The Randall–Sundrum Radion: Production Through Gluon Fusion, and Two-Photon Decay

Cand. Scient. Thesis in Theoretical Particle Physics

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Preface

Until recently, it was generally accepted that we live in a universe with four infinite (or large) dimensions, one of time and three of space. Over the last few years, several theories with large extra dimensions have been proposed. The reason why these theories get a lot of attention is that in addition to explaining why the difference between the weak scale and the Planck scale is so huge, or why the GUT scale seems so large, these theories can be tested experimentally in the next generation of hadron colliders.

One of the theories mentioned above, known as the Randall–Sundrum (RS) scenario, takes one compactified extra dimension into consideration. To stabilize the compactification radius of the extra dimension, a field called the radion field is introduced. The *radion*, which is the quantum of the radion field in the RS scenario, is a massive scalar particle, which has much in common with the more familiar (but still hypothetical) *Higgs* particle of the Standard Model. Thanks to this similarity, we can approach calculations concerning radion cross section and decay rates by calculating the corresponding Higgs cross section and decay rates.

The radion could be the first signature of the Randall–Sundrum scenario in the next generation of proton-proton colliders, and therefore it is important to determine its phenomenology. In this thesis we will calculate the cross section for Higgs production from two gluons, via a quark loop. We can then use the obtained Higgs result to find the radion production cross section through gluon fusion. By using results for the decay of the Higgs into two photons via a triangle diagram, we can also find the radion decay rate for decay into two photons. Finally we compare the two processes, where two gluons go to two photons via an intermediate radion or Higgs, by looking at some combinations of cross sections and decay rates for the two cases.

In Chapter 1 we give a historical review together with a brief introduction to the notation of relativistic quantum mechanics and field theory. We also mention some aspects of the two gauge field theories known as Quantum Electrodynamics and Quantum Chromodynamics.

Chapter 2 is meant to explain how the theory of electroweak unification leads to the prediction of the Higgs boson. In this chapter we introduce the concept of spontaneous symmetry breaking, which is crucial to the understanding of the Higgs mechanism. We also take a look at the Goldstone model, the Higgs model and the Weinberg–Salam model before we arrive at the electroweak theory. At the end of the chapter, we give the experimental and theoretical constraints on the mass of the Higgs boson.

Since we need some concepts from Einstein's general theory of relativity in our treatment of the radion, a short introduction to this topic is given in Chapter 3. Note that the notation in Chapter 3 differs from the rest of this thesis, since we use opposite sign in the metric.

The cross section for Higgs production from two gluons via a quark loop is calculated in Chapter 4. We use this to estimate the number of Higgs events produced at the LHC during one year.

In Chapter 5, we review some recently proposed models concerning large extra dimensions. To calculate the production cross section of the Randall–Sundrum radion, through gluon fusion, we use the results reviewed in Chapter 4. This radion cross section is then compared to the corresponding Higgs cross section.

The decay of the radion into two photons via a triangle diagram is considered in Chapter 6. We also compare this radion decay rate to the corresponding Higgs decay rate.

In Chapter 7, we look at the product of the cross section for radion production through gluon fusion and the decay rate for radion decay into two photons. To compare the radion to the Higgs, we take the ratio of these products for the radion case and the Higgs case. We also look at the ratio of the cross section and decay rate in the radion case, and compare it to the corresponding ratio in the Higgs case.

Some concluding remarks are given in Chapter 8.

Contents

Chapter 1

Introduction

Particle physics is the study of *elementary particles* and their *interactions*. The particles we today call elementary are six leptons and six quarks[1], together with their antiparticles and the force carriers. Some of these particles have been predicted by theorists before their discovery, whereas others, like the muon for instance, came as a surprise to the physics community[2]. Based on the outstanding success of the Standard Model and its predictions, a particle known as the *Higgs* particle is believed to exist. In this thesis we consider the most dominant Higgs production channel at proton-proton colliders, namely gluon fusion, and also its decay to two photons.

In a recently proposed scenario with five space-time dimensions, referred to as the Randall–Sundrum (RS) scenario, a particle called the *radion*, which is very similar to the Higgs particle of the Standard Model, is introduced. We will therefore take the Higgs results and transfer them to the case of radion production through gluon fusion, and its decay to two photons. There are some important differences between the Higgs and the radion cases, as will be discussed.

After giving a short historical review, we will in this chapter focus on some aspects of quantum field theory.

1.1 Historical Review

1.1.1 Fundamental Building Blocks of Matter

J. J. Thompson discovered the first elementary particle, the *electron*, in 1897. This particle was the first constituent of the atom to be identified, and is still, after more than a century of experiments, considered fundamental.

Through a series of experiments by E. Rutherford and his students, H. Geiger and E. Marsden, Rutherford concluded that most of the atomic mass, and all of the positive charge, lie in a minute central nucleus of the atom. Rutherford also found the first experimental

[1]Each quark comes in three different colors.

[2]When the muon was discovered, I. I. Rabi asked, “Who ordered that?”

evidence for the *proton* in 1919. After the discovery of the *neutron* in 1932, by J. Chadwick, there was no longer any doubt that the constituents of the atom are electrons, protons and neutrons.

In 1930 W. Pauli postulated a new particle, called the *neutrino*, to explain the apparent loss of energy in β-decay. Since the neutrino had to interact very weakly with other particles, Pauli said it could never be detected. However, the last statement was wrong. In 1956 the weakly interacting neutrino was discovered by F. Reines and C. Cowan, who obtained a very high neutrino flux by placing their detector nearby a nuclear reactor.

During the fifties and sixties, experiments at particle accelerators showed that protons and neutrons are members of a huge family of particles called hadrons. In 1963 M. Gell-Mann and G. Zweig independently proposed that hadrons are composite systems, consisting of two or three fundamental constituents called quarks[3]. Hadrons consisting of two quarks[4] are called *mesons* and those consisting of three quarks are called *baryons*. In the original quark model there were three types of quarks, *up*, *down* and *strange*, and with this model Gell-Mann and Zweig made a giant step towards taming the *hadronic zoo*.

A fourth quark, *charm*, was proposed in 1967, based on an underlying symmetry of nature, to "balance" the number of leptons[5] known at the time. This assumption was confirmed in 1974, when a new heavy meson called J/ψ was discovered independently by two groups, one led by B. Richter at the Stanford Linear Accelerator (SLAC) and the other led by S. Ting at the Brookhaven National Laboratory (BNL).

In the same spirit, the discovery of the tau lepton in 1975 by researchers at Stanford University, led to the proposal of two new quarks, *bottom* and *top*. Both of these were first observed by researchers at the Fermi National Laboratory (FNAL), the lightest one (bottom) in 1977 and the heaviest one (top) in 1995. High precision experiments at CERN and SLAC have confirmed that there only exist three (light) neutrino types. Based on these results, no more quarks are expected to show up[6].

In July 2000, the DONUT experiment at Fermilab announced the direct observation of the third neutrino, namely the tau neutrino.

1.1.2 Fundamental Forces

At the beginning of the 19th century, four forces were considered to be fundamental. These were gravity, electricity, magnetism and the forces between atoms and molecules. In 1873 J. C. Maxwell succeeded in unifying the electric and magnetic forces into one single force, known as the *electromagnetic* force. The forces between atoms and molecules were also understood to be due to electromagnetism.

When nuclear physics entered the scene, two new forces had to be taken into account,

[3]Zweig called them "*aces*". Gell-Mann borrowed the word *quark* from the passage "Three quarks for Muster Mark" in James Joyce's *Finnegan's Wake*.

[4]One quark and one antiquark.

[5]Electron, muon, electron-neutrino and muon-neutrino.

[6]For simplicity, we often use u, d, s, c, b and t to label the different quarks.

namely the nuclear force, between nucleons[7], and the weak force, which was first observed in β-decay. The nuclear force is no longer considered fundamental, but due to the strong force between quarks.

The electromagnetic and the weak force were unified into the electroweak force, by S. Glashow, S. Weinberg and A. Salam in the sixties, and had a great triumph in 1983, when C. Rubbia, S. Van der Meer and collaborators at CERN discovered the massive vector bosons predicted by the theory. The electroweak unification is widely accepted, although a last challenge still remains since the *Higgs boson*, predicted by the theory, has not yet been observed. In our summary of the forces considered to be fundamental, we will therefore keep these two forces separate. The forces are listed according to their strength.

- The *strong* force is described in a theory formulated in the seventies, called *Quantum Chromodynamics* (QCD), and is due to the exchange of massless *gluons*. There are eight gluons and they are spin-1 particles. Only particles which have color charge, i.e. quarks and gluons, feel the strong force. This force is responsible for binding quarks together, to form color-neutral hadrons, and for binding nucleons together in the nucleus. If we try to separate two colored objects, quark-antiquark pairs will be created in the strong field between them. This process is called *hadronization* since we are left with color-neutral hadrons. The range of the strong force is therefore short, of the order of 10^{-15} m.

- The quantum theory of Electromagnetism is known as *Quantum Electrodynamics* (QED), and is believed to be fully understood. All particles with electric charge feel the electromagnetic force, and they interact through the exchange of spin-1 *photons*. The electromagnetic force is what keeps the electrons and the nucleus together in atoms. It has infinite range since the photons are massless.

- The *weak* force is described by the exchange of massive spin-1 particles known as *vector bosons*. There are three massive vector bosons, two charged, $W^{\pm}$, and one neutral, Z^0. Quarks, leptons and the massive vector bosons feel this force. Since the force carriers are very heavy, the weak force has a very short range, of the order of 10^{-18} m.

- *Gravity* is felt by all massive particles. Einstein's general theory of relativity has turned out to be a very good description of this force, but every attempt at formulating a quantum theory of gravity has failed. The quanta of the gravitational field are called *gravitons*, and are massless spin-2 particles, but such particles have never been observed. Gravity is the weakest of the fundamental forces, but since it has infinite range and is always attractive, it dominates on large scales.

Note that all the force carriers have integer spin, and such particles are called *bosons*, while leptons and quarks have half-integer spin, and are called *fermions*.

[7] Protons and neutrons.

Today's *Standard Model* (SM) of particle physics consists of quantum theories for both the strong and the electroweak interactions. There have been several attempts to unify the strong and electroweak forces into a *Grand Unified Theory* (GUT) and also attempts to make a *Theory Of Everything* (TOE) which should include quantum gravity as well. Some of these theories have already been excluded through experiments, while some theories make predictions that remain to be tested.

1.1.3 Quantum Mechanics

When we observe nature on macroscopic scales, it seems obvious that the energy is part of a continuous spectrum. On microscopic scales, however, this is not true. If we consider the energy of an electron in an atom, the energy spectrum is discrete. We say that the energy is *quantized*.

To explain the photoelectric effect, A. Einstein used the idea of quantization, first proposed by M. Planck. Einstein assumed that electromagnetic waves were absorbed in energy packets, called *photons*, and not as a continuous stream as previously thought. Another phenomenon that needed explanation was why electrons do not radiate, and lose energy by orbiting around the nucleus. This problem was solved by N. Bohr in 1913, when he postulated that electrons can move in certain orbits without losing energy. Bohr's postulate together with L. de Broglie's assumption that not only the photon, but also the electron have both particle and wave nature led to the formulation of *quantum mechanics*. Quantum mechanics is based on four postulates, and one of them introduces the famous equation known as *Schrödinger's wave equation*, written down in 1926.

1.2 Relativistic Quantum Mechanics

1.2.1 Relativistic Notation

To be able to do *relativistic* quantum mechanics, it is convenient to have a relativistic notation. We will in this thesis use the notation of Mandl and Shaw [1], in natural units, which means that we put[8] $c = \hbar = 1$.

In this notation we use the four-component vectors of space-time, where the first component is the time component, t, and the last three are the space components, represented by $\mathbf{x}$. The *contravariant* four-vector, x^μ, is defined as

$$x^\mu = (x^0, x^1, x^3, x^4) = (t, \mathbf{x}) \tag{1.1}$$

where the superscript μ is called the Lorentz index.

Since we want the space component of the *covariant* vector, x_μ, to have opposite sign

[8]The constant, c, is the speed of light in vacuum and $\hbar$ is Planck's constant divided by 2π.

from the space component of x^μ, we define the *Minkowski tensor*, $\eta_{\mu\nu}$ as

$$\begin{aligned} \eta_{\mu\nu} &= \eta^{\mu\nu} \\ \eta_{00} = +1\ ,\ \eta_{11} &= \eta_{22} = \eta_{33} = -1 \\ \eta_{\mu\nu} &= 0 \text{ for } \mu \neq \nu \end{aligned} \tag{1.2}$$

Now we can define the *covariant* four-vector, x_μ, in terms of x^μ and $\eta_{\mu\nu}$ as

$$x_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} x^\nu \equiv \eta_{\mu\nu} x^\nu = (x^0, -x^1, -x^3, -x^4) = (t, -\mathbf{x}) \tag{1.3}$$

where we have used *Einstein's summation convention*: Two repeated (Greek) indices, one covariant and one contravarinat, are summed over the values[9] 0, 1, 2, 3. The scalar product of two four-vectors, which is invariant under Lorentz transformations, can now be written in different ways:

$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu = \eta_{\mu\nu} a^\mu b^\nu = \cdots = a^0 b^0 - \mathbf{a} \cdot \mathbf{b} \tag{1.4}$$

We will now introduce a generalization of ∇, the gradient operator in three dimensions. In our four-dimensional picture it is denoted ∂_μ, and has the following definition

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \tag{1.5}$$

Note that the generalization of the Laplacian operator, $\bigtriangledown^2$, called the d'Alembertian operator, $\Box$, which is defined by

$$\Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \bigtriangledown^{\mathbf{2}} \tag{1.6}$$

is a scalar.

1.2.2 Lorentz Transformation

A Lorentz transformation is a transformation between space-time coordinates of two different coordinate frames. The transformation can be written as a four-dimensional rotation in space-time

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1.7}$$

where the coefficients, $\Lambda^\mu{}_\nu$, depend on the relative velocity and spatial orientation of the two frames. Such a transformation is therefore equivalent to a transformation from one coordinate frame to another coordinate frame with a different velocity, followed by a spatial rotation.

Einstein's theory of special relativity requires that laws of motion valid in one inertial system must be valid in all inertial systems. Therefore we should demand our equations to be Lorentz covariant, which means that they have the same form in all inertial systems.

[9]Repeated latin letters indicate summation over the values 1, 2 and 3.

1.2.3 The Dirac Equation

Since the Schrödinger equation does not hold as a relativistic equation, it has to be modified. In 1927 P. Dirac proposed a relativistic wave equation for the electron, known as the *Dirac equation*. The solution of this equation led Dirac to the remarkable conclusion that there exists a particle with the same mass as the electron, but opposite charge. He called this particle, which was the first example of an antiparticle, the *positron*. The positron was discovered in 1932 by C. Anderson, and was an enormous triumph for the theory of relativistic quantum mechanics.

The Dirac equation applies to all fermions, and can be written

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = (i\not{\partial} - m)\psi(x) = 0 \tag{1.8}$$

where m is the fermion mass, γ^μ are 4×4 matrices, and $\psi(x)$ is a 4-component spinor. We have also used the "slash" notation:

$$\not{p} \equiv \gamma^\mu p_\mu = \gamma_\mu p^\mu \tag{1.9}$$

Since we want the Dirac equation to be satisfied for a free particle, we must require the four γ-matrices, $\gamma^0, \gamma^1, \gamma^2$ and γ^3, to obey the following anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{1.10}$$

We also need the condition $p^2 = m^2$ to be satisfied. If we let the four-momentum, p^μ be denoted by $p^\mu = (E, \mathbf{p})$, where E is the energy and $\mathbf{p}$ is the three-momentum, we get

$$p^2 = p_\mu p^\mu = E^2 - \mathbf{p^2} = m^2 \tag{1.11}$$

which is recognized as the relativistic energy-momentum condition.

The γ-matrices are not unique, i.e. we can choose among different representations. Independent of representation, the γ-matrices also have to satisfy the condition[10]

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{1.12}$$

where the Hermitian conjugation, $\gamma^{\mu\dagger}$, is obtained by taking the complex conjugate of the transpose of γ^μ.

It is often convenient to introduce a fifth γ-matrix, γ_5, which can be expressed in terms of the four others as

$$\gamma_5 \equiv \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1.13}$$

In this section we introduced the concept of anticommutation relation. The *anticommutator* between two operators, A and B, is defined as

$$\{A, B\} \equiv AB + BA \tag{1.14}$$

[10]This condition is to ensure Hermiticity of the Hamiltonian, which is necessary to obtain real eigenvalues.

and in the same manner, we define the *commutator* to be

$$[A, B] \equiv AB - BA \tag{1.15}$$

Two operators, A and B, are said to *anticommute* if $\{A, B\} = 0$ and they *commute* if $[A, B] = 0$.

1.3 Quantum Field Theory

The starting point in quantum field theory is the classical Lagrangian formalism. We define the *action* integral as

$$J \equiv \int_{t_1}^{t_2} \mathrm{d}t\, L = \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\, \mathcal{L} \tag{1.16}$$

where L is the Lagrangian and $\mathcal{L}$ is the *Lagrangian density.* By Hamilton's principle we require the action to be extremal, $\delta J = 0$, and from this we can obtain the Euler–Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} = 0, \qquad r = 1, 2, \ldots, n \tag{1.17}$$

where the r in ϕ_r labels independent fields.

In the procedure called *canonical quantization* we start by choosing a Lagrangian density, $\mathcal{L}$, which will reproduce the classical equations of motion for the fields when we insert it in the Euler–Lagrange equations[11], (1.17).

To be able to quantize the fields, we need to determine the *conjugate momentum*, π_r, of each field:

$$\pi_r(\mathbf{x}, t) \equiv \frac{\partial \mathcal{L}(\phi_r, \partial_\mu \phi_r)}{\partial \dot{\phi}_r(\mathbf{x}, t)} \tag{1.18}$$

where the "dot" notation indicates differentiation with respect to time.

We quantize *boson* fields by imposing equal time *commutation* relations between the fields and their conjugate momenta[12]:

$$\begin{aligned} [\phi_r(\mathbf{x}, t), \phi_s(\mathbf{x}', t)] &= 0 \\ [\pi_r(\mathbf{x}, t), \pi_s(\mathbf{x}', t)] &= 0 \\ [\pi_r(\mathbf{x}, t), \phi_s(\mathbf{x}', t)] &= -i\delta_{rs}\delta^{(3)}(\mathbf{x} - \mathbf{x}') \end{aligned} \tag{1.19}$$

The quantized fields are now operators and the coordinates are numbers, in contrast to non-relativistic quantum mechanics where the position coordinates are operators.

[11] In some situations there are no classical equations of motion. This is the case for fermion fields, where we use a Lagrangian density that reproduces the Dirac equation instead.

[12] For *fermion* fields we impose *anticommutation* relations.

It is also useful to have an expression for the Hamiltonian, H, where

$$H = \int d^3\mathbf{x}\,\mathcal{H} \tag{1.20}$$

The Hamiltonian density, $\mathcal{H}$, can be written in terms of the fields, their conjugate momenta and the Lagrangian density

$$\mathcal{H}(x) = \sum_r \pi_r \dot{\phi}_r - \mathcal{L}(x) \tag{1.21}$$

We end this section by mentioning Noether's theorem, which states that: *"Any continuous symmetry transformation that leaves the Lagrangian invariant, corresponds to a conserved current"*. This theorem implies that to each internal symmetry there corresponds a conserved charge.

1.4 Quantum Electrodynamics

In 1929 the theory called *Quantum Electrodynamics* (QED) was formulated. This theory describes interactions between the Dirac "matter" field and the electromagnetic field.

We start by considering the free field Lagrangian density for the Dirac field, $\psi(x)$,

$$\mathcal{L}_0^\psi(x) = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \tag{1.22}$$

where the *Dirac adjoint* of a Lorentz spinor is defined as

$$\overline{\psi}(x) \equiv \psi^\dagger(x)\gamma^0 \tag{1.23}$$

To include the electromagnetic interaction, we adopt the minimal substitution from non-relativistic quantum mechanics by replacing the derivative, ∂_μ, with the *covariant* derivative, D_μ,

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu(x) \tag{1.24}$$

In eq. (1.24), q is the charge annihilated by $\psi(x)$, and $A_\mu(x)$ is the electromagnetic field, also called the photon field. After this substitution, the Lagrangian density becomes

$$\begin{aligned}\mathcal{L}^\psi(x) &= \overline{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) \\ &= \mathcal{L}_0^\psi(x) - q\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) \\ &\equiv \mathcal{L}_0^\psi(x) + \mathcal{L}_I(x)\end{aligned} \tag{1.25}$$

QED was the first *gauge field theory*. In such theories, the Lagrangian has to be invariant under certain transformations called *gauge transformations*. A gauge transformation of the electromagnetic field is of the form

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\chi(x) \tag{1.26}$$

where $\chi(x)$ is an arbitrary real differentiable function. Observable quantities are *gauge independent*, which means that our choice of $\chi(x)$ does not change the predictions of the theory. Since there is no non-trivial group involved, QED is referred to as an *Abelian* gauge field theory.

A gauge transformation of the Dirac field, which together with eq. (1.26) leaves the Lagrangian density in (1.25) gauge invariant, is

$$\begin{aligned} \psi(x) &\to \psi'(x) = e^{-iq\chi(x)}\psi(x) \\ \overline{\psi}(x) &\to \overline{\psi}'(x) = \overline{\psi}(x)e^{iq\chi(x)} \end{aligned} \tag{1.27}$$

The coupled transformation (1.26) and (1.27) is called a *local* gauge transformation, since $\chi(x)$ depends on x.

It now remains to add a gauge invariant free-field Lagrangian density for the photon field, $A_\mu(x)$. It follows immediately that

$$F_{\mu\nu}(x) \equiv \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) \tag{1.28}$$

is invariant under the transformation (1.26). Since the Lagrangian density

$$\mathcal{L}_0^A(x) = -\tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \tag{1.29}$$

reproduces Maxwell's equations when no charges are present, the total Lagrangian density for QED can be written as

$$\begin{aligned} \mathcal{L}(x) &= \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) - \tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \\ &\quad -q\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) \\ &= \mathcal{L}_0(x) + \mathcal{L}_I(x) \end{aligned} \tag{1.30}$$

where the first line in (1.30) is the free Lagrangian density, and the term in the second line describes the interaction.

To avoid the problem of a vanishing conjugate momenta, $\pi^0(x) \equiv 0$, which is incompatible with the canonical quantization, Fermi suggested the following replacement in the Lagrangian density, (1.30)

$$\mathcal{L}_0^A(x) = -\tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \to -\tfrac{1}{2}(\partial_\nu A_\mu(x))(\partial^\nu A^\mu(x)) \tag{1.31}$$

This replacement is equivalent to imposing the Lorentz gauge condition, $\partial_\mu A^\mu(x) = 0$. A covariant quantization of the electromagnetic field can then be obtained by applying the so-called *Gupta–Bleuler procedure* to the modified Lagrangian density. In this procedure, the Lorentz gauge condition is replaced by a weaker condition, which results in a modification of Hilbert space.

Note that the photon field is massless, i.e. it can not be given mass by adding a mass term, $m^2A_\mu A^\mu$, since such a term would destroy the invariance of the Lagrangian density, (1.30), under gauge transformations. This is related to the infinite range of electromagnetic interactions.

If we should generalize the procedure followed in this section, we would start by the requirement of invariance with respect to local phase transformations of the matter field, $\psi(x)$. This is obtained by introducing a gauge field that couples to the matter field by the minimal substitution $\partial_\mu \psi(x) \to D_\mu \psi(x)$, where $D_\mu \psi(x)$ transforms as $\psi(x)$ itself.

The interaction between the Dirac field and the electromagnetic field is known from the classical limit, and confirmed by high precision experiments. Also in cases without any classical limit, this approach has been extremely successful.

1.5 Quantum Chromodynamics

As mentioned earlier, only colored objects feel the strong force. This force is described by Quantum Chromodynamics (QCD), which is a *non*-Abelian gauge field theory based on the $SU(3)$ group[13]. Experiments have shown that there are three color degrees of freedom, which often are labelled by the colors red, blue and green. The quarks, which come in six flavors, can carry one of these three colors, whereas there are eight gluons which carry a combination of color and anticolor. Quarks and antiquarks form objects as baryons, antibaryons and mesons, which are referred to as colorless objects because of their invariance under rotation in color-space. Since only colorless objects can be observed, the quarks are said to be *confined*.

We will now introduce some concepts used in the mathematical description of QCD. The quark field can be represented by a color triplet,

$$\psi_q(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} \tag{1.32}$$

which, under a local $SU(3)$ gauge transformation, transforms as

$$\psi_q(x) \to \psi'_q(x) = e^{-ig_S \chi(x)} \psi_q(x) \tag{1.33}$$

where g_S is the strong coupling constant. We express $\chi(x)$ as

$$\chi(x) = \sum_{a=1}^{8} T^a \chi^a(x) \equiv T^a \chi^a(x) \tag{1.34}$$

where $\chi^a(x)$ are real functions, and T^a are hermitian 3×3 matrices which satisfy the $SU(3)$ algebra

$$[T^a, T^b] = i f^{abc} T^c \tag{1.35}$$

where f^{abc} are the $SU(3)$ structure constants.

[13] A non-Abelian field theory is a field theory based on a non-Abelian group, i.e., a group with non-commuting elements.

Let us now turn to the description of QCD in terms of the Lagrangian density, which has the following definition

$$\mathcal{L}_{\mathrm{QCD}}(x) = \overline{\psi}_q(x)(i\,\not{D} - m_q)\psi_q(x) - \tfrac{1}{4}G^a_{\mu\nu}(x)G^{a\,\mu\nu}(x) \tag{1.36}$$

The gluon field-strength tensor, $G^a_{\mu\nu}(x)$, is defined as

$$G^a_{\mu\nu}(x) = \partial_\nu A^a_\mu(x) - \partial_\mu A^a_\nu(x) + g_S f^{abc} A^b_\mu(x) A^c_\nu(x) \tag{1.37}$$

where the gluon field is labelled $A^a_\mu(x)$, and the indices a, b and c run over the eight (gluon) color states. In QCD, the covariant derivative is defined, analogous to QED, as

$$D_\mu \equiv \partial_\mu + i g_S A_\mu(x) = \partial_\mu + i g_S T^a A^a_\mu(x) \tag{1.38}$$

Gauge invariance of the Lagrangian density, $\mathcal{L}_{\mathrm{QCD}}(x)$, leads to the following (local) gauge transformation of the gluon field:

$$A^a_\mu(x) \to A^{a\,\prime}_\mu(x) = A^a_\mu(x) + \partial_\mu \chi^a(x) - g_S f^{abc} A^b_\mu(x)\chi^c(x) \tag{1.39}$$

where we have dropped higher orders in $\chi^a(x)$. Note that except for the last term, which originates from the non-commutativity of QCD, the gauge transformation of the gluon field is exactly the same as in QED (see eq. (1.26)). The presence of this term gives rise to three- and four-gluon couplings, which have no analogy in QED.

To describe the running of the coupling constant, g, in a renormalizable gauge field theory, a function called the β-function is introduced. This function is defined as

$$\beta\left(g(\mu)\right) \equiv \frac{\partial g(\mu)}{\partial \ln\mu} \tag{1.40}$$

where μ is the energy scale and $g(\mu)$ is the coupling. Note that if the β-function is positive, which is the case in QED, the coupling, $g(\mu)$, increases when μ increases. It is possible to calculate the β-function in perturbation theory, and the general, lowest order expression for the group $SU(N)$ is (see [2], p. 199)

$$\beta(g) = -\left(\frac{11}{3}C_2^A - \frac{4}{3}T\right)\frac{1}{16\pi^2}g^3 = -\frac{b}{16\pi^2}g^3 \tag{1.41}$$

where[14] $C_2^A = N$, $T = \frac{1}{2}n_f$, and n_f is the number of active quark flavors.

In the case of QCD ($SU(3)$), the lowest-order term becomes

$$\beta_{\mathrm{QCD}} \equiv -\frac{b_{\mathrm{QCD}}}{16\pi^2}g_S^3 \tag{1.42}$$

where

$$b_{\mathrm{QCD}} = (11 - \tfrac{2}{3}n_f) \tag{1.43}$$

[14] C_2^A is the quadratic Casimir operator in the adjoint representation.

From (1.42) and (1.43), we see that $\beta_{\rm QCD} > 0$ would require $n_f > 33/2$, which is certainly not realized in nature. The running coupling constant of QCD, which is large at low energy, will therefore decrease when the energy increases. This is known as *asymptotic freedom*, and ensures perturbation theory to be valid at high energies.

From (1.40), (1.42) and (1.43) we see that

$$\mathrm{d}\left(\frac{1}{g^2}\right) = -\frac{2}{g^3}\,\mathrm{d}g = \frac{33-2n_f}{24\pi^2}\,\mathrm{d}(\ln\mu) \tag{1.44}$$

for $g^2 \ll 1$. After integration, we define α_S^{-1}, which is the inverse of the QCD analogue to the fine structure constant of QED, as

$$\alpha_S^{-1}(\mu) \equiv \frac{4\pi}{g_S^2} = \alpha_S^{-1}(\mu_0) - \frac{b_{\rm QCD}}{2\pi}\ln\left(\frac{\mu}{\mu_0}\right) \tag{1.45}$$

where we have introduced μ_0 as a constant of integration. In this case, μ_0 is the QCD cut-off parameter, and $\mu \gg \mu_0$. Note that this expression is linear in $\ln(\mu/\mathrm{GeV})$, since

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\alpha_S^{-1}(\mu) = -\frac{b_{\rm QCD}}{2\pi} \tag{1.46}$$

is a constant.

It is possible to express α_S in terms of the momentum transfer, Q, and the QCD cut-off parameter, Λ, which can be thought of as the scale at which α_S is $\mathcal{O}(1)$ (see [3], p. 35). To two-loop order,

$$\alpha_S(Q^2) = \frac{12\pi}{(33-2n_f)\ln(Q^2/\Lambda^2)}\left[1 - \frac{6(153-19n_f)}{(33-n_f)^2}\frac{\ln(\ln(Q^2/\Lambda^2))}{\ln(Q^2/\Lambda^2)} + \cdots\right] \tag{1.47}$$

where n_f is the number of active quark flavors. In our calculations, we use $\Lambda = 226\,\mathrm{MeV}$, since this is the value used by the CTEQ collaboration in [4], which we will use later. By inserting $n_f = 5$, and neglecting higher order terms, (1.47) becomes approximately

$$\alpha_S(Q) \simeq \frac{6\pi}{23\ln(4.425\times Q/\mathrm{GeV})}\left[1 - \frac{174\ln[2\ln(4.425\times Q/\mathrm{GeV})]}{529\ln(4.425\times Q/\mathrm{GeV})}\right] \tag{1.48}$$

Note that the sign of the β-function also determines whether α increases or decreases since α is given by $\alpha = g^2/4\pi$.

Chapter 2

The Higgs Mechanism

In the unification of the weak and electromagnetic forces into the *electroweak* force, the leptons and gauge bosons are first considered massless (see [1], pp. 264–74). This unification is based on *non-Abelian* $SU(2) \times U(1)$ gauge invariance under local phase transformations.

If we want our theory to be consistent with what we observe in nature, the assumption of massless leptons and gauge bosons is far from realistic. The big question is therefore: How can we get the particles to acquire mass, without spoiling the *renormalizability* of the theory? The reason why we are so concerned about renormalizability is that we are unable to make predictions from a non-renomalizable theory. To protect the renormalizability of the theory, it is important that the gauge invariance be preserved. The mechanism which will take care of this for us is called the *Higgs mechanism*[1], and will be discussed in this chapter. Our discussion closely follows the description given in Mandl and Shaw [1].

2.1 Spontaneous Symmetry Breaking

The concept of *spontaneous symmetry breaking* is known from the physics of phase transitions, where the best known examples are the magnetic properties of iron and the absence of magnetic fields in superconductors, also known as the Meissner effect.

We consider a system where the Hamiltonian is invariant with respect to a certain symmetry. This symmetry can be broken spontaneously if the lowest energy level is degenerate. One of these degenerate energy states is chosen spontaneously as the ground state, which no longer respects the symmetry. The underlying symmetry of the system is now "hidden" in the Hamiltonian.

To apply this to gauge field theory, we must take the Lagrangian density to be invariant under the symmetry transformation, but the vacuum to be characterized by some field which is *not* invariant. The internal symmetry we want to break must be broken by a *scalar* field $\phi(x)$, with a non-zero, but constant expectation value in vacuum

$$\langle 0|\phi(x)|0\rangle = \phi_0 \neq 0 \tag{2.1}$$

[1]Proposed independently by F. Englert and R. Brout [5], and by P. W. Higgs [6] in 1964.

where ϕ_0 is a constant. This is the condition for spontaneous symmetry breaking. On the other hand, the vacuum expectation value for spinor fields, $\psi(x)$, and vector fields, $V^\mu(x)$, must vanish, i.e.

$$\langle 0|\psi(x)|0\rangle = 0, \qquad \langle 0|V^\mu(x)|0\rangle = 0 \tag{2.2}$$

The reason for this is that a non-zero spinor or vector field would break the rotational symmetry by giving the vacuum a non-vanishing angular momentum, but that is not what we want, since the vacuum is rotational invariant.

2.2 Goldstone Model

We will now consider a model, called the *Goldstone model*, were we break a global symmetry. Let us start with a classical *complex* scalar field

$$\phi(x) \equiv \frac{1}{\sqrt{2}}[\phi_1(x) + i\phi_2(x)] \tag{2.3}$$

where $\phi_1(x)$ and $\phi_2(x)$ are real, Hermitian[2] fields. In this model, we take the Lagrangian density to be

$$\mathcal{L}(x) \equiv [\partial^\mu \phi^*(x)][\partial_\mu \phi(x)] - \mu^2|\phi(x)|^2 - \lambda|\phi(x)|^4 \tag{2.4}$$

with μ^2 and λ real parameters[3]. The Lagrangian density in (2.4) is invariant under a *global* $U(1)$ gauge transformation

$$\begin{aligned} \phi(x) &\to \phi'(x) = e^{i\alpha}\phi(x) \\ \phi^*(x) &\to \phi^{*\prime}(x) = \phi^*(x)e^{-i\alpha} \end{aligned} \tag{2.5}$$

since $\phi(x)$ and $\phi^*(x)$ in (2.4) always appear in pairs. This leads to cancellation of the phases from (2.5).

If we insert (2.4) into (1.18), we get the following expressions for the conjugate momenta

$$\begin{aligned} \pi(x) &= \dot{\phi}^*(x) \\ \pi^*(x) &= \dot{\phi}(x) \end{aligned} \tag{2.6}$$

so the Hamiltonian density, defined in (1.21), becomes

$$\begin{aligned} \mathcal{H}(x) &= \dot{\phi}^*(x)\dot{\phi}(x) + \dot{\phi}(x)\dot{\phi}^*(x) - \mathcal{L}(x) \\ &= [\partial^0\phi^*(x)][\partial^0\phi(x)] + [\nabla\phi^*(x)]\cdot[\nabla\phi(x)] + \mathcal{V}(\phi(x)) \end{aligned} \tag{2.7}$$

[2]A classical field, $\phi(x)$, is Hermitian if $\phi(x) = \phi^*(x)$, where $\phi^*(x)$ is the complex conjugate of $\phi(x)$.

[3]As we will see, the most interesting case is $\mu^2 < 0$, therefore this notation is somewhat misleading.

where the potential energy density, $\mathcal{V}(\phi(x))$, is defined by

$$\mathcal{V}(\phi(x)) \equiv \mu^2|\phi(x)|^2 + \lambda|\phi(x)|^4 \tag{2.8}$$

The first two terms in (2.7) are positive definite, and vanish for $\phi(x)$ independent of x. When $\phi(x)$ becomes large, the last term in (2.8) dominates, and therefore we must require $\lambda > 0$ to have $\mathcal{H}(x)$ bounded from below.

The vacuum is determined by the minimum value of $\mathcal{H}(x)$, and therefore corresponds to the minimum value of $\mathcal{V}(\phi(x))$. If we write out $\mathcal{V}(\phi(x))$ in terms of ϕ_1 and ϕ_2, we get

$$\mathcal{V}(\phi(x)) = \tfrac{1}{2}\mu^2[\phi_1^2(x) + \phi_2^2(x)] + \tfrac{1}{4}\lambda[\phi_1^2(x) + \phi_2^2(x)]^2 \tag{2.9}$$

Depending on the sign of μ^2, two situations may occur:

(i) $\mu^2 > 0$. Both terms of $\mathcal{V}(\phi(x))$ are positive definite, so there is a *global* minimum for the unique solution $\phi(x) = \phi_1(x) = \phi_2(x) = 0$. Spontaneous symmetry breaking cannot occur in this case, since $\langle 0|\phi(x)|0\rangle = 0$. If we now quantize the complex Klein-Gordon field, $\phi(x)$, we get charged spin-0 particles of mass μ. In the quantized theory, the λ-term represents self-interaction.

(ii) $\mu^2 < 0$. By partial differentiation of (2.9) with respect to ϕ_1 and ϕ_2 we find a local maximum at $\phi_1(x) = \phi_2(x) = 0$, and a whole circle of global minima at

$$[\phi_1^2(x) + \phi_2^2(x)] = \frac{-\mu^2}{\lambda} \tag{2.10}$$

If we use (2.3) to write this as

$$\phi(x) = \sqrt{\frac{-\mu^2}{2\lambda}}\, e^{i\theta}, \qquad 0 \le \theta < 2\pi \tag{2.11}$$

we see that the vacuum state is degenerate since a value for θ must be chosen. Since the phase, $e^{i\theta}$, can be removed by a global phase transformation, we may take $\theta = 0$, and define

$$\phi_0 = \sqrt{\frac{-\mu^2}{2\lambda}} \equiv \frac{1}{\sqrt{2}} v \quad (>0) \tag{2.12}$$

At this point we introduce new coordinates:

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \sigma(x) + i\eta(x)] \tag{2.13}$$

If we substitute these new coordinates into the Lagrangian density in (2.4), and collect terms, we can express $\mathcal{L}(x)$ as

$$\begin{aligned}\mathcal{L}(x) = {} & \tfrac{1}{2}[\partial^\mu\sigma(x)][\partial_\mu\sigma(x)] - \tfrac{1}{2}(2\lambda v^2)\sigma^2(x) \\ & + \tfrac{1}{2}[\partial^\mu\eta(x)][\partial_\mu\eta(x)] \\ & - \lambda v\sigma(x)[\sigma^2(x) + \eta^2(x)] - \tfrac{1}{4}\lambda[\sigma^2(x) + \eta^2(x)]^2\end{aligned} \tag{2.14}$$

where we have used $\mu^2 = -v^2\lambda$ and suppressed the constant $\lambda v^4/4$, which is of no significance. We are allowed to cancel constant terms from a Lagrangian density, since the Euler–Lagrange equation only contains derivatives of the Lagrangian density.

It is not possible to carry out perturbation theory around an unstable equilibrium, without getting into trouble. We easily see this if we try to use perturbation theory around $\phi(x) = 0$ in (2.4), by treating $\lambda|\phi(x)|^4$ as a perturbation. This leads to imaginary mass for $\mu^2 < 0$. Instead we can use perturbation theory around $\phi(x) = \phi_0$, and treat the terms involving λ in eq. (2.14) as a perturbation. The free Lagrangian density now becomes

$$\begin{aligned} \mathcal{L}_0(x) = &\tfrac{1}{2}[\partial^\mu\sigma(x)][\partial_\mu\sigma(x)] - \tfrac{1}{2}(2\lambda v^2)\sigma^2(x) \\ &+\tfrac{1}{2}[\partial^\mu\eta(x)][\partial_\mu\eta(x)] \end{aligned} \tag{2.15}$$

We see from $\mathcal{L}_0(x)$ that $\sigma(x)$ and $\eta(x)$ are real Klein-Gordon fields, which we quantize to get neutral spin-0 particles, with the corresponding masses m_H and m_η:

$$m_H = \sqrt{2\lambda v^2} \quad , \quad m_\eta = 0 \tag{2.16}$$

The massless quanta of the η-field, which arise from the degeneracy of the vacuum state, are called *Goldstone bosons*. Since these are not observed in nature, it is important that gauge field theories with spontaneous symmetry breaking do not generate Goldstone bosons.

Since there are no particles in the vacuum, we can see from (2.12) and (2.13) that

$$\langle 0|\phi(x)|0\rangle = \phi_0 \tag{2.17}$$

which satisfies the condition for spontaneous symmetry breaking in (2.1).

2.3 Higgs Model

Let us now try to break a local $U(1)$ gauge symmetry spontaneously by generalizing the Goldstone model. To get the Lagrangian density for this model, which is referred to as the *Higgs model*, we replace $\partial_\mu\phi(x)$ in eq. (2.4) by the covariant derivative

$$\partial_\mu\phi(x) \to D_\mu\phi(x) = [\partial_\mu + iqA_\mu(x)]\phi(x) \tag{2.18}$$

and add the free Lagrangian density of the photon field. The Lagrangian density for the Higgs model then becomes

$$\mathcal{L}(x) \equiv [D^\mu\phi(x)]^*[D_\mu\phi(x)] - \mathcal{V}(\phi(x)) - \tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \tag{2.19}$$

where $\mathcal{V}(\phi(x))$ and $F_{\mu\nu}(x)$ are defined in (2.8) and (1.28) respectively. A gauge transformation, under which the Lagrangian density in (2.19) is invariant, is

$$\begin{aligned} A_\mu(x) &\to A'_\mu(x) = A_\mu(x) + \partial_\mu\chi(x) \\ \phi(x) &\to \phi'(x) \;\; = e^{-iq\chi(x)}\phi(x) \\ \phi^*(x) &\to \phi'^*(x) = \phi^*(x)e^{iq\chi(x)} \end{aligned} \tag{2.20}$$

where $\chi(x)$ is real. Like in the Goldstone model, we start from a classical theory, and the same argument as before forces us to take $\lambda > 0$. Again two situations occur, depending on the sign of μ^2:

(i) $\mu^2 > 0$. In this case, both $\phi(x)$ and $A_\mu(x)$ vanish in the vacuum, so the condition for spontaneous symmetry breaking in (2.1) is not fulfilled.

(ii) $\mu^2 < 0$. For the vector field, A_μ, we must require $\langle 0|A_\mu(x)|0\rangle = 0$ due to Lorentz invariance, and the fact that the vacuum is invariant under rotation. We again get a circle of absolute minima for $\mathcal{V}(\phi(x))$, and choose ϕ_0, given in (2.12), as the minimum value. If we write the Lagrangian density in terms of the new coordinates in (2.13), we get

$$\begin{aligned}\mathcal{L}(x) = & \tfrac{1}{2}[\partial^\mu\sigma(x)][\partial_\mu\sigma(x)] - \tfrac{1}{2}(2\lambda v^2)\sigma^2(x) \\ & -\tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \tfrac{1}{2}(qv)^2A_\mu(x)A^\mu(x) \\ & +\tfrac{1}{2}[\partial^\mu\eta(x)][\partial_\mu\eta(x)] \\ & +qvA^\mu(x)\partial_\mu\eta(x) \\ & +\mathcal{L}_I(x)\end{aligned} \tag{2.21}$$

In this case $\mathcal{L}_I(x)$ contains cubic and quartic terms in the fields, and the constant, $\lambda v^4/4$, has been removed. Note that there is a problematic term in $\mathcal{L}(x)$, namely the one containing $A^\mu(x)\partial_\mu\eta(x)$, which tells us that these fields are not independent.

Another problem is that the number of degrees of freedom in (2.21) does not match the number of degrees of freedom in (2.19). In eq. (2.19) we have a complex scalar field, $\phi(x)$, representing two degrees of freedom, and a massless vector field, $A_\mu(x)$, also representing two degrees of freedom. The total number of degrees of freedom in eq. (2.19) is therefore four. If we now look at eq. (2.21) we see two real scalar fields, $\sigma(x)$ and $\eta(x)$, representing one degree of freedom each, and a massive vector field $A_\mu(x)$ representing three degrees of freedom. The total number of degrees of freedom in eq. (2.21) is then five. We conclude that (2.21) contains an unphysical field, which can be removed by an appropriate choice of gauge. In the gauge called *unitary gauge*, the $\eta(x)$-field is removed by requiring $\phi(x)$ to be real, and of the form

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \sigma(x)] \tag{2.22}$$

We see that the vacuum expectation value in this case is

$$\langle 0|\phi(x)|0\rangle = \phi_0 \tag{2.23}$$

which is the condition for spontaneously broken symmetry given in (2.1).

The Lagrangian density in unitary gauge is

$$\begin{aligned}\mathcal{L}_U(x) = & \tfrac{1}{2}[\partial^\mu\sigma(x)][\partial_\mu\sigma(x)] - \tfrac{1}{2}m_\sigma^2\sigma^2(x) \\ & -\tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \tfrac{1}{2}M^2A_\mu(x)A^\mu(x) \\ & +\tfrac{1}{2}q^2[2v\sigma(x) + \sigma^2(x)]A_\mu(x)A^\mu(x) \\ & -\lambda v\sigma^3(x) - \tfrac{1}{4}\lambda\sigma^4(x) \\ = & \mathcal{L}_0(x) + \mathcal{L}_I(x)\end{aligned} \tag{2.24}$$

where $\mathcal{L}_0(x)$ is the first two lines and $\mathcal{L}_I(x)$ is the third and fourth lines of eq. (2.24). Treating $\mathcal{L}_I(x)$ as a perturbation, we see that $\sigma(x)$ is a real Klein-Gordon field, and $A_\mu(x)$ is a massive vector field. If we now quantize, $\sigma(x)$ corresponds to neutral spin-0 bosons of mass $m_H = \sqrt{2\lambda v^2}$, and $A_\mu(x)$ to neutral vector bosons of mass $M = |qv|$. The scalar field $\sigma(x)$ is known as the *Higgs field*, and its quanta are called *Higgs bosons*.

To summarize: It seems possible to get a massless vector field to become massive through spontaneously broken symmetry, without destroying renormalizability, which is protected by the hidden symmetry of the Lagrangian density. The complex scalar field is through this process transformed into a massive scalar field, and it seems like the Goldstone bosons have been "eaten" by the gauge transformed vector field. This phenomenon, by which a vector boson acquires mass without destroying the gauge symmetry of the Lagrangian density, is called the *Higgs mechanism*.

To show renormalizability, G. 't Hooft used a technique similar to the one used in QED by introducing a gauge condition

$$\partial_\mu A^\mu(x) = M\eta(x), \qquad M = |qv| \tag{2.25}$$

which is known as 't Hooft gauge. This is equivalent to the following substitution in the Lagrangian density in (2.21)

$$-\tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \to -\tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \tfrac{1}{2}[\partial_\mu A^\mu(x) - M\eta(x)]^2 \tag{2.26}$$

The cross term in (2.26) is combined with the problematic term $A^\mu(x)\partial_\mu\eta(x)$ in (2.21), to give a total derivative which is removed through integration by parts.

In 't Hooft gauge, the $\eta(x)$-field is reintroduced. This is, as we have previously seen, not a physical field. The quanta of such fields are called "ghosts", and contribute only as intermediate, or virtual, "particles". Their properties are gauge dependent.

2.4 Weinberg–Salam Model

In this section, we will see how a local $SU(2) \times U(1)$ gauge symmetry can be broken spontaneously. This was done independently by Weinberg in 1967 and Salam in 1968, and it leads to the theory of electroweak interactions.

First we define the field-strength tensors in terms of the fields:

$$F^{\mu\nu}(x) = \partial^\nu A^\mu(x) - \partial^\mu A^\nu(x) \tag{2.27}$$
$$Z^{\mu\nu}(x) = \partial^\nu Z^\mu(x) - \partial^\mu Z^\nu(x) \tag{2.28}$$
$$F_W^{\mu\nu}(x) = \partial^\nu W^\mu(x) - \partial^\mu W^\nu(x) \tag{2.29}$$

where the fields A^μ, Z^μ and W^μ are related to the fields W_i^μ and B^μ by

$$A^\mu(x) = \cos\theta_W B^\mu(x) + \sin\theta_W W_3^\mu(x) \tag{2.30}$$
$$Z^\mu(x) = -\sin\theta_W B^\mu(x) + \cos\theta_W W_3^\mu(x) \tag{2.31}$$
$$W^\mu(x) = \frac{1}{\sqrt{2}}[W_1^\mu(x) - iW_2^\mu(x)] \tag{2.32}$$

Note that $W^\mu(x)$ is a non-Hermitian field, and that the weak mixing angle[4], θ_W, relates the coupling constants

$$g_W \sin\theta_W = g'_W \cos\theta_W = e \tag{2.33}$$

Let us start from the gauge-invariant Lagrangian density given in [1], pp. 270 and 274:

$$\begin{aligned}\mathcal{L}(x) &= \mathcal{L}^L(x) + \mathcal{L}^B(x)\\ &= i\Big[\overline{\Psi}{}^L_l(x)\,\not{D}\Psi^L_l(x) + \overline{\psi}{}^R_l(x)\,\not{D}\psi^R_l(x) + \overline{\psi}{}^R_{\nu_l}(x)\,\not{D}\psi^R_{\nu_l}(x)\Big]\\ &\quad+\Big[-\tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \tfrac{1}{2}F^\dagger_{W\mu\nu}(x)F^{\mu\nu}_W(x) - \tfrac{1}{4}Z_{\mu\nu}(x)Z^{\mu\nu}(x)\Big]\\ &\quad+\mathcal{L}^B_I(x)\end{aligned} \tag{2.34}$$

where $\mathcal{L}^B_I(x)$ describes interactions among gauge bosons, and the covariant derivatives are defined as[5]

$$D^\mu\Psi^L_l(x) = [\partial^\mu + ig_W\tau_j W^\mu_j(x)/2 - ig'_W B^\mu(x)/2]\Psi^L_l(x) \tag{2.35}$$
$$D^\mu\psi^R_l(x) = [\partial^\mu - ig'_W B^\mu(x)]\psi^R_l(x) \tag{2.36}$$
$$D^\mu\psi^R_{\nu_l}(x) = \partial^\mu\psi^R_{\nu_l}(x) \tag{2.37}$$

In the definition of $\psi^L(x)$ and $\psi^R(x)$, the left and right projection operators, P_L and P_R, are involved

$$\psi^L(x) = P_L\psi(x) \equiv \tfrac{1}{2}(1-\gamma_5)\psi(x) \tag{2.38}$$
$$\psi^R(x) = P_R\psi(x) \equiv \tfrac{1}{2}(1+\gamma_5)\psi(x) \tag{2.39}$$

A two-component isospinor field, $\Psi^L_l(x)$, is also defined as

$$\Psi^L_l(x) = \begin{pmatrix}\psi^L_{\nu_l}(x)\\ \psi^L_l(x)\end{pmatrix} \tag{2.40}$$

with

$$\overline{\Psi}{}^L_l(x) = \begin{pmatrix}\overline{\psi}{}^L_{\nu_l}(x) & \overline{\psi}{}^L_l(x)\end{pmatrix} \tag{2.41}$$

The Lagrangian density in (2.34) is invariant under $SU(2)\times U(1)$ gauge transformation, but in order to give the gauge bosons, $W^\pm$ and Z^0, non-vanishing masses, we must apply the Higgs mechanism. To break the $SU(2)$ symmetry, we introduce the Higgs field as a weak isospin doublet

$$\Phi(x) = \begin{pmatrix}\phi_a(x)\\ \phi_b(x)\end{pmatrix} \tag{2.42}$$

[4] Sometimes referred to as the Weinberg angle.
[5] The 2×2 Pauli-matrices are represented by τ_j.

where the components $\phi_a(x)$ and $\phi_b(x)$ are scalar fields under Lorentz transformations. To include the Higgs field, the Lagrangian density, (2.34), is generalized to

$$\mathcal{L}(x) = \mathcal{L}^L(x) + \mathcal{L}^B(x) + \mathcal{L}^H(x) \tag{2.43}$$

where $\mathcal{L}^H(x)$ is

$$\mathcal{L}^H(x) = [D^\mu\Phi(x)]^\dagger[D_\mu\Phi(x)] - \mu^2\Phi^\dagger(x)\Phi(x) - \lambda[\Phi^\dagger(x)\Phi(x)]^2 \tag{2.44}$$

and the covariant derivative is defined analogous to (2.35) as

$$D^\mu\Phi(x) = [\partial^\mu + ig_W\tau_j W_j^\mu(x)/2 + ig'_W Y B^\mu(x)]\Phi(x) \tag{2.45}$$

Note that the Lagrangian density in (2.43) is still $SU(2) \times U(1)$ invariant[6].

Analogous to the Higgs model, for $\lambda > 0$ and $\mu^2 < 0$, we get the minimum energy density for

$$\Phi(x) = \Phi_0 = \begin{pmatrix} \phi_a^0 \\ \phi_b^0 \end{pmatrix} \tag{2.46}$$

when

$$\Phi^\dagger(x)\Phi(x) = |\phi_a^0|^2 + |\phi_b^0|^2 = -\frac{\mu^2}{2\lambda} \tag{2.47}$$

This corresponds to a circle of absolute minima as before, and we may choose Φ_0 to be

$$\Phi_0 = \begin{pmatrix} \phi_a^0 \\ \phi_b^0 \end{pmatrix} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \tag{2.48}$$

where v is defined, according to (2.12), as

$$v = \sqrt{-\mu^2/\lambda} \tag{2.49}$$

To ensure that the photon remains massless, we require the Higgs field in vacuum to be invariant under a $U(1)$ symmetry. Since there is no electric charge in vacuum, we must require $\langle 0|\Phi_0|0\rangle$ to be electrically neutral.

To determine the hypercharge of the Higgs field, we focus on the scalar field, ϕ_b^0, which has a non-vanishing expectation value in vacuum. The hypercharge, Y, has the following definition

$$Y = Q/e - I_3^W \tag{2.50}$$

where Q is the electric charge and I_3^W is the weak isocharge. Since the lower component of a weak isospinor has $I_3^W = -\frac{1}{2}$, and $Q = 0$ in this case, the hypercharge of the Higgs field, Φ, is $Y = \frac{1}{2}$.

[6] The hypercharge, Y, of the Higgs field will be determined later.

To obtain non-vanishing lepton masses we introduce Yukawa couplings by adding the following term to the Lagrangian

$$\begin{aligned}\mathcal{L}^{LH}(x) = &-g_l\Big[\overline{\Psi}_l^L(x)\psi_l^R(x)\Phi(x) + \Phi^\dagger(x)\overline{\psi}_l^R(x)\Psi_l^L(x)\Big] \\ &-g_{\nu_l}\Big[\overline{\Psi}_l^L(x)\psi_{\nu_l}^R(x)\widetilde{\Phi}(x) + \widetilde{\Phi}^\dagger(x)\overline{\psi}_{\nu_l}^R(x)\Psi_l^L(x)\Big]\end{aligned} \tag{2.51}$$

where g_l and g_{ν_l} are dimensionless coupling constants, proportional to the lepton masses[7]. The definition of $\widetilde{\Phi}(x)$ is

$$\widetilde{\Phi}(x) = -i\Big[\widetilde{\Phi}^\dagger(x)\tau_2\Big]^T = \begin{pmatrix}\phi_b^*(x) \\ -\phi_a^*(x)\end{pmatrix} \tag{2.52}$$

Quark masses and quark-Higgs couplings are obtained in the same manner.

Note that in an arbitrary gauge, the Higgs field from (2.42) can be written in terms of four real fields, $\eta_1(x)$, $\eta_2(x)$, $\eta_3(x)$ and $\sigma(x)$, as

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}\eta_1(x) + i\eta_2(x) \\ v + \sigma(x) + i\eta_3(x)\end{pmatrix} \tag{2.53}$$

2.5 Standard Electroweak Theory

The theory of electroweak interactions, developed by Glashow, Weinberg and Salam, leads to predictions which are in excellent agreement with experiments. There is only one piece missing in the puzzle, and that is the Higgs boson. We will in this section see why the mass of the Higgs boson cannot be predicted from the electroweak theory.

In unitary gauge, the Higgs field, $\Phi(x)$, from the previous section can be written as

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}0 \\ v + \sigma(x)\end{pmatrix} \tag{2.54}$$

which can be used to rewrite the Lagrangian density obtained from (2.43) and (2.51). Here we shall only quote the result (see [1], p. 300):

$$\begin{aligned}\mathcal{L}(x) &= \mathcal{L}^B(x) + \mathcal{L}^H(x) + \mathcal{L}^L(x) + \mathcal{L}^{LH}(x) \\ &= \mathcal{L}_0(x) + \mathcal{L}_I(x)\end{aligned} \tag{2.55}$$

where

$$\begin{aligned}\mathcal{L}_0(x) = &\ \overline{\psi}_l(x)(i\partial\!\!\!/ - m_l)\psi_l(x) + \overline{\psi}_{\nu_l}(x)(i\partial\!\!\!/ - m_{\nu_l})\psi_{\nu_l}(x) \\ &-\tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \\ &-\tfrac{1}{2}F_{W\mu\nu}^\dagger(x)F_W^{\mu\nu}(x) + m_W^2 W_\mu^\dagger(x)W^\mu(x) \\ &-\tfrac{1}{4}Z_{\mu\nu}(x)Z^{\mu\nu}(x) + \tfrac{1}{2}m_Z^2 Z_\mu(x)Z^\mu(x) \\ &+\tfrac{1}{2}(\partial_\mu\sigma(x))(\partial^\mu\sigma(x)) - \tfrac{1}{2}m_H^2\sigma^2(x)\end{aligned} \tag{2.56}$$

[7]Summation over $l = e,\ \mu,\ \tau$ is assumed.

and

$$\mathcal{L}_I(x) = \mathcal{L}_I^{LB}(x) + \mathcal{L}_I^{BB}(x) + \mathcal{L}_I^{HH}(x) + \mathcal{L}_I^{HB}(x) + \mathcal{L}_I^{HL}(x) \tag{2.57}$$

The different terms in $\mathcal{L}_I(x)$ can be found in [1], pp. 299–300. Some new parameters have been introduced in eq. (2.56), which can be expressed as

$$m_W = \tfrac{1}{2} v g_W\,, \qquad m_Z = \frac{m_W}{\cos\theta_W}\,, \qquad m_H = \sqrt{-2\mu^2} = \sqrt{2\lambda}\, v \tag{2.58}$$

By using the relations in (2.33) and (2.58), the masses of the heavy vector bosons can be predicted in terms of three experimentally well-known quantities, which are [7]:
The fine-structure constant

$$\alpha = \frac{e^2}{4\pi} = 1/137.036... \tag{2.59}$$

the Fermi coupling constant

$$G_F = \frac{1}{\sqrt{2}\, v^2} = \frac{\sqrt{2}\, g_W^2}{8 m_W^2} = 1.166... \times 10^{-5}\,\text{GeV}^{-2} \tag{2.60}$$

and the weak mixing angle[8], θ_W, given by

$$\sin^2\theta_W = 0.2315 \pm 0.0002, \qquad 0 \le \theta_W \le \pi/2 \tag{2.61}$$

The predictions are

$$m_W = \left(\frac{\alpha\pi}{G_F\sqrt{2}}\right)^{\frac{1}{2}} \frac{1}{\sin\theta_W}\,, \qquad m_Z = \left(\frac{\alpha\pi}{G_F\sqrt{2}}\right)^{\frac{1}{2}} \frac{2}{\sin(2\theta_W)} \tag{2.62}$$

If radiative corrections are taken into account[9], these predictions are in good agreement with the experimental values, but we do not know the mass of the Higgs boson, since the self-interaction constant, λ, is unknown.

Since the electromagnetic gauge symmetry has not been broken, the photon must remain massless. The gauge bosons $W^{\pm}$, Z^0, and the leptons however acquire mass[10]. We again see that the Higgs field survives, and on quantization gives rise to massive Higgs bosons. The standard electroweak theory is an attempt to unify the electromagnetic and the weak force. If this model describes the world we live in, the Higgs boson must exist. The only way to find out is through experiments.

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[8] The value given here is obtained from measurements of m_W and m_Z. Historically, a value obtained from neutrino scattering was used.

[9] Radiative corrections take into account corrections of higher order in the coupling constant. These include self-energy corrections, vertex corrections and real (soft) emission.

[10] If quarks were included in the theory, they would acquire mass the way leptons do.

2.6 Higgs Production

Through their thorough experimental searches, physicists have excluded the existence of a neutral scalar Higgs boson with a mass less than 107.7 GeV at a confidence level of 95 % (see [7]). From the requirement of perturbation theory to be valid, theorists have established an upper mass limit of the order 1 TeV. Therefore, if the Higgs boson exists, there are strong reasons to believe it will be found in the mass region

$$107.7\,\mathrm{GeV} \le m_H \lesssim 1\,\mathrm{TeV} \tag{2.63}$$

The experimental mass limit has been obtained through experiments at LEP[11]. At LEP, one has searched for the decay $Z^0 \to HZ^{0*}$, where Z^{0*} is a virtual Z^0 boson, or $e^+e^- \to HZ^0$. If LEP fails in detecting the Higgs boson[12], the challenge will go to future experiments at the upgraded Tevatron at Fermilab. At Fermilab, where they have a proton-antiproton collider, the background is larger. The most promising processes are $p\bar{p} \to HZ^0X$ and $p\bar{p} \to HW^{\pm}X$, where X can be "anything".

From the year 2005 (or 2006), the Large Hadron Collider (LHC) at CERN is expected to collide protons at energies up to 14 TeV in the center of mass. If the Higgs boson is still undiscovered, the experiments at the LHC will determine if it exists or not. At LHC, the production process $gg \to H$, where g represents a gluon, will be dominant [8]. Since gluons are massless, and the Higgs boson couples to other particles proportional to their masses, gluons will not couple directly to the Higgs boson. Instead we have to consider one-loop Feynman diagrams as the lowest order in perturbation theory. Since gluons only couple to color-charged particles, the loop will contain (heavy) quarks. The two possible one-loop Feynman diagrams for this process are shown in Figure 4.2 on page 38.

If the Higgs boson has a relatively low mass, $m_H \lesssim 130$ GeV, one possible way of detecting it at the LHC is through two-photon decay. Although the branching ratio, which gives us the probability of a Higgs decaying into two photons, is very low, this is a very clean channel, by which we mean low background. The decay rate of a Higgs decaying into two photons is to lowest order calculated from one-loop Feynman diagrams, which we show schematically in Figure 6.1 on page 62.

So far, we have only considered the Higgs boson of the Standard Model. Extensions of this model often include several Higgs bosons. One of these extended models is the *Minimal Supersymmetric Standard Model* (MSSM). In supersymmetric models there is a symmetry between bosons and fermions, which leads to supersymmetric partners[13] for each particle. Since sparticles should have the same mass as particles, but are not observed,

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[11] Large Electron Positron collider at CERN.

[12] On the 5th of September, 2000, the "Special LEPC Seminar" was held at CERN. The preliminary combined result on the SM Higgs mass limit for the four LEP experiments, which was presented by C. Tully, LEP Higgs working group, was $m_H > 112.3$ GeV at a confidence level of 95 %. Due to some possible Higgs events observed around 115 GeV, a proposal to extend the run of LEP for some months, to improve the statistics, is now under evaluation.

[13] The superpartner of a boson is a fermion and vice versa, and they have the same name as their partner, but normally with an "s" in front, like in squark and slepton, or an "ino" at the end, like in gluino.

the symmetry has to be spontaneously broken. To avoid anomalies, two complex Higgs doublets, with together eight degrees of freedom, have to be introduced. Three degrees of freedom are absorbed in the longitudinal components of the massive electroweak gauge bosons, and the five remaining degrees of freedom lead to five scalar Higgs particles. Two of them, h^0 and H^0, are CP-even[14], two are charged, $H^\pm$, and there is one pseudoscalar, or CP-odd, A^0. The Higgs sector of the MSSM is described by two parameters, the most common choice is the mass of the pseudoscalar, m_{A^0}, and $\tan\beta \equiv v_2/v_1$, where v_1 and v_2 are the two non-vanishing vacuum expectation values. However, the discovery of a Higgs boson with SM couplings would not exclude the MSSM since in the limit of large masses for A^0 and H^0, h^0 is indistinguishable from the Standard Model Higgs particle, H_{SM}.

Let us define the R-parity for a particle of spin S as

$$R = (-1)^{3(B-L)+2S} \tag{2.64}$$

where B is the baryon number and L is the lepton number[15]. The MSSM is an R-parity conserving theory, and R-parity invariance implies that supersymmetric particles must be pair-produced. Due to R-parity conservation, the sparticles will decay to the Lightest Supersymmetric Particle (LSP), which is stable, and therefore may contribute to the "cold dark matter" in the universe. Based on cosmological observations, the LSP interacts weakly with ordinary matter, and would therefore behave like a neutrino with an enormous mass. If supersymmetry exists, but has not been discovered until 2005, the experiments at LHC will discover it.

[14] By convention, h^0 is the lightest.

[15] Note that $R = 1$ for particles, and $R = -1$ for sparticles.

Chapter 3

General Theory of Relativity

We will in this chapter give some definitions, and introduce some concepts of the general theory of relativity[1]. Some of the concepts presented in this chapter will be useful in our study of the radion in Chapter 5.

In Einstein's general theory of relativity, gravitation is understood as a consequence of the curvature of space-time. The branch of mathematics used to describe this non-Euclidean behavior of space-time is called *differential*, or *Riemannian geometry*. In 1916, after several attempts, Einstein finally succeeded in formulating the general theory of relativity. The hope was now that all forces could be unified in a geometrical description. As pointed out by S. Weinberg in [9], our present knowledge of strong, weak and electromagnetic interactions suggests that the geometrical approach is not the way to obtain a complete unification of the fundamental forces. The general theory of relativity can also be derived from the *principle of equivalence of gravitation and inertia*, which is the approach followed by Weinberg. In his approach, Riemannian geometry is only a mathematical tool, whereas the principle of equivalence, which is the fundamental basis, can be confirmed by experiment.

3.1 Notation

We will now specify the notation we will use in this chapter, since it differs from the one used in the previous chapters. In inertial coordinate systems, we use the familiar Minkowski metric, $\eta_{\alpha\beta}$, which corresponds to a "flat" space-time, but with diagonal elements[2] -1, $+1$, $+1$, $+1$. We let the indices at the beginning of the Greek alphabet run over the four coordinate labels in Minkowski space, 0, 1, 2, 3, with 0 as the time component. In a general four-dimensional coordinate system, we use $g_{\mu\nu}$ as the *metric tensor* and let the indices in the middle of the Greek alphabet run over the general coordinate labels. We will also write out the partial derivatives instead of using the ∂_μ notation, in order to clarify which coordinates we differentiate with respect to.

[1] For more details, see [9].

[2] This metric will only be used in Chapter 3.

3.2 Principle of Equivalence

The principle of the equivalence of gravitation and inertia, which was introduced by Einstein in 1907, can be formulated as follows[3]: *At every space-time point in an arbitrary gravitational field it is possible to choose a "locally inertial coordinate system" such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation.* In this context, "sufficiently small region" means that the gravitational field can be considered constant within the region, and "the same form as in ...", refers to the laws of nature given by special relativity. From Riemannian geometry, we know that at any point on a curved surface, it is possible to find a locally Cartesian coordinate system, in which distances obey the law of Pythagoras, so a connection between the laws of gravitation and the formulae of Riemannian geometry should not surprise us.

In a locally inertial coordinate frame, ξ^α, of a freely falling massive particle, the equation of motion is the equation of a straight line in space, which is

$$\frac{\mathrm{d}^2\xi^\alpha}{\mathrm{d}\tau^2} = 0 \tag{3.1}$$

where τ is the proper time

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2 = -\eta_{\alpha\beta}\mathrm{d}\xi^\alpha\mathrm{d}\xi^\beta \tag{3.2}$$

By a coordinate transformation, the equation of motion, (3.1), can be expressed in any other coordinate frame, x^μ, as [9]

$$\frac{\mathrm{d}^2x^\lambda}{\mathrm{d}\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \tag{3.3}$$

where the *affine connection*[4], $\Gamma^\lambda_{\mu\nu}$, is defined by

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu} \tag{3.4}$$

If we define the metric tensor, $g_{\mu\nu}$, by

$$g_{\mu\nu} \equiv \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} \tag{3.5}$$

we may also write the proper time as

$$\mathrm{d}\tau^2 = -g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{3.6}$$

To get the equations of motion for massless particles, we substitute the proper time, τ, by $\sigma \equiv \xi^0$ in the above equations.

[3]Quoted from [9].

[4]The elements of the affine connection are called *Christoffel symbols*.

All information of the gravitational field is contained in $\Gamma^{\lambda}_{\mu\nu}$ and $g_{\mu\nu}$, which also determine the locally inertial coordinate frame up to an inhomogeneous Lorentz transformation. In order to find a relation between the affine connection and the metric tensor, we specify the meaning of "locally inertial" in the principle of equivalence as: It is possible to choose the locally inertial coordinates, ξ^{α}_{X}, that we construct in a given point, X, such that the first derivatives of the metric tensor, $g^{X}_{\gamma\delta}$, in the locally inertial coordinates vanish at X. This means that the metric tensor at a point X' in the coordinate system ξ^{α}_{X}, which can be expressed as

$$g^{X}_{\gamma\delta}(X') = \left(\frac{\partial \xi^{\alpha}_{X'}(x)}{\partial \xi^{\gamma}_{X}(x)} \frac{\partial \xi^{\beta}_{X'}(x)}{\partial \xi^{\delta}_{X}(x)} \eta_{\alpha\beta} \right)_{x=X'} \tag{3.7}$$

is stationary in X' at $X' = X$. We can now express the affine connection, $\Gamma^{\kappa}_{\lambda\mu}$, in terms of the metric tensor and its derivatives, as [9]

$$\Gamma^{\kappa}_{\lambda\mu} = \tfrac{1}{2} g^{\nu\kappa} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right\} \tag{3.8}$$

The metric tensor can also be expressed in matrix notation as

$$g = D^{T} \eta D \tag{3.9}$$

where g and η are 4×4 matrices, with elements $g_{\mu\nu}$ and $\eta_{\alpha\beta}$ respectively, and D is given by

$$D_{\alpha\mu} \equiv \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \tag{3.10}$$

with D^{T} as its transpose

$$\left(D^{T}\right)_{\alpha\mu} \equiv D_{\mu\alpha} \tag{3.11}$$

This kind of transformation between the metric tensor, $g_{\mu\nu}$, and the Minkowski tensor, $\eta_{\alpha\beta}$, ensures that both must have one positive eigenvalue and three negative eigenvalues.

Another formulation of the principle of equivalence is the *principle of general covariance*, which states that a physical equation holds in a general gravitational field if the following two conditions are satisfied[5]:

- The equation holds in the absence of gravitation, which means that it agrees with the laws of special relativity when the metric tensor, $g_{\alpha\beta}$, equals the Minkowski tensor $\eta_{\alpha\beta}$, and when the affine connection, $\Gamma^{\alpha}_{\beta\gamma}$, vanishes.
- The equation is generally covariant, i.e. it preserves its form under a general coordinate transformation, $x \to x'$.

When the principle of general covariance is applied on a small scale compared to the scale of the gravitational field, we only expect $g_{\mu\nu}$ and its first derivatives to enter the generally covariant equations.

[5]Quoted from [9].

3.3 Tensors and Curvature

By definition, a *tensor* transforms like a product of contravariant and covariant vectors under a coordinate transformation. Tensors with two indices are called second-rank tensors, four-vectors are tensors of rank one, and scalars are tensors of rank zero. As an example, we consider the transformation of a mixed third-rank tensor[6]

$$T^{\mu}{}_{\nu\lambda} \to T'^{\mu}{}_{\nu\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\kappa}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\lambda}} T^{\kappa}{}_{\rho\sigma} \tag{3.12}$$

The transformation of a tensor of rank n is a straightforward generalization. Note that we can also have purely covariant or contravariant tensors. A theorem from tensor analysis, which is important in the general theory of relativity says that *any equation will be invariant under general coordinate transformations if it states the equality of two tensors with the same upper and lower indices.*

The affine connection is not a tensor, and another very important non-tensor is the determinant of the metric tensor

$$g \equiv \mathrm{Det}\, g_{\mu\nu} \tag{3.13}$$

Note that the metric tensor has lower indices in the definition of g. It is also possible to show that $\sqrt{-g}\,\mathrm{d}^4x$ is an invariant volume element.

We introduce the *covariant derivative*, $V^{\mu}{}_{;\lambda}$, of a contravariant four-vector, V^{μ}, which is defined as

$$V^{\mu}{}_{;\lambda} \equiv \frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}_{\lambda\kappa} V^{\kappa} \tag{3.14}$$

When covariant four-vectors are considered, a minus sign enters in front of the affine connection. We illustrate the generalization to the covariant derivative of a tensor by an example (see [9], eq. (4.6.10))

$$T^{\mu\sigma}{}_{\lambda;\rho} = \frac{\partial}{\partial x^{\rho}} T^{\mu\sigma}{}_{\lambda} + \Gamma^{\mu}_{\rho\nu} T^{\nu\sigma}{}_{\lambda} + \Gamma^{\sigma}_{\rho\nu} T^{\mu\nu}{}_{\lambda} - \Gamma^{\kappa}_{\lambda\rho} T^{\mu\sigma}{}_{\kappa} \tag{3.15}$$

If we take the covariant derivative of a tensor, the result is also a tensor, and in the absence of gravity, i.e. when $\Gamma^{\mu}_{\nu\lambda} = 0$, covariant differentiation reduces to ordinary differentiation. Equations that hold in a general coordinate system are obtained by replacing $\eta_{\mu\nu}$ with $g_{\mu\nu}$, and all derivatives with covariant derivatives in the equations from special relativity. The principle of general covariance ensures that the resulting equations will be true in the presence of gravitational fields.

From the metric tensor and its first and second derivatives, it is possible to construct only one tensor which is linear in the second derivatives. This tensor is called the *Riemann–Christoffel curvature tensor*, $R^{\lambda}{}_{\mu\nu\kappa}$, and is defined as (see [9], eq. (6.1.5))

$$R^{\lambda}{}_{\mu\nu\kappa} \equiv \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\sigma}_{\mu\nu} \Gamma^{\lambda}_{\kappa\sigma} - \Gamma^{\sigma}_{\mu\kappa} \Gamma^{\lambda}_{\nu\sigma} \tag{3.16}$$

[6] A mixed tensor has both upper and lower indices.

Related tensors are the *Ricci tensor*,

$$R_{\mu\kappa} = R^{\lambda}{}_{\mu\lambda\kappa} \tag{3.17}$$

and the *curvature scalar*,

$$R = g^{\mu\kappa} R_{\mu\kappa} \tag{3.18}$$

The necessary and sufficient conditions for a metric, $g_{\mu\nu}$, to be equivalent to the Minkowski metric, $\eta_{\alpha\beta}$, are:

- The curvature tensor calculated from $g_{\mu\nu}$ must vanish everywhere, i.e. $R^{\lambda}{}_{\mu\nu\kappa} \equiv 0$
- At some point X, the matrix $g^{\mu\nu}(X)$ has one positive eigenvalue and three negative eigenvalues.

It is important to be able to make predictions from a theory, and the theory of general relativity certainly satisfies this demand. Whether we interpret the predictions as a consequence of the curvature of space and time, or as the physical effect of gravitational fields is of less importance.

3.4 Einstein's Field Equations

Gravitational fields are generated by energy and momentum, but since the fields themselves represent energy and momentum, they must contribute to their own source, and therefore be described by nonlinear partial differential equations. In the case of the general theory of relativity, these equations, known as *Einstein's field equations*[7], are

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \tag{3.19}$$

where G is Newton's constant[8], and $T_{\mu\nu}$ is the symmetric energy-momentum tensor, which will be defined later (see eqs. (3.27) and (3.28)). The left hand side is sometimes defined as the *Einstein tensor*, $G_{\mu\nu}$, and eq. (3.19) becomes

$$G_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{3.20}$$

By contracting indices, (3.19) can be written alternatively as

$$R_{\mu\nu} = -8\pi G (T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T^{\lambda}{}_{\lambda}) \tag{3.21}$$

[7]Initially, Einstein had a term involving a cosmological constant, Λ, in his equations. This term was introduced *ad hoc* in the theory, to give a static universe. Einstein later referred to this as the biggest mistake of his life, but recent observations have suggested that maybe a cosmological constant has to be reintroduced.

[8]$G \simeq 6.71 \times 10^{-39} (\mathrm{GeV})^{-2} \simeq 6.67 \times 10^{-11} \mathrm{m}^3\,\mathrm{kg}^{-1}\mathrm{s}^{-2}$.

Since the energy-momentum tensor vanishes in vacuum, we see from (3.21) that the Einstein equations in empty space are simply

$$R_{\mu\nu} = 0 \tag{3.22}$$

In the case of *three-dimensional* space-time, the curvature tensor can be written as[9] (see [10], p. 99)

$$R_{\mu\nu\rho\sigma} = R_{\mu\rho}g_{\nu\sigma} + R_{\nu\sigma}g_{\mu\rho} - R_{\mu\sigma}g_{\nu\rho} - R_{\nu\rho}g_{\mu\sigma} - \frac{R}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{3.23}$$

From eqs. (3.18), (3.22) and (3.23), we see that $R_{\mu\nu\rho\sigma}$ vanishes in this case, which means that no gravitational fields can exist. When two-dimensional space-time is considered, the same conclusion is valid, since only the last term in eq. (3.23) enters. Four-dimensional space-time is therefore the lowest dimension in which true gravitational fields can exist in empty space.

When we consider a gravitational field far from its source, we may write the metric tensor, $g_{\mu\nu}$, as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.24}$$

and treat the gravitational field, $h_{\mu\nu}$, as a perturbation. If we neglect terms of second order in h, we must have

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{3.25}$$

in order to satisfy the relation $g^{\mu\rho}g_{\rho\nu} = \delta^{\mu}{}_{\nu}$. This approach leads to wave equations for a spin-2 field, but since Einstein's equations are nonlinear, the theory of gravitational radiation is very complicated. The fact that gravitational waves, predicted by the general theory of relativity, never have been detected is no surprise, because of the weakness of gravity.

As mentioned in Chapter 1, there does not exist any complete quantum theory of gravity. However, the force carrier of gravity is expected to be a massless spin-2 particle, called the *graviton*. The problem with such a theory is that it is non-renormalizable, i.e. it contains an infinite number of divergent integrals which can not be absorbed into redefinitions of the parameters. According to Weinberg, it seems impossible to create a Lorentz invariant quantum theory of gravity, without requiring the principle of equivalence, on which the whole of classical general relativity is based, to be satisfied.

3.5 General Definition of $T^{\mu\nu}$

The *energy-momentum tensor*, $T^{\mu\nu}$, is of particular interest to us, since we will use it in Chapter 5 to establish the relation between the Higgs and the radion couplings to

[9] The upper index in (3.16) has been lowered by the metric tensor; the indices run over three coordinate labels.

Standard Model particles[10]. We will in this section give the general definition of the energy-momentum tensor.

Under an infinitesimal variation of the metric tensor,

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu} \tag{3.26}$$

the variation of the matter action, J_M, of a system can be written

$$\delta J_M = \tfrac{1}{2} \int \mathrm{d}^4 x \, \sqrt{-g(x)}\, T^{\mu\nu}(x) \delta g_{\mu\nu}(x) \tag{3.27}$$

where the coefficient, $T^{\mu\nu}(x)$, by definition, is the energy-momentum tensor of the system. The energy-momentum tensor is symmetric, and it is also conserved in the covariant sense if and only if the matter action is a scalar. From (3.27) and the generalization[11] of eq. (1.16), we see that $T^{\mu\nu}$ can be expressed as

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\,\mathcal{L}_M\right)}{\delta g_{\mu\nu}} \tag{3.28}$$

where $\mathcal{L}_M$ is the matter Lagrangian density[12].

We will now consider the total action of a system, which can be expressed as

$$J = J_M + J_G \tag{3.29}$$

Let us take the purely gravitational part of the action, J_G, to be

$$J_G \equiv -\frac{1}{16\pi G} \int \mathrm{d}^4 x \, \sqrt{-g(x)}\, R(x) \tag{3.30}$$

where R is the curvature scalar[13]. Variation of the metric as indicated in (3.26), leads to the following change in the integrand

$$\delta(\sqrt{-g}\, R) = \sqrt{-g}\, R_{\mu\nu} \delta g^{\mu\nu} + R \delta \sqrt{-g} + \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} \tag{3.31}$$

It is possible to show that the last term in (3.31) drops out when we integrate over all space[14]. If we therefore focus on the first two terms, we see that expressions for $\delta g^{\mu\nu}$ and $\delta\sqrt{-g}$ are needed.

In order to find $\delta\sqrt{-g}$, we shall first find δg. The differential of the determinant of $g_{\mu\nu}$ is the sum of the differentials of each component, $\delta g_{\mu\nu}$, multiplied by their corresponding

[10] In this section, we follow the presentation given in [9], Chapter 12.

[11] We let $\mathrm{d}^4 x \rightarrow \sqrt{-g}\, \mathrm{d}^4 x$ to maintain an invariant volume element.

[12] The energy-momentum tensor for a gravitational field will also contain derivatives with respect to the first derivative of the metric tensor. This is because Einstein's equations are nonlinear partial differential equations.

[13] If J_G is defined with opposite sign ([11], eq. (11–18)), Einstein's equations become $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ ([11], eq. (11–34)).

[14] This is shown in [9], p. 364.

cofactors. An expression for the cofactor of the component, $g_{\mu\nu}$, can be obtained from the component $g^{\mu\nu}$. Since the matrices $g_{\mu\nu}$ and $g^{\mu\nu}$ are reciprocal, i.e. $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu$, we may use a result from linear algebra: Let A be the following 4×4 matrix

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{3.32}$$

then A^{-1}, which is the inverse of A, is given by (see [12], p. 179)

$$A^{-1} = \frac{1}{\text{Det}A} \begin{bmatrix} C_{00} & C_{10} & C_{20} & C_{30} \\ C_{01} & C_{11} & C_{21} & C_{31} \\ C_{02} & C_{12} & C_{22} & C_{32} \\ C_{03} & C_{13} & C_{23} & C_{33} \end{bmatrix} \tag{3.33}$$

where $C_{\mu\nu}$ is the cofactor of the element $a_{\mu\nu}$. From eqs. (3.32) and (3.33), we see that the component $g^{\mu\nu}$ equals the cofactor of the element $g_{\nu\mu}$, divided by the determinant, g. This allows us to express the cofactors of the determinant of $g_{\mu\nu}$ as $gg^{\nu\mu}$, but since $g^{\nu\mu}$ is symmetric, we may write

$$\delta g = gg^{\mu\nu}\delta g_{\mu\nu} = -\left(\sqrt{-g}\right)^2 g^{\mu\nu}\delta g_{\mu\nu} \tag{3.34}$$

We can now use (3.34) to express $\delta\sqrt{-g}$ as

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = \tfrac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu} \tag{3.35}$$

Differentiation of the expression $g_{\rho\sigma}g^{\sigma\nu} = \delta^\nu_\rho$, leads to

$$g^{\sigma\nu}\delta g_{\rho\sigma} + g_{\rho\sigma}\delta g^{\sigma\nu} = 0 \tag{3.36}$$

and if we multiply (3.36) by $g^{\mu\rho}$, we get

$$\delta g^{\mu\nu} = -g^{\mu\rho}g^{\sigma\nu}\delta g_{\rho\sigma} \tag{3.37}$$

Let us now use eqs. (3.31), (3.35) and (3.37) to express δJ_G as

$$\delta J_G = \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\left[R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R\right]\delta g_{\mu\nu} \tag{3.38}$$

Equations (3.27) and (3.38) can be combined to give δJ, which is the differential of the total action, $J = J_M + J_G$. By Hamilton's principle, we require $\delta J = 0$, but since $\delta g_{\mu\nu}$ is arbitrary, this corresponds to

$$R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R + 8\pi GT^{\mu\nu} = 0 \tag{3.39}$$

which we recognize as Einstein's field equations with upper indices compared to (3.19).

3.6 Vierbeins

Since spinor fields do not behave like tensors under Lorentz transformation, the formalism in the previous section can not be used. In this section, we will introduce the concept of *vierbeins*, which allow us to incorporate spinors in the theory.

The relation between the Minkowski tensor and the metric tensor, given in eq. (3.5) can be written as

$$g_{\mu\nu}(x) = V^{\alpha}{}_{\mu}(x) V^{\beta}{}_{\nu}(x) \eta_{\alpha\beta} \tag{3.40}$$

where

$$V^{\alpha}{}_{\mu}(X) \equiv \left(\frac{\partial \xi_X{}^{\alpha}(x)}{\partial x^{\mu}} \right)_{x=X} \tag{3.41}$$

Since the locally inertial coordinates, ξ_X^{α}, are fixed at each point, $V^{\alpha}{}_{\mu}$ must transform like

$$V^{\alpha}{}_{\mu} \to V'^{\alpha}{}_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} V^{\alpha}{}_{\nu} \tag{3.42}$$

We see that $V^{\alpha}{}_{\mu}$, which is called a *vierbein*, or *tetrad*, behaves like four covariant vector fields, and not as a tensor. If A^{μ} is a vector field, we can use the definition

$$*A^{\alpha} \equiv V^{\alpha}{}_{\mu} A^{\mu} \tag{3.43}$$

to replace the vector field by a set of scalars. This definition is easily generalized, to covariant vector fields and tensors. Note that the α index of a vierbein is raised or lowered by the Minkowski tensor, whereas for the μ index we use the metric tensor

$$V_{\beta}{}^{\nu} \equiv \eta_{\alpha\beta} g^{\mu\nu} V^{\alpha}{}_{\mu} \tag{3.44}$$

If we use (3.40) and (3.44), we see that

$$V_{\beta}{}^{\mu} V^{\beta}{}_{\nu} = \eta_{\alpha\beta} g^{\mu\lambda} V^{\alpha}{}_{\lambda} V^{\beta}{}_{\nu} = g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}_{\nu} \tag{3.45}$$

and

$$\begin{aligned} D^{\alpha}_{\beta} &\equiv V^{\alpha}{}_{\mu} V_{\beta}{}^{\mu} = V^{\alpha}{}_{\mu} V^{\gamma}{}_{\nu} \eta_{\beta\gamma} g^{\nu\mu} = V^{\alpha}{}_{\mu} V^{\gamma}{}_{\nu} V_{\delta}{}^{\nu} V_{\varphi}{}^{\mu} \eta_{\beta\gamma} \eta^{\delta\varphi} \\ &= (V^{\alpha}{}_{\mu} V_{\varphi}{}^{\mu})(V_{\delta}{}^{\nu} V^{\gamma}{}_{\nu}) \eta_{\beta\gamma} \eta^{\delta\varphi} = (V^{\alpha}{}_{\mu} V_{\varphi}{}^{\mu})(V^{\varphi}{}_{\nu} V_{\beta}{}^{\nu}) \\ &= D^{\alpha}_{\varphi} D^{\varphi}_{\beta} = \delta^{\alpha}_{\beta} \end{aligned} \tag{3.46}$$

which means that we can use (3.40) to express $*g_{\alpha\beta}$ as

$$*g_{\alpha\beta} \equiv V_{\alpha}{}^{\mu} V_{\beta}{}^{\nu} g_{\mu\nu} = \eta_{\alpha\beta} \tag{3.47}$$

Using this formalism, spinor fields can be incorporated in the theory, but when spinor fields are involved, we have to change the definition of the energy-momentum tensor to

$$T_{\mu\nu} \equiv V_{\alpha\mu} U^{\alpha}{}_{\nu} \tag{3.48}$$

where ${U^\alpha}_\nu$ comes from the change in the matter action as a consequence of variation in the vierbein

$$\delta J_M = \int \mathrm{d}^4x \sqrt{-g}\, {U^\alpha}_\mu \delta {V_\alpha}^\mu \tag{3.49}$$

Let us now find the vierbein relation, which corresponds to (3.37). By differentiating the expression ${V^\beta}_\nu {V_\alpha}^\nu = \delta^\beta_\alpha$ we get

$${V_\alpha}^\nu \delta {V^\beta}_\nu = -{V^\beta}_\nu \delta {V_\alpha}^\nu \tag{3.50}$$

We multiply (3.50) by ${V^\alpha}_\mu$ and find

$$\frac{\delta {V^\beta}_\mu}{\delta {V_\alpha}^\nu} = -{V^\alpha}_\mu {V^\beta}_\nu \tag{3.51}$$

where we have used (3.45).

3.7 Vielbeins

The formalism introduced in this chapter is also relevant in models with more than four space-time dimensions. When the vierbein formalism is generalized to more than four dimensions, the vierbeins change name to *vielbeins*. We will use the notation G_{MN}, where M and N run over all space-time indices, for the metric tensor in more than four dimensions.

Chapter 4

Higgs Production through Gluon Fusion

As mentioned in Chapter 2, the dominant production channel for Standard Model Higgs bosons in proton-proton colliders is $gg \to H$ (see ref. [8]). In this chapter we will calculate the cross section, σ, for this process by using the *QCD improved parton model.*

4.1 QCD Improved Parton Model

The QCD improved parton model is based on the so-called "naive" parton model, which describes the hadron in terms of distribution functions for its constituents, the *partons*[1]. We can express the momentum of a parton, k_i, as $k_i = x_i p$, where p is the momentum of the hadron, and x_i is the momentum fraction. If we sum over all momentum fractions, x_i, of the partons in a hadron, we get

$$\sum_i x_i = 1 \tag{4.1}$$

and another obvious restriction on x_i is

$$0 \leq x_i \leq 1 \quad \forall\, i \tag{4.2}$$

Consider $f_a^A(x)$ to be the distribution function of parton-type a in hadron A, then $f_a^A(x)\,\mathrm{d}x$ is the expectation value for the number of partons of type a with momentum fraction in the interval from x to $x{+}\mathrm{d}x$. It follows that $x f_a^A(x)\,\mathrm{d}x$ represents the momentum fraction carried by these partons, but this interpretation only makes sense if we require

$$\sum_a \int_0^1 x f_a^A(x)\,\mathrm{d}x = 1 \tag{4.3}$$

where we sum over all parton types, a.

[1]By partons, we mean constituents of the proton, namely quarks, antiquarks and gluons.

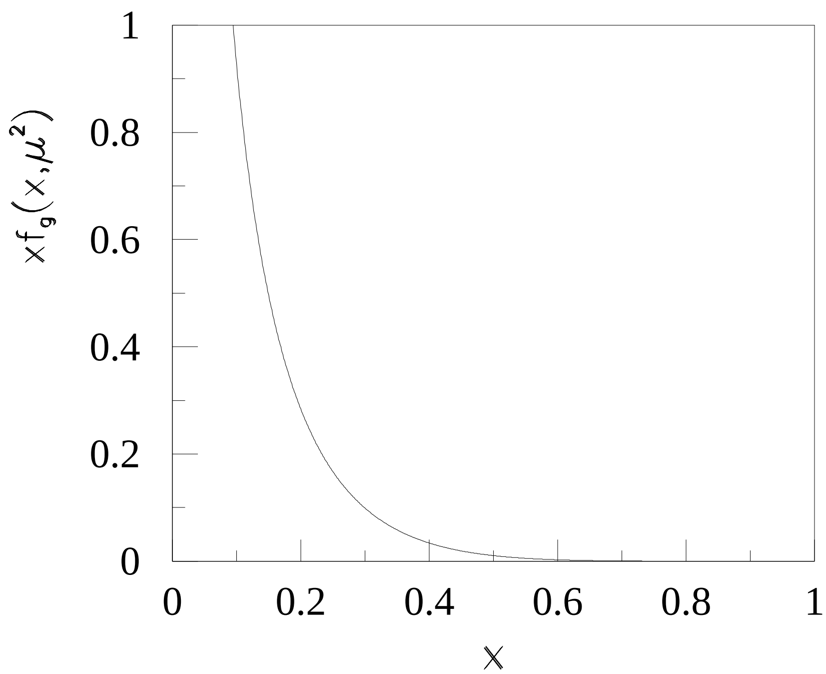


Figure 4.1: The weighted gluon distribution function, $xf_g(x, \mu^2)$, as a function of x. We obtained $f_g(x, \mu^2)$ from [4], with a factorization scale, $\mu = 100$ GeV.

In the QCD improved parton model, the distribution functions $f_a^A(x)$ are replaced by $f_a^A(x, \mu^2)$, which contain the factorization scale[2], μ. In Figure 4.1 we display the function $xf_g(x, \mu^2)$, where $f_g(x, \mu^2)$ is the gluon distribution function. Note that a gluon seldom carries a large momentum fraction[3].

According to [3], p. 238, the cross section we are interested in can be written as[4]

$$\sigma(p_1, p_2) = \int \mathrm{d}x_1 \, \mathrm{d}x_2 \, f_g^{(1)}(x_1, \mu^2) f_g^{(2)}(x_2, \mu^2) \hat{\sigma}_{gg}(k_1, k_2, \alpha_S(\mu^2), Q^2/\mu^2) \tag{4.4}$$

where α_S is the running coupling constant of QCD, and $\hat{\sigma}_{gg}$ is the short-distance cross section for the subprocess $gg \to H$. We have labelled the proton momenta by p_1 and p_2, with the momenta of the respective gluons labelled as $k_1 = x_1 p_1$ and $k_2 = x_2 p_2$. The long-distance effects can be removed from the perturbative cross section and absorbed into the parton distribution functions through factorization. Since μ is arbitrary, and can be considered as the scale which separates long- and short-distance physics, it is often set

[2]We will explain the factorization scale later.

[3]In total, the gluons carry approximately half of the momentum of a proton.

[4]To renormalize a theory, one must choose a renormalization scale, where the ultraviolet divergences are removed. In order to simplify (4.4), the renormalization scale has been set equal to the factorization scale, μ.

equal to Q, which is the scale of the parton-parton interaction. When we look at Higgs production, it is therefore natural to insert $Q = \mu = m_H$ into α_S, which is given in eq. (1.48), to get

$$\alpha_S(m_H) \simeq \frac{6\pi}{23\ln(4.425 \times m_H/\text{GeV})}\left[1 - \frac{174\ln[2\ln(4.425 \times m_H/\text{GeV})]}{529\ln(4.425 \times m_H/\text{GeV})}\right] \tag{4.5}$$

This expression for $\alpha_S(m_H)$ will be used later in our numerical calculations.

4.2 Feynman Amplitude

Before we focus on the cross section for the subprocess, $gg \to H$, we shall find the *Feynman amplitude*, $\mathcal{M}$, for this process. The Feynman amplitude can be found by applying the Feynman rules of Appendix A to all Feynman diagrams of the process under consideration. To leading order in perturbation theory, there are two Feynman diagrams, which are shown in Figure 4.2 on page 38. Since the Feynman amplitude corresponding to these diagrams is similar to the Feynman amplitude of the quark loop contribution in the process $H \to \gamma\gamma$, we will refer to some of the results given in [13].

If we look at Figure 4.2, we see that the amplitude has two contributions, $\mathcal{M}_1$ and $\mathcal{M}_2$. By using the Feynman rules in Appendix A, we can write out the mathematical expressions corresponding to each diagram[5]:

$$\begin{aligned}\mathcal{M}_1 = -\text{Tr}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\,(ig_S\gamma^\alpha\varepsilon_{r\alpha}(k_1)T^a_{ij})\frac{i(\not{q} - \frac{1}{2}\not{p} + m_q)}{(q-\frac{1}{2}p)^2 - m_q^2}\left(-\frac{i}{2}g_W\frac{m_q}{m_W}\right)\\ \times\frac{i(\not{q} + \frac{1}{2}\not{p} + m_q)}{(q+\frac{1}{2}p)^2 - m_q^2}(ig_S\gamma^\beta\varepsilon_{s\beta}(k_2)T^b_{ji})\frac{i(\not{q} + \not{k} + m_q)}{(q+k)^2 - m_q^2}\end{aligned} \tag{4.6}$$

and

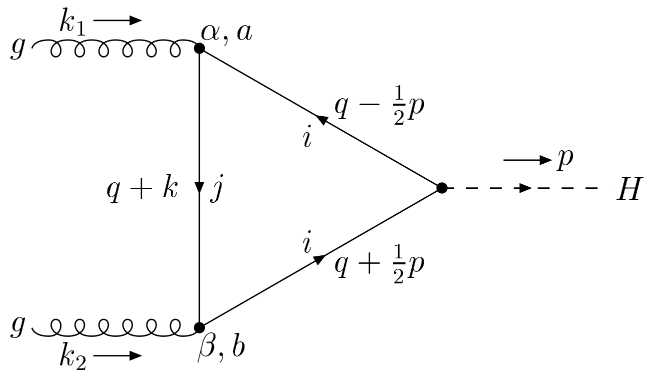
$$\begin{aligned}\mathcal{M}_2 = -\text{Tr}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\,(ig_S\gamma^\alpha\varepsilon_{r\alpha}(k_1)T^a_{ji})\frac{i(-\not{q} - \not{k} + m_q)}{(-q-k)^2 - m_q^2}(ig_S\gamma^\beta\varepsilon_{s\beta}(k_2)T^b_{ij})\\ \times\frac{i(-\not{q} - \frac{1}{2}\not{p} + m_q)}{(-q-\frac{1}{2}p)^2 - m_q^2}\left(-\frac{i}{2}g_W\frac{m_q}{m_W}\right)\frac{i(-\not{q} + \frac{1}{2}\not{p} + m_q)}{(-q+\frac{1}{2}p)^2 - m_q^2}\end{aligned} \tag{4.7}$$

where $-m_q^2$ in the denominators represents $-m_q^2 + i\varepsilon$. We also define the denominators of the propagators as

$$\begin{aligned}d_1 &= (q + \tfrac{1}{2}p)^2 - m_q^2\\ d_2 &= (q - \tfrac{1}{2}p)^2 - m_q^2\\ d_3 &= (q + k)^2 - m_q^2\end{aligned} \tag{4.8}$$

[5]For simplicity we write $\varepsilon(k)$, even tough the polarization vector only depends on the spatial part, $\mathbf{k}$.

(1)



(2)

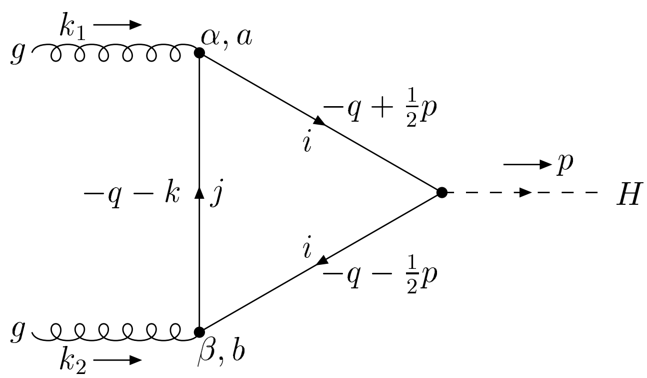


Figure 4.2: Lowest order Feynman diagrams for the production of a Higgs boson from two gluons, through a quark loop. We have used k_1 and k_2 for the gluon momenta, and p for the momentum of the Higgs boson, where $p = k_1 + k_2$ and $k = \frac{1}{2}(k_1 - k_2)$. The quarks have colors i and j, whereas the Higgs particle is color neutral.

Let us sum over quark colors, i and j, in the loop, then we get the following relation involving the color matrices T^a_{ij} and T^b_{ji}:

$$T^a_{ij} T^b_{ji} = \mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta^{ab} \tag{4.9}$$

The Feynman amplitude, $\mathcal{M}$, which is the sum of the two contributions, $\mathcal{M}_1$ and $\mathcal{M}_2$ given in eqs. (4.6) and (4.7) respectively, can now be expressed as

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = -\frac{g_S^2 g_W m_q}{4 m_W} \delta^{ab} \varepsilon_{r\alpha}(k_1) \varepsilon_{s\beta}(k_2) \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathcal{N}^{\alpha\beta}}{d_1 d_2 d_3} \tag{4.10}$$

where $\mathcal{N}^{\alpha\beta} = \mathcal{N}_1^{\alpha\beta} + \mathcal{N}_2^{\alpha\beta}$, and

$$\mathcal{N}_1^{\alpha\beta}(p,k) = \mathrm{Tr}\Big\{\gamma^\alpha(\not{q} - \tfrac{1}{2}\not{p} + m_q)(\not{q} + \tfrac{1}{2}\not{p} + m_q)\gamma^\beta(\not{q} + \not{k} + m_q)\Big\} \tag{4.11}$$

$$\mathcal{N}_2^{\alpha\beta}(p,k) = \mathrm{Tr}\Big\{\gamma^\alpha(-\not{q} - \not{k} + m_q)\gamma^\beta(-\not{q} - \tfrac{1}{2}\not{p} + m_q)(-\not{q} + \tfrac{1}{2}\not{p} + m_q)\Big\} \tag{4.12}$$

By using the cyclic property of the trace and considering $k \leftrightarrow -k$ and $p \leftrightarrow -p$, eqs. (4.11) and (4.12) can be written as

$$\mathcal{N}_1^{\alpha\beta}(-p,-k) = \mathrm{Tr}\Big\{\gamma^\alpha(\not{q} + \tfrac{1}{2}\not{p} + m_q)(\not{q} - \tfrac{1}{2}\not{p} + m_q)\gamma^\beta(\not{q} - \not{k} + m_q)\Big\} \tag{4.13}$$

$$\mathcal{N}_2^{\alpha\beta}(-p,-k) = \mathrm{Tr}\Big\{\gamma^\beta(-\not{q} + \tfrac{1}{2}\not{p} + m_q)(-\not{q} - \tfrac{1}{2}\not{p} + m_q)\gamma^\alpha(-\not{q} + \not{k} + m_q)\Big\} \tag{4.14}$$

If we compare eqs. (4.13) and (4.14) to eqs. (4.7) and (4.8) in ref. [13], respectively, we see that they are equal. This allows us to use the result (4.19) in ref. [13] directly, and substitute back $k \leftrightarrow -k$ and $p \leftrightarrow -p$, which gives[6]:

$$\begin{aligned}\mathcal{N}^{\alpha\beta}(p,k) = -8m_q\Big[&(d_3 + \tfrac{1}{4}p^2 - k^2)g^{\alpha\beta} \\ &-q^\alpha(p+2k)^\beta + (p-2k)^\alpha q^\beta - 4q^\alpha q^\beta\Big]\end{aligned} \tag{4.15}$$

The same expression for $\mathcal{N}^{\alpha\beta}$ was also obtained by using parts of the `REDUCE` program given in [13], Appendix C. If we insert (4.15) into (4.10), we get

$$\begin{aligned}\mathcal{M} = &\frac{2g_S^2 g_W m_q^2}{m_W}\delta^{ab}\varepsilon_{r\alpha}(k_1)\varepsilon_{s\beta}(k_2)\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{d_1d_2d_3} \\ &\times\Big[(d_3 + \tfrac{1}{4}p^2 - k^2)g^{\alpha\beta} + (p-2k)^\alpha q^\beta - q^\alpha(p+2k)^\beta - 4q^\alpha q^\beta\Big]\end{aligned} \tag{4.16}$$

A dimensionless tensor amplitude, $\mathcal{M}_0^{\alpha\beta}(k_1,k_2)$ is obtained by extracting some constants and the two polarization vectors, $\varepsilon_{r\alpha}(k_1)$ and $\varepsilon_{s\beta}(k_2)$, from $\mathcal{M}$:

$$\mathcal{M} = (i\pi^2)\frac{g_S^2 g_W m_H^2}{2m_W(2\pi)^4}\delta^{ab}\varepsilon_{r\alpha}(k_1)\varepsilon_{s\beta}(k_2)\mathcal{M}_0^{\alpha\beta}(k_1,k_2) \tag{4.17}$$

where $\mathcal{M}_0^{\alpha\beta}(k_1,k_2)$ is defined as

$$\begin{aligned}\mathcal{M}_0^{\alpha\beta}(k_1,k_2) = &\frac{4m_q^2}{(i\pi^2)m_H^2}\int\mathrm{d}^4q\,\frac{1}{d_1d_2d_3} \\ &\times\Big[(d_3 + \tfrac{1}{4}p^2 - k^2)g^{\alpha\beta} + (p-2k)^\alpha q^\beta - q^\alpha(p+2k)^\beta - 4q^\alpha q^\beta\Big]\end{aligned} \tag{4.18}$$

[6] A misprint in ref. [13] has been corrected.

By counting the momenta in the numerator and the denominator, we can see that the integral in (4.18) is logarithmically divergent in the UV region, i.e. as $q \to \infty$. This problem can be solved by a technique called dimensional regularization, where the calculations are carried out in n dimensions, before the limit $n \to 4$ is taken. These calculations can be found in [13], Chapter 6, and before we continue, we will quote some of the results.

The integral, $\mathcal{J}_q(n_1, n_2, n_3)$, is defined as

$$\mathcal{J}_q(\nu_1, \nu_2, \nu_3) \equiv \int \mathrm{d}^n q \, \frac{1}{d_1^{\nu_1} d_2^{\nu_2} d_3^{\nu_3}} \tag{4.19}$$

where we have used the definitions in (4.8). At the one loop level, we need the integrals where $\nu_i = -1, 0, 1$. The relation

$$\mathcal{J}_q(1,0,0) = \mathcal{J}_q(0,1,0) = \mathcal{J}_q(0,0,1) \tag{4.20}$$

is obtained by shifting the integration variable. If we substitute $p \leftrightarrow -p$ and $k \leftrightarrow -k$, our definitions of d_1, d_2 and d_3 correspond to d_1, d_2 and d_3 in [13], eq. (6.4). We use these substitutions in eqs. (6.13) – (6.15) in ref. [13], to get[7]

$$\mathcal{J}_q(-1,1,1) = \mathcal{J}_q(0,0,1) + p\cdot(\tfrac{1}{2}p + k)\mathcal{J}_q(0,1,1) \tag{4.21}$$

$$\mathcal{J}_q(1,-1,1) = \mathcal{J}_q(0,0,1) + p\cdot(\tfrac{1}{2}p - k)\mathcal{J}_q(1,0,1) \tag{4.22}$$

$$\mathcal{J}_q(1,1,-1) = \mathcal{J}_q(1,0,0) + (k^2 - \tfrac{1}{4}p^2)\mathcal{J}_q(1,1,0) \tag{4.23}$$

There are also some useful limits in eq. (6.40) in [13], concerning the two-point functions:

$$\lim_{n\to 4}(n-4)\mathcal{J}_q(1,1,0) = \lim_{n\to 4}(n-4)\mathcal{J}_q(1,0,1) = \lim_{n\to 4}(n-4)\mathcal{J}_q(0,1,1) = -2i\pi^2 \tag{4.24}$$

Finally, we have $\mathcal{J}_q(1,1,1)$ which can be written

$$\mathcal{J}_q(1,1,1) = -i\pi^2 \frac{\mu_q^2}{2m_q^2} f(\mu_q^2) \tag{4.25}$$

with $f(\mu_q^2)$ as

$$f(\mu_q^2) = \begin{cases} \arcsin^2 \sqrt{1/\mu_q^2}, & \mu_q \geq 1 \\ -\frac{1}{4}\left[\ln \frac{1+\sqrt{1-\mu_q^2}}{1-\sqrt{1-\mu_q^2}} - i\pi\right]^2, & \mu_q < 1 \end{cases} \tag{4.26}$$

and $\mu_q^2 \equiv 4m_q^2/m_H^2$.

The tensor amplitude, $\mathcal{M}_0^{\alpha\beta}(k_1, k_2)$, from eq. (4.18) can be decomposed in terms of k_1 and k_2 as

$$\mathcal{M}_0^{\alpha\beta}(k_1,k_2) = A_0\eta^{\alpha\beta} + A_1 k_1^\alpha k_1^\beta + A_2 k_1^\alpha k_2^\beta + A_3 k_2^\alpha k_1^\beta + A_4 k_2^\alpha k_2^\beta \tag{4.27}$$

[7] Since $d_1 \leftrightarrow d_2$ under these substitutions, we also have to interchange the first and second arguments in $\mathcal{J}_q(\nu_1, \nu_2, \nu_3)$.

To ensure gauge invariance of the Feynman amplitude, $\mathcal{M}$, we must require

$$\begin{aligned} k_{1\alpha}\mathcal{M}_0^{\alpha\beta}(k_1,k_2) &= 0 \\ k_{2\beta}\mathcal{M}_0^{\alpha\beta}(k_1,k_2) &= 0 \end{aligned} \tag{4.28}$$

This condition leads to (see [13], Chapter 5)

$$\mathcal{M}_0^{\alpha\beta} = A_0\left\{\eta^{\alpha\beta} - \frac{2k_2^\alpha k_1^\beta}{m_H^2}\right\} \tag{4.29}$$

If we now use the relation

$$\sum_{r,s=1}^{2} \varepsilon_{r\alpha}(k_1)\varepsilon_{s\beta}(k_2)\varepsilon_{r\rho}(k_1)\varepsilon_{s\sigma}(k_2) = \eta_{\alpha\rho}\eta_{\beta\sigma} \tag{4.30}$$

which is valid for transverse, physical gluons, together with (4.29), we find

$$\sum_{r,s=1}^{2} |\varepsilon_{r\alpha}(k_1)\varepsilon_{s\beta}(k_2)\mathcal{M}_0^{\alpha\beta}(k_1,k_2)|^2 = 2|A_0|^2 \tag{4.31}$$

The coefficient A_0, which we from now on shall label $A_0(q)$, can be expressed in terms of the integrals $\mathcal{J}_q(1,1,1)$ and $\mathcal{J}_q(1,1,0)$ as[8]

$$A_0(q) = \frac{2m_q^2}{(i\pi^2)(n-2)}\left\{\left[(n-2)-2\mu_q^2\right]\mathcal{J}_q(1,1,1) + \frac{2(n-4)}{m_H^2}\mathcal{J}_q(1,1,0)\right\} \tag{4.32}$$

If we let $n \to 4$, use the relations (4.24) and (4.25), and sum over quark flavors, q, we get

$$A_0(q) = -\sum_q \mu_q^2\left[1 + (1-\mu_q^2)f(\mu_q^2)\right] \tag{4.33}$$

The u, d, s, c and b quark contributions are negligible compared to the t quark contribution because of the quark mass dependence in μ_q^2. Therefore, we will only consider the t quark contribution,

$$A_0(t) = -\mu_t^2\left[1 + (1-\mu_t^2)f(\mu_t^2)\right] \tag{4.34}$$

in our calculations, where the definition of $f(\mu_t^2)$ is given in (4.26), and $\mu_t^2 = 4m_t^2/m_H^2$.

[8]We again checked the result by using parts of the REDUCE program in ref. [13], Appendix C.

4.3 Cross Section for the Subprocess $gg \to H$

In this section we will find the cross section, $\hat{\sigma}(x_1, x_2)$, for the subprocess of Higgs production through gluon fusion. Since we are interested in the unpolarized cross section, we must average over initial polarization and color degrees of freedom. There are two initial gluons, each with two polarization states since they are massless vector bosons, and with eight color degrees of freedom. Let us therefore define X as the average

$$X \equiv \frac{1}{2^2}\frac{1}{8^2}\sum_{r,s=1}^{2}\sum_{a,b=1}^{8}|\mathcal{M}|^2 \tag{4.35}$$

where r and s are polarization indices, whereas a and b are color indices. If we use eqs. (2.60), (4.17), (4.31) and the relation

$$\sum_{a,b=1}^{8}(\delta^{ab})^2 = \sum_{a}^{8} 1 = 8 \tag{4.36}$$

we can write X as

$$X = \frac{G_F m_H^4}{2\sqrt{2}}\left(\frac{\alpha_S}{8\pi}\right)^2 |A_0(t)|^2 \tag{4.37}$$

where $A_0(t)$ is given in (4.34), and $\alpha_S \equiv g_S^2/4\pi$.

To find the unpolarized cross section for the subprocess $gg \to H$, we start with a formula from [1], p. 139

$$\mathrm{d}^3\hat{\sigma} = (2\pi)^4\delta^{(4)}(p - k_1 - k_2)\frac{X}{4\omega_{k_1}\omega_{k_2}v_{rel}}\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E} \tag{4.38}$$

Since the gluons are massless, we have

$$4\omega_{k_1}\omega_{k_2}v_{rel} = 4(k_1 \cdot k_2) = 4(p_1 \cdot p_2 x_1 x_2) = 2s x_1 x_2 \tag{4.39}$$

in any Lorentz frame (see [1], p. 139). In (4.39), we neglected the proton masses and used $s = (p_1 + p_2)^2$, where $\sqrt{s}$ is the energy in the center of mass. By applying (4.39) to (4.38), we get

$$\mathrm{d}^3\hat{\sigma}(x_1, x_2) = \delta^{(4)}(p - x_1 p_1 - x_2 p_2)\frac{\pi X}{s x_1 x_2}\frac{\mathrm{d}^3\mathbf{p}}{2E} \tag{4.40}$$

If we require $p^0 > 0$, we have

$$\delta(p^2 - m_H^2) = \frac{1}{2E}\delta(p^0 - E) \tag{4.41}$$

which gives

$$\frac{\mathrm{d}^3\mathbf{p}}{2E} = \int \mathrm{d}p^0\, \theta(p^0)\, \delta(p^2 - m_H^2)\, \mathrm{d}^3\mathbf{p} \tag{4.42}$$

We use this to write eq. (4.40) as

$$\mathrm{d}^4\hat{\sigma}(x_1, x_2) = \frac{\pi X}{s x_1 x_2}\delta^{(4)}(p - x_1 p_1 - x_2 p_2)\theta(p^0)\delta(p^2 - m_H^2)\,\mathrm{d}^4 p \tag{4.43}$$

Finally, we integrate over p and use (4.37) to get

$$\hat{\sigma}(x_1, x_2) = \frac{\pi G_F m_H^4}{2\sqrt{2} x_1 x_2 s^2}\left(\frac{\alpha_S}{8\pi}\right)^2 |A_0(t)|^2 \delta\left(x_1 x_2 - \frac{m_H^2}{s}\right) \tag{4.44}$$

which is the unpolarized cross section for the subprocess $gg \to H$.

4.4 Cross Section for the Process $p + p \to H + X$

We will now use the results from the previous sections to find the unpolarized cross section, $\sigma(p + p \to H + X)$, for Higgs production through gluon fusion, in proton-proton colliders. By using eq. (4.4), we may write the differential cross section as

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x_1 \mathrm{d}x_2} = \hat{\sigma}(x_1, x_2) f_g^{(1)}(x_1, m_H^2) f_g^{(2)}(x_2, m_H^2) \tag{4.45}$$

where $\sigma = \sigma(p + p \to H + X)$. Let us now express the cross section, σ, in terms of new variables, y and τ. We define the *rapidity* variable as $y = \frac{1}{2}\ln(\frac{x_1}{x_2})$, and τ as $\tau = x_1 x_2$. If we express x_1 and x_2 in terms of y and τ, we get

$$x_1 = \sqrt{\tau} e^{y}\ , \qquad 0 \le x_1 \le 1 \tag{4.46}$$

$$x_2 = \sqrt{\tau} e^{-y}, \qquad 0 \le x_2 \le 1 \tag{4.47}$$

From the constraints on the variables x_1 and x_2, we see that $0 \le \tau \le 1$. Equation (4.46) gives us an upper bound on y, which is $y \le -\frac{1}{2}\ln\tau$, and from eq. (4.47) we find a lower bound, $\frac{1}{2}\ln\tau \le y$. Since the absolute value of the Jacobian determinant is $|J(\tau, y)| = 1$, the differential cross section becomes

$$\begin{aligned}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\tau \mathrm{d}y} &= \hat{\sigma}(\sqrt{\tau}e^{y}, \sqrt{\tau}e^{-y}) f_g^{(1)}(\sqrt{\tau}e^{y}, m_H^2) f_g^{(2)}(\sqrt{\tau}e^{-y}, m_H^2) \\ &= \frac{\pi G_F m_H^4}{2\sqrt{2}\tau s^2}\left(\frac{\alpha_S}{8\pi}\right)^2 |A_0(t)|^2\, \delta\left(\tau - \frac{m_H^2}{s}\right) f_g^{(1)}(\sqrt{\tau}e^{y}, m_H^2) f_g^{(2)}(\sqrt{\tau}e^{-y}, m_H^2)\end{aligned} \tag{4.48}$$

If we integrate (4.48) over τ by using the δ-function, we get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}y} = \frac{\pi G_F}{2\sqrt{2}}\left(\frac{\alpha_S}{8\pi}\right)^2 \frac{m_H^2}{s} |A_0(t)|^2 f_g^{(1)}\left(\frac{m_H}{\sqrt{s}}e^{y}, m_H^2\right) f_g^{(2)}\left(\frac{m_H}{\sqrt{s}}e^{-y}, m_H^2\right) \tag{4.49}$$

The integration over the rapidity, y, still remains, but since there is no analytical expression for the distribution functions, such integrals must be performed numerically. We may express the cross section, σ, for Higgs production through gluon fusion, in terms of an integral over y, as

$$\sigma = \frac{\pi G_F}{2\sqrt{2}}\left(\frac{\alpha_S}{8\pi}\right)^2 \frac{m_H^2}{s}|A_0(t)|^2 \int_{y_1}^{y_2} \mathrm{d}y\, f_g^{(1)}\left(\frac{m_H}{\sqrt{s}}e^{y}, m_H^2\right) f_g^{(2)}\left(\frac{m_H}{\sqrt{s}}e^{-y}, m_H^2\right) \tag{4.50}$$

where $y_1 = \ln\left(\frac{m_H}{\sqrt{s}}\right)$, $y_2 = -\ln\left(\frac{m_H}{\sqrt{s}}\right)$ and $A_0(t)$ is given in (4.34). The integral that appears in (4.50) is referred to as a *convolution integral*. By transforming the integral in (4.4) to the integral in (4.50), we have collected all the information about the partons in the distribution functions. Since the distribution functions appearing in (4.50) are multiplied by the unpolarized cross section, $\hat{\sigma}$, they yield the unpolarized gluon distribution, by which we mean unspecified color and polarization. Alternatively, we could use the polarized cross section, combined with the polarized gluon distributions, which would give the same result when appropriately summed over polarizations and colors.

We will now compare our result to the cross section given in [14]. Let us define the differential gluon luminosity, $\mathrm{d}L_{gg}/\mathrm{d}\tau_1$, in accordance with (7.12) in ref. [3], as

$$\tau_1 \frac{\mathrm{d}L_{gg}}{\mathrm{d}\tau_1} = \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2\, x_1 x_2 f_g^{(1)}(x_1, m_H^2) f_g^{(2)}(x_2, m_H^2)\delta(\tau_1 - x_1 x_2) \tag{4.51}$$

If we again use the coordinate transformation in (4.46) and (4.47) to change the integration variables from x_1 and x_2 to τ and y, we get

$$\frac{\mathrm{d}L_{gg}}{\mathrm{d}\tau_1} = \int_{y_1}^{y_2} \mathrm{d}y \int_0^1 \mathrm{d}\tau\, \frac{\tau}{\tau_1} f_g^{(1)}\left(\frac{m_H}{\sqrt{s}}e^{y}, m_H^2\right) f_g^{(2)}\left(\frac{m_H}{\sqrt{s}}e^{-y}, m_H^2\right)\delta(\tau_1 - \tau) \tag{4.52}$$

where y_1 and y_2 are the same as in (4.50). After integration over τ, the differential gluon luminosity can be expressed as

$$\frac{\mathrm{d}L_{gg}}{\mathrm{d}\tau_1} = \int_{y_1}^{y_2} \mathrm{d}y\, f_g^{(1)}\left(\frac{m_H}{\sqrt{s}}e^{y}, m_H^2\right) f_g^{(2)}\left(\frac{m_H}{\sqrt{s}}e^{-y}, m_H^2\right) \tag{4.53}$$

which is the convolution integral in (4.50). Another way to express the differential luminosity would be to integrate (4.51) over x_2, to get

$$\frac{\mathrm{d}L_{gg}}{\mathrm{d}\tau_1} = \int_{\tau_1}^1 \frac{\mathrm{d}x_1}{x_1}\, f_g^{(1)}(x_1, m_H^2) f_g^{(2)}\left(\frac{\tau_1}{x_1}, m_H^2\right) \tag{4.54}$$

which is the expression used in ref. [14]. Note that (4.54) is symmetric under the transformation $x \leftrightarrow 1/x$. If we compare the function $A_Q(\tau_Q)$ in [14], eqs. (2), (3) and (4) to our notation, we see that it corresponds to $-2A_0(t)$ (see eq. (4.34)). When we take this into account, we see that our result coincides with the result given by Spira *et al.* in [14], eqs. (31), (33) and (34).

4.5 Numerical Results

By using the result obtained in the previous section (see eq. (4.50)), we plot the cross section, $\sigma(p+p \to H+X)$, for Higgs production through gluon fusion in pp-colliders. The numerical integration was performed in C++, with a FORTRAN subroutine which provided us with the gluon distribution functions from [4]. In our calculations, we used the CTEQ5M parametrization, which corresponds to $\Lambda = 226$ MeV and five active quark flavors[9]. In Figure 4.3 we have plotted the cross section as a function of the Higgs mass. We used $\alpha_S(m_H)$ from (4.5), and neglected the loop contributions from the light quarks. Note that the $t\bar{t}$ threshold represents a sign change in the second derivative of the cross section[10].

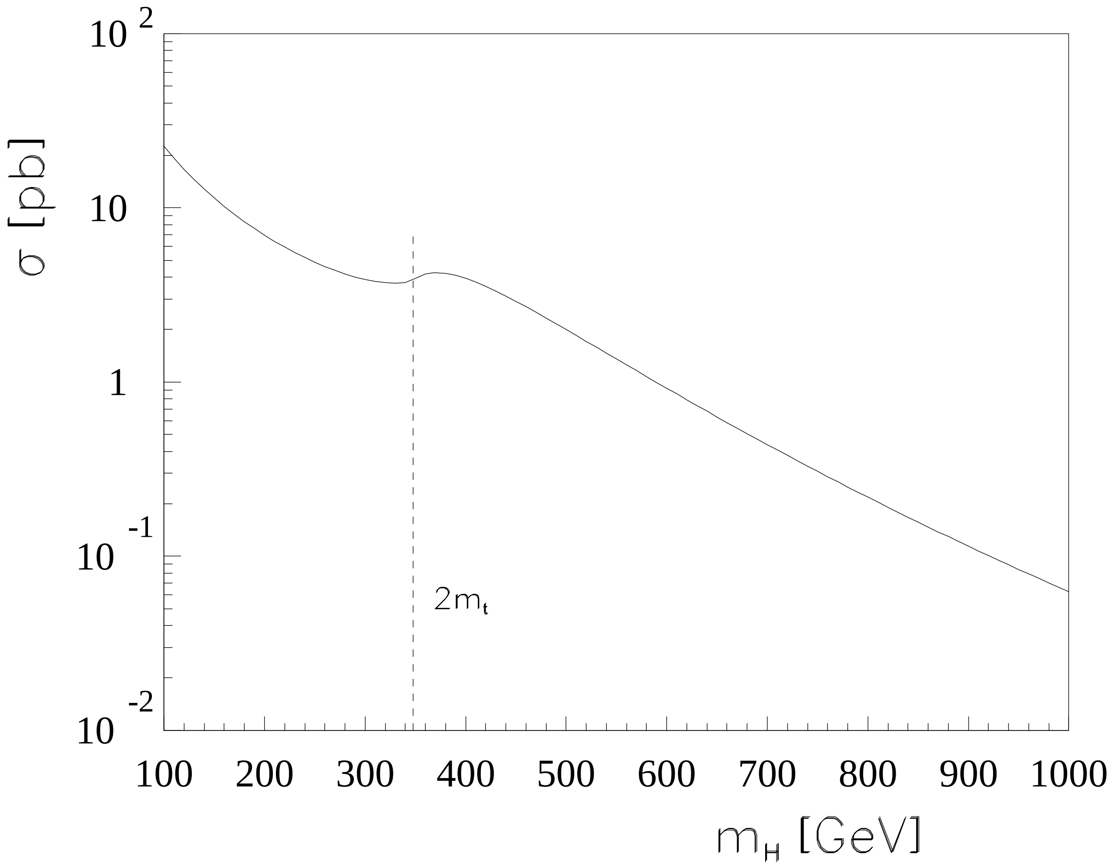


Figure 4.3: The cross section, $\sigma(p+p \to H+X)$, as a function of the Higgs mass, m_H. We have used $\sqrt{s} = 14\,\text{TeV}$, which corresponds to the expected center of mass energy at the LHC.

[9]This parameterization is chosen by assigning the value 1 to the ISET-parameter in the FORTRAN subroutine.

[10]The C++ program, used to compute the cross section, can be found in Appendix B.

The number of events, N, which are generated in an accelerator during a certain time interval, is given by the following formula

$$N = \sigma \times \int \mathrm{d}t\, L \tag{4.55}$$

where σ is the cross section for the process, and $\int \mathrm{d}t\, L$ is the integrated luminosity. Let us now use (4.55) to see how the number of Higgs particles created through gluon fusion at the LHC can be estimated. The expected instantaneous luminosity, L, at the LHC is

$$L_{\mathrm{LHC}} = 10^{34}\,\mathrm{cm}^{-2}\mathrm{s}^{-1} \tag{4.56}$$

If we integrate this expression over a time interval of $10^7\,\mathrm{s}$, which is approximately one effective year (120 days) at the LHC, and use the relation $\mathrm{pb}^{-1} = 10^{36}\,\mathrm{cm}^{-2}$, we get

$$L_{\mathrm{LHC}} = 10^5\,\mathrm{pb}^{-1} \tag{4.57}$$

By inserting (4.57) into (4.55), we see that

$$N_{\mathrm{LHC}} = 10^5 \times \sigma[\mathrm{pb}] \tag{4.58}$$

where N_{LHC} is the number of events generated through gluon fusion at the LHC during one (effective) year. From Figure 4.3 we see that N_{LHC}, which depends on the mass of the Higgs, is of the order of 10^6 for a light Higgs, and decreases to 10^4 for a heavy Higgs. Of course, only some fraction of them can be identified.

Chapter 5

Radion Production through Gluon Fusion

We will in this chapter take a look at the Randall–Sundrum (RS) scenario [15], which is a scenario with five space-time dimensions. A particle which is very similar to the Higgs particle of the Standard Model is introduced in this scenario, and we can therefore use the results obtained in the previous chapter to find the cross section for radion production through gluon fusion. Analogous to the case of Higgs production, the dominant production channel for radions in proton-proton colliders will be the process $gg \to \phi$, where ϕ is the radion. However, there is an additional term in the Feynman amplitude, caused by the trace anomaly of QCD. Before we calculate this cross section, and compare it to the cross section of the Higgs boson, we will give a short review of some of the most popular models concerning extra dimensions[1].

5.1 Extra Dimensions

There has recently been a lot of speculation concerning the possibility that there might be more than four space-time dimensions. Already in 1919, T. Kaluza solved the equations of general relativity in five dimensions. His solutions were Einstein's four-dimensional equations, together with Maxwell's equations. Some years later, O. Klein showed that if the fifth dimension is periodic, and curled up into a tiny circle, it could be real, but yet unseen.

This way of treating the extra dimensions is used in the so-called superstring theories, where the fundamental objects, instead of being point particles, are one-dimensional strings, called superstrings. Superstrings must obey supersymmetry, but since they live in a 10-dimensional space-time, six of these dimensions have to be compactified. These extra dimensions should be very small, of the order of 10^{-35} m, which is known as the Planck length[2]. Supersymmetry is however possible in up to 11 dimensions, and there is also a

[1] We will in this chapter use the index notation, but not the metric, introduced in Chapter 3.

[2] The scale at which gravity becomes strong.

theory in 11 dimensions, called M-theory. These theories also contain two- and higher-dimensional objects, called branes. By shrinking one of the 11 dimensions in M-theory to a line segment, P. Hořava and E. Witten ended up with two 10-dimensional universes. The two branes in this scenario can only communicate through the 11th dimension via gravity. It has been suggested that maybe the dark matter in the universe lies on one of these branes, whereas the visible universe lies on the other. If the goal is to unify all forces into a Theory Of Everything, a unique theory would be preferable. According to M. J. Duff, there are five different superstring theories, but two of these can be obtained from M-theory by shrinking or curling up one of the 11 dimensions, while the other three can be related to M-theory via a concept he calls dualities (see [16]). From this point of view, M-theory may be the most promising candidate to the ambitious title of a Theory Of Everything.

Recently, some new ideas about extra dimensions have brought a lot of excitement among theorists. A proposal by N. Arkani–Hamed, S. Dimopoulos and G. Dvali [17], states that gravity becomes strong at the electroweak scale. This is due to two or more extra, gravity-only, dimensions, maybe as large as a millimeter (ADD scenario). Since gravity only has been directly measured down to distances of about a millimeter, large extra dimensions are not excluded by experiment. If gravity is strong at short distances, but falls off rapidly, due to extra dimensions, it becomes very weak before it transforms to the inverse square law we know. Since the Planck scale is lowered when these large extra dimensions are introduced, the hierarchy problem can be avoided[3]. The advantage of this model, compared to models including extra dimensions of the order $M_{Pl} \sim 10^{-35}\,\mathrm{m}$, is that its predictions can be tested at accelerators like the LHC. Collider signatures would be missing energy due to emission of real gravitons into extra dimensions, or deviations from production cross sections of SM particles as a consequence of virtual graviton exchange (see [18]).

Another suggestion, by K. R. Dienes, E. Dudas and T. Gherghetta [19], is that also SM particles are allowed to propagate in extra dimensions. A consequence of this scenario is that “standing waves” in the extra dimensions, which are referred to as Kaluza-Klein (KK) excitations, would arise when the circumference of an extra dimension corresponds to a multiple of the de Broglie wavelength associated with the particle. Since the momentum in the extra dimensions is invisible in our four-dimensional world, the KK modes would appear to us as particles with larger mass. KK modes of the SM particles could be an experimental indication of this scenario. However, the size of such dimensions cannot be larger than $10^{-19}\,\mathrm{m}$, due to experimental limits on the electroweak parameters.

In contrast to the ADD scenario, where the Planck scale is lowered, it is now the GUT scale[4] which is lowered from $10^{16}\,\mathrm{GeV}$ to $10^{3}\,\mathrm{GeV}$. We will now use the result given in [19], eq. (3.12), to find the derivative of the inverse of $\alpha_i = g_i^2/4\pi$ with respect to the logarithm

[3]The hierarchy problem is the lack of explanation for why the ratio M_{Pl}/m_{EW}, where $M_{Pl} \sim 10^{19}\,\mathrm{GeV}$ is the Planck scale and $m_{EW} \sim 10^{3}\,\mathrm{GeV}$ is the electroweak scale, is so huge.

[4]Unification of the gauge couplings of the strong, electromagnetic and weak interactions occur at the GUT scale.

of the energy scale[5]. Here g_i are the gauge couplings corresponding to the gauge groups $U(1)$, $SU(2)$ and $SU(3)$. The differentiation gives

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\alpha_i^{-1}(\mu) = -\frac{b_i - \tilde{b}_i}{2\pi} - \frac{\tilde{b}_i X_\delta}{2\pi}\left(\frac{\mu}{\mu_0}\right)^\delta \tag{5.1}$$

where b_i are the one loop β-function coefficients of the Minimal Supersymmetric Standard Model (MSSM), and $\tilde{b}_i$ are β-function coefficients corresponding to contributions from KK excitations[6]. Note that since there are additional (s)particles in the MSSM, the beta-function coefficients are different from those of the Standard Model. The number of extra dimensions is labeled δ, μ_0 sets the mass scale at which these extra dimensions become significant, and X_δ is the volume of a δ-dimensional unit sphere. Since $\mu^\delta = e^{\delta\ln\mu}$, we see from the comparison of (1.46) and (5.1) that the running of α_i^{-1} versus the logarithm of the energy scale changes from linear to exponential when additional dimensions are taken into account. As a consequence of this change, the unification scale is lowered[7].

The next section is devoted to a discussion of some aspects of the Randall–Sundrum (RS) scenario, which is based on an article by L. Randall and R. Sundrum[8] [15]. They consider five dimensions with a non-factorizable metric, where the four-dimensional metric is multiplied by a "warp" factor[9]. According to G. F. Giudice *et al.* [20], the KK states of a graviton in the RS scenario will be widely separated, in contrast to the ADD scenario, where a "continuous" spectrum will appear. However, a particle called the *radion* may be the first experimental signature of the RS scenario. In the rest of this chapter, we will take a closer look at this particle, which has much in common with the Higgs particle.

5.2 The Randall–Sundrum Scenario

As mentioned in the previous section, we will now consider the Randall–Sundrum scenario [15], which is a five-dimensional scenario with a non-factorizable metric. The fifth dimension is periodic, and can be parameterized by θ, where $-\pi \leq \theta \leq \pi$. An additional constraint on the fifth dimension is to identify (x, θ) with $(x, -\theta)$. This construction, which is referred to as an $S_1/\mathbf{Z_2}$ *orbifold*, has two fixed points, $\theta = 0$ and $\theta = \pi$. At each of these fixed points, there is a 3-brane[10]. We label the 3-brane at $\theta = 0$ *hidden*, whereas we live

[5] This result gives one-loop corrected values.

[6] See discussion of the β-function on page 11.

[7] Since the higher dimensional scenario in [19] is non-renormalizable, a cut-off parameter has to be introduced. The gauge couplings receive finite, cut-off dependent, quantum corrections, and it is the one-loop corrected values which are calculated as functions of the cut-off parameter. Above μ_0, the energy scale, μ, is treated as the cut-off parameter. By truncating the infinite KK towers, it is demonstrated that there exists an effective renormalizable theory, which succeeds in describing the evolution of the gauge couplings.

[8] There is also an article by the same authors, where a scenario with an infinite extra dimension is proposed. This scenario will not be considered in this thesis.

[9] The warp factor is an exponential function, with an exponent depending on the extra dimension.

[10] Generally, a p-brane is an object with $(p + 1)$ space-time dimensions. If the brane has Dirichlet boundary conditions, it is referred to as a D-brane.

on the *visible* 3-brane, localized at $\theta = \pi$. The metric in the RS-scenario can be expressed as[11]

$$ds^2 = e^{-2kr_c|\theta|}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 d\theta^2 \tag{5.2}$$

where $e^{-2kr_c|\theta|}$ is the "warp" factor, k is a scale of the same order as the fundamental Planck mass, and r_c is the compactification radius. If G_{MN} is the five-dimensional metric tensor[12], we have

$$g_{\mu\nu}^{\text{vis}}(x^\mu) \equiv G_{\mu\nu}(x^\mu, \theta = \pi), \qquad g_{\mu\nu}^{\text{hid}}(x^\mu) \equiv G_{\mu\nu}(x^\mu, \theta = 0) \tag{5.3}$$

One of the motivations for proposing the RS scenario is that it solves the so-called gauge hierarchy problem, which means that the model should eliminate the enormous difference between the weak scale, $v \sim 10^3$ GeV, and the Planck scale, $M_{Pl} \sim 10^{19}$ GeV. In the ADD scenario, which is another proposal for solving this problem, a new hierarchy, between $\mu_c = 1/r_c$ and the vacuum expectation value of the Higgs field, v, is introduced. Let us now see why both of these hierarchies disappear in the RS scenario. In the RS scenario, the effective four-dimensional Planck scale, M_{Pl}, can be expressed as [15]

$$M_{Pl}^2 = \frac{M^3}{k}\left[1 - e^{-2kr_c\pi}\right] \tag{5.4}$$

where M is the fundamental Planck mass. Since k is of order M, we see from (5.4) that M_{Pl} and M are of the same order. Another result from [15] is that any mass parameter, m_0, corresponds, on the visible 3-brane, to a physical mass

$$m \equiv e^{-kr_c\pi} m_0 \tag{5.5}$$

We see from (5.5) that $kr_c \sim 12$ can give rise to mass scales of order TeV even though the fundamental parameters, v_0, M, k and μ_c, are all of order 10^{19} GeV. Also M_{Pl} will correspond to a physical mass in the TeV range, since $M_{Pl} \sim M$. By a rescaling of coordinates, $x^\mu \to e^{kr_c\pi}x^\mu$, one could choose the physical TeV mass scale to be the fundamental scale, instead of the five-dimensional parameter scale.

To stabilize the value of r_c, W. D. Goldberger and M. B. Wise introduce a bulk scalar field[13], ϕ, called the *radion* field (see [21] and [22]). However, the purpose of this thesis is not to discuss how r_c is stabilized, we will instead use a result from Goldberger and Wise [22] as a starting-point: There is a massive radion field in the RS scenario, which arises as a gravitational degree of freedom. Its couplings to brane matter are therefore fixed by four-dimensional general covariance, and it couples to SM particles through the trace of the energy-momentum tensor of the Standard Model[14], $T^{\mu\nu}$. The interaction Lagrangian density for the coupling of the radion field, ϕ, to ordinary matter, given in [22] is

$$\mathcal{L}_{\text{int}} = \frac{\phi}{\Lambda_\phi} T^\mu{}_\mu \tag{5.6}$$

[11] To follow the notation of Randall and Sundrum, we use s instead of τ which was used in Chapter 3.
[12] $M, N = \mu, \theta$.
[13] A bulk field is a field that propagates in all dimensions. The gravitational field is also a bulk field.
[14] G. Nordström [23] proposed a theory of gravity in 1912, with a scalar field coupled to $T^\mu{}_\mu$.

where Λ_ϕ is the non-vanishing vacuum expectation value for the radion field[15] on "our" brane. If the radion is lighter than the Kaluza-Klein excitations of bulk fields, which is likely, it may be the first experimental evidence of the Randall–Sundrum scenario.

5.3 Trace of the Energy-Momentum Tensor

To determine the radion coupling to the Standard Model particles, we will now find the trace of the energy-momentum tensor of the Standard Model fields. We shall use the expression given in (3.28), but when spinor fields are involved, another definition of the energy-momentum tensor is required (see (3.48)). At this point, we are interested in the tree-level couplings, so we consider the free Lagrangian density of the Standard Model, which is[16]

$$\begin{aligned}\mathcal{L}_0 = &\overline{\psi}_f(i\partial\!\!\!/ - m_f)\psi_f - \tfrac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &-\tfrac{1}{2}F^\dagger_{W\mu\nu}F^{\mu\nu}_W + m_W^2 W^\dagger_\mu W^\mu \\ &-\tfrac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \tfrac{1}{2}m_Z^2 Z_\mu Z^\mu \\ &+\tfrac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \tfrac{1}{2}m_H^2\sigma^2 \end{aligned} \tag{5.7}$$

where the difference from (2.56) is that we have also included quarks[17] and gluons. The gluon field-strength tensor, $G^a_{\mu\nu}$, is defined as

$$G^a_{\mu\nu} = \partial_\nu A^a_\mu - \partial_\mu A^a_\nu + g_S f^{abc} A^b_\mu A^c_\nu \tag{5.8}$$

where the gluon field is denoted A^a_μ, f^{abc} are the $SU(3)$ structure constants, and the indices a, b and c run over the eight color states. Since we are only interested in couplings at tree-level, the last term in (5.8) will be ignored. To specify what we mean by γ-matrices in a general coordinate frame, we use the vierbein formalism from section 3.6, and define

$$\gamma^\mu \equiv V_\alpha^{\ \mu}\gamma^\alpha \tag{5.9}$$

Since we in Chapter 3 followed the notation of [9], with opposite sign in the metric, we have to introduce a minus sign in the energy-momentum tensor[18], $T^{\mu\nu}$, defined in (3.28). When we consider Lagrangian densities without spinor fields, and introduce this minus sign, we see that the energy-momentum tensor can be written

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_M\right)}{\delta g_{\mu\nu}} = -2\frac{\delta\mathcal{L}_M}{\delta g_{\mu\nu}} - g^{\mu\nu}\mathcal{L}_M \tag{5.10}$$

[15] Λ_ϕ is assumed to be in the TeV range.

[16] We will not study coupling to gravitons, and have therefore not introduced covariant derivatives in the Lagrangian density (see Chapter 3).

[17] The f in ψ_f and m_f stands for "fermion".

[18] By comparing [9], eq. (5.3.7) and the two first terms of [24], eq. (19.150), we see that different sign conventions have been used.

where we have used (3.28) and (3.35).

From eq. (3.37) we find that

$$\frac{\delta g^{\mu\nu}}{\delta g_{\rho\sigma}} = -g^{\mu\rho}g^{\sigma\nu} \tag{5.11}$$

If we start by looking at the Z-boson part of (5.7), we have to find the derivative of $\mathcal{L}_Z$ with respect to $g_{\mu\nu}$. By using (5.11) we find

$$\begin{aligned}
\frac{\delta \mathcal{L}_Z}{\delta g_{\mu\nu}} &= \frac{\delta}{\delta g_{\mu\nu}}\left(-\tfrac{1}{4}g^{\rho\kappa}g^{\sigma\lambda}Z_{\rho\sigma}Z_{\kappa\lambda} + \tfrac{1}{2}g^{\rho\kappa}m_Z^2 Z_\rho Z_\kappa\right) \\
&= -\frac{1}{4}\left(\frac{\delta g^{\rho\kappa}}{\delta g_{\mu\nu}}g^{\sigma\lambda} + g^{\rho\kappa}\frac{\delta g^{\sigma\lambda}}{\delta g_{\mu\nu}}\right)Z_{\rho\sigma}Z_{\kappa\lambda} + \frac{1}{2}\frac{\delta g^{\rho\kappa}}{\delta g_{\mu\nu}}m_Z^2 Z_\rho Z_\kappa \\
&= -\tfrac{1}{4}\left(-g^{\rho\mu}g^{\nu\kappa}g^{\sigma\lambda} - g^{\rho\kappa}g^{\sigma\mu}g^{\nu\lambda}\right)Z_{\rho\sigma}Z_{\kappa\lambda} - \tfrac{1}{2}g^{\rho\mu}g^{\nu\kappa}m_Z^2 Z_\rho Z_\kappa \\
&= \tfrac{1}{2}Z^{\mu\lambda}Z^{\nu}{}_{\lambda} - \tfrac{1}{2}m_Z^2 Z^\mu Z^\nu
\end{aligned} \tag{5.12}$$

We insert (5.12) into (5.10) to get

$$T_Z^{\mu\nu} = -Z^{\mu\lambda}Z^{\nu}{}_{\lambda} + m_Z^2 Z^\mu Z^\nu - g^{\mu\nu}\left(-\tfrac{1}{4}Z_{\rho\sigma}Z^{\rho\sigma} + \tfrac{1}{2}m_Z^2 Z_\rho Z^\rho\right) \tag{5.13}$$

If we take the trace, we find

$$T_{Z\,\mu}^{\;\mu} = -m_Z^2 Z^\mu Z_\mu \tag{5.14}$$

The calculation for the photon part of (5.7) is identical to the Z-boson part, but since the photon is massless, we find $T_{A\,\mu}^{\;\mu} = 0$. Since we only consider tree level couplings, the same holds for the gluon part, so we get $T_{A^a\,\mu}^{\;\mu} = 0$.

Let us now consider the case of the W-bosons, where the derivative of $\mathcal{L}_W$ with respect to $g_{\mu\nu}$ becomes

$$\begin{aligned}
\frac{\delta \mathcal{L}_W}{\delta g_{\mu\nu}} &= \frac{\delta}{\delta g_{\mu\nu}}\left(-\tfrac{1}{2}g^{\rho\kappa}g^{\sigma\lambda}F^{\dagger}_{W\,\rho\sigma}F_{W\,\kappa\lambda} + g^{\rho\kappa}m_W^2 W^{\dagger}_{\rho}W_\kappa\right) \\
&= -\frac{1}{2}\left(\frac{\delta g^{\rho\kappa}}{\delta g_{\mu\nu}}g^{\sigma\lambda} + g^{\rho\kappa}\frac{\delta g^{\sigma\lambda}}{\delta g_{\mu\nu}}\right)F^{\dagger}_{W\,\rho\sigma}F_{W\,\kappa\lambda} + \frac{\delta g^{\rho\kappa}}{\delta g_{\mu\nu}}m_W^2 W^{\dagger}_{\rho}W_\kappa \\
&= -\tfrac{1}{2}\left(-g^{\rho\mu}g^{\nu\kappa}g^{\sigma\lambda} - g^{\rho\kappa}g^{\sigma\mu}g^{\nu\lambda}\right)F^{\dagger}_{W\,\rho\sigma}F_{W\,\kappa\lambda} - g^{\rho\mu}g^{\nu\kappa}m_W^2 W^{\dagger}_{\rho}W_\kappa \\
&= F_W^{\dagger\,\mu\lambda}F^{\nu}_{W\,\lambda} - m_W^2 W^{\dagger\,\mu}W^\nu
\end{aligned} \tag{5.15}$$

If we use (5.15), we can write (5.10) as

$$\begin{aligned}
T_W^{\mu\nu} = &-2F_W^{\dagger\,\mu\lambda}F^{\nu}_{W\,\lambda} + 2m_W^2 W^{\dagger\,\mu}W^\nu \\
&-g^{\mu\nu}\left(-\tfrac{1}{2}F^{\dagger}_{W\rho\sigma}F_W^{\rho\sigma} + m_W^2 W^{\dagger}_{\rho}W^\rho\right)
\end{aligned} \tag{5.16}$$

The trace of the energy-momentum tensor becomes

$$T_{W\,\mu}^{\;\mu} = -2m_W^2 W^{\dagger\,\mu}W_\mu \tag{5.17}$$

Next we consider the Higgs part of (5.7). Equation (5.10) will also be valid in this case, and the derivative of $\mathcal{L}_\sigma$ with respect to $g_{\mu\nu}$ is

$$\begin{aligned}\frac{\delta\mathcal{L}_\sigma}{\delta g_{\mu\nu}} &= \frac{\delta}{\delta g_{\mu\nu}}\left(\tfrac{1}{2}g^{\rho\kappa}(\partial_\rho\sigma)(\partial_\kappa\sigma) - \tfrac{1}{2}m_H^2\sigma^2\right)\\ &= \frac{1}{2}\frac{\delta g^{\rho\kappa}}{\delta g_{\mu\nu}}(\partial_\rho\sigma)(\partial_\kappa\sigma) = -\tfrac{1}{2}g^{\rho\mu}g^{\nu\kappa}(\partial_\rho\sigma)(\partial_\kappa\sigma)\\ &= -\tfrac{1}{2}(\partial^\mu\sigma)(\partial^\nu\sigma)\end{aligned} \tag{5.18}$$

When we combine (5.10) and (5.18), we get

$$T_\sigma^{\mu\nu} = (\partial^\mu\sigma)(\partial^\nu\sigma) - \tfrac{1}{2}g^{\mu\nu}(\partial_\rho\sigma)(\partial^\rho\sigma) + \tfrac{1}{2}g^{\mu\nu}m_H^2\sigma^2 \tag{5.19}$$

and the trace of the energy-momentum tensor becomes

$$T_{\sigma\ \mu}^{\ \mu} = -(\partial^\mu\sigma)(\partial_\mu\sigma) + 2m_H^2\sigma^2 \tag{5.20}$$

Finally we consider the fermion part of (5.7), where we have to use the vierbein formalism introduced in Chapter 3. From eqs. (3.48) and (3.49), we see that the energy-momentum tensor for a fermion field is[19]

$$T_{\mu\nu} \equiv -V_{\alpha\mu}U^\alpha{}_\nu = -V_{\alpha\mu}\frac{1}{V}\frac{\delta(V\mathcal{L}_\psi)}{\delta V_\alpha{}^\nu} = -V_{\alpha\mu}\frac{1}{V}\frac{\delta V}{\delta V_\alpha{}^\nu}\mathcal{L}_\psi - V_{\alpha\mu}\frac{\delta\mathcal{L}_\psi}{\delta V_\alpha{}^\nu} \tag{5.21}$$

where we have defined

$$\sqrt{-g} = \sqrt{-\det(\eta_{\alpha\beta}V^\alpha{}_\mu V^\beta{}_\nu)} = \det V^\alpha{}_\mu \equiv V \tag{5.22}$$

We need an expression for δV. If we start from (3.35) and (5.22), we find

$$\begin{aligned}\delta V &= \delta\sqrt{-g} = \tfrac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} = -\tfrac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\\ &= -\tfrac{1}{2}VV^\alpha{}_\mu V^\beta{}_\nu\eta_{\alpha\beta}\left[V_\delta{}^\nu\eta^{\gamma\delta}\delta V_\gamma{}^\mu + V_\gamma{}^\mu\eta^{\gamma\delta}\delta V_\delta{}^\nu + V_\gamma{}^\mu V_\delta{}^\nu\delta\eta^{\gamma\delta}\right]\\ &= -\tfrac{1}{2}V\eta_{\alpha\beta}\eta^{\gamma\delta}\left[V^\alpha{}_\mu(V^\beta{}_\nu V_\delta{}^\nu)\delta V_\gamma{}^\mu + (V^\alpha{}_\mu V_\gamma{}^\mu)V^\beta{}_\nu\delta V_\delta{}^\nu\right]\\ &= -\tfrac{1}{2}V\eta_{\alpha\beta}\eta^{\gamma\delta}\left[V^\alpha{}_\mu\delta^\beta_\delta\delta V_\gamma{}^\mu + \delta^\alpha_\gamma V^\beta{}_\nu\delta V_\delta{}^\nu\right]\\ &= -\tfrac{1}{2}V\left[V^\alpha{}_\mu\delta^\gamma_\alpha\delta V_\gamma{}^\mu + \delta^\delta_\beta V^\beta{}_\nu\delta V_\delta{}^\nu\right]\\ &= -\tfrac{1}{2}V\left[V^\alpha{}_\mu\delta V_\alpha{}^\mu + V^\beta{}_\nu\delta V_\beta{}^\nu\right]\end{aligned} \tag{5.23}$$

where we have used (3.46), and the relation

$$g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu} \tag{5.24}$$

[19] Also in the spinor case we have to introduce a minus sign in the definition of the energy-momentum tensor.

which is obtained by differentiating $g_{\mu\nu}g^{\mu\nu} = \delta^\mu_\mu = 4$. We have also used $\delta\eta^{\gamma\delta} = 0$, since the Minkowski metric is constant under variation of the metric tensor.

Let us now look at the second term in (5.21), and insert $\mathcal{L}_\psi$ from (5.7) to find the derivative of $\mathcal{L}_\psi$ with respect to the vierbein, $V_\alpha{}^\nu$. By using (3.51), we find

$$\begin{aligned}\frac{\delta\mathcal{L}_\psi}{\delta V_\alpha{}^\nu} &= \frac{\delta}{\delta V_\alpha{}^\nu}\overline{\psi}_f\left(\tfrac{1}{2}iV^\beta{}_\mu(\gamma^\mu\partial_\beta + \gamma_\beta\partial^\mu) - m_f\right)\psi_f = -\tfrac{1}{2}V^\alpha{}_\mu V^\beta{}_\nu\overline{\psi}_f(i\gamma^\mu\partial_\beta + i\gamma_\beta\partial^\mu)\psi_f \\ &= -\tfrac{1}{2}\overline{\psi}_f(i\gamma^\alpha\partial_\nu + i\gamma_\nu\partial^\alpha)\psi_f \end{aligned} \tag{5.25}$$

If we insert (5.23) and (5.25) into (5.21), and again use (3.51) we get

$$\begin{aligned} T_{\mu\nu} &= -V_{\alpha\mu}\frac{1}{V}\frac{\delta V}{\delta V_\alpha{}^\nu}\mathcal{L}_\psi - V_{\alpha\mu}\frac{\delta\mathcal{L}_\psi}{\delta V_\alpha{}^\nu} \\ &= \tfrac{1}{2}V_{\alpha\mu}\left[V^\alpha{}_\mu\delta^\mu_\nu + V^\beta{}_\nu\delta^\alpha_\beta\right]\mathcal{L}_\psi + \tfrac{1}{2}V_{\alpha\mu}\overline{\psi}_f(i\gamma^\alpha\partial_\nu + i\gamma_\nu\partial^\alpha)\psi_f \\ &= V_{\alpha\mu}V^\alpha{}_\nu\overline{\psi}_f(i\partial\!\!\!/ - m_f)\psi_f + \tfrac{1}{2}\overline{\psi}_f(i\gamma_\mu\partial_\nu + i\gamma_\nu\partial_\mu)\psi_f \\ &= g_{\mu\nu}\overline{\psi}_f(i\partial\!\!\!/ - m_f)\psi_f + \tfrac{1}{2}\overline{\psi}_f(i\gamma_\mu\partial_\nu + i\gamma_\nu\partial_\mu)\psi_f \end{aligned} \tag{5.26}$$

By taking the trace, and using the Dirac equation, (1.8), we obtain

$$\begin{aligned} T^{\mu}_{\psi\,\mu} &= g^\mu{}_\mu\overline{\psi}_f(i\partial\!\!\!/ - m_f)\psi_f + \overline{\psi}_f(i\partial\!\!\!/)\psi_f \\ &= m_f\overline{\psi}_f\psi_f \end{aligned} \tag{5.27}$$

We summarize our results by writing out the trace of the energy-momentum tensor of the SM fields

$$\begin{aligned} T^\mu{}_\mu &= T^{\mu}_{\psi\,\mu} + T^{\mu}_{Z\,\mu} + T_{A^a}{}^{\mu}{}_\mu + T_A{}^\mu{}_\mu + T^{\mu}_{W\,\mu} + T^{\mu}_{\sigma\,\mu} \\ &= m_f\overline{\psi}_f\psi_f - m_Z^2 Z^\mu Z_\mu - 2m_W^2 W^{\dagger\,\mu}W_\mu + 2m_H^2\sigma^2 - (\partial^\mu\sigma)(\partial_\mu\sigma) \end{aligned} \tag{5.28}$$

which is identical to the result given in [25], eq. (2).

From (5.6) and (5.28) we can find $\mathcal{L}_{\text{int}}$. If we compare this to the Higgs coupling to two fermions, or to a gauge boson pair, given in [1], p. 299, we see that if we rescale the Higgs coupling by a factor $-(v/\Lambda_\phi)$, where v is the vacuum expectation value of the Higgs field, we get the radion coupling to the same particles. Note that the last term in (5.28) gives rise to a momentum dependent Higgs-Higgs-radion vertex. If $m_\phi > 2m_H$, the decay into two Higgs bosons would be an additional decay channel for the radion.

5.4 Trace Anomaly

Classically, the trace of the energy-momentum tensor, $T^\mu{}_\mu$, vanishes for a *massless, scale invariant theory*, but when quantum corrections are included, scale invariance is violated, and we get a non-vanishing trace. In this section, we will find an expression for this non-vanishing trace, often called the *trace anomaly*.

From the definition of the β-function in (1.40), we see that a transformation of the energy scale

$$\mu \to \mu e^{\sigma} \tag{5.29}$$

where μ is the energy scale, and σ is an arbitrary constant, leads to the following expression for $\beta(g(\mu))$

$$\beta\left(g(\mu)\right) = \frac{\delta g(\mu)}{\delta \ln\mu} = \frac{\delta g(\mu)}{\sigma} \tag{5.30}$$

where $g(\mu)$ is the coupling "constant", which should not be confused with the determinant of the metric tensor[20]. We see from (5.30) that the change in the Lagrangian density can be written

$$\delta\mathcal{L} = \sigma\beta(g)\frac{\partial}{\partial g}\mathcal{L} \tag{5.31}$$

Application of Noethers theorem (see page 8) leads to a conserved current, D^{μ},

$$\partial_{\mu} D^{\mu} = \beta(g)\frac{\partial}{\partial g}\mathcal{L} \tag{5.32}$$

According to [24], p. 683, the relation between the dilatation current, D^{μ}, and the energy-momentum tensor is $D^{\mu} = T^{\mu\nu}x_{\nu}$. Since we use the same sign in $T^{\mu\nu}$ as in [24], we can use this to write (5.32) as

$$T^{\mu}{}_{\mu} = \beta(g)\frac{\partial}{\partial g}\mathcal{L} \tag{5.33}$$

As we mentioned earlier, this non-vanishing trace of the energy momentum tensor is referred to as the *trace anomaly*, and is due to the fact that in a quantum theory, the coupling "constant" is scale dependent.

5.5 Cross Section for the Process $p + p \to \phi + X$

Like Higgs particles, radions can be produced from two gluons, via a quark loop. There is however an additional term in the cross section, as mentioned in the introduction to this chapter, caused by the anomalous breaking of scale invariance in QCD. This trace anomaly term enters the cross section since the radion couples to the trace of the energy-momentum tensor. In this section, we will first find the Feynman amplitude for radion production through gluon fusion, and then use the results from Chapter 4 to find the corresponding cross section.

[20] Throughout this section, the coupling, $g(\mu)$, is labeled g.

To find the trace anomaly of QCD, we shall use the result from the previous section on the Lagrangian density of QCD, which was introduced in eq. (1.36) as

$$\mathcal{L}_{\rm QCD} = \overline{\psi}_q(i\,\not{D} - m_q)\psi_q - \tfrac{1}{4}G^a_{\alpha\beta}G^{a\,\alpha\beta} \tag{5.34}$$

Note that we choose to go to Minkowski space here, since we used the Minkowski metric in our loop calculations in Chapter 4. If we rescale the gluon field by $g_S A^a_\alpha \to A^a_\alpha$, the coupling constant is removed from the covariant derivative. After this rescaling, g_S only appears in the field strength tensor part of $\mathcal{L}_{\rm QCD}$:

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4g_S^2}(\partial_\beta A^a_\alpha - \partial_\alpha A^a_\beta + f^{abc}A^b_\alpha A^c_\beta)(\partial^\beta A^{a\,\alpha} - \partial^\alpha A^{a\,\beta} + f^{abc}A^{b\,\alpha}A^{c\,\beta}) + \cdots \tag{5.35}$$

From (5.33), we find

$$T^\alpha{}_\alpha = \frac{\beta(g)}{2g_S^3}(\partial_\beta A^a_\alpha - \partial_\alpha A^a_\beta + f^{abc}A^b_\alpha A^c_\beta)(\partial^\beta A^{a\,\alpha} - \partial^\alpha A^{a\,\beta} + f^{abc}A^{b\,\alpha}A^{c\,\beta}) \tag{5.36}$$

To recover the conventional normalization of the field strength tensors, we let $A^a_\alpha \to g_S A^a_\alpha$. From eqs. (1.42) and (1.43), we see that to lowest order in the β-function, $T^\alpha{}_\alpha$ becomes

$$T^\alpha{}_\alpha = \frac{\beta(g)}{2g_S}G^a_{\alpha\beta}G^{a\,\alpha\beta} = -(11 - \frac{2}{3}n_f)\frac{\alpha_S}{8\pi}G^a_{\alpha\beta}G^{a\,\alpha\beta} \equiv -\frac{\alpha_S}{8\pi}b_{\rm QCD}G^a_{\alpha\beta}G^{a\,\alpha\beta} \tag{5.37}$$

We see from eq. (5.6) that the trace anomaly gives the following term in the radion-gluon interaction Lagrangian density

$$\mathcal{L}^{\rm anom}_{\rm int} = -\frac{1}{\Lambda_\phi}\left(\frac{\alpha_S}{8\pi}\right)b_{\rm QCD}G^a_{\alpha\beta}G^{a\,\alpha\beta}\phi \tag{5.38}$$

Let us now try to find the contribution to the Feynman amplitude from the trace anomaly term. For incoming gluons, with momenta k_1 and k_2, we take $A^a_\alpha = \mathcal{A}^a_\alpha(k_i)e^{-ik_ix} = \mathcal{A}^a\varepsilon_\alpha(k_i)$, and since we are not interested in terms containing more than two gluon fields, we may drop the two terms involving $g_S f^{abc}$. This gives

$$G^a_{\alpha\beta}G^{a\,\alpha\beta} = \left[-ik_{1\,\beta}\mathcal{A}^a_\alpha(k_1) + ik_{1\,\alpha}\mathcal{A}^a_\beta(k_1)\right]\left[-ik_2^\beta\mathcal{A}^{a\,\alpha}(k_2) + ik_2^\alpha\mathcal{A}^{a\,\beta}(k_2)\right]e^{-i(k_1+k_2)x} \tag{5.39}$$

By multiplying out the terms in (5.39), we find

$$\begin{aligned}
G^a_{\alpha\beta}G^{a\,\alpha\beta} &= -2\left[k_{1\,\beta}\mathcal{A}^a_\alpha(k_1) - k_{1\,\alpha}\mathcal{A}^a_\beta(k_1)\right]k_2^\beta\mathcal{A}^{a\,\alpha}(k_2)e^{-i(k_1+k_2)x} \\
&= -2\left[(k_1\cdot k_2)(\mathcal{A}^a(k_1)\cdot\mathcal{A}^a(k_2)) - (k_1\cdot\mathcal{A}^a(k_2))(k_2\cdot\mathcal{A}^a(k_1))\right]e^{-i(k_1+k_2)x} \\
&= -2\left[(k_1\cdot k_2)(\varepsilon_r(k_1)\cdot\varepsilon_s(k_2)) - (k_1\cdot\varepsilon_s(k_2))(k_2\cdot\varepsilon_r(k_1))\right]\mathcal{A}^a\mathcal{A}^a \\
&= -2(k_1\cdot k_2)\varepsilon_r^\alpha(k_1)\varepsilon_s^\beta(k_2)\left[\eta_{\alpha\beta} - \frac{k_{2\,\alpha}k_{1\,\beta}}{k_1\cdot k_2}\right]\mathcal{A}^a\mathcal{A}^a
\end{aligned} \tag{5.40}$$

This term is gauge invariant, as it has to be, since

$$k_1^\alpha \left[\eta_{\alpha\beta} - \frac{k_{2\,\alpha} k_{1\,\beta}}{k_1 \cdot k_2} \right] = 0, \qquad k_2^\beta \left[\eta_{\alpha\beta} - \frac{k_{2\,\alpha} k_{1\,\beta}}{k_1 \cdot k_2} \right] = 0 \tag{5.41}$$

If we insert this into (5.38), and use (1.18) and (1.21), we get

$$\mathcal{H}_{\text{int}}^{\text{anom}} = -\mathcal{L}_{\text{int}}^{\text{anom}} \tag{5.42}$$

Since the incoming gluons are identical particles, we get a symmetry factor of 2 in the anomaly vertex. The reason for this is that A_α^a can either annihilate $|A^a(k_1, r)\rangle$ or $|A^a(k_2, s)\rangle$. If we use this, together with (5.38), (5.40), (5.42) and the relation $2k_1{\cdot}k_2 = m_\phi^2$, we find the trace anomaly contribution to the Feynman amplitude[21], $\mathcal{M}^{\text{anom}}$, which is

$$\mathcal{M}^{\text{anom}} = i\,\frac{2m_\phi^2}{\Lambda_\phi} \left(\frac{\alpha_S}{8\pi} \right) b_{\text{QCD}} \left[\eta_{\alpha\beta} - \frac{2k_{2\alpha} k_{1\beta}}{m_\phi^2} \right] \varepsilon_r^\alpha \varepsilon_s^\beta \tag{5.43}$$

The anomaly contribution should be added to the loop contribution, which can be obtained by rescaling the Higgs result with the factor $-(v/\Lambda_\phi)$, and changing the Higgs mass, m_H, to the radion mass, m_ϕ (see Section 5.3). Instead of summing (5.43) over color a, as implied in (5.38), we can instead introduce a Kronecker-delta, δ^{ab}, in (5.43) and take the sum over a and b instead. If we now rescale (4.17), and combine it with (5.43) by using (4.29) together with the relations $\alpha_S = g_S^2/4\pi$ and $1/v = g_W/2m_W$, we find

$$\mathcal{M} = i\delta^{ab}\,\frac{2m_\phi^2}{\Lambda_\phi} \left(\frac{\alpha_S}{8\pi} \right) (b_{\text{QCD}} - A_0(t)) \left[\eta_{\alpha\beta} - \frac{2k_{2\alpha} k_{1\beta}}{m_\phi^2} \right] \varepsilon_r^\alpha \varepsilon_s^\beta \tag{5.44}$$

where $A_0(t)$ is defined in (4.34) as

$$A_0(t) = -\mu_t^2 \left[1 + (1 - \mu_t^2) f(\mu_t^2) \right] \tag{5.45}$$

but now it is the radion mass which enters in the definition of $\mu_t^2 = 4m_t^2/m_\phi^2$.

Since we are interested in the unpolarized cross section, we must sum and average over polarizations and colors. We see from eqs. (4.29), (4.31) (4.35), (4.36), and (5.44) that

$$X = \frac{1}{2^2}\frac{1}{8^2} \sum_{r,s=1}^{2} \sum_{a=1}^{8} |\mathcal{M}|^2 = \frac{m_\phi^4}{4\Lambda_\phi^2} \left(\frac{\alpha_S}{8\pi} \right)^2 |b_{\text{QCD}} - A_0(t)|^2 \tag{5.46}$$

By using the same procedure as in the previous chapter[22], we obtain an expression for $\sigma(p + p \to \phi + X)$

$$\sigma = \frac{\pi}{4\Lambda_\phi^2} \left(\frac{\alpha_S}{8\pi} \right)^2 \frac{m_\phi^2}{s} |b_{\text{QCD}} - A_0(t)|^2 \int_{y_1}^{y_2} \mathrm{d}y\, f_g^{(1)}\left(\frac{m_\phi}{\sqrt{s}} e^y, m_\phi^2 \right) f_g^{(2)}\left(\frac{m_\phi}{\sqrt{s}} e^{-y}, m_\phi^2 \right) \tag{5.47}$$

[21] We have introduced a factor of $-i$, which comes from the S-matrix (see [1], p. 101).

[22] Compare eqs. (4.37) and (4.50).

where $y_1 = \ln\left(\frac{m_\phi}{\sqrt{s}}\right)$, $y_2 = -\ln\left(\frac{m_\phi}{\sqrt{s}}\right)$, $\sqrt{s}$ is the center of mass energy, and $f_g^{(i)}$ are the gluon distribution functions. The definition of $b_{\rm QCD}$, which can be found implicitly in (5.37), is $b_{\rm QCD} = (11 - \frac{2}{3}n_f)$, and $A_0(t)$ is given in (5.45).

If we now compare our result to some of the existing literature, we see that in [25], eqs. (10) and (11), we find an expression for the cross section we are interested in. The $I_q(x_q)$ in [25] corresponds to $-A_0(t)$ in our notation, provided we let $q = t$. We also recognize the differential gluon luminosity from eq. (4.54) in [25], which corresponds to the convolution integral in (5.47). The only difference is that they consider all quark flavors, and have included a K factor to account for QCD corrections.

Giudice *et al.* [20] have an expression for the effective vertex of the $gg\phi$ interaction (see eq. (51)), which we understand as the factor in front of the radion field in the effective Lagrangian density. Since the $F_{1/2}(\tau_t)$ in [20] corresponds to $2A_0(t)$ in our notation, we find that our result in (5.44) differs from [20] by a relative sign[23], which is due to opposite sign in the definition of $T^{\mu\nu}$. Since the cross section depends on the amplitude squared, a relative sign in the amplitude is of no consequence for the result. The reason why the difference in sign enters in both the trace anomaly term and the loop contribution is that the scaling factor, which we use in the loop term, is determined from the relation between trace of the energy-momentum tensor and the interaction Lagrangian density of the SM fields.

5.6 Comparing Higgs and Radion Cross Sections

We will now compare the cross sections for Higgs and radion production through gluon fusion. In the upper panel in Figure 5.1 on page 59, we show $(\Lambda_\phi/v)^2\sigma(p+p\to\phi+X)$ as a function of the radion mass, where $\sigma(p+p\to\phi+X)$ is the cross section for radion production through gluon fusion, given in eq. (5.47). To cancel the unknown vacuum expectation value, Λ_ϕ, of the radion field that appears in (5.47), we have introduced a scaling factor of $(\Lambda_\phi/v)^2$ in the radion cross section, where v is the vacuum expectation value of the Higgs field. The corresponding Higgs cross section[24], which we have plotted without the scaling factor, as a function of the Higgs mass, is shown in the same plot (the dotted curve). In both cases, we used five active quark flavors in the numerical calculations, and therefore there are some corrections at very high, or very low energies. The same parametrization, [4], as in Chapter 4 was used (see page 45). We use a center of mass energy of 14 TeV, since this is the expected energy at the LHC. If we look carefully, we see that the $t\bar{t}$ threshold is barely visible in the radion case. We also see that the trace anomaly term, which is the only difference, gives a considerable contribution to the radion cross section. This is because $b_{\rm QCD} = 23/3$ for five active quark flavors, whereas the real part of $-A_0(t)$ is less than 6/5 and the imaginary part of $-A_0(t)$ is less than 1 in the whole mass region. However, mass differences and different vacuum expectation values will also be important since they adjust the predicted value of the cross section.

[23] These equations can be compared by carrying out the same calculation as we did in (5.38)–(5.43).

[24] This is the Higgs cross section from Figure 4.3.

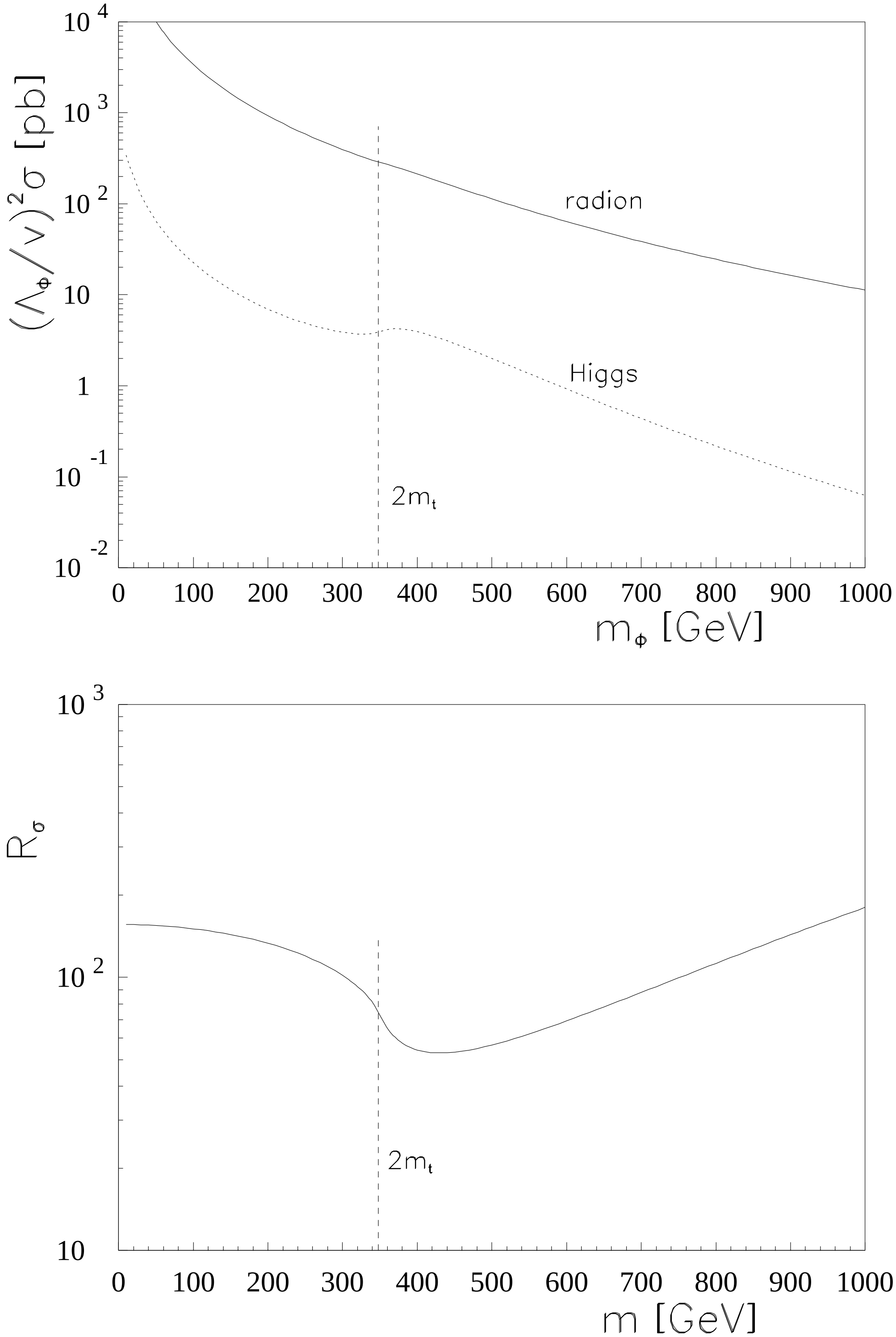


Figure 5.1: The cross sections (above), and the ratio defined in (5.48) (below), as functions of the mass of the boson (radion or Higgs). In both cross sections we use $\sqrt{s} = 14\,\text{TeV}$.

Let us now define the ratio, R_σ, as

$$R_\sigma \equiv \frac{(\Lambda_\phi/v)^2\sigma(p+p\to\phi+X)}{\sigma(p+p\to H+X)} \tag{5.48}$$

where $\sigma(p+p\to H+X)$ is given by eq. (4.50) and $(\Lambda_\phi/v)^2\sigma(p+p\to\phi+X)$ is given by eq. (5.47). In the lower panel in Figure 5.1 on page 59, we have plotted R_σ as a function of the mass of the boson (radion or Higgs). The reason why we see the $t\bar{t}$ threshold here, is that it was clearly visible in the Higgs cross section, but barely visible in the radion cross section (see Figure 5.1). Note that after rescaling, the radion cross section is $\mathcal{O}(100)$ larger than the Higgs cross section. If however $\Lambda_\phi \simeq 10v$, the cross sections would be of the same order of magnitude. We should emphasize that since Λ_ϕ is unknown, R_σ is not directly measurable, but on the other hand, if $\sigma(p+p\to\phi+X)$ was measured, one could compare with the calculated $\sigma(p+p\to H+X)$ for a Higgs boson with the same mass as the radion. By taking the ratio of the two cross sections, and compare it to R_σ, the vacuum expectation value, Λ_ϕ, of the radion field could be determined.

If we are interested in the number of radion events produced through gluon fusion at the LHC during one year, eq. (4.58) is still valid. There is however the problem with the unknown vacuum expectation value, Λ_ϕ, which enters in the radion cross section.

Chapter 6

Radion Decay to Two Photons

One possible way to detect the radion at proton-proton colliders is through its decay to two photons. We will in this chapter consider the process of radion decay into two photons to lowest order. A detailed calculation of the Higgs decay to two photons, in a notation close to ours[1], can be found in [13]. In addition to the quark loop contribution and the trace anomaly term we can also have contributions from loops of charged gauge bosons, $W^{\pm}$, and related ghost fields. We used some of the results for the quark loop in Chapter 4, and we will follow the same recipe in this chapter[2].

6.1 Loop Contributions

First we shall find the decay rate in the case of radion decay to two photons by using the corresponding Higgs result (see [13], eq. (3.7))

$$\Gamma = \frac{1}{16\pi m_H}|\mathcal{M}|^2 \tag{6.1}$$

where $\mathcal{M}$ is the Feynman amplitude for the process $H \to \gamma\gamma$. We can easily obtain the loop contributions to the corresponding radion decay from the results in [13], by the rescaling $-v/\Lambda_\phi$ (see page 54), and by changing the mass from m_H to m_ϕ. In [13], some factors are extracted from the Feynman amplitude, $\mathcal{M}$ (see [13], eqs. (3.8), (5.2) and (5.13)). By following this approach, we may write the loop contribution to the Feynman amplitude as

$$\mathcal{M}^{\text{loop}} = -i\,\frac{\alpha m_H^2}{2\pi\Lambda_\phi}A_0\left[\eta_{\alpha\beta} - \frac{2k_{2\alpha}k_{1\beta}}{m_H^2}\right]\varepsilon_r^\alpha\varepsilon_s^\beta \tag{6.2}$$

where $\alpha \equiv e^2/4\pi$ is the fine structure constant. Note that the vacuum expectation value[3], $v = 2m_W/g_W$, of the Higgs field has been replaced by the vacuum expectation value, Λ_ϕ, of the radion field through the rescaling, which also gives an extra minus sign.

[1]For references to the original literature, see [13].
[2]The calculation in [13] is done in 't Hooft–Feynman gauge.
[3]This relation can be obtained from (2.60).

Since the photon couples to all charged particles, loop diagrams with W-bosons and related ghost fields also contribute to the Feynman amplitude. In Figure 6.1, we show a schematic triangle diagram of the radion decay into two photons[4]. The A_0 in eq. (6.2) is defined as $A_0 = A_0(t) + A_0(W)$, where $A_0(t)$ is the quark loop contribution[5], and $A_0(W)$ is the contribution from W and ghost loops. If we substitute m_H by m_ϕ in the definition of μ_t^2 and μ_W^2, we can use the expression for A_0, given in [13], eq. (7.21), to find the loop contribution to the Feynman amplitude for radion decay into two photons. We get

$$\begin{aligned} A_0 &= A_0(t) + A_0(W) \\ &= -Q_t^2 N_c \mu_t^2 \left[1 + (1-\mu_t^2) f(\mu_t^2)\right] \\ &\quad + \tfrac{1}{2}\left[2 + 3\mu_W^2 + 3\mu_W^2(2-\mu_W^2) f(\mu_W^2)\right] \end{aligned} \tag{6.3}$$

where the definition of $f(\mu_t^2)$ is given in (4.26), and $\mu_t^2 = 4m_t^2/m_\phi^2$. The definitions of μ_W^2 and $f(\mu_W^2)$ are the same as in the quark case, provided we substitute m_t by m_W. Note that the quark-loop contribution to the Feynman amplitude is the same as in the two-gluon case, provided we substitute $g_S^2 \mathrm{Tr}(T^a T^b)$ by $(eQ_t)^2 N_c$, where Q_t is the charge of the t-quark, and N_c is the number of quark colors[6]. Therefore we could have got the expression for $A_0(t)$, given in (6.3), from the expression given in (5.45) by multiplying with the factor $Q_t^2 N_c$.

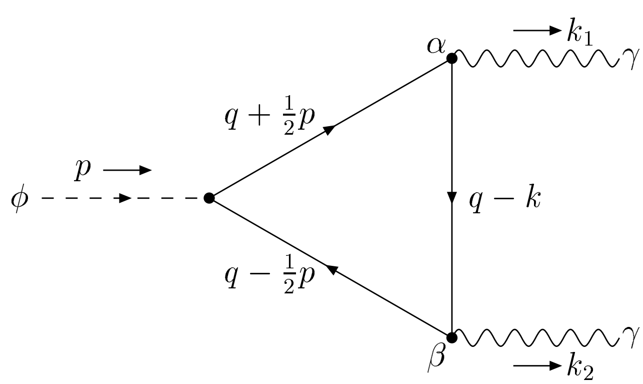


Figure 6.1: Schematic Feynman diagram for the one-loop contribution to the decay of a radion into two photons. Note that since the photon couples to all charged particles, also $W^\pm$ and their ghosts are allowed to propagate in the loop. We have used k_1 and k_2 for the photon momenta, and p for the momentum of the radion, where $p = k_1 + k_2$ and $k = \frac{1}{2}(k_1 - k_2)$.

[4] One loop diagrams with quartic couplings between e.g. $WW\gamma\gamma$ must also be taken into account.
[5] Again we leave out the light quark contributions.
[6] $N_c = 3$.

6.2 Trace Anomaly Contributions

In section 5.5, we found an expression for the trace anomaly term in the interaction Lagrangian density for radion production through gluon fusion (see eq. (5.38)). When we consider the decay of a radion to two photons, we get a similar expression, but since photons are involved instead of gluons, we have to substitute α_S by the fine structure constant, α. We also have to replace the strong β-function coefficient, b_{QCD}, by its electroweak counterparts. The group representation of the electroweak force is $SU(2)_L \times U(1)_Y$, where L stands for left-handed, and Y represents the hypercharge.

To find the respective β-function coefficients, b_2 and b_Y, we shall use the formulae given in [26], eqs. (9.2.46) and (9.2.50). Due to different ways of defining the β-function coefficients, we must multiply the coefficients from [26] by 4π in order to be consistent with the notation introduced earlier. If we also put the number of generations, $n_g = n_f/2$, we find b_2 to be

$$b_2 = \frac{22 - 2n_f}{3} - \frac{n_H}{6} = \frac{23}{6} \tag{6.4}$$

where $n_f = 5$ is the number of active quark flavors[7], and $n_H = 1$ stands for the number of Higgs particles. The first term can be found from (1.41), whereas the second term was neglected[8] in [26]. In the case of b_Y, we find

$$b_Y = \frac{5}{3}b_1 = -\frac{10}{9}n_f - \frac{n_H}{6} = -\frac{103}{18} \tag{6.5}$$

where the first term is the fermion contribution.

We may now use eqs. (6.4) and (6.5) to write down the trace anomaly contribution to the interaction Lagrangian density for radion decay into two photons, which is analogous to (5.38):

$$\mathcal{L}_{\text{int}}^{\text{anom}} = -\frac{1}{\Lambda_\phi}\left(\frac{\alpha}{8\pi}\right)(b_2 + b_Y)F_{\alpha\beta}F^{\alpha\beta}\phi \tag{6.6}$$

By following the same procedure as we did in Chapter 5, when we considered radion production through gluon fusion, we find the trace anomaly contribution to the Feynman amplitude to be (see eq. (5.43))

$$\mathcal{M}^{\text{anom}} = i\,\frac{2m_\phi^2}{\Lambda_\phi}\left(\frac{\alpha}{8\pi}\right)(b_2 + b_Y)\left[\eta_{\alpha\beta} - \frac{2k_{2\alpha}k_{1\beta}}{m_\phi^2}\right]\varepsilon_r^\alpha\varepsilon_s^\beta \tag{6.7}$$

where we, as we did in the case of gluon fusion, have introduced a symmetry factor of 2, since the outgoing photons are identical particles.

[7] In [20], $n_f = 6$ is used, and therefore they give different values for b_2 and b_Y.

[8] This term was included in the CERN 2000 Summer Student lectures given by C. Quigg.

6.3 Decay Rate for the Process $\phi \to \gamma\gamma$

We will now use the results from the previous sections to find the decay rate for radion into two photons, $\Gamma(\phi \to \gamma\gamma)$. By combining the loop contribution and the trace anomaly contribution from eqs. (6.2) and (6.7) respectively, we find the total Feynman amplitude, $\mathcal{M}$, for the decay process $\phi \to \gamma\gamma$, to be

$$\mathcal{M} = i\,\frac{2m_\phi^2}{\Lambda_\phi}\left(\frac{\alpha}{8\pi}\right)(b_2 + b_Y - 2A_0)\left[\eta_{\alpha\beta} - \frac{2k_{2\alpha}k_{1\beta}}{m_\phi^2}\right]\varepsilon_r^\alpha \varepsilon_s^\beta \tag{6.8}$$

Since we are interested in the unpolarized decay rate, summation over outgoing polarization is required. By using (4.29) and (4.31) in (6.8), we get

$$X = \sum_{r,s=1}^{2} |\mathcal{M}|^2 = \frac{8m_\phi^4}{\Lambda_\phi^2}\left(\frac{\alpha}{8\pi}\right)^2 |b_2 + b_Y - 2A_0|^2 \tag{6.9}$$

Since eq. (6.1) is valid for radion decay into two photons if we substitute m_H by m_ϕ, we see that $\Gamma(\phi \to \gamma\gamma)$ can be expressed as

$$\Gamma = \frac{m_\phi^3}{2\pi\Lambda_\phi^2}\left(\frac{\alpha}{8\pi}\right)^2 |b_2 + b_Y - 2A_0|^2 \tag{6.10}$$

where A_0, b_2 and b_Y are given in (6.3), (6.4) and (6.5) respectively.

Let us now compare the expression in (6.8) to similar expressions found in literature. The $F_{1/2}(\tau_t)$ function in [20] corresponds to $2A_0(t)$ without the $Q_t^2 N_c$ factor, and $F_1(\tau_W)$ corresponds to $2A_0(W)$, in our notation. If we use this, together with the procedure used in Chapter 5, to compare eq. (6.8) to the effective vertex given in [20], eq. (53), we find the same relative sign difference as discussed on page 58. Note the relative factor of 2 between the β-function coefficients and A_0 in (6.8), which was not present in the gluon fusion amplitude (see (5.44)). The reason why this factor did not occur in the gluon case is due to a factor of 1/2 from eq. (4.9), which we do not get in the photon case.

In Fig. 6.2 on page 65 we show $(\Lambda_\phi/v)^2\Gamma(\phi \to \gamma\gamma)$, where $\Gamma(\phi \to \gamma\gamma)$ is the decay rate of radion decay into two photons. As in the cross section plot in the previous chapter, we have introduced a scaling factor of $(\Lambda_\phi/v)^2$ to remove the unknown vacuum expectation value, Λ_ϕ, of the radion field. We have used a center of mass energy of $\sqrt{s} = 14\,\text{TeV}$, and five active quark flavors[9]. The thresholds for both $t\bar{t}$ and W-pair production are visible in the decay rates. This is because $\mu_W^2 = 1$ and $\mu_t^2 = 1$ at these thresholds. Since $f(\mu^2)$, given in (4.26), changes from one function to another at $\mu^2 = 1$, and since A_0 depends on $f(\mu^2)$, we can easily see these thresholds. In the previous chapter we found that in the cross section for radion production through gluon fusion, the trace anomaly term, b_{QCD}, gave a considerable contribution. Here we see that the trace anomaly term does not give a very large contribution in the case of two-photon decay rates, at least not for $m < 450\,\text{GeV}$.

[9] Corrections due to $n_f \neq 5$ for very high, or very low, energies have not been considered.

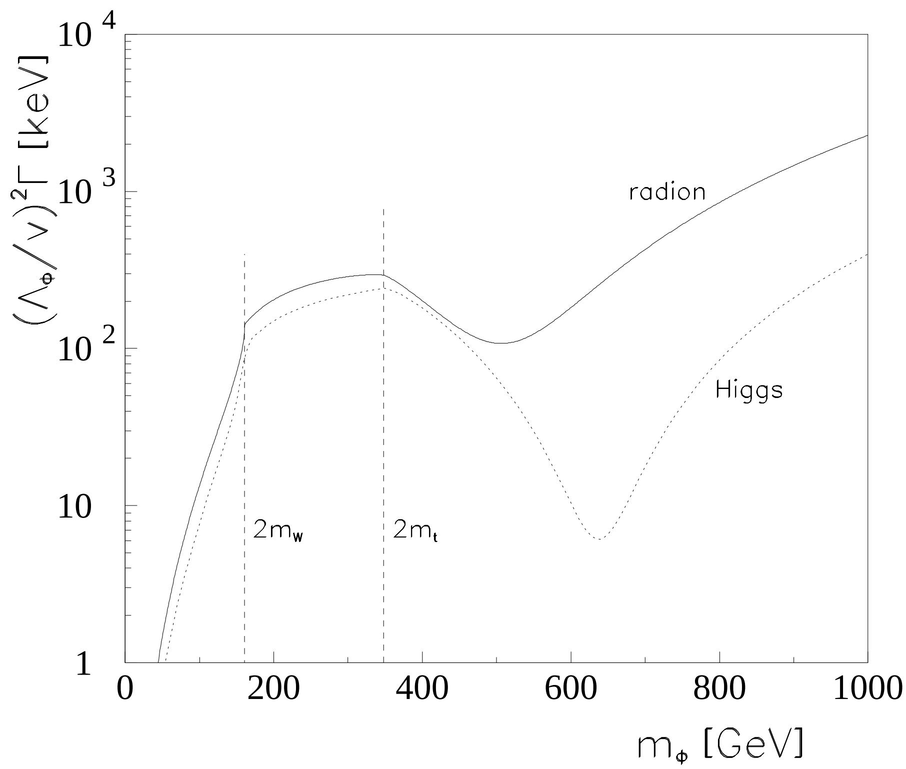


Figure 6.2: The decay rate, $\Gamma(\phi \to \gamma\gamma)$, of radion decay into two photons, with a scaling factor of $(\Lambda_\phi/v)^2$. We have also shown the corresponding Higgs decay rate (the dotted line) without the scaling factor. Five active quark flavors, and an energy of $\sqrt{s} = 14\,\text{TeV}$ in the center of mass was used.

We can understand this by noticing that the β-function coefficients in (6.4) and (6.5) have opposite sign, $b_2 + b_Y = -17/9$. There is also a relative factor of 2 between A_0 and the β-function coefficients, which makes the trace anomaly contribution less important.

If we consider the Higgs decay rate (the dotted line), we see that there is a local minimum around 650 GeV. This happens because the real part of $A_0(W)$ cancels the real part of $A_0(t)$ at almost the same mass as where the imaginary parts cancel. Note that the radion cross section also has a local minimum around 500 GeV, but because of the (real) β-function coefficients, the cancellations of the real and the imaginary parts in the amplitude do not occur near each other, and the effect is not as prominent as in the Higgs case. We also see that both decay rates increase for large masses, due to the m^3 behavior in the decay rate. Again we emphasize that mass difference and different vacuum expectation values are of importance for the relation between the two decay rates. Since Figure 6.2 is of greatest interest for a light radion, $m_\phi \leq 200\,\text{GeV}$, where the branching ratio into two photons is significant, we have shown this mass region in greater detail in Figure 6.3 on page 66 (upper panel).

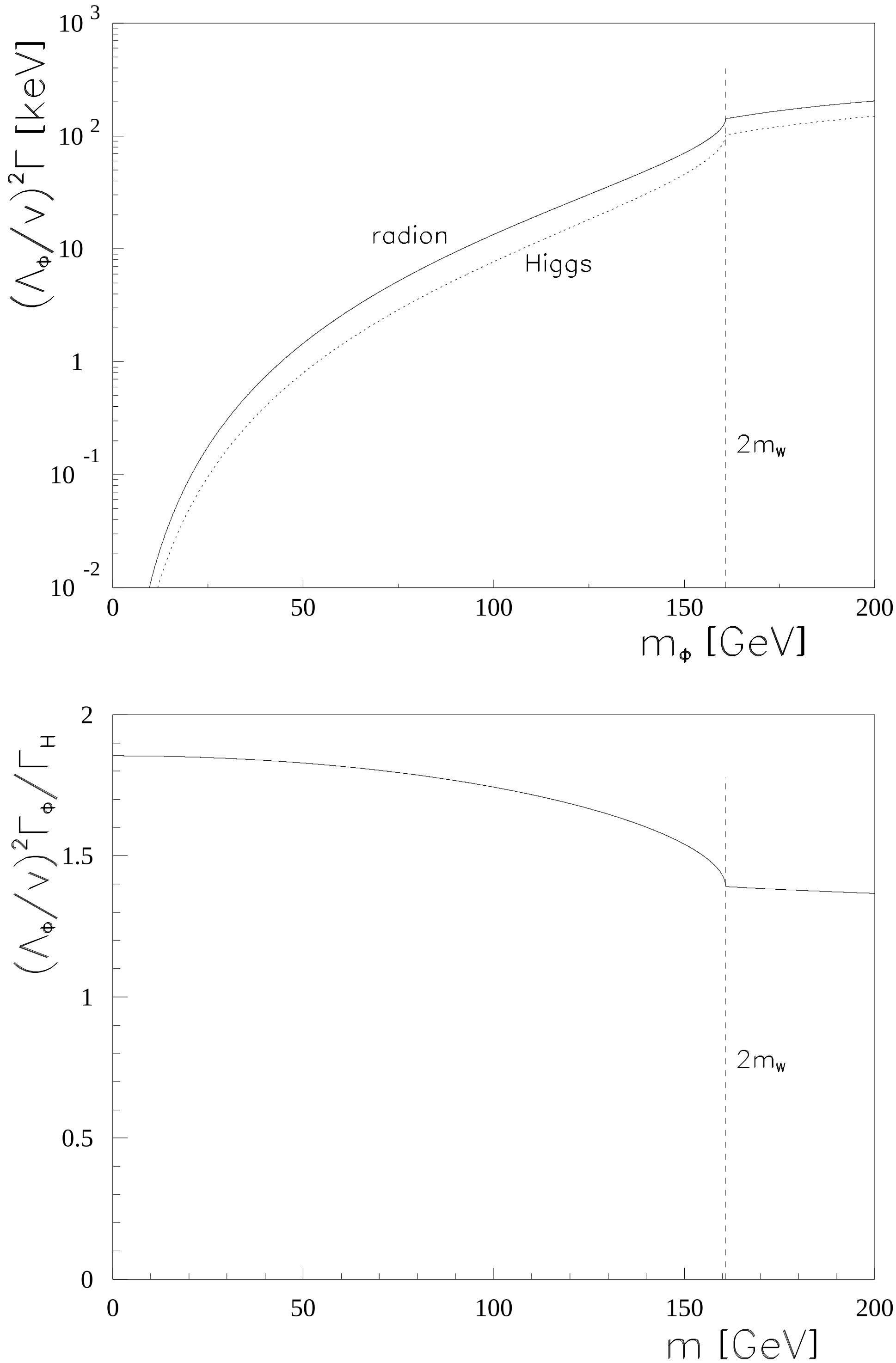


Figure 6.3: The decay rates (above), and the ratio defined in (6.11) (below), as functions of the mass of the boson (radion or Higgs). In both cross sections we use $\sqrt{s} = 14\,\text{TeV}$.

Since the upper panel on page 66, which shows the decay rates, has a logarithmic scale, we also show the ratio, R_Γ (lower panel), defined as

$$R_\Gamma = \frac{(\Lambda_\phi/v)^2\Gamma(\phi \to \gamma\gamma)}{\Gamma(H \to \gamma\gamma)} \tag{6.11}$$

as a function of the mass of the boson (radion or Higgs). We see that $R_\Gamma > 1$ in the whole mass region, which is interesting, but we have to remember that R_Γ contains an unknown scaling factor. The total cross section for radion decay will also contain two-gluon decay, which should be strongly enhanced due to the QCD trace anomaly term we discussed in the previous chapter. Therefore, the probability of a radion decaying into two photons may be lower than the probability of the corresponding Higgs decay. Note that since the radion coupling to all SM fields has a $1/\Lambda_\phi$ dependence, the probability mentioned above, which often is referred to as the branching ratio, is independent of Λ_ϕ.

Chapter 7

Radion Production and Decay

In this chapter we will use the results from the previous two chapters to study the combined process of a radion, which is produced through gluon fusion in a pp collider, decaying into two photons. In Figure 7.1 we have plotted the product of $(\Lambda_\phi/v)^2\sigma(p+p\to\phi+X)$ and

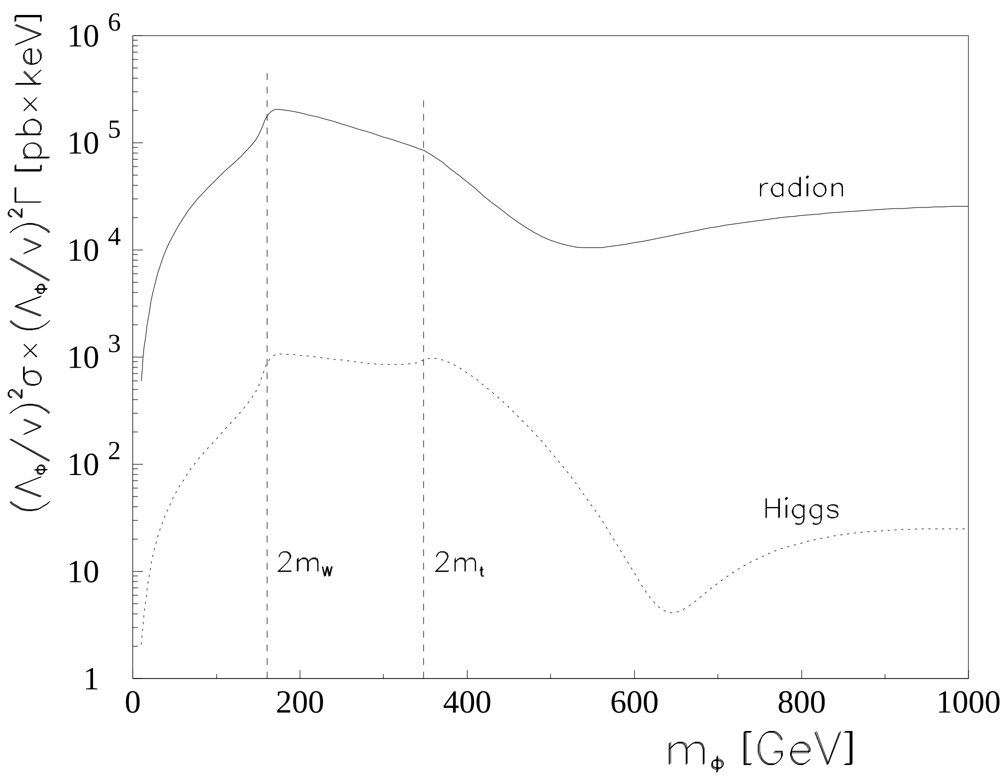


Figure 7.1: The product of the cross section, $\sigma(p+p\to\phi+X)$ and the decay rate, $\Gamma(\phi\to\gamma\gamma)$, with a scaling factor of $(\Lambda_\phi/v)^4$, and the corresponding quantities for the Higgs case (without the scaling factor). We have used five active quark flavors, and a center of mass energy of $\sqrt{s}=14\,\text{TeV}$.

$(\Lambda_\phi/v)^2\Gamma(\phi \to \gamma\gamma)$ as a function of m_ϕ, where the cross section and the decay rate are given in (5.47) and (6.10) respectively. We have also shown the corresponding quantities for the Higgs case as a function of m_H, but without the scaling factors. Since Figure 7.1 is the product of the curves shown in Figure 5.1 (upper panel) and Figure 6.2 on pages 59 and 65 respectively, we get a total scaling factor of $(\Lambda_\phi/v)^4$ in the product of the radion cross section and decay rate. This is, as we have mentioned before, to remove the unknown vacuum expectation value, Λ_ϕ, of the radion field. As we can see, the thresholds for $t\bar{t}$ and W-pair production are both visible. Note that the shapes of the two curves in Figure 7.1 are similar to those in Figure 6.2. This is since the ratio of the cross sections (see Figure 5.1, lower panel) has no strong variation.

Roughly speaking, the radion curve differs from the Higgs curve by two to three orders of magnitude. In order to better see this, we define the ratio, R, as

$$R \equiv \frac{(\Lambda_\phi/v)^4\sigma(p+p \to \phi + X) \times \Gamma(\phi \to \gamma\gamma)}{\sigma(p+p \to H + X) \times \Gamma(H \to \gamma\gamma)} \tag{7.1}$$

In Figure 7.2 on page 70 (upper panel) we have plotted the quantity R as a function of the mass of the boson (radion or Higgs). This ratio is however not an observable, since Λ_ϕ is unknown.

If the total decay rates, $\Gamma(\phi \to X)$ and $\Gamma(\phi \to X)$, had been calculated, we could have found the so-called *branching ratio*, $B_{\phi\gamma\gamma}$, defined as

$$B_{\phi\gamma\gamma} = \frac{\Gamma(\phi \to \gamma\gamma)}{\Gamma(\phi \to X)} \tag{7.2}$$

where $B_{H\gamma\gamma}$ is defined analogously. By using (4.58), the total number of events, $N_{H\gamma\gamma}$, of the process $pp \to HX \to \gamma\gamma X$, generated at the LHC during one year, could have been calculated from the formula

$$N_H = 10^5 \times \sigma[\mathrm{pb}] \times B_{H\gamma\gamma} \tag{7.3}$$

where only gluon fusion, which is the most dominant production process [8], has been taken into consideration. We could have followed the same procedure in the radion case, but since the vacuum expectation value, Λ_ϕ, is unknown, we would have to introduce a scaling factor of $(\Lambda_\phi/v)^2$ in N_ϕ to remove the Λ_ϕ dependence from the cross section. It would still have been interesting to do these calculations, and compare the two cases for different values of Λ_ϕ.

Let us now define the ratio, R_ϕ, as follows

$$R_\phi \equiv \frac{\sigma(p+p \to \phi + X)}{\Gamma(\phi \to \gamma\gamma)} \tag{7.4}$$

where R_H is defined analogously in the corresponding Higgs case. Note that the vacuum expectation values cancel, and there is no need for a scaling factor. In Figure 7.2 on page 70 (lower panel) we show R_ϕ/R_H as a function of the mass of the boson (radion or Higgs). The ratio, R_ϕ/R_H, is an observable, in contrast to R_σ and R in eqs. (5.48) and (7.1), and

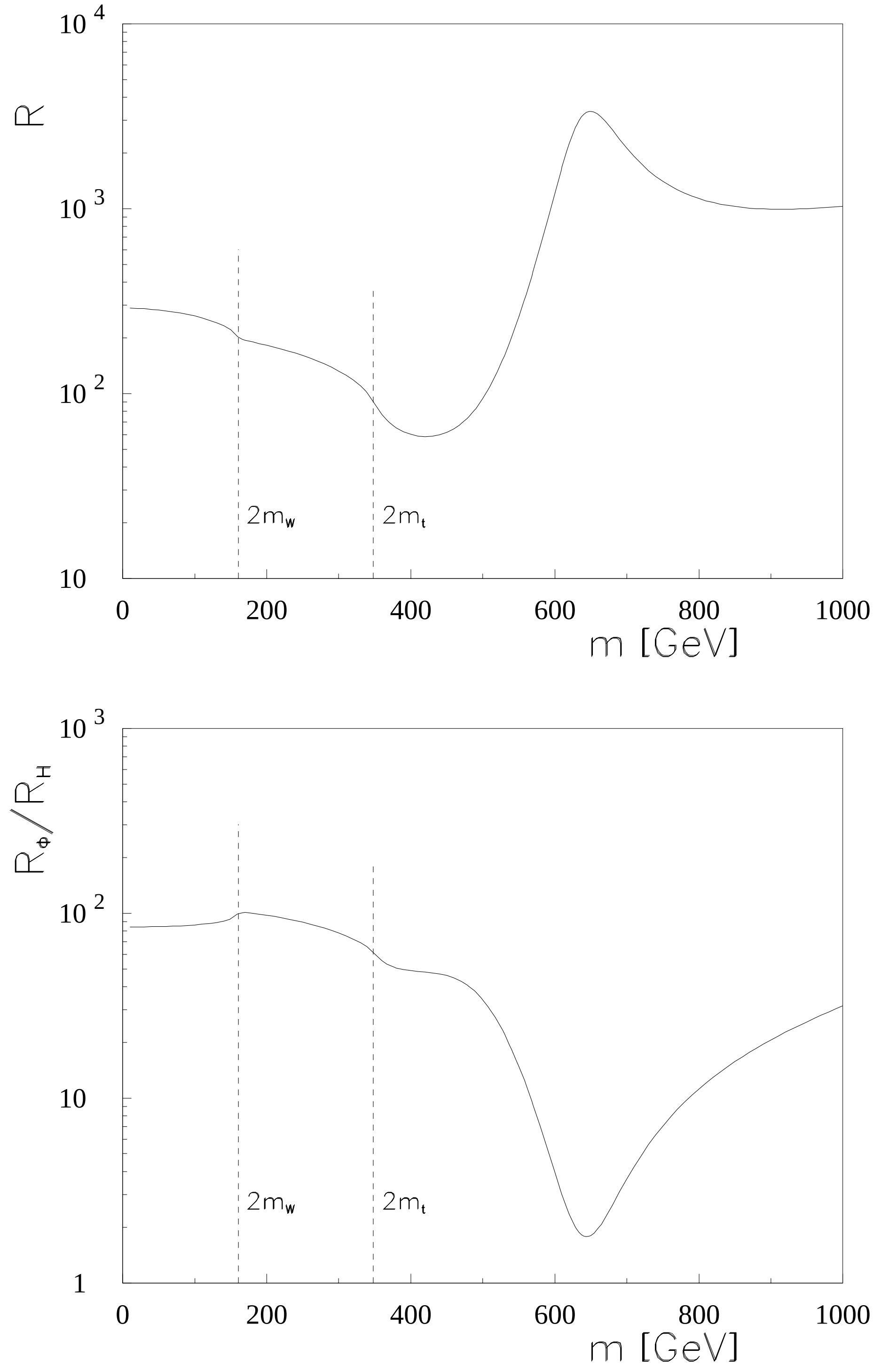


Figure 7.2: The ratio, R (above), which is defined in (7.1), and the ratio R_ϕ/R_H (below), where R_ϕ is defined in (7.4) and R_H is analogously defined, as a function of the mass, m, of the boson (radion or Higgs) at a center of mass energy of $\sqrt{s} = 14\,\text{TeV}$.

it is of greatest interest for a light radion, $m_\phi \lesssim 200\,\mathrm{GeV}$, where the branching ratio, $B_{\phi\gamma\gamma}$ is significant [25]. The fact that this ratio, given in Figure 7.2 on page 70 (lower panel) is significantly different from 1, suggests that $\sigma(p+p \to \phi/H + X)/\Gamma(\phi/H \to \gamma\gamma)$ is a very informative observable for distinguishing between a radion and a Higgs particle.

Chapter 8

Concluding Remarks

In this thesis we have studied production of the Randall–Sundrum radion through gluon fusion at the LHC. We have also investigated the decay of a radion into two photons. The radion couples to the Standard Model (SM) particles through the trace of the energy-momentum tensor. By comparing the couplings of the radion to those of the SM Higgs boson, we were able to use the calculations for the cross section of Higgs production through gluon fusion, and its decay rate for decay into two photons, to find the loop contributions in the radion case[1].

Since the vacuum expectation value, Λ_ϕ, of the radion field is unknown, we have introduced a scaling factor in our plots of the radion cross section and decay rate. After rescaling, we were able to compare the radion results to the results for the SM Higgs boson. We found that the trace anomaly terms, which enter in the radion case, give a significant contribution to the production cross section, and a more moderate contribution to the $\gamma\gamma$ decay rate. Mass differences and different vacuum expectation values of the Higgs and the radion field are of course very important, but since only one of these parameters, namely the vacuum expectation value, v, of the Higgs field, is known, we are not in position to determine these effects.

We also studied the product of the cross section for radion production through gluon fusion and the decay rate of a radion into two photons. This product was compared to the corresponding product in the Higgs case by looking at the ratio of the two products[2]. A similar comparison was made, where we first considered the ratio, R_ϕ, between the cross section for radion production through gluon fusion and the decay rate of a radion into two photons. Then we plotted the ratio between R_ϕ and the corresponding Higgs ratio, R_H. The ratio, R_ϕ/R_H, is independent of the unknown Λ_ϕ, and is an observable quantity. We found that this ratio is significantly different from 1 in the mass region $m_\phi \lesssim 200\,\text{GeV}$. This is interesting, since it is in this region the $\gamma\gamma$ channel has a significant branching ratio. In this context, it would be interesting to calculate the branching ratio for radion decay into two photons, which should be independent of the vacuum expectation value, Λ_ϕ, of

[1] All results in this thesis are one-loop results.

[2] We introduced a scaling factor of $(\Lambda_\phi/v)^4$, since the vacuum expectation value, Λ_ϕ, of the radion field is unknown.

the radion field.

The Randall–Sundrum scenario is just one out of many proposed scenarios involving extra dimensions. If it turns out that nature has chosen a different solution, the Randall–Sundrum radion would of course not exist. However, the calculations in this thesis are valid for all scalar particles which couple to SM particles through the trace of the energy-momentum tensor.

One of the reasons why scenarios involving large extra dimensions have got a lot of attention over the last years is due to the possibility that, within a few years time, one may determine experimentally if such extra dimensions exist or not. Until the next generation of hadron colliders will be turned on, these scenarios are just speculations. We do not know which scenario, if any, is correct, but the progress in this field over the last years illustrates the validity of the following statement, made by Pauli:

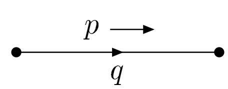
> *The question is never: Will the present theory remain as it is or not? The question is always: In what direction will it change?*
>
> *W. Pauli*

Appendix A

Feynman Rules

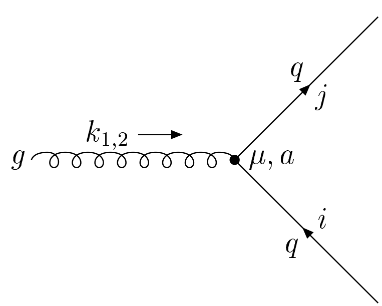
The Feynman amplitude, $\mathcal{M}$, to a given order, n, is obtained by drawing all Feynman diagrams which are connected, topologically different, have n vertices and correct external lines. Each diagram represents a mathematical expression, which is given by the Feynman rules. The Feynman amplitude is the sum of all such contributions. Below, we have listed the Feynman rules which are relevant for the calculations in this thesis:

- Follow the fermion line *against* its arrow, and write non-commuting factors (γ-matrices and fermion propagators) from left to right.
- For each closed *loop* with loop momentum q, carry out the integration: $\int \frac{\mathrm{d}^4 q}{(2\pi)^4}$
- For each closed *quark loop*, multiply by (-1) and take the trace.
- For each *initial gluon*: $\varepsilon_{r\alpha}(\mathbf{k})$
- For each *quark propagator*:



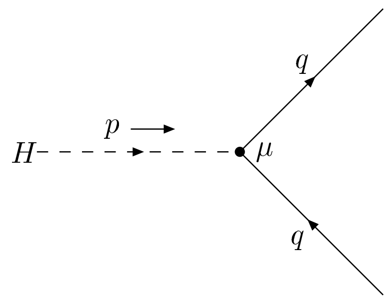
$$\frac{i(\not{p} + m_q)}{p^2 - m_q^2 + i\epsilon}$$

- For each *gluon-quark-antiquark* vertex:



$$i g_S \gamma^\mu T^a_{ij}$$

- For each *Higgs-quark-antiquark* vertex[1]:



$$-\frac{i}{2} g_W \frac{m_q}{m_W}$$

[1]Since the Higgs particle is color-neutral, the quark and the antiquark must have the same color-anticolor.

Appendix B

FORTRAN Subroutines in C++

B.1 FORTRAN Code

The following FORTRAN code was used as a subroutine in our C++ program. We choose gluon distribution by the parameter Iparton = 0.

```
**********************************************************
* Test CTEQ5 pdf's
**********************************************************
      real*8 function gluondist(x,Q)
      implicit none

      double precision x,Q,res
      double precision Ctq5Pdf
      integer Iset,Iparton

      Iset=1
      Call SetCtq5(Iset)

      Iparton=0

      res=Ctq5Pdf (Iparton, X, Q)
      gluondist=res
      return
      end
**********************************************************
      include '/gruppe/hepteori/osland/ctq5-pdf/Ctq5Pdf.f'
```

B.2 C++ Code

If we want to declare the function gluondist, from the FORTRAN code on the previous page as a subroutine in a C++ program, we write

```
extern "C"
{
  double gluondist_(double&,double&);
}
```

By using the extern "C"{· · ·} command, we can declare subroutines which are not written in C++ code. The appended underscore in the declaration of gluondist_ must also be included in a C++ call, to specify that this is a FORTRAN subroutine or function. Since FORTRAN subroutines and functions expect call-by-reference arguments, it is necessary to use the address-of (&) operator in the C++ function prototypes[1]. An example of how we used the FORTRAN code on the previous page as a subroutine in a C++ program can be found below, where we have listed the C++ code, used to compute the cross section, $\sigma(p + p \to H + X)$, in Chapter 4.

```
// This program computes the cross section of the process pp -> HX,
// by computing the convolution integral numerically. Simpson's 1/3 rule
// is used to perform the integral.
// To compile the FORTRAN file, use:
// f77 -c gluondist.f
// Compile this file by:
// g++ -o sigmafortr sigmafortr.cc gluondist.o -lfor

#include <iostream>                              // cout, cin.
#include <math.h>                                // asin(), pow(), sqrt().
#include <complex.h>                             // Complex numbers.
#include <fstream.h>                             // Writing to file.


// Define the coefficient A0t() in the Feynman amplitude:
complex <double> A0t(double mt,double mh,double PI)
{
  double mut2 = 4*pow(mt,2)/pow(mh,2);        // This is mu squared.

  double x1,y1,x2;
  x1 = log((1+sqrt(1-mut2))/(1-sqrt(1-mut2)));
```

[1] C++ passes all parameters by value (except arrays and structures), while FORTRAN passes them by reference.

```
  y1 = -PI;
  x2 = pow(asin(1/sqrt(mut2)),2);

  complex <double> z1(x1,y1);              // Declare a complex z1.
                                           // <double> means that x1
                                           // and y1 are ‘double’.

  complex <double> fmut2;                  // f(mu-squared).

  if (sqrt(mut2)<1)
    fmut2 = -0.25*pow(z1,2);
  else
    fmut2 = x2;

  complex <double> A0t;
  A0t = -mut2*(1 + (1-mut2)*fmut2);

  return A0t;
}

// Declare the FORTRAN subroutine, gluondist_():
extern "C"
{
  double gluondist_(double&,double&);
}


// Define the integrand in the convolution integral:
double integrand(double y, double ECoM, double mh)
{
  double xp = (mh/ECoM)*exp(y);
  if(xp>1)                                 // To handle round-off
    xp = 1;                                // errors.
  double xm = (mh/ECoM)*exp(-y);
  if(xm>1)                                 // To handle round-off
    xm = 1;                                // errors.

  return gluondist_(xp,mh)*gluondist_(xm,mh);
}


// Define convolution() as the convolution integral:
double convolution(double ECoM,double mh,double c)
```

```
{
  double a,b;
  double integral,integralnull,ans,diff;
  double dy,y0,y1,y2,dA;

  a = log(mh/ECoM);                              // Lower limit in integral.
  b = -log(mh/ECoM);                             // Upper limit in integral.
  dy = 0.5*(b - a);                              // First approximation.
  ans = 0;
  do {
      integralnull = ans;
      ans = 0;
      for (y0 = a; y0 <= b-2*dy; y0 = y0+2*dy)
        {
          y1 = y0 + dy;
          y2 = y0 + 2*dy;
          dA = (y2 - y0)*(integrand(y0,ECoM,mh)
                          + 4*integrand(y1,ECoM,mh)
                          + integrand(y2,ECoM,mh))/6;
          ans = ans + dA;
        }
      integral = ans;
      if(integral > integralnull)
        {
          diff = integral - integralnull;
        }
      else
        {
          diff = integralnull - integral;
        }
      dy = 0.5*dy;
  } while(diff > c);
  return ans;
}


// Define alphaS as the running coupling constant.
// nf = 5 (active colors) and Lambda = 226 MeV.
double alphaS(double mh,double PI)
{
  return (6*PI/(23*log(4.425*mh)))
         *(1 - (174*log(2*log(4.425*mh)))/(529*log(4.425*mh)));
}
```

```
// The main program:
main()
{
  cout.precision(10);             // To get 10 digits in output answer.

  double mh;                      // Higgs mass.
  double ECoM;                    // Center of mass energy (sqrt{s}).
  double c;                       // Precision in convolution().
  double PI = 3.1415927;
  double GF = 1.166e-5;           // Fermi coupling constant, (GeV)^-2.
  double mt = 173.8;              // Top-quark mass (GeV).
  double mhmin,mhmax,step;
  complex <double> i(0,1);

  // Remove comments to specify variables each time:
  // cout << "Compute until correction in convolution() is less than: ";
  // cin >> c;
  // cout<<"Set energy in center of mass (TeV): ECoM = ";
  // cin>>ECoM;
  // cout<<"Set minimum value for Higgs mass (GeV): mhmin = ";
  // cin>>mhmin;
  // cout<<"Set maximum value for Higgs mass (GeV): mhmax = ";
  // cin>>mhmax;
  // cout << "Set step value for loop (GeV) : step = ";
  // cin>>step;

  c = 0.01;
  ECoM = 14;
  mhmin = 50;
  mhmax = 1000;
  step = 10;

  ECoM = 1000*ECoM;               // To get GeV instead of TeV.

  // remove old .dat file:
  system("rm sigmafortr.dat");

  ofstream outSigmaFortrFile("sigmafortr.dat", ios::app);
  // Creates an ofstream object called outSigmaFortrFile.
  // outSigmaFortrFile creates a file sigmafortr.dat, and ios::app
  // means that all output is written to the file, without deleting
```

```
  // anything.

  for (mh = mhmin; mh <= mhmax; mh = mh+step)
  // for-loop computes cross section for mh between ‘mhmin’ and
  // ‘mhmax’ GeV, in steps of ‘step’ GeV.
    {
      // Compute cross section, sigma, as complex despite the fact that
      // it is real. This is bechause it contains complex variables.
      complex <double> sigma;
      sigma=(PI*GF*pow(alphaS(mh,PI)*mh,2)/(2*sqrt(2)*pow(8*PI*ECoM,2)))
            *A0t(mt,mh,PI)*conj(A0t(mt,mh,PI))*convolution(ECoM,mh,c);

      //Convert sigma to real form since it is a real number.
      double Sigma;
      Sigma = (389379e3)*sigma.real(); // To get cross section in pb
                                       // instead of (GeV)^-2, use
                                       // hbar*c=197.327MeV*fm

      outSigmaFortrFile << mh << " " << Sigma << endl;
      // For each loop: put mh and Sigma in sigmafortr.dat, with a space
      // between.
    }

  return 0;
}
```

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