Department
of
PURE MATHEMATICS

Geometric Transversals for Families of disjoint Translates in the Plane

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Andreas Holmsen

Report no. 67

April 2000

UNIVERSITY OF BERGEN
Bergen, Norway

Department of Mathematics
University of Bergen
5008 Bergen
Norway

ISSN 0332-5047

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ABSTRACT. A family of convex sets in the plane admits a common transversal if there exists a line that meets every member of the family.

In 1980 M. Katchalski and T. Lewis proved the following for families of (pairwise) disjoint translates of a compact convex set in the plane: There exists a positive integer k such that if F is a family satisfying the condition that any three of its members admit a common transversal, then some subfamily $G \subset F$ admits a common transversal and $|F \setminus G| \leq k$. They prove that $k \leq 603$, and conjecture that $k = 2$ is a universal constant for families of disjoint translates.

In this paper we shall improve Katchalski and Lewis' upper bound on k, and construct counterexamples to their conjecture. We also prove some results on geometric permutations, and in particular that any family consisting of more than three disjoint unit discs in the plane admits at most two geometric permutations, and with this improving a result by Smorodinsky *et. al.*

1. INTRODUCTION

Geometric transversal theory originates from Helly's theorem, when Vincensini in 1935 observed that a common point for a family of convex sets in $\mathbf{R}^n$, is a special case, $k = 0$, of a k-transversal for the family; i.e. a k-flat that meets every member of the family. It is therefore natural to ask whether there exist "Helly-type" results when $k > 0$. There are several surveys concerning geometric transversal theory, by among others, Danzer-Grünbaum-Klee [3], Goodman-Pollack-Wenger [7], Eckhoff [6], and most recently, Wenger [18].

For the main part of this paper we shall focus our attention on families in the plane, and the case of 1-transversals (or just transversals). If a family F admits a common transversal we say that F has the property T, and if every m-membered subset of F admits a common transversal, we say that F has the property $T(m)$. Examples (e.g. see [9]) show that there does not exist a Helly-type result for families of convex sets, in general. That is, there does not exist any positive integer m, such that $T(m) \Rightarrow T$, for general families of convex sets. To obtain such results one must therefore limit oneself to special families of convex sets.

Such a result was found already in 1940, by Santaló, who proved that for families of parallel rectangles, $T(6) \Rightarrow T$. This result cannot be extended, in the sense that the number 6 cannot be replaced by the number 5. Another example of a special family, for which there is a Helly-type theorem, is when F is a family of disjoint translates of a compact convex set. In 1989 Tverberg [16] proved a long-standing conjecture of Grünbaum, that $T(5) \Rightarrow T$.

A family of convex sets in the plane is said to have the property $T - k$, if there is a transversal for all but at most k of the members of the family. In 1980 Katchalski and Lewis [12] asked the following question: What happens if the number 5 is replaced by the number 3 in Grünbaum's conjecture? They proved that one will always be able to find a large subfamily that admits a common transversal. That is, $T(3) \Rightarrow T - k$, for some positive integer k, when the family in question consists of disjoint translates of a compact convex set. They proved that $k \leq 603$, but

Date: April 2000.

conjectured that $k = 2$ is a universal constant for all such families. Examples (e.g. see [2]) show that k cannot be less than 2. In 1982, Katchalski and Lewis [11] proved $T(4) \Rightarrow T-2$ for families of translates of a parallelogram. In [17] Tverberg studies the Katchalski-Lewis conjecture and the possibilities of using the methods from [16] to approach a solution of this problem.

Our paper will focus on families of disjoint translates, with special attention on the situation $T(3)$, thus continuing the work done in [12] and [17].

2. Preliminaries

We now prove some standard reductions done by Tverberg [16]. In particular we will show for families of translates, that transversal properties are in many cases preserved under symmetrization.

Let $K \subset \mathbf{R}^n$ be a compact convex set. From K we can always construct a centrally symmetric set $K' \subset \mathbf{R}^n$ by the Minkowski addition formula

$$K' = \frac{1}{2}(K - K) = \{x \mid x = \frac{1}{2}x_1 - \frac{1}{2}x_2, \forall x_1, x_2 \in K\}$$

The center of symmetry then lies in the origin.

Let F be a family of translates of K. F can be expressed as $\{K + v_i, i \in I\}$, where the v_i are translation vectors in $\mathbf{R}^n$. If we substitute K' for K, such that we get $F' = \{K' + v_i, i \in I\}$, we will say that we symmetrize F. We shall now see which properties are preserved by the symmetrization.

Proposition 2.1. *The family F is disjoint if and only F' is disjoint.*

Proof. Let $K + v_1$ and $K + v_2$ be two translates in $\mathbf{R}^n$. $K + v_1$ and $K + v_2$ have a point in common if and only if there are points $x_1, x_2 \in K$ such that $x_1 + v_1 = x_2 + v_2$. But the last statement can be rewritten as $\frac{1}{2}(x_1 - x_2) + v_1 = \frac{1}{2}(x_2 - x_1) + v_2$, stating that the symmetrized translates $K' + v_1$ and $K' + v_2$ have a common point. □

We say that F has (or admits) a *k-transversal* if there is a k-flat that meets every member of F. We then have the following.

Proposition 2.2. *F admits an $(n-1)$-transversal if and only if F' admits an $(n-1)$-transversal.*

Proof. Let H be an $(n-1)$-transversal for $F = \{K + v_i, i \in I\}$. Let H_1 and H_2 be hyperplanes that are parallel to H, and that are upper and lower supporting hyperplanes of K, respectively. Choose points $y_1 \in H_1 \cap K$ and $y_2 \in H_2 \cap K$. We may assume that the origin is placed such that $y_1 = -y_2$. Since H is an $(n-1)$-transversal for F, we have in particular for each translate $K + v_i$ in F, that the point $x_i = \lambda_i y_1 + (1 - \lambda_i) y_2 + v_i$ is in H, for some $\lambda_i \in [0, 1]$. Since

$$y_1 = \frac{1}{2}(y_1 - y_2) \in K'$$

$$y_2 = \frac{1}{2}(y_2 - y_1) \in K'$$

$$\text{we have } x_i = \lambda_i y_1 + (1 - \lambda_i) y_2 + v_i \in K' + v_i, (\forall i \in I)$$

Thus, H is an $(n-1)$-transversal for F'.

Conversely, assume that F' has an $(n-1)$-transversal H', and let H_1' and H_2' be hyperplanes that are parallel to H', and that are upper and lower supporting hyperplanes of K', respectively.

As before, choose points $y_1 \in H_1' \cap K'$ and $y_2 \in H_2' \cap K'$. By central symmetry, these points may be chosen such that $y_1 = -y_2$. For points a and b in K we have

$$y_1 = \frac{1}{2}(a-b)$$

$$y_2 = \frac{1}{2}(b-a)$$

and we may further assume that the origin is placed such that $a = -b$. This gives us $y_1 = a$ and $y_2 = b$. Since H' is an $(n-1)$-transversal for F', we have in particular for each translate $K' + v_i$ in F', that the point $x_i = \lambda_i a + (1-\lambda_i)b + v_i$ is in H', for some $\lambda_i \in [0,1]$. This gives us

$$x_i = \lambda_i a + (1-\lambda_i)b + v_i \in K + v_i, (\forall i \in I)$$

Thus H' is an $(n-1)$-transversal for F. □

Note. The above argument also shows that a family of translates and the corresponding symmetrized family F', admit $(n-1)$-transversals in the exact same directions.

Let F be a family of disjoint translates. A 1-transversal of F will meet the translates in a certain order, inducing two permutations of F, one being the reverse of the other. We call the resulting pair of permutations a *geometric permutation* (*GP*). In $\mathbf{R}^2$ we have the following.

Proposition 2.3. *The transversals of F and F' induce the same GPs.*

Proof. This follows immediately from the proof of proposition 2.2 by observing that the points $x_i \in (K + v_i)$, determine the *GP*. But since $x_i \in (K' + v_i)$, F' admits the same *GP*. □

If there exists a hyperplane H of $\mathbf{R}^n$ such that any hyperplane parallel to H meets at most one member of F, we say that F is *totally separable*. We then have the following.

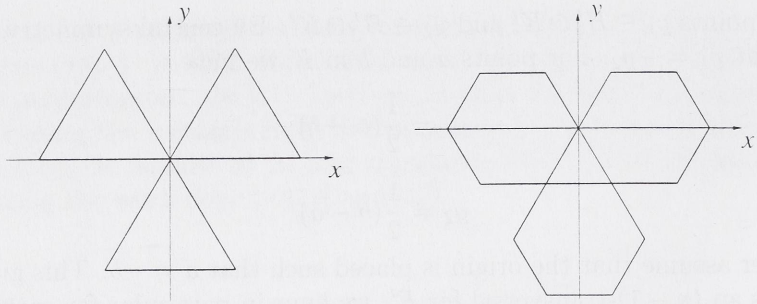
Proposition 2.4. *F is totally separable if and only if F' is totally separable.*

Proof. A family F being totally separable is equivalent to the situation in which there exists a hyperplane H such that no two members of F admit an $(n-1)$-transversal parallel to H. The result follows from the fact that F and F' admit $(n-1)$-transversals in the exact same directions. □

We have seen that transversal properties are preserved under symmetrization when the codimension of the transversal is 1. When the codimension is different from 1, transversal properties are not necessarily preserved. We give here an example of this.

Let K be an equilateral triangle in $\mathbf{R}^2$ given by the vertices $v_1 = (-1,0)$, $v_2 = (1,0)$, $v_3 = (0,\sqrt{3})$. Then K' is a regular hexagon given by the vertices $w_1 = (1,0)$, $w_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $w_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $w_i = -w_{i-3}$ $(i = 4,5,6)$. We now construct a family F of translates of K that has a 0-transversal, but such that F' does not have a 0-transversal. By Helly's theorem in the plane, we need only consider families consisting of three translates.

Let F be given as $\{K + c_i, (i = 1,2,3)\}$, where $c_1 = (-1,0)$, $c_2 = (1,0)$, and $c_3 = (0,-\sqrt{3})$. The origin is contained in each of the three translates, and the family therefore has a 0-transversal. When we replace K with K', however, we get $(K'+c_1)\cap(K'+c_2) = (0,0)$, $(K'+c_1)\cap(K'+c_3) = (-\frac{1}{2}, -\sqrt{3})$, and $(K'+c_2)\cap(K'+c_3) = (\frac{1}{2}, -\sqrt{3})$. Thus F' does not have a 0-transversal. The situation is illustrated in the figure below.



3. Incompatible geometric permutations

Geometric permutations have been a useful tool in the study of geometric transversal theory. An example of this is Tverberg's use of certain incompatible pairs of GPs for translates in the plane, in his proof of Grünbaum's conjecture [16]. In [17] Tverberg studies families of disjoint translates, that have transversals which induce three distinct GPs. He proves that the permutations must have the following form

$$(W_1T_1W_2) \quad (W_1T_2W_2) \quad (W_1T_3W_2)$$

where W_1 and W_2 are finite "words", and the triple $\{T_1, T_2, T_3\}$ has one of the following forms

$$1 : \{XABC \ , \ XBCA \ , \ XCAB\}$$
$$2 : \{XABC \ , \ XBCA \ , \ XBAC\}$$
$$3 : \{ABXC \ , \ AXCB \ , \ CAXB\}$$
$$4 : \{ABCX \ , \ BXAC \ , \ ACXB\}$$

He also mentions that some of these triples can be excluded. We prove here that only the triples 2 and 4 can exist.

Proposition 3.1. *Let A, B, C, and X be disjoint compact convex sets in the plane. The triple $\{(XABC), (XBCA), (XCAB)\}$ of geometric permutations is incompatible.*

Proof. Let l, m, and n be lines that induce the three GPs. Since parallel lines induce the same GP, l, m, and n are non-parallel. By moving the three sets a little closer to each other (or enlarging them a bit) we can ensure enough room to move the lines l, m, and n such that they do not all meet in the same point.

We now wish to distinguish the different ways l, m, and n can induce the three GPs (BAC), (ABC), and (ACB). We can assume that l is horizontal, m is ascending, and n is descending such that when we traverse l from left to right, we first meet m, and then meet n. Assume for example that m induces the GP (BAC). We then have to consider two cases, because moving downward along m, we could meet the sets in the order $..B..A..C..$ or $..C..A..B..$ The situation is similar for the other lines. When we travel downward along m, from left to right along l, and upward along n, we meet the sets A, B, C in different orders, such that we induce the three different GPs. We call this a *combination of orders.* Two combinations of orders are equivalent if one is obtainable from the other by permuting the names of the sets and lines, or/and by an appropriate congruence of the plane. It turns out that there are four different combinations of orders. This can be seen as follows. Which GPs the different lines l, m, n induce is irrelevant, since we allow permuting the names of the sets. The essential thing is that the "middle"-sets on each line are distinct. Thus we can represent each combination of orders in the following way:

Assume the lines are given as before, and when we traverse downward along m we meet the sets in the order $..x_1..x..x_2..$, when we traverse from left to right along l we meet the sets $..y_1..y..y_2..$, and when we traverse upward along n we meet $..z_1..z..z_2..$ This can be represented as the labeled graph on 6 vertices, by the edges x_2y_1, y_2z_1, and z_2x_1. Returning to our situation, we have that x, y and z are distinct. Further, for each of the three lines, the "outer"-sets on that line must be the same as the "middle"-sets on the two other lines. Thus, each combination of orders can be represented as a labeled graph on three vertices that is regular of degree 2. Let R be a given combination of orders, and let G be the corresponding labeled graph. Let π be a permutation of $\{A, B, C\}$. Then πR is an equivalent combination of orders, and the corresponding labeled graph will be πG. A congruence of the plane will not change the representing labeled graph. Thus every combination of orders corresponds to one of the three graphs on three vertices that are regular of degree 2. It turns out that one of these graphs represents two different combination of orders, while the two others are unique.

Below we have listed the different combinations of orders for the sets, when we traverse downward along m, from left to right along l, and upward along n:

$$\begin{array}{llll} 1: & ..C..A..B.. & ..A..B..C.. & ..B..C..A.. \\ 2: & ..C..A..B.. & ..A..B..C.. & ..A..C..B.. \\ 3: & ..C..A..B.. & ..C..B..A.. & ..A..C..B.. \\ 4: & ..B..A..C.. & ..C..B..A.. & ..A..C..B.. \end{array}$$

We shall now consider a special case with only two sets, prove that this cannot occur, and then we will observe that each of the cases listed above will include the special case. Let Y and Z be two disjoint compact, convex sets. Let l, m, and n be lines situated as explained above. Assume that when we traverse downward along m, from left to right along l, and upward along n, we meet the sets in the order:

$$(\star): \quad ..Y..Z.. \quad ..Y..Z.. \quad ..Y..Z..$$

We now divide the lines as follows: Let l_2 be the segment of l with endpoints $l \cap m$, and $l \cap n$. Let l_1 be the part of l that lies to the left of l_2, and let l_3 be the part of l that lies to the right of l_2. Similarly, let m_2 be the segment of m with endpoints $m \cap l$ and $m \cap n$, let m_1 be the part of m that lies above m_2, and let m_3 be the part of m that lies below m_2. Finally, let n_2 be the segment between $n \cap l$ and $n \cap m$, let n_1 be the part of n below n_2, and let n_3 be the part of n above n_2.

For each of the sets Y and Z, and each of the lines l, m, and n, we choose a point from the intersection of the set and the line. We denote the point from the intersection of the set X and the line q, as X_q. By enlarging the sets a bit (while still preserving the disjointness), we may assume that none of these points coincide with the points $l \cap m$, $l \cap n$, and $m \cap n$.

Assume Y_m and Z_m lie on $m_1 \cup m_2$, and Y_l and Z_l lie on $l_2 \cup l_3$. Then Y and Z cannot be disjoint since the segment between Y_l and Y_m will cross the segment between Z_l and Z_m. The situation is equivalent for the line pairs $\{l, n\}$ and $\{m, n\}$. Thus, by symmetry, we need only consider the case where Y_l lies on l_1, and Y_n lies on n_1. Then the segment between Y_l and Y_n must cross m_3, and Z_m must lie on m_3 below this crossing point. But Z_l must lie to the right of Y_l on l, thus we must have one of the following cases: Either the segment between Z_m and Z_l and the segment between Y_l and Y_n cross each other, or they do not cross each other. For the latter case to occur, the segment between Z_m and Z_l must cross n below the point Y_n, contradicting the fact that we meet the sets in the order $..Y..Z..$ when we traverse upward along n. Thus we have shown that the situation $(\star)$ cannot occur.

We now return to the four combinations of orders. Case 2, above, includes $(\star)$ when $A = Y$ and $B = Z$. By letting $C = Y$ and $B = Z$ in case 3, we also have $(\star)$. To exclude cases 1 and 4, we must include the set X. X is included so that the combination of orders coincide with the GPs. We then have the following cases (with the same notation):

$$1': \quad ..X..C..A..B.. \quad ..X..A..B..C.. \quad ..X..B..C..A..$$
$$2': \quad ..B..A..C..X.. \quad ..C..B..A..X.. \quad ..A..C..B..X..$$

By letting $X = Y$ and $A = Z$ in case $1'$, and letting $A = Y$ and $X = Z$ in case $4'$, both the cases include $(\star)$. This completes the proof. $\square$

We now exclude the third triple of geometric permutations.

Proposition 3.2. *Let A, B, C and Z be disjoint translates in the plane. The following triple of geometric permutations is incompatible: $\{(ABZC), (AZCB), (CAZB)\}$.*

Proof. By proposition 2.3, we may assume the translates to be centrally symmetric. Let l, m, and n be three lines that induce the different GPs. Further, the translates may be assumed to be centrally symmetric hexagons. To see this, let l_1 and l_2 be the upper and lower support lines of A that are parallel to l, respectively. By the central symmetry, there is a central chord of A that meets l_1 and l_2. This chord also meets l, and in each of the other translates the parallel central chord will also meet l. The GP that l induces on the chords is then the same as the one l induces on the translates. Doing the same for the lines m and n, we have for each translate, three central chords, where l, m, and n induce the same GP on each set of parallel chords, as on the given translates. Thus, we may cut down each translate to the centrally symmetric hexagon spanned by the three central chords. Therefore we may assume A, B, C, and Z to be translates of a centrally symmetric hexagon.

Now, label the edges of the hexagon, counterclockwise, from 1 to 6, and denote the edge j of the translate Y, as Y_j. Let X_i and Y_j be edges of X and Y, respectively. If the line that contains the edge X_i and the line that contains Y_j, are distinct and both separate X and Y, we say that X_i and Y_j are *opposing edges*. We denote this as $\{X_i, Y_j\}$. Note that if we have $\{X_i, Y_j\}$, then $|i - j| = 3$.

Before continuing, we shall make an observation concerning convex sets in the plane, in general. Let l_1 and l_2 be parallel lines in the plane, and let X, Y, and W be three disjoint convex sets in the plane. Assume that l_1 and l_2 each separate (properly) a distinct pair of the three sets. Then X, Y, W can have at most two distinct GPs. To see this, assume (for the sake of the argument) that l_1 and l_2 are horizontal, and that l_1 lies above l_2. Since l_1 separates a pair of the sets, one of the sets, say X, must lie in the upper open half-plane defined by l_1. Similarly, one of the sets, say Y, must lie in the lower open half-plane defined by l_2. Now l_1 and l_2 both separate the pair X, Y, so one of the two lines must separate another pair of the three sets. For this to occur, W must lie in the lower open half-plane defined by l_1 or in the upper open half-plane defined by l_2. In both cases we end up with a line separating one of the sets from the two other sets, and particularly separating one of the sets from the convex hull of the two other sets, excluding the existence of one of the three possible GPs. Therefore X, Y, and W can have at most two GPs.

If we consider the three hexagons A, B, and C, clearly each pair of hexagons must have at least one pair of opposing edges, by disjointness. Assume some pair of hexagons, say A and B, has more than one pair of opposing edges. Then for some edge A_i, there is a line l_i, parallel to A_i, that separates A and B. And for some other edge A_j, there is a line l_j, parallel to A_j, that also separates A and B. Clearly l_i and l_j cannot be parallel. Further, A and C must have a pair of opposing edges, so let l_m be a line, parallel to the edge A_m, that separates A and C. Similarly, let l_n be a line, parallel to the edge B_n, that separates B and C. Since the translates are centrally symmetric hexagons, the lines l_i, l_j, l_m and l_n are determined by only three distinct directions. This implies that there are two parallel lines that each separate a distinct pair of the three translates. But then the sets A, B, and C cannot have three distinct GPs. The conclusion is therefore that each pair of the three translates has exactly one pair of opposing edges. It also follows that the centers of the translates are not collinear. We may therefore assume that the centers have the cyclic order $..A..B..C..$, moving counterclockwise. All in all we have the following: $\{A_i, B_{i-3}\}, \{B_{i-4}, C_{i-1}\}, \{C_{i-2}, A_{i+1}\}$, for some $i \in \mathbf{Z}_6$. As remarked earlier, these are the only edges among A, B, and C that are opposing. For the rest of the argument we let $i = 2$.

Consider how Z is positioned relative to B. We must have $\{Z_1, B_4\}$, and no other opposing edges between Z and B. To see this, we consider all the other possibilities. If $\{Z_2, B_5\}$, there must be a line, parallel to B_5, that separates B from A and from Z, making it impossible to have the GP (ABZ). If $\{Z_3, B_6\}$, there must be a line, parallel to B_6 that separates Z from B and from C, and it follows that we cannot have the GP (BZC). If $\{Z_4, B_1\}$, a line, parallel to B_1, will separate Z from B and from C, and again we cannot have the GP (BZC). If $\{Z_5, B_2\}$, a line, parallel to B_2, will separate Z from A and from B, thus making it impossible to have the GP (AZB). Finally, consider $\{Z_6, B_3\}$. Since we cannot have $\{C_6, B_3\}$, there is a line, parallel to B_3, that separates Z from A and from B, and again we cannot have the GP (AZB). Thus, the only opposing edges between B and Z, is the pair $\{Z_1, B_4\}$. Similarly, we have that the only opposing edges between A and Z, is the pair $\{A_2, Z_5\}$, and the only opposing edges between C and Z, is the pair $\{C_6, Z_3\}$.

Let l_1 be the line through the center of A that is parallel to A_1. Assume l_1 is horizontal and A_1 lies in the lower half-plane defined by l_1. Then l_1 must intersect the interior of the edge A_2. For if not, l_1 must intersect the edge A_3, and a contradiction is obtained as follows. If l_1 intersects A_3, the center of Z must lie in the lower closed half-plane defined by l_1. If this was not the case, it would be impossible to have $\{A_2, Z_5\}$ without also having $\{A_3, Z_6\}$. But when the center of Z lies in this lower half-plane, the line that contains the edge A_1 must intersect Z. And since we also have $\{Z_1, B_4\}$, it follows that the line parallel to Z_1 that separates Z and B, also separates A and B, implying $\{A_1, B_4\}$. Thus, the line l_1 must intersect the interior of the edge A_2.

Observing the cyclic symmetry of the sets A, B, and C in the triple of GPs, we find, in general, that the line l_i through the center of one of the hexagons, which is parallel to the edge i of the hexagon, must intersect the interior of the edge $i + 1$ (numbers are read mod 3). This, however, is impossible for a centrally symmetric hexagon. The contradiction completes the proof. □

We summarize with the following:

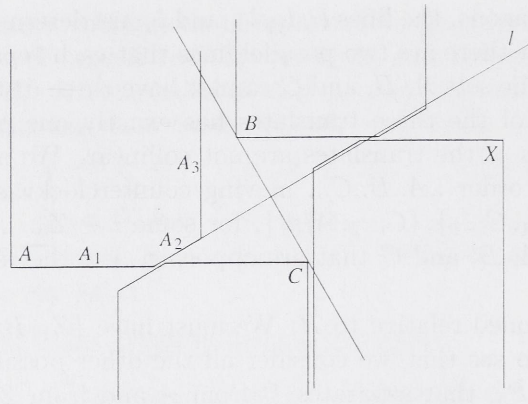
Theorem 3.3. *There are exactly two types of compatible triples of GPs for convex translates in the plane. They are*

$$\{(W_1ABCW_2), (W_1BCAW_2), (W_1BACW_2)\}$$

and

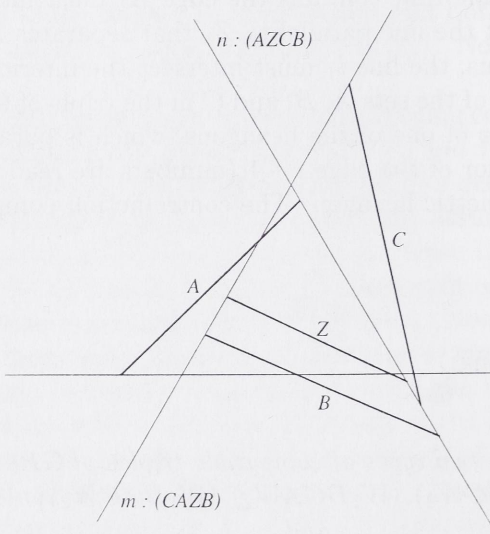
$$\{(W_1ABCXW_2), (W_1CAXBW_2), (W_1ACXBW_2)\}$$

An example of a family of disjoint translates that admits the GPs (W_1ABCW_2), (W_1BCAW_2), (W_1BACW_2) is given by Katchalski *et.al.* [13]. The figure below illustrates a family admitting the GPs of triple 4:



Let l be the line that contains the edge A_2. Let X lie such that the edge X_5 lies slightly above l, and B such that the edge B_2 lies slightly below l. C lies such that the only opposing edges between A and C are $\{A_2, C_5\}$, and such that the only opposing edges between X and C are $\{C_3, X_6\}$. Now rotate the line l, counterclockwise, about the vertex determined by A_2 and A_3, such that l still meets B and X. We rotate the line until it is about to leave X, and call the limit-line l_1. Similarly, we can rotate l, clockwise, about the vertex determined by A_1 and A_2, such that l still meets B and X. As before, we stop at the limit-line, l_2. Now, move C upward, such that the distance between A_2 and C_5 becomes small enough that C meets l_1 and l_2. Then l_1 will induce $(CAXB)$ and l_2 will induce $(ACXB)$. Finally, let l_3 be the descending line that goes through the vertex of B determined by the edges B_6 and B_1, and through the vertex of C determined by the edges C_3 and C_4. By letting the translates be tall enough, l_3 will induce $(ABCX)$.

The figure below shows that proposition 3.2 cannot be extended to apply for general families of convex sets.



We end this section with a closer look at a theorem by Smorodinsky *et.al.* [15] (also proved independently by Asinowski and Katchalski). Their theorem states that there exists a constant C, such that any family of more than C pairwise disjoint, congruent discs in the plane admits at most two GPs. We shall improve their result, and prove the following:

Theorem 3.4. *Any family of more than three pairwise disjoint, congruent discs in the plane admits at most two geometric permutations.*

Proof. By theorem 3.3 it suffices to show that a family consisting of four pairwise disjoint, congruent discs admits at most two GPs.

Let A, B, C, and D be four disjoint discs of diameter 1, and assume the discs admit three distinct GPs. By theorem 3.3 the triple of GPs must be one of the following:

$$\{(ACBD), (BADC), (ABDC)\}$$

or

$$\{(DABC), (DBAC), (DBCA)\}$$

We start with the first triple. Let a, b, c, and d be the centers of A, B, C, and D, respectively. If the centers were collinear, the discs would admit only one GP. Thus, the convex hull of the centers must be a triangle or a quadrilateral. However, the case in which the convex hull is a triangle cannot occur. To see this, let l be a transversal for the four discs. The orthogonal projections of the centers on l are contained in a segment $[x, y]$ of l, where x and y are the orthogonal projections of two of the vertices of the triangle. The order of the orthogonal projections of centers on the line then corresponds to the GP induced by l. This means that the center that is not a vertex of the triangle cannot be an "outer" element of the GP. But in the triple of GPs we are considering, each of the four discs is an "outer" element in at least one of the GPs. Thus, $conv(\{a, b, c, d\})$ must be a quadrilateral.

Now the centers must be ordered (cyclic) a, b, d, c. To see this, let l be a horizontal line parallel to some line inducing the GP $(BADC)$. We can then assume the centers lie above l, such that when we traverse l from left to right, the orthogonal projections of the centers on l have the order *..b..a..d..c..* Then d must lie to the right of the orthogonal line on l which goes through a, and to the left of the orthogonal line on l which goes through c. Thus the centers must be ordered either a, b, d, c or a, b, c, d. Let m be a line parallel to some line inducing $(ACBD)$. By the same argument the centers must be ordered a, d, b, c or a, b, d, c. The conclusion is therefore that the centers must be ordered a, b, d, c, and we assume that the centers have this order going counterclockwise.

We may assume that a and b lie on the y-axis, such that if we let $(0, y_a)$ and $(0, y_b)$ be the coordinates of a and b, respectively, then we have $y_a = -y_b$ and y_a is positive. By the disjointness of the discs we have that $y_a > \frac{1}{2}$. Let (x_c, y_c) and (x_d, y_d) be the coordinates of c and d, respectively. By the cyclic ordering of the centers, x_c and x_d must be positive. If C lies above the x-axis, the x-axis will separate B from A and C, and we cannot have the GP (ABC). The same applies to D. By symmetry C and D cannot lie below the x-axis. Thus, $y_c, y_d \in (-\frac{1}{2}, \frac{1}{2})$.

The way the coordinate axes are defined, the separating tangents of A and B cross in the origin, and the slopes have the same absolute value. Let l be the ascending separating tangent of A and B. C cannot lie above l since this would imply that C lies above the x-axis. The same applies to D. If C lies below l, we can rotate l, clockwise, about the origin, such that we get a line that separates A from B and C. It is then impossible to have the GP (BAC). Thus, C cannot lie below l. The same applies for D. We conclude that l is a common transversal for A, B, C, and D. Let m be the descending separating tangent of A and B. By symmetry m is also a common transversal for A, B, C, and D.

Now l induces $(BADC)$ and m induces $(ABDC)$. To see this, first note that since c lies in the right half plane, l cannot induce (CBA) and m cannot induce (CAB).

Assume that l induces (BAC). Now note the following: if the quadrant determined by m and l which does not contain A or B, spans an angle $\geq \frac{\pi}{2}$, then C cannot meet m at the same time as l induces (BAC). Thus, the angle must be less than $\frac{\pi}{2}$. But then it is impossible for m to induce (ACB) at the same time as l induces (BAC), without losing the disjointness of the discs. The conclusion is that it is impossible for m to induce (ACB) at the same time as l induces (BAC). Since l and m both are common transversals for A, B, C, and by the symmetry of the situation, we have the following: l induces (BAC) if and only if m induces (ABC).

Now assume l induces (BCA). It then follows from the previous observations that m also must induce (ACB). The following is then possible: we can rotate l, counterclockwise, until the line is parallel to the y-axis, while we the entire time induce (ACB). Similarly, we can rotate m, clockwise, until it is parallel to the y-axis, while we the entire time induce (ACB). We will then have traversed all directions for which A, B, C have a transversal, and since parallel lines induce the same GP, A, B, and C only have one GP. Thus, l induces (BAC) and m induces (ABC). By substituting D for C in the above argument, we have that l induces (BAD) and m induces (ABD).

From the above discussion we conclude that l must induce $(BACD)$ or $(BADC)$. But since there is a transversal inducing the GP $(ABDC)$, l must induce $(BADC)$, since $(ABDC)$ and $(BACD)$ is an incompatible pair of GPs. Similarly m must induce $(ABDC)$. From this it follows that $x_d < x_c$. This can be seen by considering the orthogonal projections of c and d on the lines l and m.

Tverberg [17] proved (also proved independently by Wenger) that if a family of disjoint translates can be ordered such that each three members of the family have a transversal that induces a GP in correspondence with the given ordering, then the family has a common transversal that induces a GP in correspondence the given ordering. We shall now modify the given family such that we obtain a critical GP, while preserving the three GPs and the disjointness of the discs. In view of Tverberg's result a GP becomes critical when some triple gets a critical GP. Since the separation lines of the discs A and B first meet D and then meet C (after leaving A and B), we can move C away from A and B, in the same direction as the line m that induces the GP $(ABDC)$. We move C in this direction, to the point where we lose one of the GPs. By the compactness of the discs and the fact that lines which induce different GPs are non-parallel, it is clear that such a point will exist. As mentioned, when a GP is critical, some triple must have a critical GP, thus a line must separate one disc, say X, from two others, say Y and Z, such that $X \cap conv(Y \cup Z) \neq \emptyset$.

Since C is the disc we are moving, C must be part of the critical GP. Also, since we move C in the same direction as the line m that induces $(ABDC)$, none of the sub-permutations of $(ABDC)$ can become critical. Now start by assuming that the sets A, B, C are the ones that get a critical GP. The critical GP cannot be (ABC) since this is contained in $(ABDC)$. Therefore it must be one of the GPs (ACB) or (BAC) that becomes critical (or both). Now assume that the sets A, C, D get a critical GP. This cannot be (ADC) since it is contained in $(ABDC)$. The only other possibility is that (ACD) becomes critical. Finally, assume that the sets B, C, D get a critical GP. Since (BDC) is contained in $(ABDC)$, it must be (CBD) that becomes critical. Clearly after obtaining a critical GP, the cyclic order of the centers must be the same as before, as we have seen that for any other cyclic ordering we cannot have the three given GPs.

We now have four possible candidates for the critical GP. We shall show that none of them can occur. Assume that (ACB) is the critical GP. Then the line $x = \frac{1}{2}$ separates C from A and B, such that the orthogonal projection of the centers of the discs on the line $x = \frac{1}{2}$ have the order $..a..c..b..d...$ But for this to happen we must have $y_d < -\frac{1}{2}$, a contradiction.

Now suppose (BAC) is the critical GP. There then exists a line l_1 that separates A from B and C such that the orthogonal projection of the centers on l_1 have the order $..b..a..d..c..$ The distance from b, a, and c to l_1 is $\frac{1}{2}$, and the distance from d to l_1 is less than or equal to $\frac{1}{2}$. In addition, d must be contained in the open strip determined by the line through a which is orthogonal on l_1, and the line through c which is orthogonal on l_1. It then follows that d must either be contained in $conv(\{a, b, c\})$, or the cyclic order of the centers is a, b, c, d. Both cases are a contradiction. Thus, (BAC) cannot be the critical GP.

Consider the case where (ACD) is the critical GP. Then there exists a line l_2 that separates C from A and D such that the orthogonal projection of the centers on l_2 have the order $..a..c..b..d...$ The distance from a, c, and d to l_2 is $\frac{1}{2}$, and the distance from b to l_2 is less than or equal to $\frac{1}{2}$. In the same way as above, we have that either b is contained in $conv(\{a, c, d\})$, or the cyclic order of the centers is a, d, b, c. Both cases are contradictions. (A different way to exclude (ACD) is observing that under the given conditions the slope of l_2 must be positive or ∞, implying $x_d \leq 0$.)

Finally, assume (CBD) is the critical GP. There then exists a line l_3 that separates B from C and D such that the distance from the centers is $\frac{1}{2}$, and the orthogonal projection of the centers on l_3, meet in the order $..a..c..b..d...$ The distance from a to l_3 cannot be larger than $\frac{1}{2}$. But the slope of l_3 must then be negative or ∞. This implies that $x_c \leq x_d$, contradicting the earlier observation $x_d < x_c$. Therefore (CBD) cannot be the critical GP. This concludes the first part of the proof.

We now assume that the discs admit the GPs $(DABC)$, $(DBAC)$, and $(DBCA)$. We assume the centers a, b, and c are ordered as before. We now move D away from A, B, and C in the same direction as one of the transversals inducing $(DABC)$. As before we continue moving D until we obtain some critical GP. Since A, B, and C, do not move during this process, D must be contained in the critical GP.

Assume first that the sets B, C, D are the ones that get the critical GP. Each of the three GPs contain (DBC), and therefore this must be the critical GP. But the unique transversal inducing (DBC) must then induce $(DABC)$ and $(DBAC)$, a contradiction.

We now assume that the sets A, B, D are the ones that get a critical GP. Since we are moving D in the same direction as one of the transversals inducing $(DABC)$, the critical GP cannot be (DAB). The only other alternative is that (DBA) is the critical GP. But the unique transversal inducing (DBA) must then induce $(DBCA)$ and $(DBAC)$, a contradiction.

It must therefore be the sets A, C, D that get a critical GP. And since (DAC) is contained in $(DABC)$, this cannot be the critical GP. Therefore (DCA) must be the critical GP, and thus there exists a unique line m that induces the GP (DCA), and that separates C from A and D.

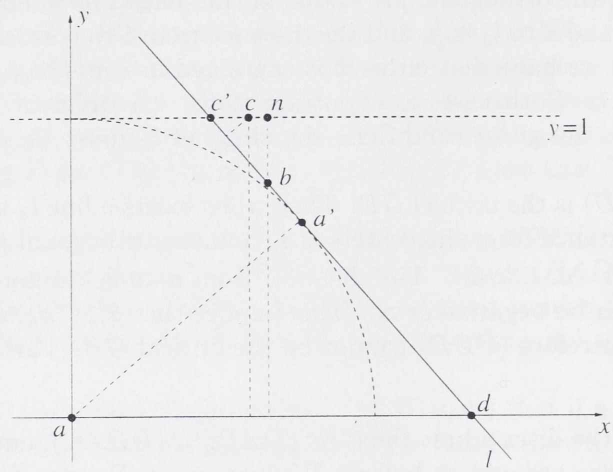
We now move D away from A, B, and C along the unique transversal m. By similar arguments as those above, it can be seen that (DAB) is the critical GP. Therefore there exists a unique line l that induces (DAB), and that separates A from B and D.

Let l_{bd} be the line through b and d. Since l separates A from B and D, the distance from a to l_{bd} is 1, the distance from c to l_{bd} is less than 1, and the orthogonal projections of the centers on l_{bd} have the order $..d..a..b..c...$ Similarly, let m_{ad} be the line through a and d. Then the distance from c to m_{ad} is 1,the distance from b to m_{ad} is less than 1, and the orthogonal projections of the centers on m_{ad} have the order $..a..c..b..d...$ As earlier we must have one of the following cases: $conv(\{a, b, c, d\})$ is either a triangle or a quadrilateral.

We start with considering the case where $conv(\{a, b, c, d\})$ is a triangle. Note that A, C, and D each are "outer" elements of at least one of the three GPs and B is not an "outer" element of any of the GPs. This means that the a, c, and d must be the vertices of $conv(\{a, b, c, d\})$.

Without loss of generality we may assume that a and d lie on the x-axis, such that a is at the origin and $x_d > 1$, where x_d denotes the x-coordinate of the point d. Let l be the line that goes through the point d and is tangent to the unit circle in the first quadrant. Let a' be the point of tangency, and let c' be the point where l intersects the line $y = 1$.

Since the distance from a to l_{bd} is 1, b must lie on l and below the line $y = 1$. If b lay above the line $y = 1$ the distance from b to m_{ad} would be greater than 1. For the order of the orthogonal projections of the centers on l_{bd} to be correct, b must lie between a' and c'. Let n be the orthogonal projection of b on the line $y = 1$. The situation is illustrated in the figure below.



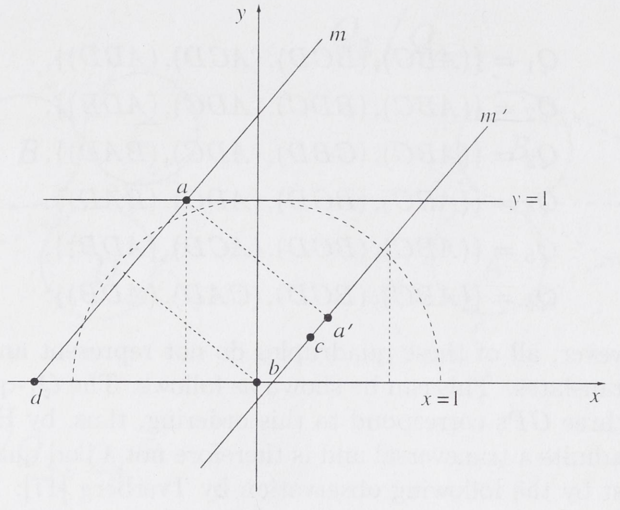
Since the distance from c to m_{ad} is 1, c must lie on the line $y = 1$. For the orthogonal projections of the centers on m_{ad} to have the correct order, and for $conv(\{a, b, c, d\})$ to be a triangle, c must lie between c' and n. By noting that $|c'd| = x_d$ and considering $\Delta aa'd$ we have the following:

$$|bc| < |a'c'| \leq x_d - \sqrt{x_d^2 - 1}$$

And since $x_d > 1$, we have $|bc| < 1$, contradicting the disjointness of the discs B and C.

Now assume $conv(\{a, b, c, d\})$ is a quadrilateral. There are three ways the centers can be ordered (cyclic). By a similar argument as the one used in the first part of the proof one sees that the cyclic order of the centers must be a, d, b, c. We can therefore assume that b and d lie on the x-axis such that b is at the origin and $x_d < -1$. For the order of the orthogonal projection of the centers on l_{bd} to be correct, c must lie in the first quadrant. By the disjointness of the discs, c must lie outside the unit circle. Further, c must lie below the line $y = 1$. The point a must lie on the line $y = 1$ since the distance from a to l_{bd} is 1. Further, if x_a denotes the x-coordinate of a, we must have $x_d < -1 < x_a < 0$.

Let m be the line through d and a. The distance from c to m must equal 1. Let m' be the line which is paralell to m and goes through the point c. Let a' be the orthogonal projection of a on the line m'. The orthogonal projections of the centers on the line m' have the order $..d..b..c..a..$ This implies that c must lie below a' on the line m'. The situation is illustrated in the figure below.



The figure above indicates that the point a' has distance less than 1 from b. In order to see this, observe first that for a fixed slope of m, the distance increases when m is moved to the right. Thus we only have to consider the limit cases when $a = (0,1)$ or $d = (-1,0)$. Easy calculations show that distance 1 is only assumed for $a = (-1,1)$, $d = (-1,0)$. Thus a' really belongs to the open unit disc around b, and so does c. But we found before that c is outside the unit disc. $\square$

4. Bad quadruples

If four disjoint translates in the plane have $T(3)$ but not T, the family is called a *bad quadruple*. Such a quadruple can be represented as $\{\pi_1, \pi_2, \pi_3, \pi_4\}$, where each π_i is a GP of one of the triples of the quadruple. In what follows we assume that each triple has only one GP. If two quadruples can be represented by the same four GPs, we say that they are equivalent. We start with showing that there exist exactly three different bad quadruples. By rearranging the names of the sets we can always assume we have the GP (ABC). Given a quadruple, we see that the four "middle" elements of the permutations are either all distinct, three are distinct, or two are distinct. Clearly an element cannot occur in all four of the GPs. Now assume all the "middle" elements are distinct. Recall that we have assumed the existence of (ABC). The quadruple then looks like

$$\{(ABC), (.A.), (.C.), (.D.)\}$$

This leads to two different quadruples. This can be seen by observing that the pair $\{A, C\}$ must be contained in one of the other GPs, and they must then either be adjacent, or not. This gives us the following quadruples

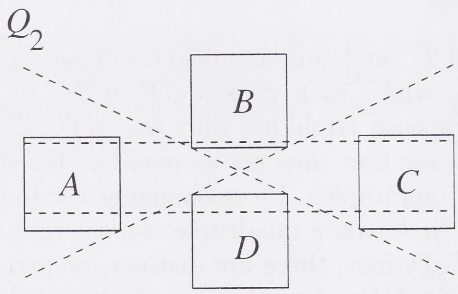
$$\{(ABC), (BAD), (BCD), (ADC)\}$$

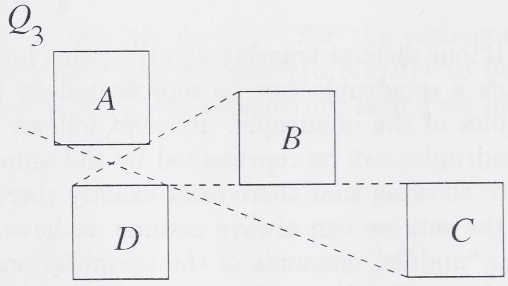
$$\{(ABC), (CAD), (BCD), (ADB)\}$$

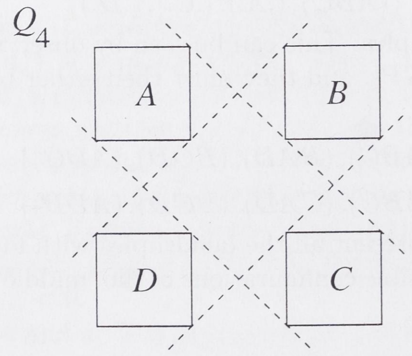
It is easy to see that these will represent all the quadruples with four distinct "middle" elements. A similar analysis of the other possible configurations of the "middle" elements yields the following list of quadruples:

$$Q_1 = \{(ABC), (BCD), (ACD), (ABD)\},$$
$$Q_2 = \{(ABC), (BDC), (ADC), (ADB)\},$$
$$Q_3 = \{(ABC), (CBD), (ADC), (BAD)\},$$
$$Q_4 = \{(ABC), (BCD), (ADC), (BAD)\},$$
$$Q_5 = \{(ABC), (BCD), (ACD), (ADB)\},$$
$$Q_6 = \{(ABC), (BCD), (CAD), (ADB)\}.$$

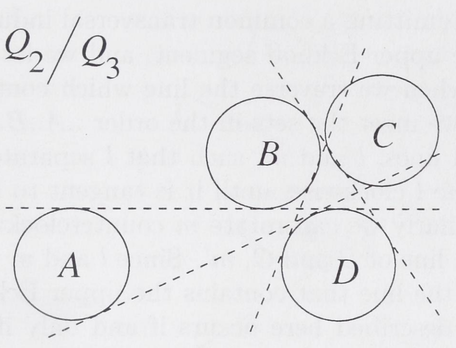
As it turns out, however, all of these quadruples do not represent an actual "geometric" bad quadruple of disjoint translates. This can be shown as follows. The Q_1-quadruple can be ordered such that each of the three GPs correspond to this ordering, thus, by Hadwiger's theorem (Th. 89, [9]) the quadruple admits a transversal and is therefore not a bad quadruple. The quadruples Q_5 and Q_6 cannot exist by the following observation by Tverberg [17]: *If a family F of disjoint translates has $T(3)$ then there exists two sets X and Y in F such that each member of $F \setminus \{X, Y\}$ meets $conv(X \cup Y) \setminus (X \cup Y)$.* The sets X and Y will be called *Eckhoff sets* (they originate from Eckhoff [4],[5]), and the segments that form the boundary of $conv(X \cup Y) \setminus (X \cup Y)$ will be called *Ekchoff segments.* For our particular situation this means that there exist two sets X and Y such that the we have the GPs (XAY) and (XBY) for the two other sets A and B of the quadruple. Thus a quadruple admitting only the GPs from either Q_5 or Q_6 cannot exist, since there does not exist Eckhoff sets. The figure below illustrates examples of the three bad quadruples.

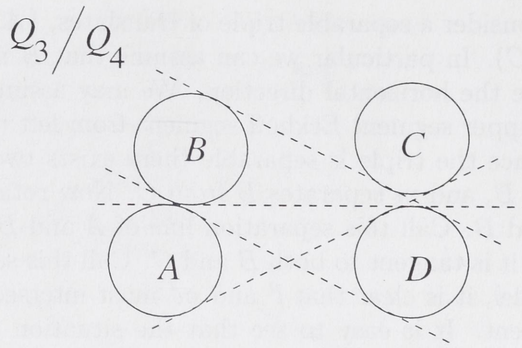


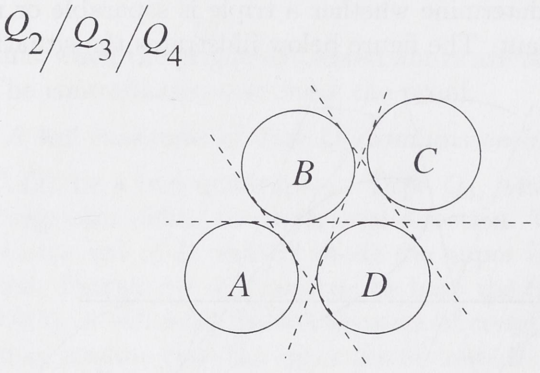




In the case when a triple has more than one GP, we say that the quadruple is for example of type Q_3/Q_4. The figure below illustrates the possible combinations. Note that the combined bad quadruple Q_2/Q_4 cannot exist without also being of type Q_3.







In connection with proposition 2.4 we defined total separability. In the plane a family of subsets is said to be *totally separable* if there exists a direction such that any line in this direction meets at most one of the sets of the family. Three sets that are totally separable will be called a *separable triple* (if it is not separable we say it is inseparable). By separation we will mean weak separation and specify the cases when we mean proper separation. A *separation line* of two closed sets is a line that is tangent to the two sets and which also separates them. We now establish some basic properties of separable triples.

Lemma 4.1. *A separable triple of compact convex sets in the plane has at most one GP.*

Proof. Let A, B, and C be a separable triple in the plane. By definition there exists a direction D (which we define to be vertical for the sake of the argument) such that any line in this direction meets at most one of the sets. It follows that the sets lie in separate parallel strips and we may assume that B is contained in the middle strip, A in the left strip, and C in the right strip. Clearly $C \cap conv(A \cup B) = \emptyset$ and $A \cap conv(C \cup B) = \emptyset$, thus we cannot have the GPs (ACB) and (BAC). □

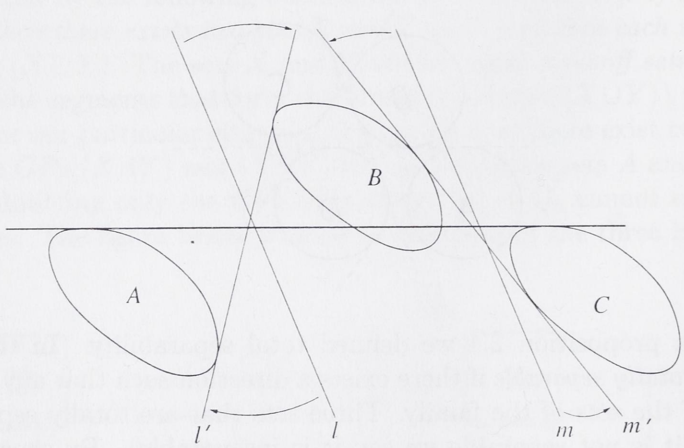
Let K be a centrally symmetric convex set with diameter d. For a family F of translates of K, let $d_c(X,Y)$, $(X,Y \in F)$ denote the distance between the centers of X and Y. Let A, B, and C be translates of K. We then have the following:

Lemma 4.2. *If A, B, and C have a common transversal and $d_c(X,Y) > \sqrt{2}d$ for each pair $X,Y \in \{A,B,C\}, X \neq Y$, then $\{A,B,C\}$ is a separable triple.*

Proof. It is easy to see that the lemma is valid when K is a disc of diameter d. For general K the result follows by inscribing K in a disc of diameter d. □

Consider a separable triple of translates, $\{A, B, C\}$, admitting a common transversal inducing (ABC). In particular we can assume that B meets the upper Eckhoff segment, and we let this define the horizontal direction. We may assume that when we traverse the line which contains the upper segment Eckhoff segment from left to right, we meet the sets in the order $..A..B..C..$

Since the triple is separable there exists two parallel lines, l and m, such that l separates A from B, and m separates B from C. Now rotate the line l clockwise, until it is tangent to both A and B. Call this separation line of A and B, l'. Similarly we can rotate m counterclockwise, until it is tangent to both B and C. Call this separation line of B and C, m'. Since l and m were parallel, it is clear that l' and m' must intersect above the line that contains the upper Eckhoff segment. It is easy to see that the situation we have described here occurs if and only if the triple $\{A, B, C\}$ is separable (if the triple were inseparable, the separation lines would either be parallel or cross each other below the upper segment of $conv(A \cup C)$). This observation will be useful in cases where we wish to determine whether a triple is separable or not. The described property is clearly an affine invariant. The figure below illustrates the situation.

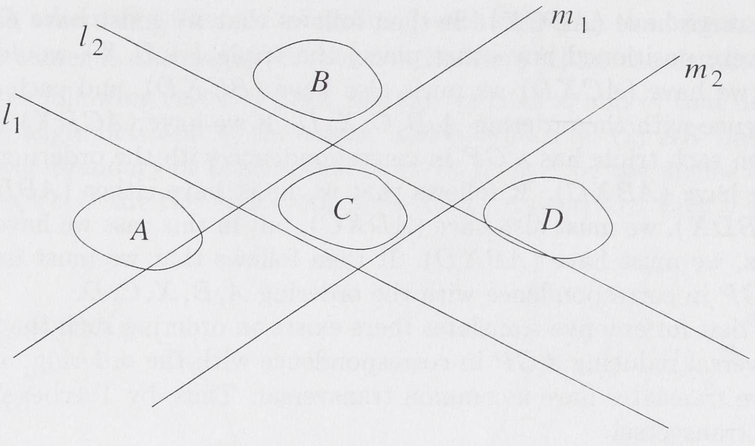


We shall now prove that every bad quadruple of translates must contain an inseparable triple. In view of lemma 4.1 we need only prove that each of the bad quadruples Q_2, Q_3, Q_4 must have an inseparable triple.

Proposition 4.3. *A bad quadruple of Type Q_2 contains an inseparable triple.*

Proof. Assume the opposite and let $\{A, B, C, D\}$ be a bad quadruple of Type Q_2, where each triple is separable. Let A and D be the Eckhoff sets, and define the direction of the Eckhoff segments to be the horizontal direction. Without loss of generality we may assume that A lies to the left of D, B meets the upper Eckhoff segment, and C meets the lower Eckhoff segment, such that we have the GPs (ABD), (ACD), (ACB), and (BCD).

Let D_1 and D_2 be the directions of total separability for the triples $\{A, C, B\}$ and $\{B, C, D\}$, respectively. Let l_1 and l_2 be the lines in the direction D_1 such that l_1 separates $\{A, C\}$ and l_2 separates $\{C, B\}$. Let m_1 and m_2 be the lines in the direction D_2 such that m_1 separates $\{B, C\}$ and m_2 separates $\{C, D\}$. The lines l_1, l_2, m_1, and m_2 exist by the total separability, and C must be contained in the interior of the parallelogram defined by these lines. B must lie in the upper open quadrant defined by the lines l_2 and m_1. Now D must lie below m_2 by the total separability and meet (or lie below) l_2 by the condition of $T(3)$. Similarly A must lie below l_1 and meet (or lie below) m_1. The situation is illustrated in the figure below.



It is easy to see that when the properties stated above are satisfied, B will lie above the upper Eckhoff segment. The contradiction completes the proof. □

Proposition 4.4. *A bad quadruple of Type Q_3 contains an inseparable triple.*

Proof. Let $\{A, B, C, D\}$ be a bad quadruple of Type Q_3, where A and D are the Eckhoff sets and let the Eckhoff segments define the horizontal direction. Without loss of generality we may assume that A lies to the left of D, that B meets the upper Eckhoff segment, and C meets the lower Eckhoff segment. Further we may assume we have the GPs (BAC) and (BCD).

Since we have (BAC), A will meet the left segment of $conv(B \cup C) \setminus (B \cup C)$, and by an affine transformation we may assume that the segments of $conv(B \cup C) \setminus (B \cup C)$ are orthogonal on the the segments of $conv(A \cup D) \setminus (A \cup D)$. Let l be a line that properly separates A and B, and let m be a line that properly separates A and C. Rotate l counterclockwise until it becomes a separation line, l', for A and B, which must have a slope that is negative or 0. Similarly, rotate m clockwise until it becomes a separation line, m', for A and C, which must have a slope that is positive or 0. It follows that l' and m' are either parallel or they cross each other to the right of the line that contains the left segment of $conv(B \cup C) \setminus (B \cup C)$, and thus $\{A, B, C\}$ is inseparable. □

Proposition 4.5. *A bad quadruple of Type Q_4 contains an inseparable triple.*

Proof. Since we did not use the GP (BCD) in the proof of proposition 4.4, the same reasoning applies for the Q_4-quadruple. Note also that by the symmetry of the Q_4-quadruple, every triple is inseparable. □

Let F be a family of sets in the plane. If F is totally separable we say that F has the property S. If every n-tuple of F is totally separable, we say that F has the property $S(n)$. We conclude this section with the following:

Theorem 4.6. *Let F be a family of disjoint translates, and assume that F has $S(3)$. Then $T(3) \Rightarrow T$.*

Proof. Let F be a family of disjoint translates, and assume each triple is separable. By propositions 4.3 - 4.5 we know that F has $T(4)$, and by lemma 4.1 each triple has exactly one GP. We shall show that for any five translates there exists an ordering such that each three translates have a transversal inducing a GP in correspondence with the ordering.

Let $\{A, B, C, D, X\} \subset F$. Each quadruple can only have one GP, or else some triple would have two GPs. We can without loss of generality assume that we have the GP $(ABCD)$. Further, by symmetry, we may assume that we have either $(ABCX)$ or $(ABXC)$.

Assume first that we have $(ABCX)$. It then follows that we must have either $(ACXD)$ or $(ACDX)$. If X were positioned any other place, the triple $\{A, C, X\}$ would have more than one GP. Now, if we have $(ACXD)$ we must also have $(BCXD)$, and each triple will have a GP in correspondence with the ordering A, B, C, X, D. If we have $(ACDX)$ we must also have $(BCDX)$, and then each triple has a GP in correspondence with the ordering A, B, C, D, X.

Now assume we have $(ABXC)$. It follows that we must have either $(ABDX)$ or $(ABXD)$. But if we have $(ABDX)$, we must also have $(ADXC)$, but in this case we have the GPs (ACD) and (ADC). Thus, we must have $(ABXD)$. It then follows that we must have $(AXCD)$ and each triple has a GP in correspondence with the ordering A, B, X, C, D.

We have shown that for any five translates there exists an ordering such that each three translates have a transversal inducing a GP in correspondence with the ordering, and by Hadwiger's theorem [9] the five translates have a common transversal. Thus, by Tverberg's theorem [16], F admits a common transversal. □

5. The Katchalski-Lewis conjecture

We say that a family F of subsets in the plane has the property $T-k$, k a nonnegative integer, if there is a straight line that intersects all but at most k members of F. In [12] Katchalski and Lewis prove the following theorem:

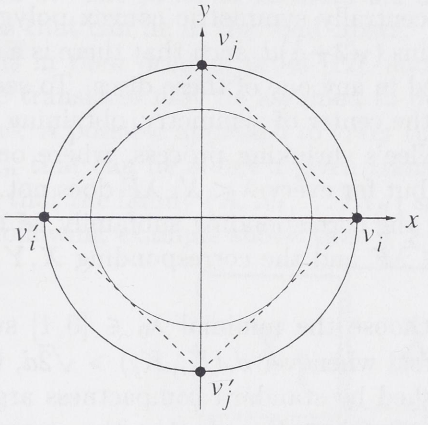
Theorem 5.1. *There exists a positive integer k such that for any family F of pairwise disjoint translates of a compact convex set K, $T(3)$ implies $T-k$.*

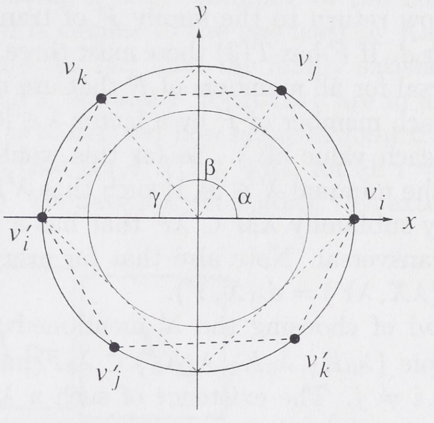
They obtain an upper bound of $k \leq 603$, but remark that there is room for improvement. They conjecture, however, that the value $k = 2$ is universal for families of disjoint translates. It is clear that the disjointness is necessary in theorem 5.1, and it is possible to construct a family of segments that shows that there is no universal value of k when one allows rotations as well as translations. We shall first improve on Katchalski and Lewis' upper bound on k and prove the following:

Theorem 5.2. *There exists a positive integer $k \leq 57$ such that for any family F of pairwise disjoint translates of a compact convex set K, $T(3)$ implies $T-k$.*

Proof. We shall mainly follow the proof of Katchalski and Lewis [12]. Clearly the transversal properties are affine invariants, and by the propositions in section 2, we may assume the sets to be translates of a centrally symmetric compact convex set K. Further we may assume that K is a polygon. This was shown by Tverberg [16], but can also be shown as follows: It is known that if F is an infinite family, then $T(3)$ implies T (e.g. see prop. 91 of [9]), thus we may assume that $|F| = N$. Let $\{K + v_{i_1}, K + v_{i_2}, K + v_{i_3}\}$ be some triple of F. Since F has $T(3)$, the triple $\{K + v_{i_1}, K + v_{i_2}, K + v_{i_3}\}$ has a transversal l, and there exists a central chord of $S \subset K$ such that l is a transversal for $\{S + v_{i_1}, S + v_{i_2}, S + v_{i_3}\}$. Doing this for every triple of F results in $\binom{N}{3}$ central chords $S_1, \ldots, S_{\binom{N}{3}}$. Let $K'' = conv(S_1 \cup \cdots \cup S_{\binom{N}{3}})$. Then K'' is a centrally symmetric convex polygon and the family $F'' = \{K'' + v_i, i \in I\}$ will have $T(3)$. Since $K'' \subset K$, the disjointness is preserved, and a subfamily $\{K'' + v_{i_1}, \ldots, K'' + v_{i_j}\} \subset F''$ has a transversal only if the subfamily $\{K + v_{i_1}, \ldots, K + v_{i_j}\} \subset F$ has one. Thus, we need only consider the case in which K is a centrally symmetric convex polygon.

Now let K be a centrally symmetric convex polygon. Then there exists an affine transformation A such that $diam(AK) = d$ and the area of AK is greater than or equal to $\frac{d^2}{2}$. This follows from the fact that there exists an affine transformation A such that AK has its center at the origin and $D' \subset AK \subset D$, where D is the disc of radius $\frac{d}{2}$ with center in the origin, and D' is the disc of radius $\frac{d}{2\sqrt{2}}$ with center in the origin. We refer to Behrend [1] for the details. Furthermore, we must have one of the following cases: (1) AK has two vertices v_i and v_j that lie on the boundary of D such that the angle between v_i, the origin, and v_j equals $\frac{\pi}{2}$. (2) AK has three vertices v_i, v_j, v_k that lie on the boundary of D such that if we let α, β, γ be the angles between v_i and v_j, v_j and v_k, v_k and v_i' (v_i' is the the vertex opposite of v_i), respectively, then $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$.





In case (1), $conv(\{v_i, v_j, v_i', v_j'\})$ is a square with sides of length $\frac{d}{\sqrt{2}}$, and by the convexity, this square must be contained in AK. Thus, the area of AK must be greater than or equal to $\frac{d^2}{2}$.

In case (2), one considers the triangles $T_1 = conv(\{v_i, v_j, 0\})$, $T_2 = conv(\{v_j, v_k, 0\})$, and $T_3 = conv(\{v_k, v_i', 0\})$. It suffices to show that the sum of the areas of these three triangles is greater than or equal to $\frac{d^2}{4}$. The area of the three triangles are, respectively, $\frac{d^2 \sin\alpha}{8}$, $\frac{d^2 \sin\beta}{8}$, $\frac{d^2 \sin\gamma}{8}$, and further we have $\gamma = \pi - (\alpha + \beta)$. Thus, the sum of the area of the three triangles is

$$\frac{d^2}{8}(\sin\alpha + \sin\beta + \sin\gamma)$$

and since $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$, we have

$$2 < (\sin\alpha + \sin\beta + \sin\gamma) \quad .$$

Thus, the area of the AK is greater than $\frac{d^2}{2}$.

We shall now show that we can assume there exist three translates $\{K_1, K_2, K_3\}$ in F such that $d_c(K_i, K_j) > \sqrt{2}d$, for $i, j \in \{1, 2, 3\}, i \neq j$. Assume first that F does not contain any pair $\{X, Y\}$ such that $d_c(X, Y) > \sqrt{2}d$. Thus the point set consisting of the centers of the translates of F has diameter $\leq \sqrt{2}d$. By the plane case of Jung's theorem, the centers can be covered by a disc of radius $\frac{\sqrt{3}}{\sqrt{2}}d$. This means that the translates of F can be covered by a disc of radius $(\frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{2})d$, and since the minimal area of a translate is $\frac{d^2}{2}$ and the translates are disjoint, we find that F contains at most

$$\left\lfloor \frac{\pi(\frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{2})^2 d^2}{\frac{d^2}{2}} \right\rfloor = 10 \text{ translates}$$

Since any family of less than 11 translates trivially satisfies $T-4$ (consider the Eckhoff segments) we may assume that F contains a pair $\{X, Y\}$ such that $d_c(X, Y) > \sqrt{2}d$.

A family of less than 51 translates trivially satisfies $T-24$, therefore let F be a family of at least 51 disjoint translates. Consider the graph G where each vertex represents a translate of F. Two vertices are connected by an edge if and only if the corresponding translates $\{X,Y\}$ have $d_c(X,Y) \leq \sqrt{2}d$. Now $|V(G)| \geq 51$ while the Ramsey number $R(3,11)$ does not exceed 51 (see [14]), thus G has a clique of size at least 11 or G has at least 3 independent vertices. But the former case was excluded above, and so G has 3 independent vertices corresponding to a triple $\{K_1,K_2,K_3\}$ in F such that $d_c(K_i,K_j) > \sqrt{2}d$, for $i,j \in \{1,2,3\}, i \neq j$.

We now return to the family F of translates of a centrally symmetric convex polygon K of diameter d. If F has $T(3)$ there exist three discs of radius $(\sqrt{2}+\frac{1}{2})d$, such that there is a common transversal for all members of F that are not contained in any one of these discs. To see this we shrink each member of F by a factor $\lambda \in [0,1]$ about the center of symmetry, obtaining a family λF for each value of λ. So far this would just be Klee's shrinking process, where one would choose the minimal $\lambda' \in [0,1]$ such that $\lambda' F$ has $T(3)$ but for every $\lambda < \lambda'$, λF does not. Clearly for every subfamily $\lambda M \subset \lambda F$ that has a transversal, the corresponding subfamily $M \subset F$ also has a transversal. Note also that for every $\lambda X, \lambda Y \in \lambda F$ and the corresponding $X, Y \in F$ we have $d_c(\lambda X, \lambda Y) = d_c(X,Y)$.

Instead of choosing the λ' mentioned above, we choose the minimal $\lambda_0 \in [0,1]$ such that each triple $\{\lambda_0 K_1, \lambda_0 K_2, \lambda_0 K_3\} \subset \lambda_0 F$ has a transversal whenever $d_c(K_i,K_j) > \sqrt{2}d$, for $i,j \in \{1,2,3\}, i \neq j$. The existence of such a λ_0 is established by standard compactness arguments. Since λ_0 is minimal there must exist some triple $\{A,B,C\} \subset F$ such that the corresponding triple $\{\lambda_0 A, \lambda_0 B, \lambda_0 C\}$ has a unique transversal l. Now, if a translate $\lambda_0 X \in \lambda_0 F$ does not meet l, then $d_c(\lambda_0 X, \lambda_0 Y) \leq \sqrt{2}d$ for some $\lambda_0 Y \in \{\lambda_0 A, \lambda_0 B, \lambda_0 C\}$. To see this, assume that $d_c(\lambda_0 X, \lambda_0 Y) > \sqrt{2}d$ for every $\lambda_0 Y \in \{\lambda_0 A, \lambda_0 B, \lambda_0 C\}$. By the choice of λ_0, the family $\{\lambda_0 A, \lambda_0 B, \lambda_0 C, \lambda_0 X\}$ has $T(3)$, but since $\lambda_0 X$ misses l, this must be a bad quadruple. By lemma 4.2 every triple is totally separable, and by theorem 4.6 the quadruple must admit a transversal, contradicting the assumption that $\lambda_0 X$ misses l. Thus, we can find three discs of radius $(\sqrt{2}+\frac{1}{2})d$ such that the members of F that are not contained in any of these discs all meet l.

For a family F of disjoint translates we can thus assume there exist three discs of radius $(\sqrt{2}+\frac{1}{2})d$ such that the translates that are not contained in any of these discs have a common transversal. Let P be the point set consisting of the centers of the translates of F. Since F has $T(3)$, each three points of P are contained in some strip of width d. It can then easily be verified that the entire point set P is contained in some strip of width $2d$. Further, this means that some strip of width $3d$ covers all the members of F (this can also be seen by considering the Eckhoff segments). Thus, the bad translates (the ones that miss the line l defined above) lie not only in one of the discs of radius $(\sqrt{2}+\frac{1}{2})d$, but also in a strip of width $3d$, and the area where the bad translates can lie must be less than (or equal to) three times the maximal area of the intersection of one of the discs and the strip. Clearly this area is maximal when the strip cuts across the middle of the disc, and the largest possible area for such a disc is:

$$max.area = 4\int_0^{\frac{3d}{2}} \sqrt{(\sqrt{2}+\frac{1}{2})^2 d^2 - x^2}\,dx \leq 10.2d^2$$

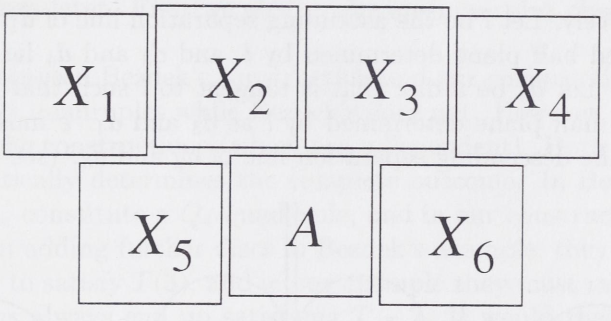
Since the translates are disjoint and the minimal area of a translate equals $\frac{d^2}{2}$, the maximal number of translates contained in one of the (reduced) discs is

$$\frac{10.2d^2}{\frac{d^2}{2}} \approx 20$$

where one of the translates is either A, B, or C. Thus each of the three discs contain at most 19 translates that do not meet l. Therefore we get $T(3) \Rightarrow T - 57$. $\square$

Remark 1. A conjecture by Eckhoff [4] states that if a point set is such that every three points can be covered by a strip of width 1, the entire point set can be covered by a strip of width $\frac{\sqrt{5}+1}{2}$. If this conjecture is true, the value of k is less than or equal to 51.

Remark 2. The proof of theorem 5.2 ends with making a crude estimate of the number of translates that can lie in the "bad" discs. This method is similar to the one used by Katchalski and Lewis in their original proof [12], and does not use the fact that the family satisfies $T(3)$. Since the translates may be assumed to be on Behrand's "standard" form they are in a certain sense "fat". A possible approach to lower the value of k is to find out how many disjoint translates $X_1, \ldots X_n$ that can lie about a given translate A such that $d_c(A, X_i) \leq \sqrt{2}d$, for all $i = 1 \ldots n$, and such that the family $\{A, X_1, \ldots, X_n\}$ satisfies $T(3)$. We have not yet studied this extensively, but the following example shows that $n \geq 7$.



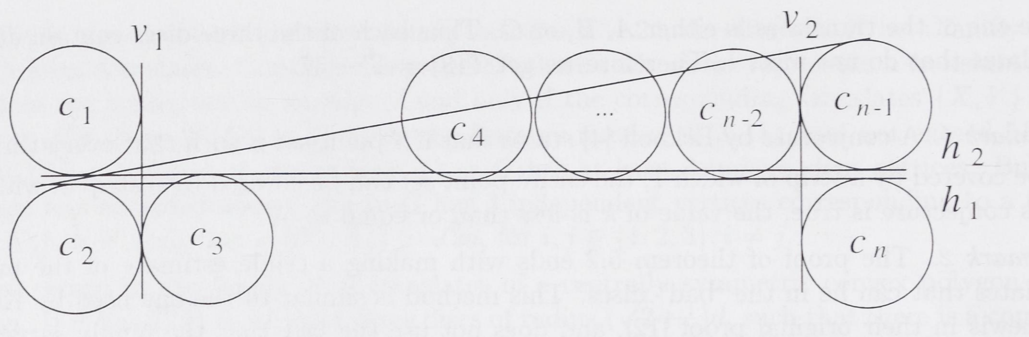
In the figure above the translates that are adjacent (e.g. X_1 and X_5) are at distance ε apart. Thus when $\varepsilon \to 0$ we have $d_c(A, X_1) \to \frac{\sqrt{13}d}{2\sqrt{2}} < \sqrt{2}d$. This shows that the best we can hope for with this approach is $T(3) \Rightarrow T - 21$.

Remark 3. I have now started to work on a modified procedure, where the translates are not shrunk, but where instead one of the Eckhoff sets is "moved" away in the direction of the Eckhoff segments (this is an idea introduced in [17]). One stops when one of the distinguished triples becomes critical. This seems promising, as the family changes very little, and the "new" Eckhoff set is contained in the critical triple, so we know more about it.

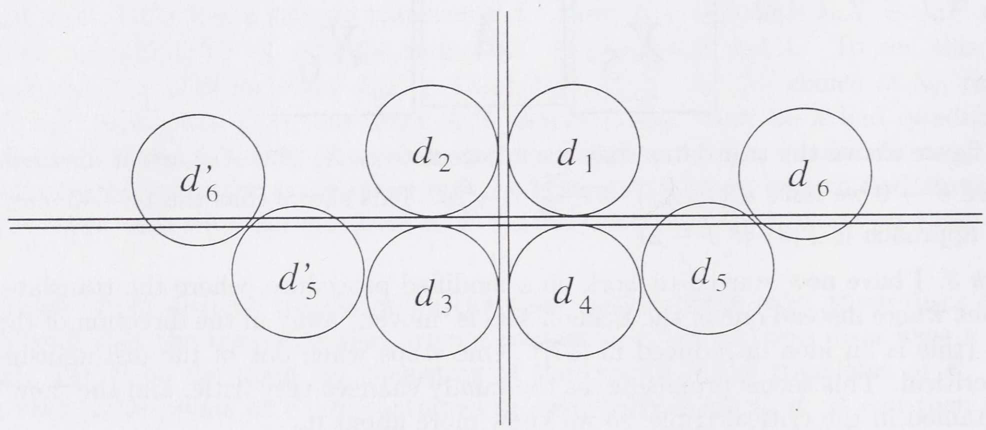
6. Examples of families of disjoint translates

We have been able to prove $T(3) \Rightarrow T - 57$, but the number 57 is obviously still too large. Tverberg [17] suggests that a possible approach to the Katchalski-Lewis conjecture would be to study smaller families with the property $T(3)$ and gain as much information about their behavior as possible, so that we can study larger families without too many geometric considerations.

We shall now take a closer look at families that satisfy $T(3)$, and we start with a construction by A. Bezdek [2] which exhibits a family of n disjoint unit discs that satisfies $T(3)$ but not $T-1$. The construction depends on four reference lines. Let v_1 and v_2 be two vertical lines that are at distance $2(n-4)$ apart, and let h_1 and h_2 be two horizontal lines that are at a distance ε apart. He continues by arranging the n discs $c_1, \ldots, c_n$ at positions described by the four reference lines. The construction is illustrated below. For more details on the construction, see [2].



We shall now construct a different family of unit discs that satisfies $T(3)$ but not $T-1$. Let d_1, d_2, d_3, and d_4 be four unit discs that lie in different quadrants such that the centers are placed at $(1,1)$, $(-1,1)$, $(-1,-1)$, and $(1,-1)$, respectively (the discs are not disjoint, but this will be taken care of). For a $\varepsilon > 0$ we move the discs away from the x- and the y-axis such that the centers now have the coordinates $(1+\varepsilon, 1+\varepsilon)$, $(-1-\varepsilon, 1+\varepsilon)$, $(-1-\varepsilon, -1-\varepsilon)$, and $(1+\varepsilon, -1-\varepsilon)$, respectively. Let l be the ascending separation line of d_1 and d_3 such that d_1 and d_2 lie in the same closed half plane determined by l, and d_3 and d_4 lie in the other closed half plane determined by l. Let d_5 be a disc that is tangent to l such that it lies to the right of d_4 and in the same closed half plane determined by l as d_3 and d_4. ε must be chosen to be small enough that d_5 meets the descending separation line of d_2 and d_3. (see the figure below)



Let m be the descending separation line of d_2 and d_4 such that d_1 and d_2 lie in the same closed half plane determined by m and d_3 and d_4 lie in the other closed half plane determined by m. Let d_6 be a disc that is tangent to m such that it lies to the right of d_5 and in the same closed half plane determined by m as d_1 and d_2. Again ε must be chosen small enough that d_6 meets the ascending separation line of d_2 and d_3. Let d'_5 and d'_6 be the reflections of d_5 and d_6 about the y-axis, respectively.

It is easily verified that the construction satisfies $T(3)$. It is also easy to see that the construction does not have $T-1$. To see this consider the ascending separation line, s_+, and the descending separation line s_- of d_2 and d_3. The family is constructed such that s_+ will meet d_6 but not d_5. s_- will meet d_5 but not d_6, thus, the quadruple $\{d_2, d_3, d_5, d_6\}$ is a bad quadruple (Type Q_3). By symmetry, $\{d_1, d_4, d'_5, d'_6\}$ is also a bad quadruple. In other words the family consists of two disjoint bad quadruples and can therefore not have $T-1$. It is also easy to see that the construction can be extended to contain any finite number of discs. This is done by adding discs to the left of d'_6 and to the right of d_6 with centers on the x-axis. The property $T(3)$ will be preserved when ε is chosen to be small enough.

The constructions above show that the Katchalski-Lewis conjecture is best possible, in the sense that we cannot have $T(3) \Rightarrow T - 1$. Tverberg [17] suggests that the following steps should be studied as a way of proving the Katchalski-Lewis conjecture (with the obvious notation):

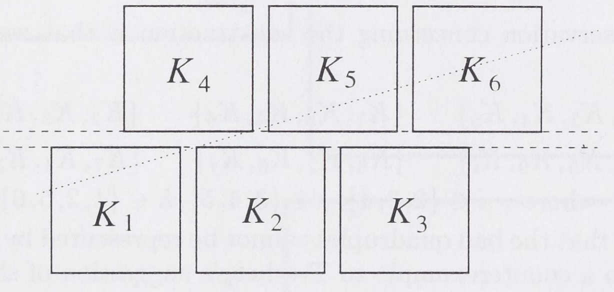
$$T(3) \Rightarrow T(4) - 1 \quad \text{and} \quad T(4) \Rightarrow T - 1$$

This would obviously imply the K.-L. conjecture, however, our construction that contains two distinct quadruples provides a counterexample to the statement $T(3) \Rightarrow T(4) - 1$. It is also easy to check that the bad quadruples of Bezdek's construction cannot be represented by one element, therefore this is also a counterexample to the statement in question. Tverberg [17] also suggests to try proving the following:

$$T(3) \Rightarrow T(4) - 2 \quad \text{and} \quad T(4) \Rightarrow T - 2$$

This would yield a weaker version of the K.-L. conjecture, namely $T(3) \Rightarrow T - 4$. We shall comment on this approach later. First we shall make an important observation concerning the constructions above.

The main difference between Bezdek's construction and our construction is that our construction contains two distinct quadruples while Bezdek's does not. However, it is more important to note the similarities of the constructions (which are independent). Both the examples contain a Q_4-quadruple that practically determines the complete outcome. In Bezdek's construction the discs c_1, c_2, c_{n-1}, and c_n constitute a Q_4-quadruple, and in our construction we find the similar sets d_1, d_2, d_3, d_4. When adding further discs in Bezdek's example, they must all meet the lines h_1 and h_2 for the family to satisfy $T(3)$, and in our example they must meet the equivalent lines. Thus, both constructions always end up satisfying $T - 2$. It would therefore be natural to try constructing a Q_4-quadruple which is such that one can add further translates which do not meet both these lines (while satisfying $T(3)$). As it turns out, this is possible, and the construction is illustrated in the figure below.



The translates K_1, K_3, K_4, K_6 form a Q_4-quadruple, and we have added two additional translates K_2 and K_5 which each only meet one of the two lines that correspond to h_1 and h_2 in Bezdek's example. In what follows we shall show that this simple arrangement can, surprisingly, be extended to a counterexample to the K.-L. conjecture.

Theorem 6.1. *There exists a family of disjoint translates in the plane which satisfies $T(3)$ but not $T - 2$.*

Proof. Let K be a square with sides of length 1, where the sides are parallel to the x- and y-axis. Let K_1, K_2, K_3 be translates of K. Let the NW-corner of K_1 lie at the origin. Now K_2 and K_3 shall be tangent to the x-axis from below such that K_2 lies directly to the right of K_1, and K_3 lies directly to the right of K_2. Further let the distance between K_1 and K_2 be ε, and let the distance between K_2 and K_3 also be ε. Let K_4, K_5, K_6 be translates of K that are tangent to the line $y = \varepsilon$ from above. K_4 lies such that the part above K_1 is as big as the part above K_2. K_5 lies such that equal parts lie above K_2 and K_3. K_6 lies such that the distance between K_5 and K_6 is ε. So far the construction consists of the example in the previous figure. Clearly the translates are disjoint for all $\varepsilon > 0$.

Define three reference lines l, m, n as follows: Let l be the ascending line that goes through the NW-corner of K_1 and the SE-corner of K_5. Let m be the ascending line that goes through the NW-corner of K_2 and the SE-corner of K_6. Finally, let n be the descending line that goes through the SW-corner of K_4 and the NE-corner of K_3. Note that l and m are parallel, and that the slope of the three reference lines is a function of ε. We shall now place three translates K_7,K_8, and K_9, such that their positions are described by the lines l, m, n.

Let K_7 lie to the left of K_1 such that n goes through its NE-corner, let K_8 lie to the left of K_7 such that m goes through its SE-corner, and let K_9 lie to the right of K_6 such that l goes through its NW-corner. Now, ε must be chosen small enough such that (1) K_9 meets the line through the SW-corner of K_4 and the NE-corner of K_1, (2) K_8 meets the line that goes through the SW-corner of K_6 and the NE-corner of K_3, and (3) K_7 meets the line that goes through the SE-corner of K_5 and the NW-corner of K_3. Clearly by choosing ε small enough, the construction will have $T(3)$. It is also easy to see that the family can be extended to contain any finite number of translates. The construction is illustrated on the next page.

We shall now prove that the family does not have $T-2$. By the construction the following triples are critical (i.e. have a unique transversal):

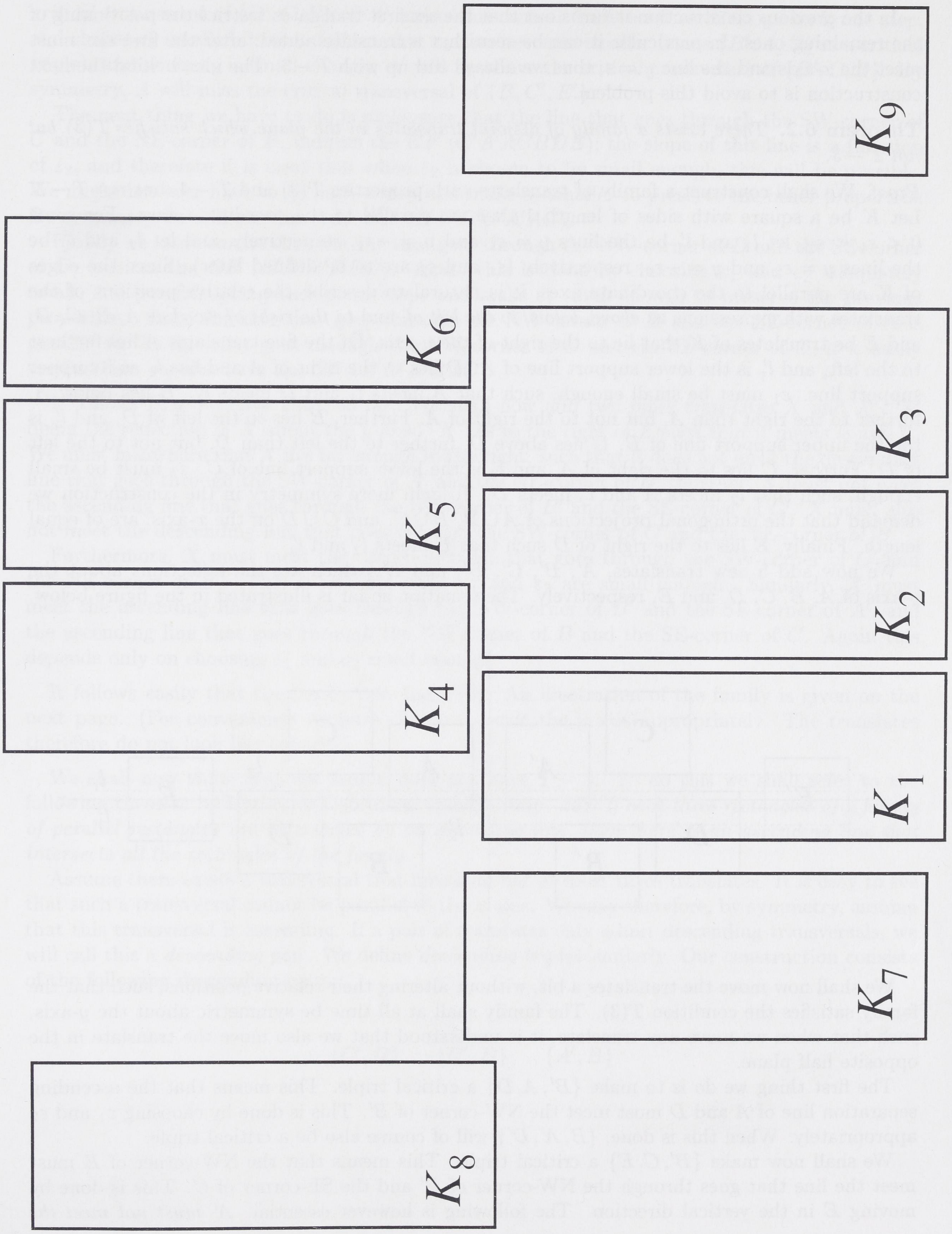
$$\{K_1, K_5, K_9\} \qquad \{K_8, K_2, K_6\} \qquad \{K_7, K_4, K_3\}$$

Clearly the critical triples constitute the entire family. The unique transversals are the reference lines l, m, and n. It can easily be checked that these lines will not give $T-2$, in other words, if there exists a line that induces $T-2$ it must be some other line than l, m, or n. But every other line must miss at least one of the sets in each of the three critical triples, and since the critical triples consist of distinct elements, the construction cannot have T_2. □

Remark: A final observation concerning the construction is that we will have the following bad quadruples:

$$\{K_1, K_3, K_4, K_5\} \qquad \{K_1, K_3, K_4, K_6\} \qquad \{K_2, K_3, K_4, K_6\}$$
$$\{K_1, K_5, K_9, K_i\} \qquad \{K_8, K_2, K_6, K_j\} \qquad \{K_7, K_4, K_3, K_k\}$$
$$\text{where },\ i \in \{2,3,4\}, j \in \{3,4,5\}, k \in \{1,2,5,6\}$$

It can easily be verified that the bad quadruples cannot be represented by two elements. Therefore the construction is also a counterexample to Tverberg's suggestion of showing $T(3) \Rightarrow T(4) - 2$ (This also shows that the family does not have $T-2$).

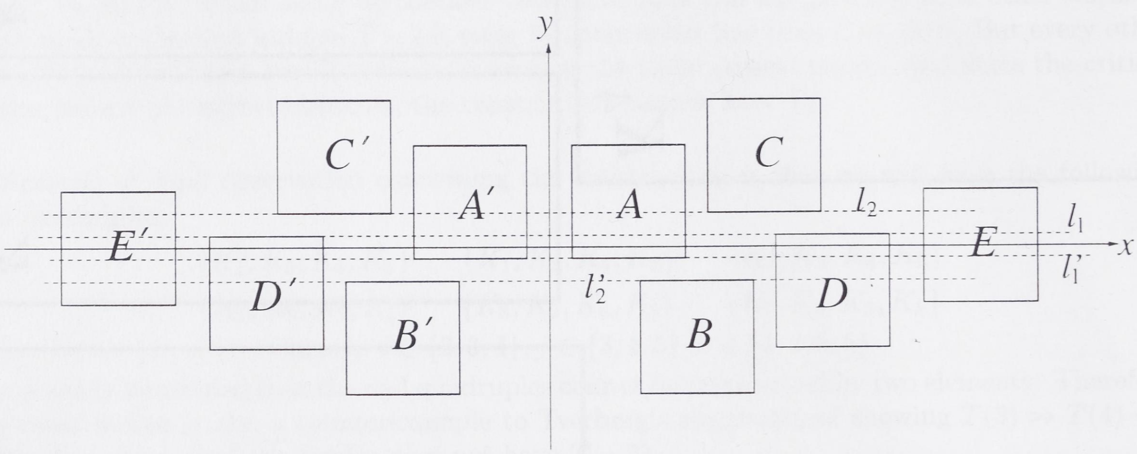

K_9
K_6
K_3
K_5
K_2
K_4
K_1
K_7
K_8

In the previous construction it turns out that the six first translates restrict the positioning of the remaining ones. In particular it can be seen that a translate added, after the first six, must meet the x-axis and the line $y = \varepsilon$, thus we always end up with $T-3$. The idea behind the next construction is to avoid this problem.

Theorem 6.2. *There exists a family of disjoint translates in the plane which satisfies $T(3)$ but not $T-3$.*

Proof. We shall construct a family of translates with properties $T(3)$ and $T-4$, but not $T-3$. Let K be a square with sides of length 1 that are parallel to the coordinate axes. For some $0 < \varepsilon_1 < \varepsilon_2$, let l_1 and l_1' be the lines $y = \varepsilon_1$ and $y = -\varepsilon_1$, respectively, and let l_2 and l_2' be the lines $y = \varepsilon_2$ and $y = -\varepsilon_2$, respectively (ε_1 and ε_2 are to be defined later). Since the edges of K are parallel to the coordinate axes, it is natural to describe the relative positions of the translates with expressions as *above*, *below*, *to the left of*, and *to the right of* etc. Let A, B, C, D, and E be translates of K that lie to the right of the y-axis. Of the five translates A lies farthest to the left, and l_1' is the lower support line of A. D lies to the right of A and has l_1 as its upper support line. ε_1 must be small enough, such that A meets l_1 and D meets l_1'. B lies below A, further to the right than A, but not to the right of A. Further, B lies to the left of D, and l_2' is the the upper support line of B. C lies above D, further to the left than D, but not to the left of D. Further, C lies to the right of A, and l_2 is the lower support line of C. ε_2 must be small enough, such that l_2 meets A and l_2' meets D. To gain more symmetry in the construction we demand that the orthogonal projections of $A \cup B$, $B \cup C$, and $C \cup D$ on the x-axis, are of equal length. Finally, E lies to the right of D such that E meets l_2 and l_2'.

We now add 5 new translates, A', B', C', D', and E', which are the reflections about the y-axis of A, B, C, D, and E, respectively. The situation so far is illustrated in the figure below.



We shall now move the translates a bit, without altering their relative positions, such that the family satisfies the condition $T(3)$. The family shall at all time be symmetric about the y-axis, such that when we move any translate, it is understood that we also move the translate in the opposite half plane.

The first thing we do is to make $\{B', A, D\}$ a critical triple. This means that the ascending separation line of A and D must meet the NW-corner of B'. This is done by choosing ε_1 and ε_2 appropriately. When this is done, $\{B, A', D'\}$ will of course also be a critical triple.

We shall now make $\{B', C, E\}$ a critical triple. This means that the NW-corner of E must meet the line that goes through the NW-corner of B' and the SE-corner of C. This is done by moving E in the vertical direction. The following is however essential: A' *must not meet the* *critical transversal of* $\{B', C, E\}$. If A' meets this critical transversal we must go back one step, and choose ε_1 smaller. It then follows that ε_2 also becomes smaller (since $\{B', A, D\}$ is a critical triple), and when ε_2 is small enough, A' will lie above the critical transversal of $\{B', C, E\}$. By symmetry, A will miss the critical transversal of $\{B, C', E'\}$.

The next thing we have to do is make sure that the line that goes through the SW-corner of C and the NE-corner of B, induces the *GP* $(C'A'ACBDE)$; the slope of this line is a function of ε_2, and therefore it is clear that when ε_2 is chosen to be small enough, this will be possible. We might however have to go back a step and make ε_1 smaller to preserve the other properties. By symmetry we will also have the *GP* $(E'D'B'C'A'AC)$.

Finally we must ensure that the line that goes through the NW-corner of D and the SE-corner of C, induces the *GP* $(E'D'B'BDC)$. Again this is done by choosing ε_1 and ε_2 adequately small, and possibly going back some steps and make ε_1 small enough to preserve the previous properties. Since the line that goes through the NW-corner of B and the SE-corner of A is parallel to the line that goes through the NW-corner of D and the SE-corner of C, it is easily verified that we also have the *GP* $(E'D'B'BAC)$.

It follows that we have $T(3)$, so far. We shall now add two more translates X and X' (X' being the reflection of X about the y-axis). X lies to the right of E such that it meets l_2 and l'_2. We must make sure that X lies far enough to the right, such that it does not meet the descending line that goes through the SW-corner of A' and the NE-corner of B'. Further, X must not meet the ascending line that goes through the NW-corner of B' and the SE-corner of C'. X must also not meet the descending line that goes through the SW-corner of C and the NE-corner of B.

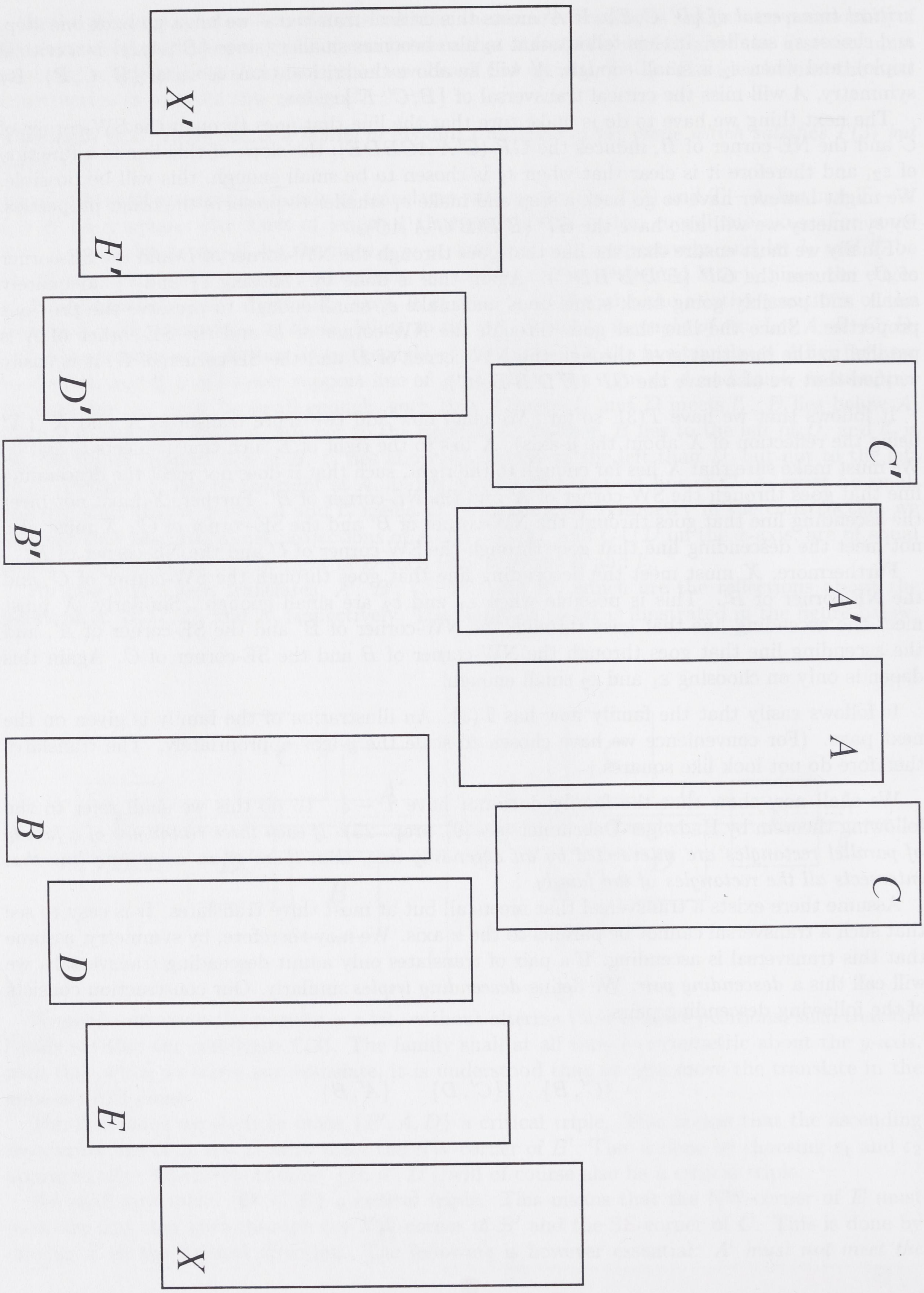
Furthermore, X must meet the descending line that goes through the SW-corner of C' and the NE-corner of B'. This is possible when ε_1 and ε_2 are small enough. Similarly, X must meet the ascending line that goes through the NW-corner of B' and the SE-corner of A', and the ascending line that goes through the NW-corner of B and the SE-corner of C. Again this depends only on choosing ε_1 and ε_2 small enough.

It follows easily that the family now has $T(3)$. An illustration of the family is given on the next page. (For convenience we have chosen to scale the y-axis appropriately. The translates therefore do not look like squares.)

We shall now show that the family does not have $T-3$. To do this we shall refer to the following theorem by Hadwiger-Debrunner (see [9], prop. 25): *If each three rectangles of a family of parallel rectangles are intersected by an ascending line, then there is an ascending line that intersects all the rectangles of the family.*

Assume there exists a transversal that meets all but at most three translates. It is easy to see that such a transversal cannot be parallel to the x-axis. We may therefore, by symmetry, assume that this transversal is ascending. If a pair of translates only admit descending transversals, we will call this a *descending pair*. We define *descending triples* similarly. Our construction consists of the following descending pairs:

$$\{C', B\} \qquad \{C', D\} \qquad \{A', B\}$$


X'
E'
D'
C'
B'
A'
A
B
C
D
E
X

Every triple that contains a descending pair will naturally be a descending triple. Apart from the triples that contain a descending pair, it is easily verified that we have the following descending triples:

$$\{A,B,D\} \quad \{A,B,E\} \quad \{A,B,X'\}$$
$$\{C,D,E\} \quad \{C,D,X'\} \quad \{C',D',E\}$$
$$\{A,C,D\} \quad \{X',C',B'\} \quad \{X,C',B'\}$$
$$\{A',B',E\} \quad \{C',B',E\} \quad \{A',C,D\}$$
$$\{A',B',D\} \quad \{B,C,E\}$$

Now, if there exists an ascending transversal that meets all but at most three translates, then by Hadwiger-Debrunner's theorem, we can remove three translates and thus "destroy" all the descending triples. Or stated otherwise, the descending triples must be represented by at most three elements. We shall show that this however is not the case.

We can start by concentrating on removing elements such that we destroy the descending pairs. For if we still have a descending pair, after removing three elements, we cannot have an ascending transversal that meets the remaining translates. It is easily verified that we must remove at least two translates to destroy the descending pairs, and that every triple representing the descending pairs contains at least one of the following pairs: $\{C',B\}$, $\{C',A'\}$, or $\{B,D\}$. But these pairs represent the descending pairs, and it therefore suffices to check what happens when these pairs have been removed.

We start by removing the pair $\{C',B\}$. We are then left with the following descending triples:

$$\{C,D,E\} \quad \{C,D,X'\} \quad \{A,C,D\}$$
$$\{A',B',E\} \quad \{A',B',D\} \quad \{A',C,D\}$$

Removing the pair $\{C',A'\}$, we are then left with the following descending triples:

$$\{A,B,D\} \quad \{A,B,E\} \quad \{A,B,X'\}$$
$$\{C,D,E\} \quad \{C,D,X'\} \quad \{A,C,D\}$$
$$\{B,C,E\}$$

Finally, if we remove the pair $\{B,D\}$, we are left with the following descending triples:

$$\{X,C',B'\} \quad \{X',C',B'\} \quad \{C',D',E\}$$
$$\{A',B',E\} \quad \{C',B',E\}$$

It is easily verified that in each of the cases above the descending triples are not representable by a single element. Thus, the family does not satisfy $T-3$. Clearly a horizontal line can be found that induces the GP $(X'E'C'A'ACEX)$, thus the family satisfies $T-4$. □

It is possible to extend the family to any finite number of translates by adding translates to the right of X and to the left of X', and choosing ε_1 and ε_2 appropriately small to preserve $T(3)$.

Our construction clearly contains a large amount of bad quadruples. The list below represents some of the bad quadruples (indicated by a row of four x's):

A	B	C	D	E	X	A'	B'	C'	D'	E'	X'
-	-	X	X	-	-	-	-	X	X	-	-
-	-	X	X	-	-	X	X	-	-	-	-
X	-	-	X	-	-	-	-	X	X	-	-
X	-	-	X	-	-	X	X	-	-	-	-
X	X	-	-	-	-	X	-	-	X	-	-
-	X	-	-	-	-	X	-	X	X	-	-
-	X	-	-	-	-	X	X	-	X	-	-
-	-	-	-	X	X	-	-	X	X	-	-
-	-	X	X	-	-	-	-	-	-	X	X
-	-	-	-	-	-	-	X	X	-	X	X
-	X	X	-	X	X	-	-	-	-	-	-
-	-	X	-	X	-	X	X	-	-	-	-
X	X	-	X	-	-	-	X	-	-	-	-
X	X	-	-	-	-	-	-	X	-	X	-

It can easily be verified that the bad quadruples cannot be represented by three elements. Thus our construction also proves the statement $T(3) \not\Rightarrow T(4)-3$, thus the best we can hope for in this sense is $T(3) \Rightarrow T(4) - 4$.

Finally we note a special property of the above construction. When we start placing the translates, we are free to chose how large the distance between A and the x-axis is. The same applies for the distance between D and E. In particular the translates B' and C can lie arbitrarily far apart. The same goes for the translates C and E. Further, the triple $\{B', C, E\}$ is, by construction, a critical triple. In the final section we shall discuss, in general, families that contain a critical triple where the distance between the translates is "large".

(*Note:* In the constructions of theorems 6.1 and 6.2 the positions of the translates could of course have been given by explicit coordinates. These coordinates have been computed, but it requires some rather tedious calculations which do not yield any extra insight to the constructions, and has therefore been omitted from this paper.)

7. Final remarks

We have shown the following:

Theorem 7.1. *Let F be a family of disjoint translates of a compact convex set in the plane. Then $T(3) \Rightarrow T - k$, where $4 \leq k \leq 57$.*

The upper bound of k is clearly too high. In fact, we believe the lower bound to be the correct value of k, although we do not have any concrete idea of how to prove this. Tverberg's suggestion of approaching the problem in the steps $T(3) \Rightarrow T(4) - x$ and $T(4) \Rightarrow T - y$, will yield a theorem $T(3) \Rightarrow T - z$, where $z = x + y$, but in view of the comments in the previous section, we must have $x \geq 4$, and simple examples show that $y \geq 1$. Thus, this approach cannot bring us closer than $T(3) \Rightarrow T - 5$. The advantage of this approach was that the the first step could be limited to a study of small families of translates ($|F| \leq 12$, see [17]); this however is no longer the case. It therefore seems necessary to find a different way of attacking the problem. (The problem of $T(4)$ should nevertheless be studied in its own right).

The main difficulty in studying geometric transversals seems to be when the sets are close to each other. When they are spread out the situation is easier to handle, as was seen in theorem

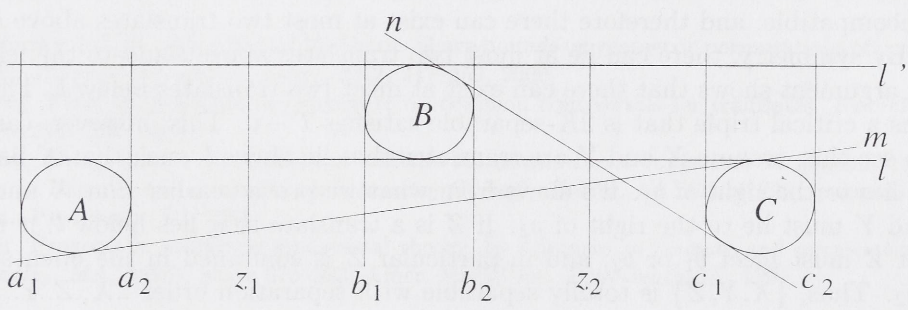
4.6. Other examples of this can be seen in corollary 4 of [8] and proposition 92 of [9]. We shall conclude this paper with showing that when a family of disjoint translates in the plane contains a critical triple where the members of the triple are spread sufficiently far apart, the conclusion $T(3) \Rightarrow T-4$ is valid.

Let a τ-*strip* be a strip of breadth τ. By the direction of a strip we mean the direction of the parallel lines that define the strip. A family of sets in the plane is said to be τ-*separable* if there exists a direction such that any τ-strip in this direction meets at most one member of the family. Thus, a family that is 0-separable is what we earlier called totally separable. Since we are studying families of translates of a convex set K, it is convenient to use the set K to define distance in the plane. We then have the following: For a $\lambda > 0$, we define a λK-strip in the direction D to be a τ-strip in the direction D, where τ is the length of the orthogonal projection of K on a line which is orthogonal to the direction D.

Let A, B, and C, be translates of a compact convex set K in the plane. Further, assume that A, B, and C have a unique transversal and that the triple is $2K$-separable. Let the critical transversal l define the horizontal direction, and assume that A lies to the left of C, and that A and C are tangent to l from below. Then B is tangent to l from above and meets l between $A \cap l$ and $C \cap l$. Since λK-separability is an affine invariant we may assume that the separation strips are orthogonal on l.

We now define eight lines that are orthogonal to l. Let a_1 be the line that is tangent to A from the left, and let a_2 be the line that is tangent to A from the right. Similarly, let b_1 be the line that is tangent to the B from the left, and b_2 be the line that is tangent to B from the right. Define c_1 and c_2 similarly. Finally, let z_1 be the line that lies at equal distance from a_2 and b_1, and let z_2 be the line that lies at equal distance between b_2 and c_1. Since $\{A, B, C\}$ is $2K$-separable, the strip defined by a_2 and z_1 is wider than a $1K$-strip. The same applies for the line pairs $\{z_1, b_2\}$, $\{b_2, z_2\}$, and $\{z_2, c_1\}$.

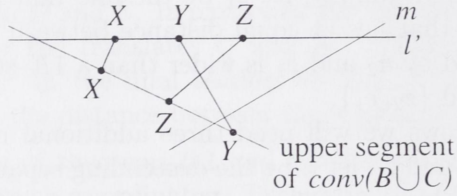
For the discussion that follows we will need three additional reference lines. Let m be the ascending separation line of A and C, let n be the descending separation line of B and C, and let l' be the parallel of l that is tangent to B from above. The figure below illustrates the situation we have described.



Now, assume that a translate X lies above l and to the right of b_1, such that $\{A, B, C, X\}$ is a disjoint family of translates satisfying $T(3)$. Note that we can always enlarge the translates a bit, then move them slightly, such that we can assume that a translate lying above l always lies either to the right of b_1 or to the left of b_2. Now, X must meet m. For if X lies above m, then $\{A, C, X\}$ cannot have a transversal, and if X lies below m, it must lie to the right of c_2 and thus $\{B, C, X\}$ is totally separable in the order $..B..C..X..$ But since X lies above l, $\{B, C, X\}$ cannot have a common transversal. Thus m is a transversal for $\{A, C, X\}$. Further, we must have one of the following *GP*s: (BCX), (BXC) and (XBC). (BCX) is excluded since X lies above l. And if we have (XBC), X must meet n after n has left B, when we traverse n upward.

But since $\{A, B, C\}$ is $2K$-separable, it is impossible for X to meet m such that we have (ACX), and meet n such that we have (XBC). Thus, we must have (BXC), and X must meet the boundary of $conv(B \cup C) \setminus (B \cup C)$. Finally, note that X must meet either c_1 or c_2, for if X lies to the left of c_1, X cannot meet m, and if X lies to the right of c_2, X cannot meet $conv(B \cup C)$.

We shall now show that there can lie at most two translates above l and to the right of b_1. Assume there are three translates X, Y, and Z, that lie above l and to the right of b_1, such that $\{A, B, C, X, Y, Z\}$ is a family of disjoint translates that satisfies $T(3)$. Note that each of the translates must meet the segment $conv(\{l' \cap n, l' \cap m\})$. For each of the translates X, Y, and Z, we mark off a point where the translate intersects the segment $conv(\{l' \cap n, l' \cap m\})$, such that each of these points represents one of the translates. Since the translates are disjoint, a single point cannot represent two of the translates, and the order in which the points lie, corresponds to the GP induced by l'.Without loss of generality, we can assume that we meet the translates in the order ..X..Y..Z.. when we traverse l' from left to right. Further, we have seen that each of the translates X, Y, and Z, must meet the upper segment of the boundary of $conv(B \cup C) \setminus (B \cup C)$, and the line m. Mark off, as above, points on the upper segment of the boundary of $conv(B \cup C) \setminus (B \cup C)$, and on the segment of m that lies above l and below l'. It is then easily verified that m must induce $(ACXYZ)$, and the upper segment of the boundary of $conv(B \cup C) \setminus (B \cup C)$ must induce $(BXYZC)$. If the points lie in any other order the translates cannot be disjoint. This can be seen by observing that the segments between the points representing the different translates will cross. An example of this is illustrated in the figure below.



The two GPs that are induced contain the following sub-permutations: $(CXYZ)$ and $(XYZC)$. These are incompatible, and therefore there can exist at most two translates above l and to the right of b_1. By symmetry, there can lie at most two translates above l and to the left of b_2.

A similar argument shows that there can exist at most two translates below l. Thus, a family that contains a critical triple that is $2K$-separable satisfies $T - 6$. This, however, can be further reduced. To see this, assume X and Y are translates that lie above l, such that X lies to the left of b_2 and Y lies to the right of b_1. It follows from what we proved earlier that X must lie to the left of z_1 and Y must lie to the right of z_2. If Z is a translate that lies below l, it is can easily be seen that Z must meet b_1 or b_2, and in particular Z is contained in the open strip defined by z_1 and z_2. Thus, $\{X, Y, Z\}$ is totally separable with separation order ..X..Z..Y.. But since l properly separates Z from X and Y, $\{X, Y, Z\}$ cannot admit a common transversal. Thus there are at most four translates that do not meet l; if four, then either four that lie above l, or two that lie above l and two that lie below l.

This shows that the construction of theorem 6.2 cannot be extended in the sense of having $T - 5$, but not $T - 4$. This follows from the fact that the critical triple $\{B', C'E\}$ can be chosen to be a $2K$-separable triple.

The condition of the $2K$-separable critical triple may seem somewhat stringent; it is intended only to show that there exists a sufficient condition. The conclusion is the following: *There exists a $\lambda_0 \leq 2$ such that if F contains a critical triple that is $\lambda_0 K$-separable, then* $T(3) \Rightarrow T - 4$.

8. Acknowledgements

This paper is based on my cand.scient.thesis (Masters thesis) [10] at the University of Bergen. I owe special thanks to my advisor, Professor Helge Tverberg, for introducing me to transversal theory and making many helpful suggestions on my work. I also wish to thank my fellow student, Oddmar Eiksund, for performing a computer check which verified that the bad quadruples of theorem 6.2 are not representable by three elements.

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