Department
of
APPLIED MATHEMATICS

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Report no. 133 November 1999

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ISSN 0084-778x

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ØYSTEIN ROGNES AND XUE-CHENG TAI

ABSTRACT. A convergence proof for general space decomposition techniques applied to an abstract non-symmetric parabolic equation is given. One of the main concerns is to give a unified convergence analysis for domain decomposition and multigrid methods for parabolic problems. The analysis is also valid for non-symmetric problems. For second order parabolic problems, the convection can dominate the diffusion. The algorithms can be applied to domain decomposition methods with or without the coarse mesh. For applications to multigrid methods, the coarsest mesh does not need to be very coarse. A relation between the coarse mesh size and the time step is needed to get a convergence rate independent of the mesh. The number of iterations at each time step for the algorithms is also estimated. Some numerical experiments are presented for domain decomposition methods with minimum overlap which support the theoretical predictions. The algorithms are able to capture the sharp traveling shocks for convection dominated problems.

1. INTRODUCTION

In this paper we will use a space decomposition method for an abstract parabolic problem:

$$\begin{cases} \left(\dfrac{\partial u}{\partial t}, v\right) + \epsilon a(u,v) + b(u,v) = (f,v), & \forall v \in \mathcal{H}, \\ u(0) = u_0 \in \mathcal{H}. \end{cases} \tag{1}$$

Above, $\mathcal{H}$ is a Hilbert space, $a(u,v)$ is a bounded, bilinear, symmetrical form on a Hilbert space $\mathcal{V}$, $b(u,v)$ is generally a non-symmetric and bounded, bilinear form on the same Hilbert space $\mathcal{V}$, and ϵ is a positive constant, possibly small. A space decomposition method refers to techniques that decompose the Hilbert space $\mathcal{V}$ into a sum of subspaces, i.e

$$\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 + \ldots + \mathcal{V}_m, \quad \mathcal{V}_i \subset \mathcal{V}. \tag{2}$$

The above decomposition means that $\forall v \in \mathcal{V}$, $\exists v_i \in \mathcal{V}_i$ (may not be unique) such that $v = \sum_{i=1}^{m} v_i$. Domain decomposition and multilevel methods have been intensively studied as iterative methods for elliptic problems. Both methods are powerful for symmetric and certain non-symmetric stationary problems. Developments by Xu [32] establish a unified theory for iterative preconditioners in an elegant framework of space decomposition, see also [21, 22, 25, 24, 27, 26]. This framework includes domain decomposition and multigrid/multilevel methods. In this work,

This work was supported by the Research Council of Norway (NFR), under grant 115837/431, and also partially by ELF Petroleum Norway AS.

we shall use this general frame to analyze iterative solvers for time dependent problems.

In Tai and Espedal [25, 24] a space decomposition iterative procedure was proposed for a class of nonlinear elliptic problems with focus on the analysis of subdomain solvers, rather than viewing it as a preconditioning method. To use these decomposition techniques for parabolic problems, there are basically two approaches. First, one could use a time-stepping scheme to discretize the time variable. Then at each time level an elliptic perturbed problem needs to be solved by using the space decomposition techniques. This approach was used for domain decomposition in the pioneering work of Lions [16] and recent advances have been made by Cai [3, 4], Dryja [12], Cai and Sarkis [6], Ewing, Lazarov, Pasciak and Vassilevski [14], Mathew, Polyakov, Russo, and Wang [17] and Tai [23], etc. Second, one could integrate the space decomposition with the time-stepping. This would give a "blockwise implicit" time-stepping scheme. For overlapping domain decomposition this approach have been used by Blum, Lisky and Rannacher [2] and Rannacher [18]. For non-overlapping domain decomposition, it has been used by Dawson, Du and Dupont [10] and Dawson and Dupont [11]. In this paper we will use the first approach.

The main concern of this work is to give a unified convergence analysis for both domain decomposition and multigrid methods for parabolic problems. One of the difficulties with the analysis is that the problems may be non-symmetric. This difficulty has been studied for elliptic problems in [3, 5, 7, 8, 29, 28, 31]. For time dependent problems, very few works have been done in this direction. The tool used in our analysis is the space decomposition and subspace correction approach which is different from the earlier approaches for the elliptic problems. It is known that a coarse mesh may not be necessary for domain decomposition method for parabolic problems when the time step or the diffusion coefficient is small. Our analysis gives an indication about when the coarse mesh may be needed and how the coarse mesh can help to improve the convergence rate. For applications to multigrid methods for parabolic problems, the analysis shows that the coarsest mesh in the multigrid methods does not need to be of the size O(1), which is often required for elliptic problems. In fact, the coarsest mesh size only needs to be of order $O(\sqrt{\tau})$ in order to get a convergence rate that is independent of the mesh sizes and the time step for second order parabolic problems. Another concern of the analysis is the number of iterations that is necessary at each time step. It is shown that only $O(|\log(\tau)|)$ steps of iterations are needed for domain decomposition and multigrid methods.

The error analysis is valid even for the case when the parameter ε is very small. In case that $\varepsilon = 0$, we will produce the same algorithm as in [30] for purely hyperbolic problems in applications to domain decomposition methods.

The analysis and the algorithms are given for a general abstract parabolic equation, hoping that they can be applied to different kind of problems. The analysis and the algorithms are applicable to some fourth order problems. However, the decomposition of the finite element spaces and the estimations for the constants C_L, C_V, C_H, C_1, C_2 and C_3 (c.f. (3) and (4)) will become more involved.

The paper is organized as follows. The conditions that are needed for the abstract equation (1) and the decomposed subspaces are stated in section 2. In section 3, the algorithms are formulated for a general space decomposition for the abstract parabolic equation (1). The convergence rate estimates for the iterative solvers at each time level are given for the algorithms in section 4. In addition, we also estimate the number of iterations that is needed at each time step. In section 5, we apply the algorithms and the error analysis to domain decomposition and multigrid methods for second order parabolic problems where the convection can be dominating. The constants needed for the error analysis are analyzed for domain decomposition and multigrid methods. In section 6, some numerical experiments are presented for domain decomposition methods which support the theoretical predictions. Minimum overlap is used. The algorithms are able to capture the sharp traveling shocks for convection dominated problems.

2. Preliminaries

We consider the abstract parabolic problem (1). Above, and also later, $(\cdot,\cdot)$ denotes the inner product $(\cdot,\cdot)_{\mathcal{H}}$ of the Hilbert space $\mathcal{H}$, into which the Hilbert space $\mathcal{V}$ can be embedded. The bilinear form $a(\cdot,\cdot)$ is symmetric, bounded and positive definite in the Hilbert space $\mathcal{V}$. Correspondingly, we define the inner product and norm for $\mathcal{V}$ as:

$$(u,v)_{\mathcal{V}} = a(u,v), \quad \forall u, v \in \mathcal{V},$$

and

$$\|u\|_{\mathcal{V}} = \sqrt{a(u,u)}, \quad \forall u \in \mathcal{V}.$$

The bilinear form $b(\cdot,\cdot)$ is in general non-symmetric, and has the following property: $\exists \sigma > 0$ such that

$$|b(u,v)| \le \sigma \|u\|_{\mathcal{V}} \|v\|_{\mathcal{H}} \quad \forall u, v \in \mathcal{V}.$$

For the space decomposition (2), we assume that there exist constants $C_L, C_H, C_V > 0$ with the following properties:

$$\begin{cases} \forall v \in \mathcal{V} : \exists v_i \in \mathcal{V}_i \text{ such that } v = \sum_{i=1}^{m} v_i, \text{ and} \\ \sum_{i=1}^{m} \|v_i\|_{\mathcal{H}}^2 \le C_L \|v\|_{\mathcal{H}}^2, \\ \sum_{i=1}^{m} \|v_i\|_{\mathcal{V}}^2 \le C_H \|v\|_{\mathcal{H}}^2 + C_V \|v\|_{\mathcal{V}}^2. \end{cases} \tag{3}$$

For general space decomposition, one can always find such constants, see Lions [16, p.7]. In practical applications, the space should be decomposed such that the constants C_L, C_H and C_V are suitable for the proposed iterative schemes, see section 6.

In addition we assume that there exist constants C_1, C_2, C_3 such that for all $u_i \in \mathcal{V}_i$ and $v_j \in \mathcal{V}_j$ we have

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=1}^{m} |(u_i, v_j)| &\le C_1 \left(\sum_{i=1}^{m} \|u_i\|_{\mathcal{H}}^2\right)^{1/2} \left(\sum_{j=1}^{m} \|v_j\|_{\mathcal{H}}^2\right)^{1/2}, \\
\sum_{i=1}^{m}\sum_{j=1}^{m} |a(u_i, v_j)| &\le C_2 \left(\sum_{i=1}^{m} \|u_i\|_{\mathcal{V}}^2\right)^{1/2} \left(\sum_{j=1}^{m} \|v_j\|_{\mathcal{V}}^2\right)^{1/2}, \qquad (4) \\
\sum_{i=1}^{m}\sum_{j=1}^{m} |b(u_i, v_j)| &\le C_3 \left(\sum_{i=1}^{m} \|u_i\|_{\mathcal{V}}^2\right)^{1/2} \left(\sum_{j=1}^{m} \|v_j\|_{\mathcal{H}}^2\right)^{1/2}.
\end{aligned}$$

Later in §5, it will be shown that the well-known domain decomposition and multigrid methods can be regarded as space decomposition techniques and the assumptions (3) and (4) are satisfied.

3. The Algorithms

After doing the decomposition of the space, we can search for a solution in each subspace $\mathcal{V}_i$ iteratively, and in the limit the sum of the solutions in the subspaces will converge to the solution of the original problem. The following algorithm is a combination of this space decomposition with the Euler time-discretization.

Algorithm 1. *(Additive Euler space decomposition)*

Step 1. *Set* $u^0 = u_0$*, and choose* α_i *such that* $0 < \alpha \le \alpha_i < 1$ *and* $\sum_{i=1}^{m} \alpha_i = 1$.

Step 2. *At time level* n*, for* $k = 1, 2, \cdots, s$*, do: for each* k*, compute* $u_i^{n+\frac{k}{s}}$ *in parallel for* $i = 1, 2, \cdots, m$ *such that* $u_i^{n+\frac{k}{s}} - u^{n+\frac{k-1}{s}} \in \mathcal{V}_i$ *and*

$$\left(\frac{u_i^{n+\frac{k}{s}} - u^n}{\tau}, v_i\right) + \epsilon a\left(u_i^{n+\frac{k}{s}}, v_i\right) + b\left(u_i^{n+\frac{k}{s}}, v_i\right) = \left(f^{n+1}, v_i\right), \qquad \forall v_i \in \mathcal{V}_i. \quad (5)$$

and set

$$u^{n+\frac{k}{s}} = \sum_{i=1}^{m} \alpha_i u_i^{n+\frac{k}{s}}.$$

Step 3. *Go to the next time level.*

In the above algorithm, τ is the time step, $f^{n+1} = f((n+1)\tau) \in \mathcal{H}$, subscript i refers to the number for the subspace $\mathcal{V}_i$, n indicates the time level, s is the number of space decomposition iteration that is performed at each time step and k is the counter for the space decomposition iteration. If $s = 1$, then it is a one-step space decomposition algorithm.

In algorithm 1, we are using the damped Jacobi method for the space decomposition. If we use the Gauss-Seidel method, the following algorithm is obtained:

Algorithm 2. *(Multiplicative-Euler-Space-decomposition)*

Step 1. *Set* $u_m^0 = u_0$.

Step 2. *At time level n, for $k = 1, 2, \cdots, s$, do: for each k, set $u_0^{n+\frac{k}{s}} = u_m^{n+\frac{k-1}{s}}$ and compute $u_i^{n+\frac{k}{s}}$ sequentially for $i = 1, 2, \cdots, m$ such that $u_i^{n+\frac{k}{s}} - u_{i-1}^{n+\frac{k}{s}} \in \mathcal{V}_i$ and*

$$\left(\frac{u_i^{n+\frac{k}{s}} - u^n}{\tau}, v_i\right) + \epsilon a\left(u_i^{n+\frac{k}{s}}, v_i\right) + b\left(u_i^{n+\frac{k}{s}}, v_i\right) = \left(f^{n+1}, v_i\right), \quad \forall v_i \in \mathcal{V}_i. \tag{6}$$

Step 3. *Set $u^{n+1} = u_m^{n+\frac{k}{s}}\big|_{k=s}$ and go to the next time level.*

For this algorithm, the subproblems need to be computed sequentially. However, by decomposing the space properly, see section 6, each of the subproblems can be computed by parallel processors.

4. Error Analysis

4.1. **Global stability.** Now, we start to estimate the error between the space decomposition solution and the true solution. In particular, we shall analyze how large s should be in order to retain the same accuracy as the global Euler scheme. In order to simplify the error analysis, we shall compare the space decomposition solution with the standard Euler scheme solution V^{n+1} of:

$$\begin{cases} \left(\dfrac{V^{n+1} - V^n}{\tau}, v\right) + \epsilon a\left(V^{n+1}, v\right) + b\left(V^{n+1}, v\right) = \left(f^{n+1}, v\right), & \forall v \in \mathcal{V}, \\ V^0 = u_0. \end{cases} \tag{7}$$

For the sake of analysis we define a τ-dependent norm:

$$\|u\|_{a_\tau}^2 = \|u\|_{\mathcal{H}}^2 + \tau\epsilon \|u\|_{\mathcal{V}}^2 = (u, u) + \tau\epsilon a(u, u).$$

In order to guarantee that the Euler scheme is stable, we need the bilinear form $\epsilon a(u, u) + b(u, u)$ to be positive. This can be guaranteed if ϵ is relatively larger than σ. But we want to consider cases where ϵ is small, independent of the size of σ. To ensure the positiveness it is therefore necessary, as in works by Johnson, Nävert and Pitkäranta [15], Rannacher and Zhou [19], Barrett and Morton [1], etc., to assume that

$$b(u, u) \geq 0, \quad \forall u \in \mathcal{V}. \tag{8}$$

Throughout the rest of this section, we shall assume that (8) is valid. With this condition, the global Euler Scheme has the following convergence estimate:

Theorem 3. . *Let the solution $u(t)$ of (1) be in $W^{2,\infty}([0, T], \mathcal{V})$, and $f \in W^{1,\infty}([0, T], \mathcal{H})$. Then*

$$\left\|u(t_{n+1}) - V^{n+1}\right\|_{a_\tau} \leq C\tau, \tag{9}$$

where C does not depend on ϵ, τ and n.

In the literature, the convergence of the Euler scheme is often proved in the $\mathcal{H}$-norm, i.e. $\left\|u(t_{n+1}) - V^{n+1}\right\|_{\mathcal{H}} \leq C\tau$. However, according to Lemma 4 and Lemma 7, we need the error estimate in the $\|\cdot\|_{a_\tau}$-norm. Due to the use of this norm, and also due to the reason that ϵ can be small, the damping property enjoyed by the Euler scheme will not be heavily used for the error estimates.

Proof of theorem 3. By (1) and a Taylor expansion we get

$$\begin{aligned}&\left(\frac{u(t_{n+1})-u(t_n)}{\tau},v\right)+\epsilon a\left(u(t_{n+1}),v\right)+b\left(u(t_{n+1}),v\right)\\&=(f^{n+1},v)-\left(u_t-\frac{u(t_{n+1})-u(t_n)}{\tau},v\right)\\&=(f^{n+1},v)+C\tau\left(\theta(\tau),v\right).\end{aligned}\tag{10}$$

Above, $\theta(\tau)$ satisfies $u_t=\frac{u(t_{n+1})-u(t_n)}{\tau}+C\tau\theta(\tau)$ and $\|\theta(\tau)\|_{\mathcal{H}}\le C$. Subtracting (10) from (7), we get

$$\left(\frac{e^{n+1}-e^n}{\tau},v\right)+\epsilon a(e^{n+1},v)+b(e^{n+1},v)=C\tau\left(\theta(\tau),v\right),$$

where $e^n=V^n-u(t_n),\quad\forall n\ge1$. Putting $v=e^{n+1}$ gives us

$$(e^{n+1}-e^n,e^{n+1})+\tau\epsilon a(e^{n+1},e^{n+1})+\tau b(e^{n+1},e^{n+1})=C\tau^2(\theta,e^{n+1}),$$

and so, using the fact $b(u,u)\ge0,\ \forall v\in\mathcal{V}$, it follows that

$$\begin{aligned}\left\|e^{n+1}\right\|_{a_\tau}^2&=(e^{n+1},e^{n+1})+\tau\epsilon a(e^{n+1},e^{n+1})\\&\le(e^{n+1},e^{n+1})+\tau\epsilon a(e^{n+1},e^{n+1})+\tau b(e^{n+1},e^{n+1})\\&=C\tau^2(\theta,e^{n+1})+(e^n,e^{n+1})\\&\le C\tau^3+\tau\left\|e^{n+1}\right\|_{\mathcal{H}}^2+\frac12\left\|e^n\right\|_{\mathcal{H}}^2+\frac12\left\|e^{n+1}\right\|_{\mathcal{H}}^2\\&=C\tau^3+(\frac12+\tau)\left\|e^{n+1}\right\|_{\mathcal{H}}^2+\frac12\left\|e^n\right\|_{\mathcal{H}}^2.\end{aligned}$$

Rearranging, observing that $\|e^n\|_{\mathcal{H}}\le\|e^n\|_{a_\tau}\ \forall n$, assuming $2\tau<1$, and putting $\beta=\frac{1}{1-2\tau}$, lead by induction to

$$\begin{aligned}\left\|e^{n+1}\right\|_{a_\tau}^2&\le C\beta\tau^3+\beta\left\|e^n\right\|_{a_\tau}^2\\&\le C\beta\tau^3+\beta(C\beta\tau^3+\beta\left\|e^{n-1}\right\|_{a_\tau}^2)\\&\le\dots\\&\le C\beta\tau^3(1+\beta+\beta^2+\ldots+\beta^n)+\beta^n\left\|e^0\right\|_{a_\tau}^2\\&=\frac{C\tau^3\beta(\beta^{n+1}-1)}{\beta-1}\le\frac{C\tau^3\beta^{n+2}}{\beta-1}\\&=\frac{1-2\tau}{2\tau}C\tau^3\beta^{n+2}\le\frac{C}{2}\tau^2\beta^{n+2}.\end{aligned}$$

Now, assuming $2\tau\le\frac12$ and $(n+2)\le2n$, using that $\ln(1+x)\le x$ for $x>0$ and noting that $n\tau=T$, we get

$$\begin{aligned}\left\|e^{n+1}\right\|_{a_\tau}^2&\le\frac{C}{2}\tau^2e^{(n+2)\ln\frac{1}{1-2\tau}}=\frac{C}{2}\tau^2e^{(n+2)\ln\left(1+\frac{2\tau}{1-2\tau}\right)}\\&\le\frac{C}{2}\tau^2e^{(n+2)\frac{2\tau}{1-2\tau}}\le\frac{C}{2}\tau^2e^{8T}.\end{aligned}$$

In case that the embedding $\mathcal{V}\subset\mathcal{H}$ is compact and the diffusion parameter ϵ is big, the exponential dependence on T can be removed. □

4.2. **The additive scheme.** Before proving the first of the two main theorems, we first give a lemma which estimates the rate of convergence, i.e. the error reduction, when the space decomposition iteration method is applied to the singular perturbed elliptic equation at each time level.

Lemma 4. *Given $G^k \in \mathcal{V}$, let the function G_i^{k+1} satisfy: $G_i^{k+1} - G^k \in \mathcal{V}_i$, and*

$$\tau^{-1}(G_i^{k+1}, v_i) + \epsilon a(G_i^{k+1}, v_i) + b(G_i^{k+1}, v_i) = 0, \quad \forall v_i \in \mathcal{V}_i, \ i = 1, \dots, m. \tag{11}$$

If we set

$$G^{k+1} = \sum_{i=1}^{m} \alpha_i G_i^{k+1}, \tag{12}$$

and assume $\tau < \tau_0$, then

$$\left\|G^{k+1}\right\|_{a_\tau}^2 \le \frac{16C_a}{16C_a + 1} \left\|G^k\right\|_{a_\tau}^2.$$

Above

$$C_a = \frac{\max(3C_L, 3C_H\tau\epsilon^2 + C_V + 3C_L\sigma^2\tau/\epsilon)}{\alpha}, \tag{13}$$

$$\tau_0 = \min\left\{\frac{\epsilon}{2\sigma^2}, \frac{\alpha\epsilon}{288\sigma^2(3C_L + C_V)}, \frac{C_L + C_V/3}{C_H\epsilon^2}, \frac{(C_L + C_V/3)\epsilon}{C_L\sigma^2}\right\}, \tag{14}$$

α is the lower bound for the relaxation parameter α_i in step 1 of algorithm 1, σ is the upper bound for the bilinear form $b(u, v)$, ϵ is the diffusion parameter, and C_L, C_V and C_H are defined in (3).

Proof. By assumption (3), there exists $\phi_i^{k+1} \in \mathcal{V}_i$ such that

$$G^{k+1} = \sum_{i=1}^{m} \phi_i^{k+1}, \tag{15}$$

$$\sum_{i=1}^{m} \left\|{\phi_i}^{k+1}\right\|_{\mathcal{H}}^2 \le C_L \left\|G^{k+1}\right\|_{\mathcal{H}}^2, \tag{16}$$

$$\sum_{i=1}^{m} \left\|{\phi_i}^{k+1}\right\|_{\mathcal{V}}^2 \le C_H \left\|G^{k+1}\right\|_{\mathcal{H}}^2 + C_V \left\|G^{k+1}\right\|_{\mathcal{V}}^2. \tag{17}$$

Thus

$$\begin{aligned}
&\frac{(G^{k+1}, G^{k+1})}{\tau} + \epsilon a(G^{k+1}, G^{k+1}) \\
&\le \frac{(G^{k+1}, G^{k+1})}{\tau} + \epsilon a(G^{k+1}, G^{k+1}) + b(G^{k+1}, G^{k+1}) \quad \textit{(By the positiveness of } b(u,u)\textit{)} \\
&= \sum_{i=1}^{m} \left[\frac{(G^{k+1}, {\phi_i}^{k+1})}{\tau} + \epsilon a(G^{k+1}, {\phi_i}^{k+1}) + b(G^{k+1}, {\phi_i}^{k+1})\right] \quad \textit{(By (15))} \\
&= \sum_{i=1}^{m} \left[\frac{(G^{k+1} - G_i^{k+1}, {\phi_i}^{k+1})}{\tau} + \epsilon a(G^{k+1} - G_i^{k+1}, {\phi_i}^{k+1})\right. \\
&\quad \left. + b(G^{k+1} - G_i^{k+1}, {\phi_i}^{k+1})\right] \quad \textit{(By using (11))}
\end{aligned}$$

$$\leq \sum_{i=1}^{m} \Big[\frac{1}{\tau} \|G^{k+1} - G_i^{k+1}\|_{\mathcal{H}} \left\|{\phi_i}^{k+1}\right\|_{\mathcal{H}} + \epsilon \|G^{k+1} - G_i^{k+1}\|_{\mathcal{V}} \left\|{\phi_i}^{k+1}\right\|_{\mathcal{V}}$$
$$+ \sigma \|G^{k+1} - G_i^{k+1}\|_{\mathcal{V}} \left\|{\phi_i}^{k+1}\right\|_{\mathcal{H}} \Big]$$

(By using Cauchy Schwarz inequality)

$$\leq \frac{1}{\tau}\Big(\sum_{i=1}^{m} \|G^{k+1} - G_i^{k+1}\|_{\mathcal{H}}^2\Big)^{\frac{1}{2}} \Big(\sqrt{C_L}\,\|G^{k+1}\|_{\mathcal{H}}\Big) \tag{18}$$
$$+ \varepsilon\Big(\sum_{i=1}^{m} \|G^{k+1} - G_i^{k+1}\|_{\mathcal{V}}^2\Big)^{\frac{1}{2}} \Big(\sqrt{C_H}\,\|G^{k+1}\|_{\mathcal{H}} + \sqrt{C_V}\,\|G^{k+1}\|_{\mathcal{V}}\Big)$$
$$+ \sigma\Big(\sum_{i=1}^{m} \|G^{k+1} - G_i^{k+1}\|_{\mathcal{V}}^2\Big)^{\frac{1}{2}} \big(\sqrt{C_L}\,\|G^{k+1}\|_{\mathcal{H}}\big). \quad \textit{(By using (15)-(17)).}$$

By applying the inequality $ab \leq \frac{1}{2p}a^2 + \frac{p}{2}b^2$, with $p > 0$ being chosen properly, on each term and using the fact that $0 < \alpha \leq \alpha_i \leq 1$, one gets that

$$\frac{1}{\tau}\|G^{k+1}\|_{\mathcal{H}}^2 + \epsilon \|G^{k+1}\|_{\mathcal{V}}^2$$
$$\leq \frac{3C_L}{2\alpha}\sum_{i=1}^{m} \frac{\alpha_i}{\tau} \|G^{k+1} - G_i^{k+1}\|_{\mathcal{H}}^2 + \frac{1}{6\tau}\|G^{k+1}\|_{\mathcal{H}}^2$$
$$+ \frac{3C_{\mathcal{H}}\tau\epsilon}{2\alpha}\sum_{i=1}^{m} \alpha_i\epsilon \|G^{k+1} - G_i^{k+1}\|_{\mathcal{V}}^2 + \frac{1}{6\tau}\|G^{k+1}\|_{\mathcal{H}}^2$$
$$+ \frac{C_V}{2\alpha}\sum_{i=1}^{m} \alpha_i\epsilon \|G^{k+1} - G_i^{k+1}\|_{\mathcal{V}}^2 + \frac{\epsilon}{2}\|G^{k+1}\|_{\mathcal{V}}^2$$
$$+ \frac{3C_L\sigma^2\tau}{2\alpha\epsilon}\sum_{i=1}^{m} \alpha_i\epsilon \|G^{k+1} - G_i^{k+1}\|_{\mathcal{V}}^2 + \frac{1}{6\tau}\|G^{k+1}\|_{\mathcal{H}}^2$$
$$\leq \frac{3C_L}{2\alpha}\sum_{i=1}^{m} \alpha_i \left(\frac{1}{\tau}\|G^{k+1} - G_i^{k+1}\|_{\mathcal{H}}^2\right)$$
$$+ \frac{3C_H\tau\epsilon + C_V + 3C_L\sigma^2\tau/\epsilon}{2\alpha}\sum_{i=1}^{m} \alpha_i \left(\frac{1}{\tau}\|G^{k+1} - G_i^{k+1}\|_{\mathcal{V}}^2\right)$$
$$+ \frac{1}{2\tau}\|G^{k+1}\|_{\mathcal{H}}^2 + \frac{\epsilon}{2}\|G^{k+1}\|_{\mathcal{V}}^2$$
$$\leq \frac{C_a}{2}\sum_{i=1}^{m} \alpha_i \left(\frac{1}{\tau}\|G^{k+1} - G_i^{k+1}\|_{\mathcal{H}}^2 + \epsilon \|G^{k+1} - G_i^{k+1}\|_{\mathcal{V}}^2\right)$$
$$+ \frac{1}{2\tau}\|G^{k+1}\|_{\mathcal{H}}^2 + \frac{\epsilon}{2}\|G^{k+1}\|_{\mathcal{V}}^2.$$

From the above inequality, it follows that

$$\frac{1}{\tau}\|G^{k+1}\|_{\mathcal{H}}^2 + \epsilon \|G^{k+1}\|_{\mathcal{V}}^2$$
$$\leq C_a \sum_{i=1}^{m} \alpha_i \left(\frac{1}{\tau}\|G^{k+1} - G_i^{k+1}\|_{\mathcal{H}}^2 + \epsilon \|G^{k+1} - G_i^{k+1}\|_{\mathcal{V}}^2\right). \tag{19}$$

By using $G^{k+1}-G_i^{k+1}=G^{k+1}-G^k+G^k-G_i^{k+1}$, relation (12), and $(a+b)^2\le 2(a^2+b^2)$, we get that

$$\begin{aligned}
&\sum_{i=1}^{m}\alpha_i\left(\frac{1}{\tau}\left\|G^{k+1}-G_i^{k+1}\right\|_{\mathcal{H}}^2+\epsilon\left\|G^{k+1}-G_i^{k+1}\right\|_{\mathcal{V}}^2\right)\\
&\le 2\Big(\frac{1}{\tau}\left\|G^{k+1}-G^k\right\|_{\mathcal{H}}^2+\epsilon\left\|G^{k+1}-G^k\right\|_{\mathcal{V}}^2\\
&\quad+\sum_{i=1}^{m}\alpha_i\left(\frac{1}{\tau}\left\|G_i^{k+1}-G^k\right\|_{\mathcal{H}}^2+\epsilon\left\|G_i^{k+1}-G^k\right\|_{\mathcal{V}}^2\right)\Big)\\
&=2\left(\frac{1}{\tau}\left\|\sum_{i=1}^{m}\alpha_i(G_i^{k+1}-G^k)\right\|_{\mathcal{H}}^2+\epsilon\left\|\sum_{i=1}^{m}\alpha_i(G_i^{k+1}-G^k)\right\|_{\mathcal{V}}^2\right)\\
&\quad+2\sum_{i=1}^{m}\alpha_i\left(\frac{1}{\tau}\left\|G_i^{k+1}-G^k\right\|_{\mathcal{H}}^2+\epsilon\left\|G_i^{k+1}-G^k\right\|_{\mathcal{V}}^2\right)\\
&\le 4\sum_{i=1}^{m}\alpha_i\left(\frac{1}{\tau}\left\|G_i^{k+1}-G^k\right\|_{\mathcal{H}}^2+\epsilon\left\|G_i^{k+1}-G^k\right\|_{\mathcal{V}}^2\right). \qquad (20)
\end{aligned}$$

Combining (19) and (20) to get

$$\begin{aligned}
&\frac{1}{\tau}\left\|G^{k+1}\right\|_{\mathcal{H}}^2+\epsilon\left\|G^{k+1}\right\|_{\mathcal{V}}^2\\
&\qquad\le 4C_a\sum_{i=1}^{m}\alpha_i\left(\frac{1}{\tau}\left\|G_i^{k+1}-G^k\right\|_{\mathcal{H}}^2+\epsilon\left\|G_i^{k+1}-G^k\right\|_{\mathcal{V}}^2\right). \qquad (21)
\end{aligned}$$

Taking $v_i=G_i^{k+1}-G^k$ in (11), and using the equality

$$(u,u-v)=\frac{1}{2}(\|u\|^2-\|v\|^2+\|u-v\|^2),$$

we get that

$$\begin{aligned}
&\frac{1}{2\tau}\left(\left\|G_i{}^{k+1}\right\|_{\mathcal{H}}^2-\left\|G^k\right\|_{\mathcal{H}}^2+\left\|G_i^{k+1}-G^k\right\|_{\mathcal{H}}^2\right)\\
&+\frac{\epsilon}{2}\left(\left\|G_i{}^{k+1}\right\|_{\mathcal{V}}^2-\left\|G^k\right\|_{\mathcal{V}}^2+\left\|G_i^{k+1}-G^k\right\|_{\mathcal{V}}^2\right)+b(G_i^{k+1},G_i^{k+1}-G^k)=0. \qquad (22)
\end{aligned}$$

This shows that

$$\begin{aligned}
&\frac{1}{\tau}\left\|G_i^{k+1}-G^k\right\|_{\mathcal{H}}^2+\epsilon\left\|G_i^{k+1}-G^k\right\|_{\mathcal{V}}^2\\
&\le\frac{1}{\tau}\left(\left\|G^k\right\|_{\mathcal{H}}^2-\left\|G_i{}^{k+1}\right\|_{\mathcal{H}}^2\right)+\epsilon\left(\left\|G^k\right\|_{\mathcal{V}}^2-\left\|G_i{}^{k+1}\right\|_{\mathcal{V}}^2\right)+2\left|b(G_i^{k+1},G_i^{k+1}-G^k)\right|\\
&\le\frac{1}{\tau}\left(\left\|G^k\right\|_{\mathcal{H}}^2-\left\|G_i{}^{k+1}\right\|_{\mathcal{H}}^2\right)+\epsilon\left(\left\|G^k\right\|_{\mathcal{V}}^2-\left\|G_i{}^{k+1}\right\|_{\mathcal{V}}^2\right)\\
&\quad+2\tau\sigma^2\left\|G_i{}^{k+1}\right\|_{\mathcal{V}}^2+\frac{1}{2\tau}\left\|G_i^{k+1}-G^k\right\|_{\mathcal{H}}^2,
\end{aligned}$$

which leads to

$$\frac{1}{\tau}\left\|G_i^{k+1}-G^k\right\|_{\mathcal{H}}^2+\epsilon\left\|G_i^{k+1}-G^k\right\|_{\mathcal{V}}^2$$
$$\leq 2\left[\frac{1}{\tau}\left(\left\|G^k\right\|_{\mathcal{H}}^2-\left\|G_i{}^{k+1}\right\|_{\mathcal{H}}^2\right)+\epsilon\left(\left\|G^k\right\|_{\mathcal{V}}^2-\left(1-\frac{2\sigma^2\tau}{\epsilon}\right)\left\|G_i{}^{k+1}\right\|_{\mathcal{V}}^2\right)\right]. \quad (23)$$

Using $\tau \leq \frac{\epsilon}{2\sigma^2}$, and combining (21) with (23), we obtain

$$\frac{1}{\tau}\left\|G^{k+1}\right\|_{\mathcal{H}}^2+\epsilon\left\|G^{k+1}\right\|_{\mathcal{V}}^2$$
$$\leq 8C_a\sum_{i=1}^m \alpha_i\left[\frac{1}{\tau}\left(\left\|G^k\right\|_{\mathcal{H}}^2-\left\|G_i{}^{k+1}\right\|_{\mathcal{H}}^2\right)+\epsilon\left(\left\|G^k\right\|_{\mathcal{V}}^2-\left(1-\frac{2\sigma^2\tau}{\epsilon}\right)\left\|G_i{}^{k+1}\right\|_{\mathcal{V}}^2\right)\right]$$
$$\leq 8C_a\left[\frac{1}{\tau}\left\|G^k\right\|_{\mathcal{H}}^2+\epsilon\left\|G^k\right\|_{\mathcal{V}}^2\right]-8C_a\left[\frac{1}{\tau}\left\|G^{k+1}\right\|_{\mathcal{H}}^2+\epsilon\left(1-\frac{2\sigma^2\tau}{\epsilon}\right)\left\|G^{k+1}\right\|_{\mathcal{V}}^2\right],$$

which clearly implies

$$8C_a\left[\frac{1}{\tau}\left\|G^{k+1}\right\|_{\mathcal{H}}^2+\epsilon\left\|G^{k+1}\right\|_{\mathcal{V}}^2\right]+\frac{1}{\tau}\left\|G^{k+1}\right\|_{\mathcal{H}}^2+\epsilon\left(1-\frac{16C_a\sigma^2\tau}{\epsilon}\right)\left\|G^{k+1}\right\|_{\mathcal{V}}^2$$
$$\leq 8C_a\left[\frac{1}{\tau}\left\|G^k\right\|_{\mathcal{H}}^2+\epsilon\left\|G^k\right\|_{\mathcal{V}}^2\right].$$

Now we require

$$1-\frac{16C_a\sigma^2\tau}{\epsilon}>\frac{1}{2}.$$

This means that

$$\tau<\frac{\epsilon}{32C_a\sigma^2}=\frac{\alpha\epsilon}{32\sigma^2\max(3C_L,3C_H\tau\epsilon^2+C_V+3C_L\sigma^2\tau/\epsilon)}$$
$$<\frac{\alpha\epsilon}{32\sigma^2(3C_L+3C_H\tau\epsilon^2+C_V+3C_L\sigma^2\tau/\epsilon)}.$$

From this we get the inequality

$$\frac{96\sigma^2\tau}{\alpha\epsilon}\left(C_L+C_H\tau\epsilon^2+\frac{C_V}{3}+C_L\sigma^2\tau/\epsilon\right)<1,$$

which certainly holds if $\tau < \tau_0$. Thus $1-16C_a\tau\sigma^2/\epsilon < 1/2$, and we finally get

$$\left(\frac{1}{\tau}\left\|G^{k+1}\right\|_{\mathcal{H}}^2+\epsilon\left\|G^{k+1}\right\|_{\mathcal{V}}^2\right)\left(\frac{1}{2}+8C_a\right)\leq 8C_a\left(\frac{1}{\tau}\left\|G^k\right\|_{\mathcal{H}}^2+\epsilon\left\|G^k\right\|_{\mathcal{V}}^2\right). \quad (24)$$

This proves the lemma. □

Theorem 5. *At each time level, for any $0<\rho\leq 1$, assume that the number s of the space decomposition iteration satisfies*

$$s \geq 2|\ln(\rho\tau)| \Big/ \left|\ln\frac{16C_a}{16C_a+1}\right|. \quad (25)$$

Then the following error estimate holds for algorithm 1:

$$\|u^n-V^n\|_{a_\tau}\leq C\tau.$$

Proof. At each time level n, let $U^{n+1} \in \mathcal{V}$ be an auxiliary function that satisfies

$$\left(\frac{U^{n+1}-u^n}{\tau}, v\right) + \epsilon a\left(U^{n+1}, v\right) + b\left(U^{n+1}, v\right) = \left(f^{n+1}, v\right), \quad \forall v \in \mathcal{V}. \tag{26}$$

Comparing (5) with (26), and using the fact that $\mathcal{V}_i \subset \mathcal{V}$, we see that:

$$\left(\frac{u_i^{n+\frac{k}{s}} - U^{n+1}}{\tau}, v_i\right) + \epsilon a\left(u_i^{n+\frac{k}{s}} - U^{n+1}, v_i\right) + b\left(u_i^{n+\frac{k}{s}} - U^{n+1}, v_i\right) = 0, \quad \forall v_i \in \mathcal{V}_i, \tag{27}$$

and $u_i^{n+\frac{k}{s}} - u^{n+\frac{k-1}{s}} \in \mathcal{V}_i$. Now, let us set

$$G_i^{k+1} = u_i^{n+\frac{k}{s}} - U^{n+1}, \quad G^k = u^{n+\frac{k-1}{s}} - U^{n+1}.$$

We note that G_i^{k+1} satisfies (11) and $G_i^{k+1} - G^k \in \mathcal{V}_i$. Clearly, G_i^{k+1} and G^k also depends on n, for notational simplicity the index n in G_i^{k+1} and G^k is omitted. According to lemma 4 there holds:

$$\begin{aligned} &\left\|u^{n+1} - U^{n+1}\right\|_{a_\tau}^2 = \left\|u^{n+\frac{s}{s}} - U^{n+1}\right\|_{a_\tau}^2 = \left\|G^{s+1}\right\|_{a_\tau}^2 \\ &\leq \frac{16C_a}{16C_a+1}\left\|G^s\right\|_{a_\tau}^2 \leq \cdots \\ &\leq \left(\frac{16C_a}{16C_a+1}\right)^s \left\|G^1\right\|_{a_\tau}^2 = \left(\frac{16C_a}{16C_a+1}\right)^s \left\|u^n - U^{n+1}\right\|_{a_\tau}^2, \end{aligned} \tag{28}$$

which means

$$\left\|u^{n+1} - U^{n+1}\right\|_{a_\tau} \leq \left(\frac{16C_a}{16C_a+1}\right)^{\frac{s}{2}} \left\|u^n - U^{n+1}\right\|_{a_\tau}. \tag{29}$$

As

$$\left\|u^{n+1} - V^{n+1}\right\|_{a_\tau} \leq \left\|u^{n+1} - U^{n+1}\right\|_{a_\tau} + \left\|U^{n+1} - V^{n+1}\right\|_{a_\tau} \tag{30}$$

and

$$\left\|u^n - U^{n+1}\right\|_{a_\tau} \leq \left\|u^n - V^n\right\|_{a_\tau} + \left\|U^{n+1} - V^{n+1}\right\|_{a_\tau} + \left\|V^n - V^{n+1}\right\|_{a_\tau}, \tag{31}$$

so, if s satisfies (25), it follows from (29)–(31) that

$$\begin{aligned} &\left\|V^{n+1} - u^{n+1}\right\|_{a_\tau} \\ &\leq \left(\frac{16C_a}{16C_a+1}\right)^{\frac{s}{2}} \Big(\left\|u^n - V^n\right\|_{a_\tau} + \left\|U^{n+1} - V^{n+1}\right\|_{a_\tau} \\ &\quad + \left\|V^n - V^{n+1}\right\|_{a_\tau}\Big) + \left\|U^{n+1} - V^{n+1}\right\|_{a_\tau} \\ &\leq \rho\tau\Big(\left\|u^n - V^n\right\|_{a_\tau} + \left\|U^{n+1} - V^{n+1}\right\|_{a_\tau} \\ &\quad + \left\|V^n - V^{n+1}\right\|_{a_\tau}\Big) + \left\|U^{n+1} - V^{n+1}\right\|_{a_\tau}. \end{aligned} \tag{32}$$

Next we estimate $\left\|V^{n+1}-U^{n+1}\right\|_{a_\tau}$ and $\left\|V^n-V^{n+1}\right\|_{a_\tau}$. Subtracting (7) from (26), it follows:

$$\left(\frac{V^{n+1}-U^{n+1}}{\tau},v\right)+\epsilon a\left(V^{n+1}-U^{n+1},v\right)$$
$$+b\left(V^{n+1}-U^{n+1},v\right)=\left(\frac{V^n-u^n}{\tau},v\right),\qquad \forall v\in\mathcal{V}.$$

Letting $v=V^{n+1}-U^{n+1}$, it gives

$$\begin{aligned}
&\left\|V^{n+1}-U^{n+1}\right\|_{\mathcal{H}}^2+\tau\epsilon\left\|V^{n+1}-U^{n+1}\right\|_{\mathcal{V}}^2\\
&\le\left\|V^{n+1}-U^{n+1}\right\|_{\mathcal{H}}^2+\tau\epsilon\left\|V^{n+1}-U^{n+1}\right\|_{\mathcal{V}}^2+\tau b\left(V^{n+1}-U^{n+1},V^{n+1}-U^{n+1}\right)\\
&\le\left\|V^n-u^n\right\|_{\mathcal{H}}\left\|V^{n+1}-U^{n+1}\right\|_{\mathcal{H}}\\
&\le\frac{1}{2}\left\|V^n-u^n\right\|_{\mathcal{H}}^2+\frac{1}{2}\left\|V^{n+1}-U^{n+1}\right\|_{\mathcal{H}}^2,
\end{aligned}$$

and thus

$$\left\|V^{n+1}-U^{n+1}\right\|_{a_\tau}^2\le\left\|V^n-u^n\right\|_{\mathcal{H}}^2. \tag{33}$$

Now, by using (9), we obtain

$$\begin{aligned}
&\left\|V^{n+1}-V^n\right\|_{a_\tau}\\
&\le\left\|V^{n+1}-u(t_{n+1})\right\|_{a_\tau}+\left\|u(t_{n+1})-u(t_n)\right\|_{a_\tau}+\left\|u(t_n)-V^n\right\|_{a_\tau}=C\tau.
\end{aligned}\tag{34}$$

From (30), (32), (33) and (34) we clearly see that

$$\begin{aligned}
&\left\|V^{n+1}-u^{n+1}\right\|_{a_\tau}\\
&\le\rho\tau\left(2\left\|V^n-u^n\right\|_{a_\tau}^2+C\tau\right)+\left\|V^n-u^n\right\|_{a_\tau}^2\\
&=(1+2\rho\tau)\left\|V^n-u^n\right\|_{a_\tau}^2+C\rho\tau^2.
\end{aligned}\tag{35}$$

By induction, one obtains:

$$\begin{aligned}
&\left\|V^n-u^n\right\|_{a_\tau}\\
&\le(1+2\rho\tau)^n\left\|V^0-u^0\right\|_{a_\tau}+C\rho\tau^2\left(1+(1+2\rho\tau)+(1+2\rho\tau)^2+\cdots+(1+2\rho\tau)^n\right)\\
&\le\frac{C\rho\tau^2}{2\rho\tau}(1+2\rho\tau)^{n+1}=\frac{C}{2}\tau e^{(n+1)\ln(1+2\rho\tau)}\le\frac{C}{2}\tau e^{2\rho(n+1)\tau}\le\frac{C}{2}\tau e^{4\rho T}.
\end{aligned}$$

□

Remark 6. *When the parameter ϵ is very small, the damping property of the Euler scheme is very weak. According to the above error estimate, in order to do long time integration, we must choose ρ small, $\rho\le CT^{-1}$. So, correspondingly, at each time step we must do more space decomposition iterations. In case that the embedding $\mathcal{V}\subset\mathcal{H}$ is compact and the diffusion parameter ϵ is big, then we can use inequality (33) in a more efficient way to remove the exponential dependence on T and get an estimate that is independent of T, see Remark 3.1 of* [23, p.34].

4.3. **The multiplicative scheme.** For Algorithm 2, the following lemma estimates the error reduction for the space decomposition iterations.

Lemma 7. *Given $G_0 \in \mathcal{V}$, let the function G_i satisfy: $G_i - G_{i-1} \in \mathcal{V}_i$, and*

$$\tau^{-1}(G_i, v_i) + \epsilon a(G_i, v_i) + b(G_i, v_i) = 0, \quad \forall v_i \in \mathcal{V}_i, \quad i = 1, 2, \dots, m. \tag{36}$$

If we assume $\tau < \tau_0$, then

$$\|G_m\|_{a_\tau}^2 \le \frac{4C_m}{4C_m + 1} \|G_0\|_{a_\tau}^2 . \tag{37}$$

Above

$$C_m = \max(3C_L {C_1}^2, 3C_H {C_2}^2 \tau\epsilon + C_V {C_2}^2 + 3C_L {C_3}^2 \tau/\epsilon),$$

$$\tau_0 = \frac{1}{m} \min\left\{ \frac{\epsilon}{144\sigma^2(3C_L + C_V)}, \frac{C_L + C_V/3}{C_H \epsilon^2}, \frac{(C_L + C_V/3)\epsilon}{C_L \sigma^2} \right\}. \tag{38}$$

Proof. By assumption (3), there exists $\phi_i \in \mathcal{V}_i$, such that for $1 \le r \le m$

$$G_r = \sum_{i=1}^m \phi_i,$$

$$\sum_{i=1}^m \|\phi_i\|_{\mathcal{H}}^2 \le C_L \|G_r\|_{\mathcal{H}}^2,$$

$$\sum_{i=1}^m \|\phi_i\|_{\mathcal{V}}^2 \le C_H \|G_r\|_{\mathcal{H}}^2 + C_V \|G_r\|_{\mathcal{V}}^2 .$$

Similar as getting (18), one can deduce

$$\begin{aligned}
&\frac{(G_r, G_r)}{\tau} + \epsilon a(G_r, G_r) \\
&\le \frac{(G_r, G_r)}{\tau} + \epsilon a(G_r, G_r) + b(G_r, G_r) \\
&= \sum_{i=1}^m \left[\frac{(G_r, \phi_i)}{\tau} + \epsilon a(G_r, \phi_i) + b(G_r, \phi_i) \right] \\
&= \sum_{i=1}^m \left[\frac{(G_r - G_i, \phi_i)}{\tau} + \epsilon a(G_r - G_i, \phi_i) + b(G_r - G_i, \phi_i) \right] \\
&= \sum_{i=1}^m \sum_{j=i+1}^r \left[\frac{(G_j - G_{j-1}, \phi_i)}{\tau} + \epsilon a(G_j - G_{j-1}, \phi_i) + b(G_j - G_{j-1}, \phi_i) \right] \\
&\le \frac{C_1}{\tau} \left(\sum_{i=1}^m \|G_i - G_{i-1}\|_{\mathcal{H}}^2 \right)^{1/2} \left(\sum_{i=1}^m \|\phi_i\|_{\mathcal{H}}^2 \right)^{1/2} \\
&\quad + C_2 \epsilon \left(\sum_{i=1}^m \|G_i - G_{i-1}\|_{\mathcal{V}}^2 \right)^{1/2} \left(\sum_{i=1}^m \|\phi_i\|_{\mathcal{V}}^2 \right)^{1/2} \qquad \text{(By using (4))} \\
&\quad + C_3 \left(\sum_{i=1}^m \|G_i - G_{i-1}\|_{\mathcal{V}}^2 \right)^{1/2} \left(\sum_{i=1}^m \|\phi_i\|_{\mathcal{H}}^2 \right)^{1/2}
\end{aligned}$$

$$\leq \frac{C_1}{\tau}\left(\sum_{i=1}^{m}\|G_i-G_{i-1}\|_{\mathcal{H}}^2\right)^{1/2}\left(\sqrt{C_L}\,\|G_r\|_{\mathcal{H}}\right)$$
$$+C_2\epsilon\left(\sum_{i=1}^{m}\|G_i-G_{i-1}\|_{\mathcal{V}}^2\right)^{1/2}\left(\sqrt{C_H}\,\|G_r\|_{\mathcal{H}}+\sqrt{C_V}\,\|G_r\|_{\mathcal{V}}\right)$$
$$+C_3\left(\sum_{i=1}^{m}\|G_i-G_{i-1}\|_{\mathcal{V}}^2\right)^{1/2}\left(\sqrt{C_L}\,\|G_r\|_{\mathcal{H}}\right).$$

By applying the inequality $ab \leq \frac{1}{2p}a^2 + \frac{p}{2}b^2$ on each term with $p > 0$ properly chosen, we get that

$$\frac{1}{\tau}\|G_r\|_{\mathcal{H}}^2+\epsilon\|G_r\|_{\mathcal{V}}^2$$
$$\leq \frac{3C_LC_1{}^2}{2}\sum_{i=1}^{m}\frac{1}{\tau}\|G_i-G_{i-1}\|_{\mathcal{H}}^2+\frac{1}{6\tau}\|G_r\|_{\mathcal{H}}^2$$
$$+\frac{3C_HC_2{}^2\tau\epsilon}{2}\sum_{i=1}^{m}\epsilon\|G_i-G_{i-1}\|_{\mathcal{V}}^2+\frac{1}{6\tau}\|G_r\|_{\mathcal{H}}^2$$
$$+\frac{C_VC_2{}^2}{2}\sum_{i=1}^{m}\epsilon\|G_i-G_{i-1}\|_{\mathcal{V}}^2+\frac{\epsilon}{2}\|G_m\|_{\mathcal{V}}^2$$
$$+\frac{3C_LC_3{}^2\tau}{2\epsilon}\sum_{i=1}^{m}\epsilon\|G_i-G_{i-1}\|_{\mathcal{V}}^2+\frac{1}{6\tau}\|G_r\|_{\mathcal{H}}^2$$
$$\leq \frac{3C_LC_1{}^2}{2}\sum_{i=1}^{m}\frac{1}{\tau}\|G_i-G_{i-1}\|_{\mathcal{H}}^2$$
$$+\frac{3C_HC_2{}^2\tau\epsilon+C_VC_3{}^2+3C_LC_3{}^2\tau/\epsilon}{2}\sum_{i=1}^{m}\epsilon\|G_i-G_{i-1}\|_{\mathcal{V}}^2$$
$$+\frac{1}{2\tau}\|G_r\|_{\mathcal{H}}^2+\frac{\epsilon}{2}\|G_r\|_{\mathcal{V}}^2$$
$$\leq \frac{C_m}{2}\sum_{i=1}^{m}\left(\frac{1}{\tau}\|G_i-G_{i-1}\|_{\mathcal{H}}^2+\epsilon\|G_i-G_{i-1}\|_{\mathcal{V}}^2\right)$$
$$+\frac{1}{2\tau}\|G_r\|_{\mathcal{H}}^2+\frac{\epsilon}{2}\|G_r\|_{\mathcal{V}}^2.$$

Then there follows

$$\|G_r\|_{a_\tau}^2 \leq C_m\sum_{i=1}^{m}\|G_i-G_{i-1}\|_{a_\tau}^2. \tag{39}$$

Putting $v_i = G_i - G_{i-1}$ in (36), we get, by the same procedure as the one leading to (22),

$$\frac{1}{\tau}\|G_i-G_{i-1}\|_{\mathcal{H}}^2+\epsilon\|G_i-G_{i-1}\|_{\mathcal{V}}^2$$
$$\leq 2\left[\frac{1}{\tau}\left(\|G_{i-1}\|_{\mathcal{H}}^2-\|G_i\|_{\mathcal{H}}^2\right)+\epsilon\left(\|G_{i-1}\|_{\mathcal{V}}^2-\|G_i\|_{\mathcal{V}}^2\right)+2\tau\sigma^2\|G_i\|_{\mathcal{V}}^2\right].$$

Thus, by (39) we get

$$\|G_i - G_{i-1}\|_{a_\tau}^2 \le 2\left(\|G_{i-1}\|_{a_\tau}^2 - \|G_i\|_{a_\tau}^2\right) + \frac{4\tau\sigma^2}{\epsilon}\|G_i\|_{a_\tau}^2$$
$$\le 2\left(\|G_{i-1}\|_{a_\tau}^2 - \|G_i\|_{a_\tau}^2\right) + \frac{4\tau\sigma^2 C_m}{\epsilon}\sum_{j=1}^{m}\|G_j - G_{j-1}\|_{a_\tau}^2.$$

Summing over i from 1 to m, we obtain

$$\sum_{i=1}^{m}\|G_i - G_{i-1}\|_{a_\tau}^2 \le 2\left(\|G_0\|_{a_\tau}^2 - \|G_m\|_{a_\tau}^2\right) + \frac{4\tau\sigma^2 C_m m}{\epsilon}\sum_{i=1}^{m}\|G_i - G_{i-1}\|_{a_\tau}^2.$$

This leads to

$$\left(1 - \frac{4C_m m\sigma^2\tau}{\epsilon}\right)\sum_{i=1}^{m}\|G_i - G_{i-1}\|_{a_\tau}^2 \le 2\left(\|G_0\|_{a_\tau}^2 - \|G_m\|_{a_\tau}^2\right).$$

Using that $1 - \frac{4\tau\sigma^2 C_m m}{\epsilon} \ge \frac{1}{2}$, which is equivalent to $\tau \le \frac{\epsilon}{8\sigma^2 C_m m}$, we get by the argument preceding (24)

$$\sum_{i=1}^{m}\|G_i - G_{i-1}\|_{a_\tau}^2 \le 4\left(\|G_0\|_{a_\tau}^2 - \|G_m\|_{a_\tau}^2\right). \tag{40}$$

Finally, using (40) in (39) and setting $r = m$, it follows

$$\|G_m\|_{a_\tau}^2 \le 4C_m\left(\|G_0\|_{a_\tau}^2 - \|G_m\|_{a_\tau}^2\right),$$

and

$$\|G_m\|_{a_\tau}^2 \le \frac{4C_m}{4C_m + 1}\|G_0\|_{a_\tau}^2.$$

□

Now we are ready to formulate the second of the two main theorems.

Theorem 8. *At each time level, for any $0 < \rho \le 1$, assume that the number s of the space decomposition iteration satisfies*

$$s \ge 2|\ln(\rho\tau)| \Big/ \left|\ln\frac{4C_m}{4C_m+1}\right|.$$

Then the following error estimate holds for algorithm 2:

$$\|u^n - V^n\|_{a_\tau} \le C\tau.$$

Proof. The proof is very similar to the proof of Theorem 5. At each time level n, let $U^{n+1} \in \mathcal{V}$ be an auxiliary function that satisfies

$$\left(\frac{U^{n+1} - u^n}{\tau}, v\right) + \epsilon a\left(U^{n+1}, v\right) + b\left(U^{n+1}, v\right) = \left(f^{n+1}, v\right), \qquad \forall v \in \mathcal{V}. \tag{41}$$

Comparing (6) with (41) and using the fact that $\mathcal{V}_i \subset \mathcal{V}$, we see that:

$$\left(\frac{u_i^{n+\frac{k}{s}} - U^{n+1}}{\tau}, v_i\right) + \epsilon a\left(u_i^{n+\frac{k}{s}} - U^{n+1}, v_i\right) + b\left(u_i^{n+\frac{k}{s}} - U^{n+1}, v_i\right) = 0, \forall v_i \in \mathcal{V}_i,$$

and $u_i^{n+\frac{k}{s}} - u_{i-1}^{n+\frac{k}{s}} \in \mathcal{V}_i$. Now, let us set

$$G_i^{k+1} = u_i^{n+\frac{k}{s}} - U^{n+1}, \quad G_{i-1}^{k+1} = u_{i-1}^{n+\frac{k}{s}} - U^{n+1}.$$

We note that G_i^{k+1} satisfies (36) and $G_i^{k+1} - G_{i-1}^{k+1} \in \mathcal{V}_i$. Clearly, G_i^{k+1} and G_{i-1}^{k+1} also depend on n, for notational simplicity, the index n in G_i^{k+1} and G_{i-1}^{k+1} is omitted. According to lemma 7 it is true that

$$\begin{aligned}
&\left\|u^{n+1} - U^{n+1}\right\|_{a_\tau}^2 = \left\|u^{n+\frac{s}{s}} - U^{n+1}\right\|_{a_\tau}^2 = \left\|u_m^{n+\frac{s}{s}} - U^{n+1}\right\|_{a_\tau}^2 = \left\|G_m^{s+1}\right\|_{a_\tau}^2 \\
&\leq \frac{4C_m}{4C_m+1}\left\|G_m^s\right\|_{a_\tau}^2 \leq \cdots \leq \left(\frac{4C_m}{4C_m+1}\right)^s \left\|G_m^1\right\|_{a_\tau}^2 \\
&= \left(\frac{4C_m}{4C_m+1}\right)^s \left\|u_m^n - U^{n+1}\right\|_{a_\tau}^2 = \left(\frac{4C_m}{4C_m+1}\right)^s \left\|u^n - U^{n+1}\right\|_{a_\tau}^2,
\end{aligned}$$

which means

$$\left\|u^{n+1} - U^{n+1}\right\|_{a_\tau} \leq \left(\frac{4C_m}{4C_m+1}\right)^{\frac{s}{2}} \left\|u^n - U^{n+1}\right\|_{a_\tau}.$$

The rest of the proof is identical to the proof of Theorem 5. □

5. Decomposition of Finite Element Spaces

In this section, we try to use a finite element method to solve the equation

$$\begin{aligned}
u_t - \nabla\cdot(a(x,t)\nabla u) + \vec{b}(x,t)\cdot\nabla u + c(x,t)u = f(x,t), \quad in \quad \Omega, \quad t \geq 0, \qquad (42)\\
u(x,0) = u_0(x), \quad u = 0 \text{ on } \partial\Omega.
\end{aligned}$$

The variational formulation for the solution of (42) is

$$\begin{aligned}
(u_t, v)_{L^2(\Omega)} + (a\nabla u, \nabla v)_{L^2(\Omega)} + (\vec{b}\cdot\nabla u, v)_{L^2(\Omega)} \\
+ (cu, v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}, \ \forall v \in H_0^1(\Omega).
\end{aligned}$$

We shall require that $a(x,t) > 0$ and that $\exists \varepsilon > 0$ such that

$$(a\nabla v, \nabla v)_{L^2(\Omega)} + (\vec{b}\cdot\nabla v, v)_{L^2(\Omega)} + (cv, v)_{L^2(\Omega)} \geq \varepsilon(a\nabla v, \nabla v)_{L^2(\Omega)}.$$

Correspondingly, by choosing

$$\begin{aligned}
\mathcal{H} = L^2(\Omega), \quad \mathcal{V} = H_0^1(\Omega), \quad a(u,v) = \varepsilon(a\nabla u, \nabla v)_{L^2(\Omega)} \quad \text{and} \\
b(u,v) = ((a-\varepsilon)\nabla u, \nabla v)_{L^2(\Omega)} + (\vec{b}\cdot\nabla u, v)_{L^2(\Omega)} + (cu, v)_{L^2(\Omega)},
\end{aligned}$$

and assuming the functions have some smoothness properties, it can be seen that equation (42) is a special case of (1). We shall show how the finite element spaces can be decomposed. Estimations for the constants C_L, C_V, C_H, C_1, C_2 and C_3 will be given.

For equation (42), the embedding $\mathcal{H} \subset \mathcal{V}$ is compact. When ε is big (i.e. $\varepsilon = O(1)$), the exponential dependence of the errors on the time T can be removed by using the damping property resulted from the compact embedding, see Remark 3.1 of [23, p.34].

5.1. **Domain Decomposition.** The domain decomposition method is used here to decompose a finite element space. To construct a finite element approximation space, we first divide Ω into coarse mesh elements $\{\Omega_i\}_{i=1}^m$ which are shape-regular, see Ciarlet [9], and have diameters of order H. For each Ω_i, we further divide it into smaller simplices with diameter of order h. We call this the fine mesh or the h-level subdivision of Ω with mesh parameter h. We denote $\Omega_h = \bigcup\{T \in \mathcal{T}_h\}$ as the fine mesh subdivision. Let $S_0^h \subset H_0^1(\Omega)$ be the continuous, piecewise linear function space, with zero trace on $\partial\Omega_h$, over the h-level subdivision of Ω. For each Ω_i, we consider an enlarged subdomain $\Omega_i^\delta = \bigcup\{T \in \mathcal{T}_h, distance(T, \Omega_i) \leq \delta\}$. The union of Ω_i^δ covers Ω_h with overlaps of size δ. Let us denote the piecewise linear finite element space with zero traces on the boundaries $\partial\Omega_i^\delta$, as $S_0^h(\Omega_i^\delta)$. Then it is true that

$$S_0^h = \sum S_0^h(\Omega_i^\delta). \tag{43}$$

For the overlapping subdomains, assume that there are m colors such that each subdomain Ω_i^δ can be marked with one color, and the subdomains with the same color will not intersect each other. For suitable overlaps, one can always choose m to be a fixed number. Let Ω_i' be the union of the subdomains with the i^{th} color, and

$$\mathcal{V} = S_0^h, \qquad \mathcal{V}_i = \{v \in S_0^h | \ \ v(x) = 0, \ \ x \notin \Omega_i'\}. \tag{44}$$

The decomposition (43) means

$$\mathcal{V} = \sum_{i=1}^m \mathcal{V}_i. \tag{45}$$

Let $\{\theta_i\}_{i=1}^m$ be a partition of unity with respect to $\{\Omega_i'\}_{i=1}^m$, $\theta_i \in C_0^\infty(\Omega' \cap \Omega)$, $0 \leq \theta_i \leq 1$ and $\sum_{i=1}^m \theta_i = 1$. It can be chosen such that $|\nabla\theta_i| \leq C/\delta$. Let I_h be the interpolation operator which uses the function values at the h-level nodes. For any $v \in \mathcal{V}$, let $v_i = I_h(\theta_i v) \in \mathcal{V}_i$. They will satisfy $v = \sum_{i=1}^m v_i$, and

$$\sum_{i=1}^m \|v_i\|_{L^2(\Omega_i)}^2 \ \leq \ C\|v\|_{L^2(\Omega_i)}^2, \tag{46}$$

$$\sum_{i=1}^m \|v_i\|_{H^1(\Omega_i)}^2 \ \leq \ \frac{C}{\delta^2}\|v\|_{L^2(\Omega_i)}^2 + C\|\nabla v\|_{L^2(\Omega_i)}^2. \tag{47}$$

The proof of (46) and (47) can be found in different papers, see Cai [3, 4]. In the literature, the overlapping size δ is often taken to be very large, i.e. $\delta = c_0 H$, see [4, 33, 13]. In order to reduce communication and computational work, we shall keep the overlapping size δ small, let us take $\delta \approx h$. Estimates (46) and (47) show that for overlapping domain decomposition, the constants in (3) are:

$$C_L = C, \ \ C_V = C, \quad C_H = \frac{C}{\delta^2}. \tag{48}$$

The constant C_a defined in (13) will then be:

$$C_a = \frac{\max(C, C\tau\epsilon^2/\delta^2 + C + C\sigma^2\tau/\epsilon)}{\alpha}. \tag{49}$$

So, when $\epsilon^2\tau \leq C\delta^2$, $\tau \leq C\epsilon$, the constant C_a does not depend on τ and the mesh parameters, which means that the rate of convergence of the space decomposition does not depend on τ and the finite element meshes.

For the multiplicative space decomposition, we have

$$C_m = \max(3C_L{C_1}^2, 3C_H{C_2}^2\tau\epsilon + C_V{C_2}^2 + 3C_L{C_3}^2\tau/\epsilon)$$

Using the Hölder inequality, it is easy to show that $C_1 = m, C_2 = m$ and $C_3 = m$ (See Tai and Xu [27]), where m is the number of colors for the subdomains. Thus,

$$C_m = \max(C, C\tau\epsilon^2/\delta^2 + C + C\tau/\epsilon). \tag{50}$$

Under the same conditions as for the additive scheme, we have a convergence rate independent of τ, the mesh size and the number of subdomains.

When the diffusion parameter is large, i.e. $\epsilon = O(1)$, we may need to add a coarse mesh to accelerate the convergence. In such a case,

$$C_L = C, \quad C_V = C, \quad C_H = \frac{CH^2}{\delta^2}. \tag{51}$$

Then we just need to choose τ, H and δ to satisfy

$$\tau H^2 \le C\delta^2, \quad \tau \le O(1)$$

to get a convergence rate not depending τ, the mesh size and the number of subdomains.

5.2. **Multigrid Method.** For the multigrid method, we assume the finite element partition Ω_h is constructed by a successive refinement process. More precisely, $\Omega_h = \mathcal{T}_J$ for some $J > 1$, and $\mathcal{T}_j$ for $j \le J$ is a nested sequence of quasi-uniform finite element partitions, i.e. $\mathcal{T}_j$ consist of finite elements $\mathcal{T}_j = \{\tau_j^i\}$ of size h_j such that $\Omega_h = \cup_i \tau_j^i$ for which the quasi-uniformity constants are independent of j (cf. [9]) and τ_{j-1}^l is a union of elements of $\{\tau_j^i\}$. We further assume that there is a constant $\gamma < 1$, independent of j, such that h_j is proportional to γ^{2j}.

In applications for elliptic equations, it is always required that the coarsest mesh size is proportional to γ^2. However, the coarsest mesh does not need to be very coarse for applications to parabolic problems. In the following, we shall assume that the coarsest mesh size is proportional to γ^{2j_0}, i.e. we only use the multilevel meshes between level j_0 and level J.

Corresponding to each finite element partition $\mathcal{T}_j$, a finite element space $\mathcal{M}_j$ can be defined by

$$\mathcal{M}_j = \{v \in W^{1,\infty}(\Omega) : v|_\tau \in \mathcal{P}_1(\tau), \quad \forall \tau \in \mathcal{T}_j\}.$$

Each finite element space $\mathcal{M}_j$ is associated with a nodal basis, denoted by $\{\phi_j^i\}_{i=1}^{n_j}$ satisfying

$$\phi_j^i(x_j^k) = \delta_{ik},$$

where $\{x_j^k\}_{k=1}^{n_j}$ is the set of all interior nodes of the elements of $\mathcal{T}_j$. Associated with each such an interior nodal basis function, we define a one dimensional subspace as follows:

$$\mathcal{M}_j^i = \text{span } (\phi_j^i).$$

On each level, the nodes can be colored so that the neighboring nodes are always of different colors. The number of colors needed for a regular mesh is always a bounded constant; call it m_c. Let $\mathcal{V}_j^k, \ k = 1, 2, \cdots m_c$ be the sum of the subspaces

$\mathcal{M}_j^i$ associated with nodes of the k^{th} color on level j. Letting $\mathcal{V} = \mathcal{M}_J$, we have the following trivial space decomposition:

$$\mathcal{V} = \sum_{j=1}^{J} \sum_{k=1}^{m_c} \mathcal{V}_j^k. \tag{52}$$

Each subspace $\mathcal{V}_j^k$ contains some orthogonal one dimensional subspaces $\mathcal{M}_j^i$, and so the equations (5) and (6) for each $\mathcal{V}_j^k$ can be done in parallel over the one dimensional subspaces $\mathcal{M}_j^i$.

Similarly as in [32, Prop. 8.6, pp.611], [20, pp.181], [35] and [27, §4.2.1], it is not difficult to show that assumption (3) is valid with

$$C_L = C, C_H = \frac{C}{h_{j_0}^2}, C_V = C.$$

We shall concentrate on estimating C_1, C_2 and C_3. The analysis for the estimates depends heavily on the following inequalities (see [32, pp.600-601], [27, section 5.2] and [34, Lemma 2.7], [35, Lemma 3.2]):

$$\begin{aligned}
(u,v) &\leq C\gamma^{|l-j|}\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}, \quad \forall u \in \mathcal{V}_j^i,\ \forall v \in \mathcal{V}_l^k, \\
a(u,v) &\leq C\gamma^{|l-j|}\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}, \quad \forall u \in \mathcal{V}_j^i,\ \forall v \in \mathcal{V}_l^k, \\
b(u,v) &\leq C\gamma^{|l-j|}\|u\|_{\mathcal{V}}\|v\|_{\mathcal{H}}, \quad \forall u \in \mathcal{V}_j^i,\ \forall v \in \mathcal{V}_l^k.
\end{aligned}$$

In proving the above inequalities, we need to use the fact that a function from $\mathcal{V}_j^i$ is a sum of one dimensional orthogonal functions, and the fact that the support set of a nodal basis function from $\mathcal{V}_j^i$ is a refined element of the support of a nodal basis function from $\mathcal{V}_l^k$ for $j > l \geq j_0$. Using Lemma 5.1 of [27], we get that

$$\begin{aligned}
&\sum_{j=j_0}^{J}\sum_{l=j_0}^{J}\sum_{i=1}^{m_c}\sum_{k=1}^{m_c} a(u_j^i, u_l^k) \\
\leq\ & \sum_{j=j_0}^{J}\sum_{l=j_0}^{J}\sum_{i=1}^{m_c}\sum_{k=1}^{m_c} C\gamma^{|l-j|}\|u_j^i\|_{\mathcal{V}}\|u_l^k\|_{\mathcal{V}} \\
\leq\ & m_c \sum_{j=j_0}^{J}\sum_{l=j_0}^{J} \gamma^{|j-l|}\left(\sum_{i=1}^{m_c}\|u_j^i\|_{\mathcal{V}}^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{m_c}\|v_l^k\|_{\mathcal{V}}^2\right)^{\frac{1}{2}} \\
\leq\ & m_c\left(\max_j \sum_{l=j_0}^{J}\gamma^{|j-l|}\right)\left(\sum_{j=j_0}^{J}\sum_{i=1}^{m_c}\|u_j^i\|_{\mathcal{V}}^2\right)^{\frac{1}{2}}\cdot\left(\sum_{l=j_0}^{J}\sum_{k=1}^{m_c}\|v_l^k\|_{\mathcal{V}}^2\right)^{\frac{1}{2}} \\
\leq\ & \frac{m_c}{1-\gamma}\left(\sum_{j=1}^{J}\sum_{i=1}^{m_c}\|u_j^i\|_{\mathcal{V}}^2\right)^{\frac{1}{2}}\cdot\left(\sum_{l=1}^{J}\sum_{k=1}^{m_c}\|v_l^k\|_{\mathcal{V}}^2\right)^{\frac{1}{2}},
\end{aligned}$$

which shows that the constant C_2 is independent of the parameters h, J and j_0. By the same technique, we see that C_1 and C_3 are also independent of h, J and j_0.

In order to get a convergence rate independent of the mesh size, the time step size and number of levels, we just need to choose j_0 and τ in such a way that

$$\varepsilon^2\tau \leq Ch_{j_0}^2 \quad \text{and} \quad \tau \leq C\varepsilon,$$

which indicate that the coarser meshes may not be needed when the diffusion parameter ε is very small. For relative large diffusion, we may need coarser meshes, but the coarsest mesh does not need a mesh size such that $h_{j_0} = O(1)$. The coarsest mesh size only needs to be in the order of

$$h_{j_0} = O\left(\sqrt{\tau}\right).$$

6. Numerical Experiments

In this section, we try to solve the equation

$$u_t - \epsilon \Delta u + u_x + u_y = 0, \quad in \ \ \Omega, \quad t \geq 0, \tag{53}$$

which has an analytical solution

$$u(x,y,t) = \frac{1}{4\epsilon(t+0.2)} e^{-\frac{x^2+y^2}{4\epsilon(t+0.2)}}, \tag{54}$$

with consistent Dirichlet boundary conditions. Let us take $\Omega = [0,1] \times [0,1]$. Equation (53) describes a diffusion process plus convection in the diagonal direction. When ϵ is small, the convection is dominating.

In the numerical tests, Ω is first divided into N coarse mesh elements, both in the x and y directions and each coarse element is then divided into M finite mesh elements, again both in the x and y directions. So the coarse mesh size is $H = \frac{1}{N}$ and the fine mesh size is $h = \frac{1}{MN}$. The Laplace operator is approximated by the 5-points finite-difference approximation and the convection term is approximated by the up-wind approximation. In the tables, s is the number of iterations performed at each time level. $u(t_n)$ is the true solution at $t = t_n$ and u_g^n is the global finite element solution without domain decomposition.

Both algorithms 1 and 2 are tested for (53) when ϵ is large, i.e. $\epsilon = 1$. Minimum overlap is used, i.e. $\delta = h$. Table 1 shows the errors for the different values of s for algorithm 1. In the computations, $\alpha_i = \frac{1}{4}$, $i = 1,2,3,4$. Table 2 shows the computational errors for algorithm 2. For algorithm 1, about 25 iterations are needed to reach the same accuracy as the global finite element solution. For algorithm 2, only 5 iterations are needed to reach the same accuracy. The error between the domain decomposition solution and the global finite element is getting smaller when the iteration number s is getting bigger.

Our numerical experiences show that the term containing τ/ε in the expressions of C_a and C_m (c.f. (49) and (50)) is negligible compared with the other terms as long as the algorithms are stable. The dominating quantity for C_a and C_m is from the term containing $\tau\varepsilon^2/\delta^2$ (c.f. (49) and (50)). Thus a smaller τ or a bigger δ will give a better convergence. Adding a coarse mesh is also making the dominating term smaller and will also lead to a better convergence. In tables 3 and 4, we try to compute the same problem as in Tables 1 and 2, but with a smaller time step. The discretization error seems to be dominated by the spacial variables when time step size is small. However, the convergence for the domain decomposition iteration is getting better. In Table 1, the error $\|u^n - u_g^n\|_\infty$ is reduced from 5.9708×10^{-2} to 1.6178×10^{-3} when s is increased from 2 to 20. In Table 3, the error $\|u^n - u_g^n\|_\infty$ is reduced from 3.6813×10^{-2} to 3.6155×10^{-4} when s is increased from 2 to 20. It is clear that the convergence for the smaller τ is better. For algorithm 2, the improvement of the convergence is even better, see Tables 2 and 4.

In Tables 5 and 6, we show some numerical results for the convection dominated case. The value of ϵ is taken as $\epsilon = 0.01$ and so the convection is dominating. Figure 1 shows the computed solution. The errors for Algorithms 1 and 2 are shown in Tables 5 and 6 respectively. Minimum overlap is used. We can see that $s = 1$

Table 1. Numerical results by algorithm 1 with $H = 1/5, h = 1/50, \tau = 1/400$.

s	$\|u^n - u(t_n)\|_\infty$	$\|u_g^n - u(t_n)\|_\infty$	$\|u^n - u_g^n\|_\infty$
2	0.0595	0.0002	5.9708e-002
4	0.0275	0.0002	2.7719e-002
6	0.0158	0.0002	1.5993e-002
8	0.0101	0.0002	1.0303e-002
10	0.0069	0.0002	7.0735e-003
12	0.0049	0.0002	5.0552e-003
14	0.0035	0.0002	3.7122e-003
16	0.0026	0.0002	2.7787e-003
18	0.0020	0.0002	2.1092e-003
20	0.0015	0.0002	1.6178e-003

Table 2. Numerical results by algorithm 2 with $H = 1/5, h = 1/50, \tau = 1/400$.

s	$\|u^n - u(t_n)\|_\infty$	$\|u_g^n - u(t_n)\|_\infty$	$\|u^n - u_g^n\|_\infty$
2	0.0023	0.0002	2.4434e-003
4	0.0001	0.0002	2.4032e-004
6	0.0002	0.0002	2.5924e-005
8	0.0002	0.0002	2.8194e-006
10	0.0002	0.0002	3.0612e-007
12	0.0002	0.0002	3.3169e-008
14	0.0002	0.0002	3.5877e-009
16	0.0002	0.0002	3.8746e-010
18	0.0002	0.0002	4.1799e-011
20	0.0002	0.0002	4.5126e-012

is already enough for Algorithm 2 to get an accuracy as good as the global finite element solution. For time dependent problems, Algorithm 2 is always much faster than Algorithm 1.

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Table 3. Numerical results by algorithm 1 with $H = 1/5, h = 1/50, \tau = 1/1000$.

s	$\|u^n - u(t_n)\|_\infty$	$\|u_g^n - u(t_n)\|_\infty$	$\|u^n - u_g^n\|_\infty$
2	0.03663	0.0001887	3.6813e-002
4	0.01458	0.0001887	1.4762e-002
6	0.007512	0.0001887	7.6987e-003
8	0.00431	0.0001887	4.4957e-003
10	0.002602	0.0001887	2.7864e-003
12	0.001605	0.0001887	1.7897e-003
14	0.0009943	0.0001887	1.1764e-003
16	0.000609	0.0001887	7.8565e-004
18	0.0003574	0.0001887	5.3072e-004
20	0.00019	0.0001887	3.6155e-004

Table 4. Numerical results by algorithm 2 with $H = 1/5, h = 1/50, \tau = 1/1000$.

s	$\|u^n - u(t_n)\|_\infty$	$\|u_g^n - u(t_n)\|_\infty$	$\|u^n - u_g^n\|_\infty$
2	0.0002339	0.0001887	4.0364e-004
4	0.0001813	0.0001887	7.5526e-006
6	0.0001886	0.0001887	1.4181e-007

Table 5. Numerical results by algorithm 1 with $H = 1/5, h = 1/50, \tau = 1/400$.

s	$\|u^n - u(t_n)\|_\infty$	$\|u_g^n - u(t_n)\|_\infty$	$\|u^n - u_g^n\|_\infty$
2	24.43	1.49	2.2982e+001
4	17.62	1.49	1.6767e+001
6	10.17	1.49	9.8991e+000
8	5.618	1.49	5.4491e+000
10	3.132	1.49	2.9548e+000
12	1.942	1.49	1.6459e+000
14	1.542	1.49	9.2742e-001
16	1.453	1.49	5.2589e-001
18	1.456	1.49	2.9940e-001
20	1.460	1.49	1.7059e-001

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Table 6. Numerical results by algorithm 2 with $H = 1/5, h = 1/50, \tau = 1/400$.

s	$\|u^n - u(t_n)\|_\infty$	$\|u_g^n - u(t_n)\|_\infty$	$\|u^n - u_g^n\|_\infty$
2	1.49	1.49	1.3121e-002
4	1.49	1.49	1.5878e-006
6	1.49	1.49	1.8820e-010
8	1.49	1.49	1.3500e-013
10	1.49	1.49	1.2079e-013
12	1.49	1.49	1.2434e-013
14	1.49	1.49	1.1724e-013
16	1.49	1.49	1.2434e-013
18	1.49	1.49	1.3145e-013
20	1.49	1.49	1.2079e-013

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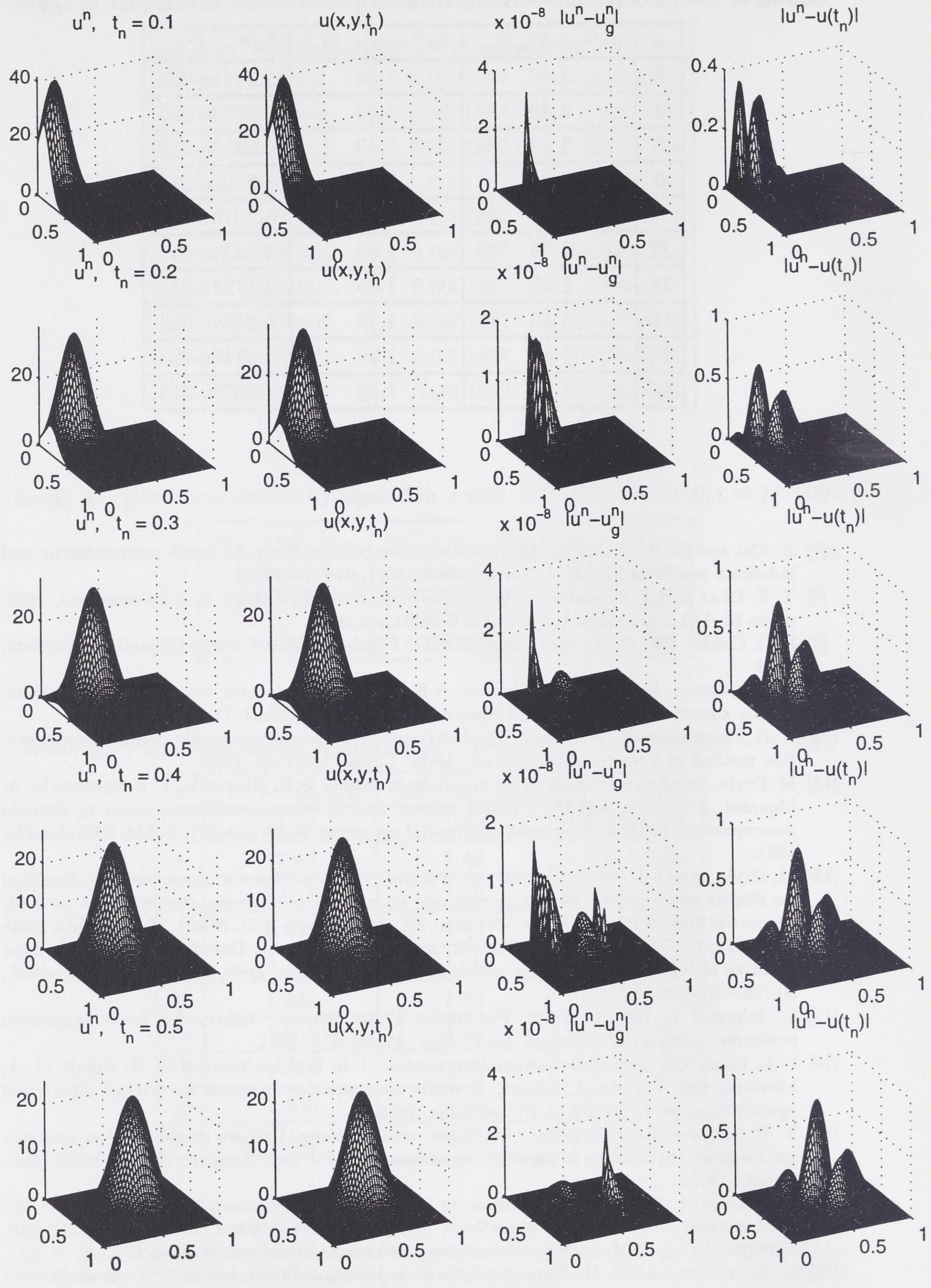

FIGURE 1. The computed solution by algorithm 2 with $\varepsilon = 0.01; H = 1/5; h = 1/50; \tau = 1/400; s = 6$.

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