.40) we can price this contract. For simplicity denote the "european" part by $C(x,K) = \max\left[x - K, 0\right]$. Now with this notation we have from (3.45) and (3.46) that

$$C_\beta(x,K) = \begin{cases} C(x,K) & \text{if } x > \beta \\ 0 & \text{if } x \leq \beta \end{cases} \tag{3.45}$$

and

$$C^\beta(x,K) = \begin{cases} C(x,K) & \text{if } x \leq \beta \\ 0 & \text{if } x > \beta \end{cases} \tag{3.46}$$

Using this it is obvious that if $\beta < K$, then $C_\beta(x,K) = C(x,K)$. If $\beta > K$ it gets more complicated but by drawing a figure it is seen that

$$C_\beta(x,K) = C(x,\beta) + (\beta - K)H(x,\beta)$$

where $H(x,K)$ is the **Heaviside function**, defined by

$$H(x,\beta) = \begin{cases} 1 & \text{if } x > \beta \\ 0 & \text{if } x \leq \beta. \end{cases} \tag{3.47}$$

Letting bold letters denote the prices of the various contracts and from the linear property of our pricing function, i.e. the linear property of expectation, and using (3.41), we get the price of a **down-and-out European call option**

(i) For $\beta < K$

$$\begin{aligned} \mathbf{C}_{\beta O}(t,s,K) &= \Pi(t,s,C(S(T),K)) - \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \Pi(t,\beta^2/s, C(S(T),K)) \\ &= \mathbf{C}(t,s,K) - \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \mathbf{C}(t,\beta^2/s,K) \end{aligned} \tag{3.48}$$

(ii) For $\beta \geq K$

$$\begin{aligned}
\mathbf{C}_{\beta O}(t,s,K) &= \Pi(t,s,C(S(T),\beta)) + (\beta - K)H(S(T),\beta)) \\
&- \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \Pi(t,\beta^2/s,C(S(T),\beta)) + (\beta - K)H(S(T),\beta)) \\
&= \mathbf{C}(t,s,\beta) + (\beta - K)\mathbf{H}(t,s,\beta) \\
&- \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \Big[\mathbf{C}(t,\beta^2/s,\beta) + (\beta - K)\mathbf{H}(t,\beta^2/s,\beta)\Big]
\end{aligned} \tag{3.49}$$

Where $\mathbf{C}(t,s,K)$ is the Black-Scholes formula for a European call option with excercise price K. See Theorem 3.1.1.
The pricing formula, $\mathbf{H}(t,s,\beta)$, of the Heaviside function $H(S(T),\beta)$ can easily be found in the same way as the price of a European call option when we recognize the Heaviside function as

$$H(S(T),\beta) := \begin{cases} 1 & \text{if } S(T) > \beta \\ 0 & \text{if } S(T) \leq \beta \end{cases} = \frac{\max[S(T)-\beta,0]}{S(T)-\beta} = \Phi(S(T)$$

We then get by section 3.1.1

$$\begin{aligned}
\mathbf{H}(t,s,\beta) &= e^{-r(T-t)}E^{Q}_{t,s}[\Phi(S(T))] \\
&= e^{-r(T-t)}\int_{-\infty}^{\infty} \frac{\max\left[se^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\cdot z} - \beta, 0\right]}{se^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\cdot z} - \beta}\varphi(z)dz \\
&= e^{-r(T-t)}\int_{z_0}^{\infty}\varphi(z)dz = e^{-r(T-t)}N[-z_0]
\end{aligned}$$

where

$$z_0 = \frac{\ln\beta - \ln s - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

which gives

$$\mathbf{H}(t,s,\beta) = e^{-r(T-t)}N\Big[\frac{\ln s - \ln\beta + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\Big]. \tag{3.50}$$

Now consider a **up-and-out contract**, $\mathcal{X}^{\beta O}$, given by

$$\mathcal{X}^{\beta O} = C^{\beta O} = \max[S(T)-K,0]\cdot\mathbf{1}_{\{\inf_{t<v\leq T} S(v)<\beta\}}$$

Using (3.41) we can now price this contract. From (3.46) we have per definition that $C^{\beta}(x,K) = 0$ when $\beta < K$, then the price also equals zero. When $\beta \geq K$ we use the same figure as for the down-and-out contract, and it is seen that

$$\begin{aligned} C^{\beta}(x,K) &= C(x,K) - C_{\beta}(x,K) \\ &= C(x,K) - ((C(x,\beta) + (\beta - K)H(x,\beta)) \end{aligned} \tag{3.51}$$

bn

which we knew from (3.45) and (3.46).
From the linearity of the pricing function and (3.41) we get the price of a **up-and-out European call** when $\beta \geq K$

$$\begin{aligned} \mathbf{C}^{\beta O}(t,s,K) &= \Pi(t,s,C(S(T),K)) - \Pi(t,s,C(S(T),\beta)) - (\beta-K)\Pi(t,s,H(S(T),\beta)) \\ &- \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \big(\Pi(t,\beta^2/s,C(S(T),K)) - \Pi(t,\beta^2/s,C(S(T),\beta)) \\ &- (\beta-K)\Pi(t,\beta^2/s,H(S(T),\beta))\big) \\ &= \mathbf{C}(t,s,K) - \mathbf{C}(t,s,\beta) - (\beta-K)\mathbf{H}(t,s,\beta) \\ &- \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \big(\mathbf{C}(t,\beta^2/s,K) - \mathbf{C}(t,\beta^2/s,\beta) - (\beta-K))\mathbf{H}(t,\beta^2/s,\beta)\big) \\ &= \mathbf{C}(t,s,K) - \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \mathbf{C}(t,\beta^2/s,K) - \mathbf{C}_{\beta O}(t,s,K) \end{aligned} \tag{3.52}$$

From the in-out parity and the prices of the out contracts, we find the prices of the in contracts

The price of a **down-and-in European call option** is given by

(i) For $\beta < K$

$$\begin{aligned} \mathbf{C}_{\beta I}(t,s,K) &= \Pi(t,s,C(S(T),K)) - \Pi_{\beta O}(t,s,C(S(T),K)) \\ &= \mathbf{C}(t,s,K) - \left(\mathbf{C}(t,s,K) - \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \mathbf{C}(t,\beta^2/s,K)\right) \\ &= \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \mathbf{C}(t,\beta^2/s,K) \end{aligned} \tag{3.53}$$

(ii) For $\beta \geq K$

$$\begin{aligned} \mathbf{C}_{\beta I}(t,s,K) &= \mathbf{C}(t,s,K) - \mathbf{C}(t,s,\beta) - (\beta-K)\mathbf{H}(t,s,\beta) \\ &+ \left(\frac{\beta}{s}\right)^{\frac{2\hat{r}}{\sigma^2}} \cdot \left[\mathbf{C}(t,\beta^2/s,\beta) + (\beta-K)\mathbf{H}(t,\beta^2/s,\beta)\right] \end{aligned} \tag{3.54}$$

And finally we get the price of a **up-and-in European call option**

(i) For $\beta < K$

$$\mathbf{C}^{\beta I}(t,s,K) = \mathbf{C}(t,s,K) \tag{3.55}$$

(ii) For $\beta \geq K$

$$\begin{aligned}
\mathbf{C}^{\beta I}(t,s,K) &= \mathbf{C}(t,s,K) - \big[\mathbf{C}(t,s,K) - \mathbf{C}(t,s,\beta) - (\beta - K)\mathbf{H}(t,s,\beta) \\
&- \Big(\frac{\beta}{s}\Big)^{\frac{2\hat{r}}{\sigma^2}} \cdot \big(\mathbf{C}(t,\beta^2/s,K) - \mathbf{C}(t,\beta^2/s,\beta) - (\beta - K))\mathbf{H}(t,\beta^2/s,\beta)\big)\big] \\
&= \mathbf{C}(t,s,\beta) + (\beta - K)\mathbf{H}(t,s,\beta) \\
&+ \Big(\frac{\beta}{s}\Big)^{\frac{2\hat{r}}{\sigma^2}} \cdot \big(\mathbf{C}(t,\beta^2/s,K) - \mathbf{C}(t,\beta^2/s,\beta) - (\beta - K))\mathbf{H}(t,\beta^2/s,\beta)\big)\big] \\
&= \Big(\frac{\beta}{s}\Big)^{\frac{2\hat{r}}{\sigma^2}} \cdot \mathbf{C}(t,\beta^2/s,K) + \mathbf{C}_{\beta O}(t,s,K) \qquad (3.56)
\end{aligned}$$

3.4.4 Geometrically Curved Boundary

We have yet only considered a constant barrierer, β, but in some applications it seems appealing to use a time-dependent boundary, i.e. a curved boundary like $\beta(t) = a\exp(bt)$. Finding the probability distribution of the absorbed process with a time-dependent boundary the pricing result in section 3.4.3 is still valid after making a slight modification.

We start by giving this probability distribution for the absorbed process $X_\beta(t)$, where $X(t) = \alpha + \mu t + \sigma W(t)$, with boundary $\beta(t) = a + bt$. Then using the transformation theorem to the transform $S(t) = \exp(X(t))$ we find the probability distribution of the absorbed process $S_\beta(t)$, where $S(t) = \exp(\ln(S_0) + \hat{r}t + \sigma W(t))$.

Instead of "killing" the drift term as is section 3.4.1 we now use Girsavov's Theorem to find an equivalent probability measure where $X(t)$ has the same "drift" as the boundary $\beta(t) = a+bt$, i.e. b. We can then make use of the **reflection principle** as before.

Let the barrier be given from above by $\beta(t) = a + bt$, $X(t) = \alpha + \mu t + \sigma W(t)$ and $M_X(t) = \sup_{0\leq s\leq t} X(s)$.

The distribution of the absorbed process is then given by

$$\begin{aligned}
F_\beta(x;t,\alpha) &= P(X_\beta(t) \leq x | X(0) = \alpha) = P(X(t) \leq x, M_X(t) < \beta(t)) \\
&= P(X(t) \leq x) - P(X(t) \leq x, M_X(t) \geq \beta(t)) \\
&= N\Big(\frac{x - \alpha - \mu t}{\sigma\sqrt(t)}\Big) - D.
\end{aligned}$$

The problem here is as before the D-term. Changing the drift of $X(t)$ to b we let

$$u(t) = \frac{\mu - b}{\sigma} \qquad (3.57)$$

and

$$Z(t) = e^{-\int_0^t u(s)dW(s) - \frac{1}{2}\int_0^t u^2(s)ds} = e^{-u(t)W(t) - \frac{1}{2}u(t)^2 t}.$$

By Girsanov we now have

$$d\widetilde{P} = Z(T)dP$$

where

$$\widetilde{W}(t) = \int_0^t u(s)ds + W(t) = \frac{\mu - b}{\sigma} \cdot t + W(t)$$

is Brownian motion w.r.t. the measure $\widetilde{P}$.

Under this new probability measure $X(t)$ has the dynamics $dX(t) = bdt + \sigma d\widetilde{W}(t)$.
And finally

$$\begin{aligned}
D &= P(X(t) \leq x, M_X(t) \geq \beta(t)) = \widetilde{E}\Big[\frac{1}{Z(T)} \cdot \mathbf{1}_{\{X(t) \leq x \bigcap M_X(t) > \beta(t)\}}\Big] \\
&= \widetilde{E}\big[e^{uW(T) + \frac{1}{2}u^2 T} \cdot \mathbf{1}_{\{X(t) \leq x \bigcap M_X(t) > \beta(t)\}}\big] \\
&= \widetilde{E}\big[e^{\frac{u}{\sigma}X(T) - \frac{u}{\sigma}\alpha - \frac{u}{\sigma}\mu T + \frac{1}{2}u^2 T} \cdot \mathbf{1}_{\{X(t) \leq x \bigcap M_X(t) > \beta(t)\}}\big] \\
&= e^{-\frac{u}{\sigma}\alpha} \cdot \widetilde{E}\big[e^{\frac{u}{\sigma}(2\beta(T) - X(T)) - \frac{u}{\sigma}\mu T + \frac{1}{2}u^2 T} \cdot \mathbf{1}_{\{X(t) \geq 2\beta(t) - x \bigcap M_X(t) > \beta(t)\}}\big] \quad \text{reflection principle} \\
&= e^{-\frac{u}{\sigma}\alpha} \cdot \widetilde{E}\big[e^{-\frac{u}{\sigma}X(T) + \frac{u}{\sigma}2\beta(T) - \frac{u}{\sigma}\mu T + \frac{1}{2}u^2 T} \cdot \mathbf{1}_{\{X(t) \geq 2\beta(t) - x \bigcap M_X(t) > \beta(t)\}}\big] \\
&= e^{-\frac{u}{\sigma}\alpha} \cdot \widetilde{E}\big[e^{-\frac{u}{\sigma}(\alpha + bT + \sigma\widetilde{W}(T)) + \frac{u}{\sigma}2\beta(T) - \frac{u}{\sigma}\mu T + \frac{1}{2}u^2 T} \cdot \mathbf{1}_{\{X(t) \geq 2\beta(t) - x\}}\big] \\
&= e^{-2\frac{u}{\sigma}\alpha} \cdot \widetilde{E}\big[e^{-u\widetilde{W}(T) - \frac{1}{2}u^2 T + u^2 T + \frac{u}{\sigma}(2a + 2bT) - \frac{u}{\sigma}bT - \frac{u}{\sigma}\mu T} \cdot \mathbf{1}_{\{X(t) \geq 2\beta(t) - x\}}\big] \\
&= e^{2\frac{u}{\sigma}(a - \alpha)} \cdot \widetilde{E}\big[e^{-u\widetilde{W}(T) - \frac{1}{2}u^2 T + \frac{u}{\sigma}T(\mu - b + 2b - b - \mu)} \cdot \mathbf{1}_{\{X(t) \geq 2\beta(t) - x\}}\big] \\
&= e^{2\frac{u}{\sigma}(a - \alpha)} \cdot \widetilde{E}\big[e^{-u\widetilde{W}(T) - \frac{1}{2}u^2 T} \cdot \mathbf{1}_{\{X(t) \geq 2\beta(t) - x\}}\big]
\end{aligned}$$

In order to calculate the above expectation we define a new equivalent probability measure $\widehat{P}$ by setting

$$\frac{d\widehat{P}}{d\widetilde{P}} := N(T) = e^{-u\widetilde{W}(T) - \frac{1}{2}u^2 T}.$$

Then there exists a process v(t) by Girsanov, which equals

$$v(t) = \frac{b - (2b - \alpha)}{\sigma} = u(t)$$

such that

$$\widehat{W}(t) = ut + \widetilde{W}(t) = \frac{X(t) - \alpha - (2b - \mu)t}{\sigma} \tag{3.58}$$

is Brownian motion under the measure $\widehat{P}$. Under this measure $X(t)$ has the following dynamics

$$dX(t) = (2b - \mu)dt + \sigma d\widehat{W}(t) \tag{3.59}$$

By Girsanov we get

$$\begin{aligned}
D &= e^{2\frac{u}{\sigma}(a-\alpha)} \cdot \widetilde{E}\big[e^{-u\widetilde{W}(T) - \frac{1}{2}u^2 T} \cdot \mathbf{1}_{\{X(t) \geq 2\beta(t) - x\}}\big] \\
&= e^{2\frac{(\mu-b)(a-\alpha)}{\sigma^2}} \cdot \widehat{P}(X(t) \geq 2\beta(t) - x) \\
&= e^{2\frac{(\mu-b)(a-\alpha)}{\sigma^2}} \cdot \widehat{P}\Big(\widehat{W}(t) \geq \frac{2a + 2bt - x - \alpha - (2b - \mu)t}{\sigma}\Big) \\
&= e^{2\frac{(\mu-b)(a-\alpha)}{\sigma^2}} \cdot \widehat{P}\Big(Z \geq \frac{2a - x - \alpha + \mu t}{\sigma\sqrt{(t)}}\Big) \\
&= e^{2\frac{(\mu-b)(a-\alpha)}{\sigma^2}} \cdot \widehat{P}\Big(Z \leq \frac{2a - x - \alpha + \mu t}{\sigma\sqrt{t}}\Big) \\
&= e^{2\frac{(\mu-b)(a-\alpha)}{\sigma^2}} \cdot N\Big(\frac{x - 2a + \alpha - \mu t}{\sigma\sqrt{t}}\Big)
\end{aligned} \tag{3.60}$$

The distribution of the absorbed process $X_\beta(t)$ is then given by (3.57) and (3.60)

$$\begin{aligned}
F_\beta(x; t, \alpha) &= P(X_\beta(t) \leq x | X(0) = \alpha) \\
&= N\Big(\frac{x - \alpha - \mu t}{\sigma\sqrt{t}}\Big) - e^{2\frac{(\mu-b)(a-\alpha)}{\sigma^2}} \cdot N\Big(\frac{x - 2a + \alpha - \mu t}{\sigma\sqrt{t}}\Big)
\end{aligned} \tag{3.61}$$

with density

$$f_\beta(x; t, \alpha) = \varphi\Big(\frac{x - \alpha - \mu t}{\sigma\sqrt{t}}\Big) - e^{2\frac{(\mu-b)(a-\alpha)}{\sigma^2}} \cdot \varphi\Big(\frac{x - 2a + \alpha - \mu t}{\sigma\sqrt{t}}\Big) \tag{3.62}$$

Recognizing that the only difference from (3.31) is b subtracted from μ in the exponent. If b is positive the term $\exp\big(2\frac{(\mu-b)(a-\alpha)}{\sigma^2}\big)$ decreases and thereby the probability of staying below the barrier increases. The price of e.g. a knock-out contract would then increase, which is quite intuitive.

We find the distribution of $S_\beta(t)$ through the transformation theorem and the transform $S(t) = e^{X(t)} = h(X(t))$. Where $X(t) = \ln S_0 + \hat{r}t + \sigma W(t)$ and $\hat{\mu} = \alpha - \frac{1}{2}\sigma^2$.
The drift-term α is not to be confused with X_0 from before. Such that $dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$.

We notice that the linear boundary is transformed into the exponential curved boundary $\beta'(t) = ce^{bt}$.

Since $\beta(t) = \ln \beta'(t)$ when $a = \ln c$ and $M_S(t) < \beta'(t)$ is equivalent to $M_X(t) < \beta(t)$, the density of the absorbed process $S_{\beta'}(t)$, when the boundary is from above, is given by

$$\begin{aligned} f_{\beta'}(s; t, S_0) &:= f_{S(t),M_S(t)}(s, \beta') = f_{X(t),M_S(t)}(h^{-1}(x), \beta') \cdot \left| \frac{\partial h^{-1}(x)}{\partial s} \right| \\ &= f_{X(t),M_X(t)}(\ln s, \beta) \cdot \left| \frac{\partial \ln s}{\partial s} \right| = f_\beta(\ln s; t, \ln S_0) \cdot \frac{1}{s}. \end{aligned}$$

The distribution function of the absorbed process or the probability that the process $S(v)$ stays below the boundary the entire period is then given by

$$\begin{aligned} F_{\beta'}(x; t, S_0) &= P(S_{\beta'}(t) \leq x | S(0) = S_0) = P(S(t) \leq x, M_S(t) < \beta') \\ &:= \int_{-\infty}^{x} f_{\beta'}(y; t, S_0) dy = \int_{-\infty}^{x} f_\beta(\ln y; t, \ln S_0) \cdot \frac{dy}{y}. \\ &= N\Big(\frac{\ln x - \ln S_0 - \hat{\mu}t}{\sigma\sqrt{t}}\Big) - e^{2\frac{(\hat{\mu}-b)(a-\ln S_0)}{\sigma^2}} \cdot N\Big(\frac{\ln x - 2a + \ln S_0 - \hat{\mu}t}{\sigma\sqrt{t}}\Big), \quad a = \ln c \\ &= N\Big(\frac{\ln x - \ln S_0 - \hat{\mu}t}{\sigma\sqrt{t}}\Big) - \Big(\frac{c}{S_0}\Big)^{2\frac{(\hat{\mu}-b)}{\sigma^2}} \cdot N\Big(\frac{\ln x - \ln(c^2/S_0) - \hat{\mu}t}{\sigma\sqrt{t}}\Big) \end{aligned} \tag{3.63}$$

Since our bondary now is not constant, in the sense that it is curved, we can not use the pricing result with a constant barrier. The put-call parity relation does not hold either with a curved boundary. In order to get pricing result we turn to the density in (3.63). By risk-neutralizing evaluation we get the price, $\mathbf{C}^{\beta_v O}$, of a **up-and-out** European claim, when assuming that $\beta(T) > K$.
The contract function is given by

$$\mathcal{X}^{\beta_v O} = C^{\beta_v O} = \max[S(T) - K, 0] \cdot \mathbf{1}_{\{\sup_{t<v\leq T} S(v) < \beta(v)\}}.$$

Evaluating this contract under a risk-neutralized probability measure we get its price, when $\hat{r} = r - \frac{1}{2}\sigma^2$

$$
\begin{aligned}
\mathbf{C}^{\beta_v O}(t,s,K) &= e^{-r\tau} E^Q_{t,s}[\max{[S(T)-K,0]} \cdot \mathbf{1}_{\{\sup_{t<v\le T} S(v)<\beta(v)\}}] \qquad \tau = T-t \\
&= e^{-r\tau}\Big\{ \int_K^{\beta(T)} s f_\beta(\ln s; t, \ln S_0)\cdot \frac{1}{s} ds - K \int_K^{\beta(T)} f_\beta(\ln s; t, \ln S_0)\cdot \frac{1}{s} ds \Big\} \\
&= e^{-r\tau} S_0 \int_K^{\beta(T)} e^{(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}z} \Big\{ \varphi\Big(\frac{\ln s - \ln S_0 - (r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\Big) \\
&\quad -\Big(\frac{c}{S_0}\Big)^{2\frac{(\hat{r}-b)}{\sigma^2}} \varphi\Big(\frac{\ln s - \ln{(c^2/S_0)} - (r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\Big)\Big\}\frac{1}{s} ds \\
&\quad - e^{-r\tau} K \int_K^{\beta(T)} \Big\{ \varphi\Big(\frac{\ln s - \ln S_0 - (r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\Big) \\
&\quad -\Big(\frac{c}{S_0}\Big)^{2\frac{(\hat{r}-b)}{\sigma^2}} \varphi\Big(\frac{\ln s - \ln{(c^2/S_0)} - (r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\Big)\Big\}\frac{1}{s} ds.
\end{aligned}
$$

Making the transformation $y = \ln s$ and letting the integral boundaries be

$$
\begin{aligned}
u_1 &= \frac{\ln\beta(T) - \ln S_0 - (r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} & u_2 &= \frac{\ln K - \ln S_0 - (r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \\
v_1 &= \frac{\ln\beta(T) - \ln{(c^2/S_0)} - (r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} & v_2 &= \frac{\ln K - \ln{(c^2/S_0)} - (r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathbf{C}^{\beta_v O}(t,s,K) &= S_0 \int_{u_2}^{u_1} e^{-\frac{1}{2}\sigma^2\tau+\sigma\sqrt{\tau}z}\cdot\frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}z^2} dz - S_0\Big(\frac{c}{S_0}\Big)^{2\frac{(\hat{r}-b)}{\sigma^2}} \int_{v_2}^{v_1} e^{-\frac{1}{2}\sigma^2\tau+\sigma\sqrt{\tau}z}\cdot\frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}z^2} dz \\
&\quad -e^{-r\tau} K \int_{u_2}^{u_1} \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}z^2} dz + e^{-r\tau} K \Big(\frac{c}{S_0}\Big)^{2\frac{(\hat{r}-b)}{\sigma^2}} \int_{v_2}^{v_1} \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}z^2} dz.
\end{aligned}
$$

Completing the squares w.r.t z in the two first integrals it follows that

$$
\begin{aligned}
\mathbf{C}^{\beta_v O}(t,s,K) &= S_0\int_{u_2}^{u_1}\frac{1}{\sqrt{\pi}}e^{-\frac{1}{2}(z-\sigma\sqrt{\tau})^2}dz - S_0\Big(\frac{c}{S_0}\Big)^{2\frac{(\hat{r}-b)}{\sigma^2}}\int_{v_2}^{v_1}\frac{1}{\sqrt{\pi}}e^{-\frac{1}{2}(z-\sigma\sqrt{\tau})^2}dz \\
&\quad -e^{-r\tau}K\int_{u_2}^{u_1}\frac{1}{\sqrt{\pi}}e^{-\frac{1}{2}z^2}dz + e^{-r\tau}K\Big(\frac{c}{S_0}\Big)^{2\frac{(\hat{r}-b)}{\sigma^2}}\int_{v_2}^{v_1}\frac{1}{\sqrt{\pi}}e^{-\frac{1}{2}z^2}dz. \\
&= S_0\big[N(u_1-\sigma\sqrt{\tau})-N(u_2-\sigma\sqrt{\tau})\big] - S_0\Big(\frac{c}{S_0}\Big)^{2\frac{(\hat{r}-b)}{\sigma^2}}\big[N(v_1-\sigma\sqrt{\tau})-N(v_2-\sigma\sqrt{\tau})\big] \\
&\quad -e^{-r\tau}K\big[N(u_1)-N(u_2)\big] + e^{-r\tau}K\Big(\frac{c}{S_0}\Big)^{2\frac{(\hat{r}-b)}{\sigma^2}}\big[N(v_1)-N(v_2)\big] \\
&= S_0\big[N(\sigma\sqrt{\tau}-u_2)-N(\sigma\sqrt{\tau}-u_1)\big] - S_0\Big(\frac{c}{S_0}\Big)^{2\frac{(\hat{r}-b)}{\sigma^2}}\big[N(\sigma\sqrt{\tau}-v_2)-N(\sigma\sqrt{\tau}-v_1)\big] \\
&\quad -e^{-r\tau}K\big[N(-u_2)-N(-u_1)\big] + e^{-r\tau}K\Big(\frac{c}{S_0}\Big)^{2\frac{(\hat{r}-b)}{\sigma^2}}\big[N(-v_2)-N(-v_1)\big] \\
&= S_0\Big[N\Big(\frac{\ln(S_0/K)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\Big)-N\Big(\frac{\ln(S_0/ce^{bt})+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\Big)\Big] \\
&\quad - S_0\Big(\frac{c}{S_0}\Big)^{2\frac{(\hat{r}-b)}{\sigma^2}}\Big[N\Big(\frac{\ln(c^2/S_0K)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\Big)-N\Big(\frac{\ln c/S_0e^{bt}+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\Big)\Big] \\
&\quad - e^{-r\tau}K\Big[N\Big(\frac{\ln(S_0/K)+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\Big)-N\Big(\frac{\ln(S_0/ce^{bt})+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\Big)\Big] \\
&\quad + e^{-r\tau}K\Big(\frac{c}{S_0}\Big)^{2\frac{(\hat{r}-b)}{\sigma^2}}\Big[N\Big(\frac{\ln(c^2/S_0K)+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\Big)-N\Big(\frac{\ln c/S_0e^{bt}+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\Big)\Big].
\end{aligned}
\tag{3.64}
$$

By letting $b \to \infty$, i.e. without barrier, we see that our price function above is the well known Black-Scholes formula.

3.4.5 Double Barrier

Now we turn to the more complex problem of pricing double-barrier options. The complexity of this problem consists of finding the probability distribution of our GBM-price-process when the boundaries are not hit. We can not immediately use the reflection principle as with a single barrier, but by repeated use this probability ditribution can be found as an infinite series of the normal distribution functions. What is important for practical purpose is the rate of convergence of this infinite series, and Kunitoma & Ikeda concludes in [16] that this series converges rather rapid such that quite few of the leading terms in the series approximates the probability well enough for practical purposes.

First we consider a process $Y(t) = W(t)$, i.e. standard Brownian motion, and the linear boundaries β_1 and β_2:

$$\beta_1(t) = a_1 + b_1 t, \qquad \beta_2(t) = a_2 + b_2 t \tag{3.65}$$

assuming that $a_1 \geq a_2$ and $\beta_1 > \beta_2$, i.e. the boundaries are not crossing after $t = 0$. We also assume that $b_1 \geq 0$ since this is intuitively the most practical. (In [2], Anderson also gives the hit probability when $b_1 < 0$).

After finding the probability function we are looking for, we proceed to a process $X(t) = X(0) + \mu t + \sigma W(t)$. Using Girsanov's theorem we find the appropriate probability function when the process is modelled with drift.
Finally we use the transform $S(t) = e^{X(t)}$ to find the probability we are looking for when modelling with geometric brownian motion and the boundaries:

$$\beta_1'(t) = c_1 e^{b_1 t}, \qquad \beta_2'(t) = c_2 e^{b_2 t} \tag{3.66}$$

letting $a_1 = \ln c_1$ and $a_2 = \ln c_2$ we have $\ln \beta_1'(t) = \beta_1(t)$ and $\ln \beta_2'(t) = \beta_2(t)$. That is the linear boundaries are transformed into the exponential curved boundaries.

Now consider the process $Y(t) = W(t)$, where $W(t)$ is Brownian motion, and the boundaries in (3.65). Let τ_1 and τ_2 be the first hitting times of the upper and lower boundary lines, respectively. Denote the interval between the boundaries by $I \subset [a_1 + b_1 t, a_2 + b_2 t]$, then the probability we are interested in is $P(Y(t) \in I, \tau_1 > t, \tau_2 > t)$. By total probability we obviously have

$$\begin{aligned} P(Y(t) \in I) &= P(Y(t) \in I, \tau_1 > t) + P(Y(t) \in I, \tau_1 < t) \\ &= P(Y(t) \in I, \tau_1 > t, \tau_2 > t) + P(Y(t) \in I, \tau_1 > t, \tau_2 < t) \\ &+ P(Y(t) \in I, \tau_1 < t, \tau_2 > t) + P(Y(t) \in I, \tau_1 < t, \tau_2 < t) \end{aligned} \tag{3.67}$$

Observing that the first probability term in the last equation is the one we are looking for we turn to the three other terms.
By total probability we again have

$$P(Y(t) \in I, \tau_1 < t, \tau_2 < t) = P(Y(t) \in I, \tau_1 < \tau_2 < t) + P(Y(t) \in I, \tau_2 < \tau_1 < t) \tag{3.68}$$

and

$$\begin{aligned} P(Y(t) \in I, \tau_1 < t, \tau_2 > t) &= P(Y(t) \in I, \tau_1 < t, \tau_1 < \tau_2) - P(Y(t) \in I, \tau_1 < t, \tau_1 < \tau_2 < t) \\ &= P(Y(t) \in I, \tau_1 < t, \tau_1 < \tau_2) - P(Y(t) \in I, \tau_1 < \tau_2 < t) \end{aligned} \tag{3.69}$$

and finally

$$\begin{aligned} P(Y(t) \in I, \tau_1 > t, \tau_2 < t) &= P(Y(t) \in I, \tau_2 < t, \tau_2 < \tau_1) - P(Y(t) \in I, \tau_2 < t, \tau_2 < \tau_1 < t) \\ &= P(Y(t) \in I, \tau_2 < t, \tau_2 < \tau_1) - P(Y(t) \in I, \tau_2 < \tau_1 < t) \end{aligned} \tag{3.70}$$

Inserting (3.68), (3.69) and (3.70) in (3.67) we get our probability

$$\begin{aligned} P(Y(t)\in I,\tau_1>t,\tau_2>t) &= P(Y(t)\in I)-P(Y(t)\in I,\tau_1<t,\tau_1<\tau_2) \\ &\quad - P(Y(t)\in I,\tau_2<t,\tau_2<\tau_1) \\ &= P(Y(t)\in I)-P_{\tau_1}(t)-P_{\tau_2}(t). \end{aligned} \tag{3.71}$$

The obvious problem now is to compute $P_{\tau_1}(t)$ and $P_{\tau_2}(t)$.
By conditional probability we can rewrite them to

$$\begin{aligned} P_{\tau_1}(t) &= P(Y(t)\in I,\tau_1<t,\tau_1<\tau_2) = P(\tau_1<t,\tau_1<\tau_2)|Y(t)\in I)\cdot P(Y(t)\in I) \\ &= \int_I P(\tau_1<t,\tau_1<\tau_2)|Y(t)=y)\cdot P(Y(t)=y)dy \\ &= \int_I P_1(t,y)\cdot\varphi(y,\sqrt{t})dy \end{aligned} \tag{3.72}$$

and

$$P_{\tau_2}(t) = \int_I P_2(t,y)\cdot\varphi(y,\sqrt{t})dy \tag{3.73}$$

(In [16] there seem to be an notational error in equation (A.1) which also appear in the following equations, but the result is right though).

The probabilities $P_1(t,y)$ and $P_2(t,y)$ can now be found in [2], Theorem 4.2, as two infinite series. We present the probabilities here and give the proof in Appendix B.

The conditional probability of hitting the upper boundary before the lower, for $0\le u\le t$, given that the $Y(t)=y$ is given by

$$\begin{aligned} P_1(t,y) &= \sum_{n=1}^{\infty}\exp\Big[-\frac{2}{t}\{n^2a_1(a_1+b_1t-y)+(n-1)^2a_2(a_2+b_2t-y) \\ &\qquad -n(n-1)[a_1(a_2+b_2t-y)+a_2(a_1+b_1t-y)]\}\Big] \\ &\quad -\sum_{n=1}^{\infty}\exp\Big[-\frac{2}{t}\{n^2[a_1(a_1+b_1t-y)+a_2(a_2+b_2t-y)] \\ &\qquad -n(n-1)a_1(a_2+b_2t-y)-n(n+1)a_2(a_1+b_1t-y)\}\Big]. \end{aligned} \tag{3.74}$$

Now the conditional probability of hitting the lower boundary before the lower, for $0\le u\le t$, can be found simply by replacing (a_1,b_1) by $(-a_2,-b_2)$, (a_2,b_2) by $(-a_1,-b_1)$ and y by $-y$ in (3.74)

$$
\begin{aligned}
P_2(t,y) = \sum_{n=1}^{\infty} \exp\Big[-\frac{2}{t}\{n^2 a_2(a_2+b_2t-y) + (n-1)^2 a_1(a_1+b_1t-y) \\
-n(n-1)[a_2(a_1+b_1t-y)+a_1(a_2+b_2t-y)]\}\Big] \\
-\sum_{n=1}^{\infty} \exp\Big[-\frac{2}{t}\{n^2[a_2(a_2+b_2t-y)+a_1(a_1+b_1t-y)] \\
-n(n-1)a_2(a_1+b_1t-y) - n(n+1)a_1(a_2+b_2t-y)\}\Big].
\end{aligned} \tag{3.75}
$$

We can now compute the probabilities $P_{\tau_1}(t)$ and $P_{\tau_2}(t)$ in (3.72) and (3.73), respectively. Completing the square with the y-term we get

$$
\begin{aligned}
P_{\tau_1}(t) &= \frac{1}{\sqrt{2\pi t}} \int_I P_1(t,y)\cdot e^{\frac{y^2}{2t}}\, dy \\
&= \frac{1}{\sqrt{2\pi t}} \int_I \Big\{ \sum_{n=1}^{\infty} \exp\Big[-\frac{1}{2t}\{[y-2a_2-2n(a_1-a_1)]^2 \\
&\quad +4t[a_2+n(a_1-a_2)]\cdot[b_2+n(b_1-b_2)]\}\Big] \\
&\quad -\sum_{n=1}^{\infty} \exp\Big[-\frac{1}{2t}\{[y-2n(a_1-a_2)]^2 \\
&\quad +4tn[a_1b_2-a_2b_1+n(a_1-a_2)(b_1-b_2)]\}\Big]\Big\} dy.
\end{aligned} \tag{3.76}
$$

Completing the square in $P_{\tau_2}(t)$ we get

$$
\begin{aligned}
P_{\tau_2}(t) &= \frac{1}{\sqrt{2\pi t}} \int_I P_2(t,y)\cdot e^{\frac{y^2}{2t}}\, dy \\
&= \frac{1}{\sqrt{2\pi t}} \int_I \Big\{ \sum_{n=1}^{\infty} \exp\Big[-\frac{1}{2t}\{[y-2a_2+2(n-1)(a_1-a_1)]^2 \\
&\quad +4t[-a_1+n(a_1-a_2)]\cdot[-b_1+n(b_1-b_2)]\}\Big] \\
&\quad -\sum_{n=1}^{\infty} \exp\Big[-\frac{1}{2t}\{[y+2n(a_1-a_2)]^2 \\
&\quad +4tn[a_2b_1-a_1b_2+n(a_1-a_2)(b_1-b_2)]\}\Big]\Big\} dy.
\end{aligned} \tag{3.77}
$$

In order to make $P_{\tau_1}(t)$ and $P_{\tau_2}(t)$ "compatible" so that we can add them nicely together we rearrange them a bit. We want the added result to be a summation from $-\infty$ to ∞. Observe that the first summation term in $P_{\tau_1}(t)$ and $P_{\tau_2}(t)$ are almost similar. Now rearrange the first summation term in $P_{\tau_2}(t)$ by replacing $n-1$ with n, then this summation term equals

$$\sum_{n=0}^{\infty} \exp\Big[-\frac{1}{2t}\{[y-2a_2+2n(a_1-a_2)]^2+4t[-a_1+(n+1)(a_1-a_2)]\cdot[-b_1+(n+1)(b_1-b_2)]\}\Big]$$
$$= \sum_{n=0}^{\infty} \exp\Big[-\frac{1}{2t}\{[y-2a_2+2n(a_1-a_2)]^2+4t[-a_2+n(a_1-a_2)]\cdot[-b_2+n(b_1-b_2)]\}\Big].$$

Rearranging the first summation term in $P_{\tau_1}(t)$ by replacing n with $-n$ this equals

$$\sum_{n=-1}^{-\infty} \exp\Big[-\frac{1}{2t}\{[y-2a_2+2n(a_1-a_2)]^2+4t[-a_2+n(a_1-a_2)]\cdot[-b_2+n(b_1-b_2)]\}\Big]$$

and we notice that these terms can be nicely added to

$$\sum_{n=-\infty}^{\infty} \exp\Big[-\frac{1}{2t}\{[y-2a_2+2n(a_1-a_2)]^2+4t[-a_2+n(a_1-a_2)]\cdot[-b_2+n(b_1-b_2)]\}\Big] \quad (3.78)$$

The second summation term in $P_{\tau_2}(t)$ is rearranged by replacing n with $-n$ and equals

$$\sum_{n=-1}^{-\infty} \exp\Big[-\frac{1}{2t}\{[y-2n(a_1-a_2)]^2+4tn[a_1b_2-a_2b_1+n(a_1-a_2)(b_1-b_2)]\}\Big]$$

now this can easily be added to the last summation term in $P_{\tau_1}(t)$. But the case $n=0$ is not included, so if we make the summation from $-\infty$ to ∞ we have to subtract the case when $n=0$. Then by adding the last term in $P_{\tau_1}(t)$ and $P_{\tau_2}(t)$ we get

$$\sum_{n=-\infty}^{\infty} \exp\Big[-\frac{1}{2t}\{[y-2n(a_1-a_2)]^2+4tn[a_1b_2-a_2b_1+n(a_1-a_2)(b_1-b_2)]\}\Big] - e^{-\frac{y}{2t}}$$

The sum $P_{\tau_1}(t)+P_{\tau_2}(t)$ then becomes

$$\frac{1}{\sqrt{2\pi t}}\int_I \Big[\sum_{n=-\infty}^{\infty} \exp\Big[-\frac{1}{2t}\{[y-2a_2+2n(a_1-a_2)]^2+4t[-a_2+n(a_1-a_2)]\cdot[-b_2+n(b_1-b_2)]\}\Big]$$
$$- \sum_{n=-\infty}^{\infty} \exp\Big[-\frac{1}{2t}\{[y-2n(a_1-a_2)]^2+4tn[a_1b_2-a_2b_1+n(a_1-a_2)(b_1-b_2)]\}\Big] + e^{-\frac{y}{2t}}\Big] dy. \quad (3.79)$$

And at last we get the probability we are seeking from (3.71) and (3.79), the probability that the process stays within the barriers for $0 \le u \le t$

$$
\begin{aligned}
& P(Y(t) \in I, \tau_1 > t, \tau_2 > t) = P(Y(t) \in I) - P_{\tau_1}(t) - P_{\tau_2}(t) \\
= & \frac{1}{\sqrt{2\pi t}} \int_I \Big[\sum_{n=-\infty}^{\infty} \exp\Big[-\frac{1}{2t}\{[y - 2n(a_1 - a_2)]^2 + 4tn[a_1 b_2 - a_2 b_1 + n(a_1 - a_2)(b_1 - b_2)]\}\Big] \\
- & \sum_{n=-\infty}^{\infty} \exp\Big[-\frac{1}{2t}\{[y - 2a_2 + 2n(a_1 - a_2)]^2 + 4t[-a_2 + n(a_1 - a_2)] \cdot [-b_2 + n(b_1 - b_2)]\}\Big]\Big] dy
\end{aligned}
$$

(This is equation (A.7) in [16]).

Now we proceed finding the above probability when modelling with a process $X(s) = X(0) + \mu s + \sigma W(s)$, i.e. with drift, and the same barriers, $\beta_1(s)$ and $\beta_2(s)$, as before.
Using Girsanov's theorem we can find the joint probability that $X(t) \in I$ and $\beta_2(s) < m_X(s) \leq M_X(s) < \beta_2(s)$ for any $0 \leq s \leq t$.
Let $u(s) = \mu/\sigma$ and

$$
Z(s) = e^{-\int_0^s u(v)dW(v) - \frac{1}{2}\int_0^s u^2(v)dv} = e^{-u(s)dW(s) - \frac{1}{2}u(s)s} = e^{-\frac{\mu}{\sigma^2}(X(s)-X(0)) + \frac{\mu^2 s}{2\sigma^2}},
$$

then we get by Girsanov's theorem that

$$
P(X(t) \in I, \tau_1 > t, \tau_2 > t) \;=\; \widetilde{E}\big[Z^{-1}(t) \cdot \mathbf{1}_{\{X(t)\in I, \tau_1 > t, \tau_2 > t\}}\big].
$$

Under this probability measure the X-process is without drift, and the difference from the Y-process is the starting point $X(0) := x_0$ and a variation change from t to $\sigma^2 t$. In order to implement this difference it is enough to change a_1 to $a_1 - x_0 := a_1'$, a_2 to $a_2 - x_0 := a_2'$, y to $x - x_0$ and t to $\sigma^2 t$.
With this change in mind we get from our previous probabability result and the above equation that

$$
\begin{aligned}
& \widetilde{E}\big[Z^{-1}(t) \cdot \mathbf{1}_{\{X(t)\in I, \tau_1 > t, \tau_2 > t\}}\big] \;=\; \int_I Z^{-1}(t) \cdot \widetilde{P}(\tau_1 > t, \tau_2 > t | X(t) = x)\widetilde{P}(X(t) = x)dx \\
= & \frac{1}{\sqrt{2\pi t}\sigma} \int_I e^{\frac{\mu}{\sigma^2}(x - x_0) - \frac{\mu^2 t}{2\sigma^2}} \Big[\\
& \quad \sum_{n=-\infty}^{\infty} \exp\Big[-\frac{1}{2t\sigma^2}\{[x - x_0 - 2n(a_1 - a_2)]^2 \\
& \qquad +4tn[(a_1 - x_0)b_2 - (a_2 - x_0)b_1 + n(a_1 - a_2)(b_1 - b_2)]\}\Big] \\
- & \quad \sum_{n=-\infty}^{\infty} \exp\Big[-\frac{1}{2t\sigma^2}\{[x - x_0 - 2(a_2 - x_0) + 2n(a_1 - a_2)]^2 \\
& \qquad +4t[-(a_2 - x_0) + n(a_1 - a_2)] \cdot [-b_2 + n(b_1 - b_2)]\}\Big]\Big] dx
\end{aligned}
$$

Completing the squares w.r.t. x and noticing that $\varphi(x, \alpha, \sigma)$ is our notation for the density of a varaiable X, normally distributed with expectation α and variance σ^2, the above equation equals

$$
\begin{aligned}
&\int_I \sum_{n=-\infty}^{\infty} \varphi\big(x, x_0 + 2n(a_1 - a_2) + \mu t, \sigma\sqrt{t}\big) \cdot \exp\Big[\frac{2n}{\sigma^2}[\mu(a_1 - a_2) - (a_1 - x_0)b_2 + (a_2 - x_0)b_1 \\
&\qquad -n(a_1 - a_2)(b_1 - b_2)]dx\Big] \\
-&\int_I \sum_{n=-\infty}^{\infty} \varphi\big(x, -x_0 + 2a_2 - 2n(a_1 - a_2) + \mu t, \sigma t\big) \cdot \exp\Big[\frac{2\mu}{\sigma^2}[a_2 - x_0 - n(a_1 - a_2)]\Big] \\
&\qquad \cdot \exp\Big[\frac{2}{\sigma^2}[a_2 - x_0 - n(a_1 - a_2)] \cdot [-b_2 + n(b_1 - b_2)]\Big] dx
\end{aligned}
\tag{3.80}
$$

And finally we use the transform $S(t) = e^{X(t)} = h(X(t))$ to find the joint probability of $S(t) \in I$ and both $\tau_1 > t, \tau_2 > t$, when the boundaries are given as β_1' and β_2' in (3.66). Using a "sloppy" notation we find the respective density

$$
\begin{aligned}
f_{S(t),\tau_1,\tau_2}(s, \beta_1', \beta_2') &= f_{X(t),\tau_1,\tau_2}(h^{-1}(x), \beta_1', \beta_2') \cdot \left|\frac{\partial h^{-1}(x)}{\partial s}\right| \\
&= f_{X(t),\tau_1,\tau_2}(\ln s, \beta_1', \beta_2') \cdot \left|\frac{\partial \ln s}{\partial s}\right| \\
&= f_{X(t),\tau_1,\tau_2}(\ln s, \beta_1, \beta_2) \cdot \frac{1}{s}.
\end{aligned}
\tag{3.81}
$$

Notice the change from β_1' and β_2' to β_1 and β_2. This is so because the event that the process $S(s)$ hits the exponential boundaries, β_1' and β_2', is equivalent to the event that the process $X(s)$ hits the linear boundaries, $\beta_1 = \ln \beta_1'$ and $\beta_2 = \ln \beta_2'$, i.e. $a_1 = \ln c_1$ and $a_2 = \ln c_2$. And notice that because of the transformation $x_0 = \ln s_0$ and that $\mu = \hat{r} = \mu_S - \frac{1}{2}\sigma^2$.

Finally, using (3.80) and (3.81), we get the joint probability that $S(t) \in I$ and $c_2 e^{b_2 t} < m_S(t) \leq M_S(t) < c_1 e^{b_1 t}$

$$
\begin{aligned}
&P(S(t) \in I, c_2 e^{b_2 t} < m_S(t) \leq M_S(t) < c_1 e^{b_1 t}) = \\
&\quad \int_I \sum_{n=-\infty}^{\infty} \varphi\big(\ln s, \ln s_0 + 2n(\ln a_1 - \ln a_2) + \hat{r}t, \sigma\sqrt{t}\big) \cdot \exp\Big[\frac{2n}{\sigma^2}[\hat{r}(\ln a_1 - \ln a_2) \\
&\qquad -(\ln a_1 - \ln s_0)b_2 + (\ln a_2 - \ln s_0)b_1 - n(\ln a_1 - \ln a_2)(b_1 - b_2)]\Big]\big)\frac{ds}{s} \\
&- \int_I \sum_{n=-\infty}^{\infty} \varphi\big(\ln s, -\ln s_0 + 2\ln a_2 - 2n(\ln a_1 - \ln a_2) + \hat{r}t, \sigma\sqrt{t}\big) \\
&\quad \cdot \exp\Big[\frac{2\hat{r}}{\sigma^2}[\ln a_2 - \ln s_0 - n(\ln a_1 - \ln a_2)]\Big] \\
&\qquad \cdot \exp\Big[\frac{2}{\sigma^2}[\ln a_2 - \ln s_0 - n(\ln a_1 - \ln a_2)] \cdot [-b_2 + n(b_1 - b_2)]\Big]\frac{ds}{s} \\
&= \int_I \sum_{n=-\infty}^{\infty} \varphi\big(\ln s, \ln(s_0 a_1^{2n}/a_2^{2n}) + (\mu_S - \sigma^2/2)t, \sigma\sqrt{t}\big) \\
&\qquad \cdot \Big(\frac{a_1^n}{a_2^n}\Big)^{2\mu_S/\sigma^2 - 1}\Big[\Big(\frac{a_2^n}{a_1^n}\Big)^{b_1 - b_2}\Big(\frac{s_0}{a_1}\Big)^{b_2}\Big(\frac{a_2}{s_0}\Big)^{b_1}\Big]^{2n/\sigma^2}\frac{ds}{s} \\
&- \int_I \sum_{n=-\infty}^{\infty} \varphi\big(\ln s, \ln(a_2^{2n+2}/s_0 a_1^{2n}) + (\mu_S - \sigma^2/2)t, \sigma\sqrt{t}\big) \\
&\qquad \cdot \Big(\frac{a_2^{n+1}}{s_0 a_1^n}\Big)^{2\mu S/\sigma^2 - 1}\Big(\frac{a_2^{n+1}}{s_0 a_1^n}\Big)^{2[-b_2 + n(b_1 - b_2)]/\sigma^2}\frac{ds}{s} \qquad \text{Equation (A.11) in [16]}
\end{aligned} \tag{3.82}
$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty}\Big(\frac{a_1^n}{a_2^n}\Big)^{2\mu_S/\sigma^2 - 1}\Big[\Big(\frac{a_2^n}{a_1^n}\Big)^{b_1 - b_2}\Big(\frac{s_0}{a_1}\Big)^{b_2}\Big(\frac{a_2}{s_0}\Big)^{b_1}\Big]^{2n/\sigma^2} \cdot N\Big(\frac{\ln s - \ln(s_0 a_1^{2n}/a_2^{2n}) - (\mu_S - \sigma^2/2)t}{\sigma\sqrt{t}}\Big)\Big|^I \\
&- \sum_{n=-\infty}^{\infty}\Big(\frac{a_2^{n+1}}{s_0 a_1^n}\Big)^{2\mu_S/\sigma^2 - 1}\Big(\frac{a_2^{n+1}}{s_0 a_1^n}\Big)^{2[-b_2 + n(b_1 - b_2)]/\sigma^2} \cdot N\Big(\frac{\ln s - \ln(a_2^{2n+2}/s_0 a_1^{2n}) - (\mu_S - \sigma^2/2)}{\sigma\sqrt{t}}\Big)\Big|^I
\end{aligned} \tag{3.83}
$$

This probability is quite messy, partly because it is represented as an infinite series, but it is shown in [16] to be rapidly converging towards the true value such that we can conclude it to be worthwhile implementing.
The last equation above follows from the assumption that the order of summation and integration can be reversed when the infinite series is bounded in absolute value by a series that converges.

3.4.6 Pricing of European Double Barrier Options

We can now construct a Barrier option which consists of e.g one or two of the already discussed options in this chapter. In this section we give two such examples of possible constructions. First we consider a fairly easy knock-out contract which knocks to zero whenever a barrier is hit within the contract period and which stays a European option if not. Secondly we try to price a combined option which is knocked to an Asian option at a barrier hit and stays a European option if the price process stays within the barriers.

Knock-Out Option

Consider a double barrier call option with curved boundaries $\beta_1 > \beta_2$

$$\mathcal{X}_{\beta_2}^{\beta_1} = \max\left[S(T) - K, 0\right] \cdot \mathbf{1}_{\{\tau_1 > T \bigcap \tau_2 > T\}} \tag{3.84}$$

In other words a European option which is knocked to zero if a barrier is hit.
Using the pricing formula (1.43) in chapter 1 we get the following price of the claim above when assuming deterministic interest rates

$$\begin{aligned}
\mathbf{C}(t, s; K) &= e^{-r(T-t)} \cdot E_{t,s}^{Q}\big[\mathcal{X}_{\beta_2}^{\beta_1}\big] = e^{-r(T-t)} \cdot E_{t,s}^{Q}\big[\max\left[S(T) - K, 0\right] \cdot \mathbf{1}_{\{\tau_1 > T \bigcap \tau_2 > T\}}\big] \\
&= e^{-r(T-t)} \cdot \Big(\int_0^\infty sf(s)ds - K \int_0^\infty f(s)ds \cdot \Big) \cdot \mathbf{1}_{\{\tau_1 > T \bigcap \tau_2 > T \bigcap S(T) \geq K\}}
\end{aligned}$$

where $f(s)$ is the risk-neutral density function of $S(T)$. Letting $g(s)$ be the risk-neutral density function derived by taking the derivative of equation (3.82), and by noticing that we must have $K \in (\beta_1(T), \beta_2(T))$ for the option not to be worthless, we get

$$\mathbf{C}(t, s; K) = e^{-r(T-t)} \cdot \Big(\int_K^{\beta_1(T)} sg(s)ds - K \int_K^{\beta_1(T)} g(s)ds\Big). \tag{3.85}$$

We have under the Q measure

$$\begin{aligned}
g(s) = & \sum_{n=-\infty}^{\infty} \varphi\big(\ln s, \ln\left(s_0 a_1^{2n}/a_2^{2n}\right) + (r - \sigma^2/2)(T-t), \sigma\sqrt{T-t}\big) \\
& \qquad \cdot \Big(\frac{a_1^n}{a_2^n}\Big)^{2r/\sigma^2 - 1} \Big[\Big(\frac{a_2^n}{a_1^n}\Big)^{b_1 - b_2} \Big(\frac{s_0}{a_1}\Big)^{b_2} \Big(\frac{a_2}{s_0}\Big)^{b_1}\Big]^{2n/\sigma^2} \frac{1}{s} \\
& - \sum_{n=-\infty}^{\infty} \varphi\big(\ln s, \ln\left(a_2^{2n+2}/s_0 a_1^{2n}\right) + (r - \sigma^2/2)(T-t), \sigma\sqrt{T-t}\big) \\
& \qquad \cdot \Big(\frac{a_2^{n+1}}{s_0 a_1^n}\Big)^{2r/\sigma^2 - 1} \Big(\frac{a_2^{n+1}}{s_0 a_1^n}\Big)^{2[-b_2 + n(b_1 - b_2)]/\sigma^2} \frac{1}{s}
\end{aligned}$$

where s is our price process.
The first integral seems to cause some trouble and we rewrite $sg(s)$. We try to "implement" s in φ. For simplicity define $d = \ln\left(s_0 a_1^{2n}/a_2^{2n}\right)$, then we have by completing the square w.r.t $\ln s$ that

$$
\begin{aligned}
& s \cdot \varphi\big(\ln s, d + \hat{r}(T-t), \sigma\sqrt{T-t}\big) = e^{\ln s} \cdot \frac{1}{\sqrt{2\pi(T-t)}\sigma} e^{-\frac{1}{2\sigma^2(T-t)}(\ln(s)-d-\hat{r}(T-t))^2} \\
= \; & \frac{1}{\sqrt{2\pi(T-t)}\sigma} e^{-\frac{1}{2\sigma^2(T-t)}(\ln(s)^2 - 2\ln s(d+\hat{r}(T-t)+\sigma^2(T-t)) + (d+\hat{r}(T-t))^2)} \\
= \; & \frac{1}{\sqrt{2\pi(T-t)}\sigma} e^{-\frac{1}{2\sigma^2(T-t)}(\ln s - d - \hat{r}(T-t) - \sigma^2(T-t))^2} \cdot e^{-\frac{1}{2\sigma^2(T-t)}((d+\hat{r}(T-t))^2 - (d+\hat{r}(T-t)+\sigma^2(T-t))^2)} \\
= \; & \varphi\big(\ln s, d + (r + \frac{1}{2}\sigma^2)(T-t), \sigma\sqrt{T-t}\big) \cdot e^{d + r(T-t)} \\
= \; & \varphi\big(\ln s, \ln(s_0 a_1^{2n}/a_2^{2n}) + (r + \frac{1}{2}\sigma^2)(T-t), \sigma\sqrt{T-t}\big) \cdot s_0 \Big(\frac{a_1^n}{a_2^n}\Big)^2 e^{-r(T-t)}.
\end{aligned}
$$

We also complete the square in the "second" density part of $g(s)$

$$
\varphi\big(\ln s, \ln(a_2^{2n+2}/s_0 a_1^{2n+2}) + (r + \frac{1}{2}\sigma^2)(T-t), \sigma\sqrt{T-t}\big) \cdot s_0 \Big(\frac{a_1^n}{a_2^n}\Big)^2 e^{-r(T-t)}.
$$

Now we can calculate the integrals and after some rearranging we get the price of the claim $\mathcal{X}_{\beta_2}^{\beta_1}$ by

$$
\begin{aligned}
\mathbf{C}_{\beta_2}^{\beta_1}(t, s; K) = \; & s_0 \sum_{n=-\infty}^{\infty} \Big\{ \Big(\frac{a_1^n}{a_2^n}\Big)^{c_{1n}^*} \Big(\frac{a_2}{s_0}\Big)^{c_{2n}} \big[N(k_{1n}) - N(k_{2n})\big] - \Big(\frac{a_2^{n+1}}{a_1^n s_0}\Big)^{c_{3n}^*} \big[N(k_{3n}) - N(k_{4n})\big] \Big\} \\
& - K e^{-r(T-t)} \sum_{n=-\infty}^{\infty} \Big\{ \Big(\frac{a_1^n}{a_2^n}\Big)^{c_{1n}^* - 2} \Big(\frac{a_2}{s_0}\Big)^{c_{2n}} \big[N(k_{1n} - \sigma\sqrt{T-t}) - N(k_{2n} - \sigma\sqrt{T-t})\big] \\
& - \Big(\frac{a_2^{n+1}}{a_1^n s_0}\Big)^{c_{3n}^* - 2} \big[N(k_{3n} - \sigma\sqrt{T-t}) - N(k_{4n} - \sigma\sqrt{T-t})\big] \Big\}
\end{aligned} \tag{3.86}
$$

where $c_{1n}^* = 2[r - b_2 - n(b_1 - b_2)]/\sigma^2 + 1$, $c_{2n} = 2n\frac{b_1 - b_2}{\sigma^2}$, $c_{3n}^* = 2[r - b_2 + n(b_1 - b_2)]/\sigma^2 + 1$, and

$$
\begin{aligned}
k_{1n} &= \frac{-\ln K + \ln(s_0 a_1^{2n}/a_2^{2n}) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
k_{2n} &= \frac{-\ln \beta_1(T) + \ln(s_0 a_1^{2n}/a_2^{2n}) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
k_{3n} &= \frac{-\ln K + \ln(a_2^{2n+2}/s_0 a_1^{2n}) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
k_{4n} &= \frac{-\ln \beta_1(T) + \ln(a_2^{2n+2}/s_0 a_1^{2n}) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}
$$

$N[\cdot]$ is the standard normal distribution function.

This is the pricing formula in Theorem 3.1 in Kunitomo & Ikeda [16]. They show how this quite massive formula is a generalization of the Black-Scholes formula. By letting $n = 0$, $a_1 \to +\infty$ and $a_2 \to 0$ the three terms k_{2n} k_{3n} and k_{4n} disappears. The result is identical to the Black-Scholes price of a European option with exercise price K and deterministic interest rate.

"Asian-Knock" Option

We now consider a double barrier option which is knocked to an Asian option with exercise price K_A if a barrier is hit and which stays a European option with exercise price K_E if not. We define the claim to be on the form

$$\mathcal{X}_{\beta_1}^{\beta_2} = \max\left[S(T) - K_E, 0\right] \cdot \mathbf{1}_{\{\tau_1 > T \bigcap \tau_2 > T\}} + \max\left[A_S(t,T) - K_A, 0\right] \cdot \mathbf{1}_{\{\tau_1 \leq T \bigcup \tau_2 \leq T\}}$$

where $A_S(t,T) = \frac{1}{T-t}\int_t^T S(v)dv$. Notice that $\mathbf{1}_{\{\tau_1 \leq T \bigcap \tau_2 \leq T\}} = 1 - \mathbf{1}_{\{\tau_1 > T \bigcup \tau_2 > T\}}$ and we then rewrite our claim to

$$\begin{aligned} \mathcal{X}_{\beta_1}^{\beta_2} &= \max\left[S(T) - K_E, 0\right] \cdot \mathbf{1}_{\{\tau_1 > T \bigcap \tau_2 > T\}} + \max\left[A_S(t,T) - K_A, 0\right] \\ &\quad - \max\left[A_S(t,T) - K_A, 0\right] \cdot \mathbf{1}_{\{\tau_1 > T \bigcap \tau_2 > T\}}. \end{aligned} \tag{3.87}$$

When trying to price this claim we face the same problem with the last term in (3.87) as with pricing the Asian claim in section 3.3, the non-lognormal feature of the Asian option makes it hard to find a closed-form solution of the problem.
Using the pricing formula in Theorem 1.4.1, equation (3.86) and equation (3.17) we find that the price of the above claim is given by

$$\begin{aligned} \mathbf{C}_{\beta_2}^{\beta_1}(t,s;K_A,K_E) &= \mathbf{C}_{\beta_2}^{\beta_1}(t,s;K_E) + \mathbf{C}^A(t,s;K_A) \\ &\quad - e^{-r(T-t)} \cdot E^Q\big[\max\left[A_S(t,T) - K_A, 0\right] \cdot \mathbf{1}_{\{\tau_1 > T \bigcap \tau_2 > T\}}\big]. \end{aligned} \tag{3.88}$$

The obvious problem is the last term which must be evaluated numerically. The Monte Carlo technique, which often is preferred, seem not to perform well in the context of barrier options. This is because of the discrete time points of simulations and the fact that this gives the underlying price process a "chance" to hit a barrier without being detected inbetween the discrete checkpoints. So standard Monte Carlo simulations seem to **overestimate** the hitting time. In [5] Baldi & Caramellino & Iovino use the hit probability in Anderson [2] to develop a procedure which provides an unbiased Monte Carlo estimator of the price of a barrier option. This procedure could be implemented here but instead we make an simplifying assumption in the last term in (3.88). The expectation in (3.88) can be rewritten to

$$
\begin{aligned}
&E^Q\big[\max\left[A_S(t,T)-K_A,0\right]\cdot\mathbf{1}_{\{\tau_1>T\bigcap\tau_2>T\}}\big] = E^Q\big[\max\left[A_S(t,T)-K_A,0\right]\big]\cdot E^Q[\mathbf{1}_{\{\tau_1>T\bigcap\tau_2>T\}}]\\
&+\mathrm{Cov}(\max\left[A_S(t,T)-K_A,0\right],\mathbf{1}_{\{\tau_1>T\bigcap\tau_2>T\}})\\
=\ &E^Q\big[\max\left[A_S(t,T)-K_A,0\right]\big]\cdot Q(\tau_1>T,\tau_2>T)+\mathrm{Cov}(\max\left[A_S(t,T)-K_A,0\right],\mathbf{1}_{\{\tau_1>T\bigcap\tau_2>T\}})
\end{aligned}
$$

where $Q(\tau_1 > T, \tau_2 > T)$ is the probability that the process stays within the barriers under the probability measure Q. This probability is given in (3.82) by changing μ_S to the risk free interest rate r. Now we make the (maybe catastrophical) assumption that the covariance above equals zero. This is obviously not the case but our pricing function gets more handy when simulating its value. Denote our new price approximation $\hat{\mathbf{C}}^{\beta_1}_{\beta_2}(t,s;K_A,K_E)$, we then have

$$
\hat{\mathbf{C}}^{\beta_1}_{\beta_2}(t,s;K_A,K_E) = \mathbf{C}^{\beta_1}_{\beta_2}(t,s;K_E)+\mathbf{C}^A(t,s;K_A)\big(1-Q(\tau_1>T,\tau_2>T)\big). \tag{3.89}
$$

Using standard Monte Carlo simulations $\mathbf{C}^A(t,s;K_A)$ can be approximated with $e^{-r(T-t)}\cdot\frac{1}{N}\sum_{i=1}^N(A_{S_i}(t,T)-K)^+$ and we get

$$
\hat{\mathbf{C}}^{\beta_1}_{\beta_2}(t,s;K_A,K_E) \approx \mathbf{C}^{\beta_1}_{\beta_2}(t,s;K_E)+e^{-r(T-t)}\cdot\frac{1}{N}\sum_{i=1}^N(A_{S_i}(t,T)-K)^+\cdot\big(1-Q(\tau_1>T,\tau_2>T)\big). \tag{3.90}
$$

Chapter 4

Valuation of Guaranteed Contracts

In this chapter we define the guaranteed contract and show that such contracts consists of bonds and options. By changing the option type we get different rate of return structures for the guaranteed contract. We will consider some standard contracts with both single-period and multi-period gurantees. We concentrate on guarantees where the underlying rate of return is the rate of return on stock-like investments but we also give some standard examples of interest rate guarantees, basicly taken from Bacinello & Persson [4], Aase & Persson [1] and Miltersen & Persson [19].

4.1 Introduction

Today guarantees are included in many financial products like life insurance contracts and saving contracts sold by banks and other financial institutions. The guarantee is connected to a rate of return specified in the contract, e.g. rate of return on stock-like investments or the short-term interest rate. Contracts with such guarantees give investors the possibility to join the financial markets without facing the risk of huge losses. Investors with less capital can not in general participate in option markets without facing a considerable risk, but investment banks offering guaranteed contracts make it possible for investors to place their savings in international financial markets. These relatively new type of contracts also widen the productspectre and give investors more flexibility w.r.t. type of investment and risk exposure. Typically these guaranteed contracts are effectice only at the time of expiration of the contract and thereby are called **maturity guarantees**.
In real-life, guaranteed contract specify a **rate of return factor** and a **guaranteed amount**. The rate of return factor decides how much "weight" the option part is given in a contract. Imagine the factor was set to two, i.e. the option is worth twice a option with factor set to one, hence the option price is set two times higher which obviously raises the price of the entire contract. For simplicity we only consider contracts with a factor of one since contracts with a different rate of return factor can easily be derived from the results presented.
The guaranteed amount in the contract depends on contract type and to some extent the investors preferences. Investment banks usually offer contracts where the entired amount is guaranteed, it lies in the nature of the name "guaranteed" that the customer should not lose money on his/her investment, at least in nominal terms. In life insurance the mortality factor

has to be included and the actual type of contract specifies the guarantee. The fact that some policyholders dies or survives the contract period makes such contracts attractive not only for pure insurance arguments but also inhabit a savings aspect.
The recent years financial institutions have offered non-standard contracts where the rate of return is bounded and where the invested amount is not fully guaranteed. Typically the option part of the contract is non-standard in the sense that exotic options, like Barrier options and Collar options, are offered.

4.1.1 Economic Model

We will consider guarantees where the underlying rate of return is either the rate of return from GBM-modelled stock-like investments or the short-term interest rate, where we for practical reasons define rate of return as log returns. All contracts are priced using the theory of arbitrage pricing in financial economics presented in the previous chapters. Consider a mutual fund or a index with market value S_t at time t, modelled with GBM dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{4.1}$$

where W_t is possibly multidimensional.
The short-term interest rate will at this point only be assumed to be **Gaussian**, or in other words all interest rates considered is modelled with deterministic volatilities. We take the forward rate dynamics defined by Heath & Jarrow & Morton as given and specify an example with the Vasiček short rate model.

4.2 Single Period Guarantees

The specified contracts in maturity guarantees are closely related to cashflows of bonds and options. When specifying the Collar contract in the previous chapter we saw that a up-and downside bounded contract (guarantee) consists of a specified number of bonds and options. Likewise we see that by considering a standard single period guarantee this is also the case.

4.2.1 Guarantees on the Stock Market Account

The standard example of a guarantee, where the underlying rate of return is the rate of return from a stock-like investment, is a contract on the form

$$C^E = \Phi(S_T) = \max\left[S_T, K\right] = K + \max\left[S_T - K, 0\right]$$

with closed form pricing solution $\Pi(t, T; \mathcal{X})$

$$\Pi(t, T; \mathcal{X}) = E^Q_{t,s}[e^{-\int_t^T r(v)dv} \cdot K] + E^Q_{t,s}[e^{-\int_t^T r(v)dv} \cdot \max\left[S_T - K, 0\right]] = KB(t, T) + \mathbf{C}(t, s; K)$$

where $\mathbf{C}(t,s;K)$ is the price of an European option with exercise price K.
For practical purposes, and in order to compare different contracts, define π_b and π_o as the price of the bond-and option-part, respectively, and normalize the bond-part by dividing it by π_b, i.e.

$$1 = E_{t,s}^{Q}[e^{-\int_t^T r(v)dv} \cdot e^{g(T-t)}]$$

where $g = \frac{1}{T-t}\ln\left(\frac{K}{\pi_b}\right)$ such that $\frac{K}{\pi_b} = e^{g(T-t)}$. Where g may be interpreted as a guaranteed average rate of return and we observe that $K/\pi_b = B^{-1}(t,T)$. Now, if one NOK is invested and $e^{g(T-t)}$ is guaranteed, the price of a **European contract** is given by

$$1 + \pi^E = e^{g(T-t)}B(t,T) + \mathbf{C}(t,s;e^{g(T-t)}) \tag{4.2}$$

where $\pi = \mathbf{C}(t,T;e^{g(T-t)})$ equals the price of the option-part. And $\mathbf{C}$ is given in section 3.1. Likewise we can consider non-standard contracts like the Collar, Asian or Barrier contracts, the only term changing is the option-part or the market based loading.

Price of a **Collar contract**:

$$1 + \pi^C = e^{g_1(T-t)}B(t,T) + \mathbf{C}(t,s;e^{g_1(T-t)}) - \mathbf{C}(t,s;e^{g_2(T-t)}) \tag{4.3}$$

where $g_1 = \frac{1}{T-t}\ln\left(\frac{K_1}{\pi_b}\right)$, $g_2 = \frac{1}{T-t}\ln\left(\frac{K_2}{\pi_b}\right)$ and $K_2 > K_1 > 0$.

Price of a **Asian contract**:

$$1 + \pi^A = e^{g(T-t)}B(t,T) + \mathbf{C}^A(t,s;e^{g(T-t)}) \tag{4.4}$$

where $\mathbf{C}^A$ is given in (3.17).

Price of a **Double Barrier European knock-out contract**:

$$1 + \pi^{DB} = e^{g(T-t)}B(t,T) + \mathbf{C}_{\beta_2}^{\beta_1}(t,s;e^{g(T-t)}) \tag{4.5}$$

where $\mathbf{C}_{\beta_2}^{\beta_1}$ is taken from (3.86).

4.2.2 Interest Rate Guarantees

A volatile or stochastic interest rate lies in the nature of interest rate guarantees, so modelling with deterministic interest rates makes no sense. We use the HJM-framework and the rate of return guarantee is connected to a guarantee on the short-term interest rate which we model by the Vasiček model in section 2.4.2.

Now, there is no GBM-modelled price process generating the randomness, but a short-term interest rate which is assumed to satisfy the HJM-framework in section 2.4.1. Many short rate models have been proposed but they all seem to turn hard to handle when becoming more realistic, so we only apply the simple Vasiček model.

Consider a contract $\mathcal{X}_T$, where the underlying rate of return is the rate of return from the short-term interest rate r_s, for $t \leq s \leq T$

$$C^r = \Phi(r_s) = \max\left[e^{\int_t^T r_s ds}, K\right] = K + \max\left[e^{\int_t^T r_s ds} - K, 0\right].$$

We see that the contract above can be rewritten to consist of a bond-part and an option-part where the underlying process is no price-process, but the a function of the volatile short-term interest rate.

Normalizing the bond-part and define π^r as the market-based loading we get the price of the contract

$$\begin{aligned}
1 + \pi^r &= E^Q_{t,r_0}\left[e^{-\int_t^T r_s ds} \cdot \max\left[e^{\int_t^T r_s ds}, e^{g(T-t)}\right]\right] = E^Q_{t,r_0}\left[\max[e^0, e^{g(T-t)-\int_t^T r_s ds}]\right] \\
&= e^{g(T-t)} \cdot E^Q_{t,r_0}\left[e^{-\int_t^T r_s ds}\right] + E^Q_{t,r_0}\left[\max[e^0 - e^{g(T-t)-\int_t^T r_s ds}, 0]\right] \\
&= e^{g(T-t)} B(t,T) + \int_{g(T-t)}^{\infty} \left(1 - e^{g(T-t)-x}\right) g(x) dx \\
&= e^{g(T-t)} B(t,T) + \int_{g(T-t)}^{\infty} g(x) dx - e^{g(T-t)} \int_{g(T-t)}^{\infty} e^{-x} g(x) dx \qquad (4.6)
\end{aligned}$$

where g is the density of $\int_t^T r_s ds$.

From the HJM-framework and the assumption of **Gaussian** models the above expectation can be calculated. Consider the variable $\int_t^T r_s ds$, we try some rearranging by using the HJM-drift condition under the Q-measure and from the predefined bond price in section 2.3 we have

$$f(t,s) = -\frac{\partial}{\partial s} \ln B(t,s)$$

this gives

$$\begin{aligned}
\int_t^T r_s ds &= \int_t^T f(t,s) ds + \int_t^T \int_t^s \alpha(u,s) du ds + \int_t^T \int_t^s \sigma(u,s) dW_u ds \\
&= -\ln B(t,T) + \ln B(t,t) + \int_t^T \int_t^s \sigma(u,s) \int_u^s \sigma(u,v) dv du ds + \int_t^T \int_t^s \sigma(u,s) dW_u ds
\end{aligned}$$

changing the order of integration we get

$$\int_t^T \int_t^s \sigma(u,s)dW_u ds = \int_t^T \int_u^T \sigma(u,s)dsdW_u$$

and since

$$\int_t^T \int_t^s \alpha(u,s)duds = \int_t^T \int_u^T \alpha(u,s)dsdu = \int_t^T \int_u^T \sigma(u,s)\int_u^s \sigma(u,v)dvdsdu$$

we have

$$\gamma(t) := \int_u^T \sigma(u,s)\int_u^s \sigma(u,v)dvds = \frac{1}{2}\int_u^T \sigma(u,v)dv \cdot \int_u^T \sigma(u,v)dv = \frac{1}{2}\Big(\int_u^T \sigma(u,v)dv\Big)^2.$$

Finally, we get

$$\int_t^T r_s ds = -\ln B(t,T) + \frac{1}{2}\int_t^T \Big(\int_u^T \sigma(u,v)dv\Big)^2 du + \int_t^T \int_u^T \sigma(u,s)dsdW_u. \tag{4.7}$$

From the Gaussian-assumption we now have that the above expression must be normally distributed with expectation and variance given by

$$\mu_I = -\ln B(t,T) + \frac{1}{2}\int_t^T \Big(\int_u^T \sigma(u,v)dv\Big)^2 du = -\ln B(t,T) + \frac{1}{2}\sigma_I^2$$

and

$$\sigma_I^2 = \int_t^T \Big(\int_u^T \sigma(u,v)dv\Big)^2 du.$$

In other words, $\int_t^T r_s ds \sim N[\mu_I, \sigma_I]$.

Completing the square in the term $e^{-x} \cdot g(x)$ we get $e^{-\mu_I + \frac{1}{2}\sigma_I^2} \cdot h(y)$, where h is the density function of a normally distributed variable, Y, with expectation $(\mu_I - \sigma_I^2)$. From the pricing formula of bonds in section 2.3 we have that $B(t,T) = E^Q[\exp(-\int_t^T r_s ds)] = \exp(-\mu_I + \frac{1}{2}\sigma_I^2)$.

We can now continue with the pricing formula above , normalizing X and Y we get

$$\begin{aligned} 1+\pi^r &= e^{g(T-t)}B(t,T) + P(X \geq g(t-t)) - e^{g(T-t)}B(t,T)P(Y \geq g(T-t)) \\ &= e^{g(T-t)}B(t,T) + N\Big[\frac{-g(T-t)+\mu_I}{\sigma_I}\Big] - e^{g(T-t)}B(t,T)N\Big[\frac{-g(T-t)+\mu_I-\sigma_I^2}{\sigma_I}\Big] \\ &= N\Big[\frac{-g(T-T)+\mu_I}{\sigma_I}\Big] + e^{g(T-t)}B(t,T)N\Big[\frac{g(T-t)-\mu_I+\sigma_I^2}{\sigma_I}\Big]. \end{aligned} \tag{4.8}$$

We have in mind that $\ln B(t,T) = -\mu_I + \sigma_I^2/2$.

4.3 Multi Period Guarantees

Contracts or policies with more than one period guarantee are common in life insurance. Other financial sectors, like investment banks etc, do not seem to offer multi period guarantees to the same extent.
Now we consider guarantees on the stock market account and on the short-term interest rate with a contract period $[t, T]$, where $\tau = T - t$ is divided into n periods of the same constant length Δ. The financial institutions offering these contracts, guarantees a deterministic amount in every subperiod $i = 1, \ldots, n$. We will continue to consider rate of return guarantees, i.e. contracts where log-prices are considered. Our single period contract was, in the interest rate guarantee case, given as $\max\left[\exp\left(\int_t^T r_s ds\right), \exp\left(g(T-t)\right)\right] := \exp\left(\int_t^T r_s ds\right) \vee e^{g(T-t)}$. Since the exp function is monotone and increasing we can rewrite the above expression to $\exp\left(\int_t^T r_s ds \vee g(T-t)\right)$. Modelling a stock-like investment with a GBM-model we can rewrite a guarantee on the stock market account a similar way: $\max\left[S_T, \exp g(T-t)\right] := S_T \vee \exp\left(g(T-t)\right) = \exp\left((r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t) \vee g(T-t)\right)$, where the Q-dynamics of the price process is given by

$$dS_s = rS_s ds + \sigma S_s dW_s, \quad t \leq s \leq T.$$

In the multi period case to now define g_i to be the guaranteed rate of return in period i.

4.3.1 Guarantees on the Stock Market Account

Constant Interest Rate

First we consider a rate of return guarantee when assuming a constant interest rate, C_n with n periods and a deterministic guaranteed amount g_i in subperiod i. Let $t = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = T$, then define the **multiperiod European contract** by

$$\begin{aligned} C_n^E &= \prod_{i=1}^{n} \max\left[\frac{S_{t_i}}{S_{t_{i-1}}}, e^{g_i\Delta}\right] := \prod_{i=1}^{n} e^{\left(\ln S_{t_i} - \ln S_{t_{i-1}}\right)} \vee e^{g_i\Delta} \\ &= \prod_{i=1}^{n} \left(e^{g_i\Delta} + \max\left[e^{(\ln S_{t_i} - \ln S_{t_{i-1}})} - e^{g_i\Delta}, 0\right]\right) \end{aligned}$$

Now if the periodical guarantee is "beaten" in every period the holder of the contract gets S_T/S_0, or a rate of return which is the relative increase of the underlying stock-like process. The holder of the contract will at least get or in other words the holder is guaranteed the amount $e^{\Delta\sum_{i=1}^n g_i}$, which equals e^{ng} when using a constant guarantee and annual periods. Letting $n = 1$ we have a normalized single period contract in the sense that $S_t := 1$. Assuming a GBM-model with constant interest rate r, the price of the n-period **European contract** above, with normalized bond-part, is given by

$$
\begin{aligned}
1+\pi_n^E &= e^{-r(T-t)}E_t^Q\Big[\prod_{i=1}^n \big(e^{g_i\Delta} + \max\big[e^{(r-\frac{1}{2}\sigma^2)(t_i-t_{i-1})+\sigma(W_{t_i}-W_{t_{i-1}})} - e^{g_i\Delta},0\big]\big)\Big] \\
&= \prod_{i=1}^n e^{-r\Delta}E_{t_{i-1}}^Q\Big[e^{g_i\Delta} + \max\big[e^{(r-\frac{1}{2}\sigma^2)(t_i-t_{i-1})+\sigma(W_{t_i}-W_{t_{i-1}})} - e^{g_i\Delta},0\big]\Big] \\
&= \prod_{i=1}^n \Big(e^{g_i\Delta}B(t_{i-1},t_i) + \mathbf{C}\big(t,1;e^{g_i\Delta}\big)\Big) \\
&= \prod_{i=1}^n \Big(N\big[\frac{-g_i\Delta - \ln F_{t_i}}{\sigma\sqrt{\Delta}} + \frac{1}{2}\sigma\sqrt{\Delta}\big] + e^{g_i\Delta}F_{t_i}N\big[\frac{g_i\Delta + \ln F_{t_i}}{\sigma\sqrt{\Delta}} + \frac{1}{2}\sigma\sqrt{\Delta}\big]\Big). \qquad (4.9)
\end{aligned}
$$

This follows from independency and since $F_{t_i} := B(t_{i-1},t_i) = B(t,t_i)/B(t,t_{i-1}) = e^{-r\Delta}$ is measurable at time t. The last equation follows from the assumption of a constant interest rate.
Similar we could consider other contract types with other, more bounded pay-off structures. Consider the **multiperiod Collar contract** defined by

$$
\begin{aligned}
C_n^C &= \prod_{i=1}^n \min\Big[\max\big[\frac{S_{t_i}}{S_{t_{i-1}}}, e^{g_i\Delta}\big], e^{h_i\Delta}\Big] \\
&= \prod_{i=1}^n \Big(e^{g_i\Delta} + \max\big[e^{\left(\ln S_{t_i} - \ln S_{t_{i-1}}\right)} - e^{g_i\Delta},0\big] - \max\big[e^{\left(\ln S_{t_i} - \ln S_{t_{i-1}}\right)} - e^{h_i\Delta},0\big]\Big)
\end{aligned}
$$

where $h_i > g_i$ and $e^{g_i\Delta} > 0$ for all $i = 1, ..., n$.
From the independency the price of the n-period **Collar contract** is given by

$$
\begin{aligned}
1+\pi_n^C &= \prod_{i=1}^n \Big(e^{g_i\Delta}B(t_{i-1},t_i) + \mathbf{C}\big(t,1;e^{g_i\Delta}\big) - \mathbf{C}\big(t,1;e^{h_i\Delta}\big)\Big) \\
&= \prod_{i=1}^n \Big(N[m_1] + e^{g_i\Delta}F_{t_i}N[m_2] - N[m_3] + e^{h_i\Delta}F_{t_i}N[m_4]\Big) \qquad (4.10)
\end{aligned}
$$

where

$$
\begin{aligned}
m_1 &= \frac{-g_i\Delta - \ln F_{t_i}}{\sigma\sqrt{\Delta}} + \frac{1}{2}\sigma\sqrt{\Delta} \qquad & m_2 &= \frac{g_i\Delta + \ln F_{t_i}}{\sigma\sqrt{\Delta}} + \frac{1}{2}\sigma\sqrt{\Delta} \\
m_3 &= \frac{-h_i\Delta - \ln F_{t_i}}{\sigma\sqrt{\Delta}} + \frac{1}{2}\sigma\sqrt{\Delta} \qquad & m_4 &= \frac{-h_i\Delta - \ln F_{t_i}}{\sigma\sqrt{\Delta}} - \frac{1}{2}\sigma\sqrt{\Delta} = m_3 - \sigma\sqrt{\Delta}
\end{aligned}
$$

We define the **multiperiod Asian contract** by

$$C_n^A = \prod_{i=1}^{n} \Big(\max \big[A_S(t_0, t_{i-1}, t_i), e^{g_i \Delta} \big] \Big)$$

where

$$A_S(t_0, t_{i-1}, t_i) = \frac{1}{S_{t_0}} A_S(t_{i-1}, t_i) = \frac{1}{\Delta} \int_{t_{i-1}}^{t_i} e^{(r - \frac{1}{2}\sigma^2)(u - t_{i-1}) + \sigma(W_u - W_{t_{i-1}})} du$$

From independency between periods we get the following pricing formula for the n-period **Asian contract**

$$\begin{aligned} 1 + \pi_n^A &= \prod_{i=1}^{n} \Big(e^{g_i \Delta} F_{t_i} + F_{t_i} E^Q_{t_{i-1},1} \big[\big(A_S(t_0, t_{i-1}, t_i) - e^{g_i \Delta} \big)^+ \big] \Big) \\ &= \prod_{i=1}^{n} \Big(e^{g_i \Delta - r\Delta} + \mathbf{C}^A \big(t_{i-1}, \Delta, e^{g_i \Delta} \big) \Big) \end{aligned} \tag{4.11}$$

where $\mathbf{C}^A\big(t_{i-1}, \Delta, e^{g_i\Delta}\big) = e^{-r\Delta} E^Q_{t_{i-1},1}\big[\big(A_S(t_0, t_{i-1}, t_i) - e^{g_i\Delta}\big)^+\big]$ can be estimated trough Monte Carlo simulations.

We define the **multiperiod Double Barrier European knock-out contract** by

$$C_n^{DB} = \prod_{i=1}^{n} \Big(e^{g_i \Delta} + \max \big[S_{t_i}/S_{t_{i-1}} - e^{g_i \Delta}, 0 \big] \cdot \mathbf{1}_{\{\tau_1(i) > \Delta \bigcap \tau_2(i) > \Delta\}} \Big)$$

where $\tau_1(i)$ and $\tau_2(i)$ are the hitting times of the upper and lower barrier, respectively, in subperiod i. From independency between periods we get the price of the n-period **Double Barrier European knock-out contract**

$$1 + \pi_n^{DB} = \prod_{i=1}^{n} \Big(e^{g_i \Delta - r\Delta} + \mathbf{C}_{\beta_2^i}^{\beta_1^i} \big(t_i, 1; e^{g_i \Delta} \big) \Big). \tag{4.12}$$

Stochastic Interest Rate

When modelling with a stochastic short-term interest rate we encounter problems with dependent periods. In other words we can not immediately take nth product of some expectation. To see this we define the simplest contract, the **n-period European contract**, and let $\int_{t_{i-1}}^{t_i} r_s ds := \delta_i$

$$\begin{aligned} C_n^E &= \prod_{i=1}^{n} \Big(e^{g_i \Delta} + \max \big[e^{(\mu - \frac{1}{2}\sigma^2)\Delta + \sigma(W_{t_i} - W_{t_{i-1}})} - e^{g_i \Delta}, 0 \big] \Big) \\ &= e^{\sum_{i=1}^{n} \{ [(\mu - \frac{1}{2}\sigma^2)\Delta + \sigma(W_{t_i} - W_{t_{i-1}})] \vee g_i \Delta \}} \end{aligned}$$

Since the integral with the short-term interest rate is not Markovian, the periods will be dependent. In [19] Miltersen & Persson give a special case with $n = 2$, i.e. two periods. Modelling with the Vasiček model or any other Gaussian interest model a covariance matrix can be defined for $2n$ multinormal variables. Since many life insurance contracts consist of numerous periods, like 30 to 40 years, the expectation of a expression including almost a hundred multinomal variables may get tedious. Instead simulation of the path of the short-term interest rate and the stochastic process driving the value of the stock-like investment is preferred. The price of the **n-period European contract** is then given by

$$\begin{aligned}
1+\pi_{n,r}^{E} &= E_t^Q\left[e^{-\int_t^T r_s ds}\cdot C_n^E\right] \\
&= E_t^Q\Big[\prod_{i=1}^{n}\left[e^{g_i\Delta-\delta_i}+\max\left[e^{-\frac{1}{2}\sigma^2\Delta+\sigma(W_{t_i}-W_{t_{i-1}})}-e^{g_i\Delta-\delta_i},0\right]\right]\Big] \\
&= E_t^Q\left[e^{\sum_{i=1}^{n}\{[-\frac{1}{2}\sigma^2\Delta+\sigma(W_{t_i}-W_{t_{i-1}})]\vee[g_i\Delta-\delta_i]\}}\right]
\end{aligned}$$

Using a numerical method on the simulated interest rate paths in each subperiod and simulation of the Brownian motion driving the stock-like process, where the Brownian motions in the respective processes may be correlated, we can get an estimat of this price.

Similarly the price if the **n-period Collar contract** is given by

$$\begin{aligned}
1+\pi_{n,r}^{C} &= E_t^Q\Big[\prod_{i=1}^{n}\left[e^{g_i\Delta-\delta_i}+\max\left[e^{-\frac{1}{2}\sigma^2\Delta+\sigma(W_{t_i}-W_{t_{i-1}})}-e^{g_i\Delta-\delta_i},0\right]\right. \\
&\qquad \left.-\max\left[e^{-\frac{1}{2}\sigma^2\Delta+\sigma(W_{t_i}-W_{t_{i-1}})}-e^{h_i\Delta-\delta_i},0\right]\right]\Big]
\end{aligned}\tag{4.13}$$

The price of the **n-period Asian contract** is given by

$$1+\pi_{n,r}^{A} = E_t^Q\Big[\prod_{i=1}^{n}\left[e^{g_i\Delta-\delta_i}+\max\left[A_S(t_0,t_{i-1},t_i)-e^{g_i\Delta-\delta_i},0\right]\right]\Big]\tag{4.14}$$

where $A_S(t_0,t_{i-1},t_i) = \frac{1}{\Delta}\int_{t_{i-1}}^{t_i} e^{-\int_u^{t_i} r_s ds-\frac{1}{2}\sigma^2(u-t_{i-1})+\sigma(W_u-W_{t_{i-1}})}du$.

Since traditional Monte Carlo simulations seem to give poor results w.r.t hit probability we do not bother to give a similar price approximation for the **n-period Double Barrier European knock-out contract**.

4.3.2 Interest Rate Guarantees

When considering multi period interest rate guarantees we face the same computational problem as in the above section. Since the integral of the short-term interest rate is not Markovian we can not claim independency between periods. Modelling with a Gaussian short-term interest rate, like the Vasiček model, we can calculate the expectation under Q of a contract function consisting of n multinormal variables. Specifying a covariance matrix this can be done but e.g. a life insurance contract consisting of 30-40 periods makes it computationally hard. In [19] a 2-period example is given. We define the n-period contract by

$$C_n^r = \prod_{i=1}^{n} \max\left[e^{\int_{t_{i-1}}^{t_i} r_s ds}, e^{g_i\Delta}\right] = \prod_{i=1}^{n}\left(e^{g_i\Delta} + \max\left[e^{\delta_i} - e^{g_i\Delta}, 0\right]\right) = e^{\sum_{i=1}^{n}\delta_i \vee g_i\Delta}$$

Using Monte Carlo simulations the following pricing function can be estimated

$$\begin{aligned} 1+\pi_n^r &= E_t^Q\left[e^{-\int_t^T r_s ds}\cdot\prod_{i=1}^{n}\left(e^{g_i\Delta} + \max\left[e^{\delta_i} - e^{g_i\Delta}, 0\right]\right)\right] \\ &= E_t^Q\left[\prod_{i=1}^{n}\left(e^{g_i\Delta-\delta_i} + \max\left[1 - e^{g_i\Delta-\delta_i}, 0\right]\right)\right] = E_t^Q\left[e^{\sum_{i=1}^{n}\{0\vee[g_i\Delta-\delta_i]\}}\right]. \end{aligned} \tag{4.15}$$

4.4 Life Insurance Applications

Traditionally, life insurance products have guaranteed a deterministic amount or a **benefit** specified by a contractual agreement. Typically an amount is to be paid to the insured if he/she survives, dies or both in a predetermined period. Other traditional contracts pay pre-described amounts as a future pension. The recent years insurance companies have expanded their product spectre by offering unit-link insurances and guaranteed contracts connected to these unit-link contracts. Also interest guarantees have been offered by the insurance companies. In insurance terms we denote the prices of such contracts as their market values, which is paid for with premium payments. As in traditional life insurance these "new" products are usually paid by periodical premiums, often determined in inception of the contract but other premium payment schemes have also been proposed. The insurance companies might want more "control" with long term contracts and may offer contracts with stochastic periodical premiums. See e.g. the unit-link contract constructed in [4] by Bacinello & Persson, where the periodical guarantees are expressed in number of units of the mutual fund giving a stochastic premium payment scheme.

Now, people wanting insurance do not fancy the financial risk of e.g. the mutual fund linked to their premiums (unit-link contract). By issuing the contract with a guarantee this downside risk of the financial markets is avoided. By using the pricing theory from the preceding chapters we can find the market value of this contract. Life insurance contracts must also consider the risk factor of mortality. Usually independence between mortality risk and financial risk is assumed, which computationally makes the valuation of the contract easier.
Guarantees in life insurance, like guaranteed unit-link products and interest rate guarantees, can easily be derived from the presented pricing results by considering this mortality risk. A variety of life insurance contracts can be constructed using these pricing results as "building blocks", so we are content with giving some numerical examples concerning these bulding blocks only.

Chapter 5

Some Numerical Examples

We will restrict ourselves to single payments, or single premiums in insurance terms, and we disregard the mortality risk which life insurance contracts would have to take in consideration. We also restrict ourselves to comparing guarantees with different rate of return structures without performing any type of sensitivity analysis w.r.t. the modelling parameters, except from the length of time , $T - t$, of the various contracts. Since we are not concerned with parameter estimation or sensitivity analysis the parameters in the models are tried being specified from the actual situation in the Norwegian financial market, while parameters concerning the stochastic interest rate process and the market price of risk, λ, Hull [14] has been useful.

In the case of guarantees on the stock market account, S_v we model it by a GBM-model with a one-dimensional Wiener process as its source of uncertainty. We specify whether we a assume constant or stochastic short-term interest rate, and in the last case we use the Vasiček model.

$$dS_v = \alpha S_v dv + \sigma S_v dW_v^1$$
$$dr_v = a(\bar{r} - r_v)dv + \sigma_r dW_v^2$$

For simplicity we assume independence between W^1 and W^2.
We use the following values for the modelling parameters throughout the examples

Parameters: $a = 0.125$, $\bar{r} = 0.06$, $r_t = 0.07$, $\sigma_r = 0.02$, $\lambda = -\sigma_B = -0.05$, $\sigma = 0.15$.

This is consistent with a long term interest rate mean of 0.06 and 0.092 under the measures P and Q, respectively. And a long term interest variance of 0.04^2. In the cases where a constant interest rate is assumed we let $r = \bar{r} = 0.06$.

Now we redefine g to be the guaranteed rate of return, and the examples given uses $g = 0$ and $g = 0.04$. Which in the first case means that only the invested amount is guaranteed while in the second case a rate of return of 4% is guaranteed on the investment.

Single Period Guarantee on the Stock Market Account

An investor wants to invest K NOK and agrees on a contract with an investment bank which places his/her money in a mutual fund with a guarantee. In the case of a European contract the investor is guaranteed his/her investment in monetary value or some positive rate of return. In other words the contract is defined as

$$\mathcal{X}^E = K \max\left[\frac{S_T}{S_t}, e^{g(T-t)}\right] = K + K \max\left[\frac{S_T}{S_t} - e^{g(T-t)}, 0\right]$$

letting π be the market value and price of this contract where π_b and π_o are the prices of the bond-part and option part, respectively, we have

$$\pi = \pi_b + \pi_o = K e^{g(T-t)} B(t,T) + K\mathbf{C}(t,1; e^{g(T-t)}).$$

Or, when K NOK is invested the investor has to pay a market based loading, denoted L. By definition we then must have

$$\pi = \pi_b + \pi_o = K + L$$

In order to express the loading in percent of the invested amount we divide the whole expression with K and get

$$1 + \pi^E = \frac{\pi_b + \pi_o}{K} = e^{g(T-t)} B(t,T) + \mathbf{C}(t,1; e^{g(T-t)}).$$

In the case where a constant interest rate, r, is assumed we have

$$1+\pi^E = N\Big[\frac{(-g+r)(T-t) + 1/2\sigma^2(T-t)}{\sigma\sqrt{T-t}}\Big] + e^{(g-r)(T-t)} N\Big[\frac{(g-r)(T-t) + 1/2\sigma^2(T-t)}{\sigma\sqrt{T-t}}\Big]$$

and

$$\begin{aligned} 1 + \pi_r^E &= N\Big[\frac{-g(T-t) - \ln B(t,T) + 1/2(\sigma-\lambda)^2(T-t)}{(\sigma-\lambda)\sqrt{T-t}}\Big] \\ &+ B(t,T)e^{g(T-t)} N\Big[\frac{g(T-t) + \ln B(t,T) + 1/2(\sigma-\lambda)^2(T-t)}{(\sigma-\lambda)\sqrt{T-t}}\Big] \end{aligned}$$

in the case of a stochastic short-term interest rate and when assuming a constant market price of risk. See Theorem 3.1.2.

For a Collar contract define $K^* = m \cdot K > K$ as an upper bound payoff, and π^C as the price in percent of the invested amount or the market based loading. We define the following contract

$$
\begin{aligned}
\mathcal{X} &= \min\Big[K\max\big[S_T/S_t, e^{g(T-t)}\big], K^*\Big] \\
&= Ke^{g(T-t)} + K\max\big[S_T/S_t - e^{g(T-t)}, 0\big] - K\max\big[S_T/S_t - m, 0\big]
\end{aligned}
$$

with market values

$$
\begin{aligned}
1+\pi^C &= e^{g(T-t)}B(t,T) + \mathbf{C}(t,1;e^{g(T-t)}) - \mathbf{C}(t,1;m) = 1+\pi^E - \mathbf{C}(t,1;m) \\
&= N[w_1] + e^{(g-r)(T-t)}N[w_2] - N[w_3] + m\cdot e^{-r(T-t)}N[w_4] \\
1+\pi_r^C &= 1+\pi_r^E - \mathbf{C}(t,1;m) = N[w_5] + e^{g(T-t)}B(t,T)N[w_6] - N[w_7] + m\cdot B(t,T)N[w_8]
\end{aligned}
$$

in the case of a constant interest rate and a stochastic interest rate , respectively. And where

$$
\begin{aligned}
w_1 &= \frac{(-g+r)(T-t) + 1/2\sigma^2(T-t)}{\sigma\sqrt{T-t}} & w_2 &= \frac{(g-r)(T-t) + 1/2\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\
w_3 &= \frac{-\ln m + r(T-t) + 1/2\sigma^2(T-t)}{\sigma\sqrt{T-t}} & w_4 &= \frac{(-\ln m + r)(T-t) - 1/2\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\
w_5 &= \frac{-g(T-t) - \ln B(t,T) + 1/2(\sigma-\lambda)^2(T-t)}{(\sigma-\lambda)\sqrt{T-t}} & w_6 &= \frac{g(T-t) + \ln B(t,T) + 1/2(\sigma-\lambda)^2(T-t)}{(\sigma-\lambda)\sqrt{T-t}} \\
w_7 &= \frac{-\ln m - \ln B(t,T) + 1/2(\sigma-\lambda)^2(T-t)}{(\sigma-\lambda)\sqrt{T-t}} & w_8 &= \frac{-\ln m - \ln B(t,T) - 1/2(\sigma-\lambda)^2(T-t)}{(\sigma-\lambda)\sqrt{T-t}}.
\end{aligned}
$$

By taking a the price of one particular benchmark bond as given we can, theoretically, decide the maket price of risk, λ. Assuming a constant market price of risk we have from section 3.1.2 that $\lambda = -\sigma_B$, where σ_B is the diffusion term of the benchmark bond.

$T-t=\Delta$	π^E	π_r^E	π^C	π_r^C
1	3.35	4.79	3.35	4.78
2	3.55	5.26	3.55	5.26
3	3.42	5.23	3.42	5.14
4	3.18	5.02	3.15	4.59
5	2.90	4.75	2.75	3.55
6	2.63	4.45	2.20	2.02
7	2.37	4.15	1.39	0.04
8	2.12	3.87	0.29	-2.30
9	1.89	3.60	-1.12	-4.94
10	1.69	3.34	-2.82	-7.78

Table 5.1: Market based loading in **percent** of invested amount when $g = 0$ and $m = K^*/K = 3$.

From Table 5.1 we consider contracts with expiration from one to ten years. We see that the loading of the European contract is higher than the loading of the "respective" Collar contract, which we knew it would be. When modelling with a stochastic interest rate we get

$T-t=\Delta$	π^E	π_r^E	π^C	π_r^C
1	4.98	6.47	4.98	6.47
2	6.47	8.29	6.47	8.29
3	7.39	9.39	7.39	9.30
4	8.02	10.15	7.99	9.72
5	8.48	10.71	8.33	9.52
6	8.82	11.14	8.38	8.71
7	9.06	11.48	8.08	7.37
8	9.24	11.76	7.41	5.60
9	9.36	12.00	6.35	3.47
10	9.44	12.20	4.93	1.07

Table 5.2: Market based loading in **percent** of invested amount when $g = 0.04$ and $m = K^*/K = 3$. I.e. an annual guaranteed rate of return of 4% (continously compunded).

a higher absolute loading than with a constant rate. This follows from the volatility of the interest rate, the a parameter in the Vasiček model and the market price of risk. In the case of a Collar contract we get negative loadings, this is because the option part "bounding" the contract from above becomes more valuable with time to expiration. From Table 5.1 and 5.2 we also observe that the loadings reaches a "top" for the various contracts. This can be explained from the fact that the option part increases in value with time, while the bond part does the opposite. From Table 5.2 we observe how a higher guarantee clearly increases the overall loadings.

We will consider a contract which is denoted an **Asian-tail contract**. The Asian-part is not to be evaluated in the whole contract period but only the last year. Such contracts have been offered in the Norwegian market. We define the contract function to be

$$\mathcal{X}^{AT} = K\cdot\max\left[\frac{1}{n-p}\sum_{i=t_p}^{t_n=T} S_i/S_t, e^{g(T-t)}\right] = Ke^{g(T-t)}+K\cdot\max\left[\frac{1}{n-p}\sum_{i=t_p}^{T} S_i/S_t - e^{g(T-t)}, 0\right]$$

where $t_p := t_{n-1}^+ = (T-1)^+$. Instead of an integral we are taking a mean of the index over the last year. We could consider the mean on e.g. a daily or a monthly basis. Letting $n-p = 12$ and observing the index value the last trading day of every month we get a contract which is less vunerable to price changes at the time of expiration. Letting the invested amount be guaranteed, we get the following percentage market value of this contract when assuming a constant interest rate, r,

$$1+\pi^{AT} = e^{g(T-t)}B(t,T) + \mathbf{C}^{AT}(t,1;e^{g(T-t)})$$

where

$$\mathbf{C}^{AT}(t,1;e^{g(T-t)}) = e^{-r(T-t)}\cdot E_t^Q\left[\max\left[\frac{1}{12}\sum_{i=t_p}^{T} S_i/S_t - e^{g(T-t)}, 0\right]\right]$$

Using Monte Carlo simulation we get an approximate value of π^{AT}.

$T-t$	1	2	3	4	5	6	7	8	9	10
$\pi^{AT}(g_1)$	-0.51	0.42	0.49	0.37	0.12	-0.13	-0.37	-0.56	-0.89	-1.02
$\pi^{AT}(g_2)$	1.28	3.53	4.59	5.43	5.87	6.17	6.53	6.76	8.86	7.00

Table 5.3: Approximate percentage loading of the Asian-tailed contract with $g_1 = 0$ and $g_2 = 0.04$ when 10^6 Monte Carlo simulations are performed. The Asian-part is the mean of the 12 annual observations made on the last trading day in every month.

In the case of a double barrier European knock-out contract we use the notation from section 3.4 and get

$$\mathcal{X}^{DB} = Ke^{g(T-t)} + K \cdot \max\left[S_T/S_t - e^{g(T-t)}, 0\right] \cdot \mathbf{1}_{\{\tau_1 > T \bigcap \tau_2 > T\}},$$

assuming a constant interest rate we get the market value

$$1 + \pi^{DB} = e^{g(T-t)}B(t,T) + \mathbf{C}^{\beta_1}_{\beta 2}(t, 1; e^{g(T-t)}).$$

	$a_1 = 1.8$ $a_2 = 0.5$	$b_1 = 0.1$ $b_2 = -0.1$	$a_1 = 2.5$ $a_2 = 0.3$	$b_1 = 0.08$ $b_2 = -0.08$
$T - t = \Delta$	g=0	g=0.04	g=0	g=0.04
1	3.35	4.98	3.35	4.98
2	3.49	6.41	3.55	6.47
3	3.13	7.12	3.41	7.38
4	2.55	7.45	3.12	7.97
5	1.89	7.56	2.75	8.33
6	1.22	7.55	2.30	8.52
7	0.59	7.47	1.82	8.56
8	0.00	7.34	1.30	8.49
9	-0.53	7.19	0.77	8.34
10	-1.00	7.03	0.24	8.12

Table 5.4: π^{DB} in **percent** of invested amount, with different choices of boundaries and guarantees.

From the tables 5.3 and 5.4 we observe that the Asian-tail- and Barrier contracts give significant lower loadings than the traditional European contract. Especially the Asian-tail seems to perform “well” when it comes to offering attractive and relative simple contracts.

Multi Period Guarantee on the Stock Market Account

Now consider a n-period guarantee which includes a positive and constant rate of return and where the periods are of one years length, i.e $g_i = g$ and $\Delta = 1$, respectively.
We define the various contracts when we assume that K NOK is invested

$$\mathcal{X}_n^E = K \cdot C_n^E \qquad \mathcal{X}_n^C = K \cdot C_n^C \qquad \mathcal{X}_n^A = K \cdot C_n^A \qquad \mathcal{X}_n^{DB} = K \cdot C_n^{DB}$$
$$\mathcal{X}_{n,r}^E = K \cdot C_{n,r}^E \qquad \mathcal{X}_{n,r}^C = K \cdot C_{n,r}^C \qquad \mathcal{X}_{n,r}^A = K \cdot C_{n,r}^A \qquad \mathcal{X}_{n,r}^{DB} = K \cdot C_{n,r}^{DB}$$

where the C contracts are defined in section 4.3, together with the respective market values in percent of the invested amount. Because of the computational aspect we do not consider the above contracts when a stochastic interest rate is assumed.

	$g = 0.04$	$h = 0.2$	$g = 0$	$h = 0.4$
$T - t = \Delta$	π_n^E	π_n^C	π_n^E	π_n^C
1	4.98	3.47	3.35	3.28
2	10.21	7.05	6.81	6.66
3	15.70	10.76	10.39	10.16
4	21.47	14.60	14.09	13.77
5	27.52	18.57	17.91	17.50
6	33.87	22.68	21.86	21.35
7	40.54	26.93	25.94	25.33
8	47.54	31.33	30.16	29.44
9	54.89	35.88	34.52	33.69
10	62.61	40.58	39.03	38.07
15	107.35	66.69	63.93	62.23
20	164.41	97.64	93.29	90.63
25	237.16	134.34	127.90	123.99
30	329.94	177.85	168.72	163.19

Table 5.5: π_n^E and π_n^C in percent of invested amount with different rate of raturn guarantees.

1	2	3	4	5	6	7	8	9	10	15	20	25	30
-0.52	-1.03	-1.55	-2.06	-2.56	-3.07	-3.57	-4.07	-4.57	-5.06	-7.50	-9.87	-12.18	-14.43
1.28	2.58	3.90	5.23	6.58	7.95	9.34	10.74	12.16	13.6	21.08	29.05	37.55	46.60

Table 5.6: Approximate percentage loading of the n-period Asian contract with $g_1 = 0$ and $g_2 = 0.04$, respectively, when 10^6 Monte Carlo simulations are performed. The contract is defined with the mean of the 12 annual observations made on the last trading day in every month.

Comparing the different values in Table 5.5 and Table 5.7 we see how the market values in percent of the Double Barrier contract are indentical (by using two decimals) with the corresponding European contract when the boundaries get "wide" enough. We also notify that by narrowing the boundaries we get relatively big differences in market value.

Interest Rate Guarantees

When modelling the short-term interest rate with the Vasiček model we get a quite sensitive model w.r.t. the a and σ parameters. These parameters should be carefully examined before

$T-t=\Delta$	$a_1=1.2$ $a_2=0.8$ $b_1=0.1$ $b_2=-0.1$	$a_1=2.0$ $a_2=0.4$ $b_1=0.2$ $b_2=-0.2$	$a_1=1.2$ $a_2=0.8$ $b_1=0.1$ $b_2=-0.1$	$a_1=2.0$ $a_2=0.4$ $b_1=0.2$ $b_2=-0.2$
1	3.56	4.98	1.78	3.35
2	7.25	10.21	3.60	6.81
3	11.06	15.70	5.44	10.39
4	15.02	21.47	7.33	14.09
5	19.11	27.52	9.24	17.91
6	23.35	33.87	11.19	21.86
7	27.74	40.54	13.17	25.94
8	32.29	47.54	15.19	30.16
9	36.00	54.89	17.24	34.52
10	41.87	62.61	19.33	39.03
15	68.99	107.35	30.36	63.93
20	101.28	161.41	42.40	93.29
25	139.75	164.41	55.55	127.90
30	185.56	329.94	69.93	168.72
	$g=0.04$	$g=0.04$	$g=0$	$g=0$

Table 5.7: π_n^{DB} in **percent** of invested amount, with different choices of boundaries and guaranteed rate of returns.

implementing the Vasiček model. Letting $g_i = g$ be constant in the n-period case and equal the guaranteed rate of return in the single period case. When defining the n-period contract with $\Delta = 1$ as above, the market values in percent of the invested amount in the single-and n-period case are given in equation 4.8 and equation 4.15 , respectively. When K NOK is invested we define the single-period interest rate guarantee contract by

$$\mathcal{X}^r = K \max\left[e^{\int_t^T r_s ds}, e^{g(T-t)}\right]$$

where g is the annual guaranteed rate of return.
Its market price in percent of the invested amount is given in (4.8).

$T-t=\Delta$	5	10	15	20	25	30	35	40	50	60
$g=0.00$	0.001	0.015	0.032	0.040	0.039	0.032	0.025	0.018	0.008	0.004
$g=0.02$	0.030	0.196	0.379	0.496	0.542	0.535	0.497	0.443	0.327	0.228
$g=0.04$	0.374	1.461	2.616	3.589	4.322	4.831	5.150	5.321	5.347	5.118
$g=0.06$	2.310	6.553	11.303	16.143	20.915	25.563	30.077	34.471	42.974	51.231

Table 5.8: π^r with different guarantees when modelling with a Vasiček model.

From Table 5.8 we see that by guaranteeing the long term interest rate , 6% in our case, substantial loadings are generated while guaranteeing only the invested amount very small but positive loadings are generated. Since the interest rate is always positive in real-life, there can not be loadings when guaranteeing only the invested amount. Since we only consider financial risk and not e.g. credit risk, administration costs etc, the positive loadings from guaranteeing the invested amount are consequences from the Vasiček model and the fact that it may generate a negative interest rate.

We define the multi-period interest rate guarantee contract by

$$\mathcal{X}_n^r = K \cdot \prod_{i=1}^{n} \max\left[e^{\delta_i}, e^{g_i \Delta}\right] = K \cdot e^{\sum_{i=1}^{n}\{\delta_i \vee g_i \Delta\}}$$

and its market price in percent of the invested amount is given by

$$1 + \pi_n^r = E_t^Q\left[e^{\sum_{i=1}^{n}\{0 \vee [g_i \Delta - \delta_i]\}}\right].$$

Monte Carlo simulations supplies us with an estimat of this last expectation.

Some Remarks

- In the multiperiod case the single premiums/payments tend to get very high. But we must remember that such contracts usually contain a periodical saving/investment plan or premium scheme, and that the premiums usually are paid in advance of eventual benefits.
- In some of the examples we get **negative** market based loadings. This does not mean that the guarantees are worthless but rather that the market values of the actual contracts are below the invested amount. The financial institutions offering these contracts still face costs when it comes to administration and hedging their risk-exposure .

Appendix A

Stochastic Calculus

A.1 Measure Theory

Definition A.1.1 σ-algebra *If Ω is a given set, then a σ-algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties*

(i) $\Omega \in \mathcal{F}$

(ii) $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$

(iii) $F_1, F_2 \ldots \in \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} F_i \in \mathcal{F}$

Definition A.1.2 Borel σ-algebra *If $\mathcal{U}$ is the collection of all open subsets of a topological space Ω and $\mathcal{B}$ is the smallest σ-algebra containing $\mathcal{U}$, then $\mathcal{B}$ is called the Borel σ-algebra on Ω*

The pair $(\Omega, \mathcal{F})$ is called a measurable space. On this we can define several different probability measures. For P to be a probability measure defined on $(\Omega, \mathcal{F})$; P must satisfy Kolmogorov's axioms.

Definition A.1.3 Kolmogorov's axioms *A probability measure P on a measurable space $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ such that*

(i) $P(\Omega) = 1$

(ii) $P(F) \geq 0 \;\; \forall F \in \mathcal{F}$

(iii) If $F_1, F_2, \ldots \in \mathcal{F}$ *and* $F_{i\,i=1}^{\infty}$ *are mutually disjoint, then* $P(\cup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} P(F_i)$

The triple $(\Omega, \mathcal{F}, P)$ defines a probability space.

Definition A.1.4 *Let P and Q be two probability measures on $\mathcal{F}$. Then P is absolute continous w.r.t. Q, and we write $P \ll Q$ iff $Q(F) = 0 \Rightarrow P(F) = 0 \;\; \forall F \in \mathcal{F}$*

Theorem A.1.1 Radon-Nikodym
$P \ll Q$ iff it exists a measurable function $f(\omega) \geq 0$ such that:

$$P(F) = \int_F 1 \cdot \mathrm{d}P(\omega) = \int_F f(\omega) \cdot \mathrm{d}Q(\omega)$$

Or

$$\mathrm{d}P(\omega) = f(\omega) \cdot \mathrm{d}Q(\omega)$$

Similary $Q \ll P$ iff:

$$\mathrm{d}Q(\omega) = g(\omega) \cdot \mathrm{d}P(\omega)$$

For a measurable function $g(\omega) \geq 0$.

For proof see [15].

If we have both $P \ll Q$ and $Q \ll P$, then P and Q are said to be equivalent and we write $P \sim Q$. In this case we have both:

$$\mathrm{d}P(\omega) = f(\omega) \cdot \mathrm{d}Q(\omega) \quad \textit{and} \quad \mathrm{d}Q(\omega) = g(\omega) \cdot \mathrm{d}P(\omega)$$

Thereby we get: $g(\omega) = \frac{1}{f(\omega)}$ and $P(\omega) = 0 \Leftrightarrow Q(\omega) = 0$.

Definition A.1.5 Random Variable *Let $(\Omega, \mathcal{F}, P)$ be an arbitrary probability space, and let $X = X(\omega)$ be a real-valued function on Ω so that:*

$$F(x) = P(X \leq x) \qquad -\infty < x < \infty$$

is defined. Then X is a random variable.

Definition A.1.6 Stochastic Process *A stochastic process X, is a parameterised collection of random variables*

$$\{X_t\}_{t \in T}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbf{R}$. The parameter space is usually the halfline $[0, \infty)$, but it may also be the interval $[a, b]$ etc. (The notation $X(t, \omega)$ or $X_t(\omega)$ is also used).

Note that for each $t \in T$ fixed, we have a random variable

$$\omega \to X_t(\omega); \quad \omega \in \Omega.$$

On the other hand, fixing $\omega \in \Omega$ we can consider the function

$$t \to X_t(\omega); \quad t \in \Omega$$

which is called the *path* of X_t.

Definition A.1.7 Adapted Process *Let $\{\mathcal{F}_t\}_{t \geq 0}$ be an increasing family of σ-algebras of subsets of Ω. A process $X_t(\omega) : [0, \infty) \times \Omega \to \mathbf{R}$ is called $\mathcal{F}_t$-adapted if for each $t \geq 0$ the function*

$$\omega \to X_t(\omega)$$

is $\mathcal{F}_t$-measurable.

$X_t(\omega)$ being $\mathcal{F}_t$-measurable means that the value of $X_t(\omega)$ is "known" from the values of the stochastic process generating the information,i.e. the family of σ-algebras.

Definition A.1.8 Filtration and Martingale *A* filtration on $(\Omega, \mathcal{F})$ *is a family* $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ *of* σ*-algebras* $\mathcal{F}_t \subset \mathcal{F}$ *such that*

$$0 \leq s < t \Rightarrow \mathcal{F}_s \subset \mathcal{F}_t$$

(i.e. $\{\mathcal{F}_t\}$ *is increasing). A stochastic process* $\{X_t\}_{t\geq 0}$ *on* $(\Omega, \mathcal{F}, P)$ *is called a* martingale *with respect to a filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *(and with respect to* P*) if*

(i) X_t *is* $\mathcal{F}$*-measurable for all* t

(ii) $E^P[|X_t|] < \infty$ *for all* $t \geq 0$

(iii) $E[X_s|\mathcal{F}_t] = X_t$ *for all* $s \geq t$

Definition A.1.9 Local martingale *A* $\mathcal{F}_t$*-adapted stochastic process* $\{Z(t)\}$ *is called a* local martingale *w.r.t. to the given filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *if there exists an increasing sequence of* $\mathcal{F}_t$*-stopping times* τ_k *such that*

$$\tau_k \to \infty \quad a.s \ as \ k \to \infty$$

and

$$Z(\tau \wedge \tau_k) \quad is \ an \ \mathcal{F}_t - martingale \ for \ all \ k.$$

Definition A.1.10 Markov Process *Let* X *be an* $\mathcal{F}$*-adapted process on the probability space* $(\Omega, \mathcal{F}, P)$*. We say that* X *is a Markov-process if*

$$P(X_s < a|\mathcal{F}) = P(X_s < a|X_t) \ \ \forall s \geq t, a \in \mathbf{R}$$

In other words, if X is a Markov process, the probability of any particular future behaviour of the process, when its current state is known exactly, is not altered by additional knowledge concerning its past behaviour.

A.2 Brownian Motion and Itô Stochastic Calculus

Brownian motion or Wienerprocesses has become very important in financial modelling the recent years and we give the following definition:

Definition A.2.1 1-dimensional Brownian motion *The only continous stochastic process* $\{W_t(\omega); t \geq 0\}$ *with values in* $\mathbf{R}$ *which satifies the following conditions*

(i) For any time points $0 = t_0 < t_1 < \ldots < t_n$ *the process increments* $\{W_{t_{k+1}} - W_{t_k}\}_{k=1}^n$ *are independent random variables.*

(ii) For any time points $t_0 = 0 < t_1 < \ldots < t_n$ *the process increments* $\{W_{t_{k+1}} - W_{t_k}\}_{k=0}^n$ *are stationary.*

(iii) Every increment $W_{t+s}(\omega) - W_t(\omega) \sim \mathcal{N}(0, s)$ *i.e. normally distributed with mean equal to zero and variance equal to s.*

(iv) $E[W_t] = 0$ *and* $E[W_{t_{k+1}} - W_{t_k}] = 0 \ \forall \ k \in [0, n]$*.*

is Brownian motion(Wiener process).

Definition A.2.2 *Let $\mathcal{V} = \mathcal{V}(S, T)$ be the class of functions*

$$f(t, \omega) : [0, \infty) \times \Omega \to \mathcal{R}$$

such that

(i) $(t, \omega) \to f(t, \omega)$ *is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ -algebra on $[0, \infty)$.*

(ii) $f(t, \omega)$ *is $\mathcal{F}_t$-adapted.*

(iii) $E\big[\int_S^T f(t, \omega)^2 dt\big] < \infty$

Definition A.2.3 The Itô Integral *For $f \in \mathcal{V}$ we define the Itô integral as*

$$\mathcal{I}[f](\omega) = \int_S^T f(t, \omega) dW_t(\omega)$$

The variations of the paths of W_t are so big that Brownian motion is nowhere differentiable, that is the total variation of the path is infinite.
Brownian motion is thereby continous but not Newtonian differentiable in the Riemann-Stieltjes sense.Because of this we need a "special" stochastic calculus, e.g. *Itô* or *Stratonovich* stochastic calculus, when we are evaluating stochastic integrals.
For the latter one the (Stratonovich) stochastic integral is <u>not</u> adapted to the filtration $\{\mathcal{F}^W\}$ generated by the Brownian motion and is thereby not a martingale.

This specific feature of the Stratonovic model, that is "looking into the future", is not very appealing. The Itô interpretation, which is adapted to the filtration, is based on <u>not</u>"looking into the future" and is much more appealing when modelling e.g. financial structures as stocks and indices.
Because of this the Itô stochastic calculus will be used and as a result Brownian motion will satisfy both the martingale and the Markov property.

By letting $f \in \mathcal{V}$ and using the Itô isometry we can make the Itô integral well-defined. For the construction and the properties of the Itô integral and more about the Stratonovich integral see [24].

Lemma A.2.1 Itô Isometry *If $f \in \mathcal{V}(T, S)$ and T and S are fixed, we have*

$$E\left[\left(\int_S^T f(t, \omega) dW_t(\omega)\right)^2\right] = \int_S^T E\Big[f^2(t, \omega)\Big] dt \tag{A.1}$$

For proof see [24] p29.

Lemma A.2.2 *If $f \in \mathcal{V}(T, S)$ and T and S are fixed, we have*

$$E\left[\int_S^T f(t,\omega)dW_t(\omega)\right] = 0 \tag{A.2}$$

For proof see [24] p30.

We can extend the Itô integral for a larger class of integrands than $\mathcal{V}$. By changing *(ii)* and *(iii)* in definition A.1.11 we get the following definition

Definition A.2.4 *Let $\mathcal{W}_\mathcal{H} = \mathcal{W}_\mathcal{H}(S,T)$ denote the class of functions*

$$f(t,\omega) : [0,\infty) \times \Omega \to \mathcal{R}$$

such that

(i) $(t,\omega) \to f(t,\omega)$ *is* $\mathcal{B} \times \mathcal{F}$*-measurable, where* $\mathcal{B}$ *denotes the Borel* σ*-algebra on* $[0,\infty)$.

(ii)' There exists an increasing family of σ*-algebras* $\mathcal{H}_t; t$ *such that*

a) W_t *is a martingale with respect to* $\mathcal{H}_t$ *and*

b) $f(t)$ *is* $\mathcal{H}_t$*-adapted*

(iii)' $P\left[\int_S^T f(s,\omega)^2 ds < \infty\right] = 1.$

Note that condition *(iii)'* weakens condition *(iii)* of Definition A.1.11 and thereby gives no guarantee for the Itô isometry.

Now follows a theorem about the Itô integral, giving it an important computational advantage w.r.t. e.g. the Stratonovich integral. The proof can be found in [24] p32-33.

Theorem A.2.1 *Let* $f(t,\omega) \in \mathcal{V}(0,T)$ *for all* T*. Then the Itô integral*

$$\mathcal{I}_t(\omega) = \int_0^t f(s,\omega)dW_s$$

is a martingale w.r.t. $\mathcal{F}_t$ *and*

$$P\big[\sup_{0\leq t\leq T} |\mathcal{I}_t| \geq \lambda\big] \leq \frac{1}{\lambda^2} E\Big[\int_0^T f(s,\omega)^2 ds\Big]; \qquad \lambda, T > 0.$$

The last part resulting from **Doob's martingale inequality** (see [24] p31) and the Itô isometry.

We now introduce **Itô processes** as sums of dW_s and ds integrals:

Definition A.2.5 *An Itô process is a stochastic process* X *on* $(\Omega,\mathcal{F},P)$ *of the form*

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dW_s,$$

where $v \in \mathcal{W}_{\mathcal{H}}$ *so that*

$$P\left[\int_0^t v(s,\omega)^2 ds < \infty\right] = 1 \quad \forall t \geq 0$$

We also assume that $u(t,\omega)$ *is* $\mathcal{H}_t$*-adapted and*

$$P\left[\int_0^t |u(s,\omega)| ds < \infty\right] = 1 \quad \forall t \geq 0$$

Usually we write the Itô process in the shorter differential form

$$dX_t = udt + vdW_t$$

To determine uniqueness for **stochastic differential equations**, we need some conditions. The following Theorem is taken from [24] p66:

Theorem A.2.2 Existence and uniqeness *Let us suppose the coefficients of the stochastic differential equation*

$$dX_t = u(t, X_t)dt + v(t, X_t)dW_t \tag{A.3}$$

satisfy the conditions

$$|u(t,x) + v(t,y)| \leq C(1 + |x|), \tag{A.4}$$

$$|u(t,x) - u(t,y)| + |v(t,x) - v(t,y)| \leq D(|x - y|), \tag{A.5}$$

for every $0 \leq t < \infty$ *and* $x \in \mathbf{R}$, $y \in \mathbf{R}$, *where* C *and* D *are constants Then uniqueness holds for the equation* (A.1).

Usually Itô processes are written on differential form. Itô's formula is an indispensable tool for evaluating functions of Itô processes. The 1-dimensional Itô's formula is given below. The general expression for higher dimensions can be found in [24].

Theorem A.2.3 (The 1-dimensional Itô's formula) *Let* dX_t *be a stochastic differential equation given by*

$$dX_t = u(t, X_t)dt + v(t, X_t)dW_t$$

where u *and* v *satisfy* (A.2) *and* (A.3). *Let* $g(t,x) \in C^{1,2}([0,\infty) \times \mathbf{R})$. *Then*

$$Y_t = g(t, X_t)$$

is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2,$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ *is computed according to the rules*

$$dt \cdot dt = 0,\ dt \cdot dW_t = 0,\ dW_t \cdot dW_t = dt.$$

For proof see [24].

Definition A.2.6 Itô diffusion *A time homogeneous, 1-dimensional Ito diffusion is a stochastic process $X_t(\omega) = X(t, \omega) : [0, \infty) \times \Omega \to \mathbf{R}$ satisfying a stochastic differential equation of the form*

$$dX_t = u(X_t)dt + v(X_t)dW_t, \quad t \geq s; \; X_s = x$$

where W_t is a Brownian motion and u and v satisfy (A.2) *and* (A.3).

The Markov property is satisfied for all diffusion processes.
Notice the difference between Itô processes and Itô diffusions, that is the diffusion is a special case of the process.

The Martingale and Itô Representation Theorem

From theorem A.1.2 we have that every stochastic integral satisfying a certain condition regarding the integrand is a martingale w.r.t. the filtration $\mathcal{F}_t$ and w.r.t. the probability measure P.
Here we will give some additional and useful results stating the the converse is also true. That is that any $\mathcal{F}_t$-martingale w.r.t. P can be represented as an stochastic integral, i.e. an Itô integral.

Theorem A.2.4 The Itô representation theorem *Let $F \in L^2(\mathcal{F}_T, P)$. Then there exists a unique stochastic process $f(s, \omega) \in \mathcal{V}(0, T)$ such that*

$$F(\omega) = E[F] + \int_0^T f(s, \omega)dW_s$$

For proof se e.g. [24] p52-53.

Theorem A.2.5 The martingale representation theorem *Suppose M_t is an $\mathcal{F}_t$-martingale w.r.t. P and that $M_t \in L^2(P)$ for all $t \geq 0$. Then there exists a unique stochastic process $g(s, \omega) \in \mathcal{V}(0, t)$ for all $t \geq 0$ and*

$$M_t(\omega) = E[M_0] + \int_0^t g(s, \omega)dW_s \qquad a.s \; \forall t \geq 0$$

For proof see e.g. [24] p53-54 .

A.3 Girsanov's Theorem

The following result is fundamental in stochastic analysis. It is an important "tool" in finance and gives us some nice analytical results.
The Girsanov theorem gives us a way to change the *drift* coefficient of a given Itô process without changing the *law* of the process dramatically:

Theorem A.3.1 The Girsanov theorem *Let $X(t)$ be an Itô process of the form*

$$dX_t = \beta(t, \omega)dt + \theta(t, \omega)dW_t; \; t \leq T$$

Suppose there exists processes $u(t,\omega)$ *and* $\alpha(t,\omega)$ *both* $\in \mathcal{W}_{\mathcal{H}}$ *such that*

$$\theta(t,\omega) \cdot u(t,\omega) = \beta(t,\omega) - \alpha(t,\omega)$$

and assume that $u(t,\omega)$ *satisfies Novikov's condition.*(See below)

$$E\Big[exp\Big(\frac{1}{2}\int_0^T u^2(s,\omega)ds\Big)\Big] < \infty.$$

Put

$$M_t = exp\Big(-\int_0^t u(s,\omega)dW_s - \frac{1}{2}\int_0^t u^2(s,\omega)ds\Big); \quad t \leq T$$

and

$$dQ(\omega) = M_T(\omega)dP(\omega) \quad on\ \mathcal{F}_T^W.$$

Then

$$\widetilde{W}_t := \int_0^t u(s,\omega)ds + W_t; \quad t \leq T$$

is a Brownian motion w.r.t. Q*. In terms of* $\widetilde{W}_t$ *the process* $X(t)$ *satisfies the following stochastic integral*

$$dX(t) = \alpha(t,\omega)dt + \theta(t,\omega)d\widetilde{W}_t.$$

For proof see [24] p156.

The law of the new process will be absolutely continous w.r.t the law of original process and we can compute explicitly the Radon-Nikodym derivative

$$M_T = \frac{dQ(\omega)}{dP(\omega)} \qquad on\ \mathcal{F}_T$$

Proposition A.3.1 Novikov's condition *A sufficient condition that* M_t *is a martingale is the* Kazamaki condition

$$E\Big[exp\Big(\frac{1}{2}\int_0^t u(s,\omega)dW_s\Big)\Big] < \infty \quad \forall\ t \leq T.$$

This is implied by the following and stronger Novikov condition

$$E\Big[exp\Big(\frac{1}{2}\int_0^T u^2(s,\omega)ds\Big)\Big] < \infty.$$

For proof see e.g. [15].

Appendix B

Proof of Anderson's Probabilities

Here we try to give the proofs of the probabilities $P_1(t,y)$ and $P_2(t,y)$, given in equation (3.74) and (3.75), respectively.
A more formal and general proof can be found in [2], in Theorem 4.1, Lemma 4.1 and Theorem 4.2.

Before specifying $P_1(t,y) = P(\tau_1 < t, \tau_1 < \tau_2 | Y(t) = y)$ and $P_2(t,y) = P(\tau_2 < t, \tau_2 < \tau_1 | Y(t) = y)$ we find the probabilities $P(\tau_1 < \tau_2)$ and $P(\tau_2 < \tau_1)$. These are given in Theorem 4.1 in [2].

We are going to consider the process $Y(t) = W(t)$, i.e. standard Brownian motion, and the linear boundaries $\beta_1(t)$ and $\beta_2(t)$, specified in (3.65), and assuming that the boundaries are not crossing in the interval $s \in [0,t]$.

Now we begin with the probability $P(\tau_1 < \tau_2 < t)$ which is the probability that the Brownian motion hits the upper boundary before the lower without specifying any time-constraint. In other words, the probability of the event $\tau_1 < \tau_2$. We can "divide" this event in an infinite series of events by letting $A_i,\ i = 1, 2, \ldots$ be the event of a path $y(s)$ hitting the upper boundary, β_1, and then alternating between hitting the lower and upper boundary $i-1$ times, regardless of other contacts with the boundaries, followed by hitting the upper boundary the i'th time. I.e. hitting the upper boundary, β_1, i times and the lower, β_2, $i-1$ times.
And let $B_i,\ i = 1, 2, \ldots$ be the event of a path $y(s)$ hitting the lower boundary and then hitting the upper boundary, and then alternating between hitting the lower and upper boundary $i-1$ times, regardless of other contacts with the boundaries. I.e. hitting both boundaries i times in sequence.

Then the event $\tau_1 < \tau_2$ is equivalent to

$$(A_1 - B_1) + (A_2 - B_2) + (A_3 - B_3) + \cdots . \tag{B.1}$$

This obviously gives us

$$
\begin{aligned}
P(\tau_1 < \tau_2) &= P\big\{(A_1 - B_1) + (A_2 - B_2) + (A_3 - B_3) + \cdots\big\} \\
&= P(A_1) - P(B_1) + P(A_2) - P(B_2) + \cdots .
\end{aligned}
$$

We now proceed by finding $P(A_k)$ and $P(B_k)$ when $k = 1, 2, \cdots$.
In order to find these probabilities we use the *reflection principle*. We have our "initial" boundaries β_1 and β_2 by

$$
\beta_1 = a_1 + b_1 s \qquad \beta_2 = a_2 + b_2 s
$$

with the conditions $a_1 > a_2$, $\beta_1 > \beta_2$ and $b_1 \geq 0$.

The probability of hitting β_1 is

$$
P(A_1) = \lim_{t\to\infty} P\Big(\sup_{0\leq s\leq t} Y(s) \geq \beta_1\Big) = \lim_{t\to\infty} P(M_Y(t) \geq \beta_1). \tag{B.2}
$$

By total probability , Girsanov's theorem and the reflection principle we have

$$
\begin{aligned}
P(M_Y(t) \geq \beta_1) &= P(M_Y(t) \geq \beta_1, Y(t) \geq \beta_1) + P(M_Y(t) \geq \beta_1, Y(t) < \beta_1) \\
&= P(Y(t) \geq \beta_1) + \widetilde{E}(Z^{-1}(t) \cdot \mathbf{1}_{\{M_Y(t)\geq\beta_1, Y(t)<\beta_1\}}) \\
&= N\Big(-\frac{\beta_1(t)}{\sqrt{t}}\Big) + \widetilde{E}(e^{-b_1 W(t) + \frac{1}{2}b_1^2 t} \cdot \mathbf{1}_{\{Y(t)\geq 2\beta_1 - \beta_1\}}) \\
&= N\Big(-\frac{a_1 + b_1 t}{\sqrt{t}}\Big) + \widetilde{E}(e^{-b_1(2\beta_1 - Y(t)) + \frac{1}{2}b_1^2 t} \cdot \mathbf{1}_{\{Y(t)\geq 2\beta_1 - \beta_1\}}) \\
&= N\Big(-\frac{a_1 + b_1 t}{\sqrt{t}}\Big) + e^{-2a_1 b_1} \cdot \widetilde{E}(e^{b_1 \widetilde{W}(t) - \frac{1}{2}b_1^2 t} \cdot \mathbf{1}_{\{Y(t)\geq 2\beta_1 - \beta_1\}}) \\
&= N\Big(-\frac{a_1 + b_1 t}{\sqrt{t}}\Big) + e^{-2a_1 b_1} \cdot \widehat{P}(Y(t) \geq 2\beta_1 - \beta_1) \\
&= N\Big(-\frac{a_1 + b_1 t}{\sqrt{t}}\Big) + e^{-2a_1 b_1} \cdot N\Big(\frac{2b_1 t - \beta_1}{\sqrt{t}}\Big). \\
&= N\Big(-\frac{a_1 + b_1 t}{\sqrt{t}}\Big) + e^{-2a_1 b_1} \cdot N\Big(\frac{b_1 t - a_1}{\sqrt{t}}\Big).
\end{aligned} \tag{B.3}
$$

Where $Z(t) = e^{-u(t)W(t) - \frac{1}{2}u(t)^2 t}$, $u(t) = 0 - b_1 = -b_1$, $dY(t) = b_1 dt + d\widetilde{W}(t)$ and $d(Y(t) = 2b_1 dt + \widehat{W}(t)$.

From (B.2) and (B.3) we have

$$
P(A_1) = \lim_{t\to\infty} P(M_Y(t) \geq \beta_1) = e^{-2a_1 b_1}. \tag{B.4}
$$

The other cases, $P(A_k)$ for $k > 1$, can now be shown to be functions of $P(A_1)$.

Consider a path hitting the boundaries β_1 and β_2 in sequence: $\beta_1, \beta_2, \beta_1, \ldots, \beta_1, \beta_2$, e.g. with $k-1$ hits of β_1 and β_2, and which starts with hitting β_1. We denote this alternating sequence by: $L_1, L_2, L_3, \ldots, L_{2k-2}$, where $L_1 = \beta_1, L_2 = \beta_2, \ldots, L_{2k-3} = \beta_1, L_{2k-2} = \beta_2$.
Let t_{2k-2} be the first value of t of which the path hits $L_{2k-2} = \beta_2$, after hitting $L_1, \ldots, L_{2k-3}$ in sequence. The conditional probability of then hitting $L_{2k-1} = \beta_1$ is

$$e^{-2b_1[(b_1+b_2)t_{2k-2}+a_1+a_2]}. \tag{B.5}$$

This because the line/boundary $L_{2k-1} = \beta_1$ has slope b_1 and has intercept $(b_2 t_{2k-2} + a_2) + (b_1 t_{2k-2} + a_1)$ when referring to $(t_{2k-2}, -b_2 t_{2k-2} - a_2)$ as our "new" origin.
Now this conditional probability equals the conditional probability of the process $Y'(s) = b_2 s + W(s)$ hitting the line $\beta_4 = 2\beta_2 - \beta_1$. Now instead of pursuing this argumentation Anderson, [2], proposes an equivalent conditional probability by rearranging (B.5)

$$e^{-2(b_1+b_2)\left[\frac{b_1[(b_1+b_2)t_{2k-2}+a_1+a_2]}{b_1+b_2}\right]}. \tag{B.6}$$

This is the conditional probability of hitting a line with slope $b_1 + b_2$ and which at time t_{2k-2} is a distance of

$$h = \frac{b_1[(b_1 + b_2)t_{2k-2} + a_1 + a_2]}{b_1 + b_2}$$

above $L_{2k-2} = \beta_2$.
Since $Y(s)$ is without drift we can immediately use the reflection principle and rephrasing the last conditional probability to the conditional probability of hitting the line with slope $-(b_1 + b_2)$, and which at time t_{2k-2} is h below $L_{2k-2} = \beta_2$.
Adding a_2 to h and letting $t_{2k-2} = 0$ we get the constant term of this new line, L^*, which is given by

$$L^* : y = -(b_1 + b_2)t - \frac{a_1 b_1 + a_2 b_2 + 2a_2 b_1}{b_1 + b_2} \tag{B.7}$$

By drawing a figure we easily see that by applying the reflection principle again we get that the probability of hitting $L_1, L_2, L_3, \ldots, L_{2k-1}$ in sequence equals the probability of hitting $L_1, L_2, L_3, \ldots, L_{2k-2}$ in sequence and then the line L^* in (B.7). But since L^* lies below $L_{2k-2} = \beta_2$ it can only be touched if the path already has touched $L_{2k-3} = \beta_1$, and since $L_{2k-2} = \beta_2$ lies entirely below $L_{2k-3} = \beta_1$ this last probability equals the probability of hitting $L_1, L_2, L_3, \ldots, L_{2k-3}$ in sequence an then L^*.

By an recursive argument this can be reduced further. "Redefine" the line L^* to

$$L^* : y = -\mu^* t - v^*. \tag{B.8}$$

Then the conditional probability of hitting L^* is

$$e^{-2\mu^*[(\mu^*+b_1)t_{2k-3}+v^*+a_1]}$$

rearranging this we get

$$e^{-2(\mu^*+b_1)\left[\frac{\mu^*[(\mu^*+b_1)t_{2k-3}+v^*+a_1]}{\mu^*+b_1}\right]}. \tag{B.9}$$

Using the reflection principle this equals the conditional probability of hitting a new line L^{**}, given by

$$\begin{aligned} L^{**}: y' &= (\mu^*+b_1)t + \frac{\mu^* v^* + a_1 b_1 + 2\mu^* a_1}{\mu^*+b_1} \\ &= \mu^{**}t + v^{**} \end{aligned} \tag{B.10}$$

But since L^{**} lies above $L_{2k-3} = \beta_1$, it can only be touched if the path already has touched $L_{2k-3} = \beta_1$. But since this lies entirely above $L_{2k-4} = \beta_2$, the conditional probability equals the conditional probability of hitting $L_1, L_2, L_3, \ldots, L_{2k-4}$ in sequence and then L^{**}.
Finally this equals the conditional probability of hitting a line $\hat{L}$ given that the process started in $Y(0) = 0$, but this is exactly the probability we are looking for, i.e. $P(A_k)$.

This last line $\hat{L}$ is given by

$$\begin{aligned} \hat{L}: \hat{y} &= \sum^{k}(b_1+b_2)t + \frac{k^2 a_1 b_1 + (k-1)^2 a_2 b_2 - k(k-1)(a_1 b_2 + a_2 b_1)}{\sum^{k}(b_1+b_2)} \\ &= \hat{\mu}t + \hat{v} \end{aligned} \tag{B.11}$$

In other words, the probability of hitting $L_1, L_2, L_3, \ldots, L_{2k-1}$ in sequence equals the probability of the process $Y(s)$, starting at $Y(0) = 0$, hitting the line $\hat{L}$. From (B.4) we then get

$$\begin{aligned} P(A_k) &= \lim_{t\to\infty} P(M_Y(t) \geq \hat{L}) = e^{-2\hat{\mu}\hat{v}} \\ &= e^{-2(k^2 a_1 b_1 + (k-1)^2 a_2 b_2 - k(k-1)(a_1 b_2 + a_2 b_1))}. \end{aligned} \tag{B.12}$$

When finding $P(B_k)$ observe the following. The events A_k and B_k consists by definition of the following sequences, respectively

$$\begin{aligned} A_k: \ & \beta_1, \beta_2, \ldots, \beta_1 \\ & L_1, L_2, L_3, \ldots, L_{2k-1} \end{aligned}$$

and

$$
\begin{aligned}
B_k : \;& \beta_2, \beta_1, \ldots, \beta_1 \\
& L_2, L_3, L_4, \ldots, L_{2k+1}
\end{aligned}
$$

now, it is easily seen that these two sequences are almost identical. The only difference is that the event A_k starts out from $Y(0) = 0$ and hits $L_1 = \beta_1$ when the event B_k starts out from $Y(0) = 0$, hits $L_2 = \beta_2$ and then hits $L_3 = \beta_1$ which corresponds to $L_1 = \beta_1$ in A_k. Adding an additional "round" with the reflection principle we find the conditional probability of hitting a line $\tilde{L}$ given that the process started in $Y(0) = 0$, which is the probability $P(B_k)$. This line is given by

$$
\begin{aligned}
\tilde{L} : \tilde{y} &= \sum^{k}(b_1 + b_2)t + \frac{k^2 a_1 b_1 + k^2 a_2 b_2 - k(k-1)a_1 b_2 - k(k+1)a_2 b_1)}{\sum^k (b_1 + b_2)} \\
&= \tilde{\mu} t + \tilde{v}.
\end{aligned} \tag{B.13}
$$

This gives

$$
P(B_k) = e^{-2(k^2 a_1 b_1 + k^2 a_2 b_2 - k(k-1)a_1 b_2 - k(k+1)a_2 b_1))}. \tag{B.14}
$$

Summing the equations in (B.12) and (B.14) for $k = 1, 2, \ldots$ when assuming $a_1 > a_2$, $\beta_1 > \beta_2$ and $b_1 \geq 0$ we get

$$
\begin{aligned}
P(\tau_1 < \tau_2) &= \sum_{k=1}^{\infty} e^{-2(k^2 a_1 b_1 + (k-1)^2 a_2 b_2 - k(k-1)(a_1 b_2 + a_2 b_1))} \\
&\quad - \sum_{k=1}^{\infty} e^{-2(k^2 a_1 b_1 + k^2 a_2 b_2 - k(k-1)a_1 b_2 - k(k+1)a_2 b_1))}
\end{aligned} \tag{B.15}
$$

(In order to find $P(\tau_2 < \tau_1)$ we simply replace (a_1, b_1) by $(-a_2, -b_2)$ and (a_2, b_2) by $(-a_1, -b_1)$ in the framework above.)

It remains to find the conditional probability of hitting one boundary first given that the process $Y(t) = y$, this is given in Theorem 4.2 in [2]. Since we are considering a pure "knock-out" contract, i.e. the contract is knocked to value zero when hitting a boundary, we consider only the situation when $y \in I$.

For $0 \leq s \leq t$ we have $Y(t) = Y(s) + (Y(t) - Y(s))$ and $Y(s)$, both normally distributed, $N(0, \sqrt{t})$ and $N(0, \sqrt{s})$, respectively. $Y(t)$ and $Y(s)$ have a joint normal distribution, with correlation coefficient $\rho = \frac{\mathrm{Cov}(Y(t),Y(s))}{\sigma_{Y(s)}\sigma_{Y(t)}} = \frac{\sqrt{s}}{\sqrt{t}}$, and it can easily be shown that the conditional density for $Y(s)$ given that $Y(t) = y$ is normal with

$$E[Y(s)|Y(t)=y] = E[Y(s)] + \rho\frac{\sigma_{Y(s)}}{\sigma_{Y(t)}}\big(y - E[Y(t)]\big) = \frac{s}{t}\cdot y \tag{B.16}$$

and

$$\mathrm{Var}\big(Y(s)|Y(t)=y\big) = \sigma^2_{Y(t)}(1-\rho^2) = s\Big(1-\frac{s}{t}\Big). \tag{B.17}$$

This gives $\big\{Y(s)|Y(t)=y\big\} \sim N\big(y\cdot s/t, \sqrt{s(1-s/t)}\big)$.
We want to find the probability of the event $\big\{Y(s)|Y(t)=y\big\} \geq \beta_1 = a_1 + b_1 s$. Finding an Wiener-process $Z(u)$ such that the event $Z(u) \geq \beta_1'(u)$ for $0 \leq u < \infty$ is equivalent to the above event, we can use the probability in equation (B.15).

Define $Z(u)$ as follows

$$Z(u) = \frac{t+u}{t}\Big[\Big\{Y\Big(\frac{tu}{t+u}\Big)|Y(t)=y\Big\} - \frac{u}{t+u}y\Big], \quad 0 \leq u < \infty$$

then

$$E[Z(u)] = 0$$

and

$$\mathrm{Var}(Z(u)) = u.$$

We easily see that $Z(u)$ for $0 \leq u < \infty$ is a Wiener-process.
The event $\big\{Y(s)|Y(t)=y\big\} \geq \beta_1 = a_1 + b_1 s$ is then equivalent to

$$\begin{aligned} Z(u) \geq \frac{t+u}{t}\Big[a_1 + b_1\frac{tu}{t+u} - \frac{u}{t+u}y\Big] &= \frac{1}{t}\Big[a_1(t+u) + b_1 tu - uy\Big] \\ = a_1 + \Big(\frac{a_1-y}{t} + b_1\Big)u &= a_1 + b_1^* u \end{aligned} \tag{B.18}$$

Correspondingly, $b_2^* = \frac{a_2-y}{t} + b_2$, and noting that

$$\frac{a_1-y}{t} + b_1 = \frac{a_1+b_1t-y}{t} \geq \frac{a_2+b_2t-y}{t} = \frac{a_2-y}{t} + b_2$$

and

$$\frac{a_1-y}{t} + b_1 = \frac{a_1+b_1t-y}{t} > 0$$

We finally find the probability $P_1(t,y)$ by

$$
\begin{aligned}
P_1(t,y) &= P(\tau_1 < t, \tau_1 < \tau_2 | Y(t) = y) = P(\tau_1^{Z(u)} < \tau_2^{Z(u)}) \\
&= \sum_{k=1}^{\infty} e^{-2(k^2 a_1 b_1^* + (k-1)^2 a_2 b_2^* - k(k-1)(a_1 b_2^* + a_2 b_1^*))} \\
&- \sum_{k=1}^{\infty} e^{-2(k^2 a_1 b_1^* + k^2 a_2 b_2^* - k(k-1) a_1 b_2^* - k(k+1) a_2 b_1^*))} \\
&= \sum_{k=1}^{\infty} e^{-\frac{2}{t}(k^2 a_1 (a_1 + b_1 t - y) + (k-1)^2 a_2 (a_2 + b_2 t - y) - k(k-1)(a_1(a_2 + b_2 t - y) + a_2(a_1 + b_1 t - y)))} \\
&- \sum_{k=1}^{\infty} e^{-\frac{2}{t}(k^2 a_1 (a_1 + b_1 t - y) + k^2 a_2 (a_2 + b_2 t - y) - k(k-1) a_1 (a_2 + b_2 t - y) - k(k+1) a_2 (a_1 + b_1 t - y)))}.
\end{aligned}
$$

$P_2(t,y)$ is found by replacing (a_1, b_1) by $(-a_2, -b_2)$ and (a_2, b_2) by $(-a_1, -b_1)$.

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