Department of APPLIED MATHEMATICS

SOLUTION OF A NON-STRICTLY HYPERBOLIC SYSTEM MODELLING NON-ISOTHERMAL TWO-PHASE FLOW IN A POROUS MEDIUM

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1. INTRODUCTION

The report discusses the solution of the non-strictly hyperbolic system

$$u_t + (gu)_x = 0$$
$$v_t + (gv)_x = 0 \qquad (1.1)$$

assuming that the equation

$$ug_u - vg_v = 0 \qquad (1.2)$$

defines two distinct curves in the (u,v)-space along which the system Eq.(1.1) has a parabolic degeneracy. As shown in the Appendix, such systems can be used modelling non-isothermal two-phase flow in a porous medium.

First, the Riemann problem associated to Eq.(1.1) is solved, i.e. the Cauchy problem defined by Eq.(1.1) and the initial data

$$(u,v)_{t=0} = \begin{cases} (u^L, v^L) & x < 0 \\ (u^R, v^R) & x > 0 \end{cases} \qquad (1.3)$$

u^L, u^R etc denote constant values. In the solution of the Riemann problem, entropy conditions valid independently of local linear degeneracies of the system Eq.(1.1) are defined. The system allows for an additional conservation law, an entropy equation, and this equation is solved explicitly. Opposite to strictly hyperbolic systems, it is not possible to construct locally a convex entropy at all points in the phase space. Results from application of the Riemann solver in the Random Choice Method for numerical solution of Eq.(1.1) is presented.

Based on the Riemann solution, it is proven that the general Cauchy problem for Eq.(1.1) has a solution. This is a generalization of a proof given by Temple [27] in the case of a single transition curve.

Hyperbolic systems with parabolic degeneracies have been studied by several authors [4,7,11-16,26,27]. The solution of the Riemann problem for Eq.(1.1) involving a single transition curve in phase space where the eigenvalues are equal, has been given by Keyfitz and Kranzer [14], assuming one of the wave families to be genuinely nonlinear. A specific application to a reservoir modelling problem, where a single local degeneracy exists, was discused by Isaacson [12]. Johansen and Winther [13] solved the Riemann problem for a system closely related to Eq.(1.1), also involving a single transition curve. Parts of the general solution of Eq.(1.1), with special significance to reservoir modelling, was first given by Hovdan [11] and by Pope [26]. The general solution presented in this report has also beed found independently by Da Mota [4], with a slightly more restrictive definition of the function g.

2. ALTERNATIVE FORMS OF THE MODEL EQUATIONS

It will be convenient to operate with several different forms of the system Eq.(1.1). As shown in the Appendix, when modelling non-isothermal flow in a porous media, the system originated through the physical variables S and T, representing saturation and temperature respectively. In these variables, the system is written in matrix form as

$$\begin{bmatrix} S \\ T \end{bmatrix}_t + \begin{bmatrix} f_S & f_T \\ g & 0 \end{bmatrix} \begin{bmatrix} S \\ T \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (2.1)$$

The relationship between (S,T) and (u,v) is given by

$$S = u - \beta \qquad\qquad T = \frac{v}{u - \beta} \qquad (2.2)$$

and the functions f and g are related as

$$g = \frac{f + \alpha}{S + \beta} \qquad (2.3)$$

α and β are positive constants representing thermodynamic parameters. In the physical model $\alpha < \beta$, but this restriction is not imposed in the following. Note that g will be used both as function of (u,v) and of (S,T).

A polar-coordinate form of the equations is written as

$$r_t + (rg)_x = 0 \qquad\qquad \theta_t + g\theta_x = 0 \qquad (2.4)$$

The dependent variables are then defined by

$$\theta = \mathrm{Arctg}\left(\frac{v}{u}\right) = \mathrm{Arctg}(\,T\,) \qquad\qquad r = \sqrt{u^2 + v^2} = (\,S + \beta\,)\sqrt{1 + T^2} \qquad (2.5)$$

3. DESCRIPTION OF THE FUNCTION $f = f(S,T)$

The function f is defined on the (S,T)-domain $[0,1]\times[0,1]$. For constant T, f is assumed to be the S-shaped fractional-flow function well-known from isothermal flow in porous media:

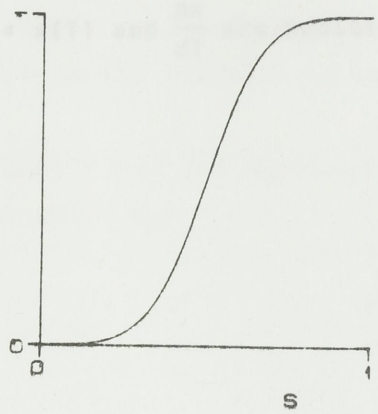


Fig 3.1 :
Example of $f(S,T=\text{const})$

Further, f will be assumed to have the following properties:

$$f_T < 0 \qquad T \in (0,1) \tag{3.1}$$

$$f_S < \frac{\alpha}{\beta} \qquad S = 0 \tag{3.2}$$

$$f_S < \frac{\alpha + 1}{\beta + 1} \qquad S = 1 \tag{3.3}$$

Eqs.(3.2-3) together assure the existence of two distinct transition curves in phase space, i.e. in (S,T)-space, where the eigenvalues of the system-matrix are equal. These properties are only possible if for each $T\in[0,1]$, there exist at least one point where $f_{SS} = 0$. As long as two and only two transition curves exist, no restriction will be made on the number of inflextion points of $f(T=\text{const})$. Hence, gravity may be included in the model equations, as described in the Appendix. It will however be assumed that $f(T=\text{const})$ is convex in the vicinety of the transition curve S_1, concave in the vicinety of S_2, $S_1(T) < S_2(T)$.

A simple example of a function f with the desired properties is given by

$$f = \frac{S^2}{S^2 + \kappa(1-S)^2} \qquad (3.4)$$

where both $\kappa = \kappa(T)$ and $\frac{d\kappa}{dT}$ are positive.

4. STRUCTURE OF THE FUNDAMENTAL WAVES

The structure of the fundamental waves is independent of the values of the constants α and β, and has previously been discussed by Keyfitz & Kranzer [14] and by Isaacson [12]. The main properties will be reviewed here for the sake of compleetness.

The eigenvalues and eigenvectors of the system-matrix given in Eq.(2.1) are given by

$$\lambda^1 = f_S \qquad \lambda^2 = g$$

$$r^1 = (1,0)^T \qquad r^2 = (f_T, g - f_S)^T \qquad (4.1)$$

$$l^1 = (f_S - g, f_T)^T \qquad l^2 = (0,1)^T$$

Consequently, the two transition-curves S_1 and S_2 where the eigenvalues are equal are defined implicitly by the following equation, equivalent to Eq.(1.2):

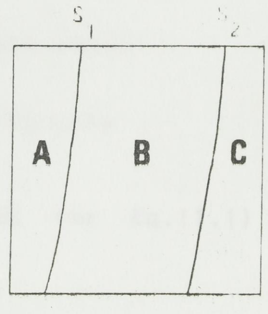
$$f_S - g = 0 \qquad (4.2)$$

The system is not diagonalizable along S_1 and S_2. Note that in the (f,S)-space, the transition curves are easily determined as points where straight lines through the point $P = (-\alpha, -\beta)$ are tangents to the curves $f = f(S, T=\text{const})$. The transition curves divide the phase space into three separate regions, defined as:

$$A = \{ (S,T) \mid T \varepsilon [0,1] \quad 0 \leqslant S \leqslant S_1(T) \}$$

$$B = \{ (S,T) \mid T \varepsilon [0,1] \quad S_1(T) \leqslant S \leqslant S_2(T) \}$$

$$C = \{ (S,T) \mid T \varepsilon [0,1] \quad S_2(T) \leqslant S \leqslant 1 \}$$



From Eqs.(3.2-3), it follows that $f_S - g$ is non-positive in A C, non-negativ in B. A point in the phase space will be denoted $U = (S,T)$.

The characteristic family belonging to the eigenvalue λ^2 is linearly degenerate, whereas the other family has a local degeneracy at each reflextion point of f(T=const):

$$r^1 \cdot \nabla\lambda^1 = f_{SS}$$
$$r^2 \cdot \nabla\lambda^2 \equiv 0 \tag{4.3}$$

The last equation also shows that $\lambda^2 = g$ is a Riemann invariant for the system, the other invariant is given by T. Due to the lack of strict hyperbolicity, no unique transformation between the Riemann invariants and the original variables (S,T) exists:

$$\frac{\partial(g,T)}{\partial(S,T)} = \begin{vmatrix} g_S & g_T \\ 0 & 1 \end{vmatrix} = g_S = \frac{1}{S+\beta}(f_S - g) \tag{4.4}$$

The discontinuities are described by the Rankine-Hugoniot conditions:

$$\sigma = \frac{[f]}{[S]} = \frac{[(f+\alpha)T]}{[(S+\beta)T]} \tag{4.5}$$

σ denotes the speed of the discontinuity and the symbol $[x]$ the jump in x across the discontinuity. The last equation may easily be transformed to a form showing that one of the Riemann invariants has to be constant across a discontinuity. Hence, a discontinuity belongs to one of the two following types:

$\sigma = \frac{[f]}{[S]}$	T = const.	"Buckley-Leverett shock"
$\sigma = g$	g = const.	Contact discontinuity

In a summary, the solution of the Riemann problem for Eq.(1.1) consists of a sequence of the follwing waves: .

1) Rarefaction waves where T is constant and the wave speed given by f_S.

2) Shocks where T is constant across the discontinuity, and the shock speed is given by the Rankine-Hugoniot condition.

3) Contact-discontinuities where g is constant across the discontinuity, and the speed of the discontinuity equals g.

The two first types will be treated together and denoted as a S-wave, the last type will be denoted as a T-wave. Consequently, if a fundamental wave is allowed to be "degenerate" in the way that the left state equals the right state, the general form of the Riemann solution is $S_1T_1S_2....T_n$. S_1 is a wave with left state U^L, T_n a wave with right state U^R. Let a J-curve denote a general contineous, piecewise smooth curve in phase space, connecting two states and consisting of a union of sements of level curves for T and g. In the phase space, the solution of the Riemann problem obviously is a J-curve. Fig 4.1 shows level curves for g, together with an example of a J-curve.

Note that changes in T can only occur through a T-wave, i.e through a contact discontinuity.

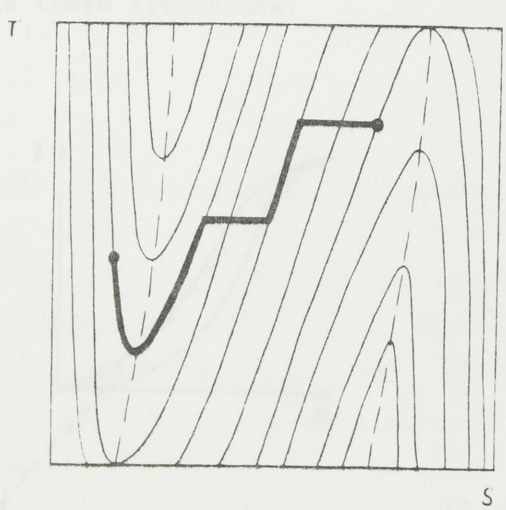


Fig 4.1 : Level curves for g, together with an example of a J-curve

5. ENTROPY CONDITIONS

An entropy condition for a non-strictly hyperbolic system was formulated by Keyfitz and Kranzer [14], generalizing the well-known Lax entropy condition [18]. For a 2×2 system, the Keyfitz-and-Kranzer entropy condition states that for a discontinuity in the solution to be admissible, either

1) 3 characteristics enter the discontinuity and 1 leaves, - or
2) 2 characteristics are tangents to the discontinuity and at least one of the remaining enters the discontinuity, - or
3) the shock may be regarded as a limit of a sequence of shocks satisfying 1) or 2)

As an example of a shock admissible according to condition 3), take a situation where one of the shock values is situated on a transition curve; then 3 characteristics are tangents, while the remaining enters or leaves the discontinuity. By a careful choice of the function f, it is also possible to construct discontinuities where all the characteristics are tangents to the line of discontinuity, confer Fig 5.1. The condition (E) given by Temple [27] does not adequately describe these situations.

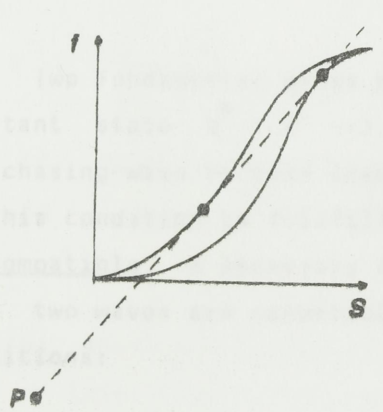


Fig.5.1 : Example of a discontinuity where all characteristics are tangents to the line of discontinuity

It is well-known that for a strictly hyperbolic system with local linear degeneracies, the Lax criterion is not restrictive enough

to resolve a unique solution [20,23]. As the function f(T=const) may have more than one inflection point, we will not use the generalized Lax criterion directly. Combining the Rankine-Hugoniot condition, Eq.(4.5), and the relation between f and g, Eq.(2.3), we have

$$g^R - \sigma = \frac{S^L + \beta}{S^R + \beta} \left(g^L - \sigma \right) \tag{5.1}$$

As (S + β) is always positive, it follows that if $\sigma \neq g^L$, one of the characteristics belonging to the second family leaves, while the other enters the discontinuity. Hence, for a "Buckley-Leverett" shock it is sufficient to study the behaviour as if only one family of waves is present. The following shock-admissibility criterion is eqivalent to the generalized Lax criterion if no local linear degeneracies is present:

Shock-admissibility criterion

A T-wave is admissible if it does not cross a transition curve. A S-wave is admissible if it satisfies the Oleinik condition [23] for constant T:

$$\frac{f - f^R}{S - S^R} \leq \frac{f^L - f^R}{S^L - S^R} \qquad T = T^L = T^R$$

Two fundamental waves may be combined into one wave through a constant state U^M ($U^L \to U^M \to U^R$) if the speed of the front of the chasing wave is less than the speed of the tail of the chased, and if this condition is fullfilled, the two fundamental waves are said to be compatible. A necessary and sufficient criterion for determing when two waves are compatible is given by the following compatibility conditions:

Compatibility conditions (CC)

The two fundamental waves in a TS-wave are compatible if and only if one of the following conditions is satisfied:

1) The S-wave contains a rarefaction part close to U^M, and $U^M \varepsilon B$.

2) The S-wave contains a shock close to U^M and, if supscript R denotes the right shock value,

$$\frac{g^M - g^R}{S^M - S^R} \geqslant 0$$

The two fundamental waves in a ST-wave are compatible if and only if one of the following conditions is satisfied:

1) The S-wave contains a rarefaction part close to U^M, and $U^M \varepsilon A \quad C$.

2) The S-wave contains a shock close to U^M and, if supscript L denotes the left shock value,

$$\frac{g^M - g^L}{S^M - S^L} \leqslant 0$$

Proof:

Only the first part concerning the TS-wave will be shown, as the second case is analogous. If S has a rarefaction part close to U^M, the waves are compatible iff $g^M \leqslant f_S^M$, i.e iff $U^M \varepsilon B$. If S has a shock close to U^M, the waves are compatible iff

$$0 \leqslant \frac{f^M - f^R}{S^M - S^R} - g^M = \frac{g^M - g^R}{S^M - S^R} (S^R + \beta) \qquad (5.2)$$

Here, the Rankine-Hugoniot expression for the shock speed has been used. As $(S + \beta)$ is always positive, the result follows.

The shock-admissibility criterion is valid independently of the number of transition curves and the number of inflextion points of f. Also, the CC is easily extended to be valid for a general number of transition curves. Together, the shock-admissibility criterion and the CC will be used to construct a solution of the Riemann problem, unique in the (x,t)-space, termed the entropy solution.

6. CONSTRUCTION OF AN ENTROPY FUNCTION

An entropy function E for the a hyperbolic system is a scalar function, having the following properties

1) E is convex, i.e the Hessian d^2E is positive definite
2) A scalar function F exists such that $\nabla E = A\,\nabla F$ where A is the system matrix and ∇ is the gradient operator in the unknown functions

For a strictly hyperbolic, genuine non-linear system, if an entropy exists, a viscous regularization must satisfy the following inequality in the topology of distributions [19]:

$$E_t + F_x \leqslant 0 \tag{6.1}$$

If E is a known function, Eq.(6.1) can be used as a shock-admissibility criterion. Using Eq.(1.1), is it easy to to show that a function $E = E(u,v)$ can have the property 2) only if it satisfies the following equation:

$$ug_v E_{uu} + (vg_v - ug_u)E_{uv} - vg_u E_{vv} = 0 \tag{6.2}$$

For a general 2x2 system, the entropy-equation is of equal type as the original 1.order system, i.e. Eq.(6.2) is hyperbolic everywhere except along the transition curves. The equation may be integrated by writing the equation on the form

$$(g_v, -g_u)\cdot\nabla[\,uE_u + vE_v - E\,] = 0 \tag{6.3}$$

As $(g_v, -g_u)$ is the tangent-vector to level-curves of g, it follows that E must satisfy Clairaut's differential equation with an inhomogeneity term φ,- φ being an arbitrary function of g:

$$uE_u + vE_v - E = \varphi(g) \tag{6.4}$$

By transforming to polar coordinates as defined in Eq.(2.5), the solution for E is written:

$$E = r \int \frac{\varphi}{r^2} dr + r\psi \qquad (6.5)$$

$\psi = \psi(\theta)$ is a new arbitrary function. The entropy-flux F is given by

$$F = gE + \int \varphi \, dg \qquad (6.6)$$

A necessary condition for the given entropy function to be convex in (r, θ) is that E_{rr} is positive:

$$E_{rr} = \frac{\varphi'}{r^2} g_r \qquad (6.7)$$

φ' denotes the derivative of φ with respect to g. As g_r changes sign across the transition curves, it follows that E_{rr} also changes sign in the phase space. Hence, it is not possible to construct a contineous entropy that is globally convex, and at the transition curves it is not possible to construct a locally convex entropy. Note that for a strictly hyperbolic 2x2 system, it was shown by Lax [19] that it is always possible to define a convex entropy locally.

If $[x]$ denotes a change in the quantity x through a discontinuity, $[x] = x^L - x^R$, then the entropy production caused by a discontinuity is given by

$$\sigma[E] - [F] = - r^L r^R \frac{g^R - g^L}{r^R - r^L} \int_{r^L}^{r^R} \frac{\varphi}{r^2} dr + \int_{g^L}^{g^R} \varphi \, dg \qquad (6.8)$$

It follows easily that for a contact discontinuity - and also for a "Buckley-Leverett" shock having $g^L = g^R$ - the entropy change across the discontinuity is zero. Expanding the right hand side of Eq.(6.8) in a Taylor series in the "shock-strenght" $(r^R - r^L)$, we have

$$\sigma[E] - [F] = \left\{ \frac{\varphi' g_r}{r^2} \left(r^2 g_r \right)_r \right\}^L \left(r^R - r^L \right)^3 + \ldots \qquad (6.9)$$

Now assume that a given wave is fully contained in <u>one</u> of the regions A, B or C and satisfies the shock-admissibility criterion given in

Chapter 5. As $(r^2 g_r)_r = (S + \beta) f_{SS}$, Eq.(6.9) shows that for a sufficiently weak shock where $f_{SS} \neq 0$, the shock-admissibility criterion is equivalent to an increase or decrease in entropy, according to whether the wave is contained in $A \cup C$ or in B. However, waves crossing a transition curve may cause an increase, a decrease or no effect in the entropy.

When deriving the property Eq.(6.1) for a strictly hyperbolic system, the choice of a convex or concave entropy is just a matter of convenience, - the inequality sign in Eq.(6.1) must however be reversed if a concave entropy is chosen [6]. Hence, for a weak shock the given entropy could be used to resolve the entropy solution if and only if it could be guaranteed that the solution does not cross a transition curve.

It will be shown in the next section that in some cases where no entropy-change is produced, the solution of the Riemann problem is not unique in the phase space.

7. CONSTRUCTION OF THE RIEMANN SOLUTION

In the following, let $R[U^L, U^R]$ denote the solution of the Riemann problem Eqs.(1.1) and (1.3). As already shown, the solution is composed of a sequence of fundamental waves $J = S_1 T_1 S_2 \ldots T_n$, where S_1 is a wave with left state U^L. The goal of this chapter is to show the following theorem

<u>THEOREM 1</u>

The Riemann problem for Eqs.(1.1) has an entropy solution of the form $S_1 T S_2$. The solution is unique in (x,t)-space, but not in the phase space.

The proof is based on a study of cases, where the solution is shown to belong to one of the following classes of J-curves:

Class 1 : $J = TS$ where $T : U^L \dashrightarrow U^M$
$S : U^M \dashrightarrow U^R$

Each fundamental wave satisfies the shock-admissibility criterion. If $U^L \in S_1 \cup S_2$, then $U^M \in B$.

Class 2 : $J = ST$ where $S: U^L \dashrightarrow U^M$
$T: U^M \dashrightarrow U^R$

Each fundamental wave satisfies the shock-admissibility criterion. If $U^R \in S_1 \cup S_2$, then $U^M \in A \cup C$.

Class 3 : $J = S_1 T S_2$ where $S_1 : U^L \dashrightarrow U^M$ $U^L \in A \cup C$
$T : U^M \dashrightarrow U^N$
$S_2 : U^N \dashrightarrow U^R$

Each fundamental wave satisfies the shock-admissibility criterion. One of the states U^M or U^N is lying on the transition curve near U^L, according to the following rule:

$U^L \varepsilon$ C and $T^L < T^R \quad => \quad U^N \varepsilon S_1 \cup S_2$

$U^L \varepsilon$ A and $T^L > T^R \quad => \quad U^N \varepsilon S_1 \cup S_2$

Else $\quad U^M \varepsilon S_1 \cup S_2$

It is easily seen from Fig 4.1 that between two arbitrary states, infinitely many J-curves not belonging to one of the classes 1, 2 or 3 exist. However, at least one curve belonging to one of the classes does exist, - and in each class, if it exists, the J-curve is unique.

In three following lemmas, criterions for when the Riemann solution belongs to one of the three classes will be given. Finally, it will be shown that the given criterions covers all possible combinations of U^L and U^R. To facilitate the statement of the criterions, define the quantities g_1 and g_2 by:

$$g_1 = g(\,S_1(T^R),T^R\,)$$
$$g_2 = g(\,S_2(T^R),T^R\,) \tag{7.1}$$

Also, let T_1 and T_2 be defined implicitly as solutions of the equations

$$g^R = g(\,S_1(T_1),T_1\,)$$
$$g^R = g(\,S_2(T_2),T_2\,) \tag{7.2}$$

If no solution for T_1 exists, define $T_1 = 0$. If no solution for T_2 exists, define $T_2 = 1$.

If a solution of Class 1-3 exists, it remains to show that also the compatibility criterion CC is satisfied, as each of the fundamental waves satisfies the shock-admissibility criterion. To show that a given J-curve consists of compatible waves, the following corollaries of the CC will be used:

<u>COROLLARY 1</u>

The fundamental waves in a Class-1 wave are compatible if both U^L and U^R are contained in B. The fundamental waves in a Class-2 wave are compatible if both U^L and U^R are contained in A or if both are contained in C.

COROLLARY 2

The fundamental waves in a J-curve of Class 1 or Class 2 are compatible if $U^M \varepsilon S_1 \cup S_2$.

Proof of Corollary 1:

When a given wave J is of Class 1 and U^L, $U^R \varepsilon B$, the whole J-curve is contained in B. As $g_S \geqslant 0$ in B, the CC is satisfied. Analogous for a Class-3 wave.

Proof of Corollary 2:

The proof is shown for $U^M \varepsilon S_1$. If the S-wave has a rarefaction part close to U^M, the result follows immediately from the CC. As f(T=const.) is convex in the vicinity of S_1, the S-wave contains a shock close to U^M only if $U^R \varepsilon A$. This gives two possibilities for a shock, according to whether the S-wave or the T-wave is contained in A Taking each case separately, it is easily checked that the CC is satisfied.

LEMMA 1

$R[U^L, U^R]$ is of Class 1 if $U^L \varepsilon B$, and in addition, one of the following conditions is satisfied:

1) $U^R \varepsilon A$ and $g^R \leqslant g^L \leqslant g_2$

2) $U^R \varepsilon B$ and $g_1 \leqslant g^L \leqslant g_2$

3) $U^R \varepsilon C$ and $g_1 \leqslant g^L \leqslant g^R$

Proof

From Fig 4.1 showing the level curves for g, it is obvious that when $U^L \varepsilon B$, $g_1 \leqslant g^L \leqslant g_2$ guarantees the existence of a J-curve of Class 1 connecting U^L and U^R. It remains to show that the fundamental waves are compatible.

If the S-wave has a rarefaction wave close to U^M, the CC is immediately satisfied, as $U^M \varepsilon B$ when $U^L \varepsilon B$. Hence, in the following, it is sufficient to study the situations where the S-wave contains a shock close to U^M:

When $U^R \varepsilon A$, the S-wave contains a shock crossing the transition curve S_1, as f(T=const) is convex in the vicinety of S_1. If U^* denotes the right shock value, then $S^* \geqslant S^R \Rightarrow g^* \leqslant g^R$, and the condition 1) in the lemma guarantees that the CC is satisfied. The argument is identical when $U^R \varepsilon C$. When $U^R \varepsilon B$, the lemma follows immediately from Corollary 1.

Lemma 2

$R[U^L, U^R]$ is of Class 2 if one of the following conditions are satisfied:

1) $U^R \varepsilon A$, $U^L \varepsilon A$ and $T^L \geqslant T_1$

2) $U^R \varepsilon A$, $U^L \varepsilon B$ and $g^L \leqslant g^R$

3) $U^R \varepsilon C$, $U^L \varepsilon C$ and $T^L \leqslant T_2$

4) $U^R \varepsilon C$, $U^L \varepsilon B$ and $g^L \geqslant g^R$

Proof :

The lemma is shown for $U^R \varepsilon A$, as the proof for $U^R \varepsilon C$ is analogous. When $U^R \varepsilon A$, the condition $T^L \geqslant T_1$ guarantees the existence of a J-curve of Class 2 connecting U^L and U^R.

When $U^L \varepsilon A$, the CC is satisfied by Corollary 1. If $U^L \varepsilon B$, the S-wave must cross S_1 by a shock, as f(T=const) is convex in the vicinety of S_1. The S-wave may still contain a rarefaction part close to U^M, and the CC is then straightforward satisfied. If the S-wave contains a shock close to U^M, the S-wave must be a pure shock wave. As $S^M \leqslant S^L$ and as $g^R = g^M \geqslant g^L$, the CC is satisfied.

Lemma 3

$R[U^L, U^R]$ is of Class 3 if one of the following conditions are satisfied:

1) $U^R \in A \cup B$ and $U^L \in C$

2) $U^R \in B \cup C$ and $U^L \in A$

3) $U^R \in A$, $U^L \in A$ and $T^L < T_1$

3) $U^R \in C$, $U^L \in C$ and $T^L > T_2$

4) $U^R \in A \cup B$, $U^L \in B$ and $g_2 < g^L$

5) $U^R \in B \cup C$, $U^L \in B$ and $g^L < g_1$

Proof :

The lemma is proven by in each case combining Lemma 1 or Lemma 2 together with Corollary 2. Only one case will be shown, the case $U^R \in A$, $U^L \in C$: In this case, it is obviously possible to construct a Class-3 J-curve with $U^N \in S_2$. Lemma 2 then gives that the fundamental waves in the S_1 T-wave are compatible. Corollary 2 gives that the combination TS_2 is compatible.

Fig 7.1-7, pp 21-22 show all the combinations of U^L and U^R covered by the Lemmas 1-3. In each figure, U^R is specified, and the phase space is divided into different regions showing the solution type if U^L is contained in the region. It is easily seen that all possible combinations have been covered by the lemmas.

If U^L is situated on the boundary between two regions, the lemmas may state that solutions of two different classes exist. The reason for this may be that the J-curve belongs to two of the classes simultaneously, as when a S-wave in a Class-3 curve is "degenerate", i.e has equal left and right states. However, in certain cases two different solutions exist in phase space,- this is when

$$U^L \in B \qquad U^R \in A \cup C$$
$$g^L = g^R \qquad (7.1)$$

It is easily checked that in (x,t)-space, the solution is unique, being a single discontinuity with speed g^L. The solution could be regarded as a pure T-wave, but this would however not satisfy the shock-admissibility criterion. The non-uniqueness in the phase-space was first pointed out by Isaacson [12] in the case when the system has a single transition curve. It follows from Eq.(7.1) and Eq.(6.8) that the change in entropy caused by the non-unique waves is zero.

Note that in some cases, a small perturbation of the initial states may change the solution drasticly. This is the case for instance if the initial states are close to a situation as described by Eq.(7.1); a perturbation may cause the solution to change between a Class-1 wave and a Class-2 wave.

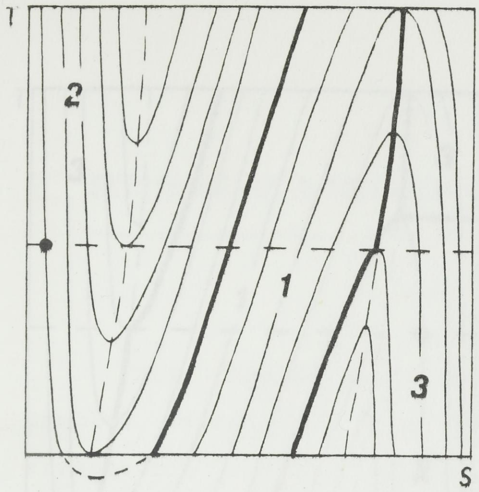


Fig 7.1

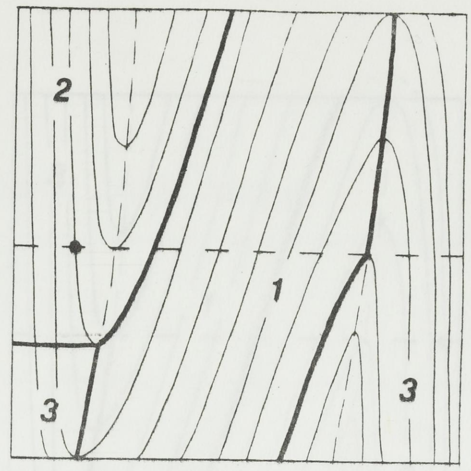


Fig 7.2

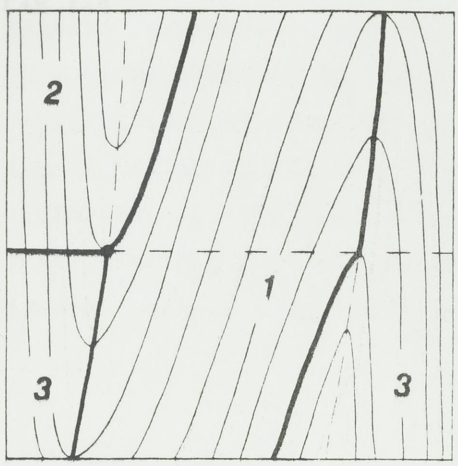


Fig 7.3

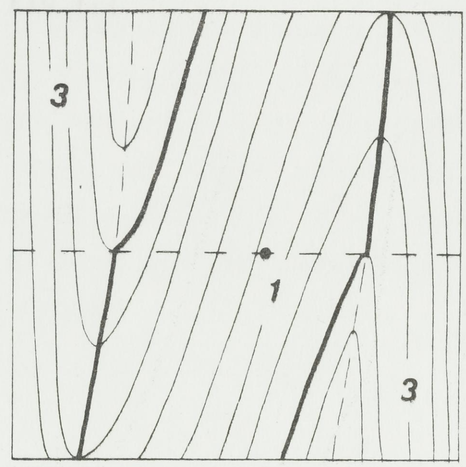


Fig 7.4

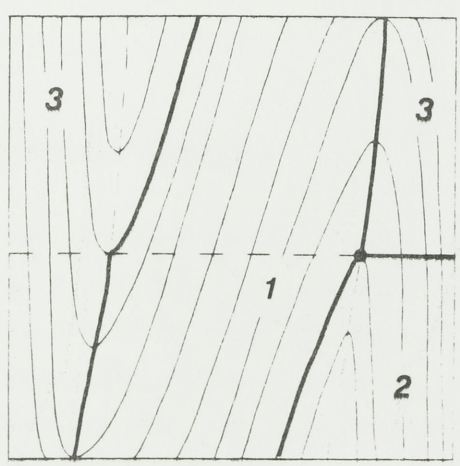


Fig 7.5

Fig 7.1-7 : Solution regions in phase space.
The position of U^R is shown by a black dot

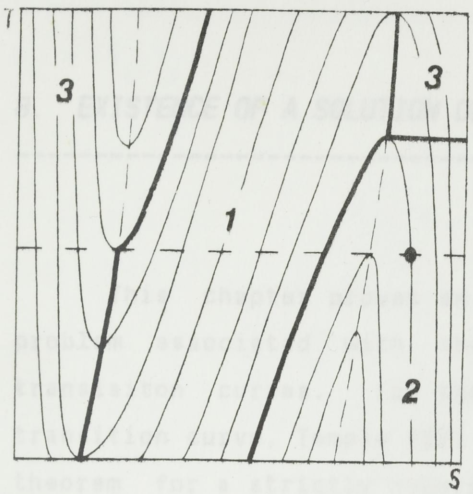


Fig 7.6

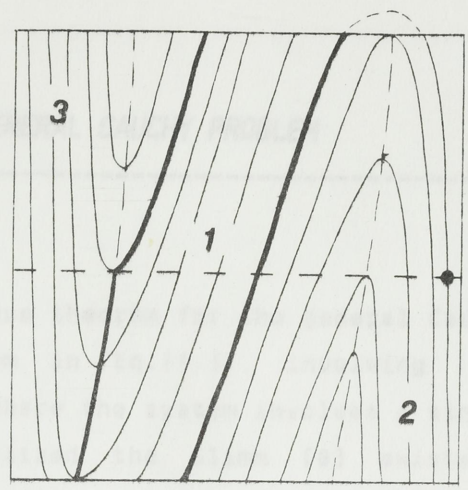


Fig 7.7

8. EXISTENCE OF A SOLUTION OF THE GENERAL CAUCHY PROBLEM

This chapter proves an existence theorem for the general Cauchy problem associated with the system in Eq.(1.1), involving two transisiton curves. For the case where the system involves a single transition curve, Temple [27] generalized the Glimm [8] existence theorem for a strictly hyperbolic system. The proof here follows the steps outlined by Temple, defining a transformation Ψ: $(S,T) \rightarrow (Z,T)$, regular everywhere except along the transisition curves, and using this transformation to construct the Glimm functional. As only the construction of the Glimm functional deviates from the proof of Temple, it is sufficient to prove that the Glimm functional is minimized by the waves of the Riemann solution $R[U^L,U^R]$. Given a certain property of the function g, the proof may be extended to a general number of distinct transition curves.

<u>THEOREM 2</u>
The Cauchy problem for the system Eq.(1.1), involving two transisition curves, has a global weak solution for arbitrary initial data of bounded variation in Z and T.

<u>Definition of the Glimm functional</u>

The variable Z is defined by first extending the definition of g into the domain $(S,T) \in [0,1]\times[-1,2]$. This is done by extending the level curves for g in a non-intersecting differentiable manner, as shown on Fig 8.1. The level curves are monotone everywhere except at the transisiton curves, and the extension of the level curves also involves an extension of the transition curves in a smooth way.

Note that to each point U in the original phase space it is possible to associate one point on each of the extended transition curves lying on the same g-level curve as U. These two point will be

termed the associated points for U and denoted U_1 and U_2 respectively, $U_i = (S_i, T_i)$.

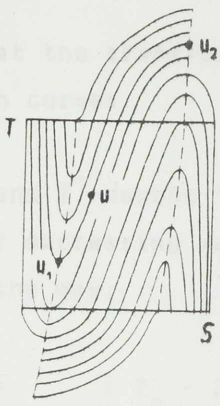


Fig 8.1 : Extension of the function g

The extension of the level curves is not unique, and to facilitate the analysis, an extension will be chosen such that if A and B are two arbitrary states,

$$|T_2^A - T_2^B| \leq 2|T_1^A - T_1^B| \tag{8.1}$$

This is always possible if the following condition is satisfied:

$$g[S_1(0),0] \leq g[S_2(1),1] \tag{8.2}$$

If it is not possible to define a extension satisfying Eq.(8.1), the significance played by the to transition curves in the following must be interchanged. In the case of n transition curves, one must assume the following condtion can be satisfied:

$$|T_n^A - T_n^B| \leq 2|T_{n-1}^A - T_{n-1}^B| \leq \ldots \leq 2(n-1)|T_1^A - T_1^B| \tag{8.3}$$

Now define the variable Z in the following manner:

$$Z(S,T) = \begin{cases} T_2 - T & U \in C \\ T - T_2 & U \in B \\ -(T - T_1) - (T_2 - T_1) & U \in A \end{cases} \tag{8.4}$$

The Jacobian of the transformation Ψ equals Z_S, and from the relation

$$\frac{\partial T_i}{\partial S} = \frac{g_S}{\left[g_S \dfrac{dS_i}{dT} + g_T \right]_{(S_i, T_i)}} \qquad i=1,2 \qquad (8.5)$$

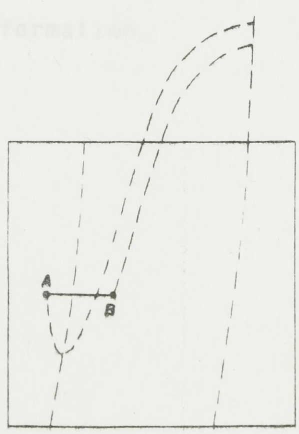
it follows that the transformation is regular everywhere, except along the transition curves.

Let T^+ and T^- denote T-waves where S, from left to right, is increasing or decreasing respectively. Then define the <u>wave strength</u> in the following way:

$$|S| = |Z_R - Z_L|$$

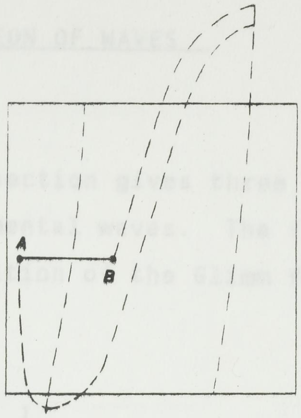
$$|T| = \begin{cases} 8|T_R - T_L| & T \in A \text{ and } T = T^- \\ 2|T_R - T_L| & T \in A \text{ and } T = T^+ \\ 6|T_R - T_L| & T \in B \text{ and } T = T^- \\ 4|T_R - T_L| & T \in B \text{ and } T = T^+ \\ 4|T_R - T_L| & T \in C \text{ and } T = T^- \\ 6|T_R - T_L| & T \in C \text{ and } T = T^+ \end{cases} \qquad (8.6)$$

Note that for a S-wave in A B, the definition of wave strenght may create two different situations, demonstrated in Fig 8.2:



$$|S| = 2(T^A - T_1) + (T_2^A - T_2^B)$$

<u>Fig 8.2A</u> : Wave strength for waves crossing S_1.



$$| S | = 2(T - T_1^A) - (T_2^B - T_2^A)$$

Fig 8.2B : Wave strength for waves crossing S_1.

The significance of the extension condition Eq.(8.1) is demonstrated in the latter possibility, as Eq(8.1) ensures that the expression given for $|S|$ is positive.

Finally, if J denotes a general J-curve as defined in Chapter 4, the Glimm functional is defined by

$$F(J) = \sum_J (|S| + |T|) \qquad (8.7)$$

To prove that F is minimized by $J = R[U^L, U^R]$, the concepts of <u>addition</u> and <u>interchange</u> of waves will be used, as introduced by Temple. Additionally, a concept of <u>reduction</u> of waves will be defined. The purpose is to use these operations successively to transform an arbitrary J-curve into the Riemann solution, and and the same time ensure that the Glimm functional decreases through each step of transformation.

ADDITION OF WAVES

This section gives three lemmas stating the behaviour of F when adding fundamental waves. The first is a straight-forward consequence of the definition of the Glimm functional, Eq.(8.7):

LEMMA 1

$F(ST) = F(S) + F(T)$

The next lemma concerns addition of S-waves. If $J = S_1 S_2$ takes U^L to U^R, define the sum $S = S_1 + S_2$ as the unique wave $S = R[U^L, U^R]$.

LEMMA 2

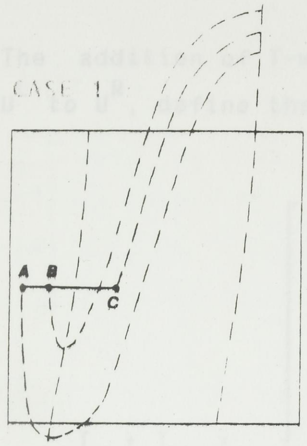
$F(S_1 + S_2) \leq F(S_1) + F(S_2)$.

If S_1 and S_2 have only one state in common, then $F(S) = F(S_1) + F(S_2)$.

Proof

A complete proof involves a study of all possible combinations of U^L and U^R, and only a few will be shown here to verify the lemma. Let A, B and C be three states such that $T = T^A = T^B = T^C$. Let S_1 be a S-wave from A to B, S_2 a S-wave from B to C, and let $S = S_1 + S_2$. The next page shows some typical cases. Note that the extension condition Eq.(8.1) is used several times.

CASE 1 :



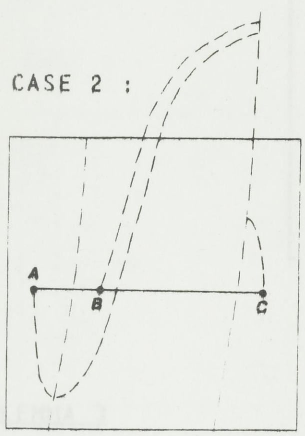
$$F(S_1) = 2(T_1^B - T_1^A) - (T_2^B - T_2^A)$$

$$F(S_2) = 2(T_2^B - T_2^C) - (T - T_1^B)$$

$$F(S) = 2(T - T_1^A) - (T_2^C - T_2^A)$$

$$F(S_1) + F(S_2) - F(S) = 0$$

CASE 2 :



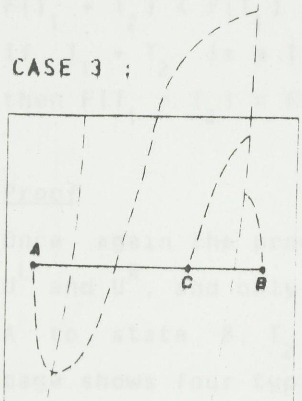
$$F(S_1) = 2(T - T_1^A) - (T_2^B - T_2^A)$$

$$F(S_2) = T_2^B + T_2^C - 2T$$

$$F(S) = T_2^A + T_2^C - 2T_1^A$$

$$F(S_1) + F(S_2) - F(S) = 0$$

CASE 3 :



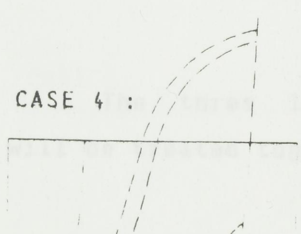
$$F(S_1) = T_2^A + T_2^B - 2T_1^C$$

$$F(S_2) = (T_2^B - T_2^C) + 2(T - T_1^B)$$

$$F(S) = T_2^A + T_2^C - T$$

$$F(S_1) + F(S_2) - F(S) = 2(T_2^B - T_2^C) + 4(T - T_2^B) \geq 0$$

CASE 4 :



$$F(S_1) = (T_2^A - T_2^B) + 2(T - T_1^A)$$

$$F(S_2) = T_2^C - T_2^B$$

$$F(S) = 2(T - T_1^A) - (T_2^C - T_2^A)$$

$$F(S_1) + F(S_2) - F(S) = 2(T_2^C - T_2^B) \geq 0$$

The addition of T-waves is a bit more complicated. If $J = T_1 T_2$ takes U^L to U^R, define the sum as following:

$T_1 + T_2 =$

- $R[U^L, U^R]$ when T_1 and T_2 both are contained in the same domain A, B or C.
- The unique TS-wave taking U^L to U^R when the two states are separated by S_1 and $T^L \geq T^R$
- The unique TS-wave taking U^L to U^R when the two states are separated by S_2 and $T^L \leq T^R$
- The unique ST-wave taking U^L to U^R when the two states are separated by S_1 and $T^L \leq T^R$
- The unique ST-wave taking U^L to U^R when the two states are separated by S_2 and $T^L \geq T^R$

LEMMA 3

$F(T_1 + T_2) \leq F(T_1) + F(T_2)$.

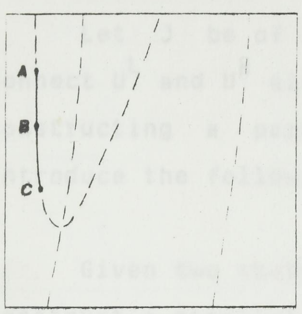
If $T_1 + T_2$ is a T-wave, and T_1 and T_2 have one point in common only, then $F(T_1 + T_2) = F(T_1) + F(T_2)$.

Proof

Once again the proof involves a study of all possible combinations of U^L and U^R, and only a few will be shown. Let T_1 be a wave from state A to state B, T_2 a wave from B to C, and let $J = T_1 + T_2$. The next page shows four typical cases.

The three lemmas stating the behaviour of F when adding waves will be treated together and termed the addition lemma.

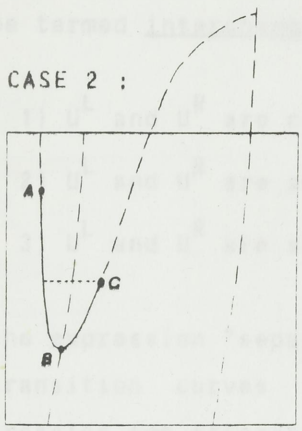
CASE 1 :



$$F(T_1) = 8(T^A - T^B)$$
$$F(T_2) = 8(T^C - T^B)$$
$$F(J) = 8(T^C - T^A)$$

$$F(T_1) + F(T_2) - F(J) = 0$$

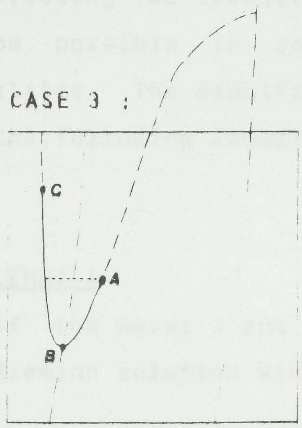
CASE 2 :



$$F(T_1) = 2(T^A - T^B)$$
$$F(T_2) = 6(T^C - T^B)$$
$$F(J) = 2(T^A - T^C) + 2(T^C - T^B)$$

$$F(T_1) + F(T_2) - F(J) = 6(T^C - T^B) \geqslant 0$$

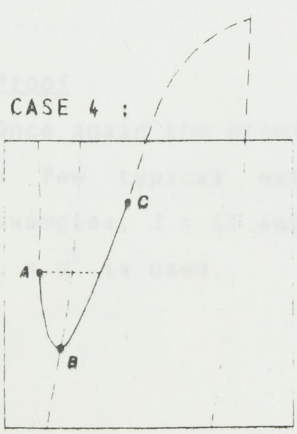
CASE 3 :



$$F(T_1) = 4(T^A - T^B)$$
$$F(T_2) = 8(T^C - T^B)$$
$$F(J) = 8(T^C - T^A) + 2(T^A - T^B)$$

$$F(T_1) + F(T_2) - F(J) = 10(T^A - T^B) \geqslant 0$$

CASE 4 :



$$F(T_1) = 2(T^A - T^B)$$
$$F(T_2) = 6(T^C - T^B)$$
$$F(J) = 2(T^A - T^B) + 6(T^C - T^A)$$

$$F(T_1) + F(T_2) - F(J) = 6(T^A - T^B) \geqslant 0$$

INTERCHANGE OF WAVES

Let J be of the form ST (TS). In many cases it is possible to connect U^L and U^R also through a wave of the form $J' = TS$ (ST), thus constructing a parallelogram of segments related to the two states. Introduce the following definition:

Given two states U^L and U^R and assume that it is possible to construct a parallelogram as described above. The waves J and J' will be termed interchangeable if

1) U^L and U^R are contained in the same domain A, B or C.
2) U^L and U^R are separated by S_1 only, and $U^R \in A$.
3) U^L and U^R are separated by S_2 only, and $U^R \in C$.

The expression "separated by S_i only" means that only one of the transition curves separates the two states. Note that no waves crossing two transition curves are interchangeable, even though it may be possible to construct a parallelogram of waves around the two states. The significance of the concept "interchange" follows from the following lemma:

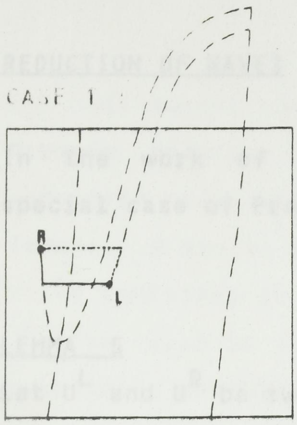
LEMMA 4

If the waves J and J' are interchangeable, then $F(J) = F(J')$, and the Riemann solution $R[U^L,U^R]$ is given by J or by J'.

Proof

Once again the proof includes a study of all possible cases, and only a few typical examples are shown on the next page. In all the examples, $J = ST$ and $J' = TS$. Also, the shorthand notation $R = U^R$, $L = U^L$ is used.

CASE 1 :



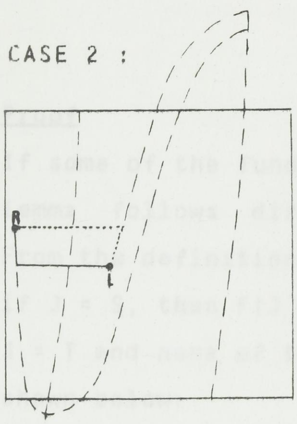
$$F(J) = 8(T^R - T^L) + (T_2^R - T_2^L) + 2(T^L - T_1^R)$$

$$F(J') = 6(T^R - T^L) + (T_2^R - T_2^L) + 2(T^R - T_1^R)$$

$$F(J) = F(J')$$

$$R[U^L, U^R] = J'$$

CASE 2 :



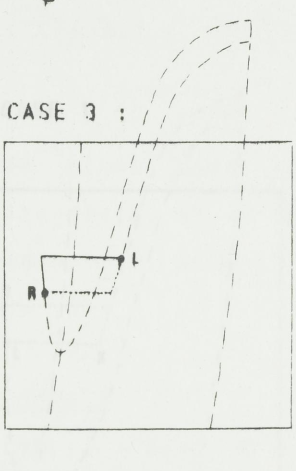
$$F(J) = 8(T^R - T^L) - (T_2^R - T_2^L) + 2(T^L - T_1^R)$$

$$F(J') = 6(T^R - T^L) - (T_2^R - T_2^L) + 2(T^R - T_1^R)$$

$$F(J) = F(J')$$

$$R[U^L, U^R] = J$$

CASE 3 :



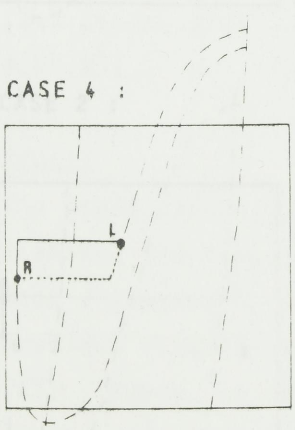
$$F(J) = 2(T^L - T^R) + (T_2^R - T_2^L) + 2(T^L - T_1^R)$$

$$F(J') = 4(T^L - T^R) + (T_2^R - T_2^L) + 2(T^R - T_1^R)$$

$$F(J) = F(J')$$

$$R[U^L, U^R] = J'$$

CASE 4 :



$$F(J) = 2(T^L - T^R) - (T_2^R - T_2^L) + 2(T^L - T_1^R)$$

$$F(J') = 4(T^L - T^R) - (T_2^R - T_2^L) + 2(T^R - T_1^R)$$

$$F(J) = F(J')$$

$$R[U^L, U^R] = J$$

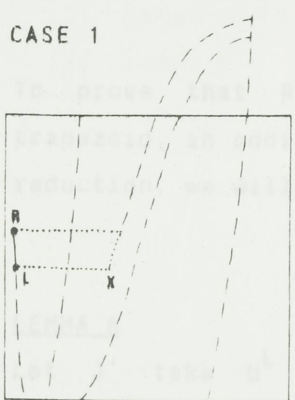
REDUCTION OF WAVES

In the work of Temple [27], the following lemma is included as a special case of Proposition 5.1:

LEMMA 5
Let U^L and U^R be two states that may be joined by a single T- or S-wave, J. Let J' be another J-curve of the form S_1TS_2 or T_1ST_2 connecting the two states. Then $F(J) \leqslant F(J')$.

Proof
If some of the fundamental waves in J' are interchangeable, then the lemma follows directly from the lemmas of addition and interchange. From the definitions of wave strenghts, it also follows readily that if J = S, then $F(J') - F(J) \geqslant 4|T^R - T^L|$. Some typical examples when J = T and none of the fundamental waves in J' are interchangeable are shown below:

CASE 1

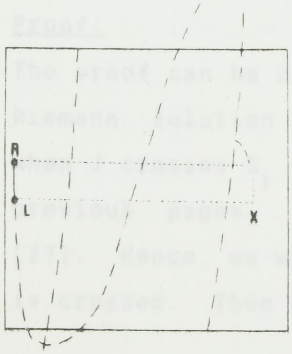


$$F(J) = 8(T^R - T^L)$$

$$\begin{aligned} F(J') = {} & 2(T^L - T_1^L) - (T_2^L - T_2^X) \\ & + 6(T^R - T^L) \\ & + 2(T^R - T_1^L) - (T_2^L - T_2^X) \end{aligned}$$

$$F(J') - F(J) = 4(T^L - T_1^L) - 2(T_2^L - T_2^X) \geqslant 0$$

CASE 2 :



$$F(J) = 8(T^R - T^L)$$

$$\begin{aligned} F(J') = {} & (T_2^R + T_2^X - 2T_1^L) \\ & + 4(T^R - T^L) \\ & (T_2^R + T_2^X - 2T_1^L) \end{aligned}$$

$$\begin{aligned} F(J') - F(J) = {} & 2(T_2^L - T^L) + 2(T_2^L - T^X) \\ & + 4(T^L - T_1^L) \geqslant 0 \end{aligned}$$

It is not possible to transform an arbitrary J-curve J into the Riemann solution through the operations addition, interchange and reduction only. However, by a successive application of the three lemmas, J may be transformed into a new curve J', $F(J') \leq F(J)$, where J' is contained on or inside a certain "trapezoid" around U^L and U^R. J' can also be restricted to follow certain "main routes" inside the trapezoid, confer Fig 8.3:

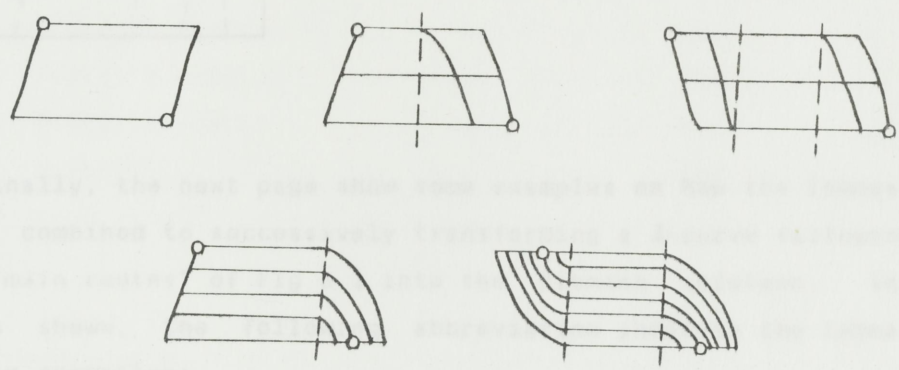


Fig 8.3 : "Main routes" after a transformation of J

To prove that $R[U^L, U^R]$ gives the minimum value of F inside the trapezoid, in addition to the lemmas of addition, interchange and reduction, we will need the following lemma:

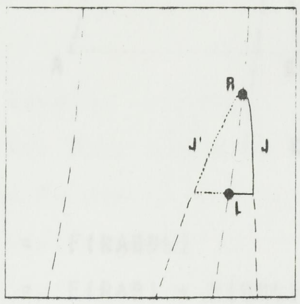
LEMMA 6

Let J' take U^L to U^R by crossing maximum one transition curve, and also assume that J' = ST or J' = TS. Then $F(J) \leq F(J')$ where $J = R[U^L, U^R]$.

Proof:

The proof can be divided into three parts, according to whether the Riemann solution crosses S_1, S_2 or no transition curve. The proof when J crosses S_1 or S_2 follows the same lines as the proofs on the previous pages, also confer the proof of Proposition 5.0 by Temple [27]. Hence, we will only study the situation when no transiton curve is crossed. Then three possibilities exist: Either J = J', J and J'

are interchangeable, or U^L and U^R are both contained on S_1 S_2. In the two first cases, obviously $F(J) = F(J')$. An example of the last case is shown by the following:



$$R[U^L, U^R] = J$$

$$F(J') = 6(T^R - T^L) + (T^R - T^L)$$

$$F(J) = 4(T^R - T^L) + (T^R - T^L)$$

$$F(J') - F(J) = 2(T^R - T^L) \geq 0$$

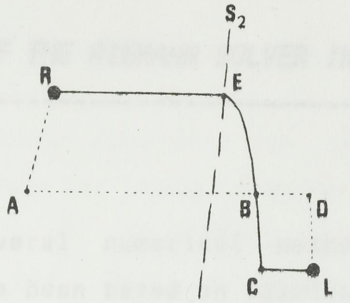
Finally, the next page show some examples on how the lemmas 1-6 can be combined to successively transforming a J-curve following one of the "main routes" of Fig 8.3 into the Riemann solution. In the examples shown, the following abbreviation indicate the lemma used during an operation:

(A) Lemma of addition

(I) Lemma of interchange

(X) Lemma 6

CASE 1 :



F(J') = F(RABDL)

= F(RAB) + F(BDL) (A)

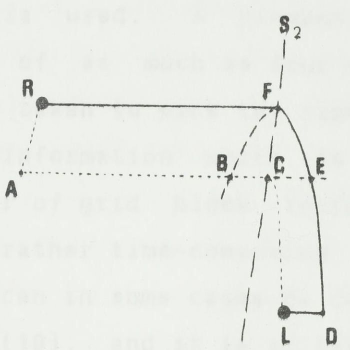
$\leq$ F(REB) + F(BDL) (X)

= F(REB) + F(BCL) (I)

= F(RECL) (A)

= F(J)

CASE 2 :



F(J') = F(RACL)

= F(RAB) + F(BC) + F(CL) (A)

= F(RFB) + F(BC) + F(CL) (I)

$\leq$ F(RF) + F(FE) + F(EC) + F(CL) (A) (X)

= F(RF) + F(FE) + F(EDL) (A) (I)

= F(RFDL) (A)

= F(J)

9. USE OF THE RIEMANN SOLVER IN NUMERICAL APPLICATIONS

Several numerical methods for solving hyperbolic conservation laws have been based on solution of local Riemann problems, examples are the Random Choice Method [2,3], Godunov-type methods [10], and the front-tracking method of Glimm et al [9]. As noted in Chapter 7, the Riemann problem of the Eqs.(1.1) is "unstable" in the sense that small perturbations in the initial states may change the solution drasticly, and this indicates that use of the Riemann solver in numerical solution of the Eqs.(1.1) is not advantageous. This chapter shows examples of the use of the solver in the Random Choice Method (RCM), also named the Uniform Sampling Method.

A common feature of methods based on exact Riemann solvers is that only a small part of the information contained in the Riemann solution is used. A Riemann problem for Eqs.(1.1) may involve solution of as much as four non-linear equations, and as great care has to be taken to pick the right solution, much time is used to produce information which is later disregarded in the solution. As the number of grid block increases, the RCM applied to Eq.(1.1) becomes rather time-consuming. For strictly hyperbolic systems, this drawback can in some cases be compensated by using <u>approximate</u> Riemann solvers [10], and it is an interesting question whether for instance Godunov-type methods could be constructed with approximate Riemann solvers in the case of non-strictly hyperbolic systems. Godunov-type methods are obviously less sensitive to the instability in the Riemann solution than the RCM, as the first tend to average the Riemann solution.

In practical applications, the function f may be represented by a table only, and as the problem is not structural stable in the same way as for a single hyperbolic equation [5,25], the method chosen for interpolating f could highly influence the solution. In all examples shown in this chapter, the function f is represented analytically, using Eq.(3.4) together with the definition

9. USE OF THE RIEMANN SOLVER IN NUMERICAL APPLICATIONS

Several numerical methods for solving hyperbolic conservation laws have been based on solution of local Riemann problems, examples are the Random Choice Method [2,3], Godunov-type methods [18], and the front-tracking method of Glimm et al [9]. As noted in Chapter 7, the Riemann problem of the Eqs.(1.1) is "unstable" in the sense that small perturbations in the initial states may change the solution drasticly, and this indicates that use of the Riemann solver in numerical solution of the Eqs.(1.1) is not advantageous. This chapter shows examples of the use of the solver in the Random Choice Method (RCM), also named the Uniform Sampling Method.

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$$\kappa(T) = \frac{1}{2 - T} \tag{9.1}$$

All non-linear equations are solved by the Newton-Raphson method. Also, in all the examples shown, $\alpha = 0.5$, $\beta = 3.5$ and a Courant number of 0.8 is used. If nothing else is specified, the number of grid blocks is 200. The solutions are shown for $t = 1$.

As g is a slowly varying function of T, the initial T-profile is convected with a minor deformation only, and in most of the examples, only the S-profile is shown. With the given function f, both the eigenvalues of the system matrix are positive, and Godunov's method is equivalent to standard upstream differencing; use of the Riemann solver is not necesarry. For comparison, results using this method is shown together with the results from the RCM.

In cases where the solution of the Eqs.(1.1) does not posess a transition state in contineous parts of the solution, the RCM behaves as for strictly hyperbolic systems. It is well-known that the method then resolves discontinuities without dispersion, but has rather low accuracy in smooth parts of the solution. Fig 9.1 shows the solution of a Riemann problem modelling injection of cold water into a hot oil reservoir: $U^L = (1,0)$, $U^R = (0,1)$. Godunov's method needs a very high number of grid blocks to resolve the plateau with constant S sufficiently.

Also, Fig 9.2-4 all show solutions of Riemann problems, these satisfying or close to satisfying the conditions of Eq.(7.1) producing non-uniqueness in phase space. The intital states are given in Table 9.1. Obviously, the upstream differencing method are not capable of resolving the abrubt changes in the solution and also reflects the non-uniqueness of the solution in the case where Eq.(7.1) is exactly satisfied. Also note that the numerical solution in this case is non-monotone, even though Godunov's method is monotone for strictly hyperbolic equations.

Fig.	S^L	T^L	S^L	T^L	g^L	g^R
9.2	0.78	0.7			0.3369940	
9.3	0.7938178	0.7	0.9319771	0.0	0.3378500	0.3378500
9.4	0.81	0.7			0.3386065	

Table 9.1 : Initial states for the solution shown in Fig 9.2-4. U^L is identical in all three cases.

In cases where the solution has a transition state in smooth parts of the solution, large instabilities may occur in the RCM, and the method converges slowly as the block lenght of the grid goes to zero. This is demonstrated in Fig 9.7-10, using the initial condition shown in Fig 9.5-6. Note the difference in the solution produced by merely changing the random-number generator involved. Except from the results of Fig 9.9, the random-number generator described in [24] is used.

Finally, two examples when all the initial states are situated on the transition curve S_2 is shown in Fig 9.11-12. T is chosen to vary linearly with x initially. Both when T increases and decreases with x, the solution seem to "avoid" the transition curve, and the solution does not involve any specific problems.

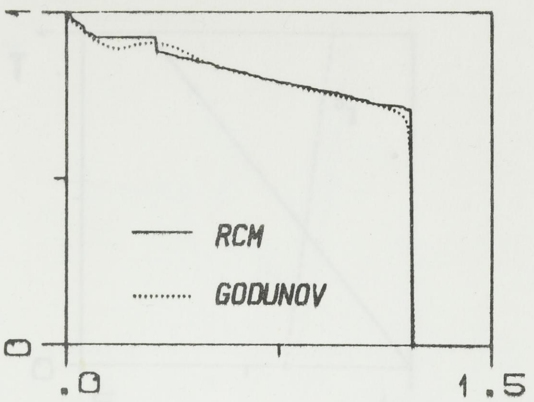


Fig 9.1 : S-profile in the solution of the Riemann Problem $U^L = (1,0)$, $U^R = (0,1)$

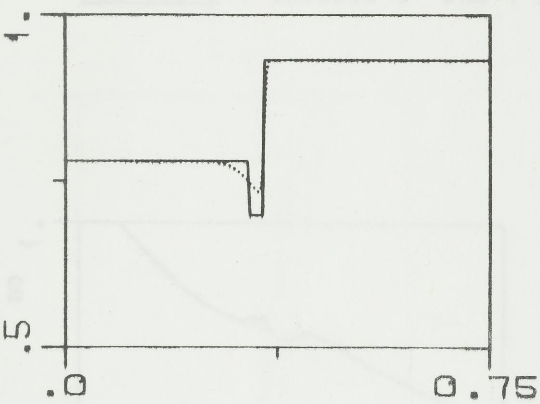


Fig 9.2 : S-profile in the solution of the Riemann problem specified in Table 9.1

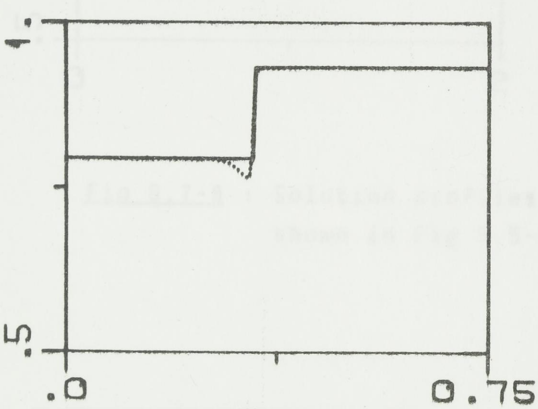


Fig 9.3 : S-profile in the solution of the Riemann problem specified in Table 9.1

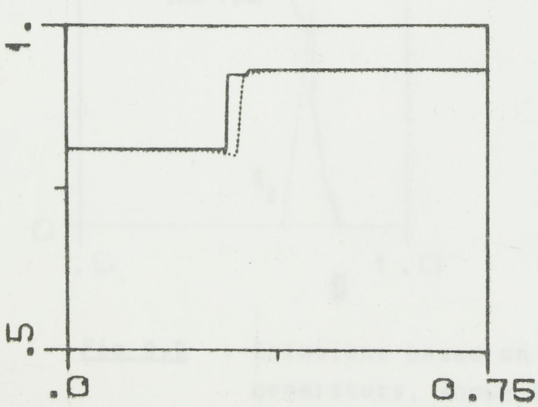


Fig 9.4 : S-profile in the solution of the Riemann problem specified in Table 9.1

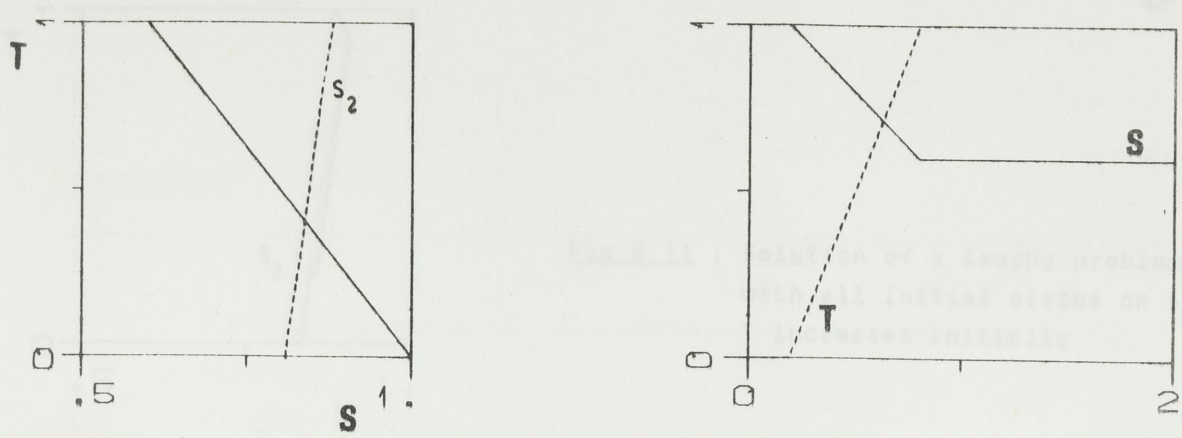


Fig 9.5-6 : Initial S- and T-profile crossing S_2

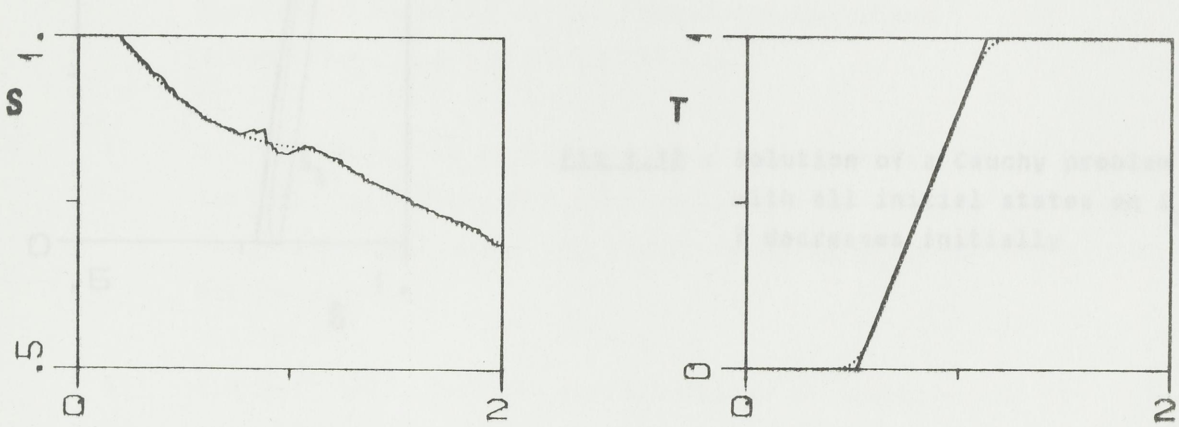
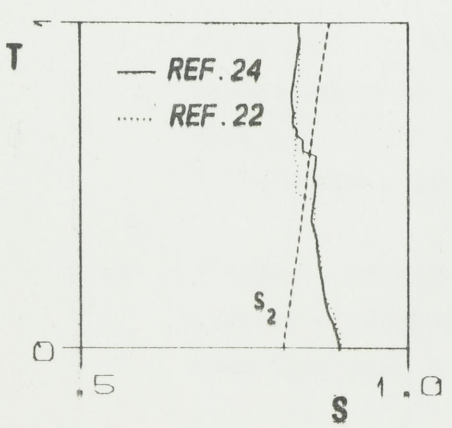


Fig 9.7-8 : Solution profiles using the initial condition shown in Fig 9.5-6



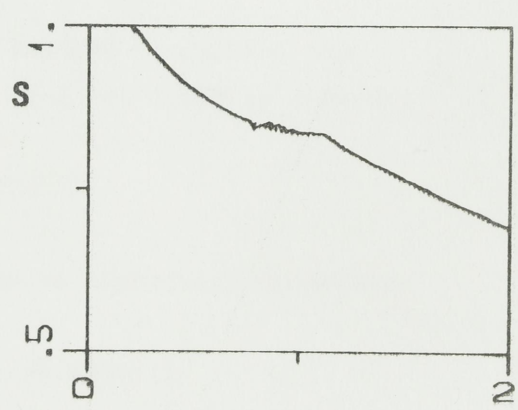


Fig 9.9 : Solutions based on two different random-number generators, shown in the phase space

Fig 9.10 : As Fig 9.7 but using 2000 grid blocks

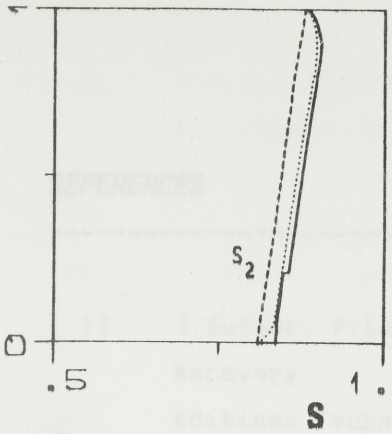


Fig 9.11 : Solution of a Cauchy problem with all initial states on S_2. T increases initially

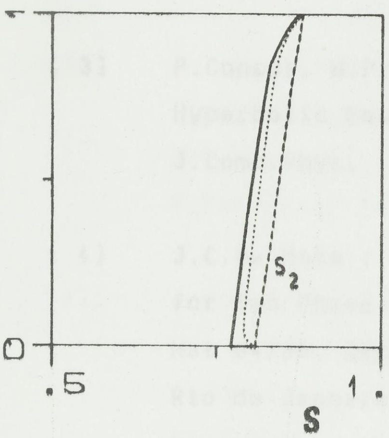


Fig 9.12 : Solution of a Cauchy problem with all initial states on S_2. T decreases initially

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APPENDIX: DERIVATION OF THE MODEL EQUATIONS

To correct a small flaw and to supplement a derivation first given by Fayers [7], this Appendix shows the derivation of the model equations describing non-isothermal two-phase flow in a porous medium.

In the following, the subscribts w,o and r will be used to denote parameters characterizing water, oil and rock respectively. Let λ_q (q=w,o) be the mobility of fluid q, i.e. the relative permeability divided by the viscosity, and let κ_q (q=w,o,r) denote the thermal capacity per unit <u>mass</u>. p_c is the capillary pressure, and Λ is the total thermal conductivity,- both are functions of water saturation S and temperature T. φ is used for porosity, k for absolute permeability and u for the total volume flux. Also introduce the notation

$$f = \frac{\lambda_w}{\lambda_w + \lambda_o}$$

$$a = -\frac{k}{u}\,\frac{\lambda_w \lambda_o}{\lambda_w + \lambda_o}\,\frac{\partial p_c}{\partial S}$$

$$b = -\frac{k}{u}\,\frac{\lambda_w \lambda_o}{\lambda_w + \lambda_o}\,\frac{\partial p_c}{\partial T} \qquad \text{(A1)}$$

$$F = f - aS_x - bT_x$$

Both the functions a and b are normally positive [1], and f_T is normally negative [17]. Assuming incompressibility and neglecting gravity, conservation of mass is expressed through the equation

$$\varphi\frac{\partial S}{\partial t} + u\frac{\partial f}{\partial x} = u\frac{\partial}{\partial x}\left[a\frac{\partial S}{\partial x} + b\frac{\partial T}{\partial x} \right] \qquad \text{(A2)}$$

Gravity is easily included by a redefinition of the function f [21].

Conservation of energy is expressed as

$$\frac{\partial}{\partial t}\{ [\varphi\kappa_w S + \varphi\kappa_o (1-S) + (1-\varphi)\kappa_r]T \}$$

$$+ u\frac{\partial}{\partial x}\{ [\kappa_w F + \kappa_o (1-F)]T \} = \frac{\partial}{\partial x}\{ \Lambda \frac{\partial T}{\partial x} \} \qquad (A3)$$

Introduce the thermodynamic functions

$$A = \kappa_w - \kappa_o$$

$$B = \kappa_o + \frac{1-\varphi}{\varphi}\kappa_s$$

$$C = \kappa_o \qquad (A4)$$

$$\alpha = \frac{C + C'T}{A + A'T} \qquad A + A'T \neq 0$$

$$\beta = \frac{B + B'T}{A + A'T}$$

In general, the thermal capacities per unit mass are functions of temperature, and the sign "'" is used to denote derivation with respect to T. After a scaling of the equations, and after a substitution of Eq.(A2) into Eq.(A3), the system of conservation laws simplifies to:

$$S_t + f_x = (aS_x + bT_x)_x$$
$$(A5)$$
$$T_t + \frac{f + \alpha}{S + \beta} T_x = \frac{1}{A + A'T}[\Lambda T_x]_x + (aS_x + bT_x)T_x$$

The functions a,b,Λ etc are now redefined as dimensionless. In his derivation, Fayers [7] seem to neglect A' B' and C' even when A, B and C are allowed to vary as function of temperature. Following Fayers, the function $(f+\alpha/S+\beta)$ is termed the thermal advance function.

As a model for high-rate conditions, the terms representing capillary pressure and thermal conduction will now be neglected and the system reduced to a first order system. If α and β are assumed constant, the equations may be written in the form given in Eq.(2.1).

This is achieved if A,B and C have the following functional form:

$$AT = A_0 T + A_1 \qquad A_0, A_1, B_0, \ldots \text{ consts.}$$

$$BT = B_0 T + B_1 \qquad \text{(A6)}$$

$$CT = C_0 T + C_1 \qquad C_0 \neq 0$$

This is obviously satisfied if the thermal capacities are independent of temperature, and from the definition of A, B and C, it then follows that $\alpha < \beta$. Note that if CT = const., the system reduces to the form studied by Johansen and Winther [13].

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