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A STUDY OF THE VERIGIN PROBLEM WITH APPLICATION TO ANALYSIS OF WATER INJECTION TESTS

by

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1. INTRODUCTION AND BACKGROUND TO THE PROBLEM

1.1 The Verigin problem

Several models with varying degree of complexity have been proposed for describing two-phase immiscible displacement in a homogeneous porous reservoir. Commonly these models are based on the assumption that both involved fluids may be treated as incompressible. Our object will be to describe the pressure distribution when water is injected into an oil reservoir with only <u>one</u> well present, and in this situation, the fluid compressibility can not be neglected. A simple mathematical model including effects of compressibility was introduced by Verigin [1,2], assuming the reservoir to consist of two distinct fluid zones separated by a moving discontinuity in fluid saturation:

$$\frac{\partial p}{\partial t}w = \frac{\eta}{r}\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r}w) \qquad 0 < r < r_f$$

$$\frac{\partial p}{\partial t}o = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r}o) \qquad r_f < r < r_e$$

(1.1)

Initial conditions: $\left.\begin{matrix} p_w = p_o = 0 \\ r_f = 0 \end{matrix}\right\} t = 0$

Boundary conditions:

$$\lim_{r\to 0} (r\frac{\partial p}{\partial r}w) = - q(t)$$

$$\lim_{r\to r_e} (r\frac{\partial p}{\partial r}o) = 0$$

Free-boundary conditions:

$$\left.\begin{aligned} p_w &= p_o \\ M\frac{\partial p}{\partial r}_w &= \frac{\partial p}{\partial r}_o \\ r_f' &= \frac{dr}{dt}_f = -\varepsilon\frac{\partial p}{\partial r}_w \end{aligned}\right\} r = r_f \tag{1.1cont}$$

All variables and parameters are dimensionless as defined on p.67. p_w and p_o is the pressure in the inner water zone and in the outer oil zone respectively. r_f is the position of the free boundary, i.e. the water front.

The model describes a piston-like displacement; the effects of capillary pressure, relative permeability variation and gravity are neglected. In addition, a line-source assumption is used. The last of the three free-boundary conditions is only valid if the connate water is immobile. The model contains three parameters, the mobility ratio M, the diffusivity ratio η, and the Peclet number ε. For water injection into an oil reservoir, M and η are both of order 1-10, ε is of order 0.001-0.01 .

Problems characterized by the given free-boundary conditions are usually called Verigin problems. These are similar to the class of Stefan problems, where the value of the dependent variable is specified on the free boundary [2]. In contrast to this class, the Verigin problems always involve diffusion in at least <u>two</u> zones. The last free-boundary condition given is common for both classes of problems and is called the Stefan condition. In the Verigin problem, this can be replaced by the following condition, which does not contain r_f explicitly:

$$\frac{\partial p}{\partial t}_w - \frac{\partial p}{\partial t}_o = \varepsilon(1 - M)\left[\frac{\partial p}{\partial r}_w\right]^2 \qquad r = r_f \tag{1.2}$$

Verigin studies constant-rate injection into an infinite reservoir ($r_e = \infty$, $q \equiv 1$), and by using the Boltzmann's transformation he is able to give exact solutions both for linear and cylindrical geometry. In the cylindrical case, the solution is given by

$$p_w = -\frac{1}{2}Ei\left(-\frac{r^2}{4\eta t}\right) + \frac{1}{2}Ei\left(-\frac{g}{4\eta}\right)$$

$$(1.3) \qquad -\frac{M}{2}e^{\frac{g}{4}\left(1-\frac{1}{\eta}\right)} Ei\left(-\frac{g}{4}\right) \qquad 0 < r < r_f$$

$$\equiv V(r,t)$$

$$p_o = -\frac{M}{2}e^{\frac{g}{4}\left(1-\frac{1}{\eta}\right)} Ei\left(-\frac{r^2}{4t}\right) \qquad r_f < r < \infty$$

$g = r_f^2/t$ is a constant determined from the Stefan condition:

$$(1.4) \qquad g = 2\varepsilon \exp\left(-\frac{g}{4\eta}\right)$$

The fact that r_f^2/t is constant will be referred to as "constant speed", in spite of the fact that the front speed is actually decreasing with time. When $\varepsilon/\eta << 1$, $g \approx 2\varepsilon$ and $r_f < r_c$, where r_c is the radius of incompressibility defined in Appendix 1. That is, the inner zone behaves as incompressible except from the first few seconds where also the line-source assumption is invalid. The logarithmic approximation to the exponential integral can be used and the expression for p_w simplifies to

$$p_w = \frac{1}{2}\ln\frac{t}{r^2} + \frac{M}{2}\ln(4e^{-\gamma}) + \frac{1-M}{2}\ln(2\varepsilon)$$

$$(1.5) \qquad = -\ln\frac{r}{r_f} + p_f$$

$$p_f = -\frac{M}{2}\ln\frac{\varepsilon e^{\gamma}}{2}$$

p_f is the pressure at the water front and is seen to be a constant, proportional to M.

A three-zone Verigin problem with linear geometry was studied by Rubinstein [2]. Both Green's functions and a quasi-stationary method were used. Rubinstein also applied Green's functions to an inverse two-zone Verigin problem [2,3]. In the inverse problem, the front

speed is given and the objective is to determine the initial pressure distribution. Kamynin [4,5] used Green's functions to prove the existence of a solution to a linear two-zone Verigin problem where the diffusivities were general functions of space and time.

The similarity with the Stefan problems is already mentioned. The huge literature existing on this class of free-boundary problems is reviewed by Rubinstein [2] and Muehlbauer & Sunderland [6]. Only a few exact solutions exist and also few general solution techniques. This report shows how three of the techniques originally developed for the Stefan problem can be applied to the Verigin problem. Chapter 2 demonstrates the use of Green's functions for a finite cylindrical reservoir. In Chapter 3, eigenfunctions are used both for linear and cylindrical geometry. These chapters are of mathematical nature and can be skipped by readers with primary interest in well-test applications. Problems encountered in injection well testing, as effects of an initial water bank, change of rates etc., are handled in Chapter 4 by a quasi-stationary method originally developed by Leibenzon [2]. The analytical results thus found are compared with results from a numerical simulator in Chapter 5.

1.2 Analysis of water injection tests

A water-injection pressure test in an oil reservoir can be run on several stages in the lifetime of a well. A general objective is to estimate characteristic fluid mobilities and wellbore parameters. Beside of this, the purpose of the tests and the conditions under which they are run can vary considerably. No single mathematical model exists that can describe this plurality, and unfortunately the distinction between different testing conditions is not always clear in the literature. One could roughly group the tests into the following categories:

1) Tests in exploration wells: An important aim is to estimate maximum injection pressure/rate without fracturing the reservoir. This is done by using step-rate injection tests [7,8].

2) Tests in developed fields where the pressure is above the saturation pressure: Estimation of the position of the fluid front, residual oil and average reservoir pressure are important objectives.

3) Tests in devloped fields where the pressure is below the saturation pressure: Three different phases, water, oil and gas, coexist in the reservoir and have to be taken into account in a theoretical model.

4) Tests in watered-out areas: Theory for one-phase tests can be employed.

A general description of a water-injection test scheme can be found in Ref.[9]. Ideally, a test includes a period of constant-rate injection and a falloff period during which the well is closed. Ref.[8] gives a general introduction to well testing.

Among the first to describe the transient history of an injection well is Muskat [10], modelling a situation where a free gas phase exists in the reservoir. He assumes that the reservoir can be divided into three distinct zones; a water bank close to the well, an oil bank ahead of this, and an outer zone uninfluenced by the injection. The zones are separated by discontinuities in the fluid saturations, and these are moving according to the condition of material balance. Also included in the model is an assumption that the water and the oil banks can be treated as incompressible. Using Darcy's law, Muskat gives an expression connecting wellbore pressure and injection rate which can be used both for constant rate and constant injection pressure.

The three-zone model was also used by Hazebroek et al. [11] who included the effects of compressibility, together with skin and afterflow. The discontinuity between the outer two zones was treated as stationary. Independently, the authors refound the solution already presented by Verigin [1], is spite of differences in the basic

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During 1950-1965 several attempts using known theory for one-phase testing were tried for analysing tests in water-injection wells [12,13,14,15]. These attempts had no stringent mathematical foundation, and there were little discussion on the validity of the assumptions involved. Some of the authors, though, report deviation between real data and single-phase theory [12]. One-phase models have also been used as basis for studying special topics connected with injection tests, as effects of fractures [16,17] and changing wellbore storage [18].

Based on results from theory describing in-situ combustion, Morse and Ott [19] claimed that plotting falloff pressure in a MDH or Horner plot will produce two straight lines that both can be used for analysis. The slope of the first of these lines is proportional to the inverse of the mobility of water, the slope of the latter proportional to the inverse of the mobility of oil. This statment was confirmed when comparing well-test data with results from core analysis.

Kazemi et al. [20,21] used a numerical simulator to test the validity of the theory developed by Morse and Ott. The simulator was based on equations describing the three-zone model used by Hazebroek et al., but was also able to handle equations for a two-zone model, as given in Eqs.(1.1). The Stefan condition was replaced by the following expression, as if the water zone behaved as incompressible:

$$r_f r_f' = \varepsilon \qquad (1.6)$$

Only the solution for the falloff period was solved numerically,- the Verigin solution was used for the injection period. The authors conclude that the first straight falloff line can be used for estimating water mobility if the discontinuity is not too close to the well. The second part will only give the oil mobility directly if the compressibility ratio is close to 1, but the authors present a general correlation between slope ratio, mobility ratio and compressiblity

ratio. Kazemi et al. also studied methods for estimating the position of the fluid front.

The discontinuities between the different fluid zones are a consequence of neglecting the influence of capillary pressure and variations in relative permeability. Sosa, Raghavan and Limon [22] present a numerical model where variations in relative permeability are included. They restrict their study to compressibility ratios equal to 1 and to reservoirs with no free gas, but are not able to find any general correlation between oil mobility and the last part of the falloff curve. In their work, though, it is difficult to distinguish between boundary effects and effects from the relative permeability variations.

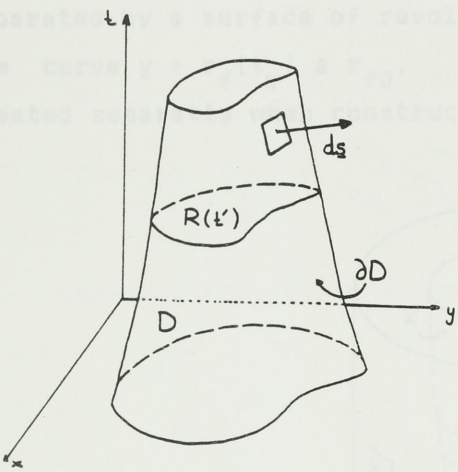
Several authors have considered effects of a difference in temperature between injection and reservoir fluids, and numerical simulators capable of handling non-isothermal effects have been created. Among these authors, Weinstein [23] is the only one concerned with problems related to two-phase well testing. His simulator includes variations in relative permeability, but since M = 0.05 in the given examples, the water front is essentially piston-like. In spite of the fact that the compressibility ratio is not equal to 1, Weinstein finds that the second part of the falloff curve reflects the mobility of the (hot) oil directly, thus in conflict with the results of Kazemi et al.

In two recent papers [24,25], Woodward and Thambynayagam present an analytical approach to the two-zone model, based on Laplace transform. Both infinite and bounded reservoirs are studied, and effects of partial penetration and heat transmission are included. Analytically, they find that the last part of the falloff curve reflects the oil mobility directly. Comparison between their analytical results and simulated data is very good. When using the Laplace transform, the authors neglect the time dependency of the front position in the transformation, but it is not clearified why this is valid. In addition, the validity of Eq.(1.6), which is used both for infinite and finite reservoirs, is not obvious.

The main part of the literature in the field concentrates on describing the falloff period, this because of problems connected with keeping a constant rate during injection. Obviously, discrepancies exist between different descriptions of this period. Much of the work is based on results from numerical simulators, and unfortunately, these results can be hard to evaluate or generalize because of lack of information about the input parameters. The author of this report has found very few descriptions of injection tests in the literature where the data given is sufficient to evaluate basic parameters as M, η and ε [21,23,24,26]. Results from studies of two-zone models will be further discussed in Chapter 4 and 5 with background in results from the quasi-stationary method.

2. REDUCTION TO A SET OF INTEGRAL EQUATIONS WITH HELP OF GREEN'S FUNCTIONS

Green's functions are one of the commonly used tools to study existence and uniqueness of solutions to free-boundary problems. The method is analogues to the use of double-layer potentials for elliptic equations; the free-boundary problem is reduced to a set of integral equations which can be used as a basis for further analytical or numerical treatment. Details of the method can be found in Refs.[28,29,30].



Let p be a solution of the heat equation $L[p] = \sigma$, defined over a finite domain D contained in $R^2 \times [\,0,t_1\,)$. Let $R(t')$ be the cross section of D and the plane $t = t'$, and let ∂D be the surface of D. If $G = G(\underline{r},t \mid \underline{r}_0,t_0)$ is a fundamental solution to the heat equation, then p is formally given by

$$p(r,t) = \int_0^t \int_{R(t_0)} G\,\sigma(\underline{r}_0,t_0)\,dV_0\,dt_0 \tag{2.1}$$
$$+ \int_{\partial D} \{\, G\nabla_0 p - p\nabla_0 G - pG\underline{e}_{t_0} \}\cdot d\underline{s}_0$$

∇_0 is the gradient operator with respect to the coordinates $(\underline{r}_0,t_0)$. Under the integral sign, p is a function of $\underline{r}_0$ and t_0. dV_0 is a volume element on $R(t_0)$, and $d\underline{s}_0$ is a surface element on ∂D. If $R(t_0)$ is varying with t_0, the last integral can generally not be made equal to zero, even if $p(t_0 = 0) = 0$ and G is chosen as the Green's function for the problem. An arbitrary fundamental solution can be used in Eq.(2.1), for instance the Green's function for 2-dimensional free

space. In polar coordinates this is given by

$$(2.2)\qquad G = \frac{1}{4\pi\eta(t-t_0)} \exp\{ -\frac{r^2 + r_0^{\,2} - 2rr_0\cos(\theta - \theta_0)}{4\eta(t-t_0)} \}$$

The notation $G^* \equiv G(\eta=1)$ will be used in the following.

When the line-source assumption is used, the well has to be included as a source term in the differential equation for the water zone. For constant-rate injection $q \equiv 1$, and the source term is given by

$$(2.3)\qquad \sigma = 2\pi q(t)\delta(\underline{r}) = 2\pi\delta(\underline{r})$$

δ is the Dirac delta function. As shown in Fig.1, the domain for the Verigin problem in Eqs.(1.1) is divided in two sub-domains, D_1 and D_2, separated by a surface of revolution K. This surface is generated by the curve $y = r_f(t_0) \equiv r_{f0}$. Each of the two sub-domains must be treated separatly when constructing the Green's solution.

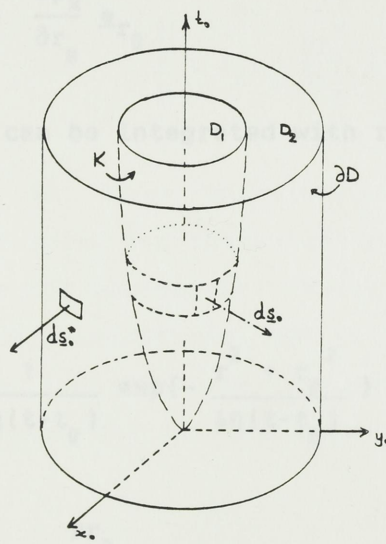


Fig.1: Integration domain for the Verigin problem

Utilizing symmetry, boundary and initial conditions, the solution of the Verigin problem can be written as

$$(2.4)\qquad p_w = 2\pi \int_0^{t_0} G(\underline{r},t|\underline{0},t_0)\,dt_0 + \int_K \{ G\nabla_0 p_w - p_w\nabla_0 G - p_w G\underline{e}_{t_0} \}\cdot d\underline{s}_0$$

$$p_o = \int_K \{ G^* \nabla_0 p_o - p_o \nabla_0 G^* - p_o G^* \underline{e}_{t_0} \} \cdot (-d\underline{s}_0)$$

(2.4cont)

$$- \int_{\partial D} p_o \nabla_0 G^* \cdot d\underline{s}_0^*$$

$d\underline{s}_0$ and $d\underline{s}_0^*$ are surface elements on K and ∂D, respectively:

$$d\underline{s}_0^* = r_e d\theta_0 dt_0 \, \underline{e}_{r_0}$$

$$ds_0 = r_{f0} d\theta_0 \, [\, 1 + r_{f0}'^{\,2} \,]^{1/2} \, dt_0$$

(2.5)

$$\underline{n} = (\, 1 + r_{f0}'^{\,2} \,)^{-1/2} \, [\, \underline{e}_{r_0} - r_{f0}' \, \underline{e}_{t_0} \,]$$

$$d\underline{s}_0 = \underline{n} \, ds_0 = r_{f0} \, [\, \underline{e}_{r_0} - r_{f0}' \, \underline{e}_{t_0} \,] \, dt_0$$

p_w and p_o are both assumed to be independent of the angle θ_0 :

(2.6) $$\nabla_0 p_x = \frac{\partial p_x}{\partial r_0} \underline{e}_{r_0} \qquad x = w, o$$

Consequently, G can be integrated with respect to θ_0:

$$\int_0^{2\pi} G \, d\theta_0$$

$$= \frac{1}{4\pi\eta(t-t_0)} \exp\{- \frac{r^2 + r_0^{\,2}}{4\eta(t-t_0)} \} \int_0^{2\pi} \exp[\frac{r r_0 \cos(\theta-\theta_0)}{2\eta(t-t_0)}] \, d\theta_0$$

(2.7)

$$= \frac{I_0 [\frac{r r_0}{2\eta(t-t_0)}]}{2\eta(t-t_0)} \exp\{- \frac{r^2 + r_0^{\,2}}{4\eta(t-t_0)} \}$$

$$\equiv E(r,t \,|\, r_0, t_0)$$

The given integral is found for instance in Ref.[50]. E is an instantaneous cylindrical heat source. Again, define $E^* \equiv E(\eta=1)$. Using Eqs.(2.5) - (2.7), Eqs.(2.4) can be written as

$$
\begin{aligned}
p_w(r,t) = & \int_0^t \frac{\exp\left[-\dfrac{r^2}{4\eta(t-t_0)}\right]}{2\eta(t-t_0)}\, dt_0 \\
& + \int_0^t \left\{ r_0 E \frac{\partial p_w}{\partial r_0} - r_0 p_w \frac{\partial E}{\partial r_0} \right\}_{r_0=r_{f0}} dt_0 \\
& + \int_0^t \left\{ p_w E \right\}_{r_0=r_{f0}} r'_{f0} r_{f0}\, dt_0
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
p_o(r,t) = & \int_0^t \left\{ r_0 p_o \frac{\partial E^*}{\partial r_0} - r_0 E^* \frac{\partial p_o}{\partial r_0} \right\}_{r_0=r_{f0}} dt_0 \\
& - \int_0^t \left\{ p_0 E^* \right\}_{r_0=r_{f0}} r'_{f0} r_{f0}\, dt_0 \\
& - \int_0^t \left\{ r_0 p_o \frac{\partial E^*}{\partial r_0} \right\}_{r_0=r_e} dt_0
\end{aligned}
$$

The right hand sides of these equations contain several unknown variables; the front speed and the values of the dependent variables and their gradients on the boundaries. Equations for these can be found from the boundary conditions, but it must then be assumed that all the integrands have continious derivatives with respect to r, such that differentiation under the integral sign is legal:

$$
\begin{aligned}
\frac{\partial p_w}{\partial r} = & \int_0^t \frac{r \exp\left[-\dfrac{r^2}{4\eta(t-t_0)}\right]}{4\eta^2 (t-t_0)^2}\, dt_0 \\
& + \int_0^t \left\{ r_0 \frac{\partial E}{\partial r} \frac{\partial p_w}{\partial r_0} - r_0 \frac{\partial^2 E}{\partial r \partial r_0} p_w \right\}_{r_0=r_{f0}} dt_0 \\
& + \int_0^t \left\{ p_w \frac{\partial E}{\partial r} \right\}_{r_0=r_{f0}} r'_{f0} r_{f0}\, dt_0
\end{aligned}
\tag{2.9}
$$

For simplicity, define

$$
\begin{aligned}
a(t) &= p_w(r_f,t) = p_o(r_f,t) \\
b(t) &= \frac{\partial p_w}{\partial r}(r_f,t) = \frac{1}{M}\frac{\partial p_o}{\partial r}(r_f,t) \qquad (2.10) \\
c(t) &= p_o(r_e,t) \\
d(t) &= r_f(t)
\end{aligned}
$$

When used under the integral sign, a,b and c are functions of t_0. If nothing else is specified, d is a function of t_0. Assuming that Eqs.(2.8) and (2.9) are satisfied on the boundaries, the boundary conditions give the following integral equations for the unknowns a,b and c:

$$
\begin{aligned}
a(t) = {} & \int_0^t \frac{\exp\left[-\dfrac{d^2(t)}{4\eta(t-t_0)}\right]}{2\eta(t-t_0)}\, dt_0 \\
& + \int_0^t \left\{ Eb - a\frac{\partial E}{\partial r_0} \right\}_{\substack{r_0=d \\ r=d(t)}} d\, dt_0 \\
& + \int_0^t \left\{ aE \right\}_{\substack{r_0=d \\ r=d(t)}} dd'\, dt_0
\end{aligned}
$$

(2.11)

$$
\begin{aligned}
b(t) = {} & \int_0^t \frac{d(t)\exp\left[-\dfrac{d^2(t)}{4\eta(t-t_0)}\right]}{4\eta^2(t-t_0)^2}\, dt_0 \\
& + \int_0^t \left\{ \frac{\partial E}{\partial r}b - \frac{\partial^2 E}{\partial r\partial r_0}a \right\}_{\substack{r_0=d \\ r=d(t)}} d\, dt_0 \\
& + \int_0^t \left\{ a\frac{\partial E}{\partial r} \right\}_{\substack{r_0=d \\ r=d(t)}} dd'\, dt_0
\end{aligned}
$$

$$c(t) = \int_0^t \left\{ a\frac{\partial E^*}{\partial r_0} - E^* M b \right\}_{\substack{r_0=d \\ r=r_e}} d\, dt_0$$

$$(2.11\text{cont}) \qquad - \int_0^t \left\{ aE^* \right\}_{\substack{r_0=d \\ r=r_e}} dd'\, dt_0$$

$$- \int_0^t \left\{ c\frac{\partial E^*}{\partial r_0} \right\}_{\substack{r_0=r_e \\ r=r_e}} r_e\, dt_0$$

The set is closed using the integrated Stefan condition

$$(2.12) \qquad d = - \varepsilon \int_0^t b\, dt_0$$

After solving the system of integral equations together with the appropriate initial conditions, the wellbore pressure can be found by setting $r=r_w$ in Eq.(2.8). The equations could easily be extended to include effects of a finite wellbore radius, but this would involve the wellbore pressure as an additional unknown.

It could be discussed whether the given system of integral equations really represents a simplification of the original problem. The equations, which are of Volterra type, are highly coupled. This type of equations are usually amenable to numerical treatment, but the system is probably too complicated to represent a basis for constructing (approximate) analytical solutions. The main advantage of the method is probably that it can be used to prove existence and uniqueness of a solution to the problem. The proof must show both that the system of integral equations is equivalent to the original problem, and that this system has a unique solution. This could be done in a way outlined by Rubinstein [2], but will be left out here.

3. USE OF EIGENFUNCTIONS

Eigenfunctions were first used to solve Stefan problems by V.G.Melamed [31,32,2], studying a problem with linear geometry and constant diffusivities. I.V.Fryazinov [33] generalized the method to include general time- and space-dependent diffusivities. The set of partial differential equations is reduced to a countable system of ordinary differential equations which have to be solved numerically. To construct the eigenfunctions, the method relies on the fact that the value of the dependent variable is known on the free boundary in the Stefan problems. This value is not given in the Verigin problems, but this chapter shows how to extend the method of Melamed to this type of problems. The trigonometric functions are more easily handled than the eigenfunctions involved in cylindrical geometry, and the extension of the method will first be demonstrated on a problem with linear geometry.

3.1 Linear geometry

Given the linear Verigin problem

$$\frac{\partial p_w}{\partial t} = \eta \frac{\partial^2 p_w}{\partial x^2} \qquad 0 < x < x_f$$

$$\frac{\partial p_o}{\partial t} = \frac{\partial^2 p_o}{\partial x^2} \qquad x_f < x < 1$$

(3.1)

Initial conditions: $p_w(x,0) = f(x) \qquad 0 < x \leqslant x_f(0) \equiv x_0$

$p_o(x,0) = f(x) \qquad x_0 \leqslant x < 1$

Boundary conditions: $\frac{\partial p_w}{\partial x}(0,t) = -q(t)$

$p_o(1,t) = p_e = f(1)$

(3.1cont)

$$\text{Free-boundary conditions:}\quad \left.\begin{aligned} p_w &= p_o \\ \frac{\partial p}{\partial x}{}_w &= \frac{1}{M}\frac{\partial p}{\partial x}{}_o = \frac{1}{\varepsilon}x_f' \end{aligned}\right\} x = x_f$$

Note that a constant-pressure outer boundary is chosen here. Introduce $p_f(t) = p_w(x_f,t)$ as a new unknown variable, and assume that the dependent variables can be written as

(3.2)

$$p_w(x,t) = p_f + q(t)(x_f - x) + \sum_{n=1}^{\infty} A_n(t)\varphi_n(\xi)$$

$$p_o(x,t) = p_f + \frac{x - x_f}{1 - x_f}(p_e - p_f) + \sum_{n=1}^{\infty} B_n(t)\psi_n(\sigma)$$

$$\xi = \frac{x}{x_f} \qquad\qquad 0 < x_f < 1$$

$$\sigma = \frac{1 - x}{1 - x_f}$$

The first terms represent the solution of the analogue problem where the effects of compressibilities are neglected, and the terms are included to make the boundary conditions for the eigenfunctions homogeneous. It will further be assumed that the infinite series can be differentiated and integrated term by term. The eigenfunctions φ_n and ψ_n have the general form

(3.3)

$$\varphi_n(x,t) = a_n \sin(\lambda_n \xi) + b_n \cos(\lambda_n \xi)$$

$$\psi_n(x,t) = c_n \sin(\mu_n \sigma) + d_n \cos(\mu_n \sigma)$$

The coefficients $a_n - d_n$ are determined from the boundary conditions:

(3.4)

$$\left.\begin{aligned}
&\frac{\partial \varphi_n}{\partial x}(0,t) = \frac{a_n\lambda_n}{x_f} = 0 &&\Rightarrow\quad a_n = 0 \\
&\varphi_n(x_f,t) = b_n \cos\lambda_n = 0 &&\Rightarrow\quad \lambda_n = (n - \tfrac{1}{2})\pi \\
& &&\qquad\;\; b_n = 1 \\
&\psi_n(1,t) = d_n = 0 && \\
&\psi_n(x_f,t) = c_n \sin\mu_n = 0 &&\Rightarrow\quad \mu_n = n\pi \\
& &&\qquad\;\; c_n = 1
\end{aligned}\right\} n = 1,2\ldots$$

Substituting Eqs.(3.2)-(3.4) into the partial differential equations in Eqs.(3.1) yields the following system of ordinary differential equations:

$$p_f' + qx_f' - q'(x - x_f) + \sum_{n=1}^{\infty} \{ A_n' \cos(\lambda_n \xi) + A_n \lambda_n \frac{x}{x_f} \frac{x_f'}{x_f} \sin(\lambda_n \xi) \}$$

$$= - \eta \sum_{n=1}^{\infty} \frac{\lambda_n^2}{x_f^2} A_n \cos(\lambda_n \xi)$$

(3.5)

$$\frac{1 - x}{1 - x_f} p_f' - \frac{x_f'(1 - x)}{(1 - x_f)^2} (p_e - p_f)$$

$$+ \sum_{n=1}^{\infty} \{ B_n' \sin(\mu_n \sigma) + B_n \mu_n \frac{x_f'(1 - x)}{(1 - x_f)^2} \cos(\mu_n \sigma) \}$$

$$= - \sum_{n=1}^{\infty} \frac{\mu_n^2}{(1 - x_f)^2} B_n \sin(\mu_n \sigma)$$

Now multiply the first of these equations by $2\cos(\lambda_m \xi)$ and then integrate with respect to ξ from 0 to 1. The second equation is multiplied with $2\sin(\mu_m \sigma)$ and integrated with respect to σ in an equal manner. The following countable set of ordinary differential equations is obtained:

$$A_m' + \eta \frac{\lambda_m^2}{x_f^2} A_m = q_1 + \sum_{n=1}^{\infty} r_{1n} A_n$$

(3.6) $m = 1,2,...$

$$B_m' + \frac{\mu_m^2}{(1 - x_f)^2} B_m = q_2 + \sum_{n=1}^{\infty} r_{2n} B_n$$

where the functions q_i are given by

$$q_1 = \frac{2(-1)^m}{\lambda_m} \{ p_f' + qx_f' + q'x_f \}$$

(3.7)

$$q_2 = \frac{2(-1)^m}{\mu_m} \{ \frac{x_f'}{1 - x_f} (p_e - p_f) - p_f' \}$$

The coefficients r_{in} are given by

$$r_{1n} = \begin{cases} -\dfrac{x_f'}{2x_f} & n = m \\[2ex] \dfrac{2x_f'}{x_f}(-1)^{n+m}\dfrac{\lambda_n\lambda_m}{\lambda_n^2 - \lambda_m^2} & n \neq m \end{cases}$$

(3.8)

$$r_{2n} = \begin{cases} -\dfrac{x_f'}{2(1 - x_f)} & n = m \\[2ex] \dfrac{2x_f'}{1 - x_f}(-1)^{n+m}\dfrac{\mu_n^2}{\mu_m^2 - \mu_n^2} & n \neq m \end{cases}$$

The Stefan condition gives the following equation for the front speed:

(3.9) $$x_f' = \varepsilon\,\{\, q - \sum_{n=1}^{\infty}\frac{\lambda_n}{x_f}(-1)^n A_n \,\}$$

The system is closed using Eq.(3.2) in the conjugation condition on the front:

(3.10) $$p_e - p_f = -\frac{M}{\varepsilon}x_f'(1 - x_f) + \sum_{n=1}^{\infty}(-1)^n \mu_n B_n$$

The appropriate initial conditions to be imposed on this set of ordinary differential equations are constructed by using Eqs.(3.2) together with the given initial conditions:

(3.11)

$0 < x < x_0$:

$$f(x_0) + q(0)(x_0 - x) + \sum_{n=1}^{\infty} A_n(0)\varphi_n(\xi_0) = f(x)$$

$x_0 < x < 1$:

$$f(x_0) + \frac{x - x_0}{1 - x_0}\{\, p_e - f(x_0)\,\} + \sum_{n=1}^{\infty} B_n(0)\psi_n(\sigma_0) = f(x)$$

where

(3.12) $$\xi_0 = \frac{x}{x_0} \qquad \sigma_0 = \frac{1 - x}{1 - x_0}$$

Again, multiplying the equations with $2\cos(\lambda_m \xi_0)$ and $2\sin(\mu_m \sigma_0)$ respectively and integrating, one finds for m = 1,2,...

$$A_m(0) = 2 \int_0^1 f(x_0 \xi) \cos(\lambda_m \xi_0)\, d\xi_0 + \frac{2}{\lambda_m} (-1)^m f(x_0) - \frac{2q(0)x_0}{\lambda_m^2}.$$

$$(3.13) \qquad B_m(0) = 2 \int_0^1 g(\sigma_0(1-x_0) + x_0) \cos(\mu_m \sigma_0)\, d\sigma_0 + \frac{2}{\mu_m} (-1)^m f(x_0) - \frac{2}{\mu_m} p_e$$

$$x_f(0) = x_0$$

$$p_f(0) = f(x_0)$$

The system of ordinary equations can now be solved numerically by truncating the infinite series after a finite number of terms. The numerical integration of the equations involve several problems that just will be pointed out here:

1) The oscillating series involved in the equations will generally converge very slowly. This is illustrated by putting $f(x) \equiv 0$ and $q(t) \equiv 1$. Combining Eqs.(3.13) and (3.9) then gives

$$(3.14) \qquad x_f'(0) = \varepsilon \left\{ 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\lambda_n} \right\} = \varepsilon \left\{ 1 - \frac{4}{\pi} \operatorname{arctg}(1) \right\} = 0$$

The series representing arctg(1) need more than 500 terms to reach 3 significant digits. It is obvious that the oscillating series have to be truncated carefully to obtain a reasonable result, and that special convergence-acceleration methods are necessary.

2) The ordinary differential equations will generally be very stiff, representing a quick damping of the coefficients A_m and B_m. This is a consequence of the fact that the liquids behave as incompressible after a short time. Special numerical methods capable of handling stiff systems must be used.

3.2 Cylindrical geometry

Now return to the cylindrical Verigin problem given in Eqs.(1.1), but assume a finite wellbore radius r_w. Write the solution of the problem as infinite series of eigenfunctions, and assume that these can be differentiated and integrated term by term:

$$p_w = p_f - q \ln \frac{r}{r_f} + \sum_{n=1}^{\infty} A_n(t)\varphi_n(\xi)$$

$$p_o = p_f + \sum_{n=1}^{\infty} B_n(t)\psi_n(\sigma)$$

(3.15)

$$\xi = \frac{r - r_w}{r_f - r_w} \qquad\qquad r_w < r_f < r_e$$

$$\sigma = \frac{r_e - r}{r_e - r_f}$$

The general form of the cylindrical eigenfunctions is

$$\varphi_n(\xi) = a_n J_0(\alpha_n \xi) + b_n Y_0(\alpha_n \xi)$$

(3.16)

$$\Psi_n(\sigma) = c_n J_0(\beta_n \sigma) + d_n Y_0(\beta_n \sigma)$$

These functions have singularities for $r = r_w$ and $r = r_e$, and it is generally not possible to satisfy boundary conditions at such points. To determine the coefficients and eigenvalues, the exact condition must be replaced with a restriction that the solution is finite in the singularities. This would have been the case if a constant-pressure outer boundary had been used, but for the present case where a no-flux condition is specified on the outer boundary, all the boundary conditions can be satisfied by choosing the eigenfunctions as

$$\varphi_n(\xi) = J_0(\alpha_n \xi)$$

(3.17) $$\Psi_n(\sigma) = J_0(\beta_n \sigma) \qquad\qquad n = 1,2,\ldots$$

$$J_0(\alpha_n) = 0 \qquad\qquad \alpha_n = \beta_n$$

Now insert Eqs.(3.15) and (3.17) into the partial differential equations as for the linear geometry. Multiply these equations with

$2zJ_0(\alpha_m z)/J_1^2(\alpha_m)$ where z is ξ and σ respectively, and then integrate with respect to z from 0 to 1. The orthogonality properties of the Bessel functions then yield the following system of ordinary differential equations:

$$A'_m + \eta \frac{\alpha_m^2}{(r_f - r_w)^2} A_m = q_1 + \sum_{n=1}^{\infty} r_{1n} A_n$$

(3.18) $m = 1,2,\ldots$

$$B'_m + \frac{\alpha_m^2}{(r_e - r_f)^2} B_m = q_2 + \sum_{n=1}^{\infty} r_{2n} B_n$$

where now

$$q_1 = -\left(p'_f + q \frac{r'_f}{r_f} + q' \ln r_f \right) \frac{2}{\alpha_m J_1(\alpha_m)}$$

$$+ \frac{2q'}{J_1^2(\alpha_m)} \int_0^1 \xi J_0(\alpha_m \xi) \ln[r_w + \xi(r_f - r_w)]\, d\xi$$

$$q_2 = -\frac{2p'_f}{\alpha_m J_1(\alpha_m)}$$

(3.19)
$$r_{1n} = \frac{2\alpha_n}{J_1^2(\alpha_m)} \left\{ -\frac{r'_f}{r_f - r_w} \int_0^1 \xi^2 J_0(\alpha_m \xi) J_1(\alpha_n \xi)\, d\xi \right.$$

$$\left. + \frac{\eta}{(r_f - r_w)^2} \int_0^1 \frac{r_w J_1(\alpha_n \xi) J_0(\alpha_n \xi)}{r_w + (r_f - r_w)\xi}\, d\xi \right\}$$

$$r_{2n} = \frac{2\alpha_n}{J_1^2(\alpha_m)} \left\{ -\frac{r'_f}{r_e - r_f} \int_0^1 \sigma^2 J_0(\alpha_m \sigma) J_1(\alpha_n \sigma)\, d\sigma \right.$$

$$\left. + \frac{1}{(r_e - r_f)^2} \int_0^1 \frac{r_f J_0(\alpha_m \sigma) J_1(\alpha_n \sigma)}{r_e - (r_e - r_f)\sigma}\, d\sigma \right\}$$

When $m \neq n$, the first integral in r_{in} $(i=1,2)$ can be found by using the following integral, given by Ref.[50]:

$$\int xJ_0(\alpha x)J_0(\beta x)\,dx =$$

$$(3.20) \qquad (\beta^2 - \alpha^2)^{-1}[\beta xJ_0(\alpha x)J_1(\beta x) - \alpha xJ_0(\beta x)J_1(\alpha x)]$$

α and β is arbitrary parameters, $\alpha \neq \beta$. Differentiating both sides with respect to β gives

$$-\int x^2 J_0(\alpha x)J_1(\beta x)\,dx =$$

$$(3.21) \qquad (\beta^2 - \alpha^2)^{-1}[\beta x^2 J_0(\alpha x)J_0(\beta x) + \alpha x^2 J_1(\alpha x)J_1(\beta x)]$$

$$- 2\beta(\beta^2 - \alpha^2)^{-2}[\beta xJ_0(\alpha x)J_1(\beta x) - \alpha xJ_1(\alpha x)J_0(\beta x)]$$

Putting $\alpha = \alpha_m$, $\beta = \alpha_n$, and using the definition of α_i, the following result is obtained:

$$(3.22) \qquad \int_0^1 x^2 J_0(\alpha_m x)J_1(\alpha_n x)\,dx = \frac{\alpha_m}{\alpha_m^2 - \alpha_n^2} J_1(\alpha_n)J_1(\alpha_m) \qquad m \neq n$$

When $m = n$, the integral is found by the simple substitution $u = xJ_1(\alpha_n x)$:

$$(3.23) \qquad \int_0^1 x^2 J_0(\alpha_n x)J_1(\alpha_n x)\,dx = \frac{1}{2\alpha_n} J_1^2(\alpha_n)$$

No explicit expressions have been found for rest of the integrals involved in the computation of the coefficients r_{in} and q_i. Consequently, in a numerical solution these integrals will have to be calculated numerically. Since this calculation must be done in <u>each</u> time step, it is obvious that the numerical solution will demand an insurmountable amount of work.

The Stefan condition now gives

$$(3.24) \qquad r_f' = \varepsilon \left\{ \frac{q(t)}{r_f} - \sum_{n=1}^{\infty} \alpha_n A_n \frac{J_1(\alpha_n)}{r_f - r_w} \right\}$$

For the linear case, the coupling between the coefficients A_n and B_n was given through the equation for the front pressure, Eq.(3.10), an equation derived from the conjugation condition. For the present

problem this condition only gives a relationship between infinite series involving these coefficients:

$$(3.25) \qquad Mq + M \sum_{n=1}^{\infty} \frac{\alpha_n A_n r_f}{r_f - r_w} J_1(\alpha_n) + \sum_{n=1}^{\infty} \frac{\alpha_n B_n r_f}{r_e - r_f} J_1(\alpha_n) = 0$$

The difference between the two cases is not due to the difference in geometry, but rather to the difference in boundary conditions. A similar summed form as Eq.(3.21) would have been found for the linear case if a closed outer boundary had been chosen. This form makes the numerical solution much more complicated than in cases where p_f are explicitly given, as in Eq.(3.10).

In the assumed form of the solution, Eqs.(3.15), terms had to be included to make the eigenfunctions satisfy homogeneous boundary conditions. These terms are of course not unique, several choices are possible. A form of the solution which could eliminate the problem with the summed form of the conjugation condition can be sought. If such a solution form exists, however, it will probably involve more integrals which cannot be calculated analytically.

4. A QUASI-STATIONARY METHOD

4.1 Discussion of the method

A quasi-stationary method for solution of free-boundary problems was introduced by Leibenzon [34]. It has not been possible to identify the details of his work, a study of the molten centre of the earth. The method is, however, reviewed by Rubinstein [2] together with several applications. Among the problems the method is applied to, are crystallization of a melt, dissolution of a gas bubble in liquid, and a three-zone Verigin problem with linear geometry.

The quasi-stationary method is based on the following algorithm:

1) Solve the associate problem with a <u>stationary</u> boundary between the zones. Let the solution of this problem be $u_i = u_i(r,t;y)$ where $r = y$ is the position of the stationary boundary.

2) Use the solution u_i in the Stefan condition to construct an explicit equation for $\varrho(t)$, which is an approximation to the position of the <u>moving</u> boundary:

$$\varrho' = -\varepsilon \frac{\partial u}{\partial r}1(\varrho,t;\varrho) \tag{4.1}$$

3) Substitute $y = \varrho(t)$ into u_i and use this as an approximation for the solution to the free-boundary problem:

$$\begin{aligned} p_w(r,t) &\approx u_1(r,t;\varrho(t)) \\ p_o(r,t) &\approx u_2(r,t;\varrho(t)) \end{aligned} \tag{4.2}$$

No criterium which can be used for testing the validity of the algorithm has been found in the literature. After comparing results produced by the method with numerical solutions, Rubinstein states that the method "gives a qualitatively correct result, although

quantitatively it contains errors."

Eqs.(4.1) and (4.2) clearly show that a basic assumption in the method is that the movement of the front does not change the gradients compared to the problem with stationary front. Only the time dependency of the two solutions is unequal:

$$(4.3)\qquad \frac{\partial p_w}{\partial r} \approx \left[\frac{\partial u_1}{\partial r}\right]_{y=\varrho} \qquad \frac{\partial p_o}{\partial r} \approx \left[\frac{\partial u_2}{\partial r}\right]_{y=\varrho}$$

Both $p_w(r,t)$ and $u_1(r,t;y)$ are solutions of the same diffusion equation, and consequently

$$L[p_w] = \left\{ \frac{\partial}{\partial t} - \frac{\eta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) \right\} p_w = 0$$

$$(4.4)\qquad L[u_1(r,t;\varrho(t))] = \{ L\,[u_1(r,t;y)] \}_{y=\varrho(t)} + \varrho'\left\{ \frac{\partial u_1}{\partial y}(r,t;y) \right\}_{y=\varrho(t)} = \varrho'\left\{ \frac{\partial u_1}{\partial y}(r,t;y) \right\}_{y=\varrho(t)} \equiv \sigma(r,t)$$

Obviously, the approximation $p_w(r,t) \approx u_1(r,t;\varrho)$ is only valid if the term $\sigma(r,t)$ can be neglected in the diffusion equation for the free-boundary problem. If the solution to the problem with stationary boundaries can be found, all the variables needed to calculate σ are known. However, there is still a problem with what criterium to use when deciding whether or not the term really can be neglected. No general criterium has been found, but it will be shown later how σ can be used to predict the validity of the results in the case studied.

The Verigin problem involves at least two time scales; the first is a fast scale corresponding to diffusion, $t_1 \sim t$, the second a slower connected to the moving front, $t_2 \sim \varepsilon t$. A multiple scale singular perturbation technique should thus be adequate for studying the problem, but it is not clear how this technique can be applied. A comparison between the Verigin solution and the associate u_i shows a significant difference in the numerical values of the two solutions, quantitatively it contains errors.

Eqs.(4.1) and (4.2) clearly show that a basic assumption in the method is that the movement of the front does not change the gradients compared to the problem with stationary front. Only the time dependency of the two solutions is unequal:

$$\frac{\partial \rho_w}{\partial r} = \left(\frac{\partial u_1}{\partial r} \right)_{y = g}$$

(4.3)

$$\frac{\partial \rho_c}{\partial r} = \left(\frac{\partial u_2}{\partial r} \right)_{y = g}$$

Both $\rho_w(r,t)$ and $u_1(r,t;y)$ are solutions of the same diffusion equation, and consequently

$$L[\rho_w] = \left(\frac{\partial}{\partial t} - \frac{a}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) \right) \rho_w = 0$$

$$L[u_1(r,t;g(t))]$$

(4.4)

$$= \left(L\,(u_1(r,t;y)) \right)_{y=g(t)} + g' \left(\frac{\partial u_1}{\partial y}(r,t;y) \right)_{y=g(t)}$$

$$= g' \left(\frac{\partial u_1}{\partial y}(r,t;y) \right)_{y=g(t)} \equiv Q(r,t)$$

Obviously, the approximation $\rho_w(r,t) \approx u_1(r,t;g)$ is only valid if the term $Q(r,t)$ can be neglected in the diffusion equation for the free-boundary problem. If the solution to the problem with stationary boundaries can be found, all the variables needed to calculate Q are known. However, there is still a problem with what criterium to use when deciding whether or not the term really can be neglected. No general criterium has been found, but it will be shown later how Q can be used to predict the validity of the results in the case studied.

The Verigin problem involves at least two time scales; the first is a fast scale corresponding to diffusion, $t_1 = t$, the second a slower connected to the moving front, $t_2 = \varepsilon t$. A multiple scale singular perturbation technique should thus be adequate for studying the problem, but it is not clear how this technique can be applied. A comparison between the Verigin solution and the quasistatic u_1 shows a significant difference in the numerical values of the two solutions.

the effect of the moving front is not merely a small perturbation of the stationary-front solution.

This chapter applies the quasi-stationary method to several problems encountered in injection well testing. A quasi-stationary approach will also be applied for estimating the validity of the different expressions developed for the wellbore pressure, i.e. validity limits will be constructed using the results in Appendix 2, replacing the parameter y with an approximation of the water-front position. Following the given algorithm, one first has to solve the associate problem with stationary boundaries. The next two sections will discuss this problem in detail.

4.2 An infinite reservoir with a lateral discontinuity in mobility and diffusivity.

Now return to the problem given in Eqs.(1.1) and let $r_e = \infty$. If in addition, the Stefan condition and the time dependency of r_f are dropped, the equations decribe the pressure in an infinite reservoir with a stationary discontinuity in mobility and diffusivity. Let r = y be the position of the discontinuity, and let u_1 and u_2 be the solution in the inner and the outer zone respectively.

The described problem is encountered when testing a reservoir with a lateral change in permeability, fluid properties etc. as discussed by several papers in the petroleum literature. An exact analytical solution is given by Hurst [35] together with a simple approximate solution valid for large time, both solutions restricted to the case $M = \eta$. Based on this work and a paper by Larkin [36], Bixel and van Poollen [37] generalize the solutions to cases where $M \neq \eta$. The derivation of the solutions can be reviewed briefly in the following way:

Let $\bar{u}_i(r;z)$ be the Laplace transform of the solution, where z is the Laplace variable. Solving the transformed equations with the associate boundary conditions, $\bar{u}_i$ are found to be

$$\bar{u}_1 = \frac{1}{z}K_0\left(\sqrt{\frac{z}{\eta}}\,y\right) + \frac{K_1\left(\sqrt{\frac{z}{\eta}}\,y\right)K_0(\sqrt{z}y) - \frac{\sqrt{\eta}}{M}K_0\left(\sqrt{\frac{z}{\eta}}\,y\right)K_1(\sqrt{z}y)}{z\left[I_1\left(\sqrt{\frac{z}{\eta}}\,y\right)K_0(\sqrt{z}y) + \frac{\sqrt{\eta}}{M}I_0\left(\sqrt{\frac{z}{\eta}}\,y\right)K_1(\sqrt{z}y)\right]}\,I_0\left(\sqrt{\frac{z}{\eta}}\,r\right)$$

(4.5)

$$\bar{u}_2 = \frac{\sqrt{\eta}}{z^{3/2}y\left[\,I_1\left(\sqrt{\frac{z}{\eta}}\,y\right)K_0(\sqrt{z}y) + \frac{\sqrt{\eta}}{M}I_0\left(\sqrt{\frac{z}{\eta}}\,y\right)K_1(\sqrt{z}y)\,\right]}\,K_0(\sqrt{z}r)$$

The inversion integral for Laplace transform can be used to give

$$u_1 = \frac{4M}{\pi^2 y^2}\int_0^{\infty}\frac{1 - e^{-s^2 t}}{s^3}\cdot\frac{J_0\left(\frac{sr}{\sqrt{\eta}}\right)}{M_1^2 + N_1^2}\,ds$$

$$u_2 = \frac{2M}{\pi y}\int_0^{\infty}\frac{1 - e^{-s^2 t}}{s^2}\cdot\frac{M_1Y_0(sr) - N_1J_0(sr)}{M_1^2 + N_1^2}\,ds$$

(4.6)

$$M_1 = \frac{M}{\sqrt{\eta}}J_1\left(\frac{sy}{\sqrt{\eta}}\right)J_0(sy) - J_0\left(\frac{sy}{\sqrt{\eta}}\right)J_1(sy)$$

$$N_1 = \frac{M}{\sqrt{\eta}}J_1\left(\frac{sy}{\sqrt{\eta}}\right)Y_0(sy) - J_0\left(\frac{sy}{\sqrt{\eta}}\right)Y_1(sy)$$

The <u>approximate</u> solution is found by expanding the modified Bessel functions in Eqs.(4.5) for small values of the arguments:

$$\bar{u}_1 \approx \frac{K_0\left(\sqrt{\frac{z}{\eta}}\,r\right)}{z} - \frac{K_0\left(\sqrt{\frac{z}{\eta}}\,y\right)}{z} + M\,\frac{K_0(\sqrt{z}y)}{z}$$

(4.7)

$$\bar{u}_2 \approx \frac{K_0(\sqrt{z}r)}{z}$$

These expressions can be inverted according to the table of Laplace transforms in Appendix 4. Using an asymptotic property of the Laplace transform [38], the result is valid for large values of t:

(4.8) $$u_1 \approx -\frac{1}{2}Ei\left(-\frac{r^2}{4\eta t}\right) + \frac{1}{2}Ei\left(-\frac{y^2}{4\eta t}\right) - \frac{M}{2}Ei\left(-\frac{y^2}{4t}\right) \equiv U(r,t;y)$$

$$(4.8\text{cont}) \qquad u_2 \approx -\frac{M}{2}Ei\left(-\frac{r^2}{4t}\right)$$

The approximate solution has a form very similiar to the Verigin solution Eq.(1.3). The time dependency of the two solutions is, however, quite different, as can be seen from figure 2.

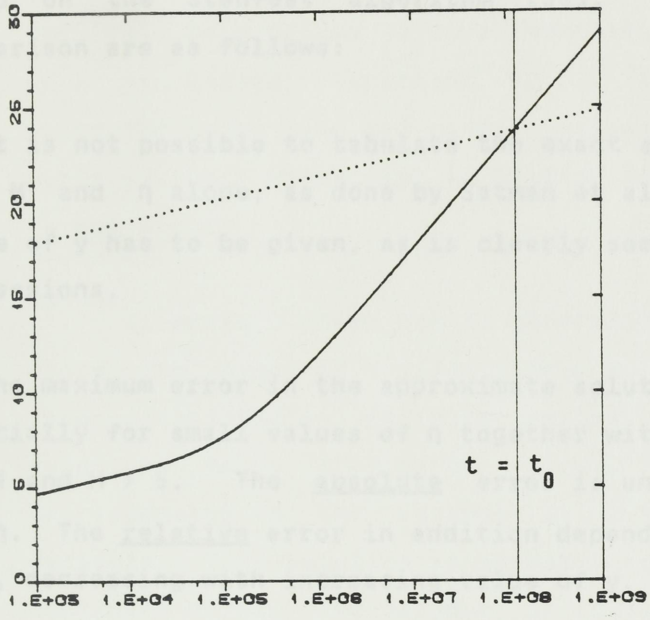


Fig.2:

Comparison between the Verigin solution (dottet line) and U.

$M = \eta = 4$

$\varepsilon = 0.001$

$r = 1 \qquad y = 500$

$t_0 = y^2/2\varepsilon$

If $y^2 >> \max(4t, 4\eta t)$, the last two exponential integrals in the expression for U can be neglected. In addidition, if $r^2 < \eta r_c^2$, where r_c is the the radius of incompressibility defined in Appendix 1, the first exponential integral can be approximated by a logarithm:

$$(4.9) \qquad U(r,t;y) \approx \frac{1}{2}\ln\left(\frac{4\eta t}{r^2} e^{-\gamma}\right) \equiv u_h(r,\eta t)$$

Note that the condition used to derive this contradicts the one used to derive $u_1 \approx U$, and it is consequently not obvious that $u_1 \approx u_h$.

When $\max(r^2, y^2, \eta y^2) < \eta r_c^2$, the logarithmic approximation can be used for all the three terms:

$$(4.10) \qquad U(r,t;y) \approx \ln\frac{y}{r} + \frac{M}{2}\ln\frac{t}{y^2} + \frac{M}{2}\ln(4e^{-\gamma})$$

Both Eqs. (4.9) and (4.10) describe a stable situation where constant-pressure "fronts" are moving outwards in the reservoir with constant

speed. (Remember that the expression "constant speed" involves cylindrical decay, as stated in the Introduction.) Between these imaginary fronts and the well, the liquid behaves as incompressible.

The relationship between the exact u_i and the approximate solutions has been investigated for r = 1, the details are shown in Appendix 2. The exact solution was represented by a numerical solution based on the Stehfest algorithm [39]. The conclusions from the comparison are as follows:

1) It is not possible to tabulate the exact solution as a function of t_y, M and η alone, as done by Satman et al. [40]. In addition, the value of y has to be given, as is clearly seen from the approximate expressions.

2) The maximum error in the approximate solution U can be quite large, especially for small values of η together with large values of M, say $\eta < 1$ and $M > 5$. The <u>absolute</u> error is uniqely determined by t_y, M and η. The <u>relative</u> error in addition depends on the absolute value of y, decreasing with increasing value of y.

3) For small values of t, u_1 can be approximated both by u_h and U, despite the fact that Eq.(4.8) was derived as an asymptotic expression. For most values of M and η, the error in $u_1 \approx U$ is mainly localized in the t_y-interval $(1/10\eta, 25)$. When t is small, $u_1 \approx u_h$ is the better approximation, generally valid for $t_y < \pi/10\eta$. The error in the expression on the right hand side of Eq.(4.10), compared to the exact solution, can be both smaller and larger than the error in U, depending on M and η. As a general rule, the lower limit $t_y = 25$ will be used also for the validity of this approximation.

The limits given for the validity of the different approximate expressions are based on the concepts of drainage radius and radius of incompressibility, defined in Appendix 1. For most values of M and η, the error was found to be less than 1% within the given limits. However, larger error may exist also within these bounds, for instance when $\eta < 1$ together with $M > 5$. It must be emphasized that they should only be used as a rough rule of thumb and <u>not</u> as basis for estimating the position of the discontinuity.

4.3 A finite reservoir with a lateral discontinuity in mobility and diffusivity

An analytical solution describing a finite reservoir with a lateral discontinuity is given by Carter [41] for the case $M = \eta$. Again, the Laplace transform was used to construct the solution. In Ref.[42], Odeh claims to have found this solution independently of the work of Carter. Hopkinson et al. [49] give an approximate solution valid for large time and general values of the parameters, but as parts of their manuscript is written by hand, details in this solution is not clear.

Appendix 3 shows how to generalize Carter's solution to cases where the value of M differs from η. The solution has a complicated form, containing an infinite series of residues. An approximate solution, valid for large values of t, can be found as the term corresponding to the residue in $z = 0$, where z is the Laplace variable. This solution is probably identical to the one presented by Hopkinson et al. The residue in $z = 0$ is given by Eqs.(A3.7) - (A3.9), and yields for the pressure in the water zone:

$$(4.11)\qquad u_1(r,t;y) \approx C\frac{2Mt}{r_e^2} + \ln\frac{y}{r}$$

$$+ C\left[-M\ln\frac{y}{r_e} + \frac{M}{2\eta}(\eta-1)\left(\frac{y}{r_e}\right)^2 - \frac{M}{2} + \frac{M}{2\eta}\left(\frac{r}{r_e}\right)^2\right]$$

$$+ C^2\left[\frac{M}{\eta}(M-\eta)\left(\frac{y}{r_e}\right)^2\ln\frac{y}{r_e} + \frac{M}{2\eta}(M-1)\left(\frac{y}{r_e}\right)^2 - \frac{M}{4\eta}\left(\frac{y}{r_e}\right)^4\left\{(M-1) + (M-\eta)\left(1+\frac{1}{\eta}\right)\right\} - \frac{M}{4}\right]$$

The factor C is defined as

$$(4.12)\qquad \frac{1}{C} = 1 + \left(\frac{M}{\eta} - 1\right)\left(\frac{y}{r_e}\right)^2$$

The product Mt in the first term of Eq.(4.11) is somewhat misleading, the factor M only being a consequence of the scaling of the variables.

Returning to the physical variables, this term vill have the form $CT/c_0R_e^2$ corresponding to usual depletion as in the homogeneous case. As the time dependency is isolated to this term, the inner zone only appears as a constant skin in the solution. If the values of M and η not differ too much and in addition $y/r_e \ll 1$, then $C \approx 1$, and Eq.(4.11) may be further simplified:

$$(4.13) \qquad u_1(r,t;y) \approx C\frac{2Mt}{r_e^2} + \ln\frac{y}{r} - M\ln\frac{y}{r_e} - \frac{3M}{4}$$

In this case, the equation shows that the inner zone behaves as incompressible, just as in the infinite-acting period.

An approximate solution similiar to Eq.(4.8) can also be constructed for a finite reservoir. First define the function $\overline{w}$ by the following equation:

$$(4.14) \qquad \overline{w} = \frac{1}{z}K_0\left(\sqrt{\frac{z}{\eta}}\,r\right) - \frac{1}{z}K_0\left(\sqrt{\frac{z}{\eta}}\,y\right) + \frac{CM}{z}K_0(\sqrt{z}y) + \frac{2CM}{z^{3/2}r_e}K_1(\sqrt{z}r_e)$$

Using the expansion of the modified Bessel functions for small values of the argument, it is possible to show that

$$(4.15) \qquad \overline{u}_1 = \overline{w} + \frac{C}{z}\left[\frac{M}{2\eta}(\eta-1)\left(\frac{y}{r_e}\right)^2 - \frac{M}{2} + \frac{M}{2\eta}\left(\frac{r}{r_e}\right)^2\right]$$

$$+ \frac{C^2}{z}\left[\frac{M}{\eta}(M-\eta)\left(\frac{y}{r_e}\right)^2\ln\frac{y}{r_e} + \frac{M}{2\eta}(M-1)\left(\frac{y}{r_e}\right)^2 - \frac{M}{4\eta}\left(\frac{y}{r_e}\right)^4\left\{(M-1) + (M-\eta)\left(1+\frac{1}{\eta}\right)\right\} - \frac{M}{4}\right]$$

$$+ O(z^0)$$

Remember that u_1 has a double pole in $z = 0$. Hence, if terms of order higher than $1/z^2$ are neglected, $\overline{u}_1 \approx \overline{w}$ for small z. The asymptotic property of the Laplace transform gives $u_1 \approx w$ valid for large t, where w is the inverse transform of $\overline{w}$. From the table of Laplace transforms in Appendix 4, it follows that

$$w = -\frac{1}{2}Ei(-\frac{r^2}{4\eta t}) + \frac{1}{2}Ei(-\frac{y^2}{4\eta t})$$

(4.16)

$$+ C[-\frac{M}{2}Ei(-\frac{y^2}{4t}) + \frac{M}{2}Ei(-\frac{r_e^2}{4t}) + \frac{2Mt}{r_e^2}\exp(-\frac{r_e^2}{4t})]$$

For small values of t, where the last two terms can be neglected, this expression differs from U defined in Eq.(4.8) by the factor C. For large t, Eq.(4.16) can be simplified further to

$$(4.17) \quad w \approx C\frac{2Mt}{r_e^2} + \ln\frac{y}{r} - MC\ln\frac{y}{r_e}$$

This should be compared with the approximate solutions in Eqs.(4.11) and (4.13). All the expressions show the same time dependency, but differs in constant values.

For $M = \eta = 1$ Eq.(4.16) reduces to an approximate solution for a finite homogeneous reservoir first given by Horner [43] who derived the expression from physical arguments.

A systematic analysis of the error in approximate solutions for a finite reservoir must include variation in three parameters; M, η and y/r_e and will consequently be rather laborious. When $C \approx 1$, Eq.(4.16) behaves approximatly as the solution U in the infinite-acting period, the error of which was investigated in Appendix 2. For large values of t, the absolute error in the approximate expressions Eqs.(4.13) and (4.17) are constant and can be found by comparing with the asymptotic solution, Eq.(4.11).

4.4 Injection into an infinite reservoir.

With basis in the solutions found in Secs. 4.2 and 4.3, we are now capable of proceeding with the second and third step in the quasi-stationary method. In the next sections, this will be done for different problems encountered in injection well testing. To test the the validity of the method, it will first be applied to injection into an infinite reservoir, where the exact Verigin solution, Eq.(1.3), is known.

To describe the pressure in an infinite reservoir with a <u>stationary</u> discontinuity, the solution U(r,t;y) given in Eq.(4.8) will be used. Substituting this into Eq.(4.1) yields the following equation for an approximation of the front speed:

$$\varrho\varrho' = -\varepsilon\varrho\frac{\partial U}{\partial r}(\varrho,t;\varrho) = \varepsilon\exp(-\frac{\varrho^2}{4\eta t}) \qquad (4.18)$$

Comparing this equation with Eq.(1.4) found by Verigin, it is seen that Eq.(4.18) is exact, although it is not imposed here that ϱ must have the form $\varrho^2 = gt$.

An approximate solution for the pressure in the water zone is now constructed as $p_w \approx U(r,t;\sqrt{gt})$:

$$(4.19) \qquad U(r,t;\sqrt{gt}) = -\frac{1}{2}Ei(-\frac{r^2}{4\eta t}) + \frac{1}{2}Ei(-\frac{g}{4\eta}) - \frac{M}{2}Ei(-\frac{g}{4})$$

This expression differs from the Verigin solution only by an exponential factor. When ε is small, this factor is approximatly equal to 1, and the solutions are identical.

It was found in Sec. 4.2 that the expression $U(1,t;y)$ can be used for the wellbore pressure in a reservoir with a stationary discontinuity if $t_y > 25$. Since now $t_y = t/r_f^2 \approx 1/2\varepsilon$, the use can be defended for oil/water where ε is of order 0.01-0.001. If ε increases, both the error in $u_1(r,t;\sqrt{gt}) \approx U(r,t;\sqrt{gt})$ and the difference between the Verigin solution and the solution found through the quasi-stationary method will grow.

Knowing ϱ and $u_1 \approx U$, it is now possible to calculate σ defined in Eq.(4.4):

$$\sigma = L[\,U(r,t;\varrho)\,] = \varrho'\{\frac{\partial}{\partial y}U(r,t;y)\}_{y=\varrho(t)}$$

(4.20)

$$= \frac{1}{2t}[\,\exp(-\frac{g}{4\eta}) - M\exp(-\frac{g}{4})\,]$$

Obviously, σ tends to zero as t increases. Also $U(r,t;y)$ is only an <u>approximate</u> solution of the equation $L[u_1] = 0$, and applying the diffusivity operator to U gives

$$(4.21)\qquad L\,[U(r,t;y)] = -\frac{1}{2t}[\,\exp(-\frac{y^2}{4\eta t}) - M\exp(-\frac{y^2}{4t})\,]$$

When $y = r_f$, the right hand side is equal to σ, except for the opposite sign. Consequently, the quasi-stationary method does not seem to introduce larger error than already introduced by using the approximation $u_1 \approx U$. Without knowledge of the Verigin solution, one should in this manner still have been able to predict the successful result of the quasi-stationary method when the Peclet number ε is small.

The previous discussion can be used to enlighten some results presented by H.J.Ramey in Ref.[44]. In this paper, Ramey used a solution for an infinite reservoir with a lateral discontinuity to study a finite homogenous reservoir. A no-flow boundary was modelled by letting the mobility in the outer region, λ_0, tend to zero and a constant-pressure boundary by letting λ_0 tend to infinity. Starting with the <u>Verigin solution</u>, Ramey found approximate solutions valid both in the transient and (semi-)stationary period. For the no-flux case, this solution is equal to Eq.(4.16) with $M = \eta = 1$. The calculations included <u>two</u> limiting operations; one involving λ_0 and a second involving the front speed. In the Verigin solution, the moving

boundary was replaced by a stationary, i.e. the front speed was put equal to zero, but it is not clear from Ramey's work why this could be done. Obviously, Ramey is doing the opposite of what is done in the quasi-stationary method, and the successful result is due to the close relationship between the Verigin solution and the solution for the problem with a stationary boundary.

For the constant-pressure case, the solution could just as well have been found starting with the approximate solution U instead of the Verigin solution. The no-flux case is somewhat more complicated, as will be described below. In both cases, the time variable t should be rescaled using the properties in the inner zone before the limit operation on λ_0.

Starting from the Verigin solution, Ramey retains the exponential factor in this solution, a factor not present in U. For the no-flux case, this factor turns out to be critical in the limit operation $\lambda_0 \to 0$, and the desired result can not be derived from U. A closer look at the restriction imposed on the time scale in the derivation of U also reveals that this solution is not valid when λ_0 tends to zero. U is an asymptotic expression, and in the derivation it was assumed that the arguments of the modified Bessel functions in Eq.(4.5) were small. Remembering that rescaling the time causes a rescaling of the Laplace variable z, it is easy to shown that this assumption is not justified as $\lambda_0 \to 0$.

For both types of boundary conditions, the limit operation on λ_0 may be carried through on the rescaled equivalent of the exact Laplace transform, Eq.(4.5), instead of using the Verigin solution or the solution U. The limit thus found is, however, just the transform of the exact solution; the pressure in a finite homogeneous reservoir. The expression can be simplified and then inverted to give the desired solution in the same manner as the solution w was derived in Sec.4.3.

4.5 Injection into a infinite reservoir with an initial water bank.

Now let $r_f(0) = r_0 > 0$, i.e. the reservoir has an initial water bank. The solution U can still be used as basis for the quasi-stationary method, and consequently, the approximation for the front speed is still given by Eq.(4.18). This equation was found to be exact when $r_f(0) = 0$, but a priori nothing is known about the validity in the present case.

Let the function φ be defined by the following equation:

$$(4.22) \qquad \varrho^2 = r_0^2 + \varphi(t)$$

Inserted into the front-speed equation, Eq.(4.18), this gives

$$(4.23) \qquad \varrho\varrho' = \varepsilon \exp[-\frac{r_0^2}{4\eta t} - \frac{\varphi(t)}{4\eta t}]$$

Assume the variation in φ with t to be of order ε. For small values of t, such that $4\eta t << r_0^2$, Eq.(4.22) and Eq.(4.23) may be consistently approximated by $\varrho \approx r_0$ and $\varrho' \approx 0$ respectively. Further, the quasi-stationary method gives for the pressure in the wellbore:

$$(4.24) \qquad p_w \approx U(1,t;r_0) \approx -\frac{1}{2}Ei(-\frac{1}{4\eta t})$$

The exponential integrals with arguments containing r_0 have been neglected as a consequence of the assumption $4\eta t << r_0^2$. Eq.(4.24) simply describes the situation before the oil zone outside the water bank is "felt" in the wellbore pressure. The pressure response in the reservoir has not yet accelerated the front between the water and oil. Based on the results found in Appendix 2, an upper limit for the validity of Eq.(4.24) is given by the concept of drainage radius:

$$(4.25) \qquad r_d^2(\eta t_A) = 10\eta t/\pi = r_0^2$$

Now assume that t is large enough to neglect $r_0^2/4\eta t$ compared to $\varphi/4\eta t$. If ϱ is assumed to have the form $\varrho^2 = \varrho_0^2 + gt$, ϱ_0 and g being

constants, a sufficient condition for this is that $r_0^2 \ll gt$. Remember that g is of order ε, hence t generally must be very large to satisfy this condition. The constant g can be determined from Eq.(4.18); neglecting terms with argument ϱ_0^2/t one finds that g must satisfy Eq.(1.4). Consequently, $g \approx 2\varepsilon$ for small values of ε, and the wellbore pressure is given by the following equation, identical to Eq.(4.19):

$$(4.26) \qquad p_w \approx U(1,t;\varrho) \approx U(1,t;\sqrt{gt})$$

Again terms with argument ϱ_0^2/t have been neglected in the last approximation. For large values of time, the equations for the front speed and for the wellbore pressure are thus both found to be independent of whether an initial water bank is present or not.

As a lower limit for the validity of Eq.(4.26), the value given by $2\varepsilon t_B = 25r_0^2$ is proposed:

$$(4.27) \qquad t_B = \frac{12.5\, r_0^2}{\varepsilon}$$

The values given for the validity of the different approximations will be further discussed in Chapter 5, with background in the numerical simulations.

4.6 Use of superposition to describe pressure during falloff

The quasi-stationary method may be applied directly to derive an expression for the pressure during falloff. However, an important problem in the modelling of the falloff period has been the use of the principle of superposition, and the use of this will be discussed before the quasi-stationary method is applied to a general change in rate in the next section.

constants, a sufficient condition for this is that $r_0^2 \ll gt$. Remember that g is of order ε, hence t generally must be very large to satisfy this condition. The constant g can be determined from Eq.(4.18); neglecting terms with argument ρ_0^2/t one finds that g must satisfy Eq.(4.4). Consequently, g ≈ 2ε for small values of ε, and the wellbore pressure is given by the following equation, identical to Eq.(4.19):

$$(4.25) \quad p_w = U(1,t;g) = U(1,\varepsilon;\sqrt{gt})$$

Again terms with argument ρ_0^2/t have been neglected in the last approximation. For large values of time, the equations for the front speed and for the wellbore pressure are thus both found to be independent of whether an initial water bank is present or not.

As a lower limit for the validity of Eq.(4.26), the value given by $2\varepsilon t_e = 25 r_0^2$ is proposed:

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The principle of superposition is strictly valid only if the problem is linear. This will again be the case only if the water front is assumed to halt immediately at shut-in. The problem is then to find expressions that can be used to construct the total solution. Due to the close relationship between the Verigin solution and the solution for a reservoir with a stationary discontinuity, such expressions are now readily found. The basic assumption that the the water front is stationary after shut-in will be discussed in the next section by using the Stefan condition.

Only falloff in an infinite reservoir will be studied. Let the well be closed at time $t = t_s$ and let the front position at this time be $y = r_f(t_s) \approx \sqrt{2\varepsilon t_s}$. The pressure during falloff can be described as a superposition of the solutions to the following problems:

1) Find the pressure in an infinite reservoir with a stationary discontinuity at $r = y$. Reservoir fluid is injected with rate $q = 1$ from time $\Delta t = t - t_s = 0$, and the pressure distribution at $\Delta t = 0$ is given by the Verigin solution $V(r,t_s)$.

2) Find the pressure in an infinite reservoir with a stationary discontinuity at $r = y$. Fluid is produced with rate $(-q)$ from time $\Delta t = 0$, and the pressure distribution at $\Delta t = 0$ is identical zero.

Both these problems are <u>purely mathematical</u> and do only involve one fluid; the reservoir fluid. An approximate solution to the first problem is given by $U(r,t;y)$, this because $U(r,t_s;y) \approx V(r,t_s)$ when ε is small. The solution of the second problem can be approximated by $-U(r,\Delta t;y)$. The principle of superposition then gives:

$$p_w(1,\Delta t) \approx U(1,t;\sqrt{2\varepsilon t_s}) - U(1,\Delta t;\sqrt{2\varepsilon t_s})$$

$$(4.28) \qquad \approx -\frac{1}{2}Ei(-\frac{1}{4\eta t}) + \frac{1}{2}Ei(-\frac{\varepsilon t_s}{2\eta t}) - \frac{M}{2}Ei(-\frac{\varepsilon t_s}{2t})$$

$$+ \frac{1}{2}Ei(-\frac{1}{4\eta\Delta t}) - \frac{1}{2}Ei(-\frac{\varepsilon t_s}{2\eta\Delta t}) + \frac{M}{2}Ei(-\frac{\varepsilon t_s}{2\Delta t})$$

For all practical applications, the logarithmic approximation can be used for the first four exponential integrals:

$$p_w \approx (1 - M)\ln y - \frac{1}{2}\ln(\eta\Delta t) + \frac{M}{2}\ln t + \frac{1}{2}(M - 1)\ln(4e^{-\gamma}) \tag{4.29}$$

$$- \frac{1}{2}Ei(- \frac{\varepsilon t_s}{2\eta\Delta t}) + \frac{M}{2}Ei(- \frac{\varepsilon t_s}{2\Delta t})$$

If Δt is small, the effect of the discontinuity is not important in the solution to problem 2), and the two last exponential integrals can be neglected:

$$p_w \approx (1 - M)\ln y - \frac{1}{2}\ln(\eta\Delta t) + \frac{M}{2}\ln t + \frac{1}{2}(M - 1)\ln(4e^{-\gamma}) \tag{4.30}$$

$$\approx (1 - M)\ln y - \frac{1}{2}\ln(\eta\Delta t) + \frac{M}{2}\ln t_s + \frac{1}{2}(M - 1)\ln(4e^{-\gamma})$$

The last expression is found using the MDH-approximation $t \approx t_s$. Note that the first equation does not contain the Horner-time $\Delta t/t$, but rather an argument of the form $\Delta t/t^M$.

If $y < \min[r_c(\eta\Delta t), r_c(\Delta t)]$, the logarithmic approximation can be used for <u>all</u> the terms. This gives an equation on the Horner-form:

$$p_w \approx \frac{M}{2}\ln\frac{t}{\Delta t} \tag{4.31}$$

From Sec.4.2 we know that an intermediate region in t_y exists where the error in the approximate solution U can be quite large, but this error region does not influence the solution of problem 1) because $t_y \approx t/2\varepsilon t_s > 25$. For problem 2), the error must be taken into account, and the error in Eq.(4.28) may be large in the Δt_y interval $(1/10\eta, 25)$.

4.7 Changes in rate

Let the dimensionless rate q(t) be given by the equation

$$(4.32)\qquad q(t) = \begin{cases} 1 & 0 < t < t_1 \\ q_1 = 1 + \Delta q & t_1 < t = t_1 + \Delta t \end{cases}$$

For a general value of q_1, the water front will continue to move also after the rate has changed. Hence, the problem is non-linear, and the principle of superposition is a priori <u>not</u> valid.

Returning to the problem with a stationary discontinuity, <u>this</u> problem is linear, and the pressure response following a change in rate can be described using the solution U and superposition:

$$(4.33)\qquad u_1 \approx U(r,t;y) + \Delta q\, U(r,\Delta t;y) \qquad\qquad t > t_1$$

Inserting Eq.(4.33) into Eq.(4.1) yields the following approximate equation for the front speed:

$$(4.34)\qquad \varrho'\varrho = \varepsilon \left\{ \exp\left(-\frac{\varrho^2}{4\eta t}\right) + \Delta q \exp\left(-\frac{\varrho^2}{4\eta\Delta t}\right) \right\}$$

Approximate solutions of this equation can be found by splitting the analysis into two time regions, as for the initial water bank case. First, assume that Δt is small enough to neglect the last exponential term in Eq.(4.34). The equation then takes the same form as for constant injection into an infinite reservoir, and the solution can be approximated by $\varrho^2 \approx 2\varepsilon t$. Remember that ε is scaled using the <u>first</u> rate, hence the water front continues to move with the same speed as before the change of rate. The pressure response following the change has not yet reached the moving front.

Once again, the pressure is determined by replacing y with ϱ, now in Eq.(4.33). By neglecting the terms with argument $\varrho^2/\Delta t$, the equation for the pressure in the wellbore is given by

$$p_w \approx U(1,t;\sqrt{2\varepsilon t}) + \Delta q \; u_h(1,\eta\Delta t)$$

$$(4.35) \qquad \approx \frac{1}{2}\ln t + \frac{\Delta q}{2}\ln(\eta\Delta t) + \frac{M + \Delta q}{2}\ln(4e^{-\gamma}) + \frac{1 - M}{2}\ln(2\varepsilon)$$

A general upper limit for the validity of Eq.(4.35) is given by

$$(4.36) \qquad r_d(\eta\Delta t_A) = \sqrt{2\varepsilon t_1}$$

We then search a solution valid for large t. Again, assume that ϱ has the asymptotic form $\varrho^2 = \varrho_0^2 + gt$, where ϱ_0 and g are constants. If $t - t_1 = \Delta t$ is large enough to neglect the terms with arguments ϱ_0^2/t or $(\varrho_0^2 + gt_1)/\Delta t$, Eq.(4.34) gives

$$(4.37) \qquad g = 2\varepsilon(1 + \Delta q)\exp(-\frac{g}{4\eta}) \approx 2\varepsilon q_1$$

Defining $\varepsilon_1 = \varepsilon q_1$, the wellbore pressure derived from Eq.(4.33) is

$$p_w \approx U(1,t;\sqrt{2\varepsilon_1 t}) + \Delta q \; U(1,\Delta t;\sqrt{2\varepsilon_1 t})$$

$$(4.38) \qquad \approx \frac{1}{2}\ln t + \frac{\Delta q}{2}\ln\Delta t + \frac{Mq_1}{2}\ln(4e^{-\gamma}) + \frac{q_1}{2}(1 - M)\ln(2\varepsilon_1)$$

Following the same arguments as for an initial water bank, an estimate of the lower limit for the validity of this equation is found from the equation $2\varepsilon_1\Delta t_B = 25(2\varepsilon t_1)$:

$$(4.39) \qquad \Delta t_B = \frac{25\, t_1}{q_1} \qquad\qquad q_1 \neq 0$$

Eqs.(4.32)-(4.37) are also valid for falloff where $\Delta q = -1$. The arguments following Eq.(4.34) show that the assumption of an immediate halting of the water front at shut-in hardly can be justified, and the principle of superposition is consequently not valid for falloff. Sec.4.6 showed that this principle could be used to produce mathematical expressions for falloff pressure, but as the basic assumption in this section is incorrect, these expressions are generally invalid.

Inserting $\Delta q = -1$ into Eq.(4.35) gives an expression for the early-time falloff pressure with the usual Horner argument $\Delta t/t$. An estimate of the validity of this equation is still given by Eq.(4.36), but note that Kazemi et al. [21] give a more detailed listing of Δt_A for different values of M, η and r_f.

For large values of Δt, the water front has halted, and y in Eq.(4.33) should be replaced by the approximate stationary value $\sqrt{2\varepsilon t_s}$. This gives a late-time approximation identical to Eq.(4.31) with a lower limit of validity given by

$$\Delta t_B = 50\varepsilon t_1 = 50\varepsilon t_s \tag{4.40}$$

In connection with changes in rate, it is usual to define an "equivalent drawdown time", t_e, by

$$q_1 \ln t_e = \ln t + \Delta q \ln \Delta t \qquad q_1 \neq 0 \tag{4.41}$$

Introducing t_e in Eqs.(4.35) and (4.38), the expressions describing a change in rate can be written as

$$p_w \approx \frac{q_1}{2} \ln t_e + \frac{\Delta q}{2} \ln \eta + \frac{M+\Delta q}{2} \ln(4e^{-\gamma}) + \frac{1-M}{2} \ln(2\varepsilon) \qquad \Delta t < \frac{\pi \varepsilon t_1}{5\eta}$$

(4.42)

$$p_w \approx \frac{q_1}{2} [\, \ln t_e + M \ln(4e^{-\gamma}) + (1-M)\ln(2\varepsilon_1)\,] \qquad \frac{25\, t_1}{q_1} < \Delta t$$

Note that the last expression is identical to a logarithmic approximation of $q_1 V(1,t_e)$, using the Peclet number ε_1.

4.8 Injection into a finite cylindrical reservoir.

When injecting into an infinite reservoir, only the compressibility in the outer zone is significant. Consequently, it is likely that the outer boundary of a finite cylindrical reservoir will start influencing the wellbore pressure at a time given by the radius of drainage concept, using oil parameters:

$$(4.43)\quad t_{eia} = \frac{\pi}{10} r_e^2$$

This value has been confirmed by numerical simulations. The front position at time t_{eia} is given approximately by

$$(4.44)\quad r_{f,eia}^2 \approx 2\varepsilon t_{eia} = \frac{\pi\varepsilon}{5} r_e^2$$

According to Eq.(4.44), the water bank is still only occupying a small part of the total volume at the end of the infinite-acting period. Hence, it is to be expected that the compressibility in the outer zone is dominating also after the boundary is felt.

Several approximate expressions are given in Sec.4.3 for describing the pressure in a finite reservoir with a stationary discontinuity, and these may all be used as a basis for the quasi-stationary method. The most exact expression is given by the asymptotic solution in Eq.(4.11) and inserting this into the front speed equation, Eq.(4.1), gives

$$(4.45)\quad \varrho'\varrho = \varepsilon\left[1 - \frac{M}{\eta + (M - \eta)\left(\frac{\varrho}{r_e}\right)^2} \left(\frac{\varrho}{r_e}\right)^2 \right]$$

This equation is separable, and the solution is given implicitly by

$$(4.46)\quad \left[1 - \left(\frac{\varrho}{r_e}\right)^2\right] \exp\left\{- \left(1 - \frac{\eta}{M}\right)\left(\frac{\varrho}{r_e}\right)^2 \right\} = \exp\left\{- \frac{2\varepsilon\eta t}{M r_e^2} \right\}$$

The value of the arguments in the exponential functions are small even for values of t following the end of the infinite-acting period.

Retaining two terms only in an expansion of these functions yields $\varrho^2 \approx 2\varepsilon t$, as for the infinite-acting period.

If the approximation $\varrho \approx r_f$ is correct, the water front continues to move with constant speed also after the outer boundary is felt in the wellbore pressure. The liquid in the inner zone still behaves as incompressible, only the compressibility in the large outer zone is significant.

Following the third step in the quasi-stationary method, the wellbore pressure can be constructed as

$$(4.47) \qquad p_w \approx u_1(1,t;\sqrt{2\varepsilon t})$$

where u_1 is given in Eq.(4.11). From Eq.(4.44) it follows that the constant C is approximatly equal to 1 also for a certain time after the end of the infinite-acting period. The simplified expression, Eq.(4.13), can thus be used for u_1:

$$(4.48) \qquad p_w \approx \frac{2Mt}{r_e^2} + (1 - M)\ln(2\varepsilon t) + M\ln r_e - \frac{3}{4}M$$

The influence of the moving front is less important than in the infinite-acting period.

An expression that could be used both before and after the boundary is felt is found by using the approximate solution w given in Eq.(4.16). Obviously, also this approach gives the solution $\varrho^2 \approx 2\varepsilon t$ when inserted in the front-speed equation. Further, the pressure during injection is given by

$$p_w \approx w(1,t;\sqrt{2\varepsilon t})$$

$$(4.49) \qquad \approx -\frac{1}{2}\mathrm{Ei}\left(-\frac{1}{4\eta t}\right) + \frac{1}{2}\mathrm{Ei}\left(-\frac{\varepsilon}{2\eta}\right) - \frac{1}{2}\mathrm{Ei}\left(-\frac{\varepsilon}{2}\right)$$

$$+ \frac{M}{2}\mathrm{Ei}\left(-\frac{r_e^2}{4t}\right) + \frac{2Mt}{r_e^2}\exp\left(-\frac{r_e^2}{4t}\right)$$

Again, the approximation $C \approx 1$ is used. For small t, this equation behaves as the Verigin solution. For large t, it differs from the asymptotic expression Eq.(4.48) by the constant 3M/4.

The validity of the results may be tested by calculating the term σ defined in Eq.(4.4). For all the approximate solutions, it is found that σ contains a term proportional to t, and hence it must be expected that the error in the approximations also will increase with time. Still, one can hope that a time interval exists close to t_{eia} where the error can be neglected and this has partly been verified by comparing the solutions with results from numerical simulations, as will be shown in the next section. The result, however, depends critically on the values of the parameters M, η and ε, and a more thorough investigation is needed to clearify the validity of the different expressions.

4.9 Comments to papers by Woodward and Thambynayagam, Refs.[24,25]

In Ref.[24] Woodward and Thambynayagam study constant-rate injection into an infinite reservoir and apply the Laplace transform directly to the system of partial differential equations in Eqs.(1.1). The Stefan condition is replaced with the approximation Eq.(1.6). The authors thus rederive the exact Verigin solution and claim that this can be made valid also when a initial water bank is present by replacing $r_f^2 \approx 2\varepsilon t$ with $r_f^2 \approx r_0^2 + 2\varepsilon t$.

When the Laplace transform is applied directly to the Verigin problem, a problem arises about how to handle the movement of the front. In an exact treatment, the time dependency of the front position has to be transformed as well as the time dependency of the pressure, but as long as r_f is an implicit variable in the problem, no

straightforward way to do this exists. In Ref.[24] this problem is handled by the following algorithm:

1) Transform the equations and boundary conditions, but neglect the time dependency of r_f in the transformation.

2) The undetermined constants in the general solution of the transformed differential equations is found by applying the boundary conditions of the moving-front problem. r_f is explicitly given, and $g = r_f^2/t$ is assumed to be a constant.

3) The resulting expression is then inverted using standard rules for the Laplace transform.

The authors seem to neglect that this is an approximation method, and the validity of the method is not discussed. The analogy with the quasi-stationary method is, however, obvious. In the latter method, the undetermined constants are found by applying the boundary conditions of the problem with a stationary front, and the front speed is determined as a part of the algorithm. These differences are small, however, and should only produce minor discrepancies between solutions produced by the algorithms. It is believed that the quasi-stationary algorithm provides a better understanding of the assumptions inherented in the methods.

The transformed problem for an infinite reservoir is in Ref.[24] solved by searching for a solution of the form

$$(4.50) \qquad \bar{p}_w = B(z)[\, A(g) + K_0(\sqrt{\frac{z}{\eta}}\, r)\,]$$

The exact solution, however, has a form given in Eq.(4.5), and Eq.(4.50) only gives an approximation to the solution. This approximation is based on the same assumptions as were used when deriving U in Sec.4.2, a fact which explaines why the two methods produce similar results.

Based on their algorithm, Woodward and Thambynayagam also present an expression for falloff pressure. This expression is based on an assumption that g is constant during the whole falloff period, an assumption that cannot be justified from the Stefan condition. For small values of Δt, however, they find an approximate expression

identical to Eq.(4.35). Surprisingly, their late-time approximation is also identical to Eq.(4.31), which was derived assuming the front to be stationary. The movement of the front does not influence the solution for large Δt.

Woodward and Thambynayagam claim that their expression for falloff also can be found by using the principle of superposition. This is only partially correct; adding Verigin solutions with different arguments does not reflect superposition, as these are solutions of non-linear problems.

Ref.[25] includes a study of a finite reservoir. g is still assumed to be constant, an assumption that now can be verified from the results in Sec.4.8. For large values of injection time, the author finds an expression identical to a late-time approximation of Eq.(4.49).

5. NUMERICAL SIMULATION OF WATER INJECTION TESTS

To verify the analytical results developed in Chapter 4, a large number of injection tests, covering a wide range of reservoir parameters, have been simulated. This chapter only shows results created with four different sets of fluid and reservoir properties, denoted Set 1-4, as these were found to be characteristic for all the simulations. These four sets are based on reservoir and fluid properties from the North Sea. All the expressions given in this chapter are written using field units, the time T in <u>hours</u>. P_w will here be used to denote the pressure in the <u>wellbore</u>.

The simulations were performed with a three phase, two dimensional, black-oil simulator named TODVARS, developed at Rogaland Research Institute [27]. Based on a more detailed model of the physical situation, TODVARS solves a system of partial differential equations different from the Verigin problem. Both effects of gravity and capillary pressure as well as variations in relative permeabilities are accounted for by the simulator, together with effects of pressure on the oil viscosity. Hence, discrepancies between analytical and numerical results may be caused both by the simplifications inherent in the Verigin model and by the approximations used when constructing the analytical solutions.

TODVARS does not include effects of variation in temperature in the reservoir.

A total number of 240 grid blocks is allowed in the simulator, with a maximum of 93 in radial direction. First, several test-runs were made keeping reservoir and fluid properties constant, but varying the number of grid blocks between 1x93, 1x30 and 8x30. For all the four sets of input parameters, the wellbore pressure was found to differ with a maximum of 6 psi after 500 hours of injection. The outer boundary was chosen so as not to influence the pressure in the wellbore during this period.

The data sets were then run with and without the force of gravity included. Still using an injection time of 500 hours, the difference in wellbore pressure was found to be of same order as that due to varying number of grid blocks. Hence, it was decided to run all the simulations without gravity included and with a maximum number of grid blocks in radial direction.

All basic input parameters used in the simulations are listed in Appendix 5. The values specific for each simulation, as rate and radius of outer boundary, are listed together with the figures in this chapter. Note that the values of absolute permeability and injection rates are very large, but the dimensionless parameters have values typical for water injection into an oil reservoir:

Set	M	η	ε
1	1.05	1.15	0.0013
2	1.58	1.73	0.0013
3	2.11	2.31	0.0013
4	3.16	3.46	0.0013

The value of ε is based on an injection rate of 7000 Stb/d, which is used in most of the examples shown. The four sets of parameters are identical except for the mobility of water, i.e. viscosity and relative permeability of water.

Fig.3 shows the pressure in the wellbore when injecting with <u>constant</u> rate Q into a small finite reservoir. In all the simulations, Eq.(4.43) was found to be a very good estimate of the end of the infinite-acting period. In field units, this estimate is given by

$$(5.1) \qquad T_{eia} = 1190.0 \frac{\varphi\mu_o c_o R_e^2}{kk_o'}$$

Including a skin factor S, the Verigin solution gives for the pressure in the wellbore

$$P_w = P_i + m_w [\lg T + \lg \frac{kk'_o}{\varphi \mu_o c_o R_w^2} - 3.23 + 0.87S$$

(5.2)

$$+ (1 - M)\{ \lg(\frac{1}{1 - S_{or} - S_{wc}} \cdot \frac{Q\mu_o c_o}{hkk'_o}) + 2.10 \}]$$

The following notation is used

$$m_w = \frac{162.6\ Q\mu_w}{hkk'_w}$$

(5.3)

$$m_o = \frac{162.6\ Q\mu_o}{hkk'_o}$$

Note that there is two possibilities of defining the dimensionless skin factor, as either water or oil parameters may be used in the scaling. From the Verigin solution and the early-time falloff approximation, the present definition is the natural, but no unique definition exists in the literature. In the simulations, the input value of S is always equal to zero.

The Verigin solution as well as the analytical expressions developed in Chapter 4 have been used to analyse simulated wellbore pressure by using a root-mean-square method for data fitting. The straight-line segments that could be used for analysis where chosen by using the validity estimates given for the different expressions. Generally, the test analysis provided very good estimates of water mobility, but a small artificial skin factor of order 0.2-0.5. This is demonstated in Fig.4, showing a comparison between the Verigin solution and the simulated result using Set 3. The results from the investigation of the effects of varying the number of grid blocks indicate that a large part of the artificial skin may be due to errors in the numerical solution.

To determine the skin factor in a test analysis, the oil mobility has to be known, and the value of S turns out to be very sensitive to the value of λ_o used. Above bubble-point pressure the variation in oil viscosity with pressure is small, but the temperature

dependency is significant. The initial value $\mu_o(P_i)$ has been used in the present analysis, causing only a minor part of the artificial skin. In a real field application, however, thermal effects may be severe in the skin analysis.

The simulated wellbore pressure following the end of the infinite-acting period is plotted in Fig.5, and Figs.6-9 show the absolute difference between the approximate expressions Eqs.(4.47)-(4.49) and the simulated result. Oscillations in the numerical solution is clearly visible. The analytical expressions are reasonable accurate in a time period following the end of the infinite-acting period, but as expected from the results in Sec.4.8, the error is increasing with time. In almost all the cases investigated, Eq.(4.49) was found to be the best approximation. The magnitude and functional form of the error have been found to vary considerably with the input parameters, but the general validity of the approximate expressions has not been investigated.

Injection into an infinite reservoir with an initial water bank is shown in Fig.10. The vertical lines show the estimates for the limits of the straight-line segments as given by Eqs.(4.25) and (4.27). In field units, these limits are given by

$$T_A = 1190.0 \frac{\varphi \mu_w c_w R_o^2}{k k'_w}$$

(5.4)

$$T_B = 335.3 \frac{\varphi h (1 - S_{or} - S_{wc}) R_o^2}{Q}$$

The time needed for the pressure to pass from the first straight line to the second, $T_B - T_A$, is proportional to R_o^2, the square of the initial position of the water front.

The effects of a change in rate is illustrated in Fig.11, the vertical lines showing the limits of the straight-line segments as given by Eqs.(4.36) and (4.39). If the rate is changed from Q_0 to Q_1 at time T_1, these limits are written dependency is significant. The initial value $\mu_o(P_i)$ has been used in the present analysis, causing only a minor part of the artificial skin. In a real field application, however, thermal effects may be severe in the skin analysis.

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Injection into an infinite reservoir with an initial water bank is shown in Fig.10. The vertical lines show the estimates for the limits of the straight-line segments as given by Eqs.(4.28) and (4.27). In field units, these limits are given by

$$T_A = 1190.0\,\frac{\phi\mu_w c_w R_o^2}{k\lambda_w}$$

$$T_B = 335.3\,\frac{\phi h(1 - S_{or} - S_{wc}) R_o^2}{Q} \qquad (5.4)$$

The time needed for the pressure to pass from the first straight line to the second, $T_B - T_A$, is proportional to R_o^2, the square of the initial position of the water front.

The effects of a change in rate is illustrated in Fig.11. The vertical lines showing the limits of the straight-line segments as given by Eqs.(4.30) and (4.33). If the rate is changed from Q_0 to Q_1 at time t_1, these limits are written

$$\Delta T_A = \frac{2\pi\varepsilon T_1}{10\eta} = 88.7 \, \frac{1}{1 - S_{or} - S_{wc}} \cdot \frac{Q_0 \mu_w c_w T_1}{hk k'_w}$$

(5.5)

$$\Delta T_B = 25 \frac{Q_0}{Q_1} T_1$$

The time needed for the pressure to "stabilize" after a change in rate is generally very long, of order 25 times the time at which the rate was changed. Hence, it is very important to keep a constant rate as long as possible before falloff.

The falloff pressure caused by a short injection time of $T_1 = T_s = 50$ hours is plotted in Fig.12. The estimate of the end of the first straight line, given by ΔT_A in Eq.(5.5), shows that the response from the water zone in these tests is the order of seconds, i.e. it would be impossible to determine water mobility from an analysis of these tests. The equation for the first straight line is found from Eq.(4.35) putting $\Delta q = -1$ and $T \approx T_s$:

$$(5.6) \qquad p_w = p_s - m_w \left[\lg\Delta T + \lg \frac{kk'_w}{\varphi \mu_w c_w R_w^2} - 3.23 + 0.87S \right]$$

p_s is the wellbore pressure at shut-in. The second straight line is given by Eq.(4.31), resulting in

$$p_w = p_s - m_o \lg \frac{\Delta T}{T_s + \Delta T}$$

$$(5.7) \qquad - m_w \Big[\lg T_s + \lg \frac{hkk'_o}{\varphi \mu_o c_o R_w^2} - 3.23 + 0.87S$$

$$+ (1 - M)\left\{ \lg \frac{1}{1 - S_{or} - S_{wc}} \cdot \frac{Q\mu_o c_o}{hkk'_o} + 2.10 \right\} \Big]$$

The estimate for the lower limit of the validity of this equation is given by Eq.(4.40) or in field units:

$$\Delta T_B = 50\varepsilon T_s = 7061 \frac{1}{1 - S_{or} - S_{wc}} \cdot \frac{Q\mu_o c_o T_s}{hkk_o} \tag{5.8}$$

The injection time used in the simulation resulting in Figs.13 and 14, T_s = 2850 hours, was chosen on the basis of the first equation in Eqs.(5.5) to give a first straight falloff line lasting for approximatly 20 minutes. This equation has been found to give a reasonable estimate of the end of the first line.

All the plots of falloff pressure show that the estimate given by Eq.(5.8) is too pessimistic. This result is typical for all the simulations, but still it has been found that the time needed for the second straight line to develop may be very long, and a significant error may be introduced if too early data points are analysed. Once again it must be emphasized that the given limits were chosen as a rough rule of thumb, valid for a large range of values for the parameters and the front position. The results given in Table 2, Appendix 2, for M = η = 2 and y = 500, indicates that the general value t_{yB} = 25, chosen as basis for the analysis, is too pessimistic in the present case. In a general situation where more exact estimates of t_{yB} are needed, an analysis equal to the one resulting in Table 2, Appendix 2, must be carried through for the actual parameter region.

Both the analytical and numerical results developed for falloff confirm the theory first presented by Morse and Ott [19]; the shut-in pressure developes two straight lines that both can be used for analysis. However, it must be empasized that effects of variations in relative permeabilities are not included in the present analytical model and that the simulated well-test examples have relatively short injection periods. In addition, the reservoir is assumed to be infinite-acting both during injection and falloff.

In Ref.[20], Kazemi et al. states that the second straight falloff line can be used for analysing the oil parameters directly only if η = 1 and, in addition, R_f/R_e < 0.1. From Eq.(4.44) it is seen that the latter condition is <u>always</u> satisfied in the infinite-acting period. Continuing the work, Kazemi et al. in Ref.[21] give a general correlation between the ratio of the slopes of the straight lines and the parameters M and η. The results are based on a

numerical simulator which uses the Verigin solution to describe the injection period, but solves the falloff period by finite differences. Hence, the reservoir is assumed to be infinite-acting during the injection period, whereas a finite outer boundary is specified during falloff. This is only correct if consistent values between injection time and radius of outer boundary is used; the injection time must at least be less than the time T_{eia} given by Eq.(5.1). The reason why Kazemi et al. find that the second line cannot be used for analysis directly is probably that their numerical result is influenced by boundary effects, but unfortunately, insufficient information is given about the input parameters used to confirm this. It must be concluded, however, that the correlation they present generally is incorrect. When boundary effects are present during falloff, the eventual development of a second straight line will depend on all the three parameters M, η and $r_f(t_s)/r_e$.

The numerical simulations show that the analytical results developed in Chapter 4 provides an accurate foundation for well-test analysis. The theory shows that several factors make it essential to design and carry through a field test very carefully, for instance are effects of changes in rate much more severe than in usual one-phase testing. When analysis of both oil and water properties is desired from falloff, the injection time is essential; it must be long enough to produce a sufficient number of data points on the first straight line, but as short as possible to minimise the time period before the second straight line starts developing. Theoretically, the mobilities can be estimated with high degree of accuracy whereas the skin factor may be more difficult to determine exactly.

numerical simulator which uses the Verigin solution to describe the injection period, but solves the falloff period by finite differences. Hence, the reservoir is assumed to be infinite-acting during the injection period, whereas a finite outer boundary is specified during falloff. This is only correct if consistent values between injection time and radius of outer boundary is used; the injection time must at least be less than the time T_{min} given by Eq.(5.11). The reason why Kazemi et al. find that the second line cannot be used for analysis directly is probably that their numerical result is influenced by boundary effects, but unfortunately, insufficient information is given about the input parameters used to confirm this. It must be concluded, however, that the correlation they present generally is incorrect. When boundary effects are present during falloff, the eventual development of a second straight line will depend on all the three parameters M, n and $r_f(t_s)/r_e$.

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Fig.3 :

Semilogarithmic plott of well pressure when injecting into a finite reservoir.

Q = 7000 Stb/d
R_e = 2000 ft
T_{eia} = 9.57 hrs

——— Set 1
.... Set 2
- - - - Set 3
——— Set 4

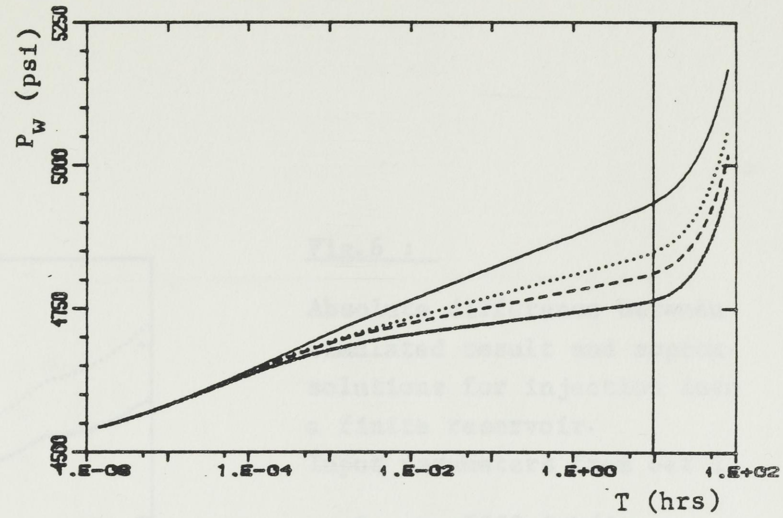


Fig.4 :

Comparison between simulated solution and the Verigin solution (dotted line) for Set 3.

Q = 7000 Stb/d
R_e = 2000 ft

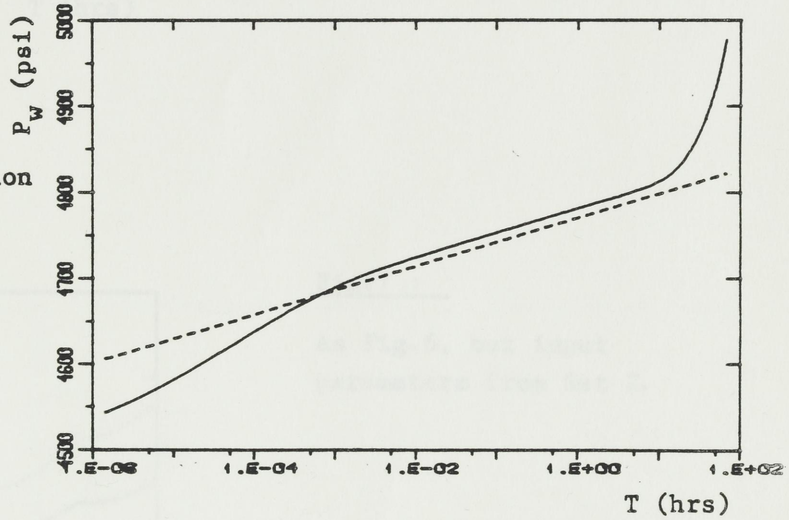
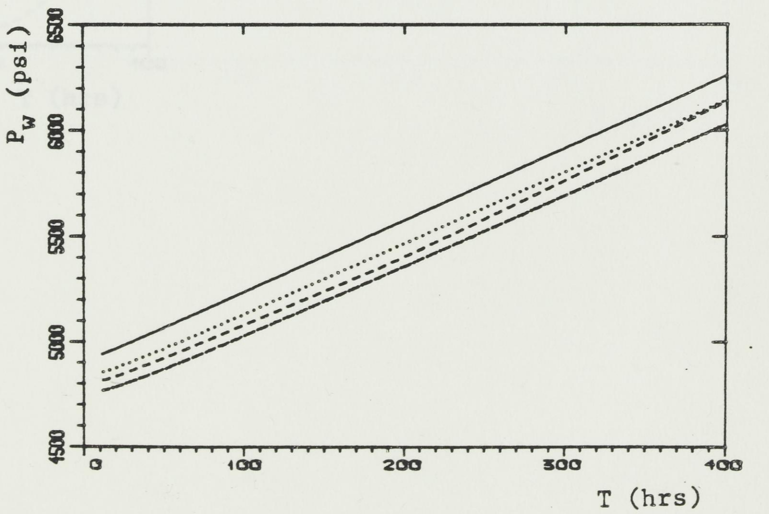


Fig.5 :

As Fig.3, but linear plot.



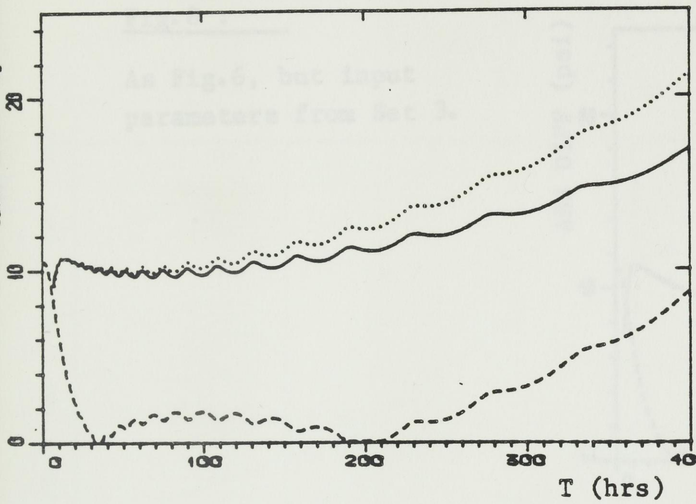


Fig.6 :

Absolute difference between simulated result and approx. solutions for injection into a finite reservoir.
Input parameters from Set 1.

Q = 7000 Stb/d
R_e = 2000 ft

——— Eq.(4.47)
.... Eq.(4.48)
------ Eq.(4.49)

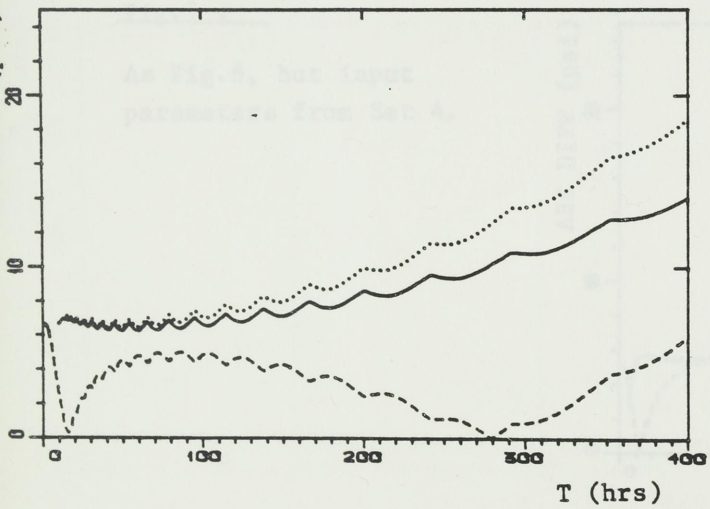


Fig.7 :

As Fig.6, but input parameters from Set 2.

Fig. 8 :

As Fig. 6, but input parameters from Set 3.

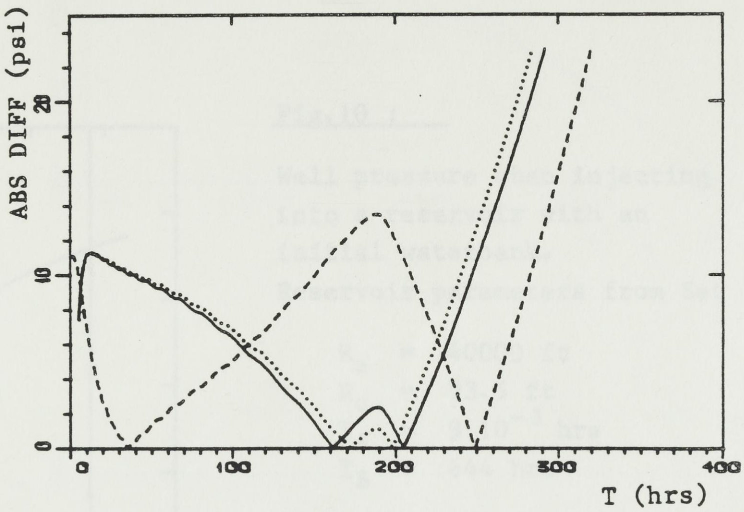
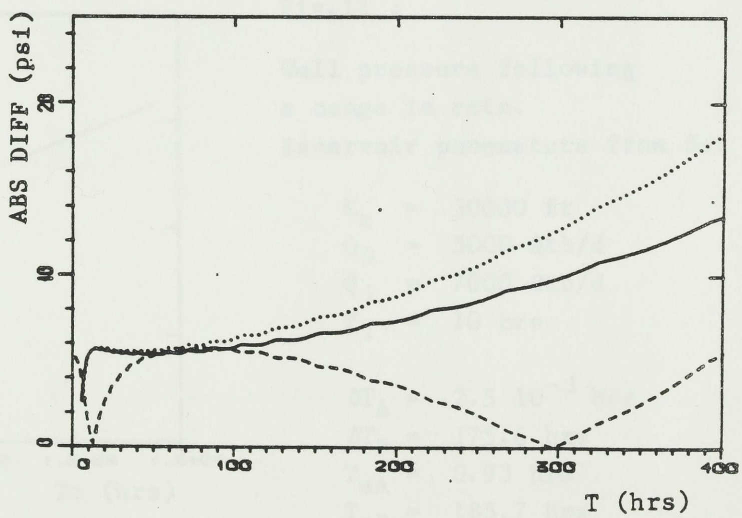


Fig. 9 :

As Fig. 6, but input parameters from Set 4.



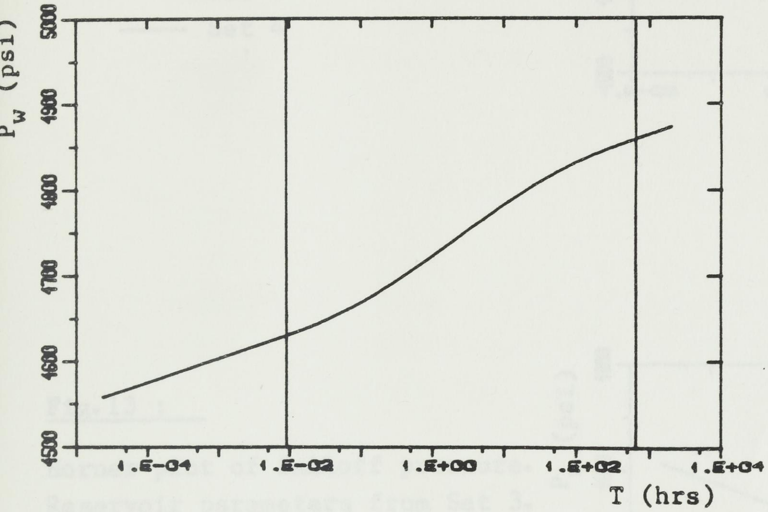


Fig.10 :

Well pressure when injecting into a reservoir with an initial waterbank. Reservoir parameters from Set 3.

R_e = 40000 ft
R_o = 93.5 ft
T_A = 9 10^{-3} hrs
T_B = 644 hrs

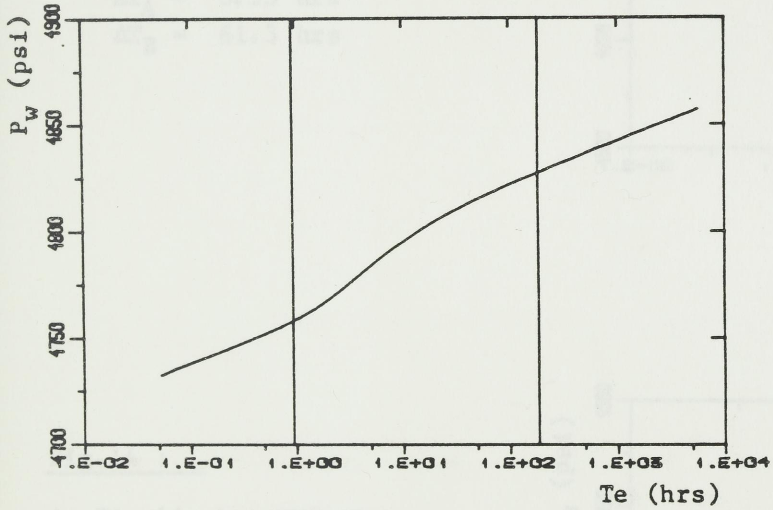


Fig.11 :

Well pressure following a cange in rate. Reservoir parameters from Set 3.

R_e = 30000 ft
Q_0 = 5000 Stb/d
Q_1 = 7000 Stb/d
T_1 = 10 hrs

ΔT_A = 2.5 10^{-3} hrs
ΔT_B = 175.6 hrs
T_{eA} = 0.93 hrs
T_{eB} = 185.7 hrs

Fig.12 :

Horner-plot of falloff pressure.

R_e = 20000 ft
T_s = 50 hrs

——— Set 1
········ Set 2
----- Set 3
——— Set 4

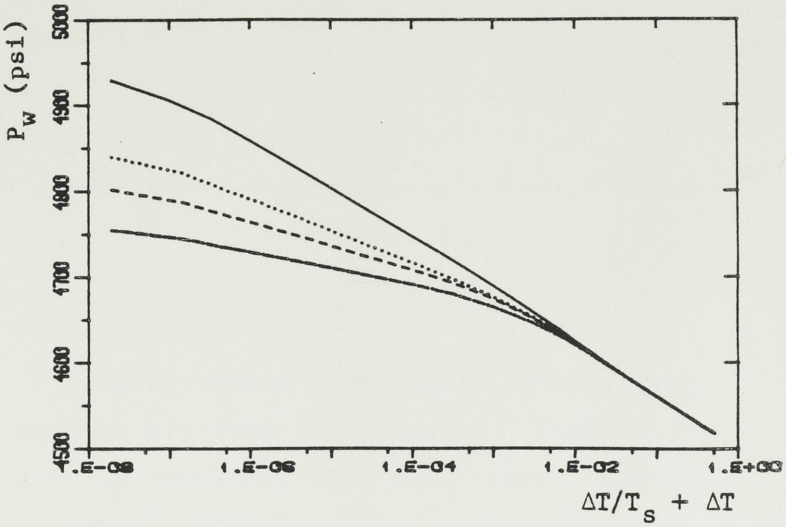


Fig.13 :

Horner-plot of falloff pressure.
Reservoir parameters from Set 3.

R_e = 30000 ft
T_s = 2850 hrs
ΔT_A = 0.33 hrs
ΔT_B = 61.3 hrs

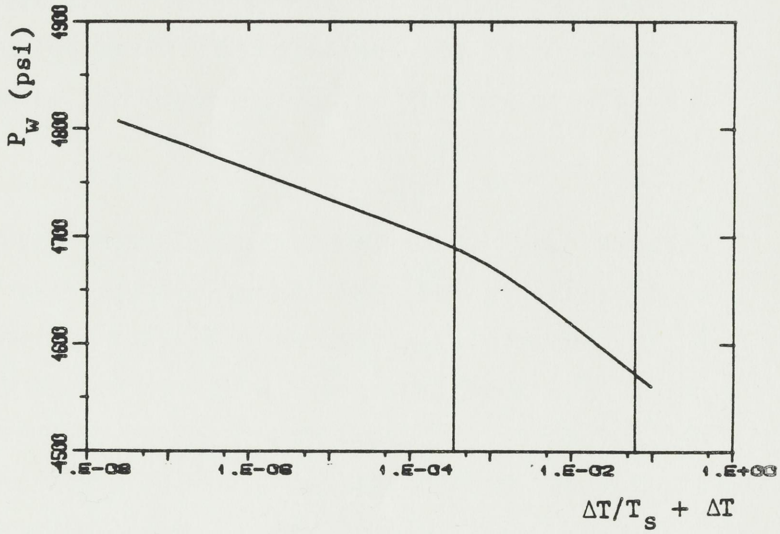
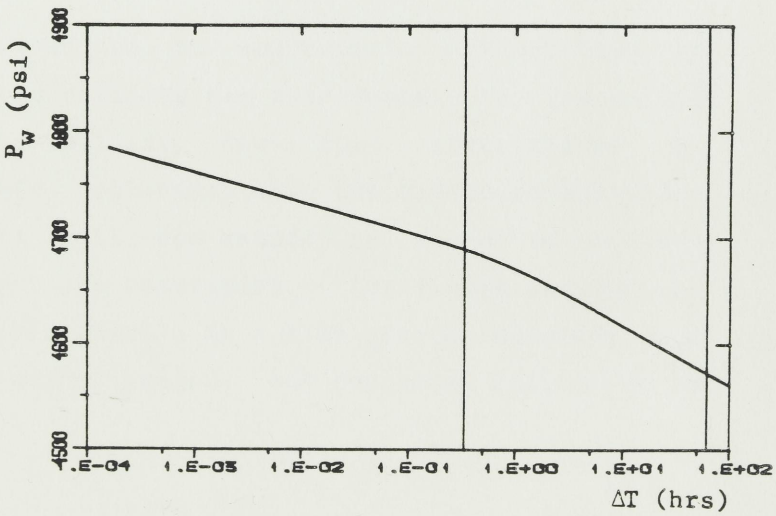


Fig.14 :

As Fig.14, but MDH-plot.



6. SUMMARY AND CONCLUSIONS

The Verigin problem, describing two-phase immiscible displacement in a homogeneous porous medium, has been investigated. Separately, three different methods originally developed for the Stefan problem have been applied to the problem. From the study of these methods, the following conclusions can be drawn:

1) Green's functions may be used to reduce the original problem to a system of integral equations. This system can be further used to prove existence and uniqueness of solutions to the original problem, but is probably too complicated for constructing simple approximate solutions.

2) By introducing the pressure at the water front as a new unknown variable, the method of eigenfunctions may be generalized to Verigin problems. The method reduces the original problem to a countable number of coupled ordinary differential equations which have to be solved numerically. The complexity of this system depends heavily on the outer boundary condition used, and in all cases, the equations are very stiff. For cylindrical geometry, the coefficients in the ordinary equations contain integrals which it has not been possible to calculate analytically.

3) The quasi-stationary method is an approximate method, but no general way of testing its validity has been found. For the actual problem, however, the validity has been investigated by substituting the produced solutions into the diffusion equation. When the Peclet number is small, the method is found to produce accurate results for infinite reservoirs. For finite reservoirs, the method gives reasonable results in a time period following the end of the infinite-acting period, but the error introduced is increasing with time.

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As basis for the quasi-stationary method, one-phase displacement in a reservoir with a stationary discontinuity has been investigated. The error in three approximate solutions for infinite reservoirs have been studied by representing the exact solution by a numerical inversion of the Laplace transform. Estimates of the validity of the different solutions are proposed. Further, exact and approximate solutions are developed for finite reservoirs, valid for all values of mobility and diffusivity ratios.

Different problems encountered in well-test analysis have been investigated by using the quasi-stationary method. The main results from this study are:

1) During injection the zone occupied by the injected fluid usually behaves as incompressible. When the reservoir is infinite, the pressure at the water front is constant and proportional to the mobility ratio.

2) Approximate analytical expressions describing effects of an initial water bank and a general change in rate, respectively, have been given, together with limits on their validity. These expressions have been confirmed by simulations of injection well tests and can be used in well-test analysis with a high degree of accuracy.

3) When injecting into an infinite reservoir with an initial water bank, the wellbore pressure developes two straight line segments with equal slopes. The approximations describing these segments are independent of the initial position of the water bank, and the late-time approximation is identical as if no initial water bank was present.

4) Except from a shut-in, a change in rate yields two straight line segments with identical slopes when wellbore pressure is plotted against "equivalent drawdown time".

5) Plotting falloff pressure in an infinite reservoir against Horner time produces two straight lines reflecting the two fluid zones. Assuming the water front to halt immediately at shut-in and using superposition gives an incorrect expression for the early-time falloff pressure. However, a correct expression can be found using

the quasi-stationary method. The late-time falloff pressure is not influenced by whether a stationary or moving water front is assumed.

6) Expressions are presented for injection into a finite cylindrical reservoir. By comparing with simulations, these have been found to be reasonable accurate in a time period after the end of the infinite-acting period. The error is, however, increasing with time, and the general validity of the expressions has not been investigated.

The quasi-stationary method provides a very good method for studying general problems in injection well testing. The method can easily be extended to problems not treated here, such as injection into a reservoir with a vertical fault.

LIST OF VARIABLES AND SYMBOLS

The variables r, t, S_w and p_o are all <u>scaled</u> variables as defined in the following list. The corresponding dimensional variables are written as capital letters, R, T etc.

The Laplace transform of a function $f(t)$ is denoted by $\bar{f}(s)$.

c_t	Total compressibility
C	Dimensionless factor defined in Eq.(4.12)
e_r, e_t	Unit vectors in r and t directions
$E = E(r,t\|r_s,t_s)$	Cylindrical heat source defined in Eq.(2.7)
$Ei(-x) = -\int_x^{\infty} \frac{1}{s}\exp(-s)\,ds$	Exponential integral function
$g = r_f^2/t$	
$G = G(x,t\|x_s,t_s)$	Green's function for 2-Dimensional free space defined in Eq.(2.2)
h	Height of reservoir
$i = \sqrt{-1}$	Imaginary unit
k	Absolute permeability
k'_x	End-point value of rel. permeability
$L = \frac{\partial}{\partial t} - \frac{\eta}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})$	Diffusivity operator
$M = \frac{k'_w}{k'_o}\frac{\mu_o}{\mu_w}$	End-point mobility ratio

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$p_x = \frac{2\pi h k k'_w}{Q(0)\mu_w}(P_x - P_i)$ — Dimensionless pressure

P_i — Initial pressure

$q = Q(t)/Q(0)$ — Dimensionless wellbore rate

Q — Wellbore rate

$r = R/R_w$ — Dimensionless radius

$r_c = \sqrt{0.04t}$ — Radius of incompressibility

$r_d = \sqrt{10t/\pi}$ — Radius of drainage

R_w — Wellbore radius

S_{or} — Residual oil saturation

S_{wc} — Connate water saturation

$t = \frac{kk'_o}{\varphi\mu_o c_o R_w^2}\cdot T$ — Dimensionless time based on wellbore radius

t_A, t_B — Lower and upper limits for the validity of approx. expressions

t_s — Injection time

$t_y = \frac{kk'_o}{\varphi\mu_o c_o Y^2}\cdot T$ — Dimensionless time based on the position of a stationary discontinuity

$u_i = u_i(r,t;y) \quad i = 1,2$ — Exact solution for a reservoir with a stationary discontinuity

$u_h = \frac{1}{2}\ln(\frac{4t}{r^2}e^{-\gamma})$ — Approx. solution for a homogeneous infinite reservoir

$U = U(r,t;y)$ — Approx. solution for an infinite reservoir with a stationary discontinuity, conf. Eq.(4.8)

$V = V(r,t)$ — Verigin solution, conf. Eq.(1.3)

w	$= w(r,t;y)$	Approx. solution for a finite reservoir with a stationary discontinuity, conf. Eq.(4.16)
y	$= Y/R_w$	Dimensionless position of a stationary discontinuity
γ	$= 0.5772...$	Euler's constant
ε	$= \dfrac{1}{1 - S_{or} - S_{wc}} \cdot \dfrac{Q(0)\mu_o c_o}{hkk'_o}$	Peclet number
η_x	$= \dfrac{kk'_x}{\varphi\mu_x c_x}$	Diffusivity
η	$= \dfrac{\eta_w}{\eta_o} = M \dfrac{c_o}{c_w}$	Diffusivity ratio
λ_x	$= \dfrac{kk'_x}{\mu_x}$	Mobility
μ_x		Viscosity
ϱ	$= \varrho(t)$	Approx. water-front position
σ	$= \sigma(r,t)$	Error term defined in Eq.(4.4)

Subscripts:

b	=	bubble point
e	=	external boundary
eia	=	end of infinite-acting period
f	=	water front
o	=	oil
s	=	shut-in
w	=	water
x	=	oil or water

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APPENDIX 1. DRAINAGE RADIUS AND RADIUS OF INCOMPRESSIBILITY.

The pressure response in an infinite reservoir during one-phase injection or production can be characterised by two radii, both functions of time. The drainage radius, denoted r_d, is a measure of how far into the reservoir the pressure response has reached. The term "radius of incompressibility", r_c, will be used here to describe an outer bound for the zone around the well where the total compressibility is negligible.

r_d and r_c do not represent physical discontinuities or boundaries, and consequently there is some arbitrariness in their definition, which is reflected in the literature. The following definition will be addopted here for the radius of incompressibility:

(A1.1) $$r_c^2 = 0.04\ t$$

For $r = r_c$, the error in the approximation $Ei(-r^2/4t) \approx \ln(r^2/4t) + \gamma$ is about 0.25%.

Some confusion exists about the term "radius of drainage", and two different types of definitions are used in the literature. The first defines r_d as the position where the value of the pressure change or fluid flow is below a certain limit, say 1% of the wellbore value [45]. This gives a definition where r_d^2 is of order 9t. The second type of definition is is concerned with the area influencing the pressure in the wellbore, or rather with the time passing before a discontinuity in the reservoir is sensed in the wellbore pressure. That is, if a discontinuity of some sort exists in the position $r = y$, this should be visible in the wellbore pressure at a time given by $r_d(t) = y$.

Whereas the first definition of drainage radius only involves the propagation speed of the pressure response out in the reservoir, the second also involves a response <u>back</u> to the well. The latter definition is perhaps the most adequate in well testing, often

referred to as "radius of investigation". Following Earlougher [8], the following expression will be used:

(A1.2) $$r_d^2 = \frac{10}{\pi} t$$

$r_d(t) = r_e$ defines the end of the infinite-acting period for a finite cylindrical reservoir. For $r = r_d$, the pressure change in an infinite homogeneous reservoir has been found to be about 4% of the wellbore value.

APPENDIX 2: COMPARISON BETWEEN THE EXACT AND APPROXIMATE SOLUTIONS FOR AN INFINITE RESERVOIR WITH A LATERAL DISCONTINUITY IN MOBILITY AND DIFFUSIVITY

The pressure distribution in a reservoir with a stationary discontinuity in M and η may be described exactly by Eqs.(4.6) or approximatly by Eqs.(4.8), (4.9) and (4.10). The error in the approximate solution U(1,t;y), defined in Eq.(4.8), has previously been investigated by Bixel and van Poollen [37] with basis in a finite difference solution of the partial differential equations describing the problem. When $\eta = 1$, good agreement between the solutions is found; when $\eta \neq 1$, "the comparisons are usually less favourable".

Bixel and van Poollen give an upper limit $t_y = t_{yA}$ for the validity of Eq.(4.9). This limit is also estimated by Odeh [42], together with a lower limit, $t_y = t_{yB}$, for the validity of the late time approximation Eq.(4.10). The investigation of Odeh is based on an analytical solution for a <u>finite</u> reservoir, but the value of t_{yA} is of course common both for finite and infinite reservoirs. No exact mathematical definition is given for any of these limits. Earlougher [8] uses the concept of drainage radius to define t_{yA}, i.e. $r_d(t_{yA}) = y$. From the definition of this concept, this is only known to be correct if the mobility in the outer zone is zero, confer Appendix 1. Comparison between the different values given for t_{yA} and t_{yB} are listed below:

		ηt_{yA}	t_{yB}
Odeh	$M = \eta < 1$:	1.5	$7.7/\eta$
	$M = \eta > 1$:	0.15	7.7
Bixel and van Poollen	All M and η :	0.25	not given
Drainage radius concept	$\lambda_0 = 0$:	$\pi/10$	-

The main reason for trying to estimate these limits has been to support a basis for calculating the position of the discontinuity.

The relationship between the exact and approximate solutions was investigated for M and η in the interval [0.5,10] and y ranging from 100 to 5000. The exact solution was represented by a numerical inversion of the Laplace transform in Eq.(4.5), following the Stehfest algorithm [39]. The modified Bessel functions involved were calculated by subroutines from the NAG library [46], but asymptotic expressions had to be used for large values of the arguments. Also the exponential integral was calculated by a NAG subroutine.

When $M = \eta = 1$, the exact analytical solution is easily calculated, and the numerical inversion was found to produce 5-6 significant digits. In addition, the inverted solution was compared with results tabulated by Satman et al [40]. These results were also based on the Stehfest algorithm, but unfortunately the authors forget to specify their values of η and y. In the comparison, the value of η was assumed to be 1, but the results in Table 1 show that the solution can not be determined uniquely by t_y, M and η; the actual value of y has to be known. This could also be seen from the approximate expressions.

The absolute and relative errors in the approximate solutions are defined in a usual manner:

$$\text{ABS ERR}(t;y) = |u_1(1,t;y) - \text{Approx. sol.}(1,t;y)|$$

(A2.1)

$$\text{REL ERR}(t;y) = \frac{\text{ABS ERR}}{u_1(1,t;y)}$$

Figs.A1-A3 show relative error in U(1,t;y) versus t_y for different values of M, η and y. For a given M and η, the maximum value of the relative error decreases for increasing value of y whereas the values of the <u>absolute</u> error were found to be independent of y. The functional form of the error shown in Fig.A2 is typical when η is slightly less than M. Keeping M fixed and increasing η, the left maximum will increase and the right decrease until only one maximum is visible. If η is decreased, the left maximum will decrease and the right increase.

As can be seen from Figs.A1-A3, the error in U(1,t;y) is mainly localized in a restricted region in t_y. Defining this region as the interval where the error exceeds 1%, the limits of the interval were found to vary considerably with the values of the parameters M, η and y. Table 2 shows some results based on this definition. In almost all the investigated cases, though, the error was found to be below 1% outside the t_y-interval (1/10η,25). These limits are shown by the vertical lines in the figures. As a consequence of the generality in the definition of this interval, the error can be negligible also for large parts <u>inside</u> the interval for certain values of M, η and y.

The maximum value of the relative error is plotted as function of M and η in Figs.A4 and A5. Note that the values can be large even when η = 1. The highest maximum values occur for smal values of η together with large values of M.

Figs.A6-A8 show a comparison between the error in all approximate solutions. The approximation $u_1 \approx u_h$ is generally valid for a <u>longer</u> time than $u_1 \approx U$. No such generality is found for the long-time approximation Eq.(4.10). Again using a 1%-defintion for the validity-bounds of Eqs.(4.9)-(4.10), these limits of the error region were found to vary considerably with M, η and y. Based on the concepts of drainage radius and radius of incompressibility defined in Appendix 1, the limiting values $\eta t_{yA} = \pi/10$ and $t_{yB} = 25$ will be chosen to give a rough rule of thumb for the validity of these expressions. It must be emphasized that a certain arbitrariness exsists in these definitions, and the values should not be used for estimating the position of the discontinuity.

y	SOL.	M=1	M=2	M=5	M=10
?	Ref.[40]	5.7023	5.7027	5.7033	5.7037
100	Auth.	4.0935	4.0939	4.0944	4.0946
500	Auth.	5.7029	5.7038	5.7038	5.7040

Table 1: Comparison between own solution (Auth.) and tabulated values from Satman et al. [40] for $t_y = 0.16$
In own solution, $\eta = 1$.

		% error for ηt_y =			Limits for 1 % error,	
					lower:	upper:
M	η	$\pi/10$,	0.25,	0.15,	ηt_y	t_y
0.75	8.0	2.9	2.4	1.2	0.1	5.1
2.0	2.0	1.3	1.1	0.5	0.2	0.7
3.0	2.25	1.2	1.1	0.5	0.2	3.6
3.0	10.0	2.5	1.8	0.6	0.2	17.5
8.0	0.75	27.	21.	10.	0.07	39.
0.75	8.0	0.9	0.4	*	0.3	6.3
2.0	2.0	0.1	*	*	0.7	2.9
3.0	2.25	0.2	0.1	*	0.5	2.0
3.0	10.0	*	*	*	1.5	21.
8.0	0.75	0.6	0.2	*	0.05	39.

Table 2: Some results from the error analysis for U (upper section) and for the logarithmic expressions Eqs.(4.9)-(4.10) (lower section).
* = error less than 0.1%.

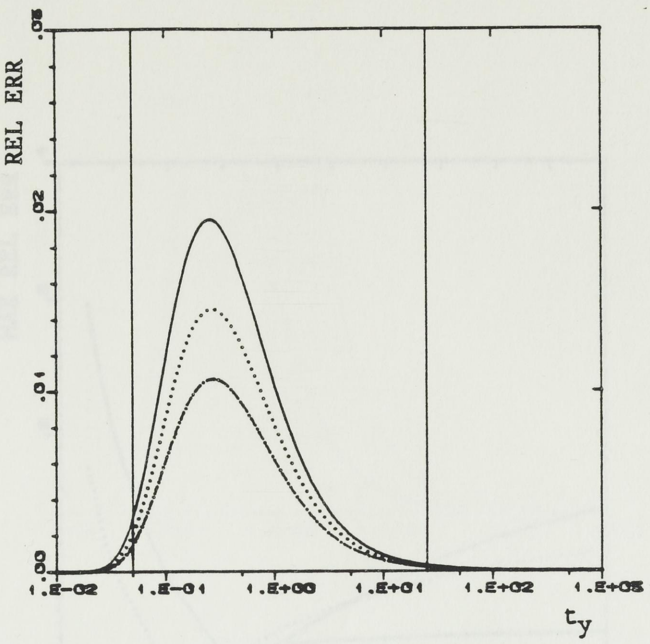


Fig. A1 :

Relative error in $U(1,t_y;y)$ when $M = \eta = 2.$

——— y = 100
········ y = 500
—·—· y = 5000

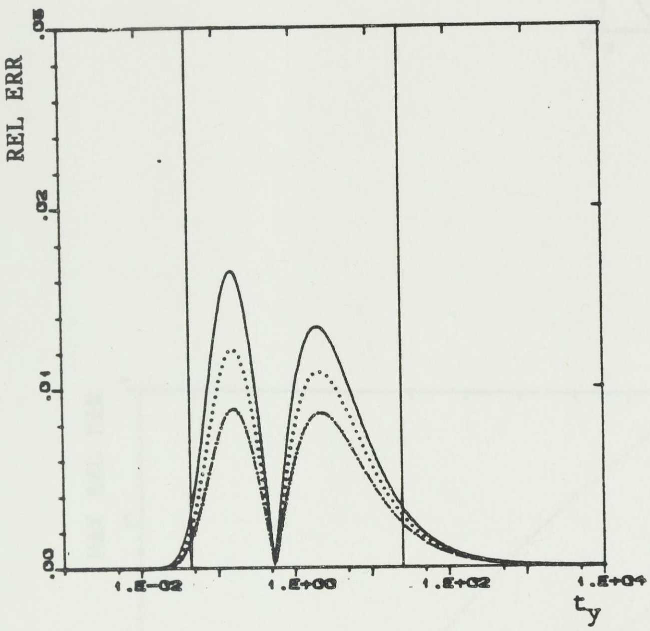


Fig. A2 :

As Fig. A1, but $M = 3.$ and $\eta = 2.25$

Fig. A3 :

As Fig. A1, but $M = 3.$ and $\eta = 10.$

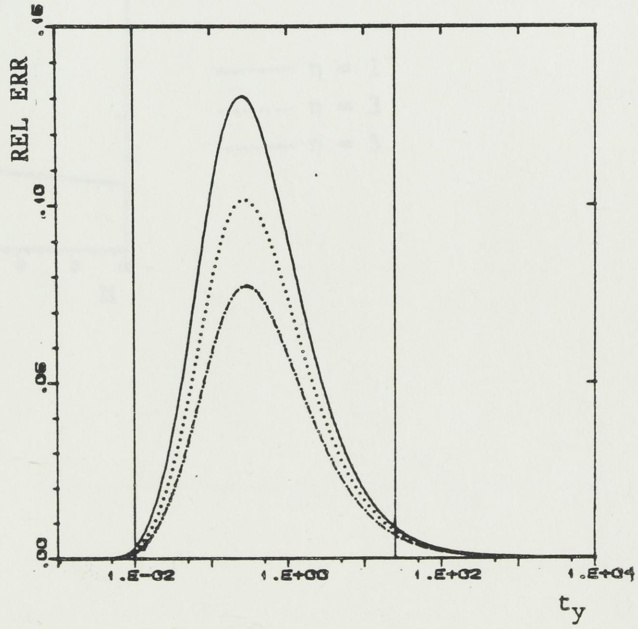
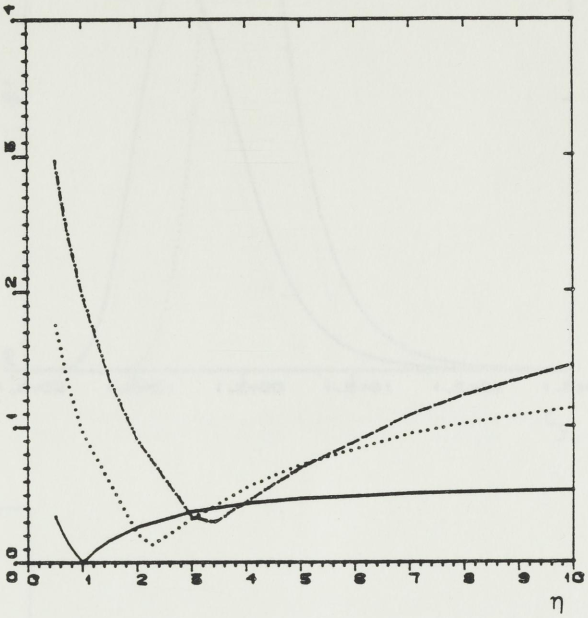


Fig. A4 :

Maximum value of the relative error in $U(1,t_y;y)$ as function of η. $y = 500$.

——— M = 1
······ M = 3
– – – M = 5



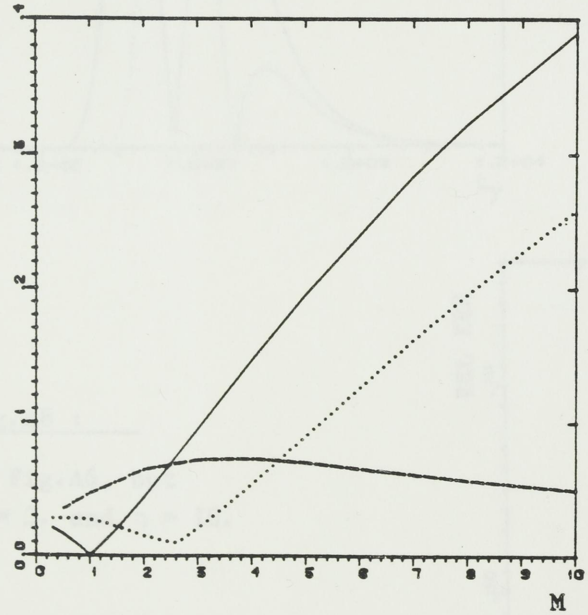


Fig. A5 :

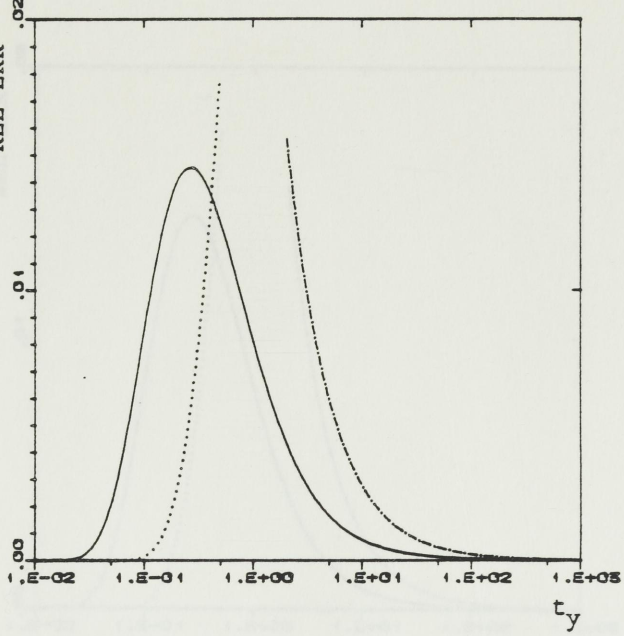
Maximum value of the relative error in $U(1,t_y;y)$ as function of M. $y = 500$.

——— η = 1
······ η = 3
– – – η = 5

Fig. A6 :

Relative error in all approximate solutions when $y = 500$ and $M = \eta = 2.$

———	$U(1,t_y;y)$	Eq. (4.8)
....	$u_h(1,\eta t)$	Eq. (4.9)
—·—·—		Eq. (4.10)



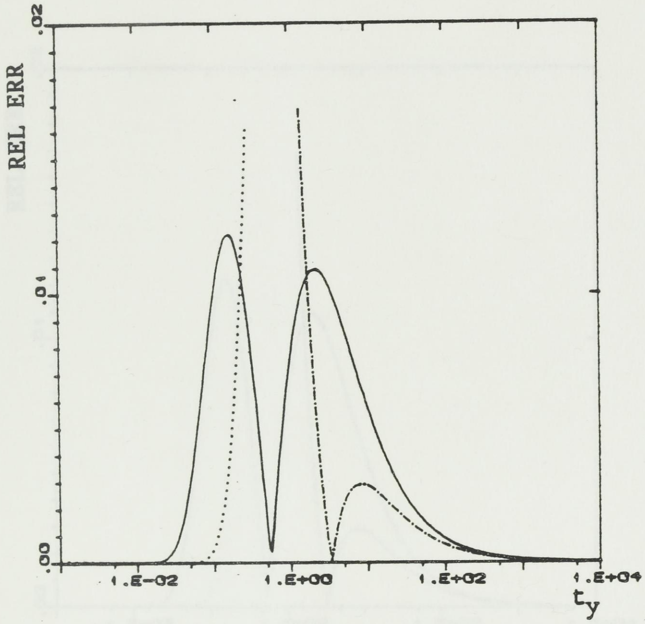
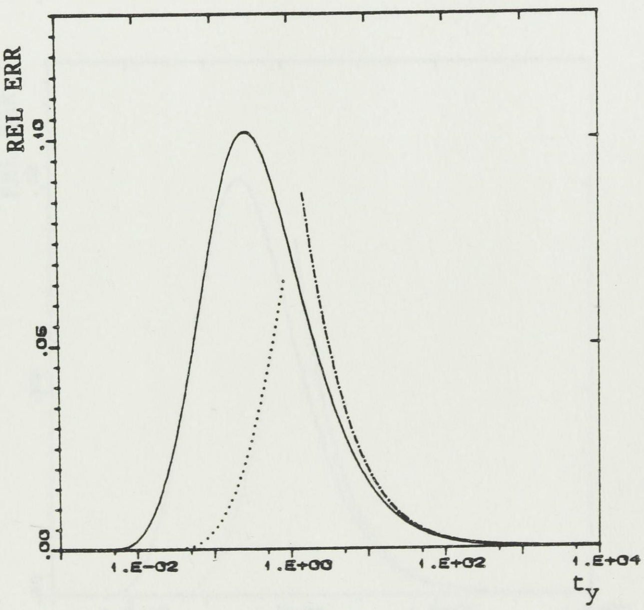


Fig. A7 :

As Fig. A6, but $M = 3.$ and $\eta = 2.25$

Fig. A8 :

As Fig. A6, but $M = 3.$ and $\eta = 10.$



APPENDIX 3. SOLUTION FOR A FINITE RESERVOIR WITH A LATERAL DISCONTINUITY IN MOBILITY AND DIFFUSIVITY

A finite reservoir with a stationary discontinuity in mobility and diffusivity can be described by Eqs.(I.1), leaving out the Stefan condition. Note that the no-flux condition is used as an outer boundary condition. Let $r_f = \gamma$ be the position of the stationary discontinuity. To separate the current problem from the one defined in Eqs.(I.1), let u_1 and u_2 be the dependent variables instead of p_w and p_o. In the following, the Laplace transform will be used to solve the problem, thus generalizing a solution for the case $M = \eta$ first given by Carter [41].

Let $\bar{u}_i(z,r)$ be the Laplace transform of $u_i(t,r)$. Transforming the equations and the boundary conditions in Eqs.(I.1) produces a system of ordinary differential equations which can be solved to give

$$\bar{u}_1 = \frac{1}{z} K_0(x_1) + \frac{A}{z\Delta} I_0(x_1)$$

$$\bar{u}_2 = \frac{\sqrt{\eta}}{\gamma z^{3/2} \Delta} \{ I_1(c) K_0(x_2) + K_1(c) I_0(x_2) \}$$

(A3.1) $$A = K_1(a)[K_0(b) I_1(c) + K_1(c) I_0(b)] - \frac{\sqrt{\eta}}{M} K_0(a)[K_1(b) I_1(c) - K_1(c) I_1(b)]$$

$$\Delta = I_1(a)[K_0(b) I_1(c) + K_1(c) I_0(b)] + \frac{\sqrt{\eta}}{M} I_0(a)[K_1(b) I_1(c) - K_1(c) I_1(b)]$$

The arguments a, b, c, x_1 and x_2 are defined by

APPENDIX 3. SOLUTION FOR A FINITE RESERVOIR WITH A LATERAL DISCONTINUITY IN MOBILITY AND DIFFUSIVITY

A finite reservoir with a stationary discontinuity in mobility and diffusivity can be described by Eqs.(1.1), leaving out the Stefan condition. Note that the no-flux condition is used as an outer boundary condition. Let $r_f = y$ be the position of the stationary discontinuity. To separate the current problem from the one defined in Eqs.(1.1), let u_1 and u_2 be the dependent variables instead of p_w and p_o. In the following, the Laplace transform will be used to solve the problem, thus generalizing a solution for the case $M = \eta$ first given by Carter [41].

Let $\bar{u}_i(z,r)$ be the Laplace transform of $u_i(t,r)$. Transforming the equations and the boundary conditions in Eqs.(1.1) produces a system of ordinary differential equations which can be solved to give

$$\bar{u}_1 = \frac{1}{z} K_0(x_1) + \frac{A}{z\Delta} I_0(x_1)$$

$$\bar{u}_2 = \frac{\sqrt{\eta}}{yz^{3/2}\Delta} \{ I_1(c)K_0(x_2) + K_1(c)I_0(x_2) \}$$

(A3.1)
$$A = K_1(a)\{ K_0(b)I_1(c) + K_1(c)I_0(b) \} - \frac{\sqrt{\eta}}{M} K_0(a)\{ K_1(b)I_1(c) - K_1(c)I_1(b) \}$$

$$\Delta = I_1(a)\{ K_0(b)I_1(c) + K_1(c)I_0(b) \} + \frac{\sqrt{\eta}}{M} I_0(a)\{ K_1(b)I_1(c) - K_1(c)I_1(b) \}$$

The arguments a, b, c, x_1 and x_2 are defined by

$$a(z) = \sqrt{\frac{z}{\eta}}\, y$$

$$b(z) = \sqrt{z}\, y$$

(A3.2) $$c(z) = \sqrt{z}\, r_e$$

$$x_1(z) = \sqrt{\frac{z}{\eta}}\, r$$

$$x_2(z) = \sqrt{z}\, r$$

The modified Bessel functions of second kind have a branch cut along the negative real axis. The expressions for $\bar{u}_i$, however, are analytical for all values of the Laplace variable z, except for a double pole in z = 0 and simple poles in the zeroes of the term Δ. These zeroes will be named z_k, (k = 1,2,...), and are situated along the negative real z-axis. The proof of these statements will be omitted here, but the consequence is that the inversion integral can be used together with the residue theorem to give the solution as an infinite series of residues:

(A3.3) $$u_i = \mathrm{Res}[\ \bar{u}_i e^{zt}\ ]_{z=0} + \sum_{k=1}^{\infty} \mathrm{Res}[\ \bar{u}_i e^{zt}\ ]_{z=z_k} \qquad i = 1,2$$

First, the residues in z = 0 will be calculated. This is a <u>double</u> pole, and the residues are given by

(A3.4) $$\mathrm{Res}[\ \bar{u}_i e^{zt}\ ]_{z=0} = [\ \frac{d}{dt}(\ z^2\ \bar{u}_i e^{zt}\ )\ ]_{z=0} \qquad i = 1,2$$

The expression on the right hand side is most easily found by expanding $\bar{u}_i$ for small values of z. Using expansions for the modified Bessel functions given for instance by Ref.[47], these result in the following expressions:

(A3.5) $$\bar{u}_1 = \frac{1}{z} \cdot \frac{N_{-1}\, z^{-1} + N_0\, z^0 + \ ....}{D_0\, z^0 + D_1\, z^1 + \ ....}$$

(A3.5cont)

$$\bar{u}_2 = \frac{1}{z} \cdot \frac{N_{-1} z^{-1} + N_0^* z^0 + \ldots\ldots.}{D_0 z^0 + D_1 z^1 + \ldots\ldots.}$$

where the first coefficients in the infinite series are given by

$$N_{-1} = \frac{\sqrt{\eta}}{y r_e}$$

$$N_0 = (1 - \frac{\eta}{M}) \frac{y}{2\sqrt{\eta} r_e} \ln \frac{y}{r} + \frac{\sqrt{\eta} r_e}{2My} \ln \frac{y}{r}$$

$$+ \frac{\sqrt{\eta} r_e}{2y} \ln \frac{r_e}{y} + \frac{r^2}{4\sqrt{\eta} y r_e}$$

$$- \frac{\sqrt{\eta} r_e}{4y} [1 + (\frac{y}{r_e})^2 (\frac{1}{\eta} - 1)]$$

(A3.6)

$$D_0 = \frac{y}{2\sqrt{\eta} r_e} (1 - \frac{\eta}{M}) + \frac{\sqrt{\eta} r_e}{2My}$$

$$D_1 = (1 - \frac{\eta}{M}) \frac{y r_e}{4\sqrt{\eta}} \ln \frac{r_e}{y} + (1 - M) \frac{y r_e}{8M\sqrt{\eta}}$$

$$+ \frac{y^3}{16\sqrt{\eta} M r_e} [(M - 1) + (M - \eta)(1 + \frac{1}{\eta})]$$

$$+ \frac{\sqrt{\eta} r_e^3}{16My}$$

$$N_0^* = \frac{\sqrt{\eta} r_e}{2y} \ln \frac{r_e}{r} - \frac{\sqrt{\eta} r_e}{4y} + \frac{\sqrt{\eta} r^2}{4 r_e y}$$

Now insert Eqs.(A3.5) into Eq.(A3.4). This gives

$$\mathrm{Res}[\,\overline{u}_1 e^{zt}\,]_{z=0}$$

$$= \frac{N_{-1}}{D_0}\,t + \frac{D_0 N_0 - N_{-1} D_1}{D_0^2}$$

(A3.7)

$$= C\,\frac{2Mt}{r_e^2} + \ln\frac{y}{r}$$

$$+ C\Big[- M \ln\frac{y}{r_e} + \frac{M}{2\eta}(\eta-1)\Big(\frac{y}{r_e}\Big)^2 - \frac{M}{2} + \frac{M}{2\eta}\Big(\frac{r}{r_e}\Big)^2 \Big]$$

$$+ C^2\Big[\frac{M}{\eta}(M-\eta)\Big(\frac{y}{r_e}\Big)^2 \ln\frac{y}{r_e} + \frac{M}{2\eta}(M-1)\Big(\frac{y}{r_e}\Big)^2 - \frac{M}{4\eta}\Big(\frac{y}{r_e}\Big)^4 \Big\{(M-1) + (M-\eta)\Big(1+\frac{1}{\eta}\Big)\Big\} - \frac{M}{4} \Big]$$

The factor C is given by

(A3.8) $$\frac{1}{C} = 1 + \Big(\frac{M}{\eta} - 1\Big)\Big(\frac{y}{r_e}\Big)^2 = \frac{2My}{\sqrt{\eta}\,r_e}\,D_0$$

The residue of $\overline{u}_2 e^{zt}$ is found by replacing N_0 with N_0^*:

$$\mathrm{Res}[\,\overline{u}_2 e^{zt}\,]_{z=0}$$

(A3.9) $$= C\,\frac{2Mt}{r_e^2} - C\Big[M \ln\frac{r}{r_e} + \frac{M}{2} - \frac{M}{2}\Big(\frac{r}{r_e}\Big)^2 \Big]$$

$$+ C^2\Big[\frac{M}{\eta}(M-\eta)\Big(\frac{y}{r_e}\Big)^2 \ln\frac{y}{r_e} + \frac{M}{2\eta}(M-1)\Big(\frac{y}{r_e}\Big)^2 - \frac{M}{4\eta}\Big(\frac{y}{r_e}\Big)^4 \Big\{(M-1) + (M-\eta)\Big(1+\frac{1}{\eta}\Big)\Big\} - \frac{M}{4} \Big]$$

The poles $z = z_k$ are simple, and the values for the residues are given from the formula for a quotient:

$$\text{Res}[\, \bar{u}_1 e^{zt} \,]_{z=z_k} = \left[\frac{\{ \frac{\Delta}{z} K_0(x_1) + \frac{A}{z} I_0(x_1) \} e^{zt}}{\frac{d\Delta}{dz}} \right]_{z=z_k}$$

(A3.10)

$$\text{Res}[\, \bar{u}_2 e^{zt} \,]_{z=z_k} = \left[\frac{\frac{\sqrt{\eta}}{yz^{3/2}} \{ I_1(c) K_0(x_2) + K_1(c) I_0(x_2) \} e^{zt}}{\frac{d\Delta}{dz}} \right]_{z=z_k}$$

The calculation of the right hand side is carried through by introducing the following new variables:

$$z = - s^2 \qquad s > 0$$

(A3.11)

$$z_k = - s_k^2 \qquad s_k > 0$$

The calculations are rather laborious, and details will be omitted here. The result is most easily written using the following notation:

$$\alpha_k = - i\, a(- s_k^2) = \frac{ys_k}{\sqrt{\eta}}$$

$$\beta_k = - i\, b(- s_k^2) = ys_k$$

(A3.12)

$$\gamma_k = - i\, c(- s_k^2) = r_e s_k$$

$$\phi_{mn}(s_k) = J_m(\beta_k) Y_n(\gamma_k) - J_n(\gamma_k) Y_m(\beta_k)$$

The value of the zeroes $z_k = - s_k^2$ are then given by the equation

(A3.13) $$J_1(\alpha_k) \phi_{01}(s_k) - \frac{\sqrt{\eta}}{M} J_0(\alpha_k) \phi_{11}(s_k) = 0$$

The values of the residues are given by

$$\mathrm{Res}\left[\,\bar{u}_1 e^{zt}\,\right]_{z=-s_k^2}$$

$$= \frac{2}{s_k^2}\cdot\frac{\eta J_0\left(\dfrac{r\,s_k}{\sqrt{\eta}}\right)\exp(-s_k^2 t)}{y^2(M-1)J_1^2(\alpha_k) - \left[\dfrac{4\eta J_0^2(\alpha_k)}{\pi^2 M s_k^2 \phi_{01}^2}\right] - y^2\left(1-\dfrac{\eta}{M}\right)J_0^2(\alpha_k)}$$

(A3.14)

$$\mathrm{Res}\left[\,\bar{u}_2 e^{zt}\,\right]_{z=-s_k^2}$$

$$= \frac{2}{s_k^2}\cdot\frac{\eta J_0(\alpha_k)\{\,J_1(\gamma_k)Y_0(rs_k) - J_0(rs_k)Y_1(\gamma_k)\,\}\exp(-s_k^2 t)}{y^2(M-1)J_1^2(\alpha_k)\phi_{01} - \left[\dfrac{4\eta J_0^2(\alpha_k)}{\pi^2 M s_k^2 \phi_{01}}\right] + y^2\left(1-\dfrac{\eta}{M}\right)J_0^2(\alpha_k)\phi_{01}}$$

By inserting the values of the residues given in Eqs.(A3.7), (A3.9) and (A3.14) into Eq.(A3.3), the exact analytical solution to the problem is given. For the case $M = \eta$, this solution differs from the one given by Carter only by the scaling of variables.

As $t \to \infty$, the transient terms given by the residues in $z = z_k$ are damped, and the first term corresponding to the residue in $z = 0$ is dominating. Consequently, this term may be used as an approximate solution for large t.

Carter also presents numerical results from a computation based on his expressions for $M = \eta$. This computation involved calculation of the roots in Eq.(A3.13) together with a calculation of the infinite series in Eq.(A3.3). It is believed that numerical inversion of the Laplace transform, Eqs.(A3.1), according to the Stehfest algorithm is an easier way to provide numerical results. This inversion has been carried through, showing that the wellbore pressure can deviate considerably from the value predicted by Carter's formula if $M \neq \eta$.

APPENDIX 4. TABLE OF LAPLACE TRANSFORMS.

Let $\bar{f}(z;a)$ be the Laplace transform of the function $f(t;a)$. a is an arbitrary positive parameter.

	$f(t;a)$	$\bar{f}(z;a)$
(A4.1)	$-\frac{1}{2}\mathrm{Ei}(-\frac{a^2}{4t})$	$\frac{K_0(\sqrt{z}a)}{z}$
(A4.2)	$\frac{1}{2t}\exp(-\frac{a^2}{4t})$	$K_0(\sqrt{z}a)$
(A4.3)	$\frac{2t}{a^2}\exp(-\frac{a^2}{4t})$	$\frac{1}{z}K_0(\sqrt{z}a) + \frac{2}{z^{3/2}a}K_1(\sqrt{z}a)$

The first two transforms can be found for instance in Ref.[48]. The last can be found from the second by using the general rule

$$t^2 f(t;a) \quad \longleftrightarrow \quad \frac{d^2}{dz^2}\bar{f}(z;a) \qquad \text{(A4.4)}$$

APPENDIX B. PARAMETERS USED IN THE NUMERICAL SIMULATIONS

This Appendix gives a listing of the input parameters used to produce the numerical results presented in Chapter 5. Four different sets of input parameters were used, based on field data from the North Sea. Where detailed information about the values was missing, such as for relative permeabilities and residual oil saturation, these values were chosen more or less arbitrary. The following reservoir and wellbore properties are common for all four sets:

Absolute permeability	k	= 432 md
Porosity	ϕ	= 0.267
Formation compressibility	c_f	= $6.8 \cdot 10^{-6}$ psi^{-1}
Height of reservoir	h	= 73.12 ft
Wellbore radius	R_w	= 0.26 ft
Residual oil saturation	S_{or}	= 0.34
Connate water saturation	S_{wc}	= 0.279
Initial pressure	P_i	= 4300 psi
Bubble-point pressure	P_b	= 3080 psi
Skin factor	S	= 0

The following table gives a listing of the PVT-properties of the saturated oil:

Pressure	B_o	μ_o	R_{so}
2596.7	1.221	1.20	412.0
4117.8	1.259	1.10	487.0
5118.7	[illegible]	0.90	734.5
7100.8	[illegible]	0.70	1101.8
(psi)	(res.bbl/STB)	(cp)	(scf/STB)

B_o = Formation volume factor of saturated oil

R_{so} = Solution gas/oil ratio

μ_o = Viscosity of saturated oil

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Absolute permeability	k	= 4621 mD
Porosity	φ	= 0.307
Formation compressibility	c_f	= $5.0\cdot10^{-6}$ psi^{-1}
Height of reservoir	h	= 13.12 ft
Wellbore radius	R_w	= 0.36 ft
Residual oil saturation	S_{or}	= 0.34
Connate water saturation	S_{wc}	= 0.279
Initial pressure	P_i	= 4500 psi
Bubble-point pressure	P_b	= 3000 psi
Skin factor	S	= 0

The following table gives a listing of the PVT-properties of the saturated oil:

Pressure	B_o	μ_o	R_{so}
2680.3	1.221	1.20	412.0
3132.8	1.250	1.10	467.0
5119.8	1.350	0.90	784.0
7106.8	1.450	0.70	1101.0
[psi]	[bbl/Stb]	[cp]	[Scf/Stb]

B_o = Formation volume factor of saturated oil

R_{so} = Solution gas/oil ratio

μ_o = Viscosity of saturated oil

Above bubble-point pressure, the volume factor and viscosity of oil are assumed to be linear functions of pressure:

$$\frac{dB_o}{dP} = -6.5 \cdot 10^{-6} \ \text{psi}^{-1}$$

$$P > P_s$$

$$\frac{d\mu_o}{dP} = 9.0 \cdot 10^{-5} \ \text{cp/psi}$$

Together with linear interpolation in the PVT-table, this gives the following initial values for compressibility and viscosity of oil:

$$c(\text{oil}) = \frac{1}{B_o}\frac{dB_o}{dP} = 5.17 \cdot 10^{-6} \ \text{psi}^{-1}$$

$$P_i = P$$

$$\mu_o = 1.26 \ \text{cp}$$

The compressibility and volume factor of water are both constants, given directly into the simulator as

$$c(\text{water}) = 3.0 \cdot 10^{-6} \ \text{psi}^{-1}$$

$$B_w = 1.0 \ \text{bbl/Stb}$$

Hence, the total compressibility in each fluid zone may be calculated as

$$c_o = c_f + S_{wc}c(\text{water}) + [1 - S_{wc}]c(\text{oil})$$

$$= 9.57 \cdot 10^{-6} \ \text{psi}^{-1}$$

$$c_w = c_f + [1 - S_{or}]c(\text{water}) + S_{or}c(\text{oil})$$

$$= 8.74 \cdot 10^{-6} \ \text{psi}^{-1}$$

The viscosity of water is assumed to be constant in each simulation, but differs for the various sets of input parameters:

Set	μ_w
1	0.80
2	0.40
3	0.40
4	0.40
	[cp]

The variations in relative permeability are given by

S_w	k_w				k_o
	Set 1	Set2	Set3	Set4	Set1-4
0.279	0.0	0.0	0.0	0.0	0.4
0.30	0.002	0.001	0.002	0.003	0.36
0.40	0.040	0.025	0.040	0.100	0.20
0.50	0.100	0.075	0.100	0.180	0.11
0.60	0.190	0.145	0.190	0.300	0.04
0.66	0.267	0.200	0.267	0.400	0.00

The capillary pressure is identically zero in all simulations. The values of rate and radius of the outer boundary are specific for each simulation and are given along with the results in Chapter 5.

When effects of gravity were included, identical values of horizontal and vertical absolute permeabilities were used. Further, the following values of the densities at <u>standard conditions</u> were applied:

$$\rho_o = 54.93 \text{ lbm/Scf}$$

$$\rho_g = 9.11 \cdot 10^{-3} \text{ lbm/Scf}$$

$$\rho_w = 62.42 \text{ lbm/Scf}$$

ρ_g is the density of the dissolved gas as given by the solution gas/oil ratio.

All simulations that did not include the effects of gravity, were run in a radial mode with 1x93 grid blocks.

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ERRATA:

The correct equations should read

(2.4)

$$p_w = 2\pi \int_0^t G(\underline{r},t|\underline{0},t_0)\, dt_0 + \int_K \{ G\nabla_0 p_w - p_w \nabla_0 G - p_w G \underline{e}_{t_0} \} \cdot d\underline{s}_0$$

(2.5) $$d\underline{s}_0 = \underline{n}\, ds_0 = r_{f0} [\, \underline{e}_{r_0} - r'_{f0}\, \underline{e}_{t_0}]\, dt_0 d\theta_0$$

(4.5)

$$\bar{u}_1 = \frac{1}{z} K_0\left(\sqrt{\frac{z}{\eta}}\, r\right) + \frac{K_1\left(\sqrt{\frac{z}{\eta}}\, y\right) K_0(\sqrt{z}y) - \frac{\sqrt{\eta}}{M} K_0\left(\sqrt{\frac{z}{\eta}}\, y\right) K_1(\sqrt{z}y)}{z\left[I_1\left(\sqrt{\frac{z}{\eta}}\, y\right) K_0(\sqrt{z}y) + \frac{\sqrt{\eta}}{M} I_0\left(\sqrt{\frac{z}{\eta}}\, y\right) K_1(\sqrt{z}y)\right]} I_0\left(\sqrt{\frac{z}{\eta}}\, r\right)$$

(4.7) $$\bar{u}_2 \approx M \frac{K_0(\sqrt{z}r)}{z}$$

The correct definition of the Peclet number in the List of Variables and Symboles, p.69, is

$$\varepsilon = \frac{1}{1 - S_{or} - S_{wc}} \cdot \frac{Q(0)\mu_o c_o}{2\pi h k k'_o}$$